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DISSERTATIONES

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Lassi Kurittu

Contents

Introduction

We consider a nonelementary Kleinian group G acting on the Riemann sphere, and the set K of the limit points of G. The geometry of K is, in general, very complicated, as we can see by the following observation.

If G does not preserve any disc, it certainly has loxodromic elements and, moreover, infinitely many. Every fixed point of such an element is a limit point. Suppose that we have loxodromic elements T and T' in G with distinct sets of fixed points. Call x the attractive fixed point of T and y a fixed point of T'. Then the sequence $\{T^n y\}$ converges to x spiralling around it. Now every $T^n y$ is a limit point of G and, in fact, a fixed point of $T^n T' T^{-n}$, which is also a loxodromic element and thus has its own spirals around itself, and so on. It is clear that, in general, this leads to a very complicated behaviour of K.

With this in mind, it is rather surprising that there is a deep theorem of Maskit [M1], based on Teichmüller theory, which asserts that if G is finitely generated and the set of the ordinary points of G has exactly two components, then K is even a quasicircle.

It remains open what happens if the assumption on a finite generation is dropped. Then K is not necessarily a quasicircle. This becomes clear from the following example.

Let $\{C_n\}_{n\in\mathbf{Z}}$ be a sequence of tangent circles as in Figure 1.

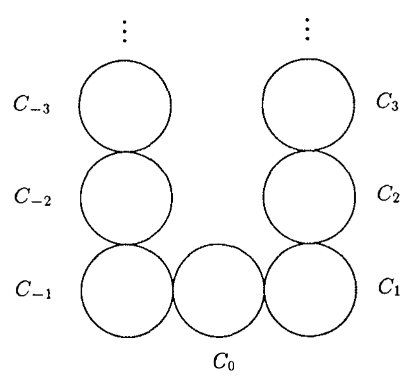

Figure 1

Let G be the group generated by the reflections on these circles (and pass to a subgroup if one wants to consider groups with only directly conformal elements).

It follows that the limit point set of G is a Jordan curve which travels through all the circles and intersects them exactly at the points of tangency. There is a cusp at infinity and so K cannot be a quasicircle.

One may ask whether K is always a Jordan curve or not, assuming that the set of the ordinary points has exactly two components. The answer is negative, as seen by following construction, due to Uri Srebro; see also Abikoff [A, pp. 3–5].

Choose an infinite tube narrowing at the both ends, filled by tangent circles and located as in Figure 2.

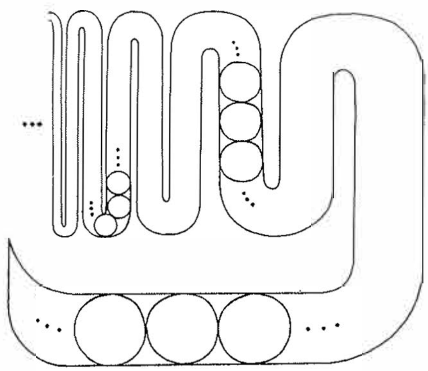

Figure 2

Again let G be the group generated by reflections on these circles. It results that the limit point set looks something like the set in Figure 3.

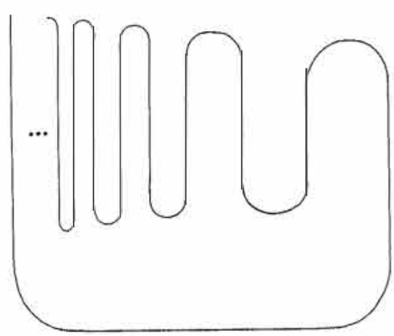

Figure 3

This is obviously not a Jordan curve. Note that the picture is greatly simplified because the bad behaviour of the set repeats itself inside every circle infinitely many times. It is, however, still true that the set of the ordinary points has exactly two components.

In this paper we state a sufficient and necessary condition under which K is a Jordan curve, see Theorem 3.1.1 and also Note 3.2.9.

To prove the theorem we will need quite a lot of preparatory material, which is developed in Sections 1 and 2. Section 1 is devoted to classical plane topology. There we also obtain a characterization of a Jordan curve which may have some interest of its own. In Section 2 we recall the basics on Kleinian and Fuchsian groups and prove some simple lemmas for Section 3, which contains the main theorem.

Our proof is rather laborious but almost completely elementary in the sense that only one deeper result is needed (see Theorem 2.3.2.). After proving the theorem we note that the theorem of Maskit mentioned above follows almost immediately from our result.

0. Notations

In this paper we denote by $\hat{\mathbf{C}}$ the Riemann sphere and by B the open unit disc in the complex plane $\mathbf{C}$.

Two non-empty domains D_1 and D_2 in $\hat{\mathbf{C}}$ are said to be *complementary*, if $\overline{D}_1 \cup \overline{D}_2 = \hat{\mathbf{C}}$, $D_1 \cap D_2 = \emptyset$ and the domains have a common boundary.

A *Jordan arc* is a homeomorphic image of the closed unit interval $[0,1]$. A *Jordan curve* is a homeomorphic image of the circle ∂B. By $\mathbf{M}$ we denote the group of all Möbius transformations acting on $\hat{\mathbf{C}}$. All the elements of $\mathbf{M}$ are assumed to be directly conformal. We use the standard classification of these elements: if $T(z) = (az+b)/(cz+d)$, $ad - bc = 1$, $T \neq \mathrm{id}$, then T is called *elliptic*, *parabolic*, *hyperbolic* or *loxodromic* if the trace square $(a+d)^2$ is in the interval $[0,4[$, exactly 4, in the interval $]4,\infty[$ or not in $\mathbf{R}_+$, respectively.

A characteristic property of parabolic elements is that they have exactly one fixed point in $\hat{\mathbf{C}}$. A hyperbolic or loxodromic element T has always two fixed points z_1 and z_2. One of them, say z_1, has the property that $T^n w \to z_1$ for all $w \in \hat{\mathbf{C}} \setminus \{z_2\}$. We call z_1 the *attractive* fixed point of T. The other fixed point z_2 is the attractive fixed point of T^{-1}. Note that if T is parabolic, then $T^n w$ converges to the sole limit point of T for all $w \in \hat{\mathbf{C}}$.

1. Plane topology

1.1. Basic notions and general results. In this chapter we suppose that D_1 and D_2 are complementary domains in $\hat{\mathbf{C}}$. It follows immediately that D_1 and D_2 are simply connected.

A point x in the boundary $\partial D_1 = \partial D_2$ is said to be *accessible from* D_1, if x can be joined to an interior point of D_1 by a curve that lies, except the endpoint at x, entirely in D_1. By selecting a suitable part of the connecting curve, we may always assume that it is a Jordan arc.

Because D_1 is simply connected, it can be mapped conformally onto B by the Riemann mapping theorem. Let k be such a mapping.

The behaviour of k near the boundary and especially near the accessible points is well known. We will list here some lemmas, proofs of which can mainly be found in Ford's classical book [F, Chapter 80].

Let the symbol A stand for the set of the points in ∂D_1, which are accessible from D_1.

Lemma 1.1.1. *Let x be a point of A, γ a curve in D_1 excluding the other endpoint at x and $\{x_n\}$ a sequence in $D_1 \cap \gamma$ such that $x_n \to x$. Then there exists a point $y \in \partial B$ such that $k(x_n) \to y$.*

Moreover, the point y does not depend on the choice of the curve γ or the sequence $\{x_n\}$.

Proof. The existence of y follows directly from [F, p. 191]. The uniqueness of y needs an argument.

Let γ_1 and γ_2 be curves in $D_1 \cup \{x\}$ such that they have a common initial point $w \in D_1$ and x is their common endpoint. Select $r > 0$ such that $r < d(x,w)$. Then γ_1 and γ_2 meet the circle $S(x,r)$; denote by w_i the last point of γ_i which meets $S(x,r)$, when we are travelling from w to x, $i = 1,2$. Select further points $z_i \in \gamma_i$ such that z_i lies between w_i and x in γ_i, $i = 1,2$. It follows that the z_i:s lie in the disc $B(x,r)$.

According to [F, pp. 190 and 192], it is now enough to show that z_1 can be connected to z_2 by a curve lying entirely in $D_1 \cap B(x,r)$.

Call α_i the parts of γ_i that lie between w_i and x, $i = 1,2$. If α_1 meets α_2 in $D_1 \cap B(x,r)$, we are done. So we may suppose that they do not meet. We may also suppose that α_1 and α_2 are Jordan arcs. The points w_1 and w_2 can be connected by a Jordan arc β in D_1, and we may suppose that β does not meet α_1 or α_2, except at the endpoints w_1 and w_2. (If $w_1 = w_2$, β is not needed.)

Then $\alpha_1 \cup \alpha_2 \cup \beta$ is a Jordan curve, which does not meet D_2. Because D_2 is connected, it lies entirely in one component of $\hat{\mathbb{C}} \setminus (\alpha_1 \cup \alpha_2 \cup \beta)$. Then the other component contains only points of D_1.

The existence of the desired connective curve follows now from the fact that $\alpha_1 \cup \alpha_2$ does not go out of $B(x,r) \cup \{w_1, w_2\}$. □

With the aid of the preceding lemma we can now define a mapping $\hat{k}\colon A \to \partial B$ as follows: if $x \in A$ and γ is a curve from D_1 to x, let $\hat{k}(x)$ be the well defined endpoint of $k(\gamma)$ in ∂B.

We will need the following lemma, which is adapted directly from [F, pp. 192 and 195].

Lemma 1.1.2. *The mapping $\hat{k}\colon A \to \partial B$ is injective and the set $\hat{k}(A)$ is dense in ∂B.*

For later use we make here a definition, again according to [F, p. 196].

Consider a point x in ∂B. Select points a_1 and b_1 in $\hat{k}(A) \subset \partial B$ different from each other and x, and recursively a_n, $b_n \in \hat{k}(A)$ such that a_n lies in the open arc of ∂B from a_{n-1} to x, which does not contain b_{n-1} and b_n in the open arc of ∂B from x to b_{n-1}, which does not contain a_{n-1}, respectively. By Lemma 1.1.2 we may additionally demand that $a_n \to x$ and $b_n \to x$.

For each n we can select a curve γ_n connecting $\hat{k}^{-1}(a_n)$ and $\hat{k}^{-1}(b_n)$ in D_1 such that $\operatorname{diam} k(\gamma_n) \to 0$. We also assume that the γ_n:s are disjoint. Each $k(\gamma_n)$ divides B into two components; let B_n be the one for which $x \in \overline{B}_n$. Then $B_{n+1} \subset B_n$ for all n.

According to the above construction $\cap_n \overline{B}_n = \{x\}$. Now the set $\cap_n \overline{k^{-1}(B_n)}$ lies in ∂D_1 and contains at least one point.

It is quite obvious that the set $\cap_n \overline{k^{-1}(B_n)}$ is independent of the choices made.

Definition 1.1.3. *Let $x \in \partial B$. The set $C(x, k^{-1}, D_1) = \cap_n \overline{k^{-1}(B_n)} \subset \partial D_1$ constructed as above is called the cluster set of x corresponding to k^{-1} and D_1.*

We will have a frequent use of the following purely topological result, known as the *Zoretti theorem* (see, e.g., Whyburn [W, p. 35]).

Theorem 1.1.4. *If K is a component of a compact set $M \subset \mathbb{C}$ and ε is any positive number, then there exists a Jordan curve γ, which circulates K and is such that $\gamma \cap M = \emptyset$, and every point of γ is at a distance less than ε from some point of K.*

1.2. Two characterizations of Jordan curves. The following characterization of a Jordan curve shall be essential in the proof of our main theorem. It appears for the first time in Kerékjártó [K], but there is a gap in his reasoning. A correct but quite long proof is given by Eilenberg [E]. We give here a short and possible new proof.

Theorem 1.2.1. *Let D_1 and D_2 be complementary domains in $\hat{\mathbb{C}}$ and K their common boundary. If there exists a homeomorphism $f: \overline{D}_1 \to \overline{D}_2$ such that $f|_K = \mathrm{id}_K$, then K is a Jordan curve.*

Proof. We may assume that $K \subset \mathbb{C}$ and (defining $f = f^{-1}$ in D_2) that f is a homeomorphism $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. According to the results in [W, pp. 33–34], it is enough to show that K is locally connected.

So, let us have $p \in K$ and $\varepsilon > 0$ given. We must find $\delta > 0$ such that every pair of points x and y in $K \cap B(p, \delta)$ lies in the same component of $\overline{K \cap B(p, \varepsilon)}$.

Because f is continuous in the compact set $\hat{\mathbb{C}}$, it is uniformly continuous, and we can find $\delta' > 0$ such that $|f(z) - f(w)| < \varepsilon/2$ whenever $|z - w| < \delta'$ (we can use the plane metric near the point $p \in K \subset \mathbb{C}$). Now we define $\delta = \min\{\delta', \varepsilon/2\}$ and show that this choice of δ will work.

Let x and y be in $K \cap B(p,\delta)$. Connect x to y by a Euclidean line segment L. The set $L \cap D_1$ consists of at most denumerably many open segments, call them M_n, $n \in \mathbf{N}$. Similarly L meets D_2 at open segments N_n, $n \in \mathbf{N}$. Now suppose that x and y are not connected in $K \cap \overline{B(p,\varepsilon)}$.

Components of the compact set $K \cap \overline{B(p,\varepsilon)}$ are compact, and thus distinct components have a positive distance. In virtue of Theorem 1.1.4 we find a Jordan curve γ, which separates the points x and y and does not meet the set $K \cap \overline{B(p,\varepsilon)}$.

By the choices made, $f(M_n) \subset B(p,\varepsilon)$ and $f(N_n) \subset B(p,\varepsilon)$ for all n. Therefore $M_n \cup f(M_n)$ and $N_n \cup f(N_n)$ can be considered as Jordan curves, each of which separates $\hat{\mathbf{C}}$ into two components, one of which lies entirely inside $B(p,\varepsilon)$. Call these components C_n and C'_n, respectively. The choice of the curve γ assures that it must intersect the line segment L, but only at the points of M_n or N_n.

It follows that we must have some M_n, say, such that γ intersects also $f(M_n)$ and, moreover, there is a part of γ, call it α, lying inside C_n and connecting a point of M_n to a point of $f(M_n)$.

Because $C_n \subset B(p,\varepsilon)$, α cannot meet K. This is impossible, for $M_n \subset D_1$ and $f(M_n) \subset D_2$. This contradiction establishes the proof. $\square$

We have still another topological characterization of a Jordan curve, suggested by Olli Martio. Let again D_1 and D_2 be complementary domains in $\hat{\mathbf{C}}$ and K their common boundary. It is well known that K is a Jordan curve if and only if every point of K is accessible from both D_1 and D_2. We get a slight improvement of this theorem.

Theorem 1.2.2. *Let D_1, D_2 and K be as above and denote by A' the set of the points in K that are accessible from both D_1 and D_2. If the set $K \setminus A'$ is compact and totally disconnected, then K is a Jordan curve.*

Proof. We may assume that $K \subset \mathbf{C}$. As in Chapter 1.1 both D_1 and D_2 are simply connected; let $k\colon D_1 \to B$ be a conformal Riemann mapping. Denote, as earlier, by A the set of the points in K that are accessible from D_1. Then $A' \subset A$. Let $\hat{k}\colon A \to \partial B$ be the mapping defined in Chapter 1.1.

Claim (i). *A' is dense in K.*

Proof. Select a point x in $K \setminus A'$ and $\varepsilon > 0$. From assumption and Theorem 1.1.4 it follows that there exists a Jordan curve $\gamma \subset \mathbf{C} \setminus (K \setminus A')$ such that γ circulates x and $\gamma \subset B(x,\varepsilon)$. We may assume that ε is small enough, so that γ contains points of both D_1 and D_2. Then there must be boundary points in γ, and, by the choice of γ, these boundary points must be in $A' \cap B(x,\varepsilon)$.

Claim (ii). *$\hat{k}(A')$ is dense in ∂B.*

Proof. By Lemma 1.1.2 $\hat{k}(A)$ is dense in ∂B. Select now any open arc I of ∂B.

Then we may choose distinct points x_1 and x_2 in A such that $\hat{k}(x_1)$, $\hat{k}(x_2) \in I$. Now connect x_1 and x_2 by a Jordan arc γ in D_1. Then $k(\gamma)$ separates B into two components. Denote by B_1 the component whose boundary consists only of $k(\gamma)$ and points of I, and denote $D_1^1 = k^{-1}(B_1)$.

Then $\gamma \subset \partial D_1^1$, but because $x_1 \neq x_2$, γ cannot be the whole of ∂D_1^1 (a Jordan arc cannot separate the plane). So we have a point $z \in \partial D_1^1 \setminus \gamma$ and then necessarily $z \in K$. z might or might not be in $K \setminus A'$, but in any case we can find a Jordan curve α that circulates z and $\alpha \subset B(x,\varepsilon) \cap \big(\mathbb{C} \setminus (K \setminus A')\big)$ for ε small enough. This follows either from Theorem 1.1.4 or from the fact that A' is open in K.

Because $D_1^1 = k^{-1}(B_1)$ is simply connected, α will contain points of both D_1^1 and $\mathbb{C} \setminus D_1^1$ for small ε. Because $d(z,\gamma) > 0$, we may also suppose that $\alpha \cap \gamma = \emptyset$.

We now select a point in $\alpha \cap D_1^1$ and travel along α until we meet the boundary ∂D_1^1 at a point y. By the choice of α, the point y must belong to A'. The point y is thus accessible from D_1^1, and recalling the definition of $\hat{k}$, $\hat{k}(y) \in \overline{k(D_1^1)} = \overline{B}_1$. Because $\overline{B}_1$ meets ∂B only at the points of I, $\hat{k}(y) \in I$ and so $\hat{k}(A') \cap I \neq \emptyset$. The claim (ii) is therefore established.

Claim (iii). *If $\{x_n\} \subset A'$ and $x_n \to x \in K$, then the sequence $\{\hat{k}(x_n)\}$ converges in ∂B.*

Proof. By the compactness of ∂B there exists a convergent subsequence, and it is enough to show that every convergent subsequence converges to the same limit point. So we assume that $\{y_n\}$, $\{z_n\} \subset A'$, $y_n \to x$, $z_n \to x$, $\hat{k}(y_n) \to y$, $\hat{k}(z_n) \to z$ and claim that $y = z$.

Assume this is not the case.

By (ii) we may select points u and v in A' such that y and z are in different components of $\partial B \setminus \{\hat{k}(u), \hat{k}(v)\}$. Because u and v are in A', we can connect them by Jordan arcs γ_1 and γ_2 in D_1 and D_2, respectively. Then $\gamma_1 \cup \gamma_2$ is a Jordan curve, which divides $\hat{\mathbb{C}}$ into two components C_1 and C_2.

On the other hand, $k(\gamma_1)$ separates B into components B_1 and B_2. We number the sets so that $y \in \overline{B}_1$, $z \in \overline{B}_2$ and $k^{-1}(B_i) \subset C_i$ for $i = 1,2$. For n large enough $\hat{k}(y_n) \in \overline{B}_1 \setminus k(\gamma_1)$ and $\hat{k}(z_n) \in \overline{B}_2 \setminus k(\gamma_1)$, from which it follows, recalling again the definition of $\hat{k}$, that $y_n \in \overline{C}_1$ and $z_n \in \overline{C}_2$.

Thus we see that $x \in \overline{C}_1 \cap \overline{C}_2$, and because $x \in K$, $x = u$ or $x = v$. This, in turn, leads to a contradiction, for the choice of u and v was quite ambiguous: we could change them a little, and get $x = u'$ or $x = v'$ for different u' and v'. This proves (iii).

Now we proceed to extend the mapping $\hat{k}\colon A' \to \partial B$ to the whole K.

Let $x \in K \setminus A'$. By (i) there exists a sequence $\{x_n\}$ in A' converging to x. By (iii) $\{\hat{k}(x_n)\}$ converges in ∂B. Now we define

$$\hat{k}(x) = \lim \hat{k}(x_n).$$

By repeating the proof of (iii) above, we see that the definition is independent of the sequence $\{x_n\}$. Thus we have a well defined mapping $\hat{k}\colon K \to \partial B$.

We will show that $\hat{k}$ is a homeomorphism, what is, of course, sufficient for the proof of the whole theorem.

Claim (iv). *$\hat{k}$ is continuous.*

Proof. Let $x \in K$ and $\{x_n\} \subset K$ such that $x_n \to x$. Select also some positive number ε.

By (i) and the definition of $\hat{k}$ we can, for each n, select a point $y_n \in A'$ such that $|x_n - y_n| < 1/n$ and $|\hat{k}(x_n) - \hat{k}(y_n)| < \varepsilon/2$. Then also $y_n \to x$. If $x \in K \setminus A'$, $|\hat{k}(x) - \hat{k}(y_n)| < \varepsilon/2$ for large n, and it follows that $|\hat{k}(x) - \hat{k}(x_n)| < \varepsilon$ for large n. If $x \subset A'$, we must use (iii) to obtain $|\hat{k}(x) - \hat{k}(y_n)| < \varepsilon/2$ for large n (consider, for example, the sequence $y_1, x, y_2, x, y_3, x, \dots$). So, in all cases, $|\hat{k}(x) - \hat{k}(x_n)| < \varepsilon$ for n large enough, and the continuity of $\hat{k}$ is clear.

Claim (v). *$\hat{k}$ is injective.*

Proof. Select distinct elements x and y in K.

First of all, we may, with the aid of Lemma 1.1.2, suppose that both x and y are not in A' at the same time.

Suppose then x is in $K \setminus A'$ and y is in A'. Because A' is open in K, we can choose a Jordan curve γ in $\mathbf{C}$, which circulates y, does not circulate x and meets K only at the points of A'. Denote by C_x (respectively, C_y) the component of $\hat{\mathbf{C}} \setminus \gamma$ that contains x (respectively, y). By the definition of $\hat{k}$, $\hat{k}(C_x \cap K) \subset \overline{k(C_x \cap D_1)}$ and $\hat{k}(C_y \cap K) \subset \overline{k(C_y \cap D_1)}$. On the other hand, the set $\overline{k(C_x \cap D_1)} \cap \partial B$ meets the set $\overline{k(C_y \cap D_1)} \cap \partial B$ only at the points of $\hat{k}(\gamma \cap K)$. This can be seen as follows.

If w is a common point of those two sets, a sequence $\{z_n\}$ can be found in $\gamma \cap D_1$ such that $k(z_n) \to w$. γ is compact, and so we may assume that $\{z_n\}$ converges to a point of z, which necessarily lies in $\gamma \cap K$. Now it must be so that $\hat{k}(z) = w$. This, in turn, may be proved by a similar idea as we proved the claim (iii) above.

We have now seen that $\hat{k}(C_x \cap K) \cap \hat{k}(C_y \cap K) \subset \hat{k}(\gamma \cap K)$. If $\hat{k}(x) = \hat{k}(y)$, it would then follow that $\hat{k}(y) = \hat{k}(z)$ for some $z \in \gamma \cap K \subset A'$. This is impossible by Lemma 1.1.2.

Finally we suppose that both x and y are in $K \setminus A'$.

As above we see that if $\hat{k}(x) = \hat{k}(y)$, then $\hat{k}(y) = \hat{k}(z)$ for some $z \in \gamma \cap K \subset A'$ (the only difference in the reasoning is that we have to use Theorem 1.1.4 when choosing γ). But this is again impossible because $y \in K \setminus A'$, $z \in A'$ and from what was said above. This finishes the proof of injectivity.

The rest is now easy. The surjectivity follows from (ii), from the compactness of K and from the continuity. Because a continuous bijection of a compact set is a homeomorphism, we have finished the proof. □

Note. There also exists a shorter proof for Theorem 1.2.2. We, however, prefer the above proof, because it provides a new argument for the classical case, i.e. $A' = K$.

2. General facts on Kleinian groups

2.1. Kleinian groups. We consider a subgroup G of $\mathbf{M}$. If there exists an open set $U \subset \hat{\mathbb{C}}$ such that $T(U) \cap U = \emptyset$ for all but finitely many $T \in G$, G is classically called a *Kleinian group.* If a point $x \in \hat{\mathbb{C}}$ has a neighbourhood U of the above type, x is an *ordinary point* of G. All the other points are called *limit points.*

The limit points are characterized by the property that for any limit point x and any ordinary point y there exist a sequence of distinct elements T_n of G such that $\{T_n(y)\}$ converges to x.

The set of the limit points is closed and is the boundary of the set of the ordinary points. The set of the ordinary points has always one, two or infinitely many components. As indicated in the introduction we are interested in the two-componented case. A component D is called *invariant*, if $T(D) = D$ for all $T \in G$.

A fixed point of a non-elliptic element is always a limit point and the set of these fixed points is dense in the set of the limit points. A fixed point of an elliptic element may be an ordinary point, but every elliptic element is of finite order.

The proof of the following lemma can be found, e.g., in Beardon [B, p. 69], and Maskit [M2, p. 19].

Lemma 2.1.1. *Two non-elliptic elements of a Kleinian group commute if and only if they have exactly the same set of the fixed points. The sets of the fixed points of two elements are always exactly the same or completely disjoint.*

It follows that a parabolic element never shares a fixed point with an element of a different type.

Another simple lemma is needed later (for a proof, see [M2, p. 22]).

Lemma 2.1.2. *Let $\{T_n\}$ be a sequence of distinct elements of a Kleinian group G. Then there exist limit points x and y and a subsequence $\{S_n\}$ of $\{T_n\}$ such that $S_n(z) \to y$ uniformly in compact subsets of $\hat{\mathbb{C}} \setminus \{x\}$.*

Note that if T is hyperbolic or loxodromic and $T_n = T^n$ in 2.1.2, then y is the attractive fixed point of T, x is the other fixed point and we can select $S_n = T_n$. If T is parabolic, then $x = y$ is the only fixed point of T.

Note 2.1.3. Let D be an invariant component of the set of the ordinary points of a Kleinian group G. If z is a fixed point of a non-elliptic element

$T \in G$, then $z \in \partial D$, but much more can be said. Namely, z is accessible from D.

This can be seen as follows. Firstly (using T^{-1} if necessary), we may suppose that $T^n(w) \to z$ for all $w \in D$ and this convergence is uniform in compact subsets of D. Then we choose a point $w \in D$. $T(w)$ lies also in D and w can be connected to $T(w)$ by a curve $\alpha \subset D$. Define now

$$\gamma = \bigcup_{n \in \mathbf{N}} T^n(\alpha) \cup \{z\}.$$

By the uniform convergence in the set α one sees immediately that γ can be considered as a curve in D excluding its other endpoint at z.

2.2. Fuchsian groups, Dirichlet fundamental domains. If every element of a Kleinian group preserves the unit disc B, the group is called *Fuchsian.* The limit points of a Fuchsian group lie in the boundary ∂B. If every point of ∂B is a limit point, the group is said to be *of the first kind*, otherwise it is *of the second kind.*

There is a classical way to construct a special fundamental domain of a Fuchsian group G, the so called *Dirichlet polygon.* Proofs of all the following facts can be found, e.g., in [B].

Let ρ be the usual hyperbolic metric in B. Select some $w \in B$, which is not fixed by any elliptic element of G. For each $T \in G$ set $H_T = \{z \in B \mid \rho(z,w) < \rho(z,T(w))\}$ and finally $\Omega = \Omega(w) = \bigcap_{T \in G \setminus \{\mathrm{id}\}} H_T$.

Ω is a hyperbolically convex polygon and a fundamental domain of G, in the sense that $\Omega \cap T(\Omega) = \emptyset$ for all $T \in G \setminus \{\mathrm{id}\}$ and for every $z \in B$ there exists an element $T \in G$ such that $T(z) \in \overline{\Omega}$.

The set $\partial\Omega \cap B$ consists of hyperbolic segments, rays or lines, all of which we call *sides.* If a side is a hyperbolic ray or line, we add to it the endpoint(s) in ∂B, so that a side is always a Jordan arc. If (and only if) G is of the second kind, $\overline{\Omega}$ meets ∂B at whole arcs of ∂B. These are called *free sides.*

For a (non-free) side s there is a uniquely determined side $s' \neq s$ and an element $S \in G$ such that $S(s) = s'$. In this case $S(\overline{\Omega}) \cap \overline{\Omega} = s'$. These elements, the so called *side pairing transformations*, generate G.

For the sake of the convexity, a point $z \in \partial\Omega$ can lie on at most two sides. If a point lies exactly on two sides, we call it a *vertex.* The vertices in ∂B are special ones; we call them *parabolic* vertices.

The definition of $\Omega = \Omega(w)$ depends on the central point w. It can be shown (see, e.g., [B]) that for almost all choices of w, the polygon Ω has the following additional properties:

(i) every parabolic vertex is a fixed point of a parabolic element,
(ii) every fixed point in $\partial\Omega \cap \partial B$ is a parabolic vertex (and thus a parabolic fixed point),
(iii) if z is a parabolic vertex and $T \in G$ such that $T(z) \in \overline{\Omega}$, then $T(z) = z$.

If a Dirichlet polygon Ω has the above three properties, we call it *regular*. Throughout the rest of this paper we assume that a Dirichlet polygon is always regular.

The following lemma can easily be established by the convexity property of Ω. See, for instance, [B, p. 219].

Lemma 2.2.1. *If $\{T_n\}$ is a sequence of distinct elements of G, then the Euclidean diameter of $T_n(\overline{\Omega})$ converges to zero.*

From now on we concentrate on Fuchsian groups of the first kind. Then there are no free sides of Ω.

Consider a point $x \in \partial\Omega \cap \partial B$. As we noted earlier x can lie on at most two sides. If x lies on two sides, it is, as defined, a parabolic vertex. It is, however, possible that x lies on only one side or outside all sides. In these cases Ω must, of course, have infinitely many sides.

We classify the points of $\partial\Omega \cap \partial B$ into three disjoint categories. Let $x \in \partial\Omega \cap \partial B$ and we say that

- $x \in I$, if x lies outside all sides,
- $x \in II$, if x lies on exactly one side,
- $x \in III$, if x is a parabolic vertex.

It is quite clear that $\partial\Omega$ is a Jordan curve and can be oriented in a natural way. So it makes sense to say that $\{x_n\}$ converges to x in $\partial\Omega$ from the right or the left.

If $x \in I$, there must be a sequence $\{s_n\}$ of sides such that $\{s_n\}$ converges to x from the right and also $\{s'_n\}$ such that $\{s'_n\}$ converges to x from the left (the precise meaning of "sides converging to a point" is obvious).

If $x \in II$, there is a sequence of sides converging to x from either side.

If we have $x \in II$, denote by s_x the unique side ending at x. There is another (again unique) side s'_x, different from s_x, and an element S_x of G such that $S_x(s'_x) = s_x$.

These notations remain fixed throughout this chapter.

Now we can naturally orient also the boundary of $\overline{\Omega} \cup S_x(\overline{\Omega})$ and we have

Lemma 2.2.2. *Let x be a point in II and $\{s_n\}$ a sequence of sides converging to x from the right (respectively, left). Then there exists a sequence $\{s'_n\}$ of sides such that $\{S_x(s'_n)\}$ converges to x from the left (respectively, right).*

Proof. If there is a sequence $\{s'_n\}$ of sides converging to $S_x^{-1}(x)$, this sequence will suit.

If such a sequence does not exist, $S_x^{-1}(x)$ must be a parabolic vertex. Because Ω is regular, $S_x^{-1}(x)$ is a parabolic fixed point. Then also x is a fixed point and thus a parabolic vertex. This is impossible, because $x \in II$. $\square$

Construction 2.2.3. For a point x in $I \cup II$ we define certain curves J_n^x, $n \in \mathbb{N}$, as follows:

For $x \in I$ select sequences $\{s_n\}$ and $\{s'_n\}$ of sides such that $\{s_n\}$ (respectively, $\{s'_n\}$) converges from the right (respectively, left) to x. Let S_n and S'_n be the side pairing transformations for which $\overline{\Omega} \cap S_n(\overline{\Omega}) = s_n$ and $\overline{\Omega} \cap S'_n(\overline{\Omega}) = s'_n$. Select an inner point z_n (respectively, z'_n) of s_n (respectively, s'_n) and connect z_n to z'_n by a hyperbolic line segment in $\overline{\Omega}$. Connect z_n (respectively, z'_n) to $S_n(x)$ (respectively, $S'_n(x)$) by a hyperbolic ray in $S_n(\overline{\Omega})$ (respectively, $S'_n(\overline{\Omega})$), and add the endpoints to the rays.

Now let J_n^x be the union of these three arcs. J_n^x is a Jordan arc, which separates B into two components; denote by B_n^x the component for which $x \in \overline{B_n^x}$. (Note that $x \notin J_n^x$ and thus B_n^x is well defined.)

The choices made and Lemma 2.2.1 ensure that $\operatorname{diam} B_n^x \to 0$, when $n \to \infty$.

If $x \in II$, we proceed similarly. The situation is slightly complicated and we have to use Lemma 2.2.2 to find the other sequence needed. The choice of the line segment above must be made in two parts; in $\overline{\Omega}$ and $S_x(\overline{\Omega})$ separately. However, it is obvious that also in this case we can obtain distinct Jordan arcs J_n^x such that $\operatorname{diam} B_n^x \to 0$.

Lemma 2.2.4. *(i) If $x \in I$, then $x \notin T(\overline{\Omega})$ for all $T \in G \setminus \{\mathrm{id}\}$,*
(ii) If $x \in II$, then $x \notin T(\overline{\Omega})$ for all $T \in G \setminus \{\mathrm{id}, S_x^{-1}\}$.

Proof. (i) Suppose $x \in T(\overline{\Omega})$ for some $T \in G \setminus \{\mathrm{id}\}$. Let s_n, s'_n, S_n, S'_n, J_n^x and B_n^x be as in 2.2.3. Because $\overline{\Omega} \cap S_n(\overline{\Omega}) = s_n$, T cannot be any of the side pairing transformations S_n. Similarly $T \neq S'_n$ for all n. Then $T(\Omega) \cap J_n^x = \emptyset$, because the points of Ω are not equivalent with any other point of $\overline{\Omega}$.

Because $x \in T(\overline{\Omega}) = \overline{T(\Omega)}$, we must have $T(\Omega) \subset B_n^x$. This is, however, impossible, because $\operatorname{diam} B_n^x \to 0$.

The proof of the case (ii) is similar. □

2.3. Isomorphisms. Suppose that G_1 and G_2 are Kleinian groups such that each element of them preserves a domain D_1 and D_2, respectively. Suppose also that there is an isomorphism $\varphi\colon G_1 \to G_2$. We say that the isomorphism φ is *geometric*, if there exists a homeomorphism $f\colon D_1 \to D_2$ such that

$$\varphi(T)(z) = f\big(T(f^{-1}(z))\big) \quad \text{for all } T \in G_1 \text{ and } z \in D_2.$$

We say that the homeomorphism f is *induced* by the isomorphism φ.

The following simple fact turns out to be extremely important in the sequel.

Note 2.3.1. Suppose that $f\colon D_1 \to D_2$ is induced by an isomorphism $\varphi\colon G_1 \to G_2$, T is a non-elliptic element with the (attractive) fixed point $z \in \partial D_1$. We saw in Note 2.1.3 that z is accessible from D_1. Let γ be a curve in D_1 (excluding the other endpoint at z) constructed by the method of 2.1.3. Then $f(\gamma)$

is a curve in D_2, the one endpoint of which is the (attractive) fixed point of $\varphi(T)$. This follows directly from the proof of 2.1.3, because now $f(\gamma)$ is constructed by the method in 2.1.3 corresponding to D_2, G_2 and $\varphi(T)$.

We now turn to Fuchsian groups G_1 and G_2, so that $D_1 = D_2 = B$. The problem whether or not a given isomorphism $\varphi\colon G_1 \to G_2$ is geometric is solved in Tukia [T] and Zieschang–Vogt–Coldewey [ZVC], but before we are able to refer to this result we need some definitions.

Let G be a Fuchsian group. If $T \in G$ is hyperbolic, both of its fixed points lie in the boundary ∂B. We call the hyperbolic line in B connecting the fixed points the *axis* of T.

If G_1 and G_2 are Fuchsian groups and $\varphi\colon G_1 \to G_2$ an isomorphism, φ is said to *preserve the relation of being crossed* if the following condition holds: the axes of hyperbolic elements T and $S \in G_1$ intersect if and only if the axes of $\varphi(T)$ and $\varphi(S)$ intersect. (Note that, in general, there is no guarantee that $\varphi(T)$ and $\varphi(S)$ are hyperbolic. If they are not, they must be parabolic. We have to define the axis of a parabolic element to be the singleton that contains the only fixed point, and, after this, the above definition makes sense.)

The following theorem is proved in [T] and, for the compact case, in [ZVC].

Theorem 2.3.2. *Let G_1 and G_2 be Fuchsian groups with infinitely many limit points. Then an isomorphism $\varphi\colon G_1 \to G_2$ is geometric if and only if it preserves the relation of being crossed.*

2.4. Kleinian groups: the two-componented case. We take up the situation that we are mainly interested in. From now on we assume that G is a Kleinian group such that the set of ordinary points consists of exactly two components D_1 and D_2. The set of the limit points, denoted by K, is our object, and it causes no restriction if we assume that D_1 (and thus also D_2) is an invariant component, because otherwise we would select a subgroup $G' = \{T \in G \mid T(D_1) = D_1\}$, which has exactly the same set of the limit points.

It follows that D_1 and D_2 are simply connected complementary domains with a common boundary K. We fix conformal bijections $k\colon D_1 \to B$ and $h\colon D_2 \to B$.

For every $T \in G$ the mapping $k \circ T \circ k^{-1}$ is the restriction of a Möbius transformation into B with $k \circ T \circ k^{-1}(B) = B$ and so the group $k_*(G) = \{k \circ T \circ k^{-1} \mid T \in G\}$ can be considered as a Fuchsian group. Denote by k_* the isomorphism that sends T to the extension of $k \circ T \circ k^{-1}$. We call the group $k_*(G)$ the *Fuchsian model of G induced by k*. Similarly we define $h_*(G)$, the Fuchsian model of G induced by h, and the isomorphism $h_*\colon G \to h_*(G)$.

Lemma 2.4.1. *An element $T \in G$ is parabolic if and only if $k_*(T)$ (respectively, $h_*(T)$) is parabolic.*

Proof. (i) Let $T \in G$ be parabolic. $k_*(T)$ cannot be elliptic, because an elliptic element in a Fuchsian group is of finite order and T is not. If $k_*(T)$ were not parabolic, it would thus be hyperbolic.

Denote, in this case, by α the axis of $k_*(T)$. The curve α can be considered as the union of two curves, which have been constructed by the method indicated in Note 2.1.3, corresponding to the elements $k_*(T)$ and $k_*(T)^{-1} = k_*(T^{-1})$. Now Note 2.3.1 ensures that $k^{-1}(\alpha)$ is the union of two curves, one of which having an endpoint at the fixed point of $k_*^{-1}\big(k_*(T)\big) = T$, and the other at the fixed point of T^{-1}.

T and T^{-1} have, of course, the same fixed point. This leads to a contradiction with the uniqueness part of Lemma 1.1.1.

(ii) Let $k_*(T)$ be parabolic. Let α be a circle inside ∂B tangent to ∂B at the fixed point of $k_*(T)$. This curve α can again be considered the union of two curves constructed as in 2.1.3, using the elements $k_*(T)$ and $k_*(T^{-1})$. As above, $k^{-1}(\alpha)$ is a curve, which connects the (attractive) fixed points of T and T^{-1}. Because of Lemma 1.1.2 these points cannot be distinct from each other. This is possible only when T is parabolic. □

Now we define an isomorphism $\varphi\colon h_*(G) \to k_*(G)$ by setting $\varphi = k_* \circ h_*^{-1}$.

Lemma 2.4.2. *The isomorphism $\varphi\colon h_*(G) \to k_*(G)$ preserves the relation of being crossed.*

Proof. Let T and S be hyperbolic elements in $h_*(G)$ such that their axes intersect. In virtue of Lemma 2.1.1 we may assume that the axes are not identical so that the sets of the fixed points of T and S are disjoint and the elements T and S do not commute.

Then $h_*^{-1}(T)$ and $h_*^{-1}(S)$ (respectively, $\varphi(T)$ and $\varphi(S)$) do not commute either and thus have disjoint sets of fixed points, again by Lemma 2.1.1.

By Lemma 2.4.1 all the elements $h_*^{-1}(T)$, $h_*^{-1}(S)$, $\varphi(T)$ and $\varphi(S)$ must have two distinct fixed points. Denote by α the axis of T and by α' the axis of $\varphi(T)$. The curve $h^{-1}(\alpha)$ connects the fixed points of $h_*^{-1}(T)$ in D_2 and $k^{-1}(\alpha')$ connects the fixed points of $k_*^{-1}\big(\varphi(T)\big) = h_*^{-1}(T)$ in D_1.

Therefore the union $h^{-1}(\alpha) \cup k^{-1}(\alpha')$ forms a Jordan curve, which separates $\hat{\mathbf{C}}$ into two components C_1 and C_2. Because the axes of T and S intersect, the fixed points of S are accessible from different components $h(C_1 \cap D_2)$ and $h(C_2 \cap D_2)$ of $B \setminus \alpha$. Taking into account that $h_*^{-1}(T)$ and $h_*^{-1}(S)$ do not have any common fixed point, we see that the fixed points of $h_*^{-1}(S)$ lie in different components C_i, $i = 1, 2$.

One of these fixed points is then accessible from $C_1 \cap D_1$ and the other from $C_2 \cap D_1$, and therefore the fixed points of $k_* \circ h_*^{-1}(S)$ must be accessible from different components $k(C_i \cap D_1)$, $i = 1, 2$. These components are precisely those of $B \setminus \alpha'$. Thus the axis of $k_* \circ h_*^{-1}(S) = \varphi(S)$ must intersect α', which is the axis of $\varphi(T)$, and we are done. □

Because K must have infinitely many points, there are infinitely many fixed points, and so the groups $h_*(G)$ and $k_*(G)$ must, by Lemma 2.1.1, have infinitely many fixed and thus limit points. The previous lemma tells us now that Theorem 2.3.2 is available. According to it φ is geometric and thus there exists a homeomorphism $f\colon B \to B$ such that

$$f\big(T(f^{-1}(z))\big) = \varphi(T)(z) \quad \text{for all } T \in h_*(G) \text{ and } z \in B.$$

The mapping f is by no means unique, but we fix now one such f.

Lemma 2.4.3. *f is orientation reversing.*

Proof. Select two distinct non-elliptic fixed points x and y in K. They can be connected, by the method of 2.1.3, in D_1 by a Jordan arc γ and similarly in D_2 by a Jordan arc η. Then $\gamma \cup \eta$ is a Jordan curve, which separates $\hat{\mathbb{C}}$ into two components C_1 and C_2. Suppose that C_1 lies left of $\gamma \cup \eta$, when we travel from x to y along γ.

The point x is the (attractive) fixed point of some element $T \in G$. Then, by 2.3.1, the curve $k(\gamma)$ has an endpoint at the (attractive) fixed point of $k_*(T)$. On the other hand, we have in D_2, by the choice of f,

$$(f \circ h)^{-1} \circ \varphi\big(h_*(T)\big) \circ (f \circ h) = T,$$

and 2.3.1 says now that the curve $f \circ h(\eta)$ has an endpoint at the (attractive) fixed point of $\varphi\big(h_*(T)\big)$. But $\varphi\big(h_*(T)\big) = k_*(T)$, and so $k(\gamma)$ and $f \circ h(\eta)$ have a common endpoint, call it x'. Similarly the other endpoint is also common, call it y'. Lemma 1.1.2 implies that $x' \neq y'$.

Now we select a third non-elliptic fixed point z in C_1 and, by 2.1.3, two more curves, one in D_1 and the other in D_2, with a common endpoint at z. As above, k and $f \circ h$ map the curves in such a way that they have a common endpoint in ∂B, distinct from x' and y', call it z'.

Because z lies left of $\gamma \cup \eta$ when we travel from x to y along γ and k is orientation preserving, z' must lie left of $k(\gamma)$ when we travel from x' to y' along $k(\gamma)$. (Note that $k(\gamma)$ is a Jordan arc, which cuts $\overline{B}$ into two distinct pieces and the above makes sense.)

Then it must be so that z' lies right of $f \circ h(\eta)$ when we travel from y' to x'. Because the point z lies left of $\gamma \cup \eta$, when we travel from y to x along η, the mapping $f \circ h$ reverses orientation.

Because h is orientation preserving, f must reverse it. $\square$

It is sometimes convenient to conjugate the group G to be TGT^{-1} with a suitable $T \in \mathbf{M}$. If O is the set of the ordinary points of G, then $T(O)$ is the set of the ordinary points of TGT^{-1} and similarly for the limit points. We call this process a *normalization*.

Consider now any subgroup Γ of G. A subset $A \subset \hat{\mathbb{C}}$ is said to be *precisely invariant* under Γ if

(i) $T(A) = A$ for all $T \in \Gamma$, and
(ii) $T(A) \cap A = \emptyset$ for all $T \in G \setminus \Gamma$.

The following lemma is probably known, but in a lack of a reference explicit enough, we prove it here.

Lemma 2.4.4. *(i) Let x in K be the fixed point of a parabolic element P of G. Then there exist open discs $H_i \subset D_i$, $i = 1,2$, such that $\overline{H_i} \cap K = \{x\}$ and H_i is precisely invariant under the subgroup $\{T \in G \mid T(x) = x\}$.*

(ii) If $r\colon D_1 \to D_2$ is a homeomorphism such that $T(z) = r^{-1}\big(T(r(z))\big)$ for all $T \in G$ and $z \in D_1$, then the discs above may be chosen so that $r(H_1) \subset H_2$.

Proof. (i) The result (i) for Fuchsian groups is proven explicitly in [M2, p. 117]. We may normalize so that $P(z) = z + 1$. Lemma 2.4.1 says that $h_*(P)$ is parabolic, denote its fixed point in ∂B by y. By the above result of [M2] we find an open disc $H \subset B$ such that $\overline{H} \cap \partial B = \{y\}$ and H is precisely invariant under the subgroup $\{T \in h_*(G) \mid T(y) = y\}$.

As in the proof of 2.4.1, ∂H can be considered as a union of two curves constructed as in 2.1.3 and it follows that $h^{-1}(\partial H) \cup \{\infty\} = \big(\cup_{n \in \mathbf{Z}} P^n(\gamma)\big) \cup \{\infty\}$ can be considered as a Jordan curve α. Here γ denotes a Jordan arc in D_2, $\gamma = h^{-1}(\eta)$, where η is a suitable part of ∂H. The Jordan curve α separates $\hat{\mathbf{C}}$ into two components, one of which lies entirely in D_2. Denote that component by H'.

Now $\gamma \subset \mathbf{C}$ is bounded and there exist numbers M_1 and M_2 such that $M_1 < \operatorname{Im}(z) < M_2$ for all $z \in \gamma$. It follows that $M_1 < \operatorname{Im}(z) < M_2$ for all $z \in \partial H' \setminus \{\infty\} = \cup_{n \in \mathbf{Z}} P^n(\gamma) = \cup_{n \in \mathbf{Z}} (\gamma + n)$.

This, in turn, means that either the set $\{a + i\,b \mid a \in \mathbf{R}, b > M_2\}$ or the set $\{a + i\,b \mid a \in \mathbf{R}, b < M_1\}$ is inside the set $H' \subset D_2$. Suppose, e.g., the former and denote $H_2 = \{a + i\,b \mid a \in \mathbf{R}, b > M_2\}$. It follows from the properties of H and Lemma 2.1.1 that this choice of H_2 is suitable, i.e. H_2 is precisely invariant under the subgroup $\{T \in G \mid T(\infty) = \infty\}$.

(ii) The assumption on r implies that $r^{-1}(z+n) = r^{-1}(z) + n$ for all $z \in D_2$ and $n \in \mathbf{Z}$. Therefore $\{\operatorname{Im}\big(r^{-1}(z)\big) \mid z \in \partial H_2 \cap \mathbf{C}\} = \{\operatorname{Im}\big(r^{-1}(z)\big) \mid z = x + i\,M_2, x \in [0,1]\}$. The continuity of r^{-1} implies that the set mentioned must have a lower bound $K \in \mathbf{R}$.

Now we set $H_1 = \{a + i\,b \mid a \in \mathbf{R}, b < K\}$. It is clear that $H_1 \subset D_1$. The fact that $r(H_1) \subset H_2$ follows from the assumption on r and the fact that H_1 is precisely invariant under $\{T \in G \mid T(\infty) = \infty\}$ follows from the corresponding property of H_2. $\square$

3. The main theorem

3.1. Formulation. We are now in a position to state the main theorem. Recall the definition of a cluster set (Definition 1.1.3), of a regular Dirichlet polygon (Chapter 2.2), of a Fuchsian model (Chapter 2.4) and of a homeomorphism induced by an isomorphism (Chapter 2.3). The existence of an induced homeomorphism was discussed in Chapter 2.4.

Theorem 3.1.1. *Let G be a Kleinian group such that the set of the ordinary points of G has exactly two components D_1 and D_2. We denote by K the set of the limit points of G, select conformal mappings k and h from D_1 and D_2, respectively, onto B and denote by $k_*(G)$ and $h_*(G)$ the Fuchsian models of G induced by k and h, respectively. We choose a regular Dirichlet polygon Ω of the Fuchsian group $k_*(G)$ and let $k_*\colon G \to k_*(G)$ and $h_*\colon G \to h_*(G)$ be the natural isomorphisms and $f\colon B \to B$ a homeomorphism induced by the isomorphism $k_* \circ h_*^{-1}\colon h_*(G) \to k_*(G)$.*

The set K is a Jordan curve if and only if $k_*(G)$ is of the first kind and for each $x \in \partial\Omega \cap \partial B$, which is not a parabolic vertex, the following condition is fulfilled: the cluster set $C(x, k^{-1}, D_1)$ consists of a single point, say y, and if $\{T_n\}$ is a sequence of distinct elements of G converging to a point uniformly in compact subsets of $\hat{\mathbf{C}} \setminus \{y\}$, then the spherical diameters of the sets $T_n\big(k^{-1}(\Omega)\big)$ and $T_n\big(h^{-1} \circ f^{-1}(\Omega)\big)$ converge to zero.

Note 3.1.2. If G is finitely generated, $k_*(G)$ must be of the first kind in this situation, and the set $\partial\Omega \cap \partial B$ is empty or consists only of parabolic vertices and the condition automatically holds.

Note 3.1.3. The assumption that $k_*(G)$ is of the first kind can be replaced by the assumption that the set $\overline{k^{-1}(\Omega)} \cap K$ is totally disconnected. It is also possible that this assumption is unnecessary. On the other side, if $k_*(G)$ is of the first kind we can prove that the mapping k has a natural continuous extension $\hat{k}\colon \overline{D_1} \to \overline{B}$. There are, however, situations (see e.g. the example in Figure 2) when there cannot exist such an extension and thus $k_*(G)$ is not of the first kind.

Note 3.1.4. With connection to the last condition in the theorem, note that in the case of a Fuchsian group, the spherical diameters of $T_n(\Omega)$ converge to zero for all sequences of distinct elements T_n.

3.2. Proof. If K is a Jordan curve, the conformal mappings $k\colon D_1 \to B$ and $h\colon D_2 \to B$ can be extended to homeomorphisms $\hat{k}\colon \overline{D_1} \to \overline{B}$ and $\hat{h}\colon \overline{D_2} \to \overline{B}$. Limit points are mapped to limit points and thus every point of ∂B is a limit point of both $k_*(G)$ and $h_*(G)$ and the groups $k_*(G)$ and $h_*(G)$ are of the first kind.

Then it follows from [T, Corollary 3.5.1] that the homeomorphism $f\colon B \to B$ can be extended to a homeomorphism $\hat{f}\colon \overline{B} \to \overline{B}$. Both of the mappings $k^{-1}\colon B \to D_1$ and $h^{-1} \circ f^{-1}\colon B \to D_2$ are thus uniformly continuous and the last condition of the theorem follows from the facts that $\operatorname{diam} k_*(T_n)(\Omega)$ converges to zero (cf. Lemma 2.2.1) and that $T\big(k^{-1}(\Omega)\big) = k^{-1}\big(k_*(T)(\Omega)\big)$ and $T\big(h^{-1} \circ f^{-1}(\Omega)\big) = h^{-1} \circ f^{-1}\big(k_*(T)(\Omega)\big)$ for all $T \in G$. The condition on the cluster set is valid for all $x \in \partial B$.

The proof of the sufficiency of the conditions is divided into several lemmas.

We use the notations and results from Chapter 2.4. So, denote by φ the isomorphism $k_* \circ h_*^{-1}: h_*(G) \to k_*(G)$ as indicated there.

First of all we concentrate on the parabolic vertices of Ω, the cluster sets of which also turn out to be singletons.

Lemma 3.2.1. *Let $z \in \partial B$ be the fixed point of a parabolic element $P \in k_*(G)$ and $\{z_n\} \subset B$ a sequence converging to z. Then*

$$\lim k^{-1}(z_n) = \lim h^{-1} \circ f^{-1}(z_n) = x,$$

where $x \in K$ is the fixed point of $k_*^{-1}(P) \in G$.

(Notethat any direct argumentation based on 2.3.1 is not available, because the convergence may be too complicated.)

Proof. We prove the result for k^{-1}. For $h^{-1} \circ f^{-1}$ the proof is similar (conformality is not needed). Now suppose the claim is not true. Then we get a point $y \in K$, y different from x, and a subsequence, denoted also by $\{z_n\}$, such that $\{k^{-1}(z_n)\}$converges to y.

By Lemmas 2.4.1 and 2.4.4 we can find an open disc $H \subset D_1$ such that $\overline{H} \cap K = \{x\}$ and H is precisely invariant under the subgroup $\{T \in G \mid T(x)= x\}$. We introduce in H the hyperbolic geometry and select a hyperbolic line ℓ, which connects x to some other point $u \in \partial H \cap D_1$.

With the aid of Note 2.3.1 we can see that H is mapped to a domain $k(H)$ such that $\overline{k(H)} \cap \partial B = \{z\}$. From this it follows that the other endpoint of $k(\ell)$ is z. So $k(\ell)$is a Jordan arc, which connects z to $k(u) \in B$. It is possible to connect $k(u)$ to a non-elliptic fixed point $v \in \partial B \setminus \{z\}$ with a Jordan arc α in $B \setminus k(H)$, constructed as in 2.1.3, in such a way that $\alpha \cap P(\alpha) = \emptyset$. We denote by γ the Jordan arc $k(\ell) \cup \alpha$ which connects the points z and v and is such that $\gamma \cap P(\gamma) = \{z\}$.

Denote now by E the domain in B that is bounded by the curves γ, $P(\gamma)$ and a part of ∂B. It is quite easy to see that E is a fundamental domain of the cyclic group $\{P^n \mid n \in \mathbf{Z}\}$. (Onemust handle the points inside and outside $k(H)$ separately.) It follows that $k^{-1}(E)$ is a fundamental domain of the subgroup $\{k_*^{-1}(P^n) \mid n \in \mathbf{Z}\}$.

Now we go back to the beginning and remember that $k^{-1}(z_n) \to y$ and $y \neq x$. Because $k^{-1}(E)$ is a fundamental domain, we can find for every n a number $q(n) \in \mathbf{Z}$ such that

$$k_*^{-1}(P^{q(n)})\big(k^{-1}(z_n)\big) \in \overline{k^{-1}(E)} \quad \text{for all } n.$$

There are two possibilities: the sequence $\{q(n)\} \subset \mathbf{Z}$ is either bounded or unbounded.

First suppose that it is bounded. Selecting a subsequence we may suppose that $q(n)$ is a constant $q \in \mathbf{Z}$. So $k_*^{-1}(P^q)\big(k^{-1}(z_n)\big) \in \overline{k^{-1}(E)}$ for all n. The set $k^{-1}(E) \cap H$ is a domain bounded by two hyperbolic lines ℓ and $k_*^{-1}(P)(\ell)$ and a part of ∂H. Now if $k_*^{-1}(P^q)\big(k^{-1}(z_n)\big) \in \overline{k^{-1}(E) \setminus H}$, then $P^q(z_n) \in \overline{E} \setminus k(H)$. This cannot be the case for infinitely many n, for $z_n \to z$ and thus $P^q(z_n) \to P^q(z) = z$. Thus we may assume that $k_*^{-1}(P^q)\big(k^{-1}(z_n)\big) \in \overline{k^{-1}(E)} \cap H$ for all n. Because of $k^{-1}(z_n) \to y$, $k_*^{-1}(P^q)\big(k^{-1}(z_n)\big) \to k_*^{-1}(P^q)(y) \in K$. The set $\overline{k^{-1}(E)} \cap H$ meets the boundary only at the point x; so $k_*^{-1}(P^q)(y) = x$ is the only possibility. Then $y = k_*^{-1}(P^{-q})(x) = x$, which contradicts our assumption.

So we may concentrate on the case that the set $\{q(n)\} \subset \mathbf{Z}$ is unbounded. By a suitable choice of a subsequence again we may assume that the mapping $n \mapsto q(n)$ is strictly increasing (the strictly decreasing case can be handled similarly). If $k_*^{-1}(P^{q(n)})\big(k^{-1}(z_n)\big) = \big(k_*^{-1}(P)\big)^{q(n)}\big(k^{-1}(z_n)\big)$ belongs to the set $\overline{k^{-1}(E)} \cap H$ for infinitely many n, we find a subsequence of $\{k^{-1}(z_n)\}$ converging to x (from the simple geometry of the set $\overline{k^{-1}(E) \cap H}$ it is easy to decide that the sequence $\{\big(k_*^{-1}(P)\big)^{-q(n)}\}$ converges uniformly to x in $\overline{k^{-1}(E)} \cap H$). This is, however, impossible according to our assumption $y \neq x$.

So we again take a subsequence to obtain the situation that

$$\big(k_*^{-1}(P)\big)^{q(n)}\big(k^{-1}(z_n)\big) \in \overline{k^{-1}(E) \setminus H}$$

for all n. Now the construction of α and Note 2.3.1 imply that $x \notin \overline{k^{-1}(E) \setminus H}$, and the sequence $\{k_*^{-1}(P)\}^{-q(n)}$ converges uniformly to x in the compact set $\overline{k^{-1}(E) \setminus H}$ (cf. the note after Lemma 2.1.2). It follows again that $k^{-1}(z_n) \to x$ and this contradiction accomplishes the proof. $\square$

From the definition of the cluster set and the condition of the theorem it is clear that if $\{x_n\} \subset B$ converges to a point $x \in \partial\Omega \cap \partial B$, which is not a parabolic vertex, then $\{k^{-1}(x_n)\}$ converges to the singleton $C(x, k^{-1}, D_1)$. By Lemma 3.2.1 a similar result holds for the parabolic vertices, too.

Thus we can extend the mapping $k^{-1}\colon \Omega \to k^{-1}(\Omega)$ to a mapping $\hat{g}\colon \overline{\Omega} \to \overline{k^{-1}(\Omega)}$ in a natural way. It is easy to see that $\hat{g}$ is continuous. It is also clear that $\hat{g}$ is surjective and the injectiviness follows from the uniqueness part of Lemma 1.1.1. Therefore $\hat{g}$ is a continuous bijection of a compact set and thus a homeomorphism.

We also want to extend the mapping $h^{-1} \circ f^{-1}\colon \Omega \to h^{-1} \circ f^{-1}(\Omega)$ to the closure $\overline{\Omega}$. For this we need a lemma.

Lemma 3.2.2. *Let $x \in \partial B$ be a point whose cluster set $C(x, k^{-1}, D_1)$ is a singleton $\{u\} \subset K$ and $\{x_n\} \subset B$ a sequence converging to x. Then the sequence $\{h^{-1} \circ f^{-1}(x_n)\}$ converges to the point u.*

Proof. Assume that the claim is not true. Then we can find a subsequence, denoted also by $\{x_n\}$, such that $\{h^{-1} \circ f^{-1}(x_n)\}$ converges to a point $y \in K$,

y distinct from u. We select a sequence $\{y_n\} \subset D_1$ converging to y and, taking a subsequence if necessary, we may suppose that the sequence $\{k(y_n)\}$ converges to a point $z \in \partial B$. (In fact, it can be proved that the whole sequence $\{k(y_n)\}$ converges). From the definition of the cluster set it follows that z must be distinct from x.

Now we use the fact that $k_*(G)$ is of the first kind. Then the set of the non-elliptic fixed points of $k_*(G)$ is dense in ∂B, and we can find fixed points v and w of $k_*(G)$ such that the points x and z lie in different components of $\partial B \setminus \{v, w\}$. We connect v and w by a Jordan arc γ using the method of 2.1.3. It follows, in the same way as in the proof of 2.4.3, that the Jordan arcs $k^{-1}(\gamma)$ and $h^{-1} \circ f^{-1}(\gamma)$ have common distinct endpoints v' and w'. It also follows that v' and w' must be distinct from y and u. The condition $k(y_n) \to z$ and the assumption $C(x, k^{-1}, D_1) = \{u\}$ imply that the Jordan curve $\eta = k^{-1}(\gamma) \cup h^{-1} \circ f^{-1}(\gamma)$ separates the points y and u.

The Jordan arc γ cuts $\overline{B}$ into two pieces, one containing x and the other containing z. Suppose that x lies left of γ when we travel from v to w along γ. Because k preserves orientation, u must lie left of the Jordan curve η, when we travel from v' to w' along $k^{-1}(\gamma)$. This means that y lies left of η, when we travel from v' to w' along $h^{-1} \circ f^{-1}(\gamma)$. Because $h^{-1} \circ f^{-1}(x_n) \to y$, $x_n \to x$ and $f \circ h$ reverses the orientation (Lemma 2.4.3), x must lie right of γ, when we travel from v to w. This is a contradiction, and the proof is established. □

With the aid of Lemmas 3.2.1 and 3.2.2 we can now naturally extend the mapping $h^{-1} \circ f^{-1} \colon \Omega \to h^{-1} \circ f^{-1}(\Omega)$ to a mapping $\hat{g}' \colon \overline{\Omega} \to \overline{h^{-1} \circ f^{-1}(\Omega)}$.

Directly from 3.2.1 and 3.2.2 we get

Remark 3.2.3. If $x \in \partial\Omega \cap \partial B$, then $\hat{g}(x) = \hat{g}'(x)$.

Like the mapping $\hat{g}$, $\hat{g}'$ is also a continuous surjection. The injectivity of $\hat{g}'$ cannot be established directly from Lemma 1.1.1, because of the lack of conformality, but it follows from 3.2.3 and the injectivity of $\hat{g}$. Thus also $\hat{g}'$ is a homeomorphism.

Next we define the sets

$$U = \bigcup_{T \in k_*(G)} T(\overline{\Omega}), \quad V = \bigcup_{T \in G} T(\hat{g}(\overline{\Omega})) \quad \text{and} \quad V' = \bigcup_{T \in G} T(\hat{g}'(\overline{\Omega})).$$

Note that $B \subset U \subsetneq \overline{B}$, $D_1 \subset V \subsetneq \overline{D_1}$ and $D_2 \subset V' \subsetneq \overline{D_2}$.

We define a mapping $g \colon U \to V$ as follows: If $x \in U$, select an element $T \in k_*(G)$ such that $T(x) \in \overline{\Omega}$ and set

$$g(x) = k_*^{-1}(T^{-1}) \circ \hat{g} \circ T(x).$$

Similarly we define $g' \colon U \to V'$ by setting

$$g'(x) = k_*^{-1}(T^{-1}) \circ \hat{g}' \circ T(x).$$

We must check that these mappings are well defined. This is done for g, the situation is similar for g'. First of all, we see immediately that in B, $g(x) = k^{-1}(x)$ and we may concentrate on the boundary.

We recall the classification of the points of $\partial\Omega \cap \partial B$ from Chapter 2.2. The examination is now divided into three parts according to $T(x) \in \partial\Omega \cap \partial B$ being in class I, II or III.

If $T(x) \in I$, it follows from Lemma 2.2.4 that the choice of T is the only possibility and the uniqueness of $g(x)$ is clear.

If $T(x) \in II$, we proceed as follows. There is a side s_0 of Ω having an endpoint at $T(x)$, call S_0 the corresponding side pairing operation in $k_*(G)$. Now if $S(x) \in \overline{\Omega}$ for some S in $k_*(G)$, then $S \circ T^{-1}\big(T(x)\big) \in \overline{\Omega}$ and Lemma 2.2.4 says that either $S \circ T^{-1} = \mathrm{id}$ or $S \circ T^{-1} = S_0^{-1}$. The former case is clear, so we consider the latter. Select a point w in $T^{-1}(s_0) \cap B$. Then $S^{-1} \circ \hat{g}^{-1} \circ k_*^{-1}(S_0^{-1}) \circ \hat{g} \circ T(w)$ makes sense and equals w. Because x is the endpoint of the arc $T^{-1}(s_0)$ and everything is continuous, we see that $S^{-1} \circ \hat{g}^{-1} \circ k_*^{-1}(S) \circ k_*^{-1}(T^{-1}) \circ \hat{g} \circ T(x) = x$, from which the desired uniqueness follows.

Lastly, let $T(x)$ be in class III, i.e. a parabolic fixed point. Let $S(x) \in \overline{\Omega}$ for some $S \in k_*(G)$. Because the Dirichlet polygon Ω is regular, it follows that $T(x)$ is the fixed point of $S \circ T^{-1}$. Then, by Lemma 3.2.1, $\hat{g}\big(T(x)\big)$ is the fixed point of $k_*^{-1}(S \circ T^{-1})$ and thus $S^{-1} \circ \hat{g}^{-1} \circ k_*^{-1}(S) \circ k_*^{-1}(T^{-1}) \circ \hat{g} \circ T(x) = x$ and we get the uniqueness again.

Remark 3.2.4. Remark 3.2.3 and the definitions of g and g' imply immediately that

$$g(x) = g'(x) \quad \text{for all } x \in U \cap \partial B.$$

Lemma 3.2.5. *The mappings g and g' are continuous.*

Proof. We may concentrate on the boundary $U \cap \partial B$. So let x be in $U \cap \partial B$ and $T \in k_*(G)$ such that $T(x) \in \partial\Omega \cap \partial B$.

If $T(x)$ is not a parabolic vertex, the assumption of the main theorem says that the cluster set $C\big(T(x), k^{-1}, D_1\big)$ is just $\{\hat{g}\big(T(x)\big)\}$. From the definition of the cluster set we see that $C(x, k^{-1}, D_1) = \{k_*^{-1}(T^{-1}) \circ \hat{g} \circ T(x)\} = \{g(x)\}$ and further that if $\{x_n\} \subset B$ and $x_n \to x$, then $g(x_n) = k^{-1}(x_n) \to g(x)$. Lemma 3.2.2 and Remark 3.2.4 also imply that $g'(x_n) \to g'(x)$.

If $T(x)$ is a parabolic vertex, it is the fixed point of some parabolic element $P \in k_*(G)$. Then x is a fixed point of $T^{-1}PT$ and $T^{-1}PT$ is also parabolic. Lemma 3.2.1 implies that $\hat{g}\big(T(x)\big) = \hat{g}'\big(T(x)\big)$ is the fixed point of $k_*^{-1}(P)$. Then $g(x) = g'(x)$ is the fixed point of $k_*^{-1}(T^{-1}PT)$. Now, if $\{x_n\} \subset B$ and $x_n \to x$, it follows from Lemma 3.2.1 that $g(x_n) = k^{-1}(x_n) \to g(x)$ and similarly $g'(x_n) = h^{-1} \circ f^{-1}(x_n) \to g'(x)$.

We have shown that if $\{x_n\} \subset B$ and $x_n \to x$, then, in every case, $g(x_n) \to g(x)$ and $g'(x_n) \to g'(x)$. The rest of the proof is an easy exercise. $\square$

Lemma 3.2.6. *The mappings $g\colon U \to V$ and $g'\colon U \to V'$ are bijective.*

Proof. The surjectivity is clear. As to the injectivity we first notice that all the points of $\hat{g}(\overline{\Omega})$ are accessible from D_1 and thus all the points of $V = \cup_{T\in G} T\big(\hat{g}(\overline{\Omega})\big)$ are accessible from D_1. After this, the injectivity of g follows from the uniqueness part of Lemma 1.1.1, and then Remark 3.2.4 implies that g' must also be injective. □

Lemma 3.2.6 says that there cxist inverse mappings $g^{-1}\colon V \to U$ and $g'^{-1}\colon V' \to U$, and we have

Lemma 3.2.7. *The mappings g^{-1} and g'^{-1} are continuous.*

Proof. We prove the assertion for g^{-1}; for g'^{-1} the proof is similar. Because $g^{-1} = k$ in D_1, we may concentrate on the set $V \cap K$. Suppose the claim is not true. Then we have a point $x \in V \cap K$ and sequences $\{x_n\}$ and $\{y_n\}$ in V such that $x_n \to x$, $y_n \to x$, but $g^{-1}(x_n) \to z$ and $g^{-1}(y_n) \to w$, where z and w are distinct points in ∂B.

Because $k_*(G)$ is of the first kind, we can select some non-elliptic fixed points u and v of $k_*(G)$ such that z and w lie in different components of $\partial B \setminus \{u, v\}$. Connect u and v by a Jordan curve γ constructed as in 2.1.3. Then $k^{-1}(\gamma)$ connects the corresponding fixed points u' and v' of G in D_1. By a suitable choice of u and v we may assure that neither u' nor v' is the point x. We connect the fixed points u' and v' also in D_2 by, e.g., $h^{-1} \circ f^{-1}(\gamma)$. Then $\alpha = k^{-1}(\gamma) \cup h^{-1} \circ f^{-1}(\gamma)$ is a Jordan curve, and x lies in one component, say C, of $\hat{\mathbf{C}} \setminus \alpha$.

Because $x_n \to x$ and $y_n \to x$, x_n and y_n also lie in C for large n. Therefore $g^{-1}(x_n)$ and $g^{-1}(y_n)$ lie in the same component of $\overline{B} \setminus \gamma$ for large n. This is, however, impossible because of the choice of u, v and γ. □

Now we define a mapping $r\colon \overline{D_1} \to \overline{D_2}$ by setting

$$r(x) = \begin{cases} g' \circ g^{-1}(x), & \text{when } x \in V \\ x, & \text{when } x \in K. \end{cases}$$

By Remark 3.2.4, r is well defined. We want to show that r is a homeomorphism. The bijectivity follows from 3.2.6, and it is enough to see that r is continuous in the compact set $\overline{D_1}$. Because $r = h^{-1} \circ f^{-1} \circ k$ in D_1, we may restrict ourselves to the boundary K. To simplify the notation we first formulate a simple lemma.

Lemma 3.2.8. *For all $x \in D_1$ and $T \in G$, $r\big(T(x)\big) = T\big(r(x)\big)$.*

Proof. Directly from the definitions we get

$$\begin{aligned} r^{-1} \circ T \circ r(x) &= g \circ g'^{-1} \circ T \circ g' \circ g^{-1}(x) \\ &= k^{-1} \circ f \circ h \circ T \circ h^{-1} \circ f^{-1} \circ k(x) \\ &= k^{-1} \circ \varphi\big(h_*(T)\big) \circ k(x) \\ &= k^{-1} \circ k \circ h^{-1} \circ h \circ T \circ h^{-1} \circ h \circ k^{-1} \circ k(x) \\ &= T(x). \quad \square \end{aligned}$$

Now we proceed to show that r is continuous at a boundary point $z \in K$. Because of 3.2.5 and 3.2.7 and the simple behaviour of r along the boundary, we may assume that $z \in K \setminus V$. Let $\{z_n\}$ be a sequence in $\overline{D_1}$ converging to z. We must show that $\{r(z_n)\}$ converges to z. We suppose now the contrary. The idea of the proof is to repeatedly extract a subsequence, so that we reach a desired contradiction.

Firstly, we may suppose that $z_n \in D_1$ for all n. Secondly, we may suppose that $r(z_n)$ converges to some point $y \in \overline{D_2}$, distinct from z. Then we select for all n an element $T_n \in G$ such that $T_n(z_n) \in g(\overline{\Omega})$. This is possible, because $z_n \in D_1 \subset V$. We may suppose that all the elements T_n are distinct. Because, if this is not the case, there is a constant n_0 such that $z_n \in T_{n_0}^{-1}\big(g(\overline{\Omega})\big)$ for infinitely many n, and thus $z \in V$, what is against the choice of z. Also we may suppose that the sequence $\{T_n(z_n)\}$ converges to some point $x \in g(\overline{\Omega})$. Now we apply Lemma 2.1.2 and, taking again a subsequence, we can assume that there are points a and b in K such that $T_n^{-1}(w) \to b$ uniformly on compact subsets of $\hat{\mathbb{C}} \setminus \{a\}$.

Now we consider the location of the point $x \in g(\overline{\Omega})$. First of all, x cannot belong to D_1. That can be seen as follows: If $x \in D_1$, it is not a limit point and cannot equal the point a above. Then $z_n = T_n^{-1}\big(T_n(z_n)\big) \to b$, so that $z = b$. Moreover $r(x) \in D_2$, and $r(x) \neq a$ as well, and now according to Lemma 3.2.8 and to the fact that $r\big(T_n(z_n)\big) \to r(x)$ we see that $r(z_n) = T_n^{-1}\big(r(T_n(z_n))\big) \to b$, and so $y = b$, too. This is against our explicit assumption $y \neq z$.

So it must be the case that $x \in g(\overline{\Omega}) \cap K$ and thus $g^{-1}(x) \in \partial\Omega \cap \partial B$.

We consider first the case that $g^{-1}(x)$ is a parabolic vertex. Then $g^{-1}(x)$ and x are fixed points of some parabolic elements of $k_*(G)$ and G, respectively. We select open discs $H_i \subset D_i$, $i = 1, 2$, as indicated in Lemma 2.4.4. Moreover, we select an open disc H_3 in B such that $\overline{H_3} \cap \partial B = \{g^{-1}(x)\}$, $g(H_3) \subset H_1$ and H_3 is precisely invariant under the subgroup $\{T \in k_*(G) \mid T\big(g^{-1}(x)\big) = g^{-1}(x)\}$. The existence of such a disc can be proved in the same way as Lemma 2.4.4.

Because $T_n(z_n) \to x$, the sequence $\{g^{-1}\big(T_n(z_n)\big)\}$ converges to $g^{-1}(x)$, and because of the convexity of Ω, $g^{-1}\big(T_n(z_n)\big)$ belongs to the disc H_3 for large n. This implies the facts that $T_n(z_n) \in H_1$ and $r\big(T_n(z_n)\big) \in H_2$ for large n. We assume that this happens for all n.

Now we may suppose that $T_m \circ T_n^{-1}(H_1) \cap H_1 = \emptyset$, when $m \neq n$. This can be seen by the following reasoning. If this is not the case, there exists some n_0 such that $T_m \circ T_{n_0}^{-1}(H_1) = H_1$ for infinitely many m. Then $T_m^{-1}(H_1) = T_{n_0}^{-1}(H_1)$ and because $T_{n_0}^{-1}(H_1)$ meets the boundary only at the point $T_{n_0}^{-1}(x)$, it must be so that $z = \lim z_m = \lim T_m^{-1}\big(T_m(z_m)\big) = T_{n_0}^{-1}(x) \in V$, a contradiction.

So we have $T_m \circ T_n^{-1}(H_1) \cap H_1 = \emptyset$ when $m \neq n$, and it follows that also $T_m \circ T_n^{-1}(H_2) \cap H_2 = \emptyset$ when $m \neq n$.

Then $T_m^{-1}(H) \cap T_n^{-1}(H_1) = \emptyset$ when $m \neq n$, and because $T_n^{-1}(H_1)$ is always a disc, it has to follow that the spherical diameter $\operatorname{diam} T_n^{-1}(H_1)$ converges to zero. Similarly $\operatorname{diam} T_n^{-1}(H_2) \to 0$.

Now $T_n(z_n) \in H_1$, $x \in \partial H_1$ and $T_n^{-1}\big(T_n(z_n)\big) = z_n \to z$, and so it must happen that $T_n^{-1}(x) \to z$. This, the fact that $r\big(T_n(z_n)\big) \in H_2$ and the diminishing of $\operatorname{diam} T_n^{-1}(H_2)$ imply that $r(z_n) = T_n^{-1}\big(r(T_n(z_n))\big) \to z$, which is a contradiction.

So we have seen that the point $g^{-1}(x)$ in $\partial\Omega \cap \partial B$ cannot be a parabolic vertex, and the last assumption of the theorem is available. Remember that we have points a and b in K such that $T_n^{-1}(w) \to b$ uniformly in compact subsets of $\hat{\mathbf{C}} \setminus \{a\}$. If it happens that $x = a$, the assumption says that $\operatorname{diam} T_n^{-1}\big(g(\overline{\Omega})\big)$ and $\operatorname{diam} T_n^{-1}\big(r(g(\overline{\Omega}))\big)$ converge to zero and it must be so that $z = \lim z_n = b = \lim r(z_n) = y$, which is impossible. So, x cannot be equal to a. Because $T_n(z_n) \to x$ and $r\big(T_n(z_n)\big) \to x$, there is some compact set in $\hat{\mathbf{C}} \setminus \{a\}$ containing $T_n(z_n)$ and $r\big(T_n(z_n)\big)$ for large n. The uniform convergence of the sequence $\{T_n\}$ then implies again that $z = b = y$, a contradiction.

All these contradictions establish the proof of the continuity of r.

It follows that r is a homeomorphism. Theorem 1.2.1 then implies the main theorem and we are done.

Note 3.2.9. We can also modify the conditions of Theorem 3.1.1 and restate it as follows (notations as in 3.1.1).

The limit point set K is a Jordan curve if and only if the following three conditions are fulfilled:

(i) $h_*(G)$ is of the first kind,

(ii) the sets $\overline{k^{-1}(\Omega)} \cap K$ and $\overline{h^{-1} \circ f^{-1}(\Omega)} \cap K$ are equal and totally disconnected and

(iii) if $x \in \partial\Omega \cap \partial B$ is not a parabolic vertex, y belongs to the cluster set $C(x, k^{-1}, D_1)$ and $\{T_n\} \subset G$ is a sequence converging to a point uniformly in compact subsets of $\hat{\mathbf{C}} \setminus \{y\}$, then the spherical diameters of the sets $T_n\big(k^{-1}(\Omega)\big)$ and $T_n\big(h^{-1} \circ f^{-1}(\Omega)\big)$ converge to zero.

The proof of the necessity of the conditions is quite immediate as in 3.1.1. As to the sufficiency we first notice that because $\overline{k^{-1}(\Omega)} \cap K$ is totally disconnected, $k_*(G)$ must be of the first kind, and moreover, using Theorem 1.2.2, we see that $k^{-1}(\Omega)$ must be a Jordan domain. Thus we can naturally extend the mapping

$k^{-1}: \Omega \to k^{-1}(\Omega)$ to a mapping $\hat{g}: \overline{\Omega} \to \overline{k^{-1}(\Omega)}$. To get the extension $\hat{g}': \overline{\Omega} \to \overline{h^{-1} \circ f^{-1}(\Omega)}$ we have to use, besides the condition that $\overline{h^{-1} \circ f^{-1}(\Omega)} \cap K$ is totally disconnected, also the condition (i), from which it follows that we can extend f to a homeomorphism $\overline{B} \to \overline{B}$. As in 3.1.1 we can define the mappings $g: U \to V$ and $g': U \to V'$ and Remark 3.2.4 follows from the condition $\overline{k^{-1}(\Omega)} \cap K = \overline{h^{-1} \circ f^{-1}(\Omega)} \cap K$. Next we show that the condition of 3.1.1 on the cluster set of a point $x \in \partial\Omega \cap \partial B$ is valid. We select curves J_n^x, $n \in \mathbf{N}$, as in Construction 2.2.3. Then $\gamma_n = g(J_n^x) \cup g'(J_n^x)$ is a Jordan curve; denote by E_n the component of $\hat{\mathbf{C}} \setminus \gamma_n$ which contains x. Then $C(x, k^{-1}, D_1) \subset E_{n+1} \subset E_n$ for all n. Now our condition (iii) applies to the pre-images $k_*^{-1}(S_n)$ of the side pairing transformations S_n in 2.2.3 and we get that $\operatorname{diam} \gamma_n \to 0$ and thus $\operatorname{diam} E_n \to 0$ at least for a subsequence. Then $\operatorname{diam} C(x, k^{-1}, D_1) = 0$ and the cluster set contains exactly one point. We are now back in the proof of 3.1.1 and the sufficiency follows.

Note 3.2.10. In the introduction we mentioned the theorem of [M1], which states that if G is finitely generated, then K is a quasicircle. Our result implies the theorem as follows:

Because $f \circ T = \varphi(T) \circ f$ in B, f induces a homeomorphism $\hat{f}$ between the Riemann surfaces $B/h^*(G)$ and $B/k^*(G)$. These surfaces are of a finite type and it is a well-known fact that there then exists a quasiconformal (orientation reversing, in this case) homeomorphism $\hat{f}'$ between them, such that the mappings $\hat{f}$ and $\hat{f}'$ are homotopic. The mapping $\hat{f}'$ can be lifted to a quasiconformal homeomorphism $f': B \to B$ satisfying $f' \circ T = \varphi(T) \circ f'$. Thus we may suppose that our original f is quasiconformal and it is readily established that $r: \overline{D_1} \to \overline{D_2}$ gives a quasiconformal reflection on the Jordan curve K. This means that K is indeed a quasicircle.

References

[A] ABIKOFF, W.: Some remarks on Kleinian groups. - In Advances in the theory of Riemann surfaces: Proceedings of the 1969 Stony Brook Conference, edited by Lars V. Ahlfors et al., Ann. of Math. Studies 66, 1971, Princeton University Press, Princeton–New Jersey, 1–5.

[B] BEARDON, A. F.: The geometry of discrete groups. - Springer-Verlag, New York–Heidelberg-Berlin, 1983.

[E] EILENBERG, S.: Sur les transformations pèriodiques de la surface de sphère. - Fund. Math. 22, 1934, 28–41.

[F] FORD, L. R.: Automorphic functions. - Chelsea, New York, 1951.

[K] VON KERÉKJÁRTÓ, B.: Über die periodischen Transformationen der Kreisscheibe und der Kugelfläche. - Math. Ann. 80, 1921, 36–38.

[M1] MASKIT, B.: On boundaries of Teichmüller spaces and on kleinian groups: II. - Ann. of Math. (2) 91, 1970, 607–639.

[M2] MASKIT, B.: Kleinian groups. - Springer-Verlag, Berlin–Heidelberg, 1988.

[T] TUKIA, P.: On discrete groups of the unit disk and their isomorphisms. - Ann. Acad. Sci. Fenn. Ser. A I Math. 504, 1972.

[W] WHYBURN, G. T.: Topological analysis. - Princeton University Press, Princeton, 1964.
[ZVC] ZIESCHANG, H., E. VOGT, and H.-D.COLDEWEY: Surfaces and planar discontinuous groups. - Springer-Verlag, Berlin–Heidelberg, 1980.

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