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Sari Kallunki

Contents

1. Introduction

Recently many papers on mappings of finite distortion have appeared. Before describing the contents of these papers relevant to this thesis let us state the definition of a mapping of finite distortion. A mapping $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$, Ω an open and connected subset of $\mathbf{R}^n$, is said to have finite distortion if:

i) The Jacobian $J_f(x)$ is locally integrable.

ii) There is a measurable function $K = K(x) \geq 1$, called a K-distortion, finite almost everywhere, such that

$$|Df(x)|^n \leq K(x) J_f(x) \qquad \text{a.e.}$$

Here $|Df(x)|$ is the operator norm of $Df(x)$. The idea of this definition is to generalize the notion of quasiregular mappings. Namely the above definition gives us a quasiregular mapping, also called a mapping of bounded distortion, if $K \in L^\infty$. The theory of quasiregular mappings is finely presented in the monograph of Rickman [17].

In order to gain regularity for a mapping of finite distortion, one needs to pose some additional conditions on K. Since, for quasiregular mappings $K \in L^\infty$, a natural way to continue is to have K in a slightly larger class, for example requiring $\exp(\lambda K) \in L^1_{loc}$ for some $\lambda > 0$ or equivalently letting K be bounded above by a BMO-function. With this assumption these mappings have been shown to have many of the nice properties of quasiregular mappings. The most relevant property, from the point of view of this thesis, is that a mapping with an exponentially integrable K-distortion is open, continuous, and discrete, just as in the quasiregular case. See the papers of Iwaniec, Koskela, and Onninen [9], and Kauhanen, Koskela, and Malý [13], and references therein for more properties of mappings of finite distortion.

Currently there is a school of thought that attempts to do analysis in a metric space setting and that has extended the notions of quasiconformality and differentiability to the abstract metric space setting. However, the definition of mappings of finite distortion is entirely analytic. This thesis attempts to extend the notion of mappings of finite distortion by approaching the subject with metric quantities. Our setting is however Euclidean here. We use limsup and liminf-distortions. The limsup-distortion is

$$H_f(x) = \limsup_{r \to 0} H_f(x, r),$$

and the liminf-distortion is

$$h_f(x) = \liminf_{r \to 0} H_f(x, r).$$

Here

$$H_f(x, r) = \frac{L_f(x, r)}{l_f(x, r)},$$

where

$$L_f(x, r) = \max\{|f(x) - f(y)| : |x - y| \leq r\},$$

and

$$l_f(x,r) = \min\{|f(x) - f(y)| : |x - y| \geq r\}.$$

The quantity $H_f(x,r)$ is well defined when r is sufficiently small and f is an open, continuous, discrete mapping. Indeed, for any open set U we have $\partial fU \subset f\partial U$. Therefore $l_f(x,r) = \min\{|f(x) - f(y)| : |x-y| = r\}$ and $B(x,2r) \cap f^{-1}(f(x))$ is finite because f is discrete. Hence for sufficiently small r we have $B(x,2r) \cap f^{-1}(f(x)) = \{x\}$ and thus $l_f(x,r) > 0$.

The metric approach is central to the theory of quasiconformal mappings. For the theory of quasiconformal mappings see the monograph of Väisälä [18]. The analytic definition of quasiconformality is the following: A quasiconformal mapping is a homeomorphism for which

$$|Df(x)|^n \leq KJ_f(x) \qquad \text{a.e.}$$

where K is a constant. The only difference to quasiregularity is that these mappings are required to be homeomorphic. Quasiconformality has an equivalent metric definition: a homeomorphism is quasiconformal if there is a constant H so that $H_f(x) \leq H$ for every $x \in \Omega$. Recently it has been showed that this can be relaxed. The best version so far is that a homeomorphism is quasiconformal if h_f is finite outside a set of σ-finite $(n-1)$-measure and $h_f(x) \leq H$ almost everywhere, see [11]. The H_f -version of the above result was already proved by Gehring in his seminal paper [5]. This metric theory of quasiconformal mappings has applications in complex dynamics, see [6], [16].

Quasiregular mappings have a metric definition as well. This was first investigated by Martio, Rickman, and Väisälä [14] in 1969. They used the limsup-distortion. Recently Cristea [2] has proved that if $h_f(x) \leq H$ for a continuous, open, discrete mapping f, with certain exceptional porous set, then f is quasiregular.

These results gave us motivation to look for metric definitions for mappings of finite distortion. We investigate open, continuous, discrete mappings for which H_f is finite outside an exceptional σ-finite set and satisfies various integrability conditions. We also examine mappings for which we have conditions for h_f. We generalize the results on homeomorphisms obtained in the papers [10], [11], and [12] to open, continuous, discrete mappings. However, contrary to the case of quasiregular mappings the metric conditions that we arrive at are not necessary. Basically this is because the σ-finiteness of the exceptional set is not guaranteed by the analytic definition when the distortion fails to be bounded.

The thesis is organized as follows. In section 2, we give basic definitions and prove useful geometric lemmas. In section 3, we give integrability conditions for H_f to guarantee that a mapping f is absolutely continuous on almost all lines parallel to the coordinate axes, shortly ACL, or that it belongs to a Sobolev class, and as the last result of the section we prove a local quasisymmetry condition. In the last section, we tackle the more difficult problem of showing analogous results for open, continuous, discrete mappings with h_f satisfying various integrability conditions. We also give a sufficient integrability condition for h_f to guarantee that f has an exponentially integrable K-distortion. The last section is mainly motivated by the paper of Cristea [2].

2. Preliminaries

We deal mostly with continuous, open, discrete, and sense-preserving mappings $f\colon \Omega \to \mathbf{R}^n$, where throughout this thesis Ω is a domain, i.e. an open, connected set of $\mathbf{R}^n$, $n \geq 2$. By openness of a mapping we mean that the image of each open set is open, and discreteness means that for every $y \in \mathbf{R}^n$ the set $f^{-1}(y)$ is discrete. "Sense-preserving" means that the local topological index is positive (see [17, p. 16]). This is actually an inessential restriction, as each mapping that is both discrete and open is either sense-preserving or sense-reserving; but it is assumed for convenience.

Openness guarantees that, for any open set U with $\overline{U}$ compact in Ω, the boundary of the image of U is a subset of the image of the boundary, i.e. $\partial fU \subset f\partial U$. This is true, because fU is open and so $fU \cap \partial fU = \emptyset$, and on the other hand $\partial fU = \partial \overline{fU} \subset \overline{fU} \subset f\overline{U} = fU \cup f\partial U$, since $\overline{U}$ is compact and f is continuous.

Since our mapping is not injective we often have to count the number of preimages of a point $y \in \mathbf{R}^n$. Let $N(y, f, A) = \operatorname{card} f^{-1}(y) \cap A$ and $N(f, A) = \sup_y N(y, f, A)$. By Proposition I.4.10 in [17] we know that if A is compact, then $N(f, A)$ is finite assuming that f is continuous, open, and discrete. This fact is very useful in the absence of injectivity.

To study continuous, open, discrete mappings we need the concept of a q-quasiadditive set function. Let U be an open set in $\mathbf{R}^n$ and $1 \leq q < \infty$. A q-quasiadditive set function in U is a function $\varphi\colon \mathcal{B}(U) \to \overline{\mathbf{R}}$ such that

(1) $\varphi(A) \geq 0$ for every $A \in \mathcal{B}(U)$,
(2) $\varphi(A) < \infty$ if $A \in \mathcal{B}(U)$ is compact,
(3) if $A, B \in \mathcal{B}(U)$ and $A \subset B$, then $\varphi(A) \leq \varphi(B)$,
(4) if $A_1, \ldots, A_k \in \mathcal{B}(U)$ are disjoint sets in $A \in \mathcal{B}(U)$, then $\sum_i \varphi(A_i) \leq q\varphi(A)$.

In addition, we say a q-quasiadditive set function is bounded if $\varphi(A) \leq M < \infty$ for every $A \in \mathcal{B}(U)$. By $\mathcal{B}(U)$ we mean the collection of the Borel subsets of U.

The following lemma guarantees that q-quasiadditive set functions are suitable for our purposes. The proof of the lemma is completely analogous to the proof of the theorem of Lebesgue – Banach. More properties of q-quasiadditive set functions can be found in [14].

Lemma 2.1. *If φ is a bounded q-quasiadditive set function in U, then*

$$\varphi'(x) < \infty \qquad \text{for a.e.} \quad x \in U.$$

Here

$$\varphi'(x) = \limsup_{h \to 0}\{\frac{\varphi(B)}{|B|} : x \in B, B \text{ a closed ball, } \operatorname{diam}(B) \leq h\},$$

is the upper derivative of φ, and $|B|$ means the Lebesgue measure of B.

By setting $\varphi(A) = |f(A)|$ for Borel sets $A \subset U \subset\subset \Omega$ (the notation $U \subset\subset \Omega$ means that $\overline{U} \subset \Omega$ is compact) gives us a bounded q-quasiadditive set function in U with $q = N(f, U)$ when f is open, continuous and discrete. Using Lemma 2.1 and the above defined φ, the proof of the next lemma is based on the Rademacher – Stepanov theorem analogously as in the case where f is a homeomorphism. The lemma gives us a sufficient condition for f to be almost everywhere differentiable.

Lemma 2.2. *Let $f\colon \Omega \to \mathbf{R}^n$ be a continuous, open, discrete, sense-preserving mapping such that*

$$H_f(x) < \infty \quad \text{for a.e. } x \in \Omega.$$

Then f is differentiable almost everywhere.

Proof. Define a bounded q-quasiadditive set function φ in $U \subset\subset \Omega$ as above. By Lemma 2.1 we have $\varphi'(x) < \infty$ for almost every $x \in U$. Now at an almost every point x of U, $\varphi'(x)$ exits and $H_f(x) < \infty$. Fix such a point x, and let $y \in U$ with $0 < |x - y| < \mathrm{d}(x, \partial U)$. Then

$$\begin{aligned}\left(\frac{|f(x) - f(y)|}{|x - y|}\right)^n &\le \left(\frac{L_f(x, |x - y|)}{l_f(x, |x - y|)}\right)^n \left(\frac{l_f(x, |x - y|)}{|x - y|}\right)^n \\ &\le H_f(x, |x - y|)^n \frac{\varphi(B(x, |x - y|))}{|B(x, |x - y|)|}.\end{aligned}$$

Letting $y \to x$ we obtain

$$\limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} < \infty.$$

By the Rademacher-Stepanov theorem the mapping f is a.e. differentiable in U. □

To prove two of our theorems we need the following technical lemma, which we will justify with the help of Corollary II.3.4 in [17]. We use the notation $[c, d] = \{td + (1 - t)c : t \in [0, 1]\}$ for $c, d \in \mathbf{R}^n$ in this lemma and throughout the thesis.

Lemma 2.3. *Let $f\colon \Omega \to \mathbf{R}^n$ be an open, discrete, continuous, sense-preserving mapping. Let $x \in \Omega$ and $r > 0$ be such that $B(x, r) \subset\subset \Omega$. Suppose $y \in B(x, r)$ and $b \in \partial f B(x, r)$ are such that $[f(y), b] \subset \overline{fB(x, r)}$. Then there exists a continuum F such that $y \in F$, $F \subset \overline{B}(x, r)$, $F \cap \partial B(x, r) \neq \emptyset$ and $f(F) \subset [f(y), b]$.*

Proof. Fix $x \in \Omega$ and $r > 0$ such that $B(x, r) \subset\subset \Omega$, and define $\beta\colon [0, 1] \to f(\Omega)$, $\beta(t) = tb + (1 - t)f(y)$. By Lemma I.4.9 in [17], for every $z \in \overline{B}(x, r)$, there is a normal neighborhood U_z i.e. $f\partial U_z = \partial f U_z$ and $U_z \cap f^{-1}(f(z)) = \{z\}$. Now $\{fU_z \cap \beta[0, 1]\}$ consist of open intervals in $\beta[0, 1]$. The line segment $\beta[0, 1]$ is compact, so we have a finite family of open intervals which already covers $\beta[0, 1]$. We denote this family the following way $\{\beta[0, b_1[, \beta]a_i, b_i[, \beta]a_p, 1], i = 2, \ldots, p - 1\}$. So we have a subfamily $\{U_i\}_{i=1}^p$ of the normal neighborhood family $\{U_z\}$. Here U_i is such that $\beta]a_i, b_i[\subset fU_i$

and $a_i, b_i \in \partial f U_i$, $i = 2, \dots, p-1$ (the cases $i = 1, p$ are slightly different). Notice that a normal neighborhood U_z can occur many times in this list under different indices.

Next we will construct F. Now $y \in U_1$, and β has a subpath $\beta|_{[0,b_1[}$ such that $\beta[0, b_1[\subset fU_1$ and $\beta(b_1) \in \partial f U_1$. By Corollary II.3.4 in [17] there is a path $\alpha_1 : [0, b_1[\to U_1$ such that $f \circ \alpha_1 = \beta|_{[0,b_1[}$ and $\alpha_1(0) = y$. Since f is continuous, $f(\overline{\alpha_1[0, b_1[}) = \beta[0, b_1]$. We denote $F_1 = \overline{\alpha_1[0, b_1[}$.

Now if $F_1 \cap \partial B(x, r) \neq \emptyset$, choose F to be the component of $F_1 \cap \overline{B}(x, r)$ that contains y. If this is not the case, then we continue the construction with $y_2 \in F_1 \cap f^{-1}(\beta(b_1)) \subset \partial U_1$. Now there is a subpath $\beta[b_1, b_2] \subset fU_2$ such that $f(y_2) = \beta(b_1)$, $\beta(b_2) \in \partial f U_2$. As above, by corollary II.3.4 in [17], there is a path $\alpha_2\colon [b_1, b_2[\to U_2$ such that $f \circ \alpha_2 = \beta|_{[b_1,b_2[}$ and $\alpha_2(b_1) = y_2$. We set $F_2 = \overline{\alpha_2[b_1, b_2[}$.

If $F_2 \cap \partial B(x, r) \neq \emptyset$, choose F to be the component of $(F_1 \cup F_2) \cap \overline{B}(x, r)$ that contains y. In the other case we repeat the construction. At some point this process will end because for at least for F_p the intersection $F_p \cap \partial B(x, r)$ is non-empty, since $f(F_p) = \beta[b_p, 1]$, and $\beta(1) \in \partial f B(x, r) \subset f \partial B(x, r)$, since f is open and continuous. $\square$

The following results, although not about continuous, open, discrete mappings, are also needed in the sequel. First we have some measure theoretical lemmas for which we need the Hausdorff (outer) measure and content. They are defined as follows: Let $0 < p < \infty$, $A \subset \mathbf{R}^n$. The p-dimensional Hausdorff content is

$$\mathcal{H}^p_\infty(A) = \inf\{\sum_{i \in \mathbf{N}} \operatorname{diam}(V_i)^p : \{V_i\} \text{ is a cover of } A\}$$

and the p-dimensional Hausdorff measure is

$$\mathcal{H}^p(A) = \lim_{t \to 0} \mathcal{H}^p_t(A),$$

where

$$\mathcal{H}^p_t(A) = \inf\{\sum_{i \in \mathbf{N}} \operatorname{diam}(V_i)^p : \{V_i\} \text{ is a cover of } A \text{ and } \operatorname{diam}(V_i) \leq t\}.$$

Clearly if $s > 0$, then we have $\mathcal{H}^p_\infty(sA) = s^p \mathcal{H}^p_\infty(A)$, and $\mathcal{H}^p(sA) = s^p \mathcal{H}^p(A)$. Throughout $sA = \{sa : a \in A\}$.

The next two lemmas provides us with tools to handle sets with a σ-finite $(n-1)$-measure. By the σ-finite $(n-1)$-measure of a set we mean that the set has a countable cover by sets of finite $\mathcal{H}^{n-1}$-measure. The first of these lemmas is a result of Gross, see [18, p. 104]. The proof is repeated here for the convenience of the reader.

Lemma 2.4. *Suppose that $E \subset \mathbf{R}^n$ has σ-finite $(n-1)$-measure and let $P\colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the projection $Px = x - x_n e_n$. Then $E \cap P^{-1}(y)$ is countable for almost every $y \in \mathbf{R}^{n-1}$.*

Proof. Let $E = \cup_i E_i$ such that $\mathcal{H}^{n-1}(E_i) < \infty$. Denote $A_i = \{y \in \mathbf{R}^{n-1} : E_i \cap P^{-1}(y)$ is uncountable$\}$. We have to show that $\mathcal{H}^{n-1}(\cup_i A_i) = 0$. For this it suffices to show that $\mathcal{H}^{n-1}(A_i) = 0$ for all i.

Next we will approximate the measure of A_i. For this let $A_{ip} = \{y \in \mathbf{R}^{n-1} : E_i \cap P^{-1}(y)$ contains at least p points$\}$, $p \in \mathbf{N}$. Now $A_i \subset A_{ip}$ and we will approximate the measure of A_{ip}.

For $k \in \mathbf{N}$ we let A_{ip}^k denote the set of points $y \in A_{ip}$ for which there exits points $x_i, \dots, x_p$ in $E \cap P^{-1}(y)$ such that $|x_j - x_l| \geq 1/k$ for each pair $x_j \neq x_l$. Then $A_{ip}^k \subset A_{ip}^{k+1}$, and $A_{ip} = \cup_k A_{ip}^k$. Consequently $\mathcal{H}^{n-1}(A_{ip}) = \lim_k \mathcal{H}^{n-1}(A_{ip}^k)$.

Next we will show that $p\mathcal{H}^{n-1}(A_{ip}^k) \leq \mathcal{H}^{n-1}(E_i)$ for an arbitrary fixed k. Let $\{G_j\}_{j\in\mathbf{N}}$ be an open covering of E_i such that $\operatorname{diam}(G_j) < 1/k$. If $y \in A_{ip}^k$, then $P^{-1}(y)$ meets at least p sets G_j. Thus we have $\sum \chi_{PG_j}(y) \geq p$ for all $y \in A_{ip}^k$. We denote by χ_A the characteristic function of A. Since each PG_j is open, there is an open set U containing A_{ip}^k such that $\sum \chi_{PG_j}(y) \geq p$ for $y \in U$. Hence we obtain

$$
\begin{aligned}
p\mathcal{H}^{n-1}(A_{ip}^k) \leq p\mathcal{H}^{n-1}(U) &\leq \int_{\mathbf{R}^{n-1}} \Big(\sum_j \chi_{PG_j}\Big)\, d\mathcal{H}^{n-1} \\
&= \sum_j \mathcal{H}^{n-1}(PG_j) \leq \sum_j \operatorname{diam}(PG_j)^{n-1} \\
&\leq \sum_j \operatorname{diam}(G_j)^{n-1}.
\end{aligned}
$$

This implies that $p\mathcal{H}^{n-1}(A_{ip}^k) \leq \mathcal{H}^{n-1}_{\frac{1}{k}}(E_i)$, and so

$$
\mathcal{H}^{n-1}(A_i) \leq \mathcal{H}^{n-1}(A_{ip}) = \lim_k \mathcal{H}^{n-1}(A_{ip}^k) \leq \frac{1}{p}\mathcal{H}^{n-1}(E_i)
$$

for every p. Thus $\mathcal{H}^{n-1}(A_i) = 0$, and the lemma is proved. □

The next lemma is actually Lemma 2.2 in [10]. For convenience we state the lemma also here and give the idea of its proof.

Lemma 2.5. *If $E \subset \mathbf{R}^n$ is a set of σ-finite $(n-1)$-measure, then for $\mathcal{H}^{n-1}$-a.e. $w \in S^{n-1}(0,1)$ the intersection of E with the radius $[0,w]$ is at most countable.*

Proof. The standard argument used in the proof of the preceding lemma can be easily modified to show that $[w/k, w] \cap E$ is at most countable for a.e. $w \in S^{n-1}(0,1)$ for every $k \in \mathbf{N}$. This gives the claim. □

We continue with a capacity type estimate which allows us to estimate from below the L^p-integrals of certain Borel functions.

Lemma 2.6. *Let u be a non-negative Borel function in $\mathbf{R}^n$ such that for each y in a continuum $F \subset \mathbf{R}^n$*

$$
\int_{[y,w)} u\, ds \geq 1
$$

for each $w \in S_y \subset S^{n-1}(y,1)$, *where* S_y *satisfies* $\mathcal{H}^{n-1}(S_y) \geq a > 0$. *Then*

$$\int_{\mathbf{R}^n} u^p\,dx \geq a\,C(n,p,\varepsilon)\,\mathcal{H}_\infty^{n-p+\varepsilon}(F),$$

where $n-1 < p < n$ *and* $\varepsilon > 0$.

The notation $C(a,b,\dots)$ means a constant which depends only on $a, b, \dots$.

Proof of Lemma 2.6. The proof is an improvement on the proof of Lemma 2.1 in [10]. Fix $y \in F$. For each $0 < r < R < 1$, and $w \in S^{n-1}(y,1)$ we have by the Hölder inequality

$$\begin{aligned}\int_r^R u(y+tw)\,dt &= \int_r^R u(y+tw)t^{(n-1)/p}t^{(1-n)/p}\,dt \\ &\leq \left(\int_r^R u(y+tw)^p t^{n-1}\,dt\right)^{1/p}\left(\frac{p-1}{n-p}(r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}})\right)^{(p-1)/p}.\end{aligned}$$

Set $r_j = 2^{-j}$. For $j = 0,1,2,\dots$ write $A_j(y) = B(y,r_j) \setminus B(y,r_{j+1})$ and set $I_j(w) = (\int_r^R u(y+tw)\,dt)^p$, where $R = r_j$ and $r = r_{j+1}$. Then integration of the above inequality with respect to w over $S^{n-1}(y,1)$ gives

$$\int_{S^{n-1}(y,1)} I_j(w)\,d\sigma(w) \leq \left(\frac{p-1}{n-p}\right)^{p-1} 2^{j(n-p)}(2^{\frac{n-p}{p-1}} - 1)^{p-1}\int_{A_j(y)} u^p(x)\,dx.$$

Suppose now that for each j

$$\int_{A_j(y)} u^p(x)\,dx \leq a\,C(p,\varepsilon)\left(\frac{p-1}{n-p}\right)^{1-p} 2^{-j(n-p)}(2^{\frac{n-p}{p-1}} - 1)^{-(p-1)}2^{-j\varepsilon},$$

where the constant $C(p,\varepsilon)$ will be chosen later. Write $\mathrm{Bad}_j(s) = \{w \in S^{n-1}(y,1) : I_j(w) \geq s\}$. Then

$$\mathcal{H}^{n-1}(\mathrm{Bad}_j(s)) \leq a\,C(p,\varepsilon)s^{-1}2^{-j\varepsilon},$$

and

$$\mathcal{H}^{n-1}(\cup_j \mathrm{Bad}_j(c'2^{-j\varepsilon/2})) \leq a\sum_j \frac{C(p,\varepsilon)}{c'2^{-j\varepsilon/2}}2^{-j\varepsilon} = a\frac{C(p,\varepsilon)}{c'}\frac{2^{\varepsilon/2}}{2^{\varepsilon/2}-1}. \tag{2.1}$$

For each $w \in S^{n-1}$ not in $\cup_j \mathrm{Bad}_j(c'2^{-j\varepsilon/2})$ we have

$$\int_0^1 u(y+tw)\,dt = \sum_j I_j(w)^{1/p} \leq \sum_j (c'\,2^{-j\varepsilon/2})^{1/p} \leq \frac{1}{2}, \tag{2.2}$$

when we choose a suitable c' depending on p and ε. Define $C(p,\varepsilon) = \frac{c'}{2}\frac{2^{\varepsilon/2}-1}{2^{\varepsilon/2}}$. By (2.1) there is some w outside the bad set so that the segment of length 1 in the direction from y to w intersects S_y. This contradicts (2.2). We conclude that there is an index j such that

$$\int_{A_j(y)} u^p\,dx \geq aC(p,\varepsilon)\left(\frac{p-1}{n-p}\right)^{1-p} 2^{-j(n-p)}(2^{\frac{n-p}{p-1}}-1)^{-(p-1)}2^{-j\varepsilon}$$

and thus

$$\begin{aligned}\int_{B(y,2^{-j})} u^p\,dx &\geq a\,C(p,\varepsilon)\left(\frac{p-1}{n-p}\right)^{1-p} 2^{-j(n-p)}(2^{\frac{n-p}{p-1}}-1)^{-(p-1)}2^{-j\varepsilon}\\ &= a\,C(p,n,\varepsilon)2^{-j(n-p+\varepsilon)}.\end{aligned}$$

By the Besicovitch covering theorem we may then cover F with balls $B(y_i,r_i)$ of the above type and so that only a bounded number (depending on n) of these balls overlap. Then

$$\mathcal{H}^{n-p+\varepsilon}_\infty(F) \leq \sum_i r_i^{n-p+\varepsilon} \leq \sum_i \frac{1}{a\,C(n,p,\varepsilon)}\int_{B(y_i,r_i)} u^p\,dx \leq \frac{1}{a\,C'(n,p,\varepsilon)}\int_{\mathbf{R}^n} u^p\,dx,$$

as desired. □

By scaling we obtain a more useful version of the previous lemma.

Lemma 2.7. *Let $t>0$ and u be a non-negative Borel function in $\mathbf{R}^n$ such that for each y in a continuum $F\subset\mathbf{R}^n$*

$$\int_{[y,w)} u\,ds \geq 1$$

for each $w\in S_y\subset S^{n-1}(y,t)$, where S_y satisfies $\frac{1}{t^{n-1}}\mathcal{H}^{n-1}(S_y)\geq a>0$. Then

$$\int_{\mathbf{R}^n} u^p\,dx \geq a\,C(n,p,\varepsilon)t^{-\varepsilon}\mathcal{H}^{n-p+\varepsilon}_\infty(F),$$

where $n-1<p<n$ and $\varepsilon>0$.

Remark 2.8. In the case $n=2$ with the assumptions of the previous lemma we immediately obtain the following estimate

$$\int_{\mathbf{R}^n} u\,dx \geq \int_{S_y}\int_{[y,w)} u\,ds\,dw \geq \int_{S_y} dw \geq at.$$

This leads to better results in the planar case. See Remark 3.7 and Remark 4.3.

The following lemma allows us to perform changes of variables in rather general settings. We give a proof for the convenience of the reader.

Lemma 2.9. *Let $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$ have the property that $N(f,Q) < \infty$ for every open $Q \subset\subset \Omega$, and let $h\colon \mathbf{R}^n \to \mathbf{R}$ be a non-negative Borel measurable function such that $h \circ f$ is locally integrable. Then*

$$\int_Q h(f(x))|J_f(x)|\,dx \le N(f,Q)\int_{f(Q)} h(y)\,dy.$$

Proof. Fix open $Q \subset\subset \Omega$. Now $f \in W^{1,1}(Q, \mathbf{R}^n)$. By standard arguments, see [4, p. 254], we find sets $\{F_j\}$ such that i) $F_j \subset F_{j+1} \subset Q$, ii) $|Q \setminus \cup_j F_j| = 0$, and iii) $f|_{F_j}$ is Lipschitz for every j. Now there are Lipschitz extensions $g_j\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g_j|_{F_j} = f|_{F_j}$. By a simple argument one sees that $Dg_j(x) = Df(x)$ for almost every point $x \in F_j$, and thus $J_{g_j}(x) = J_f(x)$ for almost every $x \in F_j$. Now by [4, Theorem 2, p. 99]

$$\begin{aligned}\int_{F_j} h(f(x))|J_f(x)|\,dx &= \int_{\mathbf{R}^n} h(g_j(x))\chi_{F_j}(x)|J_{g_j}(x)|\,dx\\ &= \int_{\mathbf{R}^n}\Big(\sum_{x\in g_j^{-1}(y)} h(g_j(x))\chi_{F_j}(x)\Big)\,dy\\ &\le N(f,Q)\int_{f(Q)} h(y)\,dy.\end{aligned}$$

By the monotone convergence theorem we conclude with the claim. □

Remarks 2.10. (a) By choosing $h \equiv 1$ we see that the Jacobian determinant of each continuous mapping which satisfies the assumptions of the lemma above is locally integrable.

(b) For a continuous, open, discrete, almost everywhere differentiable mapping we do not need the Sobolev class assumption. The Jacobians of these mappings are locally integrable by Lemma I.4.11 in [17].

The following well known covering lemma is also needed in proofs. This lemma is Lemma 31.1 in Väisälä's monograph [18].

Lemma 2.11. *Let $F \subset \mathbf{R}$ be a compact set and $\varepsilon > 0$. There exists $\delta > 0$ with the following property: For $0 < r < \delta$ there is a finite covering of F with open intervals $\Delta_1, \dots, \Delta_p$ such that*

(1) *$\operatorname{diam}(\Delta_i) = 2r$ for $1 \le i \le p$.*
(2) *The center of Δ_i belongs to F.*
(3) *Each point of F belongs to at most two Δ_i.*
(4) *$pr \le \mathcal{H}^1(F) + \varepsilon$.*

Proof. Choose an open set G such that $F \subset G$ and $\mathcal{H}^1(G) < \mathcal{H}^1(F) + \varepsilon$. Now we show that $\delta = \mathrm{d}(F, \mathbf{R} \setminus G)$ has the desired property. The notation $\mathrm{d}(A,B)$ means the distance between the sets A and B.

Suppose $0 < r < \delta$. For $x \in F$ set $\triangle(x) = (x-r, x+r)$. Then there exists a finite covering $\{\triangle(x_1), \ldots, \triangle(x_p)\}$ of F such that $x_1 < \cdots < x_p$. This covering satisfies the conditions (1) and (2). If $\triangle(x_i)$ meets $\triangle(x_{i+2})$ we may leave out $\triangle(x_{i+1})$ and obtain a covering which still satisfies (1) and (2). After a finite number of steps we obtain a covering, say $\{\triangle_1, \ldots, \triangle_p\}$, which satisfies (1), (2), and (3). Next we show that it also satisfies (4). Since $\triangle_i \subset G$, we obtain

$$\begin{aligned} 2pr = \sum_i \mathcal{H}^1(\triangle_i) &= \int_{\mathbf{R}} \sum_i \chi_{\triangle_i}\, d\mathcal{H}^1 \\ &\leq 2\mathcal{H}^1(G) < 2\mathcal{H}^1(F) + 2\varepsilon. \end{aligned}$$

□

Finally we will introduce the Hardy-Littlewood maximal function. For a locally integrable function f the maximal function is

$$Mf(x) = \sup_{\substack{B(y,r) \\ x \in B(y,r)}} \fint_{B(y,r)} |f(z)|\, dz.$$

Here and throughout the thesis we denote $\fint_E g = \frac{1}{|E|} \int_E g$. The known fact is that if $f \in L^p(\mathbf{R}^n)$, $p > 1$, then

$$\int_{\mathbf{R}^n} (Mf(x))^p\, dx \leq \big(5^n \frac{p}{p-1}\big)^p \int_{\mathbf{R}^n} |f(x)|^p\, dx. \tag{2.3}$$

For this see [15] for example.

We already use this maximal function in the next lemma. The lemma is from Bojarski [1].

Lemma 2.12. *Let $\{B_j\}$ be a sequence of balls in $\mathbf{R}^n$ and $\{a_j\}$ a sequence of non-negative real numbers. Then*

$$\int_{\mathbf{R}^n} \Big(\sum_j a_j \chi_{2B_j}(x) \Big)^p dx \leq C(n,p) \int_{\mathbf{R}^n} \Big(\sum_j a_j \chi_{B_j}(x) \Big)^p dx$$

whenever $1 \leq p < \infty$.

Proof. The case $p = 1$ is trivial, and so we can assume that $p > 1$. Let $\varphi \in L^q(\mathbf{R}^n)$, $q = \frac{p-1}{p} = p'$. Now by the monotone convergence theorem

$$\begin{aligned} \int_{\mathbf{R}^n} |\varphi(x) \sum_j a_j \chi_{2B_j}(x)|\, dx &\leq \sum_j a_j |2B_j| \fint_{2B_j} |\varphi(x)|\, dx \\ &\leq \sum_j a_j |2B_j| \inf_{x \in B_j} M\varphi(x) \\ &\leq 2^n \int_{\mathbf{R}^n} \sum_j a_j \chi_{B_j}(z) M\varphi(z)\, dz. \end{aligned}$$

Now by Hölder's inequality and the property (2.3) of the maximal function

$$\int_{\mathbf{R}^n} |\varphi(x) \sum_j a_j \chi_{2B_j}(x)|\, dx \le 2^n || \sum_j a_j \chi_{B_j} ||_p || M\varphi ||_{p'}$$
$$\le C(n,p)\, || \sum_j a_j \chi_{B_j} ||_p || \varphi ||_{p'}.$$

Thus by duality we know that

$$|| \sum_j a_j \chi_{2B_j} ||_p \le C(n,p) || \sum_j a_j \chi_{B_j} ||_p.$$

□

3. Conditions for H_f

In this chapter we give sufficient integrability conditions on H_f to guarantee that the mapping f is ACL, that is absolutely continuous on almost every line parallel to the coordinate axes, or that f is a Sobolev mapping. We also establish a kind of a local quasisymmetry condition for f.

All but the last result in this section are generalizations of theorems in [12], where the case of homeomorphisms is treated. The proofs are essentially the same as there but for completeness most of them are repeated here.

First we state an easy result. This is an analog of Theorem 2.2 in [12].

Theorem 3.1. *Let $f : \Omega \to \mathbf{R}^n$ be an open, discrete, continuous, sense-preserving mapping. If $H_f \in L^s_{loc}(\Omega)$, $s \in [1,\infty]$, then $|Df| \in L^p_{loc}(\Omega)$, where $p = sn/(n-1+s)$ and $p = n$ if $s = \infty$.*

Proof. The proof of this is same as in [12], because by Lemma 2.2 we have the almost everywhere differentiability and then by a simple linear algebra argument

$$(3.1) \qquad |Df(x)|^n \le H_f^{n-1}(x) J_f(x) \quad \text{for a.e. } x \in \Omega.$$

Since for continuous, open, discrete, sense-preserving, almost everywhere differentiable mappings J_f is locally integrable, see Remark 2.10, the result follows from (3.1) by Hölder's inequality. □

The most difficult step is to show that under suitable integrability conditions for H_f the mapping f is ACL. This is done in the theorem below. The proof of Theorem 3.2 is almost the same as in [12]. Originally the method of the proof is due to F. W. Gehring [5].

Theorem 3.2. *Let $f : \Omega \to \mathbf{R}^n$ be an open, discrete, continuous, sense-preserving mapping. If $E \subset \Omega$ and $s \in (1,\infty]$ satisfy the conditions*

(1) *$s > n/(n-1)$,*
(2) *$H_f(x) < \infty$ for each $x \in \Omega \setminus E$,*

(3) $H_f \in L^s_{loc}(\Omega)$,
(4) *E has σ-finite* $(n-1)$*-Hausdorff measure,*

then f is ACL.

Remarks 3.3. (a) We do not know the optimal value for s in Theorem 3.2, but at least one cannot take $s < 1$. Consider the following mapping in the plane: the positive real axis, and lines parallel to it, are mapped to the curve $x\sin(1/x)$ and to its translations. This mapping is a non-ACL homeomorphism of $\mathbf{R}^2$ which is the identity on the left half plane and satisfies $H_f(x) \in L^s_{loc}(\mathbf{R}^2)$ for any $s < 1$.

(b) The condition (4) is crucial. For example if $g\colon [0,1] \to [0,1]$ is the Cantor staircase function, then $f\colon]0,1[\times]0,1[\to]0,2[\times]0,1[$ defined as $f(x,y) = (g(x)+x, y)$ is a homeomorphism with $H_f(z) = 1$ almost everywhere, but f is not ACL.

Proof of Theorem 3.2. Pick a closed cube $Q \subset\subset \Omega$ whose sides are parallel to the coordinate axes. Assume that $Q = Q_0 \times J_0$, where Q_0 is $(n-1)$-interval in $\mathbf{R}^{n-1}$, and $J_0 = [a,b] \subset \mathbf{R}$. In order to show that f is ACL it suffices to show that f is absolutely continuous on almost every line segment of Q parallel to the coordinate axes, and by symmetry it is sufficient to consider segments parallel to the x_n-axis.

Next for a Borel set $A \subset Q_0$ write

$$\Phi(A) = |f(A \times J_0)| \leq |f(Q)| < \infty.$$

Then Φ is a bounded q-quasiadditive set function in Q_0, here $q = N(f,Q) < \infty$, and hence by Lemma 2.1 it has a finite upper derivative $\Phi'(y)$ for almost all $y \in Q_0$. We choose $y \in Q_0$ such that (i) $\Phi'(y)$ exists, (ii) $H_f \in L^s(\{y\} \times J_0)$, and (iii) $S = E \cap (\{y\} \times J_0)$ is countable. The last assertion comes from Lemma 2.4. Almost every y in Q_0 satisfies conditions (i) - (iii). For our theorem it now suffices to show that f is absolutely continuous on $J = \{y\} \times J_0$.

To this end let $F \subset J \setminus S$ be a compact set. For each $k = 0,1,2,\dots$ write

$$F_k = \{x \in F : 2^k \leq H_f(x) < 2^{k+1}\}.$$

Then F_k is a Borel set and $F = \bigcup F_k$ because of (2). We first derive the following estimate

$$\mathcal{H}^1(fF_k) \leq C2^k \mathcal{H}^1(F_k)^{\frac{n-1}{n}} \tag{3.2}$$

where $C = (2^{2n+1} q\Omega_{n-1}/\Omega_n \Phi'(y))^{1/n}$. Here and in the sequel $\Omega_n = |B^n(0,1)|$, and $q = N(f,Q)$.

For (3.2) fix k and for each $j = 1,2,\dots$ consider the set

$$F_{k,j} = \{x \in F_k : \frac{L_f(x,r)}{l_f(x,r)} \leq 2^{k+1} \text{ for } 0 < r < 1/j\}.$$

The sets $F_{k,j}$ are Borel sets and $F_{k,j} \subset F_{k,j+1}$ with $F_k = \cup_{j=1}^{\infty} F_{k,j}$.

By the above it suffices to prove (3.2) for $F_{k,j}$ instead of F_k. Fix j and let F' be an arbitrary compact subset of $F_{k,j}$. Let $\varepsilon > 0$ and $t > 0$. The continuity of the mapping $(x,r) \mapsto L_f(x,r)$ gives δ, $0 < \delta < 1/j$, such that $L_f(x,r) < t/2$ for $0 < r < \delta$ and for all $x \in F'$. By Lemma 2.11 for each sufficiently small $r > 0$, $0 < r < \delta$, there exists a covering of F' by a finite number of open balls $B_i = B(x_i, r)$, $i = 1, \dots, l$, where

(a) $x_i \in F'$, $i = 1, \dots, l$,
(b) each point of $\mathbf{R}^n$ lies in at most two B_i,
(c) $lr \le H^1(F') + \varepsilon$ and
(d) $B_i \subset B^{n-1}(y, r) \times J_0$.

In the following denote $B = B^{n-1}(y, r)$. Now the union of $f(B_i)$'s covers $f(F')$, and since the boundary of the image is included in the image of the boundary we have that $\operatorname{diam} f(B_i) \le \operatorname{diam} f(\partial B_i) \le 2L_f(x_i, r)$.

Because r was small enough

$$\mathcal{H}^1_t(fF') \le \sum_{i=1}^{l} \operatorname{diam}(fB_i),$$

and the Hölder inequality together with the definition of $F_{k,j}$ yields

$$\begin{aligned}\mathcal{H}^1_t(fF')^n &\le \left(\sum_{i=1}^{l} \operatorname{diam}(fB_i)\right)^n \le l^{n-1} \sum_{i=1}^{l} \operatorname{diam}(fB_i)^n \\ &\le l^{n-1} 2^n \sum_{i=1}^{l} L_f(x_i, r)^n \le \frac{l^{n-1} 2^n 2^{n(k+1)}}{\Omega_n} \sum_{i=1}^{l} |fB_i|.\end{aligned} \tag{3.3}$$

Since $q = N(f, Q) < \infty$, we obtain from (b) that

$$\sum_{i=1}^{l} |fB_i| \le 2q \left|\bigcup_{i=1}^{l} fB_i\right| \le 2q |f(B \times J_0)| = 2q\Phi(B).$$

Thus (3.3) and (c) yield

$$\begin{aligned}\mathcal{H}^1_t(fF')^n &\le q 2^{n(k+2)+1} \Omega_{n-1}/\Omega_n (\mathcal{H}^1(F') + \varepsilon)^{n-1} \frac{\Phi(B)}{\mathcal{H}^{n-1}(B)} \\ &\le q 2^{n(k+2)+1} \Omega_{n-1}/\Omega_n (\mathcal{H}^1(F_{k,j}) + \varepsilon)^{n-1} \frac{\Phi(B)}{\mathcal{H}^{n-1}(B)}.\end{aligned}$$

Since $\mathcal{H}^1_t(fF') \to \mathcal{H}^1(fF')$ as $t \to 0$, letting first $r \to 0$, then $\varepsilon \to 0$, and finally $t \to 0$ we obtain

$$\mathcal{H}^1(fF')^n \le q 2^{n(k+2)+1} \Omega_{n-1}/\Omega_n \mathcal{H}^1(F_{k,j})^{n-1} \Phi'(y). \tag{3.4}$$

Now F' is an arbitrary compact subset of $F_{k,j}$ and hence (3.4) holds for $F_{k,j}$ on the left hand side of (3.4). This leads to the estimate (3.2).

Since $fF = \cup fF_k$, (3.2) implies

$$\mathcal{H}^1(fF) \le \sum_k \mathcal{H}^1(fF_k) \le C \sum_k 2^k \mathcal{H}^1(F_k)^{\frac{n-1}{n}}. \tag{3.5}$$

The sets F_k, $k = 1, \dots,$ are disjoint and hence the integral estimate

$$\sum_{k=0}^{\infty} 2^{ks}\mathcal{H}^1(F_k) \leq \int_F H(x,f)^s \, dx_n \tag{3.6}$$

is elementary. From (3.5), (3.6) and from the Hölder inequality we obtain

$$\begin{aligned} \mathcal{H}^1(fF) &\leq C_1 \left(\sum_{k=0}^{\infty} 2^{ks}\mathcal{H}^1(F_k) \right)^{\frac{n-1}{n}} \left(\sum_{k=0}^{\infty} 2^{k(n-s(n-1))} \right)^{\frac{1}{n}} \\ &\leq C_2 \left(\int_F H_f(x)^s \, dx_n \right)^{\frac{n-1}{n}} \end{aligned} \tag{3.7}$$

where C_2 depends only on n, s, q, and $\Phi'(y)$. Note that the series

$$\sum_{k=0}^{\infty} 2^{k(n-s(n-1))}$$

converges because $s > n/(n-1)$ and hence $n - s(n-1) < 0$.

Next we will show that the absolutely continuity follows from (3.7). First it is enough to show that every coordinate function is absolutely continuous. Let f_j be any coordinate function. Now from (3.7) for every compact $F \subset J \setminus S$ we have

$$\mathcal{H}^1(f_jF) \leq \mathcal{H}^1(fF) \leq C_2 \left(\int_F H_f(x)^s \, dx_n \right)^{\frac{n-1}{n}}.$$

Let $E \subset J$ be a compact set. We will show that the above holds also for E. Let $\varepsilon > 0$. Now there is a compact set $F' \subset f_jE \setminus f_jS$ such that $\mathcal{H}^1(f_jE \setminus F') < \varepsilon$. This is true because f_jE is compact, $\mathcal{H}^1(f_jE) < \infty$, and f_jS is countable (actually, we only need that $\mathcal{H}^1(f_jS) = 0$). Let $F = f_j^{-1}(F') \cap E \cap J$. Now F is compact, since $E \cap J$ is compact and $f_j^{-1}(F')$ is closed. Also $F \cap S = \emptyset$. Thus

$$\begin{aligned} \mathcal{H}^1(f_jE) &\leq \mathcal{H}^1(f_jF) + \varepsilon \leq C_2 \left(\int_F H_f(x)^s \, dx_n \right)^{\frac{n-1}{n}} + \varepsilon \\ &\leq C_2 \left(\int_E H_f(x)^s \, dx_n \right)^{\frac{n-1}{n}} + \varepsilon \end{aligned}$$

for any $\varepsilon > 0$. So

$$\mathcal{H}^1(f_jE) \leq C_2 \left(\int_E H_f(x)^s \, dx_n \right)^{\frac{n-1}{n}}.$$

This guarantees that f_j is absolutely continuous; recall that absolute continuity deals with unions of closed intervals whose interiors are mutually disjoint, that $H_f \in L^s(J)$, and that always $\operatorname{diam}(f_jI) \leq \mathcal{H}^1(f_jI)$ when I is an interval. □

Remark 3.4. If we replace H_f with

$$\widetilde{H}_f(x) = \limsup_{r\to 0} \frac{\max\{|f(x)-f(y)| : |x-y| = r\}}{\mathrm{d}(\partial f B(x,r), f(x))}$$

in previous theorem, the claim of the theorem is still true.

Combining Theorems 3.1 and 3.2 we obtain the following corollary.

Corollary 3.5. *With the hypothesis of Theorem 3.2 the function f is almost everywhere differentiable and $|Df| \in L^p_{loc}(\Omega)$, $p = sn/(n-1+s)$. In particular $f \in W^{1,p}_{loc}(\Omega, \mathbf{R}^n)$.*

As a last result of this section we prove a local quasisymmetry condition.

Theorem 3.6. *Let $f\colon \Omega \to \mathbf{R}^n$ be an open, discrete, continuous, sense-preserving mapping for which $H_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-measure, and $H_f \in L^{p^*\frac{n-1}{n}}_{loc}(\Omega)$, where $n-1 < p < n$ and $p^* = \frac{pn}{n-p}$. Then*

$$L_f(x,r) \le d(f(x), \partial f B(x,r)) \exp\left(C(n,p,q) \left(\int_{B(x,2r)} H_f^{p^*\frac{n-1}{n}}(y)\, dy \right)^{\frac{1}{p^*}\frac{n}{n-1}} \right) \tag{3.8}$$

where $B(x,2r) \subset \Omega$ and $q = N(f, B(x,2r))$.

We do not know if (3.8) is sharp.

Proof of Theorem 3.6. Fix $x \in \Omega$, $r > 0$ such that $B(x,2r) \subset \Omega$. Denote $L = L_f(x,r)$ and $l = \mathrm{d}(f(x), \partial f B)$, and define

$$v(y) = \begin{cases} 0 & \text{when } |y| \ge L \\ 1 & \text{when } |y| \le l \\ \frac{\log\frac{1}{|y|} - \log\frac{1}{L}}{\log\frac{L}{l}} & \text{when } l < |y| < L \end{cases}.$$

Now

$$\int_{\mathbf{R}^n} |\nabla v(y)|^n \, dy \le \frac{\Omega_{n-1}}{(\log\frac{L}{l})^{n-1}}.$$

Since f is differentiable almost everywhere an elementary linear algebra argument (see e.g. [18, p.44]) shows that

$$|Df(y)|^n \le H_f(y)^{n-1} J_f(y) \quad \text{for a.e. } y \in \Omega.$$

Thus we have the estimate

$$\begin{aligned}\int_{B(x,2r)} |\nabla(v\circ f)(y)|^p\,dy &\le \int_{B(x,2r)} |Df(y)|^p|\nabla v(f(y))|^p\,dy\\ &\le \int_{B(x,2r)} H_f(y)^{\frac{(n-1)p}{n}} J_f(y)^{\frac{p}{n}}|\nabla v(f(y))|^p\,dy\\ &\le \Big(\int_{B(x,2r)} H_f(y)^{\frac{(n-1)p}{n-p}}\,dy\Big)^{\frac{n-p}{n}}\Big(\int_{B(x,2r)}|\nabla v(f(y))|^n J_f(y)\,dy\Big)^{\frac{p}{n}}\\ &\le \Big(\int_{B(x,2r)} H_f(y)^{\frac{(n-1)p}{n-p}}\,dy\Big)^{\frac{n-p}{n}}\Big(q\int_{\mathbf{R}^n}|\nabla v(y)|^n\,dy\Big)^{\frac{p}{n}},\end{aligned}$$

where the last inequality comes from Lemma 2.9 since by Corollary 3.5 we know that $f\in W^{1,1}_{loc}(\Omega,\mathbf{R}^n)$. Thus

$$\text{(3.9)}\qquad \int_{B(x,2r)} |\nabla(v\circ f)(y)|^p\,dy \le C(n,p,q)\Big(\log\frac{L}{l}\Big)^{\frac{(1-n)p}{n}}\Big(\int_{B(x,2r)} H_f(y)^{\frac{(n-1)p}{n}\frac{n}{n-p}}\,dy\Big)^{\frac{n-p}{n}}.$$

To get a lower bound for this integral, we argue as follows. Let $a\in\partial fB(x,r)$ be such that $\mathrm{d}(a,f(x))=l$, $b\in\overline{\partial B(x,r)}$ be such that $\mathrm{d}(f(x),f(b))=L$, and $c\in\partial f(B(x,2r))$ such that $[f(b),c]\subset\overline{f(B(x,2r))}\setminus B(f(x),L)$. By applying Lemma 2.3 to the line segments $[f(x),a]$, $[f(b),c]$ and to the balls $B(x,r)$ and $B(x,2r)$, we obtain continuums F_1 and F_2 such that

(i) $x\in F_1$, $b\in F_2$,
(ii) $\operatorname{diam}F_1,\operatorname{diam}F_2=\frac{r}{4}$, (these can be assumed by taking subcontinuums),
(iii) $F_2\cap B(x,r)=\emptyset$, and
(iv) $f(F_1)\subset B(f(x),l)$, $f(F_2)\subset\mathbf{R}^n\setminus B(f(x),L)$.

Since, by Corollary 3.5, $v\circ f\in W^{1,s}_{loc}(\Omega)$, for some $s>1$, the Fuglede theorem [18, Theorem 28.2] tells us that the s-modulus of those paths, where $v\circ f$ is not absolutely continuous, is zero. By continuity we have $\delta>0$ such that $|v\circ f(z)-v\circ f(z')|<\frac14$ when $|z-z'|\le\varepsilon$ and $z,z'\in\overline{B(x,2r)}$. Fix $z\in F_1$, $y\in F_2$. We consider paths which are composed of two radial segments $[z,w]$, $[w,y]$, where $w\in S^{n-1}(z,R)$, $R>0$. Now for $\mathcal{H}^{n-1}$-a.e. $w\in S^{n-1}(z,R)\cap B(x,2r)$ the function $v\circ f$ is absolutely continuous on $([z,w]\cup[w,y])\setminus(B(z,\delta)\cup B(y,\delta))$. This is true because if $A\subset S^{n-1}(z,R)\cap B(x,2r)$ is a set with positive $\mathcal{H}^{n-1}$-measure, then the s-modulus of the path family $\{[z,w]\setminus B(z,\delta):w\in A\}$, or the path family $\{[w,y]\setminus B(y,\delta):w\in A\}$, is positive. Thus the desired absolute continuity follows from the Fuglede theorem.

If $v \circ f$ is absolutely continuous on $([z,w] \cup [w,y]) \setminus (B(z,\delta) \cup B(y,\delta))$, where $w \in B(z,R)$, then by the property (iv)

$$\begin{aligned}\frac{1}{2} \le |v \circ f(z') - v \circ f(y')| &\le \int_{[z',w]\cup[w,y']} |\nabla(v \circ f)(u)|\, du \\ &\le \int_{[z,w]\cup[w,y]} |\nabla(v \circ f)(u)| \chi_{B(x,2r)}(u)\, du,\end{aligned} \tag{3.10}$$

where $z' \in S^{n-1}(z,\varepsilon) \cap [z,w]$ and $y' \in S^{n-1}(y,\varepsilon) \cap [y,w]$.

Next we will apply Lemma 2.7. First we assume that for every $z \in F_1$ there is a set $S_z \subset S(z,\frac{r}{4})$ such that $\mathcal{H}^{n-1}(S_z) \ge \frac{1}{2}\mathcal{H}^{n-1}(S(z,\frac{r}{4})) = C(n)r^{n-1}$ and for every $w \in S_z$

$$\int_{[z,w[} 4|\nabla(v \circ f)(s)| \chi_{B(x,2r)}(s)\, ds \ge 1.$$

Then, we take in Lemma 2.7, $t = r/4$, and choose $\varepsilon = p - (n-1)$ there. This gives us the estimate

$$\int_{B(x,2r)} 4^p |\nabla(v \circ f)(s)|^p\, ds \ge C(n,p) r^{-(p-(n-1))} \mathcal{H}^1_\infty(F_1).$$

Suppose then that there exists $z \in F_1$ such that for every $w \in S_z \subset S(z,\frac{r}{4})$, where $\mathcal{H}^{n-1}(S_z) \ge \frac{1}{2}\mathcal{H}^{n-1}(S(z,\frac{r}{4})) = C(n)r^{n-1}$,

$$\int_{[z,w[} 4|\nabla(v \circ f)(s)| \chi_{B(x,2r)}(s)\, ds < 1.$$

Then, because of (3.10) and the properties (i), (ii) of F_1 and F_2, for every $y \in F_2$

$$\int_{[y,w[} 4|\nabla(v \circ f)(s)| \chi_{B(x,2r)}(s)\, ds \ge 1$$

for every $w \in S_y \subset S(y,2r) \cap B(x,2r)$, where $\mathcal{H}^{n-1}(S_y) \ge C(n)r^{n-1}$ and then we have again the situation as above. So by Lemma 2.7

$$\int_{B(x,2r)} |\nabla(v \circ f)(s)|^p\, ds \ge C(n,p) r^{-(p-(n-1))} \min\{\mathcal{H}^1_\infty(F_1), \mathcal{H}^1_\infty(F_2)\}.$$

Now F_1 and F_2 are continuums and $\operatorname{diam}(F_1) = \operatorname{diam}(F_2) = r/4$. If $\{V_i\}_{i\in\mathbf{N}}$ is any cover of F_1 then

$$\sum_{i\in\mathbf{N}} \operatorname{diam}(V_i) \ge \frac{r}{4}.$$

Thus $\mathcal{H}^1_\infty(F_1) \ge r/4$. Similarly $\mathcal{H}^1_\infty(F_2) \ge r/4$. We have obtained

$$\int_{B(x,2r)} |\nabla(v \circ f)(u)|^p\, du \ge C(n,p) r^{n-p}.$$

Combining this with the inequality (3.9) we conclude with the claim. □

Remark 3.7. In the plane we can actually choose $p = 1$ because, by Remark 2.8, we have the lower bound

$$\int_{B(x,2r)} |\nabla(v \circ f)(s)| \, ds \geq Cr$$

when $p = 1$, $n = 2$. By the same argument as in the above proof we obtain the upper bound (3.9) also with $n = 2$, $p = 1$. Combining these we have a nicer inequality in the planar case

$$L_f(x,r) \leq d(f(x), \partial f B(x,r)) \exp\left(C(q) \fint_{B(x,2r)} H_f(y) \, dy \right)$$

where $B(x,2r) \subset \Omega$ and $q = N(f, B(x,2r))$.

4. Conditions for h_f

In practice it is easier to estimate h_f from above than H_f. This motivates one to work with h_f. The results in this section are generalizations of similar results in [11]. The work of Cristea [2] for the lim inf-definition of quasiregular mappings suggested that the results in [11] could also be true for noninjective mappings.

The approach, which we used for the H_f-case, does not work for h_f. We first proved that with suitable integrability assumptions on H_f it follows that f is ACL, and then by requiring more integrability we obtained more regularity. The proof of Theorem 3.2 does not work for the h_f-case directly, since we cannot define the sets $F_{k,j}$. However, the proof would work out if we knew that

$$L_f(x,r) \leq \mathrm{d}(f(x), \partial f B(x,r)) \varphi(x) \tag{4.1}$$

for all sufficiently small r's with a suitably integrable φ. This is Lemma 4.6 below. Thus our strategy is to prove (4.1) first. This is essentially the content of the first theorem of this section. All our other results in this section rely on this theorem. One way to describe this theorem is to say that it is a local quasisymmetry condition. Compare this theorem also to Theorem 3.6.

The following theorem is a generalization of Theorem 1.1 in [11]. The idea of the proof is also the same, but the lack of injectivity causes some technical problems.

Theorem 4.1. *Let $f\colon \Omega \to \mathbf{R}^n$ be an open, discrete, continuous, sense-preserving mapping for which $h_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-measure, and suppose $h_f \in L^{p^*}_{loc}(\Omega)$, where $n - 1 < p < n$ and $p^* = \frac{pn}{n-p}$. Then*

$$L_f(x,r) \leq d(f(x), \partial f B(x,r)) \exp\left(C(n,p,q) \left(\fint_{B(x,2r)} h_f^{p^*}(y) \, dy \right)^{\frac{1}{p^*} \frac{n}{n-1}} \right) \tag{4.2}$$

when $Q \subset\subset \Omega$ and $B(x,2r) \subset Q$ and $q = N(f,Q)$. In particular, f is differentiable almost everywhere.

Remarks 4.2. (a) The similar result for H_f, that is Theorem 3.6, is stronger than this. This is due to the fact that in the H_f-case we were able to prove the differentiability before the local quasisymmetry condition and we then used the differentiability in the proof. In the h_f-case we have to work without it.

(b) We do not know how sharp the above exponents are. We have not been able to find any good examples to check sharpness.

Proof of Theorem 4.1. Fix $Q \subset\subset \Omega$ and denote $q = N(f, Q) < \infty$. Let us first prove the differentiability assuming (4.2). The proof is basically the same as in the case when $H_f < \infty$ almost everywhere. Denote $\phi(A) = |f(A)|$ for every Borel set $A \subset Q$. Then ϕ is a bounded q-quasiadditive set function in Q. By Lemma 2.1, $\phi'(z) < \infty$ for almost every $z \in Q$. Fix a Lebesgue point $x \in Q$ of h_f^{p*} such that $\phi'(x) < \infty$. Let $y \in Q$ be such that $0 < |y - x| < \mathrm{d}(x, \partial Q)$. Now

$$\left(\frac{|f(x) - f(y)|}{|x-y|}\right)^n \leq \left(\frac{L_f(x, |x-y|)}{\mathrm{d}(f(x), \partial f B(x, |x-y|))}\right)^n \frac{\phi(B(x, |x-y|))}{|B(x, |x-y|)|}.$$

By inequality (4.2) it follows that

$$\limsup_{y \to x} \frac{|f(x) - f(y)|}{|x-y|} < \infty.$$

Thus by the Rademacher-Stepanov theorem f is differentiable a.e. in Q.

To prove the theorem we have to show that inequality (4.2) holds. The argument is an improvement on the techniques in [7], [10] and [11].

Fix $x_0 \in Q$ and $r > 0$ with $B(x_0, 2r) \subset Q$. We can assume that

$$L = L_f(x_0, r) > 4\mathrm{d}(f(x_0), \partial f B(x_0, r)) = 4l.$$

Define

$$A = \overline{B}(f(x_0), L) \setminus B(f(x_0), l),$$

and construct continuums F_1 and F_2 similarly as in the proof of Theorem 3.6.

Let $n - 1 < p < n$ and $\varepsilon > 0$. For each $k = 0, 1, 2, \ldots$ write

$$A_k = \{y \in B(x_0, 2r) \setminus (F_1 \cup F_2) : 2^k \leq h_f(y) < 2^{k+1}\}.$$

The set A_k is a Borel set because h_f is a Borel function, and

$$B(x_0, 2r) \setminus (F_1 \cup F_2 \cup E) = \cup_k A_k.$$

For every k there exists open U_k such that $A_k \subset U_k$ and

$$|U_k| \leq |A_k| + \frac{\varepsilon}{2^k (2^{\frac{np}{n-p}})^k}.$$

Fix k. Now for every $y \in A_k$ there is $r_y > 0$ such that

(i) $0 < r_y < \frac{1}{10} \min\{\mathrm{d}(F_1, F_2), \mathrm{d}(y, \partial B(x_0, 2r))\}$,

(ii) $\mathrm{diam}\,(f B_y) < 2^{-j_0 - 3} L$,

(iii) $H_f(y, r_y) < 2^{k+1}$, and

(iv) $B_y \subset U_k$.

Here $B_y = B(y, r_y)$ and j_0 is the least positive integer with $2^{-j_0}L < l$.

By the Besicovitch covering theorem we find balls $\overline{B}_1, \overline{B}_2, \ldots$ from balls $\overline{B}(y, r_y)$ so that

$$B(x_0, 2r) \setminus (F_1 \cup F_2 \cup E) \subset \bigcup_j \overline{B}_j \subset B(x_0, 2r)$$

and $\sum_j \chi_{\overline{B}_j}(x) \le C(n)$ for every $x \in \mathbf{R}^n$. For these balls we know that

$$|f\overline{B}_j| \le \Omega_n \operatorname{diam}(f\overline{B}_j)^n$$

and when $y_j \in A_k$ (here y_j is the center of $\overline{B}_j$)

$$|f\overline{B}_j| \ge \frac{\Omega_n}{2^n(2^{k+1})^n} \operatorname{diam}(f\overline{B}_j)^n.$$

Define

$$\rho(x) = (\log \frac{L}{l})^{-1} \sum_{f\overline{B}_j \cap A \ne \emptyset} \frac{\operatorname{diam}(fB_j)}{\mathrm{d}(fB_j, f(x_0))} \frac{1}{\operatorname{diam}(B_j)} \chi_{2B_j}(x).$$

The function ρ is Borel measurable, because it is a countable sum of simple functions.

By Lemma 2.12 the L^p-norms, $1 \le p < \infty$, of ρ are comparable to the corresponding norms of the function where the characteristic functions χ_{2B_j} are replaced with χ_{B_j}. Thus, knowing that $\sum \chi_{B_j} \le C(n)$, we arrive at the estimate

$$\int_{\mathbf{R}^n} \rho^p(x)\,dx \le C(n,p) \left(\log \frac{L}{l}\right)^{-p} \sum_{f\overline{B}_j \cap A \ne \emptyset} \left(\frac{\operatorname{diam}(fB_j)}{\mathrm{d}(fB_j, f(x_0))} \frac{1}{\operatorname{diam}(B_j)} \right)^p |B_j|.$$

Using the fact that $\operatorname{diam}(fB_j)^n \le C(n)|fB_j|(2^{k+1})^n$ when $y_j \in A_k$ and Hölder's inequality, we thus obtain

$$\int_{\mathbf{R}^n} \rho^p(x)\,dx \le C(n,p)(\log \frac{L}{l})^{-p} \Big(\sum_{f\overline{B}_j \cap A \ne \emptyset} \frac{|fB_j|}{\mathrm{d}(fB_j, f(x_0))^n} \Big)^{\frac{p}{n}} \Big(\sum_k \sum_{\substack{f\overline{B}_j \cap A \ne \emptyset \\ y_j \in A_k}} |B_j|(2^k)^{p^*} \Big)^{\frac{n-p}{n}}.$$

Next we will approximate the sums separately. Let us start from the first sum, which we denote by S_1. First we regroup the balls depending on their distance from $f(x_0)$. For this, define $R_s = B(f(x_0), 2^{s+1}l) \setminus B(f(x_0), 2^s l)$, $s = 0, \ldots, j_0 - 1$. Since $\operatorname{diam}(fB_j) < 2^{-j_0-3}L < 2^{-3}l$ we have

$$\begin{aligned} S_1 &= \sum_{s=0}^{j_0-1} \sum_{f\overline{B}_j \cap R_s \ne \emptyset} \frac{|fB_j|}{\mathrm{d}(fB_j, f(x_0))^n} \\ &\le \sum_{s=0}^{j_0-1} \sum_{f\overline{B}_j \cap R_s \ne \emptyset} \frac{|fB_j|}{(2^{s-1}l)^n}. \end{aligned}$$

Since the overlapping of the balls B_j is uniformly bounded and $q = N(f,Q) < \infty$ we obtain the estimate

$$S_1 \leq \frac{2^n}{l^n} \sum_{s=0}^{j_0-1} \frac{1}{2^{sn}} C(n) q |B(f(x_0), 2^{s+2}l)| = qC(n)j_0$$
$$\leq qC(n) \log \frac{L}{l}.$$

The last inequality comes from the fact that $2^{-j_0+1}L > l$.

Now we approximate the second term. First $|B_j| = |B_j \cap A_k| + |B_j \setminus A_k|$. The double sum over the $|B_j \cap A_k|$-terms can be estimated by the integral of $h_f^{p^*}$ due to the definition of A_k, and the double sum over the $|B_j \setminus A_k|$-terms turns out to be no more than a constant times ε, because $\cup_{y_j \in A_k} B_j \subset U_k$ and $|U_k| \leq |A_k| + \frac{\varepsilon}{2^k (2^{p^*})^k}$. Therefore

$$\sum_k \sum_{\substack{f\overline{B}_j \cap A \neq \emptyset \\ y_j \in A_k}} (2^k)^{p^*} |B_j| \leq C(n) \int_{B(x_0,2r)} h_f^{p^*}(x)\, dx + C(n)\varepsilon.$$

Because ε was arbitrary we conclude that

$$\text{(4.3)} \qquad \int_{\mathbf{R}^n} \rho^p(x)\, dx \leq C(n,p,q)(\log \frac{L}{l})^{p(1-n)/n} \left(\int_{B(x_0,2r)} h_f^{p^*}(x)\, dx \right)^{(n-p)/n}.$$

Our next goal is to find a lower bound on the integral of ρ^p. Next let $x \in F_1$, $w \in B(x_0, 2r)$, and $y \in F_2$ be such that $([x,w] \cup [w,y]) \cap E$ is countable. This is possible due to Lemma 2.5 for every x, y, and for any sphere $\partial B(z,r) \subset \Omega$ for $\mathcal{H}^{n-1}$-a.e. direction $w \in \partial B(z,r)$ from these points. Then

$$\int_{[x,w]\cup[w,y]} \rho\, ds = \left(\log \frac{L}{l}\right)^{-1} \sum_{f\overline{B}_j \cap A \neq \emptyset} \frac{\operatorname{diam}(fB_j)}{\operatorname{d}(f(x_0), fB_j)} \frac{1}{\operatorname{diam}(B_j)} \int_{[x,w]\cup[w,y]} \chi_{2B_j}\, ds.$$

Now if $([x,w] \cup [w,y]) \cap B_j \neq \emptyset$ then $\int_{[x,w]\cup[w,y]} \chi_{2B_j}\, ds \geq r_j$. So we obtain

$$\int_{[x,w]\cup[w,y]} \rho\, ds \geq \frac{1}{2} \left(\log \frac{L}{l}\right)^{-1} \sum_{\substack{f\overline{B}_j \cap A \neq \emptyset \\ ([x,w]\cup[w,y]) \cap B_j \neq \emptyset}} \frac{\operatorname{diam}(fB_j)}{\operatorname{d}(f(x_0), fB_j)}.$$

Denote $R_s = B(f(x_0), 2^{s+1}l) \setminus \overline{B}(f(x_0), 2^s l)$, $s = 0, 1, \ldots, j_0 - 2$. The sets $f\overline{B}_j$ cover $f([x,w] \cup [w,y]) \cap A$ up to a countable set and $f([x,w] \cup [w,y])$ connects $B(f(x_0), l)$

and $\mathbf{R}^n \setminus B(f(x_0), L)$. So

$$\int_{[x,w]\cup[w,y]} \rho\, ds \geq \frac{1}{2}\left(\log\frac{L}{l}\right)^{-1} \sum_{s=0}^{j_0-2} \sum_{\substack{f\overline{B}_j \cap R_s \neq \emptyset \\ ([x,w]\cup[w,y])\cap B_j \neq \emptyset}} \frac{\operatorname{diam}(fB_j)}{2^{s+1}l}$$

$$\geq \frac{1}{8}\left(\log\frac{L}{l}\right)^{-1} j_0 \geq \frac{1}{16}.$$

The last inequality comes from the fact that $j_0 > \frac{1}{\log 2}\log\frac{L}{l}$.

Now we can apply Lemma 2.7 the same way as in the proof of Theorem 3.6, and we obtain

$$\int_{\mathbf{R}^n} \rho^p(x)\, dx \geq C(n,p) r^{n-p}.$$

Combining this with inequality (4.3) we conclude with the claim. □

Remark 4.3. In the planar case we can actually choose $p = 1$ because by Remark 2.8 we obtained the lower bound $\int_{B(x,2r)} \rho\, dx \geq Cr$ for a suitable ρ. Combining this with (4.3) in the proof above gives us the nicer estimate in the plane

$$L_f(x,r) \leq \mathrm{d}(f(x), \partial f B(x,r)) \exp\left(C(q) \fint_{B(x,2r)} h_f(y)^2\, dy\right)$$

when $Q \subset\subset \Omega$, $B(x,2r) \subset Q$ and $q = N(f,Q)$. Compare this with Theorem 1.1 in [11].

Our next goal is to find a metric condition which guarantees that the mapping has an exponentially integrable K-distortion. This is done in Theorem 4.5, which is a generalization of Theorem 1.3 in [11]. For more about these mappings see [8] and also references therein. There is no real hope to find an equivalent metric condition since under the integrability conditions on K the size of the set where h_f is infinite might be larger than σ-finite; see the following example.

Example 4.4. For our example we need a quasiconformal mapping constructed by David and Toro [3]. This quasiconformal mapping $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ has some special properties: $\varphi(\mathbf{R}^{n-1}) =: \Sigma$ is the $(n-1)$-dimensional analogue in $\mathbf{R}^n$ of the usual modifications of the von Koch snowflakes in $\mathbf{R}^2$ with the Hausdorff dimension $(1-a)^{-1}(n-1)$, here $a > 0$ is sufficiently small. Furthermore, if $x, y \in \mathbf{R}^n$ and $|x-y| < 1$, then there is a constant C such that

$$\frac{1}{C}|x-y|^{1-a} \leq |\varphi(x) - \varphi(y)| \leq C|x-y|^{1-a}. \tag{4.4}$$

Let us consider the mapping $f: \mathbf{R}^n \to \mathbf{R}^n$ that is constructed with the help of the mapping φ and the mapping $g: \mathbf{R}^n \to \mathbf{R}^n$,

$$g(x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_{n-1}, x_n \log\log\log \frac{1}{x_n}) & \text{when } 0 < x_n \leq e^{-e^e} \\ (x_1 \ldots, x_n) & \text{when } x_n \leq 0 \\ (x_1, \ldots, x_{n-1}, bx_n) & \text{when } x_n > e^{-e^e} \end{cases}$$

Here $b = D(y \log\log\log \frac{1}{y})(e^{-e^e})$. The mapping g is a homeomorphism and when $0 \leq x_n \leq e^{-e^e}$, we have $K_g(x) \approx (\log\log\log \frac{1}{x_n})^{n-1}$ and the limsup-distortion is $H_g(x) \gtrsim \log\log\log \frac{1}{x_n}$.

We define the homeomorphism $f = g \circ \varphi^{-1}: \mathbf{R}^n \to \mathbf{R}^n$. Since φ^{-1} is quasiconformal, all significant distortion near $\mathbf{R}^{n-1} = \{x_n = 0\}$ comes from g. Because $\varphi^{-1}(\Sigma) = \{x_n = 0\}$, we have $H_f(x) = \infty$ for every $x \in \Sigma$. Thus H_f is infinite in a set whose dimension is larger than $(n-1)$.

Despite this fact the K-distortion K_f is exponentially integrable. To see this take a compact set K which touches Σ. If we were away from Σ, the distortion would be bounded. Divide K using $A_j = \{x \in K : 2^{-j} \leq d(x, \Sigma) < 2^{-(j-1)}\}$. In the set A_j the distortion $K_f \approx (\log\log\log 2^{\frac{j}{1-a}})^{n-1}$ by (4.4). To integrate K_f we need an approximation for the size of A_j. Let us cover A_j with mutually disjoint cubes, whose volumes are comparable to 2^{-jn}. To see how many cubes there are, we return the situation to $\varphi^{-1}(\Sigma) \times \mathbf{R} = \mathbf{R}^{n-1} \times \mathbf{R}$. There $\varphi^{-1}(A_j)$ is almost an n-rectangle. We can assume that it has a constant $(n-1)$-bottom area for all j, since K is compact and since the distance from $\varphi^{-1}(\Sigma)$ is the most relevant. The height of $\varphi^{-1}(A_j)$ is $2^{-\frac{j}{1-a}}$ by (4.4). So there are $B 2^{-\frac{j}{1-a}} 2^{\frac{jn}{1-a}}$ cubes, where B is the bottom area. Thus $|A_j| \approx 2^{-j\frac{1-n}{1-a}} 2^{-jn}$. Now we conclude that

$$\begin{aligned} \int_K \exp(K_f)\,dx &\lesssim \sum_j 2^{-j\frac{1-n}{1-a}} 2^{-jn} \exp(\log\log\log 2^{\frac{j}{1-a}})^{n-1} \\ &\lesssim \sum_j 2^{-j\frac{1-n}{1-a}} 2^{-jn} 2^{j\varepsilon} \end{aligned}$$

with any $\varepsilon > 0$. For any $a < \frac{1}{n}$ one can choose a sufficiently small ε so that the series converges.

Theorem 4.5. *Let $f: \Omega \to \mathbf{R}^n$ be an open, discrete, continuous, sense-preserving mapping such that $h_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-measure. There is a constant C', independent of E and f, such that*

$$\exp(C' h_f^{\frac{n}{n-1}}) \in L^1_{loc}(\Omega)$$

implies that $f \in W^{1,q}_{loc}(\Omega, \mathbf{R}^n)$, for any $1 < q < n$, and that

$$|Df(x)|^n \leq K(x) J_f(x) \qquad \textit{for a.e.} \quad x \in \Omega$$

with $exp(C'K^{\frac{n}{(n-1)^2}}) \in L^1_{loc}(\Omega)$. *Furthermore, if we assume that* $\exp(C'h_f^{(1+\varepsilon)(n-1)}) \in L^1_{loc}(\Omega)$, *for some* $\varepsilon > 0$, *then* $f \in W^{1,n}_{loc}(\Omega, \mathbf{R}^n)$ *and* $\exp(\lambda K) \in L^1_{loc}(\Omega)$ *for every* $\lambda > 0$.

As in the article [11] we need two lemmas for the proof of Theorem 4.5. The first lemma gives us a sufficient condition for an open, discrete, continuous, sense-preserving mapping to be ACL.

Lemma 4.6. *Let* $f\colon \Omega \to \mathbf{R}^n$ *be an open, discrete, continuous, sense-preserving mapping such that*

$$L_f(x,r) \le d(f(x), \partial f B(x,r))\varphi(x) \tag{4.5}$$

whenever $0 < r < r_0$ *and* $B(x,r) \in \Omega$. *If* $\varphi \in L^{\frac{n}{n-1}}_{loc}(\Omega)$, *then* f *is ACL* .

Proof. The idea of this proof is from [18, p. 107] and as for the proof of Theorem 3.2, the idea of the argument can be traced back at least to Gehring [5]. The proof we give is almost the same as the proof of Lemma 3.1 in [11]. Pick a closed cube $Q \subset\subset \Omega$ whose sides are parallel to the coordinate axes. In order to show that f is ACL it suffices to show that f is absolutely continuous on almost every line segment of Q parallel to the coordinate axes. By symmetry it is sufficient to consider segments parallel to the x_n -axis.

Assume $Q = Q_0 \times J_0$, where Q_0 is an $(n-1)$-interval in $\mathbf{R}^{n-1}$ and $J_0 = [a,b] \subset \mathbf{R}$. Next for each Borel set $E \subset Q_0$ set

$$\eta(E) = |f(E \times J_0)|.$$

Then η is a bounded q-quasiadditive set function in Q_0, $q = N(f,Q) < \infty$. Hence it has a finite upper derivative $\eta'(y)$ for almost every $y \in Q_0$ by Lemma 2.1. Choose $y \in Q_0$ such that

(i) $\eta'(y) < \infty$, and
(ii) $\varphi \in L^{\frac{n}{n-1}}(\{y\} \times [a-d, b+d])$, $d = \frac{1}{2}\mathrm{d}(Q, \partial\Omega)$.

The latter is possible due to the Fubini theorem. We will prove that f is absolutely continuous on the segment $\{y\} \times J_0$ which will prove the claim.

Let $J \subset \{y\} \times J_0$ be compact. We wish to estimate $\mathcal{H}^1(fJ)$. Choose $0 < \varepsilon < d$ and $t > 0$. Let $0 < \delta_1 \le r_0$ be the number given by Lemma 2.11 for the set J. Choose $\delta_2 > 0$ such that, if $0 < r < \delta_2$, then $|f(x) - f(z)| < t$ whenever $x, z \in Q_0 \times [a-d, b+d]$ and $|x - z| \le 2r$. Denote $\delta = \min\{\delta_1, \delta_2, \varepsilon\}$. Fix $0 < r < \delta$. Now the covering lemma 2.11 gives a covering $\triangle_1, \dots, \triangle_p$ of J with intervals in $\{y\} \times [a-d, b+d]$ so that

(i) $\mathrm{diam}(\triangle_i) = r$ for $1 \le i \le p$,
(ii) each point of $\{y\} \times [a-d, b+d]$ belongs to at most two different $\triangle_i$, and
(iii) each $\overline{\triangle}_i$ is contained in the ε-neighborhood of J in $\{y\} \times [a-d, b+d]$.

Now, because $\varphi(x) \geq 1$ for every x and $\varphi \in L^{\frac{n}{n-1}}(\{y\} \times [a-d, b+d])$, there are points $x_i \in \overline{\triangle}_i$ such that

$$\varphi(x_i) \leq 2 \inf_{x \in \overline{\triangle}_i} \varphi(x) < \infty. \tag{4.6}$$

Define balls $A_i = B^n(x_i, r)$. Now $\triangle_i \subset A_i$ and $A_i \subset \overline{B}^{n-1}(y,r) \times J$. Because $\operatorname{diam}(fA_i) < t$ we have that $\mathcal{H}^1_t(fJ) \leq \sum \operatorname{diam}(fA_i) \leq 2\sum L_i$, where $L_i = L_f(x_i, r)$. Denote $l_i = \mathrm{d}(f(x_i), \partial f B(x_i, r))$. Using (4.5) we obtain the estimate

$$\begin{aligned}\mathcal{H}^1_t(fJ)^n &\leq 2^n \left(\sum_i L_i\right)^n \leq 2^n \left(\sum_i l_i \varphi(x_i)\right)^n \\ &= \frac{2^n}{r^{n-1}} \left(\sum_i l_i r^{\frac{n-1}{n}} \varphi(x_i)\right)^n.\end{aligned}$$

By Hölder's inequality we further conclude that

$$\mathcal{H}^1_t(fJ)^n \leq \frac{2^n}{\Omega_n r^{n-1}} \sum_i |f(A_i)| \left(\sum_i r \varphi^{\frac{n}{n-1}}(x_i)\right)^{n-1}.$$

Because no point belongs to more than two of the sets A_i, and $q = N(f, Q) < \infty$,

$$\sum_i |fA_i| \leq 10 q\, \eta(\overline{B}^{n-1}(y,r)).$$

Since the points x_i satisfy (4.6) we arrive at

$$\mathcal{H}^1_t(fJ)^n \leq \frac{10 \cdot 2^{2n-1} q\, \eta(\overline{B}^{n-1}(y,r))}{\Omega_n \quad r^{n-1}} \left(\int_{J+\varepsilon} \varphi^{\frac{n}{n-1}}(z)\, dz_n\right)^{n-1}.$$

Here $J + \varepsilon$ is the ε-neighborhood of J in $\{y\} \times [a-d, b+d]$. Letting first $r \to 0$ and then $\delta_1 \to 0$ and finally $\varepsilon \to 0$ and $t \to 0$ we deduce that

$$\mathcal{H}^1(fJ)^n \leq C(n,q)\eta'(y) \left(\int_J \varphi^{\frac{n}{n-1}}(z)\, dz_n\right)^{n-1}.$$

The absolute continuity of f on $\{y\} \times J_0$ follows from this estimate similarly as at the end of the proof of Theorem 3.2. □

The other lemma which we need for the proof of Theorem 4.5 is the following.

Lemma 4.7. *Let $u \colon \Omega \to \mathbf{R}$ be a non-negative function so that*

$$\exp(C'u^s) \in L^1_{loc}(\Omega)$$

where $s > 0$, and let $p > 1$. Then, for each compact set $F \subset \Omega$ and $q > 0$

$$\exp(\beta C'(M(\chi_F u^q))^{\frac{s}{q}}) \in L^p_{loc}(\Omega)$$

where β depends only on p, s, q and n, and M is the usual Hardy-Littlewood maximal operator.

Proof. Let $Q \subset\subset \Omega$. By expanding the exponential function as a power series we have that

$$\int_Q \left(\exp\left(\beta C'(M(\chi_F u^q))^{\frac{s}{q}}\right)\right)^p (x)\,dx = |Q| +$$

$$\sum_{1\le k\le \frac{q}{s}} \frac{(\beta C'p)^k}{k!} \int_Q \left(M\chi_F u^q\right)^{k\frac{s}{q}}(x)\,dx + \sum_{k>\frac{q}{s}} \frac{(\beta C'p)^k}{k!} \int_Q \left(M\chi_F u^q\right)^{k\frac{s}{q}}(x)\,dx.$$

By using Hölder's inequality for the first finite sum S_1 we have

$$S_1 \le \sum_{1\le k\le \frac{q}{s}} \frac{(\beta C'p)^k}{k!} \left(\int_Q \left(M\chi_F u^q\right)^{\frac{ks}{q}+1}(x)\,dx\right)^{\frac{\frac{ks}{q}}{\frac{ks}{q}+1}} |Q|^{\frac{1}{\frac{ks}{q}+1}}.$$

Now by the fact that the Hardy-Littlewood maximal operator M is bounded from L^r to L^r, $1 < r < \infty$, see (2.3), we see that the first sum is finite. The characteristic function χ_F guarantees that the local integrability property of u is enough for finiteness.

For the other sum, denoted by S_2, we use directly (2.3) and we have

$$S_2 \le \sum_{k>\frac{q}{s}} \frac{(\beta C'p)^k}{k!} \left(5^n \frac{ks}{ks-q}\right)^{ks/q} \int_F u^{ks}(x)\,dx.$$

Let k_1 be the first integer such that $k_1 > \frac{q}{s}$. Then

$$S_2 \le \sum_{k>\frac{q}{s}} \frac{(\beta C'p)^k}{k!} \left(5^n \frac{s}{s-q/k_1}\right)^{ks/q} \int_F u^{ks}(x)\,dx$$

$$\le \int_F \exp\left(\beta C'p \left(5^n \frac{k_1 s}{k_1 s - q}\right)^{s/q} u^s\right) dx < \infty$$

by choosing $\beta = (p(5^n \frac{k_1 s}{k_1 s-q})^{s/q})^{-1}$. □

Now we are ready for the proof of the theorem.

Proof of Theorem 4.5. Our first job is to show that f is ACL. For this we need Lemma 4.6. First we have to fix a cube $Q \subset\subset \Omega$. Let us show that f is ACL in the cube Q, which will guarantee that f is ACL.

Let $d = \frac{1}{2}\mathrm{d}(Q, \partial\Omega) > 0$. Now from Theorem 4.1 we obtain

$$L_f(x,r) \le \mathrm{d}(f(x), \partial f B(x,r)) \exp\left(C(n,p,q) \left(⨍_{B(x,2r)} h_f^{p^*}(y)\,dy\right)^{\frac{1}{p^*}\frac{n}{n-1}}\right)$$

when $B(x,2r) \subset Q + d = \{x \in \Omega : \mathrm{d}(x,Q) < d\}$. Here $Q+d$ is a slightly bigger cube and $q = N(f, Q+d) < \infty$. So

$$L_f(x,r) \le \mathrm{d}(f(x), \partial f B(x,r)) \exp\left(C(n,p,q) \left(M(\chi_{Q+d} h_f^{p^*})(x)\right)^{\frac{1}{p^*}\frac{n}{n-1}}\right)$$

for every $x \in Q$ and $0 < r < \frac{1}{2}d$. So by Lemma 4.6 f is ACL in Q if

$$\exp\left(C(n,p,q)\Big(M(\chi_{Q+d}h_f^{p^*})(x)\Big)^{\frac{1}{p^*}\frac{n}{n-1}}\right) \in L^{\frac{n}{n-1}}(Q).$$

This is true by Lemma 4.7 if C' is chosen correctly. Thus we can conclude that f is ACL.

Our mapping f is differentiable almost everywhere. Now in the set $G = \{x \in \Omega : f$ differentiable at x, $\min_{|e|=1} |Df(x)e| > 0\}$ we have that $H_f(x) = h_f(x)$ everywhere and furthermore

$$|Df(x)|^n \leq H_f(x)^{n-1}J_f(x) \quad \text{a.e. in G.}$$

On the other hand, if f is differentiable at x with $\min_{|e|=1} |Df(x)e| = 0$, and $h_f(x) < \infty$, then also $\max_{|e|=1} |Df(x)e| = 0$, because otherwise the dilatation would be infinite. Thus

$$|Df(x)|^n \leq K(x)J_f(x) \quad \text{a.e. in } \Omega,$$

where

$$K(x) = \begin{cases} h_f(x)^{n-1} & \text{when } \min_{|e|=1} |Df(x)e| > 0 \\ 1 & \text{otherwise} \end{cases}$$

Now $\exp(C'K^{\frac{n}{(n-1)^2}}) \in L^1_{loc}(\Omega)$. By Hölder's inequality we see that $f \in W^{1,q}_{loc}(\Omega, \mathbf{R}^n)$, $1 < q < n$, because $h_f \in L^s_{loc}(\Omega)$, for any $s \geq 1$, and for an open, discrete, continuous, almost everywhere differentiable mapping $J_f \in L^1_{loc}(\Omega)$, see Remarks 2.10.

If we further assume that $\exp(C'h_f^{(1+\varepsilon)(n-1)}) \in L^1_{loc}(\Omega)$, then $\exp(C'K^{1+\varepsilon}) \in L^1_{loc}(\Omega)$. This implies that for every $\lambda > 0$ we have $\exp(\lambda K) \in L^1_{loc}(\Omega)$. Now from Theorem 1 in [8] we deduce that $f \in W^{1,n}_{loc}(\Omega, \mathbf{R}^n)$. Theorem 1 in [8] says that there is a constant $\lambda(n) \geq 1$ such that if $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$, $J_f \in L^1_{loc}(\Omega)$, and $|Df(x)|^n \leq K(x)J_f(x)$ almost everywhere with $\exp(\lambda K) \in L^1_{loc}(\Omega)$ for some $\lambda \geq \lambda(n)$, then $|Df| \in L^n_{loc}(\Omega)$. □

Our last theorem is an improvement on Theorem 1.4 in [11]. It gives a second condition in terms of h_f that guarantees $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$. The author thanks Jeremy Tyson for pointing out a slight improvement to this theorem.

Theorem 4.8. *Suppose that* $p^* = \frac{pn}{n-p}$, *with* $n - 1 < p < n$. *Let* $f : \Omega \to \mathbf{R}^n$ *be an open, discrete, continuous, sense-preserving mapping such that* $h_f(x) < \infty$ *and*

$$\limsup_{r\to 0} \fint_{B(x,r)} h_f^{p^*}(x)\, dx < \infty \tag{4.7}$$

outside a set S *of* σ*-finite* $(n-1)$*-measure, and* $h_f \in L^{p^*}_{loc}(\Omega)$. *Then* $f \in W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$.

Proof. Taking the $\limsup_{r\to 0}$ in inequality (4.2) we see that

$$\widetilde{H}_f(x) = \limsup_{r\to 0} \frac{L_f(x,r)}{\mathrm{d}(f(x), \partial f B(x,r))} < \infty$$

outside a set S of σ-finite $(n-1)$-measure. Moreover, by Theorem 4.1, f is differentiable almost everywhere. Next we show that f is ACL. By an elementary argument one sees that $H_f(x) = h_f(x)$ everywhere in the set where both f is differentiable and $\min_{|e|=1} |Df(x)e| > 0$. To employ Theorem 3.2 and Remark 3.4 after it (notice that $\widetilde{H}_f \leq H_f$) we would like to have $H_f = h_f$ almost everywhere in Ω. The exceptional set E where f is differentiable, $h_f(x) < \infty$, and $\min_{|e|=1} |Df(x)e| = 0$ thus causes a potential danger. On the other hand, in this set, $|Df(x)| = 0$. This saves the situation, because then the $\mathcal{H}^1$-measure of the image of the intersection of E with any line l is zero. To see this let $\varepsilon > 0$ and let us cover $E \cap l$ with balls $B(x, r_x)$, where $x \in E$ and $0 < r_x < 1$ is so small that $\operatorname{diam}(fB(x, r_x)) < \varepsilon$. The latter is possible because $Df(x) = 0$. By the Besicovitch covering theorem we can further use a boundedly overlapping covering $\{B(x_i, r_i)\}$. Thus

$$\mathcal{H}^1_\infty(f(E \cap l)) \leq \sum_i \operatorname{diam}(fB(x_i, r_i)) \leq \varepsilon P(n),$$

where $P(n)$ is the multiplicity of the overlap. So we have $\mathcal{H}^1_\infty(f(E \cap l)) = 0$, since ε was arbitrary.

Now the proof of Theorem 3.2 goes through if we replace the set E there with the set $S \cup E$ of this theorem. Thus f is ACL. And by Theorem 3.1 we see that $|Df|$ is locally integrable. We conclude that f belongs to the Sobolev class $W^{1,1}_{loc}(\Omega, \mathbf{R}^n)$. □

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