Application of novel relaxation time approximation for the Boltzmann equation in relativistic fluid dynamics

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Abstract

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Ultrarelativistic heavy-ion collisions are modelled with relativistic fluid dynamics. Since quark-gluon plasma formed in collider experiments cannot be directly measured, the investigation of the substance is heavily based on comparing experimental data to predictions of theoretical models. In this thesis we review a novel relaxation time approximation for the relativistic Boltzmann equation, which is fully compatible with the macroscopic conservation laws. We calculate approximations for the temperature dependence of bulk viscosity, particle diffusion coefficient and shear viscosity of a fluid employing this model. The calculation is carried out by matching the fluid dynamical quantities with the underlying microscopic theory, where the single particle momentum distribution function is expressed using first order Chapman–Enskog expansion. In this way, even with energy-dependent relaxation times, we achieve consistent approximations for the fluid dynamical transport coefficients that are in full agreement with the second law of thermodynamics. The novel relaxation time approximation can be utilized to construct effective kinetic descriptions for matter in heavy-ion collisions.

Keywords: relaxation time, relativistic, Boltzmann equation, kinetic theory, hydrodynamics, fluid dynamics, Navier–Stokes, viscosity, relativity

Tiivistelmä

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Boltzmannin yhtälön uuden relaksaatioaika-approksimaation sovellus relativistisessa virtausmekaniikassa
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Ultrarelativististen raskasionitörmäysten mallintamisessa hyödynnetään relativistista virtausmekaniikkaa. Törmäyskokeissa syntyvän kvarkkigluoniplasman viskositeettia ei voida suoraan mitata, joten tämän tutkimus painottuu kokeellisen datan vertaamiseen teoriamallin ennustukseen. Tässä tutkielmassa tutustaan uudenlaiseen relaksaatioaika-approksimaatioon relativistiselle Boltzmannin yhtälölle, joka on yhteensopiva makroskooppisten säilymislakien kanssa. Tämän avulla lasketaan teoreettiset arviot fluidin puristusviskositeetin, diffuusiovakion ja leikkausviskositeetin lämpötilariippuvuudelle. Lasku toteutetaan sovittamalla virtausmekaniikan suureet taustalla olevaan mikroskooppiseen teoriaan, jossa yksihiukkastiheysfunktio esitetään ensimmäisen kertaluokan Chapman–Enskog -ekspansion avulla. Näin saadaan energiariippuvilla relaksaatioajoilla tuotettua virtausmekaniikan kuljetuskertoimille johdonmukaisia arvioita, jotka ovat yhteensopivia termodynamiikan toisen lain kanssa. Uuden relaksaatioaika-approksimaation avulla on mahdollista luoda efektiivisiä kineettisiä malleja kuvaamaan materiaa raskasionitörmäyksissä.

Avainsanat: relaksaatioaika, relativistinen, Boltzmannin yhtälö, kineettinen teoria, hydrodynamiikka, virtausmekaniikka, Navier–Stokes, viskositeetti, suhteellisuusteoria

Foreword

I thank my supervisor Harri Niemi for his time and knowledge over the course of the last few years. Although some of our discussions may have gone beyond the core topic of this thesis, I have found it especially interesting to follow his office lectures on fluid dynamics. I also want to thank my fellow students Tero Lappeteläinen and Kosmo Kukkonen for discussions concerning related topics, and my relatives for the hard task of trying to keeping my mind off physics.

Jyväskylä 12.5.2024

Mika Piipponen

Contents

1 Introduction

The strong force is the fundamental interaction between quarks and gluons described by quantum chromodynamics (QCD). The interaction is so strong that as the separation between two quarks becomes large enough, new quark-antiquark pairs are formed, which leads to the quarks confining into color neutral hadrons. This means that no free quarks can be detected. However, lattice QCD calculations predict a phase change at a temperature of the order of 158 MeV [1]. At temperatures higher than this, matter is expected to exist in a form where color charges are free, called *quark-gluon plasma* (QGP).

The primordial universe and possibly even the cores of neutron stars are believed to be so hot or dense that matter could exist as QGP. In order to strobe light on the moments just after Big Bang, astrophysical research and to test our understanding of the strong interaction, *ultrarelativistic heavy-ion collisions* are carried out at the Brookhaven National Laboratory's Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC) at CERN.

Heavy-ion collision are conducted by stripping large atoms, usually lead (Pb) or gold (Au), of their electrons, accelerating the bare nuclei near lightspeed and colliding them head on (see Fig. 1). In the effective collision area, a large number of particle-antiparticle pairs are produced [2, 3], drastically increasing particle density. The remnants of the collided nuclei outside the overlap area continue on their paths, leaving a strongly interacting droplet of QGP between each other. Due to the shape of the colliding nuclei, the effective collision area is oval shaped, causing an anisotropic pressure gradient that forces the QGP to flow in to the transverse direction from the

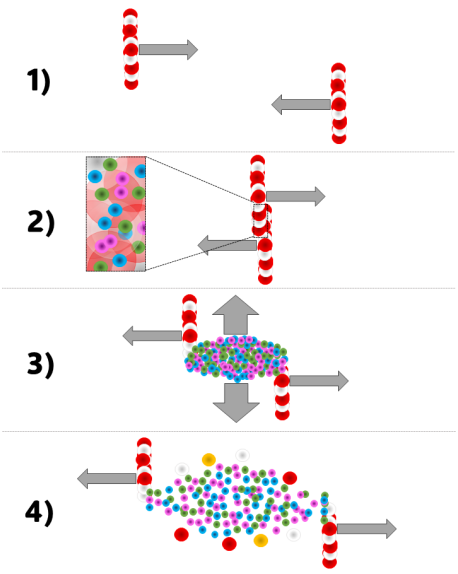

Figure 1. As two length contracted nuclei (1) collide, the constituent hadrons in the overlap area (2) break into a mix of quarks and gluons (3). Then the QGP expands and confines into hadrons (4).

beam axis. Once the QGP has expanded and cooled enough, the quarks will once again confine into hadrons that eventually reach the detectors.

For a good understanding of this extreme state of matter, the experimental results should be somehow linked to theory. Ideally we would have a theoretical understanding that would reproduce the collision exactly. It is practically impossible to know the initial stage of each collision exactly, the matter undergoes a phase transition during the event and the state of matter we are interested in cannot even be directly detected. Therefore the theoretical model used to describe these events is quite complex, consisting of several stages, each using an effective theory describing the dynamics of the matter in the best possible way.

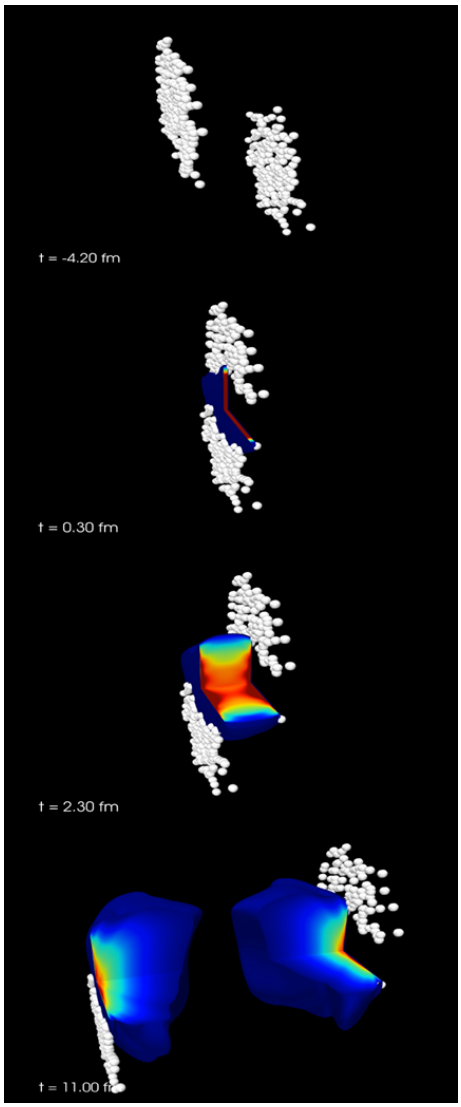

Figure 2. Time evolution of quark gluon plasma created in a heavy-ion collision simulated using transient relativistic fluid dynamics. (Harri Niemi)

One way to construct such a model is to recover the initial stage energy distributions of the colliding nuclei from a perturbative QCD model [4]. The evolution of the formed QGP is simulated by Israel–Stewart-type relativistic dissipative fluid dynamics [5, 6, 7] (see Fig. 2). This way the complicated microscopic degrees of freedom have been integrated out of the calculations and replaced by evolution of macroscopic thermodynamical quantities. The fluid dynamical simulation is carried out from the formation of the QGP all the way until the matter has cooled into a hadron gas phase. It is straightforward to model the QCD phase transition with fluid dynamics as the properties of both states of matter are embedded into an equation of state and transport coefficients of the fluid. The fluid dynamical evolution is ended e.g. at a constant temperature spacetime hypersurface, where the fluid dynamical quantities are converted into a particle distribution by a Cooper–Frye integral procedure [8]. Under these considerations, relativistic fluid dynamics is an essential tool in

understanding the properties of QCD matter in high energy collisions.

Experimentally the formation and properties of QGP can be studied by examining the Fourier decomposition of the final state particle azimuthal distribution

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_{n=1}^{\infty} 2v_n \cos\big(n\Delta\phi\big)\right) . \tag{1.1}$$

The Fourier coefficients v_n represent the flow harmonics present in the spectrum, the second of which represents elliptic flow. The fact that the final particle distribution shows signs of elliptic flow and is consistent with fluid dynamical predictions is a strong indication that QGP is infact formed in heavy-ion collisions. Even though stretching the boundaries of applicability of fluid dynamics, surprisingly even proton-proton collisions have shown collective behavior as a fluid [9].

This thesis will revolve around relativistic fluid dynamics and kinetic theory. We will review a novel relaxation time approximation [10] for the relativistic Boltzmann equation by Rocha, Denicol and Noronha. Phenomenological models like relaxation time approximations are of interest, because they offer a way to give effective kinetic descriptions of matter without necessarily knowing the exact microscopic dynamics in question. The revised collision model by Rocha et. al. will be used to calculate the transport coefficients appearing in relativistic fluid dynamics. Transport coefficients are usually obtained by matching the fluid dynamical variables to kinetic theory. Calculations incorporating the full collision integral become increasingly difficult [11] and the ones for a massless gas with constant scattering cross sections [7, 12] do not necessarily have much to do with the evolution of QGP. The commonly used relaxation time approximation [13] for the Boltzmann equation by Anderson and Witting yields inappropriate values for the bulk viscosity and particle diffusion coefficient, and is inconsis-

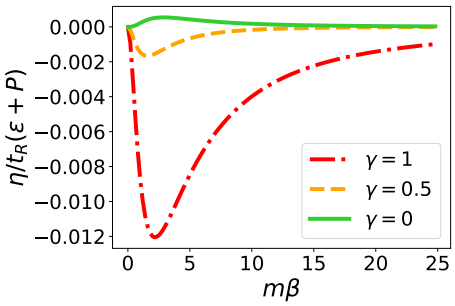

Figure 3. Dimensionless bulk viscosity for a Boltzmann gas as a function of $m\beta$ in the AW relaxation time approximation.

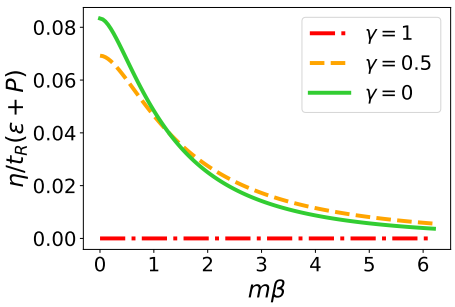

Figure 4. Dimensionless particle diffusion coefficient for a Boltzmann gas as a function of $m\beta$ in the AW relaxation time approximation.

tent with basic thermodynamics. For example relaxation time $\tau \propto \sqrt{E_p}$, that is argued to give effective descriptions of QCD matter [14, 15], suggests negative values for the bulk viscosity and $\tau \propto E_p$ vanishing particle diffusion (see Figs. 3 and 4). Also the macroscopic conservation laws are not intrinsic in the Anderson–Witting approximation, which is an essential feature of the Boltzmann equation. The Novel relaxation time approximation eliminates these problems by taking into account these homogeneous solutions of the Boltzmann equation. This method yields updated values for the bulk viscosity and particle diffusion coefficient, while the shear viscosity remains unchanged.

In section 2 of this text, we will familiarize with the relativistic kinetic theory and define the needed machinery and variables. First we define the net particle 4-current, energy-momentum tensor, introduce the relativistic version of the Boltzmann transport equation and the concept of the relaxation time approximation. The effect of modifying the microscopic dynamics in the Boltzmann equation can be tested by observing how it affects the macroscopic dynamics of matter and its material properties. Naturally in this case we want to see how altering the underlying kinetic theory affects the fluid dynamical description.

In section 3 we dive into relativistic fluid dynamics and derive the equations of motion for relativistic fluids keeping careful track of the microscopic definitions of our variables. Specifically we are interested in the expression for the bulk pressure, particle diffusion current and the shear stress tensor.

Finally in sections 4 and 5 we calculate the bulk viscosity, particle diffusion coefficient and shear viscosity by identifying them from the above mentioned quantities. This is done by employing the first order Chapman–Enskog expansion and solving the resulting integral expressions numerically to obtain the temperature dependence of the transport coefficients.

2 Relativistic kinetic theory

Microscopically matter consists of countless number of particles. Kinetic theory of gases models the properties of a gas as a consequence of the motion of the constituent particles. This way kinetic theory gives the quantities of a macroscopic state a microscopic definition. For example the temperature of a gas is the mean kinetic energy of the particles and pressure the overall force from the particles colliding to the surface of a volume element. Since a gas consists of almost innumerable number of particles, using the molecular dynamics, that are often of quantum nature, for individual particles is practically impossible and the framework of statistical physics is needed.

Kinetic theory concerns the statistical properties of a local ensemble of particles using a *single particle momentum distribution function* $f(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{x}$ and $\boldsymbol{p}$ are the particle's position and momentum, respectively (see Fig. 5). $f(\boldsymbol{x}, \boldsymbol{p})$ can be interpreted as a probability density, which tells one the number of particles within the infinitesimal interval $\boldsymbol{x} \in [\boldsymbol{x}, \boldsymbol{x} + d\boldsymbol{x}]$ with momentum $\boldsymbol{p} \in [\boldsymbol{p}, \boldsymbol{p} + d\boldsymbol{p}]$. [16, 17, 18]

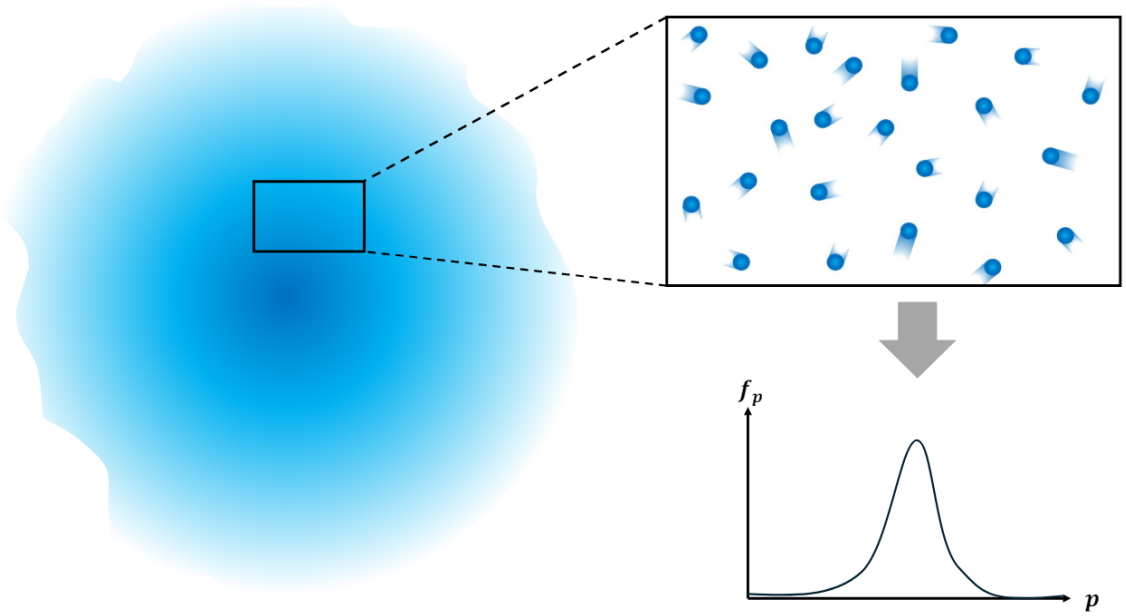

Figure 5. The function $f(\boldsymbol{x}, \boldsymbol{p})$ describes the propability of finding a particle of momentum $\boldsymbol{p}$ at the position $\boldsymbol{x}$.

In this section we will see how the macroscopic state variables can be expressed in terms of the first two moments of the distribution function $f(\boldsymbol{x}, \boldsymbol{p})$ and heuristically derive a transport equation, which $f(\boldsymbol{x}, \boldsymbol{p})$ satisfies. In most cases the form of $f(\boldsymbol{x}, \boldsymbol{p})$ is a priori unknown but in principle it can be solved from the transport equation. Finally we will consider an approximation valid for a gas sufficiently close to the state of thermodynamic equilibrium and introduce relaxation time approximations on phenomenological arguments.

2.1 Conventions

Throughout the thesis we will be using natural units $c = \hbar = k_B = 1$ and the plus sign metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \tag{2.1}$$

customary in particle physics. Here c is the speed of light, $\hbar$ the reduced Planck's constant and k_B the Boltzmann constant. We are using Einstein summation convention with indices labeled by greek letters running from 0 to 3 and latin letters from 1 to 3, unless stated otherwise. Bold letters generally refer to 3-component vectors. Position in spacetime is represented by a contravariant coordinate 4-vector

$$x^\mu = (t, \boldsymbol{x}) . \tag{2.2}$$

4-velocity of an object is defined as the proper time derivative of it's coordinate vector

$$u^\mu = \frac{dx^\mu(\tau)}{d\tau} = \gamma(1, \boldsymbol{v}) , \tag{2.3}$$

where $\boldsymbol{v}$ is the 3-velocity and the Lorentz gamma factor is $\gamma = [1 - \boldsymbol{v}^2]^{-1/2}$. We will be decomposing 4-vectors into irreducible time-like and space-like components with respect to the 4-velocity frequently through this text. Time-like components (parallel to u^μ) of variables can be recovered by contracting them with u^μ and space-like components (orthogonal to u^μ) by contracting with the *3-space projector*

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu . \tag{2.4}$$

The *4-momentum* is defined as

$$p^\mu = \gamma(E, \boldsymbol{p}) = Eu^\mu + p^{\langle\mu\rangle} \ , \tag{2.5}$$

where we introduced a decomposition into components parallel and orthogonal to u^μ with the notation $p^{\langle\mu\rangle} = \Delta^\mu_\nu p^\nu$.

Furthermore we will for the most part adopt a more suppressed notation where the *4-gradient operator* is denoted as

$$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right) \equiv \partial_\mu = u_\mu \frac{d}{d\tau} + \nabla_\mu \ , \tag{2.6}$$

and the momentum dependence of a function as

$$f(\boldsymbol{p}) \equiv f_p \ . \tag{2.7}$$

In (2.6) ∇ is the usual gradient operator and $\nabla_\mu = \Delta^\nu_\mu \partial_\nu$ the *3-gradient*, the spatial part of 4-gradient operator, equivalent to ∇ in the local rest frame.

2.2 Particle 4-current N^μ

In terms of the single particle distribution function, the total particle number of a system can be expressed as an integral of f_p over all space and momenta

$$N = \int d^3x \, \frac{d^3p}{(2\pi)^3} \, f_p \, . \tag{2.8}$$

Quantities of the macroscopic state used in fluid dynamics, such as particle and energy density, can be linked to the single particle distribution function as well [16, 18]. The particle density can by equation (2.8) be represented as

$$N^0 = \int \frac{d^3p}{(2\pi)^3} \, f_p \, . \tag{2.9}$$

Particle flux to direction i depends on the particle density and the spatial velocity $v^i = p^i/p^0$ of the particles

$$N^i = \int \frac{d^3p}{(2\pi)^3 p^0} \, p^i f_p \, . \tag{2.10}$$

Equations (2.9) and (2.10) can be combined as a single 4-vector

$$N^\mu = \int \frac{d^3p}{(2\pi)^3 p^0} \, p^\mu f_p \, , \tag{2.11}$$

which is the *particle 4-current*. The particle current N^μ can be seen as the first moment of f_p, similarly as in the non-relativistic case.

Particle density in the fluids rest frame, where $u^\mu = (1, \mathbf{0})$, can be identified as

$$u_\mu N^\mu = \int \frac{d^3p}{(2\pi)^3 p^0} \, u_\mu p^\mu f_p = n \, . \tag{2.12}$$

2.3 Energy-momentum tensor $T^{\mu\nu}$

Energy density can be written as

$$T^{00} = \int \frac{d^3p}{(2\pi)^3}\, p^0 f_p \,. \tag{2.13}$$

Momentum density and energy flux into direction i

$$T^{0i} = T^{i0} = \int \frac{d^3p}{(2\pi)^3}\, p^i f_p \,, \tag{2.14}$$

and i-component of momentum flux into direction j

$$T^{ij} = \int \frac{d^3p}{(2\pi)^3 p^0}\, p^i p^j f_p \,. \tag{2.15}$$

These quantities can be combined in a 2nd rank tensor

$$T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0}\, p^\mu p^\nu f_p \,, \tag{2.16}$$

which is the second moment of f_p and is called the *energy-momentum tensor*.

Furthermore, the energy density in the fluids rest frame and isotropic pressure can be identified as

$$u_\mu u_\nu T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3 p^0}\, u_\mu p^\mu u_\nu p^\nu f_p = \varepsilon \,, \tag{2.17}$$

$$-\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} = \frac{1}{3}\int \frac{d^3p}{(2\pi)^3 p^0}\, (E_p^2 - m^2) f_p = \frac{1}{3}\int \frac{d^3p}{(2\pi)^3 p^0}\, |\boldsymbol{p}|^2 f_p = P \,. \tag{2.18}$$

It is to be noted that (2.12), (2.17) and (2.18) describe the quantities in the local rest frame of the fluid in an arbitrary state, that are not necessarily the same as the respective equilibrium quantities.

From now on it is convenient to denote the Lorentz invariant integration measure as

$$\frac{d^3p}{(2\pi)^3 p^0} \equiv d\boldsymbol{P} \,. \tag{2.19}$$

2.4 Conservation laws

Using N^μ and $T^{\mu\nu}$, let's consider the net flow of particles, energy and momentum through a arbitrary spacetime volume V_4, which is enclosed by a 3-dimensional surface $\mathcal{S}$ (see Fig. 6). If particle number is conserved, the particle flux through $\mathcal{S}$ must be zero

$$\int_{\mathcal{S}} d\mathcal{S}\ \mathbf{n}_\mu N^\mu = 0\ , \tag{2.20}$$

Energy and momentum fluxes through $\mathcal{S}$ must also vanish

$$\int_{\mathcal{S}} d\mathcal{S}\ \mathbf{n}_\mu T^{\mu\nu} = 0\ . \tag{2.21}$$

By the divergence theorem (2.20) and (2.21) become

$$\int_{V_4} d^4x\ \partial_\mu N^\mu = 0\ , \tag{2.22}$$

$$\int_{V_4} d^4x\ \partial_\mu T^{\mu\nu} = 0\ . \tag{2.23}$$

Since the conservation must hold for all space, the divergences of N^μ and $T^{\mu\nu}$ must vanish

$$\partial_\mu N^\mu = 0\ , \tag{2.24}$$

$$\partial_\mu T^{\mu\nu} = 0\ . \tag{2.25}$$

Equations (2.24) and (2.25) are the macroscopic conservation laws that must hold for any system in order to behave physically. They will yield the fluid dynamical equations of motion and play a crucial role in the following sections.

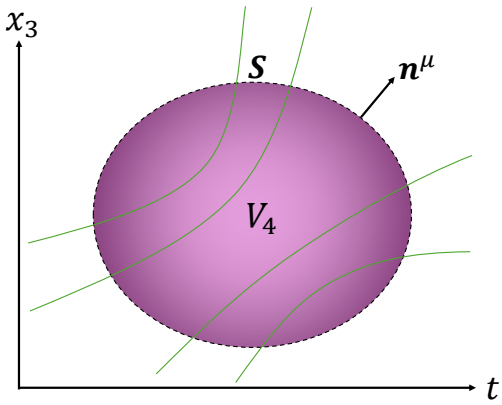

Figure 6. The net particle, energy and momentum flow through the volume V_4 must be conserved.

2.5 Boltzmann transport equation

In this subsection we will familiarize to the microscopic transport of particles and derive the simplest kinetic transport equation that the distribution function f_p satisfies in the case of sufficiently rarefied gases.

Following references [16, 17, 19], let us again consider a 4-dimensional spacetime volume V_4 enclosed by a 3-dimensional surface $\mathcal{S}$. We assume V_4 is big enough to consider the statistical properties of the ensemble of particles inside it, but at the same time small enough for f_p not to vary significantly over its region. The microscopic transport of particles and instantaneous values of f_p will depend on free streaming and scattering particles entering and exiting the phase space volume.

The flow of particles through $\mathcal{S}$ near the momentum state $\boldsymbol{p}$ can be expressed as

$$\Delta N_p = \int_{\mathcal{S}} d\mathcal{S}\ \mathbf{n}_\mu N_p^\mu = \int_{\mathcal{S}} d\mathcal{S}\ \mathbf{n}_\mu \int_{\Delta \boldsymbol{p}} d\boldsymbol{P}\ p^\mu f_p = 0\ , \tag{2.26}$$

and by using the divergence theorem

$$\int_{V_4} d^4x\ \partial_\mu \int_{\Delta \boldsymbol{p}} d\boldsymbol{P}\ p^\mu f_p = 0\ . \tag{2.27}$$

Since V_4 and $\Delta \boldsymbol{p}$ are arbitrary and p^μ does not have any position dependence, particle conservation can be represented as

$$p^\mu \partial_\mu f_p = 0\ . \tag{2.28}$$

However, this only applies to very sparse set of free streaming particles where the mean free path is larger than the size of V_4 and no contribution of interparticle collisions is taken into account.

In reality particles scatter around chaotically and a gas is driven towards thermodynamic equilibrium by the interparticle collisions. Particles can scatter in or out of the momentum state $\boldsymbol{p}$, or the Minkowski space volume V_4. This is accounted by introducing a collision term that describes the loss and gain of particles in the phase space volume $V_4 \cdot \Delta \boldsymbol{p}$ due to the scattering events. Assuming the the gas is dilute enough, such that the mean free time between collisions is much greater that the actual collision events and that the microscopic dynamics is dominated by 2 to 2 processes, it suffices to account only for elastic binary particle collisions with incoming momenta $\boldsymbol{p}$ and $\boldsymbol{p'}$ and outgoing momenta $\boldsymbol{k}$ and $\boldsymbol{k'}$ (see Fig. 7). With the

further assumption that the particle momenta are uncorrelated, the collision term can be written as

$$\mathcal{C}_f = \frac{1}{2}\int d\boldsymbol{P}'\, d\boldsymbol{K}\, d\boldsymbol{K}'\, W_{pp'\leftrightarrow kk'}\left(f_k f_{k'}\bar{f}_p\bar{f}_{p'} - f_p f_{p'}\bar{f}_k\bar{f}_{k'}\right) . \tag{2.29}$$

The factor $\frac{1}{2}$ prevents from double counting identical particles, $W_{pp'\leftrightarrow kk'}$ is the transition rate between the momentum states that contains the information of the individual scattering processes and the term $\bar{f}_p = 1 - af_p$ accounts for quantum statistics, where the parameter a can get values -1, 1 or 0 for bosons, fermions and classical particles, respectively. The form (2.29) assumes detailed balance, or microscopic time reversibility of the transition rate ($W_{pp'\leftrightarrow kk'} = W_{kk'\leftrightarrow pp'}$).

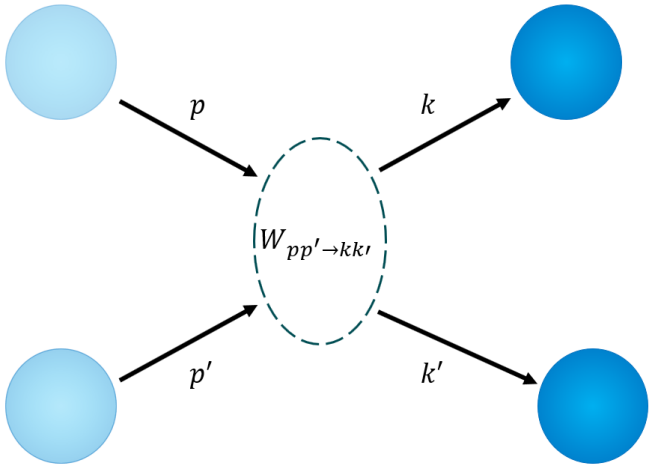

Figure 7. The Boltzmann equation accounts for elastic binary particle collisions of momenta $\boldsymbol{p} + \boldsymbol{p'} \to \boldsymbol{k} + \boldsymbol{k'}$.

Introducing the collision term changes equation (2.28) into the primary statement of kinetic theory, the *relativistic Boltzmann transport equation*

$$p^\mu \partial_\mu f_p = \frac{1}{2}\int d\boldsymbol{P}'\, d\boldsymbol{K}\, d\boldsymbol{K}'\, W_{pp'\leftrightarrow kk'}\left(f_k f_{k'}\bar{f}_p\bar{f}_{p'} - f_p f_{p'}\bar{f}_k\bar{f}_{k'}\right) , \tag{2.30}$$

which describes the balance between free streaming particles and collisions that drive the gas towards thermodynamic equilibrium. The Boltzmann equation is a 9-fold integro-differential equation of the single particle momentum distribution function f_p, and the transition rate contains momentum dependent scattering amplitudes of quantum nature, which generally make it very hard to solve. For this reason there are many simplified models, which allow the efficient use of the Boltzmann equation in its applications.

2.5.1 Equilibrium distribution

In thermodynamic equilibrium the initial and final state particles have the same momentum distribution function and the collision term vanishes, $\mathcal{C}_f = 0$, which has a solution of the form

$$f_{0p} = \frac{1}{e^{\beta u_\mu p^\mu - \alpha} + a} \,, \tag{2.31}$$

which corresponds to the Maxwell–Jüttner, Fermi–Dirac or Bose–Einstein distributions for the different values of a. Here $u_\mu p^\mu = E_p$ is the energy of the particle in the fluids rest frame and

$$\alpha = \frac{\mu}{T} \,, \tag{2.32}$$

$$\beta = \frac{1}{T} \,, \tag{2.33}$$

are the thermal potential (chemical potential/temperature) and inverse temperature, respectively. The covariant formulation differs from its classical counterparts as it prevents the particles from exceeding the speed of light and thus their velocity distribution now has also an upper limit.

2.5.2 Collision invariants

Lets consider the following contraction of the collision term with a function ψ_p

$$\begin{aligned}
\int d\boldsymbol{P}\, \psi_p \mathcal{C}_f &= \frac{1}{2} \int d\boldsymbol{P}\, d\boldsymbol{P'}\, d\boldsymbol{K}\, d\boldsymbol{K'}\, \psi_p W_{pp'\leftrightarrow kk'} \Big(f_k f_{k'} \bar{f}_p \bar{f}_{p'} - \underbrace{f_p f_{p'} \bar{f}_k \bar{f}_{k'}}_{p\leftrightarrow k \;\&\; p'\leftrightarrow k'} \Big) \\
&= \frac{1}{2} \int \underbrace{d\boldsymbol{P}\, d\boldsymbol{P'}\, d\boldsymbol{K}\, d\boldsymbol{K'}\, \Big(\psi_p - \psi_k\Big) W_{pp'\leftrightarrow kk'} f_k f_{k'} \bar{f}_p \bar{f}_{p'}}_{p\leftrightarrow p' \;\&\; k\leftrightarrow k'} \\
&= \frac{1}{2} \int d\boldsymbol{P}\, d\boldsymbol{P'}\, d\boldsymbol{K}\, d\boldsymbol{K'}\, \Big(\psi_{p'} - \psi_{k'}\Big) W_{pp'\leftrightarrow kk'} f_k f_{k'} \bar{f}_p \bar{f}_{p'} \,,
\end{aligned} \tag{2.34}$$

where we interchanged the integration variables in the latter integral on the first line and took advantage of the microscopic time reversal symmetry of the transition rate. Now summing the last two lines together and dividing by two we get

$$\int d\boldsymbol{P}\, \psi_p \mathcal{C}_f = \frac{1}{4} \int d\boldsymbol{P}\, d\boldsymbol{P'}\, d\boldsymbol{K}\, d\boldsymbol{K'}\, \Big(\psi_p + \psi_{p'} - \psi_k - \psi_{k'}\Big) W_{pp'\leftrightarrow kk'} f_k f_{k'} \bar{f}_p \bar{f}_{p'} \,. \tag{2.35}$$

From here we see that in the case that $\psi_p \propto 1$ or $\psi_p \propto p^\mu$ and using the 4-momentum conservation, the contraction (2.35) gives zero. This means the collision term has

five zero eigenvalue eigenfunctions, or so called collision invariants, namely 1 and p^μ [16, 17]. Now contracting the Boltzmann equation with 1 and p^ν gives

$$\partial_\mu \underbrace{\int d\boldsymbol{P}\; p^\mu f_p}_{=N^\mu} = \underbrace{\int d\boldsymbol{P}\; 1\, \mathcal{C}_f}_{=0,(2.35)} \,, \tag{2.36}$$

$$\partial_\mu \underbrace{\int d\boldsymbol{P}\; p^\mu p^\nu f_p}_{=T^{\mu\nu}} = \underbrace{\int d\boldsymbol{P}\; p^\nu \mathcal{C}_f}_{=0,(2.35)} \,, \tag{2.37}$$

which reproduces the conservation laws (2.24) and (2.25) exactly. To conclude, the collision invariants ensure the local conservation of the particle number, energy and momentum, and by that, this kinetic description of a gas by the Boltzmann equation is in perfect agreement with the macroscopic laws.

2.5.3 Linearized Boltzmann equation

If the system can be considered to be sufficiently close to equilibrium it is possible to approximate the collision term to linear order. The single particle momentum distribution in an arbitrary state can be expressed as

$$f_p = f_{0p} + \delta f \equiv f_{0p}(1 + \phi_p) \,, \tag{2.38}$$

where ϕ_p is an auxiliary correction term characterizing the deviation from equilibrium. Expanding f_p this way introduces 4-velocity, chemical potential and temperature into the Boltzmann equation. For a system close to equilibrium ($\phi_p \ll 1$), it is possible to linearize the Boltzmann equation in the deviations from equilibrium by using (2.38) and dropping off all the second order terms in ϕ_p and other first order small quantities. Linearizing the right hand side of the Boltzmann equation (of classical gas for simplicity) gives the linearized collision term

$$\begin{aligned}
\mathcal{C}_f &= \int d\boldsymbol{P}\,' d\boldsymbol{K}\; d\boldsymbol{K}'\; W_{pp'\leftrightarrow kk'} \Big[f_{0k}(1+\phi_k) f_{0k'}(1+\phi_{k'}) - f_{0p}(1+\phi_p) f_{0p'}(1+\phi_{p'}) \Big] \\
&= \int d\boldsymbol{P}\,' d\boldsymbol{K}\; d\boldsymbol{K}'\; W_{pp'\leftrightarrow kk'} \Big[f_{0k} f_{0k'}(1 + \phi_k + \phi_{k'} + \phi_k\phi_{k'}) \\
&\qquad\qquad - f_{0p} f_{0p'}(1 + \phi_p + \phi_{p'} + \phi_p\phi_{p'}) \Big] \\
&\approx \int d\boldsymbol{P}\,' d\boldsymbol{K}\; d\boldsymbol{K}'\; W_{pp'\leftrightarrow kk'} f_{0p} f_{0p'} \Big(\phi_k + \phi_{k'} - \phi_p - \phi_{p'} \Big) \equiv \hat{L}\phi_p \,,
\end{aligned} \tag{2.39}$$

where we used the 4-momentum conservation in the last line in order to see $f_{0k}f_{0k'} = f_{0p}f_{0p'}$ and defined the linearized collision operator $\hat{L}$.

From (2.39) we notice the linearized collision operator $\hat{L}$ satisfies the properties

$$\hat{L}1 = 0 \,, \tag{2.40}$$

$$\hat{L}p^{\mu} = 0 \,, \tag{2.41}$$

and, due to similar considerations as for (2.35), is self-adjoint

$$\int d\boldsymbol{P}\ \psi_p\ \hat{L}\phi_p = \int d\boldsymbol{P}\ \phi_p\ \hat{L}\psi_p \,, \tag{2.42}$$

and as such satisfies also

$$\int d\boldsymbol{P}\ 1\ \hat{L}\phi_p = 0 \,, \tag{2.43}$$

$$\int d\boldsymbol{P}\ p^{\mu}\hat{L}\phi_p = 0 \,. \tag{2.44}$$

2.5.4 Relaxation time approximation

A further simplification, the relaxation time approximation, was originally proposed Bhatnagar, Gross and Krook [20] and has seen wide use in numerical lattice simulations. A relativistic version was proposed by Marle, [21] and later refined by Anderson and Witting [13], with the advantage of acquiring fluid dynamical transport coefficients more easily. The idea of relaxation time approximation is that the momentum distribution function f_p relaxes towards the equilibrium distribution f_{0p} on a characteristic time scale τ

$$\hat{L}_{\mathrm{RTA}}\phi_p = \frac{E_p}{\tau}\Big(f_{0p} - f_p\Big) = -\frac{E_p}{\tau}\delta f \,. \tag{2.45}$$

This is a phenomenological approach in representing the collision term, where the microscopic dynamics are parametrized by the relaxation time. Increasing τ leads to slower relaxation into equilibrium which is the same as saying the mean free time between collisions is longer and viscosity of the fluid greater, and vice versa.

The Anderson–Witting formulation (2.45) is a drastic simplification over the original integral term, but as such it is flawed. Formally the Anderson–Witting

approximation is a linear operator

$$\hat{L}_{\mathrm{RTA}}|P_m\rangle \sim -\mathbb{1}|P_m\rangle \neq 0 \ , \tag{2.46}$$

that gives only non-zero values. A similar consideration as in (2.36) and (2.37), using the kinetic definitions for N^μ and $T^{\mu\nu}$ in equations (2.11) and (2.16) the Boltzmann equation now gives

$$\partial_\mu N^\mu = -\int d\boldsymbol{P}\ \frac{E_p}{\tau}\delta f \ , \tag{2.47}$$

$$\partial_\mu T^{\mu\nu} = -\int d\boldsymbol{P}\ \frac{E_p}{\tau}p^\nu \delta f \ , \tag{2.48}$$

which is a clear violation of the macroscopic conservation laws unless one assumes momentum independent relaxation time and chooses to use the *Landau matching conditions*

$$\int d\boldsymbol{P}\, E_p \delta f = 0 \ , \tag{2.49}$$

$$\int d\boldsymbol{P}\ E_p p^\mu \delta f = 0 \ . \tag{2.50}$$

This is a problem since the conservation laws should be an intrinsic part of the theory and not require any further assumptions, and as the simple relaxation time approximation does not contain any zero eigenvalue eigenfunctions, it fails to account for the homogenous solutions of the Boltzmann equation. One also should expect some energy dependence for the relaxation time because it is directly related to the mean free time between collisions that should change for different momenta. For this reason some of the transport coefficients recovered from this approximation yield improper values. Despite of violating the basic properties of the Boltzmann equation, the Anderson–Witting approximation has been used in order to obtain expressions for transport coefficients in fluid dynamics.

2.5.5 Novel relaxation time approximation

In a recent publication Rocha, Denicol and Noronha [10] proposed a *novel relaxation time approximation* (NRTA) for the collision term. The NRTA preserves the properties of the linearized collision operator by adding counterterms in its five orthonormal zero eigenvalue eigenfunctions. This forces the collision operator to give zero when contracted with any of the orthonormal eigenfunctions. This can be formally represented as

$$\hat{L}_{\text{NRTA}}|P_m\rangle \sim \left(-\mathbb{1} + \sum_{n=1}^{5} |P_n\rangle\langle P_n|\right)|P_m\rangle = -|P_m\rangle + |P_n\rangle\delta_{nm} = 0 \,. \tag{2.51}$$

Following Rocha et al. [10], we manage to construct $\hat{L}_{\text{NRTA}}$ by expressing the eigenfunctions 1 and p^μ, or equivalently 1, $p^{\langle\mu\rangle}$ and E_p, in an orthogonal basis demanding the vectors are orthogonal in the inner product

$$\int d\boldsymbol{P}\, \frac{E_p}{\tau} P_n P_m f_{0p} \propto \delta_{nm} \,. \tag{2.52}$$

It is enough to orthogonalize 1 with E_p, which is possible using the Gram–Schmidt orthogonalization method (see appendix A). With the help of (A.11) one gets the orthogonal basis

$$P_0^{(0)} = 1 \,, \quad P_1^{(0)} = 1 - \frac{\langle E_p/\tau\rangle_0}{\langle E_p^2/\tau\rangle_0} E_p \,, \quad p^{\langle\mu\rangle} = \Delta^\mu_\nu p^\nu \,, \tag{2.53}$$

where we introduced a further suppressed notation for brevity

$$\int d\boldsymbol{P}\, \cdots f_{0p} \equiv \langle\cdots\rangle_0 \,. \tag{2.54}$$

With the basis described in (2.53), a general form for the equation (2.51) can be written as

$$\hat{L}_{\text{NRTA}}\phi_p = -\frac{E_p}{\tau} f_{0p}\Big(\phi_p - A P_0^{(0)} - B P_1^{(0)} - C_\mu p^{\langle\mu\rangle}\Big) \,, \tag{2.55}$$

where A, B and C_μ are some momentum independent coefficients. Using the orthogonality of the basis elements

$$\Big\langle (E_p/\tau) P_0^{(0)} P_1^{(0)} \Big\rangle_0 = \Big\langle (E_p/\tau) P_0^{(0)} p^{\langle\mu\rangle} \Big\rangle_0 = \Big\langle (E_p/\tau) P_1^{(0)} p^{\langle\mu\rangle} \Big\rangle_0 = 0 \,, \tag{2.56}$$

the coefficients A, B and C_μ can be found (see Appendix B)

$$A = \frac{\left\langle (E_p/\tau) P_0^{(0)} \phi_p \right\rangle_0}{\left\langle (E_p/\tau) P_0^{(0)} P_0^{(0)} \right\rangle_0} , \tag{2.57}$$

$$B = \frac{\left\langle (E_p/\tau) P_1^{(0)} \phi_p \right\rangle_0}{\left\langle (E_p/\tau) P_1^{(0)} P_1^{(0)} \right\rangle_0} , \tag{2.58}$$

$$C_\mu = 3 \frac{\left\langle (E_p/\tau) p_{\langle\mu\rangle} \phi_p \right\rangle_0}{\left\langle (E_p/\tau) p_{\langle\nu\rangle} p^{\langle\nu\rangle} \right\rangle_0} . \tag{2.59}$$

Here we made use of the fact that integrals of projected tensors of momenta satisfy

$$\int d\boldsymbol{P}\; F_p\, p^{\langle\mu_1} \cdots p^{\mu_m\rangle} p_{\langle\mu_1} \cdots p_{\mu_n\rangle} = \frac{m! \delta_{mn}}{(2m+1)!!} \Delta^{\mu_1 \cdots \mu_m}_{\mu_1 \cdots \mu_m} \int d\boldsymbol{P}\; F_p \Big(\Delta_{\alpha\beta} p^\alpha p^\beta \Big)^m , \tag{2.60}$$

where F_p is an arbitrary function of energy [16].

This leads to

$$\hat{L}_{\mathrm{NRTA}} \phi_p = -\frac{E_p}{\tau} f_{0p} \left(\phi_p - \frac{\left\langle \frac{E_p}{\tau} \phi_p \right\rangle_0}{\left\langle \frac{E_p}{\tau} \right\rangle_0} - \frac{\left\langle \frac{E_p}{\tau} P_1^{(0)} \phi_p \right\rangle_0}{\left\langle \frac{E_p}{\tau} P_1^{(0)} P_1^{(0)} \right\rangle_0} P_1^{(0)} - 3 \frac{\left\langle \frac{E_p}{\tau} p_{\langle\mu\rangle} \phi_p \right\rangle_0}{\left\langle \frac{E_p}{\tau} p_{\langle\nu\rangle} p^{\langle\nu\rangle} \right\rangle_0} p^{\langle\mu\rangle} \right) . \tag{2.61}$$

This is the novel relaxation time approximation, constructed in such a way that it gives zero when contracted with any of the collision invariants, thus accounting for the homogeneous solutions of the Boltzmann equation. Furthermore, the NRTA preserves the properties (2.40)-(2.42). With this form of the collision term, the transport coefficients of the fluid can be consistently approximated without solving the full collision integral [10, 22, 23].

Up to this point, we have seen how the microscopic dynamics of a gas can be cast into a dynamical equation of the distribution function f_p, and how it describes the balance between free streaming particles and interparticle collisions. The collision term was then simplified to its final form by the phenomenological argument that the collisions drive the gas towards equilibrium, while explicitly restoring the five collision invariants of the full theory. In the following section we will concentrate on the macroscopic dynamics of the fluid, where we will see the emergence of transport coefficients and how the information of the microscopic dynamics is embedded into their definitions.

3 Relativistic fluid dynamics

Fluid dynamics is a very practical approach in studying flow of matter in wide range of situations from aerodynamics to weather forecasts or fluid flow in industrial applications. In contrast to the kinetic theory of gases, fluid dynamics gives a macroscopic description of flowing matter using spatially averaged field quantities such as flow velocity, density, pressure and temperature. This however imposes limitations on the applicability of fluid dynamics because all of the above are statistical properties of an ensemble of particles. In order for them to be well defined, the system in question must generally show separation between the microscopic and macroscopic scales, usually referred to by the *Knudsen number* $\mathrm{Kn} = l_{\mathrm{mfp}}/L$, where l_{mfp} is the mean free path and L the characteristic macroscopic length of the system.

The separation between length scales in the system is clear at small Knudsen numbers and the matter can be considered as a continuum of small fluid elements. This guarantees each fluid element has enough particles to be described by statistical averaged quantities, governed by differential relations manifesting the conservation of mass, momentum and energy. These are namely the Navier–Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \,, \tag{3.1}$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \big(\boldsymbol{u} \cdot \nabla\big)\boldsymbol{u} \right) = -\nabla P + \nabla \cdot \mathcal{T} + \rho \boldsymbol{g} \,, \tag{3.2}$$

$$\rho \left(\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \boldsymbol{u}) \right) = -P\big(\nabla \cdot \boldsymbol{u}\big) + \kappa \nabla^2 T + \nabla \cdot \big(\mathcal{T} \cdot \boldsymbol{u}\big) \,, \tag{3.3}$$

where the components of the viscous stress tensor $\mathcal{T}$ are

$$\mathcal{T}^{ij} = \eta \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} - \frac{2}{3}\big(\nabla \cdot \boldsymbol{u}\big)\delta^{ij} \right) + \zeta \big(\nabla \cdot \boldsymbol{u}\big)\delta^{ij} \,. \tag{3.4}$$

Equations (3.1)-(3.3) really only apply to relatively small flow velocities. If one were to use fluid dynamics in an application where the system is likely to exhibit

relativistic velocities, the classical theory is not applicable anymore and relativistic fluid dynamics is needed. This kind of systems are common in cosmology, astrophysics and particle physics. As already mentioned, the main motivation concerning this thesis is modeling the quark-gluon-plasma created in ultrarelativistic heavy-ion collisions as a droplet of fluid.

In relativity the mass-energy equivalence renders the mass continuity equation (3.1) spurious. Mass is not necessarily conserved in a relativistic system where energy and momentum are. However, in a relativistic theory, it is possible to formulate a continuity equation for a conserved particle number, as Eckart [24] proposed. The net particle number is not conserved in heavy-ion collisions, but the net baryon number can be taken as a conserved quantity, for which we use the equation (2.24). The conserved quantities in relativistic fluid dynamics are thus baryon number, energy and momentum, that were already discussed in the previous section.

The spacetime evolution of a relativistic fluid can be completely described by the energy-momentum tensor $T^{\mu\nu}$, baryon 4-current N^{μ}_{B} (the subscript will be omitted in the following discussion) together with information of the material properties encoded in the equation of state and transport coefficients. In this section we will derive the equations of motion considering fluids with and without viscosity, and see the emergence of transport coefficients in the case where the fluid is no longer considered to be in local thermodynamic equilibrium.

3.1 Ideal relativistic fluid dynamics

Considering an ideal fluid where thermodynamic limit could be reached in the fluid elements, every fluid element could be considered to be in a local thermodynamic equilibrium. This assumption greatly simplifies the equations of motion of the fluid and assigns a well defined temperature for each fluid element. In fact the dynamics of an ideal fluid can be closed in just 5 equations, but it turns out that such fluids do not have any viscosity, the material property of fluid encoding its resistance to flow which we are particularly interested in the case of QGP.

Generally the thermodynamics of out of equilibrium processes are not very well known and the theories of viscous fluids will be expanded within relatively small deviations from thermal equilibrium, so ideal fluid dynamics will form a solid ground for relativistic fluid dynamics.

3.1.1 Tensorial form of N^μ and $T^{\mu\nu}$

For an ideal fluid, the assumption of local thermal equilibrium guarantees, that the single particle distribution function f_p in the definitions (2.11) and (2.16) can be represented as the corresponding equilibrium distribution

$$f_p = f_{0p} = \frac{1}{e^{\beta p_\mu u^\mu - \alpha} + a} \, . \tag{3.5}$$

Since now f_p only depends only p^μ, u^μ and scalar quantities and the integration in (2.11) and (2.16) is done over momenta, N^μ and $T^{\mu\nu}$ can only depend on u^μ (and the metric $g^{\mu\nu}$). This means that the tensorial form of N^μ and $T^{\mu\nu}$ must be

$$N^\mu = n u^\mu \, , \tag{3.6}$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} = \Big(\varepsilon + P\Big) u^\mu u^\nu - P g^{\mu\nu} \, , \tag{3.7}$$

where we identify n, ε and P as the particle density, energy density and isotropic pressure in the fluids rest frame, respectively.

3.1.2 Equations of motion

The five conservation laws (2.24) and (2.25) and the expressions for particle 4-current (3.6) and the energy-momentum tensor (3.7) produce the equations of motion for an ideal relativistic fluid. The energy and momentum equations can be extracted from the energy-momentum conservation (2.25) considering the components parallel and orthogonal to u^μ. This is done by contracting the conservation law with the 4-velocity and the 3-space projector, respectively. The continuity equation for the particle number can be recovered directly from (2.24), the energy equation from

$$u_\nu \partial_\mu T^{\mu\nu} = 0 \,, \tag{3.8}$$

and momentum equations from

$$\Delta^\mu_\beta \partial_\alpha T^{\alpha\beta} = 0 \,, \tag{3.9}$$

with the help of (3.6) and (3.7).

For an ideal fluid the equations of motion read

$$\dot{n} + n\theta = 0 \,, \tag{3.10}$$

$$\Big(\varepsilon + P\Big)\dot{u}^\mu = \nabla^\mu P \,, \tag{3.11}$$

$$\dot{\varepsilon} + \Big(\varepsilon + P\Big)\theta = 0 \,, \tag{3.12}$$

where the divergence of 4-velocity, or the fluid element expansion rate, was denoted $\theta \equiv \partial_\mu u^\mu$ and the upper dot for proper time derivative, $\dot{n} \equiv \frac{dn}{d\tau} = u^\mu \partial_\mu n$. These are the relativistic generalization of Euler's equations.

The conservation laws yield 5 equations and there are 6 independent variables, u^μ, n, ε and P. However, in equilibrium, pressure can be related to the other other thermodynamic variables through an equation of state, $P = P(n, \varepsilon)$, which closes the set of equations governing non-viscous fluid flow.

3.1.3 Entropy conservation

The first law of thermodynamics can be written in covariant form [5, 6, 25, 26], which allows to write differential changes to an *entropy 4-current* S^μ as

$$dS^\mu = \beta u_\nu dT^{\mu\nu} - \alpha dN^\mu \ . \tag{3.13}$$

It is to be noted that covariant thermodynamics do not contain any additional information as projecting to the subspace parallel to u^μ returns traditional thermodynamical relations and contracting with Δ^α_μ gives trivially zero. Equation (3.13) suggests that the divergence of S^μ is

$$\partial_\mu S^\mu = \beta u_\nu \partial_\mu T^{\mu\nu} - \alpha \partial_\mu N^\mu \ . \tag{3.14}$$

Now the conservation laws (2.24) and (2.25) lead to entropy conservation in ideal fluids

$$\partial_\mu S^\mu = 0 \ . \tag{3.15}$$

3.2 Viscous relativistic fluid dynamics

In the case of heavy-ion collisions the total number of constituent particles makes the assumption of thermal equilibrium at least questionable and as discussed in the previous section, we are interested specifically in the viscosity of QGP, we need a theory where viscosity and dissipative processes occur. A viscous fluid has friction between the fluid layers which leads to dissipative processes and heat production. It means that the fluid will no longer be in local thermodynamic equilibrium in contrast to the ideal fluid. Concerning the mathematical theory this leads to a more complex system which can not be described by only 5 equations, but altogether 14 equations are needed to describe the dynamics of a viscous relativistic fluid.

3.2.1 Off-equilibrium corrections to N^μ and $T^{\mu\nu}$

To formulate a relativistic theory of a viscous fluid, the fluid is assumed to be sufficiently close to equilibrium. This allows to define a fictitious equilibrium state in which the equilibrium thermodynamic relations are assumed to still be valid. The off-equilibrium state introduces small deviations from this fictitious equilibrium state in the particle 4-current and energy-momentum tensor

$$N^\mu = \int d\boldsymbol{P}\; p^\mu \Big(f_{0p} + \delta f \Big) = n_0 u^\mu + j^\mu \;, \tag{3.16}$$

$$T^{\mu\nu} = \int d\boldsymbol{P}\; p^\mu p^\nu \Big(f_{0p} + \delta f \Big) = \varepsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu} + \tau^{\mu\nu} \;. \tag{3.17}$$

The off-equilibrium parts j^μ and $\tau^{\mu\nu}$ can be identified as a particle diffusion current and viscous stress tensor, respectively. The 4-vector j^μ can be split into projections parallel and orthogonal to u^μ

$$j^\mu = \underbrace{\big(u_\nu j^\nu\big)}_{\equiv \delta n} u^\mu + \underbrace{\Delta^\mu_\nu j^\nu}_{\equiv n^\mu} = \delta n u^\mu + n^\mu \,, \tag{3.18}$$

where δn is the *off-equilibrium particle density correction* and n^μ the *particle diffusion current* orthogonal to u^μ.

As $T^{\mu\nu}$ is symmetric, $\tau^{\mu\nu}$ can be decomposed into a part parallel to u^μ, a symmetric traceless part and a separate term containing the trace

$$\begin{aligned}\tau^{\mu\nu} &= \underbrace{\left(u_\alpha u_\beta \tau^{\alpha\beta}\right)}_{\equiv \delta e} u^\mu u^\nu + \underbrace{\left(\frac{1}{3}\Delta_{\alpha\beta}\tau^{\alpha\beta}\right)}_{\equiv -\Pi} \Delta^{\mu\nu} + \underbrace{\left(u_\beta \Delta^\mu_\alpha \tau^{\alpha\beta}\right)}_{\equiv h^\mu} u^\nu \\ &\quad + \underbrace{\left(u_\beta \Delta^\nu_\alpha \tau^{\alpha\beta}\right)}_{\equiv h^\nu} u^\mu + \underbrace{\left(\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right) - \frac{1}{3}\Delta_{\alpha\beta}\Delta^{\mu\nu}\right)\tau^{\alpha\beta}}_{\equiv \pi^{\mu\nu}} \\ &= \delta\varepsilon u^\mu u^\nu - \Pi\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} \,,\end{aligned} \tag{3.19}$$

where $\delta\varepsilon, \Pi, h^\mu$ and $\pi^{\mu\nu}$ are *off-equilibrium energy density correction*, *bulk pressure*, *energy diffusion current* orthogonal to u^μ and *shear-stress tensor*, respectively. In the last term, we defined shear-stress tensor as

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\tau^{\alpha\beta} \,, \tag{3.20}$$

where

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right) - \frac{1}{3}\Delta_{\alpha\beta}\Delta^{\mu\nu} \,, \tag{3.21}$$

is the *doubly symmetric traceless projection operator* orthogonal to u^μ.

By choosing the fictitious equilibrium state (denoted by subscript 0) such that the fluid particle and energy densities are still defined as

$$n_0(\alpha, \beta) = u_\mu N^\mu \,, \tag{3.22}$$

$$\varepsilon_0(\alpha, \beta) = u_\mu u_\nu T^{\mu\nu} \,, \tag{3.23}$$

one gets rid of the off-equilibrium corrections to the particle and energy density, i.e

$$\begin{aligned}\delta n &= \int d\boldsymbol{P}\, E_p \delta f = 0 \,, \\ \delta\varepsilon &= \int d\boldsymbol{P}\, E_p^2 \delta f = 0 \,.\end{aligned} \tag{3.24}$$

These are the Landau matching conditions [25] that already appeared in (2.49) and (2.50), which can now be seen as an arbitrary definition of the equilibrium state that fixes the chemical potential and temperature of the fluid. Using these off-equilibrium terms and leaving the subscript 0 under the equilibrium quantities implicit, the general form of the particle 4-current and energy-momentum tensor for a viscous

fluid are

$$N^\mu = nu^\mu + n^\mu \ , \tag{3.25}$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Big(P + \Pi\Big)\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} \ , \tag{3.26}$$

where the dissipative currents are defined as

$$\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int d\boldsymbol{P}\ p^\alpha p^\beta \delta f \ , \tag{3.27}$$

$$n^\mu = \Delta^\mu_\alpha \int d\boldsymbol{P}\ p^\alpha \delta f \ , \tag{3.28}$$

$$h^\mu = u_\alpha \Delta^\mu_\beta \int d\boldsymbol{P}\ p^\alpha p^\beta \delta f \ , \tag{3.29}$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int d\boldsymbol{P}\ p^\alpha p^\beta \delta f \ . \tag{3.30}$$

3.2.2 Choice of frame

At this point $u^\mu, n, n^\mu, \varepsilon, \Pi, h^\mu$ and $\pi^{\mu\nu}$ together contain 17 independent components while N^μ and symmetric $T^{\mu\nu}$ contain only contain 14 independent components so some additional constraints are needed.

In ideal fluid dynamics the fluid 4-velocity was a uniquely defined quantity, but in the case of a viscous fluid, u^μ has yet been an arbitrary normalized time-like 4-vector. There are two clever choices for the frame which fix u^μ such that the form of the particle 4-current and energy-momentum tensor simplify conveniently. First of these, called *Eckart frame*, was proposed by Eckart [24] in which the 4-velocity is intuitively defined to be parallel to the total particle current

$$u^\mu \equiv \frac{N^\mu}{\sqrt{N_\nu N^\nu}} \ , \tag{3.31}$$

and the latter, *Landau frame*, proposed by Landau and Lifshitz [25], where the 4-velocity is parallel to the total energy current

$$u^\mu \equiv \frac{1}{\varepsilon} u_\nu T^{\mu\nu} \ . \tag{3.32}$$

These definitions for u^μ are useful, because in Eckart's particle frame, the particle diffusion current n^μ by definition vanishes because it is accounted in the 4-velocity and in Landau's energy frame the energy diffusion current h^μ in $T^{\mu\nu}$ vanishes. With

either of these choices, one of the dissipative currents is interlinked with the 4-velocity, thus reducing the number of degrees of freedom to 14.

However, the Eckart frame definition for u^μ is ill-defined when particle density vanishes. For example, in a region with equal number of particles and antiparticles the net baryon number vanishes, which could leave the regions of the fluid without a well defined velocity. In particle collisions this kind of situations are not uncommon and thus one should avoid Eckart's definition in these cases. From now on we shall use the Landau frame in the calculations which leaves the energy-momentum tensor as

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Big(P + \Pi\Big)\Delta^{\mu\nu} + \pi^{\mu\nu} \ . \tag{3.33}$$

3.2.3 Equations of motion

From the conservation of net particle number (2.24) and the form of net particle 4-current we get the continuity equation

$$\dot{n} + n\theta + \partial_\mu n^\mu = 0 \ . \tag{3.34}$$

By projecting energy-momentum conservation law (3.9) onto the subspace orthogonal to u^μ one obtains the momentum equation

$$\Big(\varepsilon + P + \Pi\Big)\dot{u}^\mu - \nabla^\mu\Big(P + \Pi\Big) + \Delta^\mu_\beta \partial_\alpha \pi^{\alpha\beta} = 0 \ . \tag{3.35}$$

Finally the energy equation is recovered by by projection of the energy-momentum conservation law to the subspace parallel to u^μ

$$\dot{\varepsilon} + \Big(\varepsilon + P + \Pi\Big)\theta + u_\nu \partial_\mu \pi^{\mu\nu} = 0 \ . \tag{3.36}$$

This form of the energy equation can be further refined. Due to $\pi^{\mu\nu}$ being orthogonal to 4-velocity, the last term in (3.36) can be recast as

$$u_\nu \partial_\mu \pi^{\mu\nu} = -\pi^{\mu\nu}\partial_\mu u_\nu \ . \tag{3.37}$$

The velocity gradient $\partial_\mu u_\nu$ can be decomposed into terms depicting different physical deformations

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \nabla_\mu u_\nu = u_\mu \dot{u}_\nu + \omega_{\mu\nu} + \sigma_{\mu\nu} + V_{\mu\nu} \ , \tag{3.38}$$

where we defined the tensors $\sigma^{\mu\nu}$, $\omega^{\mu\nu}$ and $V^{\mu\nu}$ representing shear deformations, rotation and expansion of the fluid element, respectively (see Fig. 8).

$$\sigma^{\mu\nu} = \frac{1}{2}\Big(\nabla^\mu u^\nu + \nabla^\nu u^\mu\Big) - \frac{1}{3}\Delta^{\mu\nu}\theta \ , \tag{3.39}$$

$$\omega^{\mu\nu} = \frac{1}{2}\Big(\nabla^\mu u^\nu - \nabla^\nu u^\mu\Big) \ , \tag{3.40}$$

$$V^{\mu\nu} = \frac{1}{3}\Delta^{\mu\nu}\theta \ . \tag{3.41}$$

Product between the symmetric $\pi^{\mu\nu}$ and both the antisymmetric $\omega^{\mu\nu}$ and $V^{\mu\nu}$ parallel to 4-velocity is zero, which means (3.36) can be represented as

$$\dot{\varepsilon} + \Big(\varepsilon + P + \Pi\Big)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0 \ . \tag{3.42}$$

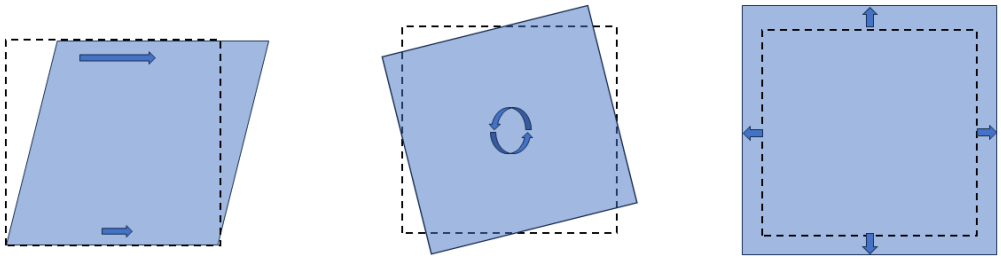

Figure 8. A tensor decomposition of the gradient of 4-velocity reveals components depicting deformations due to shear stress, rotation and expansion of the fluid element.

Equations (3.34), (3.35) and (3.42) are the general equations of motion for relativistic viscous fluid dynamics in Landau frame. As already discussed, dissipative fluid dynamics has 14 degrees of freedom but the conservation laws provide only 5 equations. The additional 9 equations are recovered by finding expressions for the dissipative currents Π, n^μ and $\pi^{\mu\nu}$.

3.3 Relativistic generalization of Navier–Stokes

The general fluid dynamical equations of motion (3.34), (3.35) and (3.42) do not yet make any reference to what the bulk pressure Π, particle diffusion current n^μ or the shear stress tensor $\pi^{\mu\nu}$ should be and do not form a closed system of equations. For a direct relativistic generalization of the Navier–Stokes theory, the dissipative quantities are related to the fluid dynamical gradients by taking advantage of the second law of thermodynamics. This will give rise to fluid dynamical transport coefficients, which are the proportionality factors between the corresponding dissipative currents and fluid dynamical gradients.

3.3.1 Entropy production

Because the fictitious state is assumed to be close to equilibrium, equation (3.14) for the equilibrium parts of N^μ and $T^{\mu\nu}$ (denoted by subscript 0) is still valid

$$\partial_\mu S_0^\mu = \beta u_\nu \partial_\mu T_0^{\mu\nu} - \alpha \partial_\mu N_0^\mu \ . \tag{3.43}$$

Now the divergence of the equilibrium parts can be expressed as

$$\partial_\mu N_0^\mu = \partial_\mu N^\mu - \partial_\mu n^\mu \ , \tag{3.44}$$

$$\partial_\mu T_0^{\mu\nu} = \partial_\mu T^{\mu\nu} - \partial_\mu \Big(- \Pi \Delta^{\mu\nu} + \pi^{\mu\nu} \Big) \ . \tag{3.45}$$

The first terms on the right hand side of (3.44) and (3.45) vanish due to the conservation laws and (3.43) can be recast as

$$\begin{aligned} \partial_\mu S_0^\mu &= \beta u_\nu \partial_\mu \Big(\Pi \Delta^{\mu\nu} - \pi^{\mu\nu} \Big) + \alpha \partial_\mu n^\mu \\ &= \beta \Big(- \Pi \theta + \pi^{\mu\nu} \sigma_{\mu\nu} \Big) + \partial_\mu \Big(\alpha n^\mu \Big) - n^\mu \partial_\mu \alpha \ . \end{aligned} \tag{3.46}$$

Following Landau and Lifshitz [25], by assuming the particle diffusion current to be the main contribution to off-equilibrium entropy 4-current, we can nominate the entropy 4-current as

$$S^\mu \equiv S_0^\mu - \alpha n^\mu \ , \tag{3.47}$$

which leads to

$$\begin{aligned}\partial_\mu S^\mu &= \partial_\mu\Big(S_0^\mu - \alpha n^\mu\Big) \\ &= \beta\Big(-\Pi\theta + \pi^{\mu\nu}\sigma_{\mu\nu}\Big) - n^\mu\partial_\mu\alpha \geq 0\ ,\end{aligned} \tag{3.48}$$

where we required the divergence of S^μ to be non-negative by the second law of thermodynamics. This is guaranteed by demanding each of the terms to be non-negative separately such that they satisfy the constitutive relations

$$\Pi = -\zeta\theta\ , \tag{3.49}$$

$$n^\mu = \kappa\nabla^\mu\alpha\ , \tag{3.50}$$

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}\ , \tag{3.51}$$

where the positive transport coefficients ζ, κ and η are the *bulk viscosity*, *particle diffusion coefficient* and *shear viscosity*, respectively. These are the only transport coefficients emerging in the relativistic Navier–Stokes theory.

From (3.49)-(3.51) we can see that the resistance to flow of a viscous fluid is manifested via two different parameters, η and ζ (see Fig. 9). Shear viscosity is the fluids resistance to shear stresses and essentially resists velocity gradients between parallel fluid layers. Microscopically, collisions of the particles in the fluid and spreading of their momentum either slows down or speeds up adjacent layers of fluid. Bulk viscosity is the resistance to compression or expansion of the fluid. This stems from energy dissipation between translational and microscopic degrees of freedom while the system is off-equilibrium [25, 27].

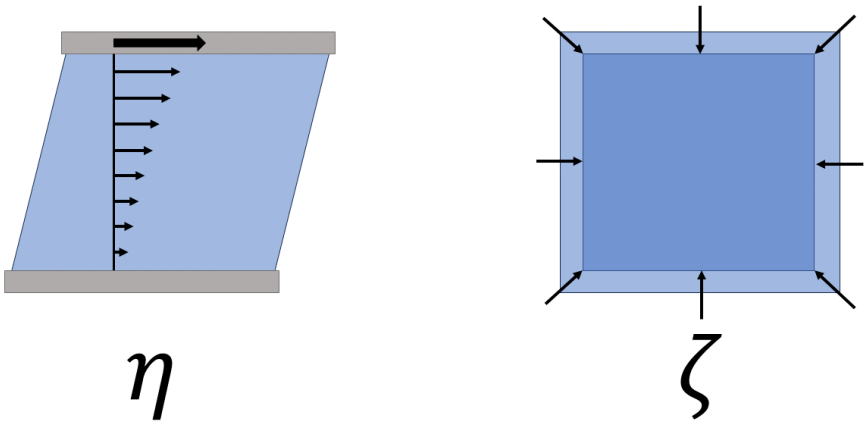

Figure 9. Shear and bulk viscosities, η and ζ, are the fluids resistance to shear rate and compression/expansion.

3.3.2 Relativistic Navier–Stokes equations

Now the relativistic Navier-Stokes equations can be derived from (3.34), (3.35), (3.42) and (3.49)-(3.51)

$$\dot{n} + n\theta = -\partial_\mu\Big(\kappa\nabla^\mu\alpha\Big) \,, \tag{3.52}$$

$$\Big(\varepsilon + P - \zeta\theta\Big)\dot{u}^\mu = \nabla^\mu\Big(P - \zeta\theta\Big) - 2\eta\Delta^\mu_\beta\partial_\alpha\sigma^{\alpha\beta} \,, \tag{3.53}$$

$$\dot{\varepsilon} + \Big(\varepsilon + P - \zeta\theta\Big)\theta = 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} \,. \tag{3.54}$$

These equations show a clear resemblance of the classical equations (3.1)-(3.3), although they have some corrections due to relativity and different definition of flow velocity. However, these equations have a fundamental flaw, they are parabolic. The terms involving the dissipative quantities are of first order in time derivatives and of second order in spatial derivatives. This leads to problems when considering boosted frames of reference, because in a relativistic theory space and time should be treated equally. The parabolic nature of the equations leads to instantaneous signal propagation of the dissipative quantities and allow entropy to grow even in equilibrium. It has been shown that perturbations to the fluids equilibrium state causes growing Fourier modes within extremely small timescales, thus labeling the direct relativistic generalization of Navier-Stokes acausal and unstable [28].

In the Navier–Stokes theory the dissipative currents satisfy linear relations (3.49)-(3.51) to the fluid dynamical gradients, but this is not necessarily the case. Restoring terms of second order in the dissipative currents into the entropy 4-current (3.47) [5], or a more rigorous approach [6, 7] based on manipulating the Boltzmann equation in kinetic theory leads to dynamical equations for Π, n^μ and $\pi^{\mu\nu}$. This kind of theories lead to causal behavior with the cost of introducing a plethora of transport coefficients and coupled terms. Further discussion of these so called second order, or Israel–Stewart type theories, will however be omitted here.

In the following we will be finding expressions for the transport coefficients appearing in the relativistic Navier–Stokes equations, by employing the relaxation time approximations introduced in section 2.

4 Transport coefficients in the RTA

A widely used technique to obtain the fluid dynamical transport coefficients ζ, κ and η is done by matching the macroscopic fluid dynamics to the underlying microscopic theory and extract them using the Boltzmann equation. As we know the microscopic expressions for the dissipative currents Π, n^μ and $\pi^{\mu\nu}$, it is possible to recover them once δf, the off-equilibrium correction to the single particle momentum distribution function, is known. Approximate solutions are usually recovered by using the so called *Grad moment method* [18] or *Chapman–Enskog expansion* [29]. In this chapter we will obtain the first order Chapman-Enskog expansion, where the single particle momentum distribution function can be related to the fluid dynamical gradients $\theta, \nabla^\mu \alpha$ and $\sigma^{\mu\nu}$. Then we calculate the transport coefficients of the relativistic Navier–Stokes theory using the traditional Anderson–Witting relaxation time approximation. The novel relaxation time approximation will be considered in section 5.

4.1 Chapman–Enskog expansion

Macroscopic dynamics of the fluid is expected to arise at small Knudsen numbers. When a fluid close to equilibrium, this means that the microscopical kinematics is dominated by the interparticle collisions which drive the fluid towards equilibrium and is the same as writing the Boltzmann equation as

$$\mathrm{Kn}\, p^\mu \partial_\mu f_p = \mathcal{C}_f \ , \tag{4.1}$$

where the streaming term is now suppressed by the factor Kn, the Knudsen number. Chapman–Enskog theory is a gradient expansion in where the single particle momentum distribution function is expanded as a perturbative series in Knudsen number

$$f_p = f_p^{(0)} + \mathrm{Kn} f_p^{(1)} + \mathrm{Kn}^2 f_p^{(2)} + \dots \ . \tag{4.2}$$

This also means the Chapman-Enskog method is only valid near the fluid dynamical regime.

Substituting (4.2) into (4.1) gives the Chapman–Enskog expansion to desired order in Knudsen number

$$\begin{aligned}
&\mathrm{Kn}\; p^\mu \partial_\mu \Big(f_p^{(0)} + \mathrm{Kn} f_p^{(1)} + \mathcal{O}(\mathrm{Kn}^2) \Big) = \\
&\frac{1}{2} \int d\boldsymbol{P'}\, d\boldsymbol{K}\, d\boldsymbol{K'}\, W_{pp' \Rightarrow kk'} \Big(f_p^{(0)} + \mathrm{Kn} f_p^{(1)} + \mathcal{O}(\mathrm{Kn}^2) \Big) \Big(f_{p'}^{(0)} + \mathrm{Kn} f_{p'}^{(1)} + \mathcal{O}(\mathrm{Kn}^2) \Big) \\
&-\frac{1}{2} \int d\boldsymbol{P'}\, d\boldsymbol{K}\, d\boldsymbol{K'}\, W_{pp' \Rightarrow kk'} \Big(f_k^{(0)} + \mathrm{Kn} f_k^{(1)} + \mathcal{O}(\mathrm{Kn}^2) \Big) \Big(f_{k'}^{(0)} + \mathrm{Kn} f_{k'}^{(1)} + \mathcal{O}(\mathrm{Kn}^2) \Big) \; .
\end{aligned} \tag{4.3}$$

Truncating this expansion to zeroth order leads to $0 = \mathcal{C}_{f^{(0)}}$, which yields the equilibrium function f_{0p} itself. The first order approximation is nontrivial and is obtained from

$$\mathrm{Kn}\; p^\mu \partial_\mu f_{0p} = \underbrace{\mathcal{C}_{f_{0p}}}_{=0} + \mathrm{Kn}\; \hat{L} \phi_p^{(1)} + \mathcal{O}(\mathrm{Kn}^2) \; , \tag{4.4}$$

where the collision term with equilibrium distribution is identically zero and the first order contribution to f_p was expressed as previously defined in (2.38)

$$f^{(1)} = f_{0p} \phi_p^{(1)} \; . \tag{4.5}$$

For clarity, the index will also be omitted since we are only discussing the first order expansion and leaving out all the $\mathcal{O}(\mathrm{Kn}^2)$ terms here.

After dividing (4.4) by the Knudsen number, the left hand side can be opened as

$$p^\mu \partial_\mu f_{0p} = p^\mu f_{0p} \Big(- E_p \partial_\mu \beta - \beta u_\nu \partial_\mu p^\nu - \beta p^\nu \partial_\mu u_\nu + \partial_\mu \alpha \Big) \; . \tag{4.6}$$

The 4-momentum vector has no coordinate dependence so its gradient vanishes. Furthermore, the 4-gradients can be decomposed to their timelike and spacelike components

$$f_{0p} \Big(- E_p^2 \dot{\beta} - E_p p^\mu \nabla_\mu \beta - \beta E_p p^\nu \dot{u}_\nu - \beta p^\mu p^\nu \nabla_\mu u_\nu + E_p \dot{\alpha} + p^\mu \nabla_\mu \alpha \Big) \; . \tag{4.7}$$

Using the equations of motion recovered from the macroscopic conservation laws allows one to express the time derivatives with respect to spacelike gradients, since the particle density, energy density and pressure are functions of temperature (β) and chemical potential (α).

Derivatives of n, ε and P in (3.34), (3.35) and (3.42) can thus be expressed with partial derivatives with respect to α and β

$$\dot{n} = \frac{\partial n}{\partial \alpha}\dot{\alpha} + \frac{\partial n}{\partial \beta}\dot{\beta} \,, \tag{4.8}$$

$$\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial \alpha}\dot{\alpha} + \frac{\partial \varepsilon}{\partial \beta}\dot{\beta} \,, \tag{4.9}$$

$$\nabla_\mu P = \frac{\partial P}{\partial \alpha}\nabla_\mu \alpha + \frac{\partial P}{\partial \beta}\nabla_\mu \beta \,. \tag{4.10}$$

For the partial derivatives we will use the kinetic definitions (2.12), (2.17) and (2.18). It is useful to define auxiliary thermodynamic integrals

$$I_{nq} = \frac{1}{(2q+1)!!}\int d\boldsymbol{P}\; E_p^{n-2q}\left(E_p^2 - m^2\right)^q f_{0p} \,, \tag{4.11}$$

$$J_{nq} = \frac{\partial I_{nq}}{\partial \alpha} \,, \tag{4.12}$$

so that $n = I_{10}, \varepsilon = I_{20}$ and $P = I_{21}$. Now the partial derivatives can be calculated

$$\begin{aligned}
\frac{\partial n}{\partial \alpha} &= \int d\boldsymbol{P}\; E_p \frac{\partial f_{0p}}{\partial \alpha} = \int d\boldsymbol{P}\; E_p \frac{\partial}{\partial \alpha}\left(e^{\beta E_p - \alpha} + a\right)^{-1} = J_{10} \,, \\
\frac{\partial n}{\partial \beta} &= \int d\boldsymbol{P}\; E_p \frac{\partial f_{0p}}{\partial \beta} = -J_{20} \,, \\
\frac{\partial \varepsilon}{\partial \alpha} &= \int d\boldsymbol{P}\; E_p^2 \frac{\partial f_{0p}}{\partial \alpha} = J_{20} \,, \\
\frac{\partial \varepsilon}{\partial \beta} &= \int d\boldsymbol{P}\; E_p^2 \frac{\partial f_{0p}}{\partial \beta} = -J_{30} \,, \\
\frac{\partial P}{\partial \alpha} &= \frac{1}{3}\int d\boldsymbol{P}\; \left(E_p^2 - m^2\right)\frac{\partial f_{0p}}{\partial \alpha} = J_{21} = \frac{I_{10}}{\beta} \,, \\
\frac{\partial P}{\partial \beta} &= \frac{1}{3}\int d\boldsymbol{P}\; \left(E_p^2 - m^2\right)\frac{\partial f_{0p}}{\partial \beta} = -J_{31} = -\frac{I_{20} + I_{21}}{\beta} \,,
\end{aligned} \tag{4.13}$$

where in the last two lines the relations between J_{nq} and I_{nq} integrals were derived

in appendix C. Now inserting (4.13) into (4.8)-(4.10) gives two expressions for $\dot{\alpha}$ and $\dot{\beta}$, and $\nabla_\mu P$ with respect to the gradients of α and β

$$\dot{\alpha} = \frac{\dot{n} + J_{20}\dot{\beta}}{J_{10}} = \frac{\dot{\varepsilon} + J_{30}\dot{\beta}}{J_{20}} \,, \tag{4.14}$$

$$\dot{\beta} = \frac{J_{10}\dot{\alpha} - \dot{n}}{J_{20}} = \frac{J_{20}\dot{\alpha} - \dot{\varepsilon}}{J_{30}} \,, \tag{4.15}$$

$$\nabla_\mu P = \frac{I_{10}}{\beta}\nabla_\mu \alpha - \frac{I_{20}+I_{21}}{\beta}\nabla_\mu \beta \,. \tag{4.16}$$

Using (4.14)-(4.15) $\dot{\alpha}$ and $\dot{\beta}$ can be solved in terms of the thermodynamic integrals and fluid dynamical variables from (3.34) and (3.42) as

$$\dot{\alpha} = \frac{1}{J_{20}^2 - J_{30}J_{10}}\Big(- J_{20}\left[(\varepsilon + P + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right] + J_{30}\left[n\theta + \partial_\mu n^\mu\right]\Big) \,, \tag{4.17}$$

$$\dot{\beta} = \frac{1}{J_{20}^2 - J_{30}J_{10}}\Big(- J_{10}\left[(\varepsilon + P + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}\right] + J_{20}\left[n\theta + \partial_\mu n^\mu\right]\Big) \,. \tag{4.18}$$

With (4.16) equation (3.35) can be recast as

$$\dot{u^\mu} = \frac{1}{\varepsilon + P}\left(\frac{I_{10}}{\beta}\nabla_\mu\alpha - \frac{I_{20}+I_{21}}{\beta}\nabla_\mu\beta + \nabla^\mu\Pi - \Pi\dot{u^\mu} - \Delta^\mu_\beta\partial_\alpha\pi^{\alpha\beta}\right) \,. \tag{4.19}$$

Now (4.17)-(4.19) can be inserted into (4.7). Since both the fluid dynamical gradients and dissipative currents are $\mathcal{O}(1)$ small quantities, their products are of $\mathcal{O}(2)$ and thus can be neglected in a linearized theory [16, 30]. Finally the equation (4.7) becomes

$$\begin{aligned} & f_{0p}\Bigg(- E_p^2\frac{J_{10}(I_{20}+I_{21}) - J_{20}I_{10}}{I_{30}I_{10} - I_{20}^2}\theta - E_p p^\mu\nabla_\mu\beta - \beta E_p p^\nu\left[\frac{I_{10}}{\beta(I_{20}+I_{21})}\nabla_\nu\alpha - \frac{1}{\beta}\nabla_\nu\beta\right] \\ & \quad - \beta p^\mu p^\nu\left[\sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}\theta\right] + E_p\frac{J_{20}(I_{20}+I_{21}) - J_{30}I_{10}}{I_{30}I_{10} - I_{20}^2}\theta + p^\mu\nabla_\mu\alpha\Bigg) \\ & \equiv f_{0p}\Big(A_p\theta + B_p p^\mu\nabla_\mu\alpha - \beta p^\mu p^\nu\sigma_{\mu\nu}\Big) \,, \end{aligned} \tag{4.20}$$

where we defined the momentum dependent coefficients

$$A_p = \frac{J_{20}(I_{20}+I_{21}) - J_{30}I_{10}}{I_{30}I_{10} - I_{20}^2} E_p - \frac{J_{10}(I_{20}+I_{21}) - J_{20}I_{10}}{I_{30}I_{10} - I_{20}^2} E_p^2 - \frac{\beta}{3}\Big(m^2 - E_p^2\Big) \ , \tag{4.21}$$

$$B_p = 1 - \frac{I_{10}}{(I_{20}+I_{21})} E_p \ . \tag{4.22}$$

The first order Chapman–Enskog expansion (4.4) can now be written as

$$\Big(A_p\theta + B_p p^\mu \nabla_\mu \alpha - \beta p^\mu p^\nu \sigma_{\mu\nu}\Big) f_{0p} = \hat{L}\phi_p \ . \tag{4.23}$$

With the help of (4.23) it is now possible to identify expressions for the fluid dynamical transport coefficients in order to solve their energy dependence using the kinetic definitions of Π, n^μ and $\pi^{\mu\nu}$ derived in the last section.

4.2 Calculation of ζ, κ and η

We have now expressed the Boltzmann equation in terms of the fluid dynamical gradients, which allows one to link them to the non-equilibrium part of the single particle momentum distribution function. Recalling the microscopic expressions for the dissipative currents in the Landau frame

$$\Pi = -\frac{1}{3}\Delta_{\mu\nu} \int d\boldsymbol{P}\ p^\mu p^\nu \delta f \ , \tag{4.24}$$

$$n^\mu = \Delta^\mu_\nu \int d\boldsymbol{P}\ p^\nu \delta f \ , \tag{4.25}$$

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int d\boldsymbol{P}\ p^\alpha p^\beta \delta f \ . \tag{4.26}$$

In the traditional Anderson–Witting relaxation time approximation ($\hat{L} \to \hat{L}_{\mathrm{RTA}}$) it is straightforward to solve δf directly from the Boltzmann equation

$$\Big(A_p\theta + B_p p^\mu \nabla_\mu \alpha - \beta p^\mu p^\nu \sigma_{\mu\nu}\Big) f_{0p} = -\frac{E_p}{\tau} \delta f \ . \tag{4.27}$$

In the Navier–Stokes theory bulk viscosity, particle diffusion coefficient and shear viscosity can then be exactly identified from equations (3.49)-(3.51) as

$$\zeta = -\frac{1}{3}\int d\boldsymbol{P}\ \frac{A_p\tau}{E_p}\Delta_{\mu\nu}p^\mu p^\nu f_{0p}\ , \tag{4.28}$$

$$\kappa = \frac{1}{3}\int d\boldsymbol{P}\ \frac{B_p\tau}{E_p}\Delta_{\mu\nu}p^\mu p^\nu f_{0p}\ , \tag{4.29}$$

$$\eta = \frac{1}{15}\int d\boldsymbol{P}\ \frac{\beta\tau}{E_p}\Big(\Delta_{\mu\nu}p^\mu p^\nu\Big)^2 f_{0p}\ . \tag{4.30}$$

The relaxation time τ itself will be parametrized as

$$\tau = t_R(\beta E_p)^\gamma\ , \tag{4.31}$$

where t_R sets the relaxation timescale and the value of γ sets its energy dependence. Once the parametrization for the energy dependence is chosen, the only free parameter left will be t_R, which is a common factor in the integrals and can be thus normalized out. The transport coefficients are exponentially suppressed thermodynamic functions getting high values when the argument $m\beta$ is small and they quickly tend to zero as $m\beta$ increases. For illustrative purposes they shall be plotted in a dimensionless form independent of the chosen relaxation timescale t_R. The transport coefficients are calculated using three different relaxation schemes, namely a constant momentum independent relaxation time $\tau = t_R$ ($\gamma = 0$), a QCD inspired relaxation time $\tau \propto \sqrt{E_p}$ ($\gamma = 0.5$) and a linear ansatz $\tau \propto E_p$ ($\gamma = 1$) [14, 15].

The results are shown in Fig. 10. We can immediately notice negative values for the bulk viscosity and vanishing particle diffusion coefficient with $\gamma = 1$, which are in direct contradiction with the arguments based on the second law of thermodynamics that were presented in section 3.3.1. This is precisely the reason why we are motivated to introduce the novel relaxation time approximation.

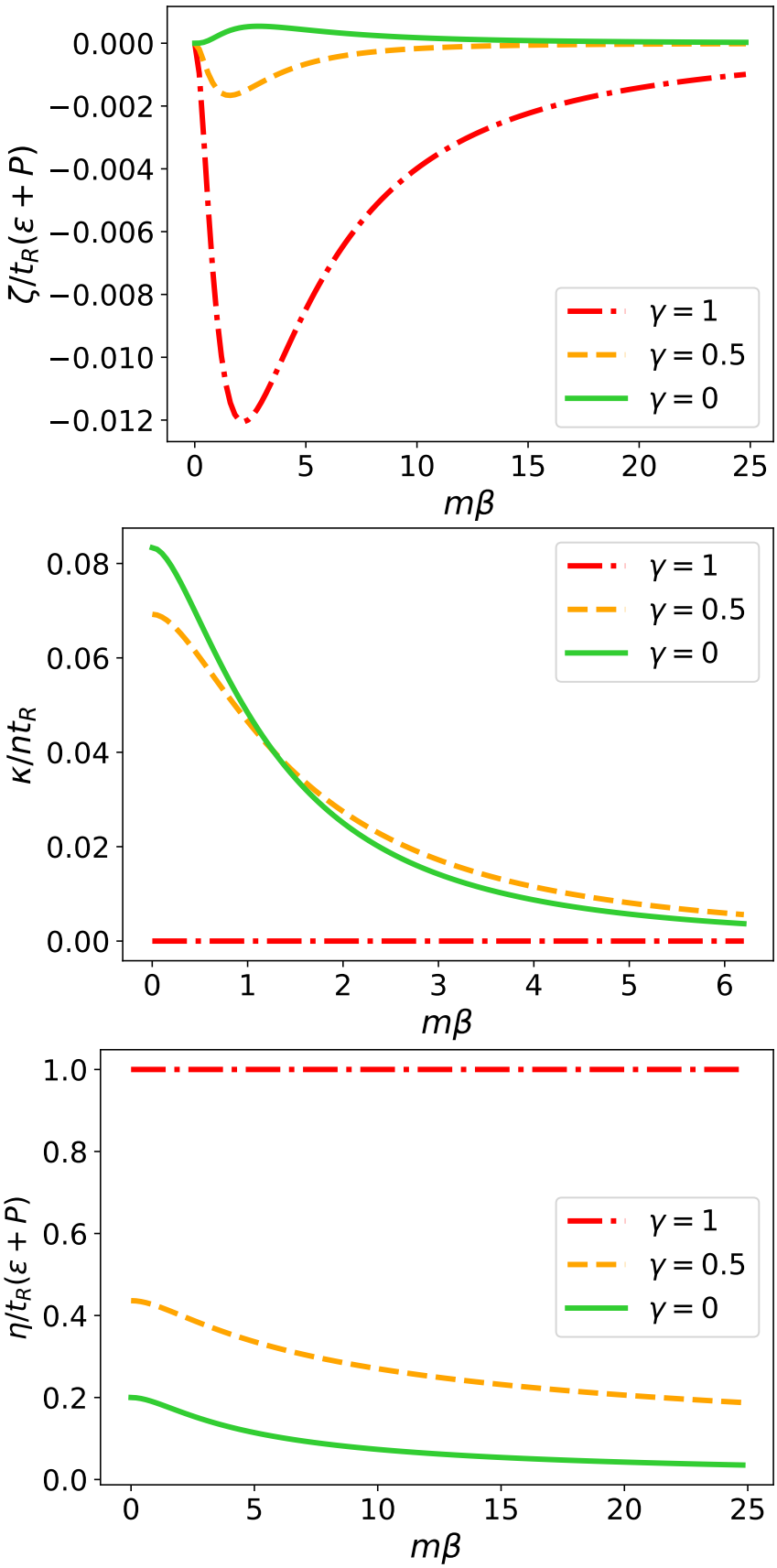

Figure 10. Dimensionless bulk viscosity, particle diffusion and shear viscosity coefficients of a Boltzmann gas as a function of $m\beta$ in the AW relaxation time approximation.

5 Transport coefficients in the NRTA

Inverting the Boltzmann equation in the novel relaxation time approximation ($\hat{L} \to \hat{L}_{\mathrm{NRTA}}$),

$$\Big(A_p\theta + B_p p^\mu \nabla_\mu \alpha - \beta p^\mu p^\nu \sigma_{\mu\nu}\Big) f_{0p} = \\ -\frac{E_p}{\tau} f_{0p} \left(\phi_p - \frac{\left\langle \frac{E_p}{\tau}\phi_p \right\rangle_0}{\left\langle \frac{E_p}{\tau} \right\rangle_0} - P_1^{(0)} \frac{\left\langle \frac{E_p}{\tau} P_1^{(0)} \phi_p \right\rangle_0}{\left\langle \frac{E_p}{\tau} P_1^{(0)} P_1^{(0)} \right\rangle_0} - p^{\langle\mu\rangle} \frac{\left\langle \frac{E_p}{\tau} p_{\langle\mu\rangle} \phi_p \right\rangle_0}{\left\langle \frac{E_p}{\tau} p_{\langle\nu\rangle} p^{\langle\nu\rangle} \right\rangle_0} \right) , \tag{5.1}$$

is more involved than in the previous section since ϕ_p appears now also inside the integral coefficients, and we are yet one step away from being able to write down expressions for the transport coefficients.

5.1 Gradient expansion for ϕ_p

From (5.1) we can see that the first order Chapman–Enskog expansion of the Boltzmann equation has three linearly independent components, namely the fluid dynamical gradients $\theta, p^\mu \nabla_\mu \alpha$ and $p^\mu p^\nu \sigma_{\mu\nu}$. Since also the novel relaxation time approximation is linear in its argument, the solution for ϕ_p can be written as a linear combination of these independent components with some momentum dependent coefficients. The general form of ϕ_p can thus be written as

$$\phi_p = S_p^{(0)}\theta + S_p^{(1)} p^\mu \nabla_\mu \alpha - S_p^{(2)} p^\mu p^\nu \sigma_{\mu\nu} \, . \tag{5.2}$$

The momentum dependence of each of the coefficients $S_p^{(l)}$ will be expressed as a formal power series in energy. Here we expand them in a complete basis of orthogonal polynomial functions as

$$S_p^{(l)} = \sum_{n=0}^{\infty} s_n^{(l)} P_n^{(l)}, \tag{5.3}$$

where the coefficients $s_n^{(l)}$ are momentum independent. The momentum dependence is captured in the polynomial functions

$$P_n^{(l)} = \sum_{r=0}^{n} a_{nr}^{(l)} E_p^n \, . \tag{5.4}$$

They are similar to the associated Laguerre polynomials, now constructed to be orthogonal in the inner product

$$\int d\boldsymbol{P} \, \frac{E_p}{\tau} \frac{W^{(l)}}{(2l+1)!!} \big(\Delta_{\mu\nu} p^\mu p^\nu\big)^l P_n^{(l)} P_m^{(l)} f_{0p} = \delta_{nm} \, , \tag{5.5}$$

such that the orthogonality can be used conveniently to extract the exact values of the coefficients $s_n^{(l)}$. Hence the problem of inverting the Boltzmann equation has been converted into a task of finding the coefficients $s_n^{(l)}$ and $a_{nr}^{(l)}$. In order to find coefficients $a_{nr}^{(l)}$ for the calculations of the transport coefficients, the polynomial functions have to be separately constructed to satisfy (5.5) up to the desired order (see Appendix A).

The coefficients $s_n^{(l)}$ can be extracted by inverting the Boltzmann equation (see Appendix D)

$$s_n^{(0)} = -\frac{\big\langle A_p P_n^{(0)} \big\rangle_0}{\big\langle (E_p/\tau) P_n^{(0)} P_n^{(0)} \big\rangle_0} \, , \quad n \geq 2 \tag{5.6}$$

$$s_n^{(1)} = -\frac{\big\langle B_p \Delta_{\mu\nu} p^\mu p^\nu P_n^{(1)} \big\rangle_0}{\big\langle (E_p/\tau) \Delta_{\mu\nu} p^\mu p^\nu P_n^{(1)} P_n^{(1)} \big\rangle_0} \, , \quad n \geq 1 \tag{5.7}$$

$$s_n^{(2)} = \frac{\big\langle \beta \big(\Delta_{\mu\nu} p^\mu p^\nu\big)^2 P_n^{(2)} \big\rangle_0}{\big\langle (E_p/\tau) \big(\Delta_{\mu\nu} p^\mu p^\nu\big)^2 P_n^{(2)} P_n^{(2)} \big\rangle_0} \, . \tag{5.8}$$

In the Anderson–Witting approximation these would be valid for all n, but in the NRTA extracting $s_0^{(0)}, s_1^{(0)}$ and $s_0^{(1)}$ is not possible by this procedure but only after employing matching conditions, since they incorporate the homogeneous solutions of the Boltzmann equation. On a further note the coefficients $s_n^{(2)}$, related to shear deformations, are the same in both the Anderson–Witting RTA and in the NRTA. This means that the NRTA will not affect the coefficient of shear viscosity, however calculating it this way will grant us some information about the validity of the method used here. Strictly speaking the traditional RTA satisfies the conservation laws only using momentum independent relaxation times and the NRTA should

validate also the momentum dependent ones.

For obtaining the missing coefficients $s_0^{(0)}, s_1^{(0)}$ and $s_0^{(1)}$, consider general matching conditions as described in [10]

$$\int d\boldsymbol{P}\; g_p \phi_p f_{0p} = 0 \;, \tag{5.9}$$

$$\int d\boldsymbol{P}\; h_p \phi_p f_{0p} = 0 \;, \tag{5.10}$$

$$\int d\boldsymbol{P}\; q_p p^{\langle\mu\rangle} \phi_p f_{0p} = 0 \;, \tag{5.11}$$

where g_p and h_p are two linearly independent functions of energy and q_p is an arbitrary function of energy. Landau matching conditions (2.49) and (2.50) can be identified to correspond to setting $g_p = q_p = E_p$ and $h_p = E_p^2$. Using the orthogonality condition (2.60) one gets

$$s_0^{(0)}\langle g_p\rangle_0 + s_1^{(0)}\Big\langle g_p P_1^{(0)}\Big\rangle_0 = -\sum_{n=2}^{\infty} s_n^{(0)}\Big\langle g_p P_n^{(0)}\Big\rangle_0 \;, \tag{5.12}$$

$$s_0^{(0)}\langle h_p\rangle_0 + s_1^{(0)}\Big\langle h_p P_1^{(0)}\Big\rangle_0 = -\sum_{n=2}^{\infty} s_n^{(0)}\Big\langle h_p P_n^{(0)}\Big\rangle_0 \;, \tag{5.13}$$

$$s_0^{(1)}\Big\langle q_p \Delta_{\mu\nu} p^\mu p^\nu\Big\rangle_0 = -\sum_{n=1}^{\infty} s_n^{(1)}\Big\langle q_p \Delta_{\mu\nu} p^\mu p^\nu P_n^{(1)}\Big\rangle_0 \;. \tag{5.14}$$

Solving $s_0^{(0)}, s_1^{(0)}$ and $s_0^{(1)}$ from these constraints gives

$$s_0^{(0)} = \frac{\sum_{n=2}^{\infty} s_n^{(0)}\big(\Big\langle h_p P_1^{(0)}\Big\rangle_0\Big\langle g_p P_n^{(0)}\Big\rangle_0 - \Big\langle g_p P_1^{(0)}\Big\rangle_0\Big\langle h_p P_n^{(0)}\Big\rangle_0\big)}{\langle h_p\rangle_0\Big\langle g_p P_1^{(0)}\Big\rangle_0 - \langle g_p\rangle_0\Big\langle h_p P_1^{(0)}\Big\rangle_0} \;, \tag{5.15}$$

$$s_1^{(0)} = \frac{\sum_{n=2}^{\infty} s_n^{(0)}\big(\langle g_p\rangle_0\Big\langle h_p P_n^{(0)}\Big\rangle_0 - \langle h_p\rangle_0\Big\langle g_p P_n^{(0)}\Big\rangle_0\big)}{\langle h_p\rangle_0\Big\langle g_p P_1^{(0)}\Big\rangle_0 - \langle g_p\rangle_0\Big\langle h_p P_1^{(0)}\Big\rangle_0} \;, \tag{5.16}$$

$$s_0^{(1)} = -\frac{\sum_{n=1}^{\infty} s_n^{(1)}\Big\langle q_p \Delta_{\mu\nu} p^\mu p^\nu P_n^{(1)}\Big\rangle_0}{\langle q_p \Delta_{\mu\nu} p^\mu p^\nu\rangle_0} \;. \tag{5.17}$$

Now equations (5.6)-(5.8) together with (5.15)-(5.17) give the full prescription of ϕ_p within terms of the polynomial functions $P_n^{(l)}$.

5.2 Calculation of ζ, κ and η

So far we have obtained an expression for the Boltzmann equation in the fluid dynamical regime using the Chapman-Enskog method and presented the off-equilibrium contribution to the single particle distribution function expanding it in fluid dynamical gradients with coefficients in a basis of orthogonal polynomials. Finally it is possible to identify expressions for the fluid dynamical transport coefficients in order to solve their energy dependence using the expansion coefficients derived in the last subsection.

Bulk viscosity, particle diffusion coefficient and shear viscosity can now be exactly identified from equations (3.49)-(3.51) as

$$\zeta = \frac{1}{3}\int d\boldsymbol{P}\ \Delta_{\mu\nu}p^\mu p^\nu S_p^{(0)} f_{0p}\ , \tag{5.18}$$

$$\kappa = \frac{1}{3}\int d\boldsymbol{P}\ \Delta_{\mu\nu}p^\mu p^\nu S_p^{(1)} f_{0p}\ , \tag{5.19}$$

$$\eta = \frac{1}{15}\int d\boldsymbol{P}\ \left(\Delta_{\mu\nu}p^\mu p^\nu\right)^2 S_p^{(2)} f_{0p}\ . \tag{5.20}$$

The functions $S_p^{(l)}$ express the solutions as an infinite series in the orthogonal polynomials of energy but in practice the series has to be truncated. As seen above, in the bulk viscosity and particle diffusion coefficient, one must take into account the homogenous solution of the Boltzmann equation and thus the lowest order approximations read

$$\zeta = \frac{1}{3}\int d\boldsymbol{P}\ \Delta_{\mu\nu}p^\mu p^\nu \left(s_0^{(0)} + s_1^{(0)}P_1^{(0)} + s_2^{(0)}P_2^{(0)}\right) f_{0p}\ , \tag{5.21}$$

$$\kappa = \frac{1}{3}\int d\boldsymbol{P}\ \Delta_{\mu\nu}p^\mu p^\nu \left(s_0^{(1)} + s_1^{(1)}P_1^{(1)}\right) f_{0p}\ , \tag{5.22}$$

$$\eta = \frac{1}{15}\int d\boldsymbol{P}\ \left(\Delta_{\mu\nu}p^\mu p^\nu\right)^2 s_0^{(2)} f_{0p}\ . \tag{5.23}$$

The truncation of the polynomial series also affects the homogeneous solutions incorporated in (5.15)-(5.17) since they are written using all the polynomials. Higher order estimates can be achieved by accounting for more polynomials in the $S_p^{(l)}$ functions and observing whether the solution converges. It is not self-evident, at least to the writer, that this should always converge with the basis truncated only to the first few polynomials.

Nevertheless, in order to obtain at least next to lowest order approximations, here we truncate the polynomial series after the fourth polynomial such that

$$\zeta = \frac{1}{3}\int d\boldsymbol{P}\ \Delta_{\mu\nu}p^\mu p^\nu \left(s_0^{(0)} + s_1^{(0)}P_1^{(0)} + s_2^{(0)}P_2^{(0)} + s_3^{(0)}P_3^{(0)}\right) f_{0p}\ , \tag{5.24}$$

$$\kappa = \frac{1}{3}\int d\boldsymbol{P}\ \Delta_{\mu\nu}p^\mu p^\nu \left(s_0^{(1)} + s_1^{(1)}P_1^{(1)} + s_2^{(1)}P_2^{(1)} + s_3^{(1)}P_3^{(1)}\right) f_{0p}\ , \tag{5.25}$$

$$\eta = \frac{1}{15}\int d\boldsymbol{P}\ \left(\Delta_{\mu\nu}p^\mu p^\nu\right)^2 \left(s_0^{(2)} + s_1^{(2)}P_1^{(2)} + s_2^{(2)}P_2^{(2)} + s_3^{(2)}P_3^{(2)}\right) f_{0p}\ , \tag{5.26}$$

where

$$P_1^{(l)} = a_{10}^{(l)} + a_{11}^{(l)}E_p\ , \tag{5.27}$$

$$P_2^{(l)} = a_{20}^{(l)} + a_{21}^{(l)}E_p + a_{22}^{(l)}E_p^2\ , \tag{5.28}$$

$$P_3^{(l)} = a_{30}^{(l)} + a_{31}^{(l)}E_p + a_{32}^{(l)}E_p^2 + a_{33}^{(l)}E_p^3\ . \tag{5.29}$$

All of the coefficients $s_n^{(l)}$ and $a_{nr}^{(l)}$ are complicated thermodynamic functions as derived in appendices A and D.

In principle the integrals (5.24)-(5.26) could be integrated numerically by changing integration variable to energy. For numerical convenience, here we convert the momentum integrals to a dimensionless form by extracting the physical variables out by substituting $x \equiv \beta E_p$

$$\begin{aligned}\int d\boldsymbol{P}\ E_p^n f_{0p} &= \frac{1}{2\pi^2}\int_m^\infty dE_p\ E_p^n\sqrt{E_p^2 - m^2}\frac{1}{e^{\beta E_p - \alpha} + a} \\ &= \frac{1}{2\pi^2}\left(\frac{1}{\beta}\right)^{n+2}\int_{m\beta}^\infty dx\ x^n\sqrt{x^2 - (m\beta)^2}\frac{1}{e^{x-\alpha} + a}\ .\end{aligned} \tag{5.30}$$

In the case of a Boltzmann gas ($a = 0$), it has been customary to express such integrals in terms of the integral representations of modified Bessel functions of the second kind [6, 13, 19]

$$\begin{aligned}K_n(z) &= \frac{2^n n!}{(2n)!}z^{-n}\int_z^\infty dx\ \left(x^2 - z^2\right)^{n-\frac{1}{2}}e^{-x} \\ &= \frac{2^{n-1}(n-1)!}{(2n-2)!}z^{-n}\int_z^\infty dx\ x\left(x^2 - z^2\right)^{n-\frac{3}{2}}e^{-x}\ .\end{aligned} \tag{5.31}$$

Now thermodynamic integrals used in the calculation can be written as

$$n = I_{10} = \frac{e^\alpha}{2\pi^2}\frac{m^2}{\beta}K_2(m\beta)\ , \tag{5.32}$$

$$\varepsilon = I_{20} = \frac{e^\alpha}{2\pi^2}\left(\frac{3m^2}{\beta^2}K_2(m\beta) + \frac{m^3}{\beta}K_1(m\beta)\right)\ , \tag{5.33}$$

$$P = I_{21} = \frac{e^\alpha}{2\pi^2}\frac{m^2}{\beta^2}K_2(m\beta)\ , \tag{5.34}$$

$$I_{30} = \frac{e^\alpha}{2\pi^2}\left(\frac{3m^3}{\beta^2}K_3(m\beta) + \frac{m^4}{\beta}K_2(m\beta)\right)\ . \tag{5.35}$$

As a sidenote, from (5.32) and (5.34) one can also see that the thermodynamic equation of state for relativistic classical particles is $P = nT$, just as in the non-relativistic case. Using (5.31), equations (5.24)-(5.26) in terms of the Bessel K_n -functions are

$$\zeta = -\frac{e^\alpha}{2\pi^2}\frac{m^2}{\beta^2}\left(C_1^{(0)}K_2(m\beta) + C_2^{(0)}K_3(m\beta) + \frac{5m^2}{\beta}s_3^{(0)}a_{33}^{(0)}K_4(m\beta)\right)\ , \tag{5.36}$$

$$\kappa = -\frac{e^\alpha}{2\pi^2}\frac{m^2}{\beta^2}\left(C_1^{(1)}K_2(m\beta) + C_2^{(1)}K_3(m\beta) + \frac{5m^2}{\beta}s_3^{(1)}a_{33}^{(1)}K_4(m\beta)\right)\ , \tag{5.37}$$

$$\eta = \frac{e^\alpha}{2\pi^2}\frac{m^3}{\beta^3}\left(C_1^{(2)}K_3(m\beta) + C_3^{(2)}K_4(m\beta) + \frac{7m^2}{\beta}s_3^{(2)}a_{33}^{(2)}K_5(m\beta)\right)\ , \tag{5.38}$$

where

$$C_1^{(l)} = s_0^{(l)} + s_1^{(l)}a_{10}^{(l)} + s_2^{(l)}\left(a_{20}^{(l)} + a_{22}^{(l)}m^2\right) + s_3^{(l)}\left(a_{30}^{(l)} + a_{32}^{(l)}m^2\right)\ , \tag{5.39}$$

$$C_2^{(l)} = m\left(s_1^{(l)}a_{11}^{(l)} + s_2^{(l)}a_{21}^{(l)} + s_3^{(l)}\left[a_{31}^{(l)} + a_{33}^{(l)}m^2\right]\right) + \frac{5m}{\beta}\left(s_2^{(l)}a_{22}^{(l)} + s_3^{(l)}a_{32}^{(l)}\right)\ , \tag{5.40}$$

$$C_3^{(l)} = m\left(s_1^{(l)}a_{11}^{(l)} + s_2^{(l)}a_{21}^{(l)} + s_3^{(l)}\left[a_{31}^{(l)} + a_{33}^{(l)}m^2\right]\right) + \frac{7m}{\beta}\left(s_2^{(l)}a_{22}^{(l)} + s_3^{(l)}a_{32}^{(l)}\right)\ . \tag{5.41}$$

The relaxation time is parametrized similarly as in (4.31), which now affects all the coefficients $s_n^{(l)}$ and $a_{nr}^{(l)}$.

Finally using (5.32)-(5.41) we get

$$\frac{\zeta}{t_R(\varepsilon+P)} = -\frac{C_1^{(0)}K_2(m\beta) + C_2^{(0)}K_3(m\beta) + \dfrac{5m^2}{\beta}s_3^{(0)}a_{33}^{(0)}K_4(m\beta)}{4K_2(m\beta) + m\beta K_1(m\beta)}\,, \tag{5.42}$$

$$\frac{\kappa}{nt_R} = -\frac{C_1^{(1)}K_2(m\beta) + C_2^{(1)}K_3(m\beta) + \dfrac{5m^2}{\beta}s_3^{(1)}a_{33}^{(1)}K_4(m\beta)}{\beta K_2(m\beta)}\,, \tag{5.43}$$

$$\frac{\eta}{t_R(\varepsilon+P)} = \frac{m\left(C_1^{(2)}K_3(m\beta) + C_3^{(2)}K_4(m\beta)\right) + \dfrac{7m^3}{\beta}s_3^{(2)}a_{33}^{(2)}K_5(m\beta)}{4\beta K_2(m\beta) + m\beta^2 K_1(m\beta)}\,. \tag{5.44}$$

The integrals inside the coefficients $s_n^{(l)}$ and $a_{nr}^{(l)}$ are calculated numerically using SciPy's general purpose quadrature method, while the Bessel functions are handled by pre-defined special functions.

The dimensionless bulk viscosity, particle diffusion coefficient and shear viscosity obtained this way are shown in Fig. 11. Here we truncated the polynomial expansion for the $S_p^{(l)}$ coefficients to the fourth order. Convergence of the transport coefficients using all the available polynomial functions is in turn shown in Fig. 12, where the dashed line corresponds to the respective plots from Fig. 11 and the solid lines are approximations of lower order. The results of this calculation are compared to the original results of Rocha et. al. in Fig. 13. Further discussion of Figs. 11-13 is located in the following conclusions.

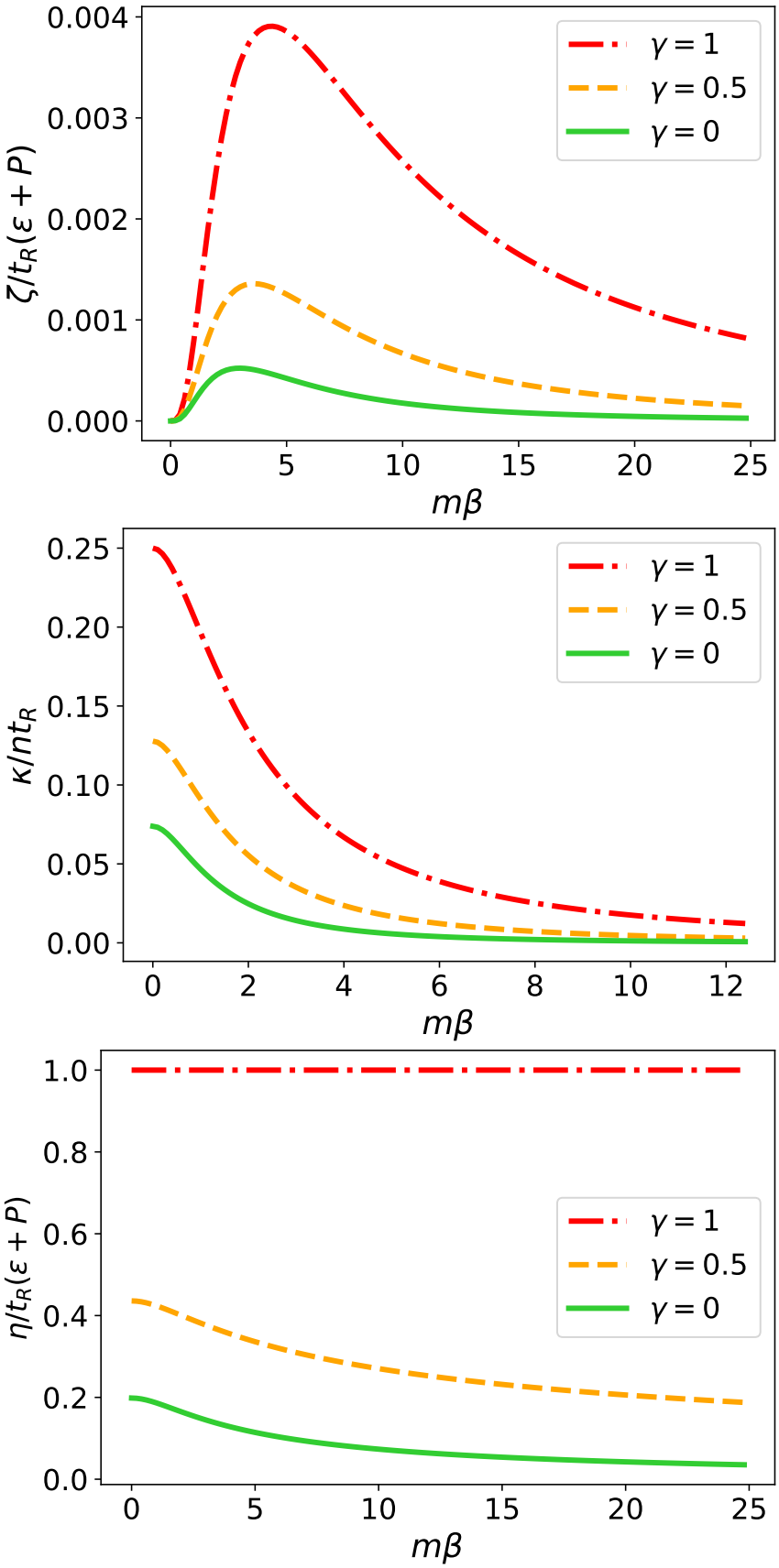

Figure 11. Dimensionless bulk viscosity, particle diffusion and shear viscosity coefficients of a Boltzmann gas as a function of $m\beta$ in the novel relaxation time approximation.

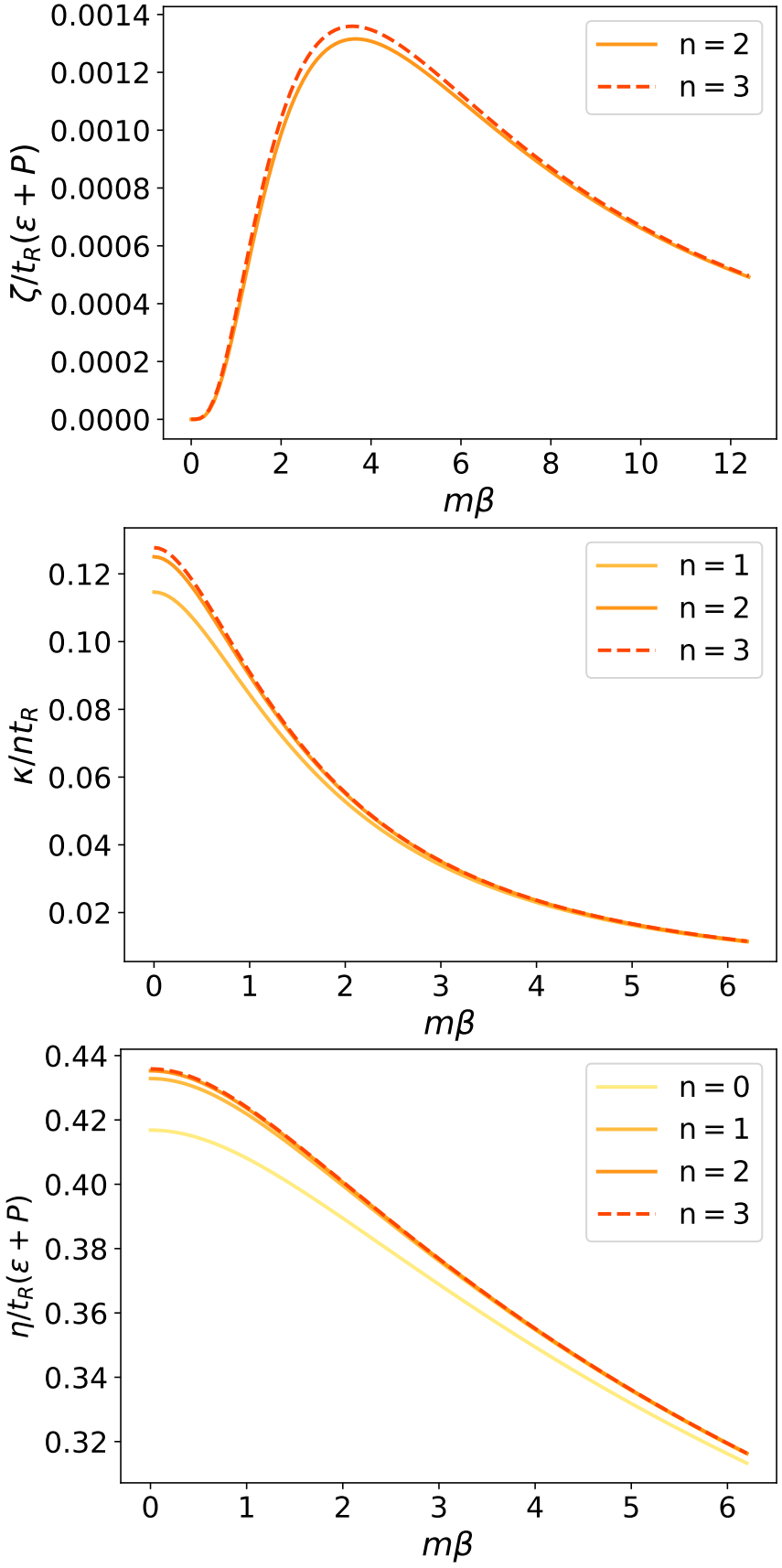

Figure 12. The behavior of transport coefficients with a QCD inspired relaxation time $\tau \propto \sqrt{E_p}$ with basis polynomials up to order n.

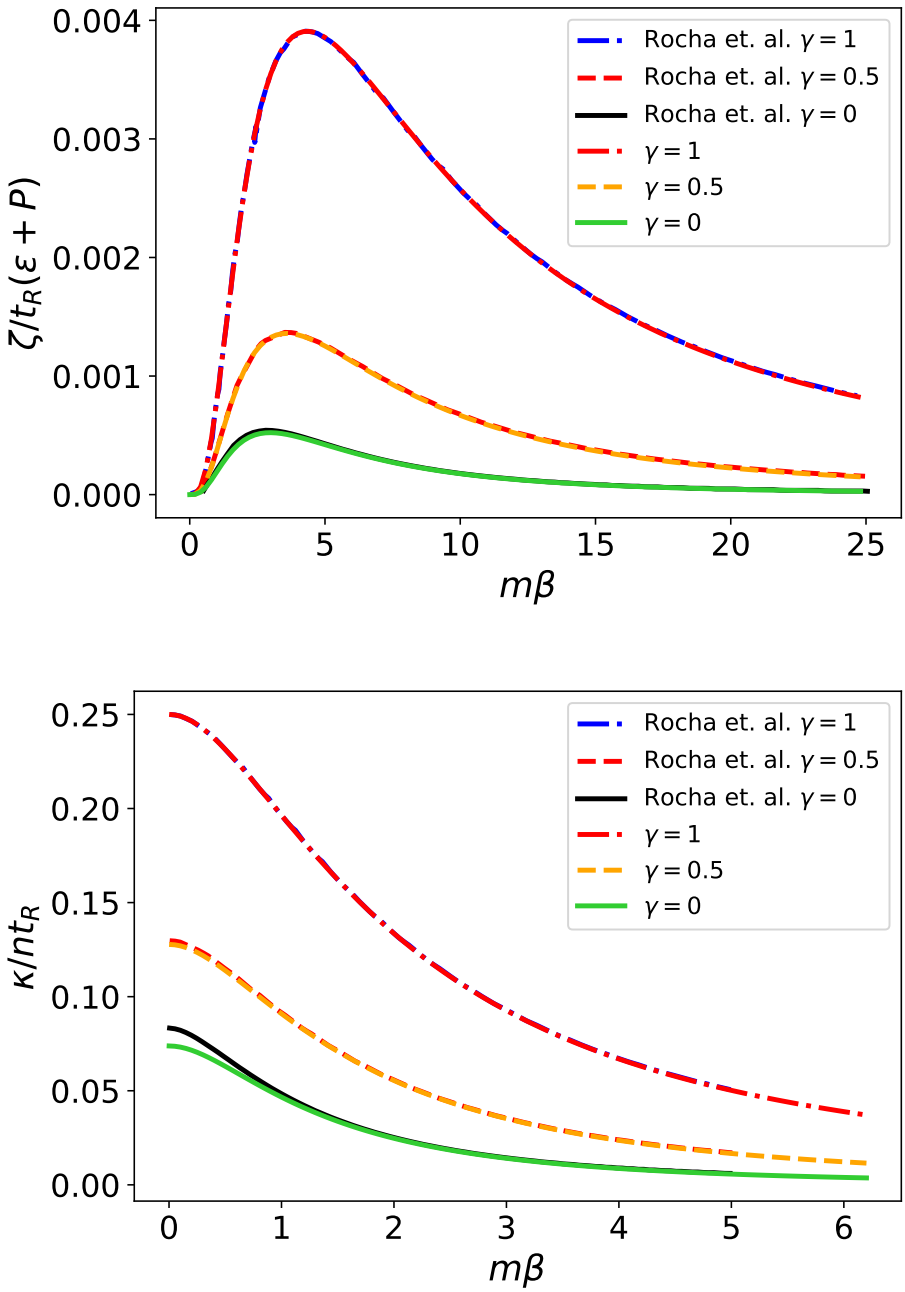

Figure 13. Comparison between the results of the original paper by Rocha et al. [10] and this work.

6 Conclusions

In this thesis we have now thoroughly gone through the basic principles of obtaining fluid dynamical transport coefficients using the novel relaxation time approximation for the Boltzmann equation. The main motivation was that while the Anderson–Witting approximation delivers appropriate estimates for the shear viscosity, its fundamental properties disagree with the linearized Boltzmann equation and as such it fails to deliver suitable estimates for the bulk viscosity and particle diffusion coefficient.

The results from the novel relaxation time approximation seem to give consistent approximations for both, the bulk viscosity and particle diffusion coefficient that are in good agreement with the second law of thermodynamics. Considering high energy physics such as heavy-ion collision, we are particularly interested in the behavior of these transport coefficients in the ultrarelativistic limit, $m\beta \ll 1$. When inspecting the behavior of the obtained transport coefficients in Fig. 11, bulk viscosity seems to vanish in the ultrarelativistic limit while dimensionless particle diffusion coefficient and shear viscosity tend to a constant. As discussed before, in theory the expressions for the transport coefficients are valid for only an infinite series of the basis polynomials, but their behavior seems to converge quickly. Already the lowest order approximation seems to capture the temperature behavior well, although accounting more polynomial functions changes the value of particle diffusion coefficient and shear viscosity at the ultrarelativistic limit, as seen in Fig. 12.

Comparing the results obtained in this work show a small discrepancy between the original work of Rocha et al. [10], which is visualized in Fig. 13. The results seem to differ mostly in the constant relaxation time plots. This however could be just due to different orders of approximation, since here we did the calculations with a basis of just four polynomial functions. Accounting more polynomial functions in the calculation seems to change the ultrarelativistic behavior of the coefficients more with relaxation times that have weaker energy dependency.

It is especially interesting to compare the results for the coefficient of shear viscosity obtained from inverting δf directly from the Boltzmann equation to the

ones obtained by expanding it in fluid dynamical gradients with coefficients expressed as power series in energy. This shows how powerful tool the method of expanding functions in a complete basis of orthogonal polynomials here is, since already the lowest order approximation seems to pick the true behavior well.

Here all the calculations are carried out assuming a gas of classical particles, which is per se not the case in heavy-ion collisions, where the overall size of the system already indicates the occurrence of quantum dynamics. In the future the calculation could be extended accounting for boson and fermion statistics. This would lead to more involved integral expressions which would be represented by series of Bessel functions. These calculations are also just an illustration how the novel relaxation time can be used to capture the underlying microscopic dynamics in a consistent way to derive plausible results for the macroscopic quantities. Obtaining realistic transport coefficients for heavy-ion collisions should incorporate information of the multicomponent quantum nature of the QGP and hadronic gas phases.

The true power of the novel relaxation time approximation could be utilized in constructing effective descriptions of matter in the future. The relaxation time approximation, now consistent with the basic properties of the Boltzmann equation, could be used to capture the prominent behavior of matter without necessarily knowing the details of individual interactions. In heavy-ion collisions the properties of matter change considerably on the course of the events as the QGP slowly confines into hadron gas, which means the nature of the particle interactions also change. In the kinetic theory this leads to changes the transition rate and scattering cross sections. Calculations considering varying momentum dependent scattering cross sections inside the collision integrals quickly become cumbersome and computationally heavy. In practise the transport coefficients customarily enter the simulations in a parametric form, which are tried to be fitted in the experimental results. The transport coefficients matched to the experimental data should contain real information of the material properties. Relaxation times, sufficiently molded to recreate these transport coefficients, could give decent effective kinetic descriptions of bulk QCD and hadronic matter present in high energy collisions.

References

[1] S. Borsanyi, Z. Fodor, J.N. Guenther, R. Kara, S.D. Katz, P. Parotto, A. Pasztor, C. Ratti and K.K. Szabo, "QCD Crossover at Finite Chemical Potential from Lattice Simulations", *Phys. Rev. Lett.* **125** (2020) no. 5, 052001. DOI: 10.1103/PhysRevLett.125.052001

[2] E. Fermi, "High Energy Nuclear Events", *Prog. Theor. Phys.* **5** (1950) 570–583. DOI: 10.1143/ptp/5.4.570

[3] S.Z. Belenkji and L.D. Landau, "Hydrodynamic theory of multiple production of particles", *Nuovo Cim. Supp.* **3**(S1) (1956) 15–31. DOI: 10.1007/bf02745507

[4] K.J. Eskola, K. Kajantie, P.V. Ruuskanen and K. Tuominen, "Scaling of transverse energies and multiplicities with atomic number and energy in ultra-relativistic nuclear collisions", *Nucl. Phys. B* **570**(1–2) (2000) 379–389. DOI: 10.1016/S0550-3213(99)00720-8

[5] W. Israel, "Nonstationary Irreversible Thermodynamics: A Causal Relativistic Theory", *Ann. Phys.* **100**(1–2) (1976) 310–331. DOI: 10.1016/0003-4916(76)90064-6

[6] W. Israel and J.M. Stewart, "Transient Relativistic Thermodynamics and Kinetic Theory", *Ann. Phys.* **118**(2) (1979) 341–372. DOI: 10.1016/0003-4916(79)90130-1

[7] G.S. Denicol, H. Niemi, E. Molnar and D.H. Rischke, "Derivation of transient relativistic fluid dynamics from the Boltzmann equation", *Phys. Rev. D* **85** (2012) 114047. DOI: 10.1103/PhysRevD.85.114047

[8] F. Cooper and G. Frye, "Single-particle distribution in the hydrodynamic and statistical thermodynamic models of multiparticle production", *Phys. Rev. D* **10** (1974) 186. DOI: 10.1103/PhysRevD.10.186

[9] J. Nagle and W. Zajc, "Small System Collectivity in Relativistic Hadronic and Nuclear Collisions", *Annu. Rev. Nucl. Part. Sci.* **68** (2018) 211–235. DOI: 10.1146/annurev-nucl-101916-123209

[10] G.S. Rocha, G.S. Denicol and J. Noronha, "Novel Relaxation Time Approximation to the Relativistic Boltzmann Equation", *Phys. Rev. Lett.* **127** (2021) no. 4, 042301. DOI: 10.1103/PhysRevLett.127.042301

[11] P.B. Arnold, G.D. Moore and L.G. Yaffe, "Transport coefficients in high temperature gauge theories. 2. Beyond leading log", *JHEP* **05** (2003) 051. DOI: 10.1088/1126-6708/2003/05/051

[12] E. Molnar, H. Niemi, G.S. Denicol and D.H. Rischke, "Relative importance of second-order terms in relativistic dissipative fluid dynamics", *Phys. Rev. D* **89** (2014) no. 7, 074010. DOI: 10.1103/PhysRevD.89.074010

[13] J.L. Anderson and H.R. Witting, "A relativistic relaxation-time model for the Boltzmann equation", *Physica* **74** (1974) 466. DOI: 10.1016/0031-8914(74)90355-3

[14] K. Dusling, G.D. Moore and D. Teaney, "Radiative energy loss and v2 spectra for viscous hydrodynamics", *Phys. Rev. C* **81** (2010) 034907. DOI: 10.1103/PhysRevC.81.034907

[15] A. Kurkela and U.A. Wiedemann, "Analytic structure of nonhydrodynamic modes in kinetic theory", *Eur. Phys. J. C* **79** (2019) no. 9, 776. DOI: 10.1140/epjc/s10052-019-7271-9

[16] S.R. de Groot, W.A. van Leeuwen and Ch. G. van Weert, *Relativistic kinetic theory - Principles and applications*, North-Holland Publishing Company (1980). ISBN: 0444854533

[17] L.E. Reichl, *A Modern Course in Statistical Physics*, Wiley-VCH (1999). ISBN: 9783527690480

[18] H. Grad, "On the kinetic theory of rarefied gases", *Comm. Pure Appl. Math.* **2** (1949) 331–407. DOI: 10.1002/cpa.3160020403

[19] L.P. Csernai, *Introduction to Heavy ion Collisions*, Wiley (2008). ISBN: 0471934208

[20] P.L. Bhatnagar, E.P. Gross and M. Krook, "A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems", *Phys. Rev.* **94** (1954) 511–525. DOI: 10.1103/PhysRev.94.511

[21] C. Marle, "Sur l'établissement des équations de l'hydrodynamique des fluides relativistes dissipatifs", *Ann. de l'IHP Physique Théorique* **10** (1969) no. 1, 67–126; no. 2, 127–194. URL: http://www.numdam.org/item/AIHPA_1969__10_1_67_0/, http://www.numdam.org/item/AIHPA_1969__10_2_127_0/

[22] G.S. Rocha, G.S. Denicol, M.N. Ferreira and J. Noronha, "Novel Relaxation Time Approximation: A Consistent Calculation of Transport Coefficients with QCD-inspired Relaxation Times", *Acta Phys. Polon. Supp.* **16** (2023) no. 1, 1–A29. DOI: 10.5506/APhysPolBSupp.16.1-A29

[23] G.S. Rocha and G.S. Denicol, "Transient fluid dynamics with general matching conditions: A first study from the method of moments". *Phys. Rev. D* **104** (2021) 9 096016. DOI: 10.1103/PhysRevD.104.096016

[24] C. Eckart, "The Thermodynamics of Irreversible Processes III: Relativistic Theory of the Simple Fluid", *Phys. Rev.* **58** (1940) 919–924. DOI: 10.1103/PhysRev.58.919

[25] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Elsevier (1987). ISBN: 9781483161044

[26] G.S. Denicol, "Microscopic Foundations of Relativistic Dissipative Fluid Dynamics", Ph.D. dissertation, Goethe University Frankfurt (2012).

[27] B. Sharma and R. Kumar, "A brief introduction to bulk viscosity of fluids", arXiv:2303.08400 (2023). DOI: 10.48550/arXiv.2303.08400

[28] W.A. Hiscock and L. Lindblom, "Generic instabilities in first-order dissipative relativistic fluid theories". *Phys. Rev. D* **31**(4) (1985) 725–733. DOI: 10.1103/physrevd.31.725

[29] S. Chapman and T.G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Cambridge University Press (1952). ISBN: 9780521075770

[30] G.S. Rocha and G.S. Denicol, "Perturbative approaches in relativistic kinetic theory and the emergence of first-order hydrodynamics", *Phys. Rev. D* **106** (2022) 036010. DOI: 10.1103/PhysRevD.106.036010

A Orthogonal polynomials $P_n^{(l)}$

An explicit calculation of the first few energy polynomials is presented in this section. We construct the orthogonal polynomials to satisfy the relation

$$\int dP\ \frac{E_p}{\tau}\frac{W^{(l)}}{(2l+1)!!}\Big(\Delta_{\mu\nu}p^\mu p^\nu\Big)^l P_n^{(l)} P_m^{(l)} f_{0p} = \delta_{nm}\ . \tag{A.1}$$

The normalization factor $W^{(l)}$ is recovered from the condition

$$\int dP\ \frac{E_p}{\tau}\frac{W^{(l)}}{(2l+1)!!}\Big(\Delta_{\mu\nu}p^\mu p^\nu\Big)^l P_0^{(l)} P_0^{(l)} f_{0p} = 1\ . \tag{A.2}$$

All the polynomials of order 0 can be set to 1 without loss of generality

$$P_0^{(l)} = a_{00}^{(l)} \equiv 1\ , \tag{A.3}$$

which means the normalization factor can be written as

$$W^{(l)} = \frac{1}{\langle 1\rangle^{(l)}}\ , \tag{A.4}$$

where we defined generalized integrals in a similar manner as in the main text

$$\langle\cdots\rangle^{(l)} = \int dP\ \frac{E_p}{\tau}\frac{(\cdots)}{(2l+1)!!}\Big(\Delta_{\mu\nu}p^\mu p^\nu\Big)^l f_{0p}\ . \tag{A.5}$$

The first order polynomial function is

$$P_1^{(l)} = a_{10}^{(l)} + a_{11}^{(l)} E_p\ . \tag{A.6}$$

The coefficients $a_{10}^{(l)}$ and $a_{11}^{(l)}$ can be solved from the constraints

$$\Big\langle P_0^{(l)} P_1^{(l)}\Big\rangle^{(l)} = 0\ , \tag{A.7}$$

$$\Big\langle P_1^{(l)} P_1^{(l)}\Big\rangle^{(l)} = 1\ . \tag{A.8}$$

Simplifying equations (A.7) and (A.8) gives

$$a_{11}^{(l)} = -\frac{\langle 1\rangle^{(l)}}{\langle E_p\rangle^{(l)}} a_{10}^{(l)} \,, \tag{A.9}$$

$$a_{10}^{(l)} = \left(\frac{\left[\langle E_p\rangle^{(l)}\right]^2}{\langle E_p^2\rangle^{(l)}\langle 1\rangle^{(l)} - \left[\langle E_p\rangle^{(l)}\right]^2} \right)^{\frac{1}{2}} \,, \tag{A.10}$$

which allows $P_1^{(l)}$ to be expressed as

$$P_1^{(l)} = \left(\frac{\left[\langle E_p\rangle^{(l)}\right]^2}{\langle E_p^2\rangle^{(l)}\langle 1\rangle^{(l)} - \left[\langle E_p\rangle^{(l)}\right]^2} \right)^{\frac{1}{2}} \left(1 - \frac{\langle 1\rangle^{(l)}}{\langle E_p\rangle^{(l)}} E_p \right) \,. \tag{A.11}$$

This polynomial function is used in the novel relaxation time formula, since it is first order in energy and orthogonal to both 1 and $p^{\langle\mu\rangle}$ in the inner product under consideration.

The polynomial function of second order is

$$P_2^{(l)} = a_{20}^{(l)} + a_{21}^{(l)} E_p + a_{22}^{(l)} E_p^2 \,, \tag{A.12}$$

and the coefficients are constrained by

$$\left\langle P_0^{(l)} P_2^{(l)} \right\rangle^{(l)} = 0 \,, \tag{A.13}$$

$$\left\langle P_1^{(l)} P_2^{(l)} \right\rangle^{(l)} = 0 \,, \tag{A.14}$$

$$\left\langle P_2^{(l)} P_2^{(l)} \right\rangle^{(l)} = 1 \,. \tag{A.15}$$

After some manipulation the coefficients are found

$$a_{22}^{(l)} = -\frac{\langle 1\rangle^{(l)} a_{20}^{(l)} + \langle E_p\rangle^{(l)} a_{21}^{(l)}}{\langle E_p^2\rangle^{(l)}} \,, \tag{A.16}$$

$$a_{21}^{(l)} = -\frac{A^{(l)}}{B^{(l)}} a_{20}^{(l)} \,, \tag{A.17}$$

$$a_{20}^{(l)} = \left(\frac{\langle E_p^4 \rangle^{(l)} \left[\langle 1 \rangle^{(l)} - \langle E_p \rangle^{(l)} \frac{A^{(l)}}{B^{(l)}} \right]^2}{\langle 1 \rangle^{(l)} (\langle E_p \rangle^{(l)})^2} \right. \\ \left. - \frac{\left[\langle E_p^2 \rangle^{(l)})^2 + \langle E_p^3 \rangle^{(l)} \langle E_p^2 \rangle^{(l)} \frac{A^{(l)}}{B^{(l)}} \right] \left[\langle 1 \rangle^{(l)} - \langle E_p \rangle^{(l)} \frac{A^{(l)}}{B^{(l)}} \right]}{\langle 1 \rangle^{(l)} (\langle E_p \rangle^{(l)})^2} \right)^{-\frac{1}{2}} , \tag{A.18}$$

where we defined the following coefficients

$$A^{(l)} = \langle E_p \rangle^{(l)} \langle E_p^2 \rangle^{(l)} - \langle 1 \rangle^{(l)} \langle E_p^3 \rangle^{(l)} , \tag{A.19}$$

$$B^{(l)} = \left(\langle E_p^2 \rangle^{(l)} \right)^2 - \langle E_p \rangle^{(l)} \langle E_p^3 \rangle^{(l)} . \tag{A.20}$$

The polynomial function of third order is

$$P_3^{(l)} = a_{30}^{(l)} + a_{31}^{(l)} E_p + a_{32}^{(l)} E_p^2 + a_{33}^{(l)} E_p^3 , \tag{A.21}$$

and the coefficients are constrained by

$$\left\langle P_0^{(l)} P_3^{(l)} \right\rangle^{(l)} = 0 , \tag{A.22}$$

$$\left\langle P_1^{(l)} P_3^{(l)} \right\rangle^{(l)} = 0 , \tag{A.23}$$

$$\left\langle P_2^{(l)} P_3^{(l)} \right\rangle^{(l)} = 0 , \tag{A.24}$$

$$\left\langle P_3^{(l)} P_3^{(l)} \right\rangle^{(l)} = 1 . \tag{A.25}$$

Using (A.22)-(A.25) we find

$$a_{30}^{(l)} = -\frac{\langle E_p \rangle^{(l)} a_{31}^{(l)} + \langle E_p^2 \rangle^{(l)} a_{32}^{(l)} + \langle E_p^3 \rangle^{(l)} a_{33}^{(l)}}{\langle 1 \rangle^{(l)}} , \tag{A.26}$$

$$a_{31}^{(l)} = -\frac{A^{(l)} a_{32}^{(l)} + C^{(l)} a_{33}^{(l)}}{D^{(l)}} , \tag{A.27}$$

$$a_{32}^{(l)} = \frac{A^{(l)}C^{(l)} - D^{(l)}E^{(l)}}{\left(A^{(l)}\right)^2 - D^{(l)}F^{(l)}} a_{33}^{(l)} \,, \tag{A.28}$$

$$a_{33}^{(l)} = \left(\frac{\left(\langle 1 \rangle^{(l)}\right)^2 D^{(l)}}{\frac{\left(A^{(l)}C^{(l)} - D^{(l)}E^{(l)}\right)^2}{\left(A^{(l)}\right)^2 - B^{(l)}F^{(l)}} + \left(C^{(l)}\right)^2 - G^{(l)}D^{(l)}} \right)^{\frac{1}{2}} \,, \tag{A.29}$$

where we further defined

$$C^{(l)} = \langle E_p \rangle^{(l)} \langle E_p^3 \rangle^{(l)} - \langle 1 \rangle^{(l)} \langle E_p^4 \rangle^{(l)} \,, \tag{A.30}$$

$$D^{(l)} = \left(\langle E_p \rangle^{(l)}\right)^2 - \langle 1 \rangle^{(l)} \langle E_p^2 \rangle^{(l)} \,, \tag{A.31}$$

$$E^{(l)} = \langle E_p^2 \rangle^{(l)} \langle E_p^3 \rangle^{(l)} - \langle 1 \rangle^{(l)} \langle E_p^5 \rangle^{(l)} \,, \tag{A.32}$$

$$F^{(l)} = \left(\langle E_p^2 \rangle^{(l)}\right)^2 - \langle 1 \rangle^{(l)} \langle E_p^4 \rangle^{(l)} \,, \tag{A.33}$$

$$G^{(l)} = \left(\langle E_p^3 \rangle^{(l)}\right)^2 - \langle 1 \rangle^{(l)} \langle E_p^6 \rangle^{(l)} \,. \tag{A.34}$$

The calculation of the transport coefficients in section 5.2 are carried out with the polynomial functions characterized by equations (A.3), (A.9)-(A.11), (A.16)-(A.20) and (A.26)-(A.34).

B Constructing NRTA

In order to find the full expression for the collision term in the novel relaxation time approximation, we consider the Boltzmann equation with the ansatz (2.55)

$$p^\mu \partial_\mu f_p = -\frac{E_p}{\tau} f_{0p} \left(\phi_p - A P_0^{(0)} - B P_1^{(0)} - C_\mu p^{\langle \mu \rangle} \right) . \tag{B.1}$$

The goal is to construct the coefficients A, B and C_μ in such a way that the NRTA reproduces the macroscopic conservation laws $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ when contracted with the collision invariants 1 and p^μ.

To find the coefficient A, we contract (B.1) with $P_0^{(0)}$ and integrate over momenta

$$\int d\boldsymbol{P}\; P_0^{(0)} p^\mu \partial_\mu f_p = -\int d\boldsymbol{P}\; \frac{E_p}{\tau} f_{0p} \Big(P_0^{(0)} \phi_p - A P_0^{(0)} P_0^{(0)} - \underbrace{B P_0^{(0)} P_1^{(0)}}_{=0,(2.56)} - \underbrace{C_\mu P_0^{(0)} p^{\langle \mu \rangle}}_{=0,(2.56)} \Big) , \tag{B.2}$$

where the last two terms in the right hand side can be eliminated using the orthogonality of the basis elements. As the zeroth order polynomial function $P_0^{(0)}$ was set to 1, the left hand side of (B.2) is just the divergence of particle 4-current

$$\underbrace{\partial_\mu N^\mu}_{\equiv 0} = -\int d\boldsymbol{P}\; \frac{E_p}{\tau} f_{0p} \Big(P_0^{(0)} \phi_p - A P_0^{(0)} P_0^{(0)} \Big) , \tag{B.3}$$

which we now explicitly force to zero for the NRTA to fulfill the macroscopic laws. From here it is straightforward to solve for the coefficient A

$$A = \frac{\int d\boldsymbol{P}\; (E_p/\tau) P_0^{(0)} \phi_p f_{0p}}{\int d\boldsymbol{P}\; (E_p/\tau) P_0^{(0)} P_0^{(0)} f_{0p}} . \tag{B.4}$$

The coefficient B can be found similarly, now contracting (B.1) with $P_1^{(0)}$ and

integrating over momenta

$$\int d\boldsymbol{P}\; P_1^{(0)} p^\mu \partial_\mu f_p = -\int d\boldsymbol{P}\; \frac{E_p}{\tau} f_{0p}\Big(P_1^{(0)}\phi_p - \underbrace{A P_0^{(0)} P_1^{(0)}}_{=0,(2.56)} - B P_1^{(0)} P_1^{(0)} - \underbrace{C_\mu P_1^{(0)} p^{\langle\mu\rangle}}_{=0,(2.56)}\Big)\,. \tag{B.5}$$

Recalling (2.53) for the first order polynomial function $P_1^{(0)}$ and we get

$$\underbrace{\partial_\mu N^\mu}_{\equiv 0} - \frac{\langle E_p/\tau\rangle_0}{\langle E_p^2/\tau\rangle_0} u_\nu \underbrace{\partial_\mu T^{\mu\nu}}_{\equiv 0} = -\int d\boldsymbol{P}\; \frac{E_p}{\tau} f_{0p}\Big(P_1^{(0)}\phi_p - B P_1^{(0)} P_1^{(0)}\Big)\,, \tag{B.6}$$

where we now explicitly set also the divergence of the energy-momentum tensor to zero. This leads to the following expression for the coefficient B

$$B = \frac{\int d\boldsymbol{P}\; (E_p/\tau) P_1^{(0)} \phi_p f_{0p}}{\int d\boldsymbol{P}\; (E_p/\tau) P_1^{(0)} P_1^{(0)} f_{0p}}\,. \tag{B.7}$$

Finally the coefficient C_μ is recovered by contracting (B.1) with $p_{\langle\nu\rangle}$ and integrating over momenta

$$\int d\boldsymbol{P}\; p_{\langle\nu\rangle} p^\mu \partial_\mu f_p = -\int d\boldsymbol{P}\; \frac{E_p}{\tau} f_{0p}\Big(p_{\langle\nu\rangle}\phi_p - \underbrace{A P_0^{(0)} p_{\langle\nu\rangle}}_{=0,(2.56)} - \underbrace{B P_1^{(0)} p_{\langle\nu\rangle}}_{=0,(2.56)} - C_\mu p_{\langle\nu\rangle} p^{\langle\mu\rangle}\Big)\,. \tag{B.8}$$

Making use of the properties of projected tensors we get

$$\underbrace{\partial_\mu T^\mu_\nu}_{\equiv 0} - u_\nu u_\alpha \underbrace{\partial_\mu T^{\mu\alpha}}_{\equiv 0} = -\int d\boldsymbol{P}\; \frac{E_p}{\tau} p_{\langle\nu\rangle}\phi_p f_{0p} + \frac{1}{3}\Delta^\mu_\nu C_\mu \int d\boldsymbol{P}\; \frac{E_p}{\tau}\Delta_{\alpha\beta} p^\alpha p^\beta f_{0p}\,, \tag{B.9}$$

which leads to the coefficient C_μ as

$$C_\mu = \frac{\int d\boldsymbol{P}\; (E_p/\tau) p_{\langle\mu\rangle}\phi_p f_{0p}}{(1/3)\int d\boldsymbol{P}\; (E_p/\tau) p_{\langle\nu\rangle} p^{\langle\nu\rangle} f_{0p}}\,. \tag{B.10}$$

C Thermodynamic integrals

In order to obtain the first Chapman-Enskog approximation in section 4.1, we would like to represent the partial derivatives in terms of the integrals I_{nq}. The auxiliary thermodynamic integrals are defined as

$$I_{nq} = \frac{1}{(2q+1)!!}\int d\boldsymbol{P}\,\left(u_\mu p^\mu\right)^{n-2q}\Big(-\Delta^{\mu\nu}p_\mu p_\nu\Big)^q f_{0p}\,, \tag{C.1}$$

$$J_{nq} = \frac{\partial I_{nq}}{\partial\alpha}\,. \tag{C.2}$$

From this relation one can show that J_{nq} can be expressed solely in terms of the I_{nq} integrals. Since the integral is written in a Lorentz invariant way, the calculation (C.2) can be carried out in LRF

$$J_{nq} = \frac{1}{(2q+1)!!}\int \frac{d^3p}{(2\pi)^3 E_p}\,E_p^{n-2q}\Big(E_p^2-m^2\Big)^q\frac{\partial f_{0p}}{\partial\alpha}\,. \tag{C.3}$$

Expressing the momentum integrals in polar coordinates and changing integration variables from momentum to energy gives

$$\begin{aligned} J_{nq} &= \frac{1}{(2q+1)!!}\int d\Omega\int_0^\infty \frac{dp}{(2\pi)^3}\,|\bar{p}|^2 E_p^{n-2q-1}\Big(E_p^2-m^2\Big)^q\frac{e^{\beta E_p-\alpha}}{(e^{\beta E_p-\alpha}+a)^2} \\ &= \frac{1}{(2q+1)!!}\int d\Omega\int_m^\infty \frac{dE_p}{(2\pi)^3}\,E_p^{n-2q}\Big(E_p^2-m^2\Big)^{q+\frac{1}{2}}\frac{e^{\beta E_p-\alpha}}{(e^{\beta E_p-\alpha}+a)^2}\,. \end{aligned}$$

Next partially integrating the last line

$$\begin{aligned} J_{nq} = \frac{1}{(2q+1)!!}\int d\Omega\,\underbrace{\Big/_m^\infty \frac{1}{(2\pi)^3}\,E_p^{n-2q}\Big(E_p^2-m^2\Big)^{q+\frac{1}{2}}\left(-\frac{1}{\beta}\frac{1}{e^{\beta E_p-\alpha}+a}\right)}_{=0} \\ -\frac{1}{(2q+1)!!}\int d\Omega\int_m^\infty \frac{dE_p}{(2\pi)^3}\,(n-2q)E_p^{n-2q-1}\Big(E_p^2-m^2\Big)^{q+\frac{1}{2}}\left(-\frac{1}{\beta}\frac{1}{e^{\beta E_p-\alpha}+a}\right) \\ -\frac{1}{(2q+1)!!}\int d\Omega\int_m^\infty \frac{dE_p}{(2\pi)^3}\,(2q+1)E_p^{n-2q+1}\Big(E_p^2-m^2\Big)^{q-\frac{1}{2}}\left(-\frac{1}{\beta}\frac{1}{e^{\beta E_p-\alpha}+a}\right)\,. \end{aligned}$$

Then converting back to 3-dimensional momentum integrals

$$J_{nq} = \frac{n-2q}{\beta}\frac{1}{(2q+1)!!}\int \frac{d^3p}{(2\pi)^3 E_p}\, E_p^{(n-1)-2q}\Big(E_p^2 - m^2\Big)^q f_{0p} + \frac{2q+1}{\beta(2q+1)}\frac{1}{(2\,[q-1]+1)!!}\int \frac{d^3p}{(2\pi)^3 E_p}\, E_p^{(n-1)-2(q-1)}\Big(E_p^2 - m^2\Big)^{q-1} f_{0p}\ , \tag{C.4}$$

where the integrals can be identified as $I_{n-1,q}$ and $I_{n-1,q-1}$

$$J_{nq} = \frac{n-2q}{\beta} I_{n-1,q} + \frac{1}{\beta} I_{n-1,q-1}\ . \tag{C.5}$$

Equation (C.5) shows that the partial derivatives of pressure appearing in (4.13) can be expressed as

$$J_{21} = \frac{I_{10}}{\beta}\ , \tag{C.6}$$

$$J_{31} = \frac{I_{20} + I_{21}}{\beta}\ , \tag{C.7}$$

solely in terms of I_{10}, I_{20} and I_{21}, which were identified to represent particle density, energy density and pressure, respectively.

D Expansion coefficients $s_n^{(l)}$

In the first order Chapman–Enskog expansion and using the novel relaxation time approximation, the Boltzmann equation is

$$
\begin{aligned}
&-\frac{E_p}{\tau}\left(\phi_p-\frac{\left\langle\frac{E_p}{\tau}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}\right\rangle_0}-P_1^{(0)}\frac{\left\langle\frac{E_p}{\tau}P_1^{(0)}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}P_1^{(0)}P_1^{(0)}\right\rangle_0}-p^{\langle\mu\rangle}\frac{\left\langle\frac{E_p}{\tau}p_{\langle\mu\rangle}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}p_{\langle\nu\rangle}p^{\langle\nu\rangle}\right\rangle_0}\right)f_{0p} \\
&=\Big(A_p\theta+B_pp^\mu\nabla_\mu\alpha-\beta p^\mu p^\nu\sigma_{\mu\nu}\Big)f_{0p}\,,
\end{aligned}
\tag{D.1}
$$

where ϕ_p is expressed as (5.2) with energy dependent coefficients expanded in orthogonal polynomials. Extracting the coefficients $s_n^{(l)}$ can be done by abusing the orthogonality relations (5.5) and (2.60) by conveniently contracting (D.1) with the corresponding polynomial $P_n^{(l)}$ and integrating over momenta.

The scalar coefficients $s_n^{(0)}$ for $n\geq 2$ can be thus recovered from

$$
\begin{aligned}
&-\int d\boldsymbol{P}\;\frac{E_p}{\tau}P_n^{(0)}\left(\phi_p-\underbrace{\frac{\left\langle\frac{E_p}{\tau}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}\right\rangle_0}}_{\Rightarrow 0,(5.5)}-\underbrace{P_1^{(0)}\frac{\left\langle\frac{E_p}{\tau}P_1^{(0)}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}P_1^{(0)}P_1^{(0)}\right\rangle_0}}_{\Rightarrow 0,(5.5)}-\underbrace{p^{\langle\mu\rangle}\frac{\left\langle\frac{E_p}{\tau}p_{\langle\mu\rangle}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}p_{\langle\nu\rangle}p^{\langle\nu\rangle}\right\rangle_0}}_{\Rightarrow 0,(2.60)}\right) \\
&=\int d\boldsymbol{P}\;P_n^{(0)}\Big(A_p\theta+\underbrace{B_pp^\mu\nabla_\mu\alpha}_{\Rightarrow 0,(2.60)}-\underbrace{\beta p^\mu p^\nu\sigma_{\mu\nu}}_{\Rightarrow 0,(2.60)}\Big)f_{0p}\,.
\end{aligned}
\tag{D.2}
$$

For coefficients where $n\geq 2$ the last 3 terms on the left hand side vanish due to the orthogonality relations. Note that in the Anderson–Witting model this would be valid for all n. In the NRTA, the coefficients $s_0^{(0)}$ and $s_1^{(0)}$ cannot be recovered by by inverting the Boltzmann equation, because in this case the left hand side is no longer invertible for ϕ_p. On the right hand the last two terms are space like and vanish

when contracted with the energy polynomials inside the integral. This leads to

$$\begin{aligned} &-\int d\boldsymbol{P}\ \frac{E_p}{\tau}P_n^{(0)}\left(\left[\sum_{m=0}^{\infty}s_m^{(0)}P_m^{(0)}\right]\theta+\underbrace{S_m^{(1)}p^\mu\nabla_\mu\alpha}_{\Rightarrow 0,(2.60)}+\underbrace{S_m^{(2)}p^\mu p^\nu\sigma_{\mu\nu}}_{\Rightarrow 0,(2.60)}\right)f_{0p} \\ &=\int d\boldsymbol{P}\ P_n^{(0)}A_p\theta f_{0p}\ . \end{aligned} \tag{D.3}$$

On the left hand side orthogonality picks only the term with $P_n^{(0)}$ such that

$$-\int d\boldsymbol{P}\ \frac{E_p}{\tau}s_n^{(0)}P_n^{(0)}P_n^{(0)}\theta f_{0p}=\int d\boldsymbol{P}\ A_pP_n^{(0)}\theta f_{0p}\ . \tag{D.4}$$

Now the momentum independent coefficient $s_n^{(0)}$ can be removed from the integral and solved

$$s_n^{(0)}=-\frac{\int d\boldsymbol{P}\ A_pP_n^{(0)}f_{0p}}{\int d\boldsymbol{P}\ (E_p/\tau)P_n^{(0)}P_n^{(0)}f_{0p}}\ . \tag{D.5}$$

The vector coefficients $s_n^{(1)}$ for $n\geq 1$ are recovered from

$$\begin{aligned} &-\int d\boldsymbol{P}\ \frac{E_p}{\tau}P_n^{(1)}p_{\langle\sigma\rangle}\left(\phi_p-\underbrace{\frac{\left\langle\frac{E_p}{\tau}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}\right\rangle_0}}_{\Rightarrow 0,(2.60)}-\underbrace{P_1^{(0)}\frac{\left\langle\frac{E_p}{\tau}P_1^{(0)}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}P_1^{(0)}P_1^{(0)}\right\rangle_0}}_{\Rightarrow 0,(2.60)}-\underbrace{p^{\langle\mu\rangle}\frac{\left\langle\frac{E_p}{\tau}p_{\langle\mu\rangle}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}p_{\langle\nu\rangle}p^{\langle\nu\rangle}\right\rangle_0}}_{\Rightarrow 0,(5.5)}\right)f_{0p} \\ &=\int d\boldsymbol{P}\ P_n^{(1)}p_{\langle\sigma\rangle}\Big(\underbrace{A_p\theta}_{\Rightarrow 0,(2.60)}+B_pp^\mu\nabla_\mu\alpha-\underbrace{\beta p^\mu p^\nu\sigma_{\mu\nu}}_{\Rightarrow 0,(2.60)}\Big)f_{0p}\ . \end{aligned} \tag{D.6}$$

$$\begin{aligned} \Longrightarrow\ &-\int d\boldsymbol{P}\ \frac{E_p}{\tau}P_n^{(1)}p_{\langle\sigma\rangle}\left(\underbrace{S_p^{(0)}\theta}_{\Rightarrow 0,(2.60)}+\left[\sum_{m=0}^{\infty}s_m^{(1)}P_m^{(1)}\right]p^\mu\nabla_\mu\alpha+\underbrace{S_p^{(2)}p^\mu p^\nu\sigma_{\mu\nu}}_{\Rightarrow 0,(2.60)}\right)f_{0p} \\ &=\int d\boldsymbol{P}\ P_n^{(1)}p_{\langle\sigma\rangle}B_pp^\mu\nabla_\mu\alpha f_{0p}\ . \end{aligned} \tag{D.7}$$

From the left hand side orthogonality picks only the term with $P_n^{(1)}$. Including tensors like $p^{\langle\mu\rangle}p_{\langle\nu\rangle}$ the integrals can be further manipulated with (2.60) into

$$\begin{aligned} &-\frac{1}{3}\Delta_\sigma^\mu\int d\boldsymbol{P}\ \frac{E_p}{\tau}\Delta_{\alpha\beta}p^\alpha p^\beta s_n^{(1)}P_n^{(1)}P_n^{(1)}\nabla_\mu\alpha f_{0p} \\ &=\frac{1}{3}\Delta_\sigma^\mu\int d\boldsymbol{P}\ \Delta_{\alpha\beta}p^\alpha p^\beta B_pP_n^{(1)}\nabla_\mu\alpha f_{0p}\ . \end{aligned} \tag{D.8}$$

By contracting with $g^{\mu\sigma}$ and isolating the momentum independent terms outside the

integral the vector coefficients for $n \geq 1$ become

$$s_n^{(1)} = -\frac{\int d\boldsymbol{P}\ \Delta_{\alpha\beta}p^\alpha p^\beta B_p P_n^{(1)} f_{0p}}{\int d\boldsymbol{P}\ (E_p/\tau)\Delta_{\alpha\beta}p^\alpha p^\beta P_n^{(1)} P_n^{(1)} f_{0p}}\ . \tag{D.9}$$

Finally the tensor coefficients $s_n^{(2)}$

$$\begin{aligned}
&-\int d\boldsymbol{P}\ \frac{E_p}{\tau}p_{\langle\rho}p_{\sigma\rangle}P_n^{(2)}\left(\phi_p - \underbrace{\frac{\left\langle\frac{E_p}{\tau}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}\right\rangle_0}}_{\Rightarrow 0,(2.60)} - P_1^{(0)}\underbrace{\frac{\left\langle\frac{E_p}{\tau}P_1^{(0)}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}P_1^{(0)}P_1^{(0)}\right\rangle_0}}_{\Rightarrow 0,(2.60)} - p^{\langle\mu\rangle}\underbrace{\frac{\left\langle\frac{E_p}{\tau}p_{\langle\mu\rangle}\phi_p\right\rangle_0}{\left\langle\frac{E_p}{\tau}p_{\langle\nu\rangle}p^{\langle\nu\rangle}\right\rangle_0}}_{\Rightarrow 0,(2.60)}\right) f_{0p} \\
&= \int d\boldsymbol{P}\ p_{\langle\rho}p_{\sigma\rangle}P_n^{(2)}\Big(\underbrace{A_p\theta}_{\Rightarrow 0,(2.60)} + \underbrace{B_p p^\mu\nabla_\mu\alpha}_{\Rightarrow 0,(2.60)} - \beta p^\mu p^\nu\sigma_{\mu\nu}\Big)f_{0p}\ .
\end{aligned} \tag{D.10}$$

$$\begin{aligned}
\Longrightarrow\ &-\int d\boldsymbol{P}\ \frac{E_p}{\tau}p_{\langle\rho}p_{\sigma\rangle}P_n^{(2)}\left(\underbrace{S_p^{(0)}\theta}_{\Rightarrow 0,(2.60)} + \underbrace{S_p^{(1)}p^\mu\nabla_\mu\alpha}_{\Rightarrow 0,(2.60)} + \sum_{m=0}^{\infty} s_m^{(2)}P_m^{(2)}p^\mu p^\nu\sigma_{\mu\nu}\right)f_{0p} \\
&= -\int d\boldsymbol{P}\ p_{\langle\rho}p_{\sigma\rangle}P_n^{(2)}\beta p^\mu p^\nu\sigma_{\mu\nu}f_{0p}\ .
\end{aligned} \tag{D.11}$$

$$\begin{aligned}
\Longrightarrow\ &\frac{2}{15}\Delta^{\mu\nu}_{\rho\sigma}\int d\boldsymbol{P}\ \frac{E_p}{\tau}\Big(\Delta_{\alpha\beta}p^\alpha p^\beta\Big)^2 s_n^{(2)}P_n^{(2)}P_n^{(2)}\sigma_{\mu\nu}f_{0p} \\
&= \frac{2}{15}\Delta^{\mu\nu}_{\rho\sigma}\int d\boldsymbol{P}\ \Big(\Delta_{\alpha\beta}p^\alpha p^\beta\Big)^2\beta P_n^{(2)}\sigma_{\mu\nu}f_{0p}\ .
\end{aligned} \tag{D.12}$$

This leads to

$$s_n^{(2)} = \frac{\int d\boldsymbol{P}\ \beta\Big(\Delta_{\alpha\beta}p^\alpha p^\beta\Big)^2 P_n^{(2)}f_{0p}}{\int d\boldsymbol{P}\ (E_p/\tau)\Big(\Delta_{\alpha\beta}p^\alpha p^\beta\Big)^2 P_n^{(2)}P_n^{(2)}f_{0p}}\ . \tag{D.13}$$