JYU DISSERTATIONS 780

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UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Abstract

This thesis focuses on studying inverse problems for nonlinear elliptic partial differential equations and in particular inverse problems for the minimal surface equation and semilinear elliptic equations. It is shown that one can recover information about the coefficients of the equation or some geometric information from boundary measurements of solutions. The main tool used is linearization, both first order and higher order linearization.

The introduction describes inverse problems for partial differential equations in the context of the Calderón problem and gives a survey of the literature related to the linearization methods. Main theorems of the included articles are presented and the methods to prove them are also discussed.

The articles (A) and (C) focus on inverse problems for the minimal surface equation. In both articles we look at the minimal surface equation in Euclidean space that is equipped with a Riemannian metric. Then from boundary measurements we determine information about the metric. In (A) the metric is conformally Euclidean and in (C) the metric will be in a class of admissible metrics. The main method used in both articles is the higher order linearization method.

The remaining articles (B) and (D) study inverse problems for semilinear elliptic equations. In (B) the equation has a power type nonlinearity and the aim is to determine an unbounded potential from boundary measurements. Also in (B) the method used is the higher order linearization method. In (D) the focus is on recovering a general zeroth order nonlinearity from boundary measurements. Here the first linearization is used and we improve previous results for this method in the case of semilinear equations.

Foreword

I wish to thank my advisor Professor Mikko Salo for giving me the opportunity to test my wings as a researcher and guiding me through my doctoral studies. He has been a very kind and understanding advisor and I could not have hoped for a better mentor. I also want to thank the Department of Mathematics and Statistics of University of Jyväskylä for providing a welcoming work environment and great colleagues to work with.

I want to thank Yavar Kian for agreeing to be my opponent at my defense and the pre-examiners Tracey Balehowsky and Cătălin I. Cârstea for their encouraging words.

I also wish to thank my fellow doctoral students for all their support in a PhD students life both inside and outside of work. I also wish to express personal thanks to my other collaborator David Johansson for all the enlightening discussions we had.

I wish to thank the former and present members of the inverse problems research in Jyväskylä for the great, friendly atmosphere we have had. It has been a great joy to work with you all.

Finally I want to thank my family for being there for me. Special thanks go to Maria for always being there for me through all the highs and lows. She has believed in me when I did not and most certainly I would not be here without her support.

Jyväskylä, April 2024
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Tiivistelmä

Tässä väitöskirjassa tutkitaan inversio-ongelmia epälineaarisille osittaisdifferentiaaliyhtälöille, joista erityisesti keskitytään inversio-ongelmiin minimipintayhtälölle ja semilineaarisille yhtälöille. Tässä työssä näytetään, että ratkaisujen reunamittauksista voidaan saada tietoa geometriasta tai yhtälön kertoimista. Työn tärkeimpiä työkaluja ovat ensimmäisen asteen ja korkeamman asteen linearisaatio.

Johdannossa kuvaillaan inversio-ongelmia osittaisdifferentiaaliyhtälöille Calderónin ongelman kontekstissa ja annetaan katsaus linearisaatiotekniikoihin liittyvään kirjallisuuteen. Lisäksi esitellään tutkielmaan sisältyvien artikkeleiden päätulokset sekä todistuksiin käytetyt tekniikat.

Artikkelit (A) ja (C) keskittyvät inversio-ongelmiin minimipintayhtälölle. Molemmissa artikkeleissa minimipintayhtälöä katsotaan euklidisessa avaruudessa, joka on varustettu Riemannin metriikalla ja reunamittauksista saadaan tietoa tästä metriikasta. Artikkelissa (A) metriikka on konformisesti euklidinen ja artikkelissa (C) metriikka kuuluu hyväksyttäviin metriikoihin. Päätyökalu molemmissa artikkeleissa on korkeamman asteen linearisaatio.

Artikkeleissa (B) ja (D) tutkitaan inversio-ongelmia semilineaarisille elliptisille yhtälöille. Artikkelin (B) yhtälössä on potenssityylinen epälineaarisuus ja tarkoituksena on määrittää rajoittamaton potentiaali reunamittauksista. Jälleen päätyökaluna on korkeamman asteen linearisaatio. Artikkelin (D) tarkoituksena on määrittää reunamittauksista yleinen nollannen asteen epälineaarisuus. Tärkein työkalu on ensimmäinen linearisaatio ja työssä parannetaan aikaisempia tuloksia tälle tekniikalle semilineaaristen yhtälöiden tapauksessa.

List of included articles

This dissertation consists of an introduction and the following four articles:

(A) Janne Nurminen. *An inverse problem for the minimal surface equation. Nonlinear Analysis* Volume 227 (2023), 19 pp.

(B) Janne Nurminen. *Determining an unbounded potential for an elliptic equation with a power type nonlinearity. Journal of Mathematical Analysis and Applications* Volume 523, Issue 1 (2023), 11 pp.

(C) Janne Nurminen. *An inverse problem for the minimal surface equation in the presence of a Riemannian metric.* Preprint (April 2023), arXiv: 2304.05808.

(D) David Johansson, Janne Nurminen and Mikko Salo. *Inverse problems for semilinear elliptic PDE with a general nonlinearity* $a(x, u)$. Preprint (December 2023), arXiv: 2312.12196.

The author of this dissertation has actively taken part in the research of the joint article (D).

Contents

1. Introduction

In inverse problems one is interested in studying the properties of a medium by making indirect measurements. One of the classical inverse problems for partial differential equations is the Calderón problem that arises from oil detection. The name comes from the mathematician Alberto Calderón who in a famous paper from the 80's [Cal80] studied the mathematical model of determining the conductivity of a medium by making measurements on the boundary of this medium. The inverse problem of finding the conductivity is called electrical impedance tomography or EIT for short. In EIT one makes measurements by placing electrodes on the surface of a medium and then inserts a voltage on these electrodes. This voltage then induces a current inside the medium and one can then measure the current on the boundary of this medium. Then with the knowledge of voltage and current on the boundary one tries to determine the conductivity inside the medium.

This problem is modelled by a partial differential equation. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with smooth boundary $\partial\Omega$ and γ be a smooth positive function that is the conductivity of Ω. The potential will be a function u that solves the following boundary value problem

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u_f(x)) = 0, & \text{in } \Omega \\ u_f = f, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and here the boundary value f is the known potential on the boundary. In fact u is induced by f. The measurements that one makes are encoded in the so called Dirichlet-to-Neumann map, or DN map for short:

$$\Lambda_\gamma \colon f \mapsto \gamma\partial_\nu u_f|_{\partial\Omega}.$$

Here the function u_f is the unique solution to (1.1) with boundary value f.

By doing the so called Liouville reduction [Sal08] that is doing the substitution $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$ in (1.1) we get a reformulation of the Calderón problem by a Schrödinger equation:

$$\begin{cases} \Delta u_f + qu_f = 0, & \text{in } \Omega \\ u_f = f, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Then the corresponding DN map will be

$$\Lambda_q \colon f \mapsto \partial_\nu u_f|_{\partial\Omega}.$$

Now the inverse problem would be to determine the potential q from the boundary measurements encoded in the DN map Λ_q.

There are different questions that one may study for this inverse problem, and we mention some of them here:

I. Boundary uniqueness: Can one determine γ or q on the boundary $\partial\Omega$ from the DN map?
II. Interior uniqueness: Can one determine γ or q in the set Ω from the DN map?
III. Regularity: What is the lowest regularity needed to achieve uniqueness either on $\partial\Omega$ or in Ω?
IV. Stability: If there is some noise in the measurements, how does this affect the recovery of the conductivity?
V. Reconstruction: Can one give a constructive procedure to get recovery?
VI. Partial data: If one can only make measurements on a part of the boundary, can uniqueness still be achieved? One can also study the above questions with partial data.

These questions can be asked with inverse problems related to any other partial differential equation. In this thesis, we study mainly interior uniqueness (II.), regularity (III.) and partial data (VI.).

For the Calderón problem these have been extensively studied. A standard method to show interior uniqueness (II.) is to construct complex geometric optics solutions [AP06], [SU86], [SU87]. For more results on this, we refer to [Uhl14].

The results mentioned above are for the full data. When dealing with partial data interior uniqueness there is a difference in the results depending on the dimension. For $n = 2$ one can measure on an arbitrary subset $\Gamma \subset \partial\Omega$ and get uniqueness [IUY10]. When $n \geq 3$ there are only partial results for example in [BU02], [Isa07], [KSU07], [KS13]. For further references, see [KS14].

When studying regularity (III.) we again distinguish the dimensions $n = 2$ and $n \geq 3$. When $n \geq 3$ and we have full data, then in [Nac92], [Cha90], [DKS13] is shown that for $q \in L^{\frac{n}{2}}$ one has interior uniqueness and this is considered optimal (in the sense of standard well-posedness and the strong unique continuation principle). One can show uniqueness also when $q \in W^{-1,n}$ [Hab18]. For partial data similar results for $q \in L^{\frac{n}{2}}$ have been obtained in [CT20], [Tzo18] for specific geometric assumptions. In the full generality the partial data case is still open. When $n = 2$ the optimal regularity would be $q \in L^p$ where $p > 1$ is as close to 1 as possible. The best known result so far is for $q \in L^{\frac{4}{3}}$ [BTW20], [Ma20b]. For partial data the best known result is for $q \in W^{1,p}$, $p > 2$, with an arbitrary subset of the boundary [IY12].

Another variant of the Calderón problem is when the conductivity γ is a matrix. This is a realistic assumption, since many materials are not isotropic which would be the case with a positive real valued function γ as before. When γ is a matrix the inverse problem is called the anisotropic Calderón problem. This problem is closely related to a geometric problem (see for example [LU89], [Uhl14]) of recovering a Riemannian metric g from measurements done on the boundary of a manifold (M, g). More specifically one would study the boundary value problem

$$\begin{cases} \Delta_g u_f = 0, & \text{in } M \\ u_f = f, & \text{on } \partial M, \end{cases} \tag{1.3}$$

and the related DN map

$$\Lambda_g \colon f \mapsto \partial_\nu u_f|_{\partial M}.$$

Here Δ_g is the Laplace-Beltrami operator and $\partial_\nu u$ is the Riemannian boundary normal derivative.

The above mentioned partial differential equations are linear. In this thesis we study inverse problems for nonlinear partial differential equations. We study two types of equations; first is a semilinear equation

$$\Delta u + a(x, u) = 0,$$

where a is nonlinear in u, for example $a(x,u) := q(x)u^m$, where $m \geq 2$ is an integer and q a potential. This equation is a nonlinear version of the Schrödinger equation (1.2). The second type that we study is the minimal surface equation. We will study this in a Riemannian setting and the inverse problem will have similar features as the one for (1.3). These similarities will be highlighted in Section 3.

One possible way of studying these inverse problems for nonlinear partial differential equations is to linearize the DN map. Recently many authors have been using a method called the higher order linearization for these problems. We will describe this method in more detail in Section 2 and also review the literature on this subject. In Section 3 we go through articles (A) and (C) where we study inverse problems for the minimal surface equation. Finally in Sections 4 and 5 we discuss articles (B) and (D) where inverse problems for $\Delta u + a(x,u) = 0$ are studied. In article (B) we focus on the case when $a(x,u) = q(x)u^m$, $m \geq 2$, and study the question III mentioned above. In (D) we try to see what assumptions on the nonlinearity a are needed in order to solve the inverse problem of recovering the nonlinearity from boundary measurements.

2. The method of higher order linearization

In inverse problems for nonlinear partial differential equations one can try to linearize the DN map and then try to extract information from this linearization. In [Isa93] the first linearization was used in a parabolic case where the linearized equation is a linear one and hence one can use the theory of inverse problems for linear PDEs. When dealing with elliptic semilinear PDEs of the type $\Delta u + a(x,u) = 0$ the inverse problem of determining the nonlinearity $a(x,u)$ was studied in [IS94], [IN95], [Sun10], [IY13]. In all of these articles they use the (first) linearization method, that is, they study the (first) Fréchet derivative of the DN map and use results for linear equations.

In [IS94] the authors show that for $n > 2$ one can recover the nonlinearity a from boundary measurements in a certain reachable set. For the nonlinearity a they assume that a, $\partial_u a$ and $\partial_u^2 a$ are bounded and that $\partial_u a \leq 0$. In [IN95] the author shows a similar result for $n = 2$. In that article it is assumed that the first derivative $\partial_u a$ is bounded from above by a function q_* so that the smallest Dirichlet eigenvalue of $\Delta + q_*$ is strictly negative (this assumption guarantees that for example the maximum principle holds) and that the constant function $u \equiv 0$ is a solution to $\Delta u + a(x,u) = 0$. That is they assume that $a(x,0) = 0$.

In [Sun10] the author shows that one can recover the nonlinearity a for $n \geq 2$ in the reachable set without the assumption $a(x,0) = 0$ but still keeping essentially the same assumptions that have been made in [IS94], [IN95]. In [Sun10] it is also shown that one cannot recover the nonlinearity uniquely since there is a gauge invariance. That is if one has two nonlinearities a_1 and a_2 whose DN maps agree, then one can show that $a_2 = T_\varphi a_1$ on the reachable set. Here $(T_\varphi a)(x,u) = a(x, u + \varphi(x)) + \Delta\varphi(x)$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = \nabla\varphi|_{\partial\Omega} = 0$. If however there is a common solution, then this gauge invariance disappears.

In [IY13] the dimension is $n = 2$ and the equation involves also a linear zeroth order term, that is the equation is of the form $\Delta u + q(x)u - a(x,u) = 0$. Also this article is different from the others mentioned before since they focus on partial data. The authors also make some stronger assumptions on the nonlinearity in order to determine the nonlinearity. Some growth conditions are made on $a, \partial_u a$ and $\partial_u^2 a$ and in addition they assume $a(x,0) = \partial_u a(x,0) = 0$.

Since in this thesis we study the minimal surface equation, which is a quasilinear equation, we give some remarks on previous results for quasilinear equations. The Euclidean minimal surface equation is

$$\operatorname{div}\left(\frac{\nabla u}{(1 + |\nabla u|^2)^{\frac{1}{2}}}\right) = 0,$$

which can be thought of as a special case of the equation $\operatorname{div}_x(a(x, \nabla u(x))) = 0$. Results for these equations can be found for example in [HS02], [MU20] where they make strong assumptions on the nonlinearity a e.g. some growth conditions. For other types of quasilinear equations we refer to [CF21] and the references there.

Recently there has been an increasing interest towards inverse problems for nonlinear partial differential equations and a new method was introduced to solve these problems. In [KLU18] the method of *higher order linearization* was first used for nonlinear hyperbolic wave equations

$$\Box u(x) + a(x)u(x)^2 = f(x)$$

on Lorentzian manifolds. The aim was to recover the nonlinearity a. After this, in two articles, [LLLS21a] and [FO20], the method was introduced for semilinear elliptic equations of the type

$$\Delta u + a(x, u) = 0.$$

These two articles were independently done and published at the same time in the same preprint server. In [LLLS21a] they consider the special case of a power type nonlinearity, that is when $a(x, u) = q(x)u(x)^m$, $m \in \mathbb{N}, m \geq 2$. They continue to a more general nonlinearity a in [LLLS21b] with partial data results. They also prove results on the simultaneous recovery of an unknown cavity or boundary and the nonlinearity. In [LLST22] the authors generalize this for general power type nonlinearities. The inverse problem with partial data was also treated in [KU20].

In these works there are also assumptions on the nonlinearity, but they are more general than in the works before. In [LLLS21b] it is assumed that $a(x, 0) = 0, \partial_u a(x, 0) = 0$ and 0 is not a Dirichlet eigenvalue for the first linearization. Article [LLLS21a] is a special case of this. In [FO20] they also assume $a(x, 0) = 0$ and that 0 is not a Dirichlet eigenvalue for the first linearization. In addition they assume that $\partial_u a, \partial_u^2 a$ are known, a is analytic in the second variable and with these assumptions there is a power series representation. Somewhat similarly in [KU20] a is assumed to be holomorphic in the second variable, $a(x, 0) = 0, \partial_u a(x, 0) = 0$ and from these it follows that a has a power series representation.

Also in [LL23] the authors discuss semilinear elliptic equations with a source term, that is equations of type $\Delta u + a(x, u) = F$, and they study the inverse problem of recovering both the nonlinearity and the source term. They remark that in general one cannot determine both since there is a gauge invariance. They then address some examples where it is possible to break the gauge by making some assumptions on the nonlinearity.

Other types of equations have also been considered. Quasilinear conductivity equations with an isotropic nonlinear conductivity have been treated in [CFKKU21], [KKU23]. In [CFKKU21] they consider a conductivity equation $\mathrm{div}(\gamma(x, u, \nabla u)\nabla u) = 0$ such that $\gamma(\cdot, 0, 0) > 0$ is a smooth function and that γ is holomorphic in the last two variables. In [KKU23] they study partial data with conductivities depending on x and u or additionally depending on some fixed direction of the gradient ∇u. They assume holomorphicity, as before, and that the linearized equation at 0 is the Laplace equation. These assumptions are similar to the ones made for the equation $\Delta u + a(x, u) = 0$.

There are other inverse problems for nonlinear partial differential equations that have been studied. For example the minimal surface equation in a geometric setting [CLLO24], [CLLT23] (more on these in Section 3) and the nonlinear magnetic Schrödinger equation have been studied in [KU23], [LZ20], [Ma20a], [KUY21] and in the fractional case in [LZ23]. Also when dealing with the fractional Laplace operator, some nonlinear lower order perturbations of this have been treated in [LO22], [LL22].

An interesting phenomenon is that the nonlinearity actually helps. For example if you consider an inverse problem for the Schrödinger equation $\Delta u + qu = 0$, then in dimension $n \geq 3$ the partial data in the general case remains open, that is when $\Gamma \subset \partial\Omega$ is an arbitrary open set. For the nonlinear variant $\Delta u + qu^m = 0$ the partial data is solved for an arbitrary open set ([LLLS21b], [KU20]) and even for Neumann data measured at a single point [ST23]. For the nonlinear isotropic conductivity in [KKU23] the authors show partial data interior uniqueness in the cases mentioned above for an arbitrary open subset of the boundary. For the linear conductivity equation in dimensions $n \geq 3$ the partial data recovery is not known. For $n = 2$ it is known and we refer to [IUY10].

Next we will present some details about the method, mainly we will discuss well-posedness for the nonlinear problems and give an example of the method in a model case.

2.1. **Well-posedness.** We are dealing with elliptic nonlinear partial differential equations and it is not clear if we will have well-posedness for boundary value problems of this form. In some cases we would also like to have a well-defined DN map. In [LLLS21a] this issue was solved by showing that if the boundary values are small enough, then there exists a unique small solution to the boundary value problem. This was done by using the implicit function theorem for Banach spaces (see for example [RR04, Theorem 10.6.]).

We describe here a general setting for the well-posedness that is used in articles (A), (B) and (C) although in (B) some modifications are done to the function spaces. In (B) we use Sobolev spaces instead of Hölder spaces. In (D) the well-posedness is more delicate and we postpone the discussion to Section 5. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary and let $F\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ be a smooth function so that $F(x, 0, 0, 0) = 0$. We will consider the boundary value problem

$$\begin{cases} F(x, u, \nabla u, \nabla^2 u) = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f \in C^s(\partial\Omega)$, $s > 0$ and $s \notin \mathbb{N}$. Then we can show the existence of small solutions to (2.1) in the following sense:

Proposition 2.1 ((A), Proposition 2.1)**.** *Let $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$, $F = F(x, u, p, P)$, be a C^∞ mapping with $F(x, 0, 0, 0) = 0$. Furthermore assume that the map*

$$v \mapsto L(v) := \partial_u F(x, 0, 0, 0)v + \nabla_p F(x, 0, 0, 0) \cdot \nabla v + \nabla_P F(x, 0, 0, 0) : \nabla^2 v$$

is injective on $H^1_0(\Omega)$ and that the operator L is strictly elliptic. Let $s > 3, s \notin \mathbb{N}$. Then there exist $C, \delta > 0$ such that for any

$$f \in U_\delta := \{h \in C^s(\partial\Omega) : ||h||_{C^s(\partial\Omega)} < \delta\}$$

the boundary value problem

$$\begin{cases} F(x, u, \nabla u, \nabla^2 u) = 0, & \textit{in } \Omega \\ u = f, & \textit{on } \partial\Omega, \end{cases}$$

has a unique small solution $u = u_f$ which satisfies

$$||u||_{C^s(\overline{\Omega})} \leq C||f||_{C^s(\partial\Omega)}.$$

Moreover the following mappings are C^∞ maps

$$S\colon U_\delta \to C^s(\overline{\Omega}), \qquad f \mapsto u_f,$$
$$\Lambda\colon U_\delta \to C^{s-1}(\partial\Omega), \qquad f \mapsto \partial_\nu u_f|_{\partial\Omega}.$$

This proposition gives a unique small solution and continuous dependence on the boundary values. It also gives that the DN map associated to (2.1) is a smooth mapping in the Fréchet sense. This is used to linearize the DN map consecutively to obtain information about the object that we are trying to recover from boundary measurements.

As mentioned before, the proof relies on the implicit function theorem for Banach spaces and this is where the smallness of the solution u_f comes from. That is because the implicit function theorem is proved using the Banach fixed point theorem, so we in a way rely on a contraction principle. The proof also uses that for $v, w \in C^s(\overline{\Omega})$ we have $||vw||_{C^s(\overline{\Omega})} \leq ||v||_{C^s(\overline{\Omega})}||w||_{C^s(\overline{\Omega})}$. Finally a key thing is that the linearized equation behaves well since we assume the injectivity of the map $v \mapsto L(v)$.

2.2. **Description of the method in a model case.** We will next describe this method in a model case. Consider an inverse problem of recovering a potential $q \in C^\infty(\overline{\Omega})$ from boundary measurements, that is from the knowledge of the DN map

$$\Lambda_q \colon f \mapsto \partial_\nu u_f|_{\partial\Omega}$$

associated to the boundary value problem

$$\begin{cases} \Delta u + qu^2 = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

First of all, one can show that this boundary value problem has a unique *small solution* (see Proposition 2.1 and [LLLS21a, Proposition 2.1]). More precisely there is a $\delta > 0$ such that for any $f \in C^s(\partial\Omega)$, $s > 2$, $s \notin \mathbb{N}$, with $||f||_{C^s(\partial\Omega)} \leq \delta$ there is a unique solution to (2.2) with $||u||_{C^s(\overline{\Omega})} \leq ||f||_{C^s(\partial\Omega)}$. Thus the solution u_f depends smoothly on the boundary value f and the same applies for the DN map.

Relying on these, one can do the following. By setting the boundary value $f = \varepsilon_1 f_1 + \varepsilon_2 f_2$ for $\varepsilon_1, \varepsilon_2$ small enough and $f_1, f_2 \in C^s(\partial\Omega)$ we have that the solution u_f will depend smoothly on ε_1 and ε_2. The same applies for the DN map. By taking the derivative $\frac{\partial}{\partial\varepsilon_1}|_{\varepsilon_1=\varepsilon_2=0}$ of the DN map, one gets a mapping

$$(D\Lambda_q)_0 \colon f_1 \mapsto \partial_\nu v_{f_1}|_{\partial\Omega}$$

where $v_{f_1} := \frac{\partial}{\partial\varepsilon_1} u_f|_{\varepsilon_1=\varepsilon_1=0}$ is the solution of the linearized problem. In the case of (2.2) this means a solution of

$$\begin{cases} \Delta v_{f_1} = 0, & \text{in } \Omega \\ v_{f_1} = f_1, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

One can obtain this boundary value problem also by taking the derivative $\frac{\partial}{\partial\varepsilon_1}|_{\varepsilon_1=\varepsilon_1=0}$ of (2.2).

By taking the mixed derivative $\frac{\partial}{\partial\varepsilon_2\partial\varepsilon_1}|_{\varepsilon_1=\varepsilon_2=0}$ of (2.2) one gets the second linearization of (2.2):

$$\begin{cases} \Delta w = -2qv_{f_1}v_{f_2}, & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $w := \frac{\partial}{\partial\varepsilon_2\partial\varepsilon_1} u_f|_{\varepsilon_1=\varepsilon_2=0}$. Notice that when compared to the first linearization (2.3) this equation contains the potential q. This is used below to recover q. Since the DN map can also be differentiated with the same mixed derivative, we have

$$(D^2\Lambda_q)_0 \colon (f_1, f_2) \mapsto \partial_\nu w|_{\partial\Omega}.$$

This second linearization of the DN map is a symmetric bilinear map.

Now if we would have two potentials q_1 and q_2 and assume that $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ for all f sufficiently small, then we could use the second linearizations above to get $q_1 = q_2$. We would begin by subtracting the equations (2.4) for $j = 1, 2$ and integrating the difference against a third solution v_{f_3} to (2.3). This yields

$$\int_\Omega \Delta(w_1 - w_2)v_3\, dx = -2\int_\Omega (q_1 - q_2)v_{f_1}v_{f_2}v_{f_3}\, dx. \tag{2.5}$$

Now applying the derivative $\frac{\partial}{\partial\varepsilon_2\partial\varepsilon_1}|_{\varepsilon_1=\varepsilon_2=0}$ to $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ gives $\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}$. Thus using integration by parts in (2.5) we have

$$\int_\Omega (q_1 - q_2)v_{f_1}v_{f_2}v_{f_3}\, dx = 0.$$

Choosing $v_{f_3} = 1$, f_1 and f_2 to be the boundary values of Calderón's exponential solutions [Cal80] would then give $q_1 = q_2$ in Ω.

In general when using the higher order linearization, one tries to reduce the inverse problem to a problem of determining if a product of solutions to the first linearization is dense in some suitable space (for example in L^1). In some cases it is reduced to a product of some kind that involves the solution and gradients of solutions to the first linearization (for example in [CFKKU21]).

The articles in this thesis use this method in different cases. Articles (A) and (C) (see Section 3) focus on inverse problems for the minimal surface equation and the articles (B) and (D) (Sections 4 and 5, respectively) will consider inverse problems for semilinear partial differential equations of type $\Delta u + a(x, u) = 0$.

3. Inverse problems for the minimal surface equation

In this section we discuss inverse problems for the minimal surface equation in a setting where we equip $\mathbb{R}^n$ with a Riemannian metric.

Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain and let us consider the graph of a function $u\colon \Omega \to \mathbb{R}$, that is, let us consider the set

$$\Sigma = \{(x', u(x'))\colon x' \in \Omega\} \subset \mathbb{R}^n.$$

If the metric on $\mathbb{R}^n$ would be Euclidean, then Σ is a minimal surface if the function u solves the following quasilinear partial differential equation

$$\operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right) = 0 \quad \text{for all } x \in \Omega.$$

This is equivalent with saying that the mean curvature of Σ vanishes on all points of Σ. If one compares this equation with the conductivity equation (1.1) and the inverse problem for that equation, then it is not clear what the inverse problem would be. There is no "conductivity" to recover. Also it is not clear if the boundary value problem for the minimal surface equation is well-posed or not.

The other difficulty is that if the underlying metric is *not* Euclidean, then the appearance of the minimal surface equation differs from case to case. In the case where we would have a product structure $\hat{g} \oplus e$, with a Riemannian metric $\hat{g}$ on $\mathbb{R}^{n-1}$, that is the nth direction only would be Euclidean, then Σ would be minimal if

$$\operatorname{div}_{\hat{g}}\left(\frac{\nabla_{\hat{g}} u}{(1+|\nabla_{\hat{g}} u|_{\hat{g}}^2)^{\frac{1}{2}}}\right) = 0 \quad \text{for all } x \in \Omega.$$

In this case it would be natural to ask that if one makes boundary measurements on Σ, could one then identify the metric $\hat{g}$ on Ω. This question has an answer in [CLLO24] where they show that it is possible to determine $\hat{g}$ up to an isometry that is the identity on $\partial\Omega$ when $n = 2$.

Recently in [CLLT23] the authors considered an even more general setting. They showed that under a certain topological condition a two dimensional minimal surface embedded in a three dimensional Riemannian manifold can be recovered up to an isometry from the DN map associated to the minimal surface equation. If the topological condition is removed, then one can recover the conformal factor of a general minimal surface.

There is a link from minimal surfaces to a geometric inverse problem. In [ABN20] the authors show that under some geometric conditions on a Riemannian manifold (M, g) the knowledge of least areas circumscribed by simple closed curves on the boundary ∂M determine the metric g. There are also restrictions on the metric. This inverse problem can be thought of as a generalization of the boundary rigidity problem that asks if a metric can be determined by knowing the distances between boundary points (see e.g. [BI13], [PU05], [PSU23] and the references therein). In this problem the data would be lengths of geodesics which can be thought of as one dimensional minimal surfaces.

3.1. On the recovery of a conformal factor in the presence of a conformally Euclidean metric.

Let us then look at a case when we equip $\mathbb{R}^n$ with a conformally Euclidean metric, that is we look at the manifold $(\mathbb{R}^n, c\cdot \mathrm{Id})$, where $c\in C^\infty(\mathbb{R}^n)$, $c(x)>0$ for all $x\in\mathbb{R}^n$. Then the minimal surface equation becomes

$$
\begin{aligned}
(3.1)\qquad & \operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right) \\
& + \frac{\Delta u(1-c^{-1}) + \frac{1}{2c}\nabla u\cdot\nabla_{x'}c(1+|\nabla u|^2-c^{-2}) + \frac{n-1}{2c^3}\partial_{x_n}c(1+|\nabla u|^2)}{(1+|\nabla u|^2)^{\frac{3}{2}}} = 0.
\end{aligned}
$$

Note that if $c\equiv 1$ then we get the more familiar Euclidean minimal surface equation (3). In this case one natural question is to ask that if one makes boundary measurements on Σ can one then identify the conformal factor c. These boundary measurements are encoded in the DN map

$$\Lambda_c^\Gamma\colon f\mapsto \partial_\nu u_f\big|_\Gamma$$

where $\Gamma\subset\partial\Omega$ and f is small (in the sense defined below) with $\mathrm{spt}(f)\subset\Gamma$. We will denote $\Lambda_c := \Lambda_c^{\partial\Omega}$ when $\Gamma=\partial\Omega$. This question is studied in the first article of this thesis (A) and the main result is the following:

Theorem 3.1 ((A), Theorem 1.1)**.** *Let $\Omega\subset\mathbb{R}^{n-1}$, $n\geq 3$, be a bounded domain with C^∞ boundary, $(\mathbb{R}^n,g_1)$, $(\mathbb{R}^n,g_2)$ be two Riemannian manifolds with $(g_j)_{ik}(x)=c_j(x)\delta_{ik}$, where $c_j\in C^\infty(\mathbb{R}^n)$, $c_j(x)>0$ for $j=1,2$ and for all $x\in\mathbb{R}^n$. Assume that $\partial_{x_n}c_j(x',0)=\partial^2_{x_n}c_j(x',0)=0$ for $x'\in\Omega$. We have four cases:*

(1) Let $n>3$ and Λ_{c_j} be the DN maps associated to c_j, $j=1,2$, and assume that

$$\Lambda_{c_1}(f)=\Lambda_{c_2}(f)$$

for all $f\in U_\delta := \{h\in C^s(\partial\Omega) : ||h||_{C^s(\partial\Omega)}<\delta\}$, where $\delta>0$ is sufficiently small.

(2) Assume either that
- *(a) $n=3$, $\Gamma\subset\partial\Omega$ be open and $\Gamma\neq\emptyset$ or*
- *(b) $n>3$, $\Omega\subset\{x_{n-1}>0\}$, $\Gamma\subset\partial\Omega$ be open, $\Gamma\neq\emptyset$ and that $\partial\Omega\setminus\Gamma\subset\{x_{n-1}=0\}$ or*
- *(c) $n>3$, Ω is a strict subset of some ball $B\subset\mathbb{R}^{n-1}$, $\Gamma\subset\partial\Omega$ be open, $\Gamma\neq\emptyset$ and that $\partial\Omega\setminus\Gamma\subset\partial B$.*

In addition assume that

$$\Lambda^\Gamma_{c_1}(f)=\Lambda^\Gamma_{c_2}(f)$$

for all $f\in U_\delta$, $\mathrm{spt}(f)\subset\Gamma$, where $\delta>0$ is sufficiently small and $\Lambda^\Gamma_{c_j}$ are the partial DN maps associated to c_j for $j=1,2$.

Then in the cases (1) or (2) we have for $\lambda\neq 0$

$$\partial^m_{x_n}c_1(x',0)=\lambda\partial^m_{x_n}c_2(x',0),\quad \text{in } \Omega,\ m\geq 0.$$

As seen in the result, one can get some results with measurements done on only a part of the boundary.

As mentioned in Section 2 this is proved using the higher order linearization method. We will use small boundary data of the form $f=\varepsilon_1 f_1+\cdots+\varepsilon_k f_k$ for e_j small and we denote $\varepsilon=(\varepsilon_1,\dots,\varepsilon_k)$ and by u_f^j the unique small solution to (3.1) with c replaced by c_j and with boundary value f. A key step is the first linearization of the DN map and what is the corresponding boundary value problem for that. The first linearization of the DN map at $f=0$ is

$$(D\Lambda_{c_j})_0\colon C^s(\partial\Omega)\to C^{s-1}(\partial\Omega),\quad f\mapsto \partial_\nu v_j^l\big|_{\partial\Omega}.$$

Here $v_j^l := \frac{\partial}{\partial\varepsilon_l}\big|_{\varepsilon=0}u_f^j$ is a solution to

$$
(3.2)\qquad \begin{cases} \operatorname{div}\left(c_j(x',0)^{\frac{n-1}{2}}\nabla v_j^l\right)=0, & \text{in } \Omega\\ v_j^l=f_l, & \text{on } \partial\Omega. \end{cases}
$$

Using boundary determination from [BS06] and the results in [IUY10], [SU87] or [Isa07] (depending on the dimension and if we are dealing with partial data or not) we get

$$c_1(x',0) = \lambda c_2(x',0), \quad x' \in \Omega. \tag{3.3}$$

With this knowlegde we will move to higher order linearizations of the DN map. This is done by an induction on the order of linearization, but we will describe how the method works in the second linearization.

First of all, since solutions to (3.2) are unique we get that (3.3) implies $v^l := v_1^l = v_2^l$. Then we linearize the boundary value problem for the equation (3.1) twice to get

$$\begin{cases} \operatorname{div}\left(c_j(x',0)^{\frac{n-1}{2}} \nabla w_j^{(al)}\right) - \frac{n-1}{2} c_j(x',0)^{\frac{n-1}{2}-1} \partial_{x_n}^3 c_j(x',0) v^l v^a = 0, & \text{in } \Omega \\ w_j^{(al)} = 0, & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

where v^a, v^l solve the first linearization with corresponding boundary values f_a, f_l and $w_j^{(al)} := \frac{\partial^2}{\partial\varepsilon_a \partial\varepsilon_l}\big|_{\varepsilon=0} u_f^j$. Using (3.3), subtracting (3.4) for $j = 1, 2$ and using integration by parts we obtain an integral identity

$$\int_\Omega \frac{n-1}{2} c_1(x',0)^{\frac{n-1}{2}-1} \left(\partial_{x_n}^3 c_1(x',0) - \lambda \partial_{x_n}^3 c_2(x',0)\right) v^l v^a \, dx' = 0. \tag{3.5}$$

Solutions to the first linearization (3.2) are also solutions to a Schrödinger equation. Then we use the property that products of these solutions are dense in L^1 (when $n \geq 3$ [SU87], when $n = 2$ [Buk08], see also [Blå11] and [LLLS21b, Proposition 2.1]) and thus obtain

$$\partial_{x_n}^3 c_1(x',0) = \lambda \partial_{x_n}^3 c_2(x',0), \quad x' \in \Omega.$$

This would then imply that $w_1^{(al)} = w_2^{(al)}$. The induction would then proceed in a similar fashion.

When dealing with partial data, the proof goes in almost the same way. The main difference comes when we try to obtain the integral identity (3.5). There one needs to take care of boundary integrals on $\partial\Omega \setminus \Gamma$ when using integration by parts. This is done by constructing a positive solution to the first linearization (3.2) by using the maximum principle.

Some comments on the assumptions made in Theorem 3.1. The assumption $\partial_{x_n} c_j(x',0) = 0$ is needed for the well-posedness of the boundary value problem for (3.1), that is, it ensures that 0 is a solution. The second derivative $\partial_{x_n}^2 c_j(x',0)$ is assumed to be 0 in order for the method to work, and it is not known if it could be removed. The assumption in part (*b*) comes from the fact that we use partial data results for inverse problems for the Schrödinger equation. When $n = 3$ we use [IUY10] and there they do not pose any constraints on $\Gamma \subset \partial\Omega$. When $n > 3$ we use the results of [Isa07] and there are some conditions needed for Ω, $\partial\Omega$ and Γ. These conditions are described in the statement of Theorem 3.1.

Then we would like to make some remarks about the conclusion of Theorem 3.1. Firstly, if one would assume that both c_1 and c_2 were real analytic with respect to x_n, then we would have

$$c_1(x) = \lambda c_2(x), x \in \Omega \times \mathbb{R}.$$

Secondly, there is a small gauge invariance with the conformal factor. If we replace c by μc, μ a constant such that $\mu \neq 0$, then the DN maps Λ_c and $\Lambda_{\mu c}$ will be the same and also the minimal surface equation for c and μc will be the same (this can be seen from (3.1)). This is the reason for $\lambda \neq 0$ to appear in the conclusion.

3.2. **On the recovery of a metric in a fixed conformal class in the presence of an admissible metric.** In the third article (C) we try to combine the setting of articles (A) and [CLLO24]. Thus we will look at the case where we equip $\mathbb{R}^n$ with a metric

$$g(x', x_n) = c(x', x_n) \begin{pmatrix} \hat{g}(x') & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

Here $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $\hat{g}$ is a Riemannian metric on $\mathbb{R}^{n-1}$ and $c \in C^\infty(\mathbb{R}^n)$, $c(x) > 0$ for all $x \in \mathbb{R}^n$. With this metric the minimal surface equation is

$$\begin{aligned}(3.7)\qquad &\operatorname{div}_g \left(\frac{\nabla_g u}{\left(1 + |\nabla_g u|_g^2\right)^{\frac{1}{2}}} \right) \\ &+ \frac{\Delta_g u(1 - c^{-1}) + \frac{1}{2c}(\nabla_g u)^j \partial_{x_j} c(1 + |\nabla_g u|_g^2 - c^{-2}) + \frac{n-1}{2c^3}\partial_{x_n} c(1 + |\nabla_{\hat{g}} u|_{\hat{g}}^2)}{\left(1 + |\nabla_g u|_g^2\right)^{\frac{3}{2}}} = 0.\end{aligned}$$

Note that when $\hat{g} = \mathrm{Id}$ we would have the metric in article (A) and when $c \equiv 1$ we would be in the setting of [CLLO24]. In the third article we consider the inverse problem of identifying the Riemannian metric in a fixed conformal class, that is if we know the DN maps for two metrics in the same conformal class, are these two metrics then the same. The main result of (C) gives a partial answer to this question in the setting described above. This problem is studied also when dealing with the inverse problem for the Laplace equation on a manifold (see equation (1.3)). More on this problem can be found for example in [DKSU09], [CFO23].

Before we state the result, we need to introduce some concepts. Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded open set and $\Gamma \subset \partial\Omega$. First of all the DN map in this situation will be

$$(3.8)\qquad \Lambda_g^\Gamma \colon f \mapsto \partial_\nu u_f|_\Gamma$$

for small f with $\mathrm{spt}(f) \subset \Gamma$ and when $\Gamma = \partial\Omega$ we denote $\Lambda_g := \Lambda_g^\Gamma$. Here $\partial_\nu u_f = \hat{g}^{ij}\partial_{x_i} u_f \nu_j$. Then we will introduce the set of admissible metrics:

Definition 3.2. *A Riemannian metric g is called admissible if on $\mathbb{R}^n$ it is of the form* (3.6) *and $(\Omega, \hat{g})$ is a simple manifold.*

Definition 3.3. *A compact manifold $(\hat{M}, \hat{g})$ with boundary is called simple if for any $p \in \hat{M}$ the exponential map $\exp_p$ with its maximal domain of definition in $T_p\hat{M}$ is a diffeomorphism onto $\hat{M}$, and if $\partial\hat{M}$ is strictly convex in the sense that its second fundamental form is positive definite.*

The reason to introduce this class of Riemannian metrics is that we will use results for the inverse problem for an advection diffusion equation that is in the setting of admissible metrics [KU18].

Now we are ready to state the main results of (C).

Theorem 3.4 ((C), Theorem 1.3)**.** *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a Riemannian manifold where g is as in* (3.6) *with $\partial_{x_n} c(x', 0) = \partial_{x_n}^2 c(x', 0) = 0$ and let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain with C^∞ boundary. When $n = 3$ assume that Ω is simply connected and when $n > 3$ assume that $\hat{g}$ is admissible. Let $\tilde{c} \in C^\infty(\mathbb{R}^n)$ be such that $\partial_{x_n}\tilde{c}(x', 0) = \partial_{x_n}^2\tilde{c}(x', 0) = 0$ and let $\partial_{x_n}^k\tilde{c}(x', 0)|_{\partial\Omega} = 0$ for $k \geq 3$. Assume that $\Lambda_g(f) = \Lambda_{\tilde{c}g}(f)$ for all $f \in U_\delta$, where $\delta > 0$ is sufficiently small. Then*

$$\tilde{c}(x', 0) = \lambda, \quad \partial_{x_n}^k \tilde{c}(x', 0) = 0$$

for some $\lambda > 0$ and for all $k > 2$.

Thus, if $\tilde{c}$ is real analytic with respect to x_n, then $\tilde{c}(x) = \lambda$ for all $x \in \Omega \times \mathbb{R}$.

We also consider a partial data result when $n = 3$:

Theorem 3.5 ((C), Theorem 1.4)**.** *Let $(\mathbb{R}^3, g)$, $\tilde{c}$ and Ω be as in Theorem 3.4 and let $\Gamma \subset \partial\Omega$, $\Gamma \neq \emptyset$, be open. Assume that $\Lambda_g(f) = \Lambda_{\tilde{c}g}(f)$ for all $f \in U_\delta$, where $\delta > 0$ is sufficiently small. Then*

$$\tilde{c}(x', 0) = \lambda, \quad \partial_{x_3}^k \tilde{c}(x', 0) = 0$$

for some $\lambda > 0$ and for all $k > 2$.

Thus, if $\tilde{c}$ is real analytic with respect to x_3, then $\tilde{c}(x) = \lambda$ for all $x \in \Omega \times \mathbb{R}$.

These results extend the result from (A). The proof follows a similar pattern as for Theorem 3.1. The differences come in with the first linearization of the DN map and the linearized equation for (3.7). For the metric g the linearized equation is

$$\begin{cases} -\Delta_{\hat{g}} v^l + X v^l = 0 & \text{in } \Omega \\ v^l = f_l, & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $Xv^l = \frac{1-n}{2c(x',0)} \sum_{i,j=1}^{n-1} \hat{g}^{ij} \partial_{x_i} c(x',0) \partial_{x_j} v^l$. For the metric $\tilde{c}g$ it is

$$\begin{cases} -\Delta_{\hat{g}} \tilde{v}^l + \tilde{X} \tilde{v}^l = 0 & \text{in } \Omega \\ \tilde{v}^l = f_l, & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where $\tilde{X}h = \frac{1-n}{2} \hat{g}^{ij} \left(\frac{\partial_{x_i} c(x',0)}{c(x',0)} + \frac{\partial_{x_i} \tilde{c}(x',0)}{\tilde{c}(x',0)} \right) \partial_{x_j} h$. Both equations (3.9) and (3.10) are called advection diffusion equations.

Since $\Lambda_g = \Lambda_{\tilde{c}g}$, we can use results for inverse problems for advection diffusion equations ([GT11] for $n = 3$, [KU18] for $n > 3$) to get that $\tilde{X} = X$. This implies that $\tilde{c}(x',0) = \lambda \in (0,\infty)$ and $v^l = \tilde{v}^l$.

Next we use this information together with the second linearization of the DN map and the second linearization of (3.7). The second linearizations are

$$\begin{cases} -\Delta_{\hat{g}} w^{kl} + \frac{1-n}{2c(x',0)} \sum_{i,j=1}^{n-1} \hat{g}^{ij} \partial_{x_i} c(x',0) \partial_{x_j} w^{kl} + \frac{n-1}{2c(x',0)} \partial_{x_n}^3 c(x',0) v^k v^l = 0 & \text{in } \Omega \\ w^{kl} = 0, & \text{on } \partial\Omega \end{cases} \tag{3.11}$$

for the metric g and

$$\begin{cases} -\Delta_{\hat{g}} \tilde{w}^{kl} + \frac{1-n}{2c(x',0)} \sum_{i,j=1}^{n-1} \hat{g}^{ij} \partial_{x_i} c(x',0) \partial_{x_j} \tilde{w}^{kl} + \frac{n-1}{2c(x',0)} \partial_{x_n}^3 c(x',0) v^k v^l & \\ + \frac{n-1}{2\tilde{c}(x',0)} \partial_{x_n}^3 \tilde{c}(x',0) v^k v^l = 0 & \text{in } \Omega \\ \tilde{w}^{kl} = 0, & \text{on } \partial\Omega \end{cases} \tag{3.12}$$

for the metric $\tilde{c}g$. Now fix $x_0' \in \Omega$. We construct a solution $v^{(0)}$ of the adjoint of the first linearization (3.9) so that $v^{(0)}(x_0') \neq 0$. Then subtracting equations (3.11), (3.12) and integrating this against the solution $v^{(0)}$ gives

$$\int_\Omega \frac{n-1}{2\tilde{c}(x',0)} \partial_{x_n}^3 \tilde{c}(x',0) v^k v^l v^{(0)} \, dV_{\hat{g}} = 0.$$

Since products of solutions to advection diffusion equation are dense in L^1 (see Lemma 4.1 in (C)), we get $\partial_{x_n}^3 \tilde{c}(x',0) = 0$ for $x' \in \Omega$. As before, we would proceed by induction.

The assumptions made in these two theorems are similar to those done in Theorem 3.1. As in Theorem 3.1 we assume $\partial_{x_n} c(x',0) = \partial_{x_n} \tilde{c}(x',0) = 0$ for the minimal surface equation to be well-posed for small data with both metrics $g, \tilde{c}g$ and the assumption $\partial_{x_n}^2 c(x',0) = \partial_{x_n}^2 \tilde{c}(x',0) = 0$ is made in order to make the method work. The assumption $\partial_{x_n}^k \tilde{c}(x',0)|_{\partial\Omega} = 0$ for $k \geq 3$ might possibly be removed by boundary determination. When $n > 3$ we assume that the metric $\hat{g}$ is admissible. As mentioned before, this is done in order to use results from [KU18] where the metric is of the same form as $\hat{g}$. When $n = 3$ we assume that Ω is simply connected. The reason for this is that we use results for inverse problems for the magnetic Schrödinger equation ([GT11] and [Tzo17] for partial data). In order to get information about the vector fields X and $\tilde{X}$ we need an additional argument that uses the Poincaré Lemma and thus we need the simply connectedness of Ω (see Lemma 4.2. of (C) for this argument).

The gauge invariance that appeared in Theorem 3.1 is present in (3.7) and in the DN map (3.8) and it plays a role here also. It shows in the conclusion of Theorems 3.4 and 3.5, that is we cannot hope to say anything more than $\tilde{c}(x',0) = \lambda$ because of the gauge invariance.

The partial data result, Theorem 3.5, is proved in the same way as the full data case. One of the main differences is to use results from [Tzo17] instead of [GT11]. Also when constructing this special solution $v^{(0)}$ one needs to construct it so that it also takes care of the boundary integral terms on the inaccessible part of the boundary, that is integrals on $\partial\Omega \setminus \Gamma$.

4. Recovery of an unbounded potential

In article (B) we study an inverse problem for an elliptic equation with a power type nonlinearity. The PDE in question is

$$\Delta u + qu^m = 0 \quad \text{in } \Omega, \qquad \text{where } m \geq 2, m \in \mathbb{N}.$$

It was shown in [LLLS21a], [KU20] that with measurements made on an open subset $\Gamma \subset \partial\Omega$, which can be the whole boundary, one can recover a smooth potential q or a Hölder continuous potential q, respectively. It is also enough to make measurements on a single point on the boundary to recover a Hölder continuous potential [ST23]. In (B) we show that one can recover an unbounded potential in the same situations as mentioned before. The potential will be in $L^{\frac{n}{2}+\varepsilon}(\Omega)$ for the partial and full data and in $L^{n+\varepsilon}(\Omega)$ for the single point measurement. Here $\varepsilon > 0$ for both cases. In order to state these results, we must define the correct boundary data now that the potential is not bounded. For this let $U_\delta := \{h \in W^{2-\frac{1}{p},p}(\partial\Omega)\colon ||h||_{W^{2-\frac{1}{p},p}(\partial\Omega)} < \delta\}$, where $p := \frac{n}{2} + \varepsilon$. Thus we have the following results:

Theorem 4.1 ((B), Theorem 1.1.)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, $\varepsilon > 0$ and $q_1, q_2 \in L^{\frac{n}{2}+\varepsilon}(\Omega)$. Let Λ_{q_j} be the DN maps associated to the boundary value problems*

$$\begin{cases} \Delta u + q_j u^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$, and assume that $\Lambda_{q_1} f = \Lambda_{q_2} f$ for all $f \in U_\delta$ with $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω.

Theorem 4.2 ((B), Theorem 1.2.)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected open and bounded set with C^∞ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial\Omega$. Let $\varepsilon > 0$, $q_1, q_2 \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ and $\Lambda^\Gamma_{q_j}$ be the partial DN maps associated to the boundary value problems*

$$\begin{cases} \Delta u + q_j u^m = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \setminus \Gamma \\ u = f, & \text{on } \Gamma \end{cases}$$

for $j = 1, 2$. Assume that

$$\Lambda^\Gamma_{q_1} f = \Lambda^\Gamma_{q_2} f$$

for all $f \in U_\delta$ with $\operatorname{spt}(f) \subset \Gamma$, where $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω.

Theorem 4.3 ((B), Theorem 1.3.)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected open and bounded set with C^∞ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial\Omega$. Suppose that $\mu \not\equiv 0$ is a fixed measure on $\partial\Omega$ and let $\varepsilon > 0$. Assume that $q_1, q_2 \in L^{n+\varepsilon}(\Omega)$ satisfy*

$$\int_{\partial\Omega} \Lambda_{q_1}(f)\, d\mu = \int_{\partial\Omega} \Lambda_{q_2}(f)\, d\mu \tag{4.1}$$

for all $f \in U_\delta$ with $\operatorname{spt}(f) \subset \Gamma$, where $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω. Thus when choosing $\mu = \delta_{x_0}$ for some fixed $x_0 \in \partial\Omega$ the condition

$$\Lambda_{q_1}(f)(x_0) = \Lambda_{q_2}(f)(x_0) \quad \text{for all } f \in U_\delta \text{ with } \operatorname{spt}(f) \subset \Gamma$$

gives $q_1 = q_2$ in Ω.

As mentioned already in Section 1 one can study this kind of low regularity for the linear Schrödinger equation. Again the nonlinearity helps since we do not need to pose any restrictions on the set $\Gamma \subset \partial\Omega$ as they do in [CT20], [Tzo18]. But we are not able to get the case that is considered optimal for the linear Schrödinger equation in dimension $n = 3$, we only get arbitrarily close to it. For $n = 2$ our results have lower regularity than those that exist for the linear counterpart, since the regularity is $q \in L^{\frac{4}{3}}$ for full data ([BTW20], [Ma20b]) and $q \in W^{1,r}$, $r > 2$, for partial data ([IY12]).

The proof for all three theorems is essentially the same, and the main idea is already presented in Section 2. The main difference to the earlier works comes up in the well-posedness since we now work with Sobolev spaces. First of all, the unique small solution will be in $W^{2,p}(\Omega)$ and the boundary values are in $W^{2-\frac{1}{p},p}(\partial\Omega)$. The proof of the well-posedness is similar to the one in (A) (see Proposition 2.1) but one needs to use various embeddings for Sobolev spaces.

In Theorem 4.3 we assume $q_1, q_2 \in L^{n+\varepsilon}(\Omega)$ instead of $q_1, q_2 \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ as in Theorems 4.1 and 4.2 and here is an explanation. Since we assume (4.1), we then get for the solutions of the m-th order linearization that

$$\int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, d\mu = 0.$$

We would then like to integrate by parts against this measure μ. From Lemma 5.1 in (B) we have that

$$0 = \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, d\mu = \int_{\Omega} \Delta(w_1 - w_2)\Psi\, dx.$$

Here $\Psi \in L^{(n+\varepsilon)'}(\Omega)$, where $(n+\varepsilon)'$ is the dual exponent of $n+\varepsilon$, solves

$$\begin{cases} \Delta\Psi = 0, & \text{in } \Omega \\ \Psi = \mu, & \text{on } \partial\Omega. \end{cases}$$

To use Lemma 5.1 we need that $(n+\varepsilon)' \leq \frac{n}{n-1}$. For $\frac{n}{2}+\varepsilon$ and its dual exponent $\left(\frac{n}{2}+\varepsilon\right)'$ this is not true for every $\varepsilon > 0$.

5. Inverse problems for semilinear elliptic PDE with a general nonlinearity $a(x,u)$

In article (D) we study an inverse problem for the semilinear elliptic partial differential equation

$$\Delta u + a(x,u) = 0 \quad \text{in } \Omega. \tag{5.1}$$

Our aim is to recover the nonlinearity a from boundary measurements corresponding to solutions of (5.1).

As mentioned in Section 2, in previous works similar inverse problems have been studied under various assumptions. In (D) we remove some of the assumptions made before. The techniques can be divided into two categories: using the first linearization only and using the higher order linearization method discussed in Section 2.

When using the first linearization, the most general result known to us is that of [Sun10] which had the assumption $\partial_u a(x,u) \leq 0$. This assumption guarantees that for example the maximum principle holds and thus one has well-posedness for the boundary value problem

$$\begin{cases} \Delta u + a(x,u) = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases}$$

Similar assumptions have been made when using the higher order linearization method. In previous works (e.g. [FO20], [LLLS21b]) the assumptions have been that

(5.2) $\quad a(x,0) = 0$

(5.3) $\quad 0$ is not a Dirichlet eigenvalue of the linearized equation $\Delta u + \partial_u a(x,0)u = 0$.

The first one guarantees that $u \equiv 0$ is a solution and the second one gives that the boundary value problem is well-posed for sufficiently small boundary values.

With the above mentioned assumptions one would have a well defined DN map for small data. Without these assumptions the boundary value problem may not be well-posed but one may still work with the *Cauchy data set*:

$$C_a := \{(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) : u \in C^{2,\alpha}(\overline{\Omega}) \text{ solves } \Delta u + a(x,u) = 0\}.$$

Even without the above mentioned assumptions, we can still construct solutions to (5.1) close to some known solution w to (5.1). This is shown in Lemma 2.4 in (D) using a Banach fixed point argument and the implicit function theorem together with the fact that Fredholm theory guarantees the existence of a unique solution to the linearized equation that is L^2-orthogonal to a finite dimensional set.

The fact that we can construct solutions near some fixed solution w of (5.1) leads us to look at local Cauchy data sets such as

$$C_a^{w,\delta} := \{(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) \,:\, u \in C^{2,\alpha}(\overline{\Omega}) \text{ solves } \Delta u + a(x,u) = 0 \text{ and } \|w-u\|_{C^{2,\alpha}(\overline{\Omega})} \leq \delta\}$$

for some $\delta > 0$.

Using the first order linearization as our main tool we prove the following:

Theorem 5.1 ((D), Theorem 1.2)**.** *Let $a_1, a_2 \in C^3(\mathbb{R}^n, C^{1,\alpha}(\overline{\Omega}))$, and let $w_1 \in C^{2,\alpha}(\overline{\Omega})$ solve $\Delta w_1 + a_1(x, w_1) = 0$ in Ω. If*

$$C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$$

for some $\delta, C > 0$, then there is $\varepsilon > 0$ such that

$$a_1(x, w_1(x) + \lambda) = T_\varphi a_2(x, w_1(x) + \lambda)$$

whenever $x \in \overline{\Omega}$ and $|\lambda| \leq \varepsilon$. Here $\varphi := w_1 - w_2$ where $w_2 \in C^{2,\alpha}(\overline{\Omega})$ is the unique solution of $\Delta w_2 + a_2(x, w_2) = 0$ in Ω with $w_1|_{\partial\Omega} = w_2|_{\partial\Omega}$ and $\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}$.

Before sketching the proof, let us discuss a little bit about the result and its assumptions. First of all the mapping T_φ was already introduced in Section 2 and is defined as

$$T_\varphi a(x,u) := \Delta\varphi(x) + a(x, u + \varphi(x)).$$

To recall, there is a gauge invariance described by T_φ when trying to recover the nonlinearity a. The gauge invariance disappears if the equations $\Delta u + a_j(x,u) = 0$ have a common solution $w = w_1 = w_2$. Comparing this result with previous ones, we actually manage to recover the nonlinearity (up to gauge) in some neighborhood of w_1 without a sign condition on $\partial_u a_j(x,u)$ and without assuming (5.2), (5.3). We still assume that the equation has at least one solution, but we do not require this solution to be the zero solution. The assumption $C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$ on the Cauchy data sets is there to ensure that when we construct a solution to $\Delta u + a_2(x,u) = 0$ then that solution will be close to w_2 and that the constructed solutions $u_{j,v}$ to $\Delta u + a_j(x,u) = 0$, $j = 1, 2$, are parametrized smoothly by the same v solving $(\Delta + \partial_u a_1(x,u))v = 0$.

Here is a sketch of the proof. First we show by a Banach fixed point argument together with the implicit function theorem (similar to the one in (A) and [OSSU20, Section 6] for example) that there are solutions $u_v = w + v + O\left(||v||^2\right)$ to $\Delta u + a(x,u) = 0$ where w and v solve $\Delta w + a(x,w) = 0$ and $\Delta v + \partial_u a(x,w)v = 0$, respectively. This is Lemma 2.4 in (D) and it gives a bijective map $S_{a,w} : v \mapsto u_v$ with $DS_{a,w}(0) = \mathrm{Id}$. More importantly the solution u_v depends smoothly on v.

Now we construct a solution $u_{1,v}$ for a_1 and use the Cauchy data inclusion to obtain a solution $u_{2,v}$ for a_2 with the same Cauchy data as $u_{1,v}$. Since we constructed $u_{1,v}$ with Lemma 2.4 in (D), we know that it depends smoothly on v solving $\Delta v + \partial_u a_1(x, w_1)v = 0$. Apriori we do not know if $u_{2,v}$ depends smoothly on v also. To show this, we prove quantitative estimates to show that bounded solutions to (5.1) close to a given solution depend smoothly on their Cauchy data. Using this, we can show that $u_{2,v}$ depends smoothly on v.

Let us then look at $\varphi_v = u_{2,v} - u_{1,v}$. Now

$$\Delta\varphi_v = a_1(x, u_{1,v}) - a_2(x, u_{1,v} - \varphi_v)$$

or in other words

$$a_1(x, u_{1,v}(x)) = T_{\varphi_v} a_2(x, u_{1,v}(x)).$$

We would like to show that actually $\varphi_v = \varphi_0 = w_2 - w_1$. For this, we look at the first linearization of (5.1). Because if we can show that $\partial_u a_1(x, u_{1,v}) = \partial_u a_2(x, u_{2,v})$ for any v small enough, then the derivative of φ_v with respect to v will solve a linear elliptic equation with zero Cauchy data. Thus we would have the claim.

Now let v, h be solutions to $\Delta v + \partial_u a_1(x, w_1)v = 0$ with v sufficiently small, define $v_t = v + th$ and construct solutions u_{j,v_t} as above. We then differentiate in t the equations $\Delta u_{j,v_t} + a(x, u_{j,v_t}) = 0$, subtract them and integrate against any solution v_2 of $\Delta v + \partial_u a_2(x, w_2)v = 0$ to obtain an integral identity:

$$\int_\Omega (\partial_u a_1(x, u_{1,v}) - \partial_u a_2(x, u_{2,v}))\partial_t u_{1,v_t}|_{t=0} v_2 \, dx = 0.$$

Here $\partial_t u_{1,v_t}|_{t=0} = DS_{a,w}(v)h$. Now Lemma 2.5 in (D) shows that for small v the map $DS_{a,w}(v)$ is an isomorphism between the solutions of $\Delta h+\partial_u a_1(x, w_1)h = 0$ and $\Delta h+\partial_u a_1(x, S_{a,w}(v))h = 0$. Since $S_{a,w}(v) = u_{1,v}$, any solution v_1 to $\Delta h + \partial_u a_1(x, u_{1,v})h = 0$ can be written as $v_1 = DS_{a,w}(v)h$ for a suitable h. Thus we get an integral identity

$$\int_\Omega (\partial_u a_1(x, u_{1,v}) - \partial_u a_2(x, u_{2,v}))v_1 v_2 \, dx = 0.$$

Now it follows from the density of products of solutions as in the standard Calderón problem (see [SU87] for $n \geq 3$ and [BU02], [BTW20] for $n = 2$) that $\partial_u a_1(x, u_{1,v}) = \partial_u a_2(x, u_{2,v})$.

What is left is to verify the existence of some $\varepsilon > 0$ so that

$$a_1(x, w_1(x) + \lambda) = T_\varphi a_2(x, w_1(x) + \lambda)$$

whenever $x \in \overline{\Omega}$ and $|\lambda| \leq \varepsilon$. We begin by fixing $x_0 \in \overline{\Omega}$. Since $u_{1,v} = w_1 + v + O(\|v\|^2)$ and by Runge approximation we can generate solutions v with $v(x_0) \neq 0$, we get the desired result by varying v.

For the higher order linearization method we show that it works without the assumptions (5.2) and (5.3) and one can prove results such as

Theorem 5.2 ((D), Theorem 1.3)**.** *Let $a_1, a_2 \in C^{k+1}(\mathbb{R}^n, C^{1,\alpha}(\overline{\Omega}))$ with $k \geq 2$, let $w_1 \in C^{2,\alpha}(\overline{\Omega})$ solve $\Delta w_1 + a_1(x, w_1) = 0$ in Ω, and suppose that*

$$C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$$

for some $\delta, C > 0$. Let $w_2 \in C^{2,\alpha}(\overline{\Omega})$ be the unique solution of $\Delta w_2 + a_2(x, w_2) = 0$ in Ω with $w_1|_{\partial\Omega} = w_2|_{\partial\Omega}$ and $\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}$. Assume further that

$$\partial_u^l a_1(x, w_1) = \partial_u^l a_2(x, w_2), \qquad 1 \leq l \leq k-1. \tag{5.4}$$

Then

$$\int_\Omega (\partial_u^k a_1(x, w_1) - \partial_u^k a_2(x, w_2))v_1 \dots v_{k+1} \, dx = 0$$

for any v_j solving the linear equation $\Delta v_j + \partial_u a_1(x, w_1)v_j = 0$ in Ω.

The proof uses similar arguments as described in Sections 2, 3 and 4. Mainly we differentiate k times the equations $\Delta u_{j,v}+a(x, u_{j,v}) = 0$ with respect to v, subtract the k-th order linearizations, use (5.4) and integrate against a special solution to the first linearization. This results in the integral identity in Theorem 5.2.

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ORIGINAL PAPERS

(A)

AN INVERSE PROBLEM FOR THE MINIMAL SURFACE EQUATION

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ABSTRACT

We use the method of higher order linearization to study an inverse boundary value problem for the minimal surface equation on a Riemannian manifold $(\mathbb{R}^n, g)$, where the metric g is conformally Euclidean. In particular we show that with the knowledge of Dirichlet-to-Neumann map associated to the minimal surface equation, one can determine the Taylor series of the conformal factor $c(x)$ at $x_n = 0$ up to a multiplicative constant. We show this both in the full data case and in some partial data cases.

1. Introduction

This article focuses on an inverse problem for the minimal surface equation (MSE), which is a quasilinear elliptic PDE. In particular we consider MSE on a manifold $(\mathbb{R}^n, g), n \geq 3$, where $g_{ij}(x) = c(x)\delta_{ij}$, $1 \leq i, j \leq n$, with $c \in C^\infty(\mathbb{R}^n)$, $c(x) > 0$ for all $x \in \mathbb{R}^n$, that is the metric is conformally Euclidean. The aim is to use the method of higher order linearization to recover information about the conformal factor c from boundary measurements. This method, which uses the nonlinearity of the partial differential equation as a tool, was first introduced in [18] in the case of a nonlinear wave equation and was further developed in [7,22] for nonlinear elliptic equations.

The novelty of this work is that we use higher order linearization in the case of MSE. For a sufficiently smooth function $u : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$, Ω a bounded domain with C^∞ boundary, consider $\mathrm{Graph}_u := \{(x', u(x')) : x' \in \Omega\} \subset \mathbb{R}^n$. If $c \equiv 1$ we would call Graph_u a minimal surface if and only if the function u solves the Euclidean MSE

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \quad \text{in } \Omega.$$

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Define then a function $F:\mathbb{R}^{n^2}\to\mathbb{R}$,

$$F(x',u,p,P) := -\sum_{i=1}^{n-1} P_{ii} - \frac{n-1}{2c(x',u)}\left(\sum_{i=1}^{n-1} p_i\partial_{x_i}c(x',u) - \partial_{x_n}c(x',u)\right) + \frac{1}{1+|p|^2}\sum_{i,j=1}^{n-1} P_{ij}p_ip_j, \tag{1.1}$$

where $p=(p_1,\dots,p_{n-1})\in\mathbb{R}^{n-1}, P=(P_{ij})$ is an $(n-1)\times(n-1)$ matrix and $x'\in\mathbb{R}^{n-1}$. With the conformally Euclidean metric, MSE takes the form

$$\begin{aligned} F(x',u,\nabla u,\nabla^2 u) &= -\Delta u - \frac{n-1}{2c}\left(\nabla_{x'}c\cdot\nabla u - \partial_{x_n}c\right) + \frac{\nabla u^T\nabla^2 u\nabla u}{1+|\nabla u|^2} \\ &= -\operatorname{div}_{g_{n-1}}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right) + \frac{(n-1)\partial_{x_n}c}{2c(1+|\nabla u|^2)^{1/2}} \\ &= 0. \end{aligned} \tag{1.2}$$

for $x'\in\Omega$. Here $\operatorname{div}_{g_{n-1}}(a)=\sum_{i=1}^{n-1}\left(\partial_{x_i}a_i+\sum_{j=1}^{n-1}a_j\Gamma^i_{ij}\right)$ is the Riemannian divergence with respect to the first $n-1$ variables and Γ^i_{ij} is the Christoffel symbol corresponding to the metric g. The derivation of this equation is done in Section 3.

In this work we consider a boundary value problem

$$\begin{cases} F(x',u,\nabla u,\nabla^2 u)=0 & \text{in } \Omega \\ u=f, & \text{on } \partial\Omega, \end{cases}$$

and prove that it is well-posed (Section 2) for a certain class of small boundary values f. To be more precise, we show that there is $\delta>0$ such that whenever $f\in C^s(\partial\Omega)$, $s>3$, $s\notin\mathbb{N}$, with $\|f\|_{C^s(\partial\Omega)}\le\delta$, there exists a unique small solution $u\in C^s(\bar\Omega)$ with sufficiently small norm. Let $U_\delta:=\{h\in C^s(\partial\Omega):\|h\|_{C^s(\partial\Omega)}<\delta\}$. Thus the Dirichlet-to-Neumann (DN) map can now be defined for these small solutions as

$$\Lambda_c: U_\delta\to C^{s-1}(\partial\Omega), \quad f\mapsto \partial_\nu u_f\big|_{\partial\Omega}. \tag{1.3}$$

Here $C^s=C^{k,\alpha}$, $k\in\mathbb{Z}$, $0<\alpha<1$, is the standard Hölder space (see for example [6, Section 5.1]) and $\partial_\nu u_f$ is the Euclidean boundary normal derivative. One can think of the normal derivative on the boundary as tension on the boundary caused by the minimal surface. From the knowledge of the DN map, can we recover information about the metric g?

It is worth noting that there is a small gauge invariance for Eq. (1.2) and thus for the DN map. That is, if you instead of c put λc, $\lambda\neq 0$, into (1.2), the equation stays the same. Thus also the DN maps Λ_c and $\Lambda_{\lambda c}$ are the same.

We also consider partial data cases, that is, if we have knowledge of the DN map in an open subset Γ of the boundary $\partial\Omega$. In this case the partial DN map is defined for $f\in U_\delta$, $\operatorname{spt}(f)\subset\Gamma$, as

$$\Lambda_c^\Gamma: U_\delta\to C^{s-1}(\partial\Omega), \quad f\mapsto \partial_\nu u_f\big|_{\Gamma}. \tag{1.4}$$

Can we recover information about the metric g if we have knowledge of this partial DN map?

These are our inverse problems for the MSE and our main result gives the following answers. Before stating it, we denote by F^j the function F with c replaced by c_j.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^{n-1}$, $n\ge 3$, be a bounded domain with C^∞ boundary, $(\mathbb{R}^n,g_1)$, $(\mathbb{R}^n,g_2)$ be two Riemannian manifolds with $(g_j)_{ik}(x)=c_j(x)\delta_{ik}$, where $c_j\in C^\infty(\mathbb{R}^n)$, $c_j(x)>0$ for $j=1,2$ and for all $x\in\mathbb{R}^n$. Assume that $\partial_{x_n}c_j(x',0)=\partial^2_{x_n}c_j(x',0)=0$ for $x'\in\Omega$. We have four cases:*

(1) Let $n > 3$ and Λ_{c_j} be the DN maps associated to

$$\begin{cases} F^j(x', u, \nabla u, \nabla^2 u) = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

$j = 1, 2$, and assume that

$$\Lambda_{c_1}(f) = \Lambda_{c_2}(f)$$

for all $f \in U_\delta := \{h \in C^s(\partial\Omega) : \|h\|_{C^s(\partial\Omega)} < \delta\}$, where $\delta > 0$ is sufficiently small.

(2) Assume either that

(a) $n = 3$, $\Gamma \subset \partial\Omega$ be open and $\Gamma \neq \emptyset$ or

(b) $n > 3$, $\Omega \subset \{x_{n-1} > 0\}$, $\Gamma \subset \partial\Omega$ be open, $\Gamma \neq \emptyset$ and that $\partial\Omega \setminus \Gamma \subset \{x_{n-1} = 0\}$ or

(c) $n > 3$, Ω is a strict subset of some ball $B \subset \mathbb{R}^{n-1}$, $\Gamma \subset \partial\Omega$ be open, $\Gamma \neq \emptyset$ and that $\partial\Omega \setminus \Gamma \subset \partial B$.

In addition assume that

$$\Lambda^\Gamma_{c_1}(f) = \Lambda^\Gamma_{c_2}(f)$$

for all $f \in U_\delta$, $\operatorname{spt}(f) \subset \Gamma$, where $\delta > 0$ is sufficiently small and $\Lambda^\Gamma_{c_j}$ are the partial DN maps associated to (1.5) for $j = 1, 2$.

Then in the cases (1) and (2) we have for $\lambda \neq 0$

$$\partial^m_{x_n} c_1(x', 0) = \lambda \partial^m_{x_n} c_2(x', 0), \quad \text{in } \Omega,\ m \geq 0.$$

The assumption $\partial_{x_n} c_j(x', 0) = 0$ is needed in order for $u \equiv 0$ to be a solution to (1.5), and this is used to prove the well-posedness. The condition $\partial^2_{x_n} c_j(x', 0) = 0$ is assumed in order for the method to work and it is not known if it could be removed.

As an immediate corollary of Theorem 1.1 we get the following.

Corollary 1.2. *Assume the conditions in Theorem 1.1 and assume additionally that c_j are real analytic with respect to x_n. Then for $\lambda \neq 0$ we have*

$$c_1(x) = \lambda c_2(x), \quad x \in \Omega \times \mathbb{R}.$$

In Section 5 we give a full proof of Theorem 1.1 and as mentioned, it will use higher order linearization together with complex geometric optics (CGO) solutions. In the proof we first linearize (1.5) at $u \equiv 0$ and the DN map at $f = 0$. We see that the linearization of (1.5) correspond to a conductivity equation where the conductivity is $c_j(x', 0)$. The first linearization of the DN map maps a boundary value f to $\partial_\nu v|_{\partial\Omega}$ where v is a solution to the conductivity equation. We will show that $c_j(x', 0)$ can be recovered up to a multiplicative constant with the knowledge of this (partial) DN map with the help of boundary determination for a first order perturbation of the Laplacian from [2] ([10] for $n = 3$ and [12,29] for $n > 3$). In the full data case the higher order linearizations lead to an integral equality

$$\int_\Omega \left(\partial^{m+4}_{x_n} c_1(x', 0) - \lambda \partial^{m+4}_{x_n} c_2(x', 0)\right) \prod_{N=1}^{m+3} v^{l_N}\, dx' = 0.$$

where v^{l_N} are solutions to the first linearization. For the partial data cases, we need a special solution $v^{(0)}$ which is positive in Ω and vanishes on $\partial\Omega \setminus \Gamma$. With the help of this function we get the integral identity

$$\int_\Omega \left(\partial^{m+4}_{x_n} c_1(x', 0) - \lambda \partial^{m+4}_{x_n} c_2(x', 0)\right) v^{(0)} \prod_{N=1}^{m+3} v^{l_N}\, dx' = 0.$$

Again v^{l_N} are solutions to the first linearization. In both cases, choosing two of v^{l_N} to be real or imaginary parts of CGO solutions and the rest equal to 1 we get that $\partial_{x_n}^{m+4} c_1(x',0) = \lambda \partial_{x_n}^{m+4} c_2(x',0)$ (for $n = 3$ [3], for $n > 3$ [29]).

This method has received a lot of attention in various situations lately. Linearization has already been used in a parabolic case in [11] where the author shows that the first linearization of the nonlinear DN map is the DN map of a linear equation. Thus one can use the theory of inverse problems for linear equations. Also nonlinear elliptic cases have been studied, for example in [13,28]. As mentioned above, the method of higher order linearization was first used in [18] for a nonlinear wave equation. After that there were two simultaneously published articles [7,22] in which higher order linearization was introduced to nonlinear elliptic equations of the type $\Delta u + a(x,u) = 0$. The important thing in this method was that it used the nonlinearity as a tool. In [16,23] the method was further developed for the case $\Delta u + a(x,u) = 0$ in inverse problems with partial data. See also [24,27] for more results on the special case of a power type nonlinearity.

After these, there have been several articles using this method for different nonlinear elliptic equations. Different cases of nonlinear conductivity equations have had a treatment in [4,14]. This method has also been used in the case of a nonlinear magnetic Schrödinger equation [21] and in inverse transport and diffusion problems [20]. See also [17] for a semilinear elliptic equation with gradient nonlinearities and [19] for the case of fractional semilinear elliptic equations.

There are also works in inverse problems that have considered the minimal surface equation. The Euclidean case has had a treatment in [25] where the authors consider a quasilinear conductivity depending on a function u and its gradient.

Also while writing this article we have learned that Cătălin I. Cârstea, Matti Lassas, Tony Liimatainen and Lauri Oksanen are working on an inverse problem involving minimal surface equation on a Riemannian manifold in their upcoming preprint [5]. They simultaneously and independently prove a result similar to Theorem 1.1. In their work it is shown that from the knowledge of the DN map of the minimal surface equation it is possible to determine a 2-dimensional Riemannian manifold (Σ, g). We agreed with them to publish our preprints at the same time on the same preprint server.

This article is organized as follows. In Section 2 we prove well-posedness for a general nonlinear boundary value problem and we describe the first and second order linearizations for the general case. Section 3 is dedicated to the derivation of the minimal surface equation on a manifold with conformally Euclidean metric. Section 4 consists of describing the setting for Theorem 1.1 and then calculating the first and second order linearizations in this setting. Finally, we will use higher order linearization to prove Theorem 1.1 in Section 5.

2. Well-posedness and linearizations

In this section, we consider general equations $F(x,u,\nabla u,\nabla^2 u) = 0$ and in later sections apply these methods. Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded domain with C^∞ boundary and let $F\colon \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$, be a C^∞ function. Consider next the boundary value problem

$$\begin{cases} F(x,u,\nabla u,\nabla^2 u) = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f \in C^s(\partial\Omega)$ and $\nabla u, \nabla^2 u$ denote the gradient and Hessian of u, respectively. In addition let $F(x,0,0,0) = 0$ which guarantees that $u \equiv 0$ is a solution to (2.1) with $f = 0$.

Next we prove well-posedness for (2.1) using the implicit function theorem on Banach spaces [26, Theorem 10.6 and Remark 10.5]. In what follows, we denote for $m \times n$ matrices $A = (a_{ij}), B = (b_{ij})$ the matrix product

$$A : B = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

and $\nabla_P F$ is the matrix with elements $\partial_{P_{ij}} F$. Also a linear differential operator $Lu = A(x) : \nabla^2 u + b(x) \cdot \nabla u + c(x)u$ is strictly elliptic [8] in Ω if for some constants $\lambda, \Lambda > 0$ we have

$$\lambda|\xi|^2 \leq \xi^T A\xi \leq \Lambda|\xi|^2, \quad x \in \bar{\Omega},$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Here A is a symmetric $n \times n$ matrix.

Proposition 2.1. *Let $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ be a C^∞ mapping with $F(x, 0, 0, 0) = 0$. Furthermore assume that the map*

$$v \mapsto L(v) := \partial_u F(x, 0, 0, 0)v + \nabla F(x, 0, 0, 0) \cdot \nabla v + \nabla_P F(x, 0, 0, 0) : \nabla^2 v$$

is injective on $H_0^1(\Omega)$ and that the operator L is strictly elliptic. Let $s > 3, s \notin \mathbb{N}$. Then there exists $C, \delta > 0$ such that for any

$$f \in U_\delta := \{h \in C^s(\partial\Omega) : \|h\|_{C^s(\partial\Omega)} < \delta\}$$

the boundary value problem

$$\begin{cases} F(x, u, \nabla u, \nabla^2 u) = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases}$$

has a unique small solution $u = u_f$ which satisfies

$$\|u\|_{C^s(\bar{\Omega})} \leq C\|f\|_{C^s(\partial\Omega)}.$$

Moreover the following mappings are C^∞ maps

$$\begin{aligned} &S: U_\delta \to C^s(\bar{\Omega}), && f \mapsto u_f, \\ &\Lambda: U_\delta \to C^{s-1}(\partial\Omega), && f \mapsto \partial_\nu u_f|_{\partial\Omega}. \end{aligned}$$

Proof. Let $X = C^s(\partial\Omega), Y = C^s(\bar{\Omega}), Z = C^{s-2}(\bar{\Omega}) \times C^s(\partial\Omega)$ and

$$T: X \times Y \to Z, \quad T(f, u) = \big(F(x, u, \nabla u, \nabla^2 u), u|_{\partial\Omega} - f\big)$$

Since $u|_{\partial\Omega}, f \in C^s(\partial\Omega)$, $u \in C^s(\bar{\Omega})$ and $F \in C^\infty$, the map T really has this mapping property.

Next we show that the map $u \mapsto F(x, u, \nabla u, \nabla^2 u)$ is a C^∞ map $C^s(\bar{\Omega}) \to C^{s-2}(\bar{\Omega})$. This is done by using a Taylor expansion. Write $\lambda = (z, p, P) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ and expand $F(x, \cdot)$ at $\mu \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$:

$$F(x, \lambda + \mu) = \sum_{|\alpha| \leq k} \frac{D_\mu^\alpha F(x, \lambda)}{\alpha!}\mu^\alpha + \sum_{|\beta|=k+1} R_\beta(\lambda + \mu)\mu^\beta,$$

where

$$R_\beta(\lambda + \mu) = \frac{|\beta|}{\beta!}\int_0^1 (1-t)^{|\beta|-1} D_\mu^\beta F(x, \lambda + t\mu)\, dt.$$

Now let $u \in C^s(\bar{\Omega})$ be fixed, $\lambda = (u, \nabla u, \nabla^2 u)$ and let $\mu = (h, \nabla h, \nabla^2 h), h \in C^s(\bar{\Omega})$ be such that $\|\mu\|_{C^{1,\alpha}(\bar{\Omega})} \leq 1$. It is enough to show that the map $u \mapsto D_\mu^\alpha F(x, u, \nabla u, \nabla^2 u)$ is continuous for all α and

$$R_\beta(\lambda + \mu) = o(\mu^k) \quad \text{in} \quad C^s(\bar{\Omega}).$$

Firstly, since the composition of a C^∞ function F with a C^{s-2} function is again a C^{s-2} function [9, Theorem A.8], we have the continuity. The space $C^s(\bar{\Omega})$ is an algebra under pointwise multiplication

[9, Theorem A.7], and thus

$$\begin{aligned}\|R_\beta(\lambda+\mu)\mu^\beta\|_{C^s(\bar\Omega)} &\le C\left(\|R_\beta(\lambda+\mu)\|_{C(\bar\Omega)}\|\mu^\beta\|_{C^s(\bar\Omega)}+\|R_\beta(\lambda+\mu)\|_{C^s(\bar\Omega)}\|\mu^\beta\|_{C(\bar\Omega)}\right)\\ &\le C\|R_\beta(\lambda+\mu)\|_{C^s(\bar\Omega)}\|\mu\|^{|\beta|}_{C^s(\bar\Omega)}\\ &\le C\|\mu\|^{|\beta|}_{C^s(\bar\Omega)}\frac{|\beta|}{\beta!}\int_0^1(1-t)^{|\beta|-1}\|D^\beta_\mu F(x,\lambda+t\mu)\|_{C^s(\bar\Omega)}\,dt\\ &\le C_{F,u}\|\mu\|^{|\beta|}_{C^s(\bar\Omega)}\frac{|\beta|}{\beta!}\int_0^1(1-t)^{|\beta|-1}\,dt\end{aligned}$$

where $\|D^\beta_\mu F(x,\lambda+t\mu)\|_{C^s(\bar\Omega)}$ is uniformly bounded in $t\in(0,1)$ and the bounding constant may depend on u and F. This is due to F being a C^∞ function and that $u\in C^s(\bar\Omega)$. Now the remainder satisfies

$$\Big\|\sum_{|\beta|=k+1}R_\beta(\lambda+\mu)\mu^\beta\Big\|_{C^s(\bar\Omega)}\le C\|(h,\nabla h,\nabla^2h)\|^{k+1}_{C^s(\bar\Omega)}$$

and hence the map $u\mapsto F(x,u,\nabla u,\nabla^2u)$ is a C^∞ map $C^s(\bar\Omega)\to\mathbb{R}$.

By the assumption $F(x,0,0,0)=0$ we have $T(0,0)=0$. Also $D_uT(0,0)$ is linear and

$$D_uT(0,0)v=(\partial_uF(x,0,0,0)v+\nabla_pF(x,0,0,0)\cdot\nabla v+\nabla_PF(x,0,0,0):\nabla^2v,v|_{\partial\Omega}).$$

The mapping $v\mapsto L(v)$ is injective and $v\equiv0$ is a solution to

$$\begin{cases}\partial_uF(x,0,0,0)v+\nabla_pF(x,0,0,0)\cdot\nabla v+\nabla_PF(x,0,0,0):\nabla^2v=H, & \text{in } \Omega\\ v=g, & \text{on } \partial\Omega,\end{cases}\tag{2.2}$$

when $H=g=0$. Using Fredholm alternative [8, Theorem 6.15] the boundary value problem (2.2) has a unique solution for all H and g. Thus $D_uT(0,0)$ is surjective.

Then by the implicit function theorem there exist $\delta>0$ and $U_\delta:=B(0,\delta)\subset X=C^s(\partial\Omega)$ and a C^∞ map $S:U_\delta\to Y=C^s(\bar\Omega)$ such that $T(f,S(f))=0$. Also, for small enough $f\in U_\delta$ (not necessarily the same δ) and $u_f\in C^s(\bar\Omega)$, $S(f)=u_f$ is the only solution of $T(f,u_f)=0$. Moreover, since S is Lipschitz continuous and $S(0)=0$, for $u=S(f)$ we have

$$\|u\|_{C^s(\bar\Omega)}\le C\|f\|_{C^s(\partial\Omega)}.$$

Also the mapping Λ is a well defined C^∞ map between U_δ and $C^{s-1}(\partial\Omega)$ since taking a normal derivative is a linear map from $C^s(\Omega)$ to $C^{s-1}(\partial\Omega)$. □

In order to use the method of higher order linearization, we calculate formally the first and second order linearizations of (2.1) and the corresponding DN map. This formal looking calculation can be justified as in [22].

Let us begin by assuming that for

$$\begin{cases}F^j(x,u,\nabla u,\nabla^2u)=0, & \text{in } \Omega\\ u=f, & \text{on } \partial\Omega,\end{cases}\tag{2.3}$$

$j=1,2$, we have $\Lambda_{F^1}(f)=\Lambda_{F^2}(f)$ for all $f\in C^s(\partial\Omega)$ with $\|f\|_{C^s(\partial\Omega)}\le\delta$, for $\delta>0$ sufficiently small. In order to find the linearizations, let $\varepsilon_1,\dots,\varepsilon_k$ be sufficiently small numbers and $f_1,\dots,f_k\in C^s(\partial\Omega)$. Let $u_j(x,\varepsilon_1,\dots,\varepsilon_k)$ be the unique small solution to

$$\begin{cases}F^j(x,u_j,\nabla u_j,\nabla^2u_j)=0, & \text{in } \Omega\\ u_j=\sum_{m=1}^k\varepsilon_mf_m, & \text{on } \partial\Omega,\end{cases}\tag{2.4}$$

for $j = 1, 2$. Differentiate this with respect to ε_l, $l \in \{1, \dots, k\}$, and evaluate at $\varepsilon_1 = \cdots = \varepsilon_k = 0$ to get

$$\begin{cases} \partial_u F^j(x,0,0,0)v_j^l + \nabla_p F^j(x,0,0,0) \cdot \nabla v_j^l + \nabla_P F^j(x,0,0,0) : \nabla^2 v_j^l = 0, & \text{in } \Omega \\ v_j^l = f_l, & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $v_j^l := \partial_{\varepsilon_l} u_j(x, \varepsilon_1, \dots, \varepsilon_k)\big|_{\varepsilon_1=\cdots=\varepsilon_k=0}$. The boundary value problem (2.5) has a unique solution if we assume that the map

$$v \mapsto L(v) = \partial_u F^j(x,0,0,0)v + \nabla_p F^j(x,0,0,0) \cdot \nabla v + \nabla_P F^j(x,0,0,0) : \nabla^2 v$$

is injective on $H_0^1(\Omega)$ and assume strict ellipticity of the operator L. At this point, we would like to see what exactly is the first linearization and see if some information can be recovered about the coefficients $\partial_u F^j(x,0,0,0)$, $\nabla_p F^j(x,0,0,0)$, $\nabla_P F^j(x,0,0,0)$ from the knowledge of the DN maps corresponding to (2.3) for $j = 1, 2$. What actually can be recovered depends on the equation at hand.

Let us next differentiate (2.4) first with respect to ε_l and then with respect to ε_a, $a \neq l$:

$$\begin{aligned} I_j &:= \partial^2_{\varepsilon_a \varepsilon_l} F^j(x, u_j, \nabla u_j, \nabla^2 u_j) \\ &= \partial_{\varepsilon_a} \left(\partial_u F^j(x, u_j, \nabla u_j, \nabla^2 u_j) \partial_{\varepsilon_l} u_j \right) \\ &+ \partial_{\varepsilon_a} \left(\sum_{i=1}^n \partial_{p_i} F^j(x, u_j, \nabla u_j, \nabla^2 u_j) \partial_{x_i} \partial_{\varepsilon_l} u_j \right) \\ &+ \partial_{\varepsilon_a} \left(\sum_{j,k=1}^n \partial_{P_{jk}} F^j(x, u_j, \nabla u_j, \nabla^2 u_j) \partial^2_{x_j x_k} \partial_{\varepsilon_l} u_j \right) \\ &:= I_{j,1} + I_{j,2} + I_{j,3}. \end{aligned}$$

Then we expand these one by one:

$$\begin{aligned} I_{j,1} &= \partial_u F^j \partial^2_{\varepsilon_a \varepsilon_l} u_j + \partial_u^2 F^j \partial_{\varepsilon_a} u_j \partial_{\varepsilon_l} u_j + \sum_{i=1}^n \partial_{p_i} \partial_u F^j \partial_{x_i} \partial_{\varepsilon_a} u_j \partial_{\varepsilon_l} u_j \\ &+ \sum_{j,k=1}^n \partial_{P_{jk}} \partial_u F^j \partial^2_{x_j x_k} \partial_{\varepsilon_a} u_j \partial_{\varepsilon_l} u_j, \\ I_{j,2} &= \sum_{i=1}^n \Bigg(\partial_{p_i} F^j \partial_{x_i} \partial^2_{\varepsilon_a \varepsilon_l} u_j + \partial_u \partial_{p_i} F^j \partial_{\varepsilon_a} u_j \partial_{x_i} \partial_{\varepsilon_l} u_j \\ &+ \sum_{r=1}^n \partial_{p_r} \partial_{p_i} F^j \partial_{x_r} \partial_{\varepsilon_a} u_j \partial_{x_i} \partial_{\varepsilon_l} u_j + \sum_{j,k=1}^n \partial_{P_{jk}} \partial_{p_i} F^j \partial^2_{x_j x_k} \partial_{\varepsilon_a} u_j \partial_{x_i} \partial_{\varepsilon_l} u_j \Bigg), \\ I_{j,3} &= \sum_{j,k=1}^n \Bigg(\partial_{P_{jk}} F^j \partial^2_{x_j x_k} \partial^2_{\varepsilon_a \varepsilon_l} u_j + \partial_u \partial_{P_{jk}} F \partial_{\varepsilon_a} u_j \partial^2_{x_j x_k} \partial_{\varepsilon_l} u_j \\ &+ \sum_{i=1}^n \partial_{p_i} \partial_{P_{jk}} F^j \partial_{x_i} \partial_{\varepsilon_a} u_j \partial^2_{x_j x_k} \partial_{\varepsilon_l} u_j + \sum_{r,t=1}^n \partial_{P_{rt}} \partial_{P_{jk}} F^j \partial^2_{x_r x_t} \partial_{\varepsilon_a} u_j \partial^2_{x_j x_k} \partial_{\varepsilon_l} u_j \Bigg). \end{aligned}$$

Evaluate I_j at $\varepsilon_1 = \cdots = \varepsilon_k = 0$ and denote $w_j^{(al)} = (\partial^2_{\varepsilon_a \varepsilon_l} u_j)(x, \varepsilon_1, \dots, \varepsilon_k)|_{\varepsilon_1=\cdots=\varepsilon_k=0}$ to have

$$\begin{aligned} I_j &= \partial_u F^j(x,0,0,0) w_j^{(al)} + \partial_u^2 F^j(x,0,0,0) v^l v^a \\ &+ \left(\left(\nabla_p (\partial_u F^j) \right)(x,0,0,0) \cdot \nabla v^a + \left(\nabla_P (\partial_u F^j) \right)(x,0,0,0) : \nabla^2 v^a \right) v^l \\ &+ \nabla_p F^j(x,0,0,0) \cdot \nabla w_j^{(al)} + \left(\nabla_p (\partial_u F^j) \right)(x,0,0,0) \cdot \nabla v^l v^a \end{aligned} \tag{2.6}$$

$$+\sum_{i=1}^{n}\left(\left(\nabla_p(\partial_{p_i}F^j)\right)(x,0,0,0)\cdot\nabla v^a+\left(\nabla_P(\partial_{p_i}F^j)(x,0,0,0):\nabla^2 v^a\right)\right)\partial_{x_i}v^l$$
$$+\nabla_P F^j(x,0,0,0):\nabla^2 w_j^{(al)}+\left(\nabla_P(\partial_u F^j)\right)(x,0,0,0):\nabla^2 v^l v^a$$
$$+\sum_{j,k=1}^{n}\left(\left(\nabla_p(\partial_{P_{jk}}F^j)\right)\cdot\nabla v^a+\left(\nabla_P(\partial_{P_{jk}}F^j)\right)(x,0,0,0):\nabla^2 v^a\right)\partial^2_{x_ix_j}v^l$$

Thus $w_j^{(al)}$ satisfies the boundary value problem

$$\begin{cases} I_j=0, & \text{in } \Omega \\ w_j^{(al)}=0, & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Next we would like to integrate I_1-I_2 against a solution to the adjoint of

$$\partial_u F^j(x,0,0,0)v_j^l+\nabla_p F^j(x,0,0,0)\cdot\nabla v_j^l+\nabla_P F^j(x,0,0,0):\nabla^2 v_j^l=0$$

and use the assumption that the DN maps associated to (2.3) coincide for $j=1,2$ together with a completeness result to recover information about the coefficients of I_1 and I_2. Again the information that can be recovered depends on the equation and below this method is applied in the case of the minimal surface equation.

What we would do next is to use an induction argument to show that from higher order linearizations it is possible to recover more information. This too will be specified below.

3. Mimimal surface equation on a Riemannian manifold

In this section we derive Eq. (1.2). Let (M,g), $M=\mathbb{R}^n$, $n\geq 3$, be a Riemannian manifold with the metric

$$g_{ij}(x',x_n)=c(x',x_n)\delta_{ij}, \tag{3.1}$$

where

$$(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R},\ c\in C^\infty(\mathbb{R}^n),\ c(x)>0\quad\text{for all}\quad x\in\mathbb{R}^n.$$

These assumptions are valid for the rest of the article, unless otherwise stated.

Let $u\colon\Omega\subset\mathbb{R}^{n-1}\to\mathbb{R}$, $u\in C^2(\bar{\Omega})$, and consider the graph of the function u

$$\mathrm{Graph}_u=\{(x',u(x'))\colon x'\in\Omega\}\subset M.$$

This graph is a minimal surface if and only if its mean curvature H is equal to zero at all points on the graph. By defining

$$f\colon\Omega\times\mathbb{R}\to\mathbb{R},\quad f(x',x_n)=x_n-u(x'),$$

the graph of u is the surface

$$\Sigma:=\{(x',x_n)\in\Omega\times\mathbb{R}:f(x',x_n)=0\}.$$

The mean curvature of Σ at $x\in\Sigma$ is the sum of principal curvatures. We omit the normalizing factor $\frac{1}{n-1}$ when calculating the mean curvature. In order to calculate the principal curvatures, we introduce the Riemannian gradient and Hessian of a function $f\colon M\to\mathbb{R}$:

$$\nabla_g f=g^{ij}\partial_{x_i}f\partial_{x_j},\quad \nabla_g^2 f=\left(\partial^2_{x_ix_j}f-\Gamma^m{}_{ij}\partial_{x_m}f\right)_{i,j=1}^{n},$$

where g^{ij} is the inverse of g_{ij} and ${\Gamma^m}_{ij} = \frac{1}{2}g^{ml}(\partial_{x_i}g_{jl} + \partial_{x_j}g_{il} - \partial_{x_l}g_{ij})$ is the Christoffel symbol related to the metric g. Define also the Laplace–Beltrami operator, which is a trace of the Hessian (this is one way of defining it), and the norm of the gradient:

$$\Delta_g f = \mathrm{Tr}(\nabla_g^2 f) = g^{ij}\left(\partial^2_{x_ix_j}f - {\Gamma^m}_{ij}\partial_{x_m}f\right), \quad |\nabla_g f|_g^2 = g^{ij}\partial_{x_i}f\partial_{x_j}f.$$

Now the principal curvatures of Σ at $x\in\Sigma$ are the eigenvalues of $\nabla_g^2 f(x)$ restricted to the tangent space $T_x\Sigma$ at x. Since $\frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}$ is a normal to Σ at the point x, we have $T_x\Sigma = \{\nabla_g f(x)\}^\perp$, or in other words, the tangent space $T_x\Sigma$ is the orthogonal complement of the vector $\nabla_g f(x)$.

Let $\{E_1,\dots,E_{n-1}\}$ be an g-orthonormal basis of $T_x\Sigma$. Then $\left\{E_1,\dots,E_{n-1},\frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}\right\}$ is an orthonormal basis of $\mathbb{R}^n$. Now the mean curvature of Σ at $x\in\Sigma$ is the trace of $\nabla_g^2 f(x)|_{\{\nabla_g f(x)\}^\perp}$:

$$\begin{aligned}
H(x) &= \sum_{i=1}^{n-1}\langle\nabla_g^2 f(x)E_i, E_i\rangle \\
&= \sum_{i=1}^{n-1}\langle\nabla_g^2 f(x)E_i, E_i\rangle + \left(\nabla_g^2 f(x)\right)\left(\frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}, \frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}\right) \\
&- \left(\nabla_g^2 f(x)\right)\left(\frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}, \frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}\right) \\
&= \mathrm{Tr}(\nabla_g^2 f(x)) - |\nabla_g f(x)|_g^{-2}\left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x)) \\
&= \Delta_g f(x) - |\nabla_g f(x)|_g^{-2}\left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x)).
\end{aligned}$$

Thus Graph_u is a minimal surface if and only if

$$|\nabla_g f(x)|_g^2\Delta_g f(x) - \left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x)) = 0 \quad \text{for all } x\in\mathrm{Graph}_u. \tag{3.2}$$

Next we will calculate the minimal surface equation more explicitly using the conformally Euclidean metric (3.1). Now $g^{ij} = c^{-1}\delta_{ij}$ is the inverse matrix of (3.1) and thus

$$\nabla_g f = c^{-1}\sum_{j=1}^n \partial_{x_j}f\partial_{x_j}, \quad |\nabla_g f|_g^2 = c^{-1}\sum_{i=1}^n\partial_{x_i}f\partial_{x_i}f.$$

Also the Christoffel symbol can be simplified by letting $\lambda = \frac{1}{2}\log c$ and hence $\partial_{x_i}\lambda = \frac{1}{2}c^{-1}\partial_{x_i}c$. Then

$$\begin{aligned}
{\Gamma^m}_{ij} &= \frac{1}{2}g^{ml}\left(\partial_{x_i}g_{jl} + \partial_{x_j}g_{il} - \partial_{x_l}g_{ij}\right) \\
&= \frac{1}{2}c^{-1}\left(\partial_{x_i}c\delta_{jm} + \partial_{x_j}c\delta_{im} - \partial_{x_m}c\delta_{ij}\right) \\
&= \partial_{x_i}\lambda\delta_{jm} + \partial_{x_j}\lambda\delta_{im} - \partial_{x_m}\lambda\delta_{ij}.
\end{aligned}$$

Let us next calculate the two parts of (3.2) separately, starting from

$$\begin{aligned}
c^2|\nabla_g f|_g^2\Delta_g f &= \left(\sum_{i=1}^n\partial_{x_i}f\partial_{x_i}f\right)\left(\sum_{i=1}^n\partial^2_{x_ix_i}f - \sum_{m=1}^n\left(\sum_{i,j=1}^n\delta_{ij}{\Gamma^m}_{ij}\right)\partial_{x_m}f\right) \\
&= \left(\sum_{i=1}^n\partial_{x_i}f\partial_{x_i}f\right)\left(\sum_{i=1}^n\partial^2_{x_ix_i}f - (2-n)\sum_{m=1}^n\partial_{x_m}\lambda\partial_{x_m}f\right),
\end{aligned}$$

and the other part becomes

$$
\begin{aligned}
&c^2\nabla_g^2 f(\nabla_g f, \nabla_g f)\\
&= \left(\partial^2_{x_i x_j} f - \Gamma^m{}_{ij}\partial_{x_m} f\right) c g^{ia}\partial_{x_a} f c g^{jb}\partial_{x_b} f\\
&= \sum_{i,j=1}^n \left(\partial^2_{x_i x_j} f - \sum_{m=1}^n \left(\partial_{x_i}\lambda\delta_{jm} + \partial_{x_j}\lambda\delta_{im} - \partial_{x_m}\lambda\delta_{ij}\right)\partial_{x_m} f\right)\partial_{x_i} f\partial_{x_j} f\\
&= \sum_{i,j=1}^n \left(\partial^2_{x_i x_j} f - \partial_{x_i}\lambda\partial_{x_j} f - \partial_{x_j}\lambda\partial_{x_i} f + \left(\sum_{m=1}^n \partial_{x_m}\lambda\partial_{x_m} f\right)\delta_{ij}\right)\partial_{x_i} f\partial_{x_j} f\\
&= \sum_{i,j=1}^n \left(\partial^2_{x_i x_j} f - 2\partial_{x_i}\lambda\partial_{x_j} f\right)\partial_{x_i} f\partial_{x_j} f + \left(\sum_{i=1}^n \partial_{x_i} f\partial_{x_i} f\right)\left(\sum_{m=1}^n \partial_{x_m}\lambda\partial_{x_m} f\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
&c^2|\nabla_g f|_g^2 \Delta_g f - c^2\nabla_g^2 f\left(\nabla_g f, \nabla_g f\right)\\
&= \left(\sum_{i=1}^n \left(\partial_{x_i} f\right)^2\right)\left(\sum_{i=1}^n \partial^2_{x_i x_i} f + (n-3)\sum_{m=1}^n \partial_{x_m}\lambda\partial_{x_m} f\right)\\
&- \sum_{i,j=1}^n \left(\partial^2_{x_i x_j} f - 2\partial_{x_i}\lambda\partial_{x_j} f\right)\partial_{x_i} f\partial_{x_j} f.
\end{aligned}
$$

Plugging the above to (3.2), we get that Σ is a minimal surface if and only if

$$
\begin{aligned}
&\left(\sum_{i=1}^n \left(\partial_{x_i} f\right)^2\right)\left(\sum_{i=1}^n \partial^2_{x_i x_i} f + (n-3)\sum_{m=1}^n \partial_{x_m}\lambda\partial_{x_m} f\right)\\
&- \sum_{i,j=1}^n \left(\partial^2_{x_i x_j} f - 2\partial_{x_i}\lambda\partial_{x_j} f\right)\partial_{x_i} f\partial_{x_j} f = 0
\end{aligned}
\tag{3.3}
$$

for $x \in \Sigma$.

Insert next $f(x', x_n) = x_n - u(x')$ to the above in order to get an equation in terms of the function u. Then the first line of (3.3) becomes (note that $\partial_{x_n} u = 0$)

$$
\begin{aligned}
&\left(\sum_{i=1}^n \delta_{in} - 2\delta_{in}\partial_{x_i} u + \left(\partial_{x_i} u\right)^2\right)\left(-\sum_{i=1}^n \partial^2_{x_i x_i} u + (n-3)\sum_{m=1}^n \partial_{x_m}\lambda(\delta_{mn} - \partial_{x_m} u)\right)\\
&= \left(1 + |\nabla u|^2\right)\left(-\Delta u + (n-3)(\partial_{x_n}\lambda - \nabla_{x'}\lambda\cdot\nabla u)\right).
\end{aligned}
$$

The second line is equal to

$$
\begin{aligned}
&- \sum_{i,j=1}^n (-\partial^2_{x_i x_j} u - 2\partial_{x_i}\lambda(\delta_{jn} - \partial_{x_j} u))(\delta_{in}\delta_{jn} - \delta_{in}\partial_{x_j} u - \delta_{jn}\partial_{x_i} u + \partial_{x_i} u\partial_{x_j} u)\\
&= \sum_{i,j=1}^n \partial^2_{x_i x_j} u\partial_{x_i} u\partial_{x_j} u + 2\partial_{x_n}\lambda - 2\sum_{i=1}^{n-1} \partial_{x_i}\lambda\partial_{x_i} u + 2\sum_{j=1}^{n-1} \partial_{x_n}\lambda(\partial_{x_j} u)^2\\
&- 2\sum_{i,j=1}^{n-1} \partial_{x_i}\lambda\partial_{x_i} u(\partial_{x_j} u)^2\\
&= \nabla u^T\nabla^2 u\nabla u + 2\partial_{x_n}\lambda - 2\nabla_{x'}\lambda\cdot\nabla u + 2\partial_{x_n}\lambda|\nabla u|^2 - 2\nabla_{x'}\lambda\cdot\nabla u|\nabla u|^2\\
&= \nabla u^T\nabla^2 u\nabla u + 2\left(\partial_{x_n}\lambda - \nabla_{x'}\lambda\cdot\nabla u\right)(1 + |\nabla u|^2).
\end{aligned}
$$

Combining these two, we get that Graph_u is a minimal surface if and only if the function u satisfies the following minimal surface equation

$$-\Delta u+\frac{\nabla u^T\nabla^2 u\nabla u}{1+|\nabla u|^2}-\frac{n-1}{2c(x',u(x'))}\left(\nabla_{x'}c(x',u(x'))\cdot\nabla u-\partial_{x_n}c(x',u(x'))\right)=0 \tag{3.4}$$

for all $x'\in\Omega$. Multiplying both sides with $(1+|\nabla u|^2)^{-1/2}$ gives

$$\begin{aligned}
&-\operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right)-\sum_{i,j=1}^{n-1}\frac{\partial_{x_j}u}{(1+|\nabla u|^2)^{1/2}}\frac{\partial_{x_j}c\delta_{ii}}{2c}+\frac{(n-1)\partial_{x_n}c}{2c(1+|\nabla u|^2)^{1/2}}\\
&=-\operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right)-\sum_{i,j=1}^{n-1}\frac{\partial_{x_j}u}{(1+|\nabla u|^2)^{1/2}}\Gamma^i{}_{ij}+\frac{(n-1)\partial_{x_n}c}{2c(1+|\nabla u|^2)^{1/2}}\\
&=-\operatorname{div}_{g_{n-1}}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right)+\frac{(n-1)\partial_{x_n}c}{2c(1+|\nabla u|^2)^{1/2}}\\
&=0.
\end{aligned}$$

In the Euclidean setting, $c\equiv 1$, this is the more familiar Euclidean minimal surface equation.

4. Preliminaries for the higher order linearization

In this section we use the method of higher order linearization on the minimal surface equation derived in the previous section. From now on, assume that $\Omega\subset\mathbb{R}^{n-1}$ is a bounded domain. Let us start by looking at the assumptions of Proposition 2.1, where it is assumed that $u\equiv 0$ is a solution to (3.4). This leads to the condition that

$$\partial_{x_n}c(x',0)=0,\quad x'\in\Omega, \tag{4.1}$$

which can be seen as follows. For a constant function $u\colon\Omega\to\mathbb{R}, u(x')=d$ to be a solution to (3.4) is equivalent with

$$-\frac{n-1}{2c(x',d)}\partial_{x_n}c(x',d)=0\quad\text{for all }\ x'\in\Omega,$$

which is equivalent with $\partial_{x_n}c(x',d)=0$ for all $x'\in\Omega$. The assumption (4.1) comes by setting $d=0$.

Next we will focus on the boundary value problem (2.1) in the setting described above and calculate the first and second order linearizations of (2.1) and the corresponding linearizations of the DN map. This could be done directly from (3.4) but we will follow the general method in Section 2 and begin by defining a function $F\colon\mathbb{R}^{n^2}\to\mathbb{R}$,

$$\begin{aligned}
F(x',u,p,P):=&-\sum_{i=1}^{n-1}P_{ii}-\frac{n-1}{2c(x',u)}\left(\sum_{i=1}^{n-1}p_i\partial_{x_i}c(x',u)-\partial_{x_n}c(x',u)\right)\\
&+\frac{1}{1+|p|^2}\sum_{i,j=1}^{n-1}P_{ij}p_ip_j.
\end{aligned}$$

Here $p=(p_1,\dots,p_{n-1})\in\mathbb{R}^{n-1}, P=(P_{ij})$ is an $(n-1)\times(n-1)$ matrix and $x'\in\mathbb{R}^{n-1}$. Then (3.4) is equivalent with $F(x',u,\nabla u,\nabla^2 u)=0$ for all $x'\in\Omega$.

Let us start with the first linearization. For this, let $x'\in\Omega$. As shown in Section 2, we need to differentiate F with respect to u,p and P. The first derivatives with respect to variable P are

$$\partial_{P_{kl}}F(x',u,p,P)=-\delta_{kl}+\frac{p_kp_l}{1+|p|^2}.$$

When evaluated at $\varepsilon_1 = \cdots = \varepsilon_k = 0$, we get

$$\partial_{P_{kl}} F(x', 0, 0, 0) = -\delta_{kl}.$$

Next calculation is $\nabla_p F$:

$$\begin{aligned}\partial_{p_k} F(x', u, p, P) &= -\frac{n-1}{2c}\partial_{x_k} c - \frac{2p_k}{(1+|p|^2)^2}\sum_{i,j=1}^{n-1} P_{ij}p_ip_j \\ &+ \frac{1}{1+|p|^2}\sum_{i,j=1}^{n-1} P_{ij}(\delta_{ik}p_j + \delta_{jk}p_i) \\ &= -\frac{n-1}{2c}\partial_{x_k} c - \frac{2p_k}{(1+|p|^2)^2}\sum_{i,j=1}^{n-1} P_{ij}p_ip_j \\ &+ \frac{1}{1+|p|^2}\left(\sum_{j=1}^{n-1} P_{kj}p_j + \sum_{i=1}^{n-1} P_{ik}p_i\right).\end{aligned}$$

Setting $\varepsilon_1 = \cdots = \varepsilon_k = 0$, this becomes

$$\partial_{p_k} F(x', 0, 0, 0) = -\frac{n-1}{2c(x', 0)}\partial_{x_k} c(x', 0).$$

Since the solution operator S from Proposition 2.1 is smooth, the solution $u(x', \varepsilon)$ depends smoothly on ε and thus $u(x', \varepsilon)\big|_{\varepsilon_1=\cdots=\varepsilon_k=0} = 0$. Hence the coordinate x_n is 0 since we are on the graph of u.

What is left to calculate is the derivative $\partial_u F$:

$$\begin{aligned}\partial_u F(x', u, p, P) &= -\frac{n-1}{2c^2}(\partial_{x_n} c)^2 + \frac{n-1}{2c}\partial_{x_n}^2 c \\ &+ \frac{n-1}{2c^2}\partial_{x_n} c\sum_{i=1}^{n-1}\partial_{x_i} c p_i - \frac{n-1}{2c}\sum_{i=1}^{n-1}\partial_{x_i}\partial_{x_n} c p_i.\end{aligned}$$

Hence, when evaluated at $\varepsilon_1 = \cdots = \varepsilon_k = 0$

$$\partial_u F(x', 0, 0, 0) = \frac{n-1}{2c(x', 0)}\partial_{x_n}^2 c(x', 0),$$

since $\partial_{x_n} c(x', 0) = 0$.

In Theorem 1.1 the condition $\partial_{x_n}^2 c(x', 0) = 0$ is assumed and thus

$$\partial_u F(x', 0, 0, 0) = 0. \tag{4.2}$$

Now the first linearization (2.5) is the following boundary value problem

$$\begin{cases} \Delta v^l + \dfrac{n-1}{2c(x', 0)}\nabla_{x'} c(x', 0) \cdot \nabla v^l = 0, & \text{in } \Omega \\ v^l = f_l, & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

By multiplying the first equation in (4.3) with $c(x', 0)^{\frac{n-1}{2}}$ we see that (4.3) is equivalent with

$$\begin{cases} \operatorname{div}\left(c(x', 0)^{\frac{n-1}{2}}\nabla v^l\right) = 0, & \text{in } \Omega \\ v^l = f_l, & \text{on } \partial\Omega. \end{cases}$$

Hence the first linearization of the DN-map (1.3) at $f = 0$ is

$$(D\Lambda_c)_0 : C^s(\partial\Omega) \to C^{s-1}(\partial\Omega), \quad f \mapsto \partial_\nu v_f\big|_{\partial\Omega}. \tag{4.4}$$

In dimension $3-1 = 2$ one can recover $c(x', 0)$ up to a multiplicative constant using a boundary determination result [2] together with the knowledge of the partial DN-map [10]. When $n > 3$, $c(x', 0)$ can be recovered, again up to a multiplicative constant, combining the same boundary determination result and the DN map [29] or the partial DN map (when Ω is as described in Theorem 1.1 part (2) [12]). Details will be shown in the proof of Theorem 1.1.

For the second linearization, as can be seen from (2.7), second derivatives of the map F need to be calculated. Firstly

$$\partial_{P_{rs}}\partial_{P_{kl}}F(x', u, p, P) = 0,$$
$$\partial_{P_{kl}}\partial_u F(x', u, p, P) = \partial_u\partial_{P_{kl}}F(x', u, p, P) = 0$$

and hence, when evaluated at $\varepsilon_1 = \cdots = \varepsilon_k = 0$, these vanish. Let us next calculate other mixed derivatives. Now

$$\partial_{p_s}\partial_{P_{kl}}F(x', u, p, P) = \frac{-2p_s}{(1+|p|^2)^2}p_k p_l + \frac{1}{1+|p|^2}(\delta_{ks}p_l + \delta_{ls}p_k)$$

and when evaluated at $\varepsilon_1 = \cdots = \varepsilon_k = 0$

$$\partial_{p_s}\partial_{P_{kl}}F(x', 0, 0, 0) = \partial_{P_{kl}}\partial_{p_s}F(x', 0, 0, 0) = 0.$$

Also

$$\partial_{p_k}\partial_u F(x', u, p, P) = \frac{n-1}{2c^2}\partial_{x_n}c\partial_{x_k}c - \frac{n-1}{2c}\partial_{x_k}\partial_{x_n}c.$$

Setting $\varepsilon_1 = \cdots = \varepsilon_k = 0$ we have

$$\partial_{p_k}\partial_u F(x', 0, 0, 0) = \partial_u\partial_{p_k}F(x', 0, 0, 0) = 0,$$

since $\partial_{x_k}\partial_{x_n}c(x', 0) = 0$ for $k = 1, \ldots, n-1$.

What is left are the second derivatives with respect to the variables p and u. Let us start from the variable p:

$$\begin{aligned}\partial_{p_s}\partial_{p_k}F(x', u, p, P) &= \frac{-2\delta_{sk}}{(1+|p|^2)^2}\sum_{i,j=1}^{n-1}P_{ij}p_i p_j \\ &- 2p_k\left(\frac{-4p_s}{(1+|p|^2)^3}\sum_{i,j=1}^{n-1}P_{ij}p_i p_j + \frac{1}{(1+|p|^2)^2}\sum_{i,j=1}^{n-1}P_{ij}(\delta_{is}p_j + \delta_{js}p_i)\right) \\ &- \frac{2p_s}{(1+|p|^2)^2}\left(\sum_{j=1}^{n-1}P_{kj}p_j + \sum_{i=1}^{n-1}P_{ik}p_i\right) + \frac{1}{1+|p|^2}(P_{ks} + P_{sk}).\end{aligned}$$

Hence when evaluating at $\varepsilon_1 = \cdots = \varepsilon_k = 0$

$$\partial_{p_r}\partial_{p_k}F(x', 0, 0, 0) = 0.$$

For the variable u the second derivative reads

$$\begin{aligned}\partial_u\partial_u F(x', u, p, P) &= \frac{n-1}{c^3}(\partial_{x_n}c)^3 - \frac{3(n-1)}{2c^2}\partial_{x_n}c\partial^2_{x_n}c \\ &+ \frac{n-1}{2c}\partial^3_{x_n}c - \frac{n-1}{c^2}(\partial_{x_n}c)^2\sum_{i=1}^{n-1}\partial_{x_i}c p_i\end{aligned}$$

$$+\frac{n-1}{2c}\left(\partial^2_{x_n}c\sum_{i=1}^{n-1}\partial_{x_i}cp_i+\partial_{x_n}c\sum_{i=1}^{n-1}\partial_{x_i}\partial_{x_n}cp_i\right)$$
$$+\frac{n-1}{2c^2}\partial_{x_n}c\sum_{i=1}^{n-1}\partial_{x_i}\partial_{x_n}cp_i-\frac{n-1}{2c}\sum_{i=1}^{n-1}\partial_{x_i}\partial^2_{x_n}cp_i.$$

Thus, letting at $\varepsilon_1=\cdots=\varepsilon_k=0$

$$\partial^2_u F(x',0,0,0)=\frac{n-1}{2c(x',0)}\partial^3_{x_n}c(x',0).$$

Let us plug the calculated derivatives in (2.6) to find out what is the second linearization:

$$I=\tfrac{n-1}{2c(x',0)}\partial^2_{x_n}c(x',0)w^{(al)}+\tfrac{n-1}{2c(x',0)}\partial^3_{x_n}c(x',0)v^lv^a-\tfrac{n-1}{2c(x',0)}\nabla_{x'}c(x',0)\cdot\nabla w^{(al)}-\Delta w^{(al)}.$$

Here v^a,v^l satisfy (4.3) with corresponding boundary values. Now the function $w^{(al)}=(\partial^2_{\varepsilon_a\varepsilon_l}u)(x',0,\dots,0)$ solves

$$\begin{cases}\Delta w^{(al)}+\frac{n-1}{2c(x',0)}\nabla_{x'}c(x',0)\cdot\nabla w^{(al)} & \\ +\frac{1-n}{2c(x',0)}\partial^2_{x_n}c(x',0)w^{(al)}+\frac{1-n}{2c(x',0)}\partial^3_{x_n}c(x',0)v^lv^a=0, & \text{in } \Omega \\ w^{(al)}=0, & \text{on } \partial\Omega.\end{cases}\tag{4.5}$$

In Theorem 1.1 there is an assumption that $\partial^2_{x_n}c(x',0)=0$ and thus the term $\frac{n-1}{2c(x',0)}\partial^2_{x_n}c(x',0)w^{(al)}$ vanishes. Now the boundary value problem (4.5) is equivalent with

$$\begin{cases}\operatorname{div}\left(c(x',0)^{\frac{n-1}{2}}\nabla w^{(al)}\right)-\frac{n-1}{2}c(x',0)^{\frac{n-1}{2}-1}\partial^3_{x_n}c(x',0)v^lv^a=0, & \text{in } \Omega \\ w^{(al)}=0, & \text{on } \partial\Omega.\end{cases}\tag{4.6}$$

5. Proof of Theorem 1.1

Now we use the method of higher order linearization to prove our main result. This will also make use of the linearizations calculated in the previous section. Before the proof we state a proposition which says that products of solutions to the Schrödinger equation form a complete set in $L^1(\Omega)$ for $n\geq 2$ (when $n\geq 3$ [29], when $n=2$ [3], see also [1] and [23, Proposition 2.1] where this result is stated in the following form).

Proposition 5.1. *Let $\Omega\subset\mathbb{R}^n, n\geq 2$, be a bounded domain with C^∞ boundary, $q_1,q_2\in C^\infty(\bar\Omega)$ and let $f\in L^\infty(\Omega)$. Assume that*

$$\int_\Omega fv_1v_2\,dx=0,$$

for all v_j solving $(-\Delta+q_j)v_j=0$ in Ω. Then $f\equiv 0$ in Ω.

Proof of Theorem 1.1. The assumptions of Proposition 2.1 hold for our case and thus (1.5) is well-posed. Assume now that we have two conformal factors c_1,c_2 on the manifold M. As in Section 2, let $\varepsilon_1,\dots,\varepsilon_{N+1}$ be sufficiently small numbers, $\varepsilon=(\varepsilon_1,\dots,\varepsilon_{N+1})$, $f_1,\dots,f_{N+1}\in C^s(\partial\Omega)$ and $u_j(x,\varepsilon)$ be the unique small solution to

$$\begin{cases}F^j(x,u_j,\nabla u_j,\nabla^2u_j)=0, & \text{in } \Omega \\ u_j=\sum_{m=1}^{N+1}\varepsilon_mf_m, & \text{on } \partial\Omega,\end{cases}$$

for $j=1,2$, where F^j is (1.1) with c replaced by c_j.

The proof now divides to the cases (1) and (2) and we will prove first (1). It is the most straightforward of these cases and the other cases are proven similarly with only minor modifications.

Case (1): Assume now that

$$\Lambda_{c_1}(f) = \Lambda_{c_2}(f) \tag{5.1}$$

for all $f \in C^s(\partial\Omega)$ sufficiently small. Now we have the corresponding F^1, F^2 and the first linearization of Λ_{c_j} is (4.4), with $c(x', 0)$ replaced by $c_j(x', 0)$, which corresponds to the conductivity equation

$$\begin{cases} \operatorname{div}\left(c_j(x',0)^{\frac{n-1}{2}} \nabla v_j^l\right) = 0, & \text{in } \Omega \\ v_j^l = f_l, & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

for $j = 1, 2$. Using boundary determination from [2] for the case of Laplacian with a convection term we get for $x' \in \partial\Omega$

$$\frac{\nabla_{x'} c_1(x',0)}{c_1(x',0)} = \frac{\nabla_{x'} c_2(x',0)}{c_2(x',0)} \quad \Longleftrightarrow \quad \nabla_{x'}\big(\ln(c_1(x',0))\big) = \nabla_{x'}\big(\ln(c_2(x',0))\big).$$

From this we get that $\nabla_{x'}(\ln(c_1(x',0)) - \ln(c_2(x',0))) = \nabla_{x'}\left(\ln \frac{c_1(x',0)}{c_2(x',0)}\right) = 0$ which then implies that $c_1(x',0) = \lambda c_2(x',0)$ for $x' \in \partial\Omega$ and $\lambda \neq 0$. (We cannot use boundary determination for the conductivity equation (e.g. [15]) because the DN maps are different: here $f \mapsto \partial_\nu v_f\big|_{\partial\Omega}$ instead of $f \mapsto c_1(x',0)\partial_\nu v_f\big|_{\partial\Omega}$.) It is known that the knowledge of this linearized DN map combined with $c_1(x',0)|_{\partial\Omega} = \lambda c_2(x',0)|_{\partial\Omega}$ gives us $c_1(x',0) = \lambda c_2(x',0)$ in Ω [29].

By the gauge invariance of (4.3) (replacing $c_2(x',0) = \lambda^{-1} c_1(x',0)$) we have that v_j^l solves the equation

$$\begin{cases} \operatorname{div}\left(c_1(x',0)^{\frac{n-1}{2}} \nabla v_j^l\right) = 0, & \text{in } \Omega \\ v_j^l = f_l, & \text{on } \partial\Omega. \end{cases}$$

Since solutions to this are unique, we define $v^l := v_1^l = v_2^l$.

For recovering the higher order derivatives of $c_j(x',0)$ we can use the second linearizations (from (4.6))

$$\begin{cases} \operatorname{div}\left(c_j(x',0)^{\frac{n-1}{2}} \nabla w_j^{(al)}\right) - \frac{n-1}{2} c_j(x',0)^{\frac{n-1}{2}-1} \partial_{x_n}^3 c_j(x',0) v^l v^a = 0, & \text{in } \Omega \\ w_j^{(al)} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

corresponding to $j = 1, 2$. Notice that if we replace $c_2(x',0)$ by $\lambda^{-1} c_1(x',0)$ in (4.5), except in the third order derivative, we get that w_2^{al} solves

$$\begin{cases} \operatorname{div}\left(c_1(x',0)^{\frac{n-1}{2}} \nabla w_2^{(al)}\right) - \lambda \frac{n-1}{2} c_1(x',0)^{\frac{n-1}{2}-1} \partial_{x_n}^3 c_2(x',0) v^l v^a = 0, & \text{in } \Omega \\ w_2^{(al)} = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Subtract now (5.3) for $j = 1$ from (5.4), integrate against $v \equiv 1$ (solution to the first linearization) over Ω

$$\begin{aligned} &\int_\Omega \operatorname{div}(c_1(x',0)^{\frac{n-1}{2}} \nabla w_1^{(al)} - c_1(x',0)^{\frac{n-1}{2}} \nabla w_2^{(al)}) \\ &- \left(\frac{n-1}{2} c_1(x',0)^{\frac{n-1}{2}-1} \partial_{x_n}^3 c_1(x',0) - \lambda \frac{n-1}{2} c_1(x',0)^{\frac{n-1}{2}-1} \partial_{x_n}^3 c_2(x',0)\right) v^l v^a \, dx' \\ &= 0 \end{aligned}$$

and use integration by parts to have

$$\begin{aligned}
0 &= \int_{\partial\Omega} c_1(x',0)^{\frac{n-1}{2}} \left(\nabla w_1^{(al)} \cdot \nu - \nabla w_2^{(al)} \cdot \nu\right) dS \\
&= \int_{\Omega} \mathrm{div}(c_1(x',0)^{\frac{n-1}{2}} \nabla w_1^{(al)} - c_2(x',0)^{\frac{n-1}{2}} \nabla w_2^{(al)})\, dx' \\
&= \int_{\Omega} \frac{n-1}{2} c_1(x',0)^{\frac{n-1}{2}-1} \left(\partial_{x_n}^3 c_1(x',0) - \lambda \partial_{x_n}^3 c_2(x',0)\right) v^l v^a \, dx'.
\end{aligned}$$

This is true since by (5.1)

$$\partial_\nu u_1\big|_{\partial\Omega} = \partial_\nu u_2\big|_{\partial\Omega}$$

and applying $\partial_{\varepsilon_a}\partial_{\varepsilon_l}\big|_{\varepsilon=0}$ to this implies

$$\partial_\nu w_1^{(al)}\big|_{\partial\Omega} = \partial_\nu w_2^{(al)}\big|_{\partial\Omega}, \quad \text{for } a,l \in \{1,\dots,k\}.$$

Thus

$$\int_{\Omega} c_1(x',0)^{\frac{n-1}{2}-1} \left(\partial_{x_n}^3 c_1(x',0) - \lambda \partial_{x_n}^3 c_2(x',0)\right) v^l v^a \, dx' = 0 \tag{5.5}$$

for any v^a, v^l solving the conductivity equation (5.2). A solution to (5.2) is equivalently a solution to

$$\begin{cases} \left(-\Delta_{x'} + \frac{\Delta_{x'} c_j(x',0)^{\alpha/2}}{c_j(x',0)^{\alpha/2}}\right) g^l = 0, & \text{in } \Omega \\ g^l = c_j(x',0)^{\alpha/2} f_l, & \text{on } \partial\Omega, \end{cases}$$

where $\alpha = \frac{n-1}{2}$, $g^l = c_j(x',0)^{\alpha/2} v^l$ and $\Delta_{x'}$ denotes the Laplacian with respect to the first two variables. Hence by using the fact that a product of a pair of solutions (Proposition 5.1) is dense in $L^1(\Omega)$, we get

$$\partial_{x_n}^3 c_1(x',0) = \lambda \partial_{x_n}^3 c_2(x',0), \quad x' \in \Omega.$$

Also (5.3), (5.4) together with the previous equality and $c_1(x',0) = \lambda c_2(x',0)$, gives the following boundary value problem

$$\begin{cases} \mathrm{div}\left(c_1(x',0)^{\frac{n-1}{2}} \nabla \left(w_1^{(al)} - w_2^{(al)}\right)\right) = 0, & \text{in } \Omega \\ w_1^{(al)} - w_2^{(al)} = 0, & \text{on } \partial\Omega. \end{cases}$$

This has a unique solution and thus $w_1^{(al)} = w_2^{(al)}$.

Next we use induction to show $\partial_{x_n}^k c_1(x',0) = \lambda \partial_{x_n}^k c_2(x',0)$ for all $k \in \mathbb{N}$. By the above this already holds for $k = 0,1,2,3$. Our assumption now is

$$\partial_{x_n}^k c_1(x',0) = \lambda \partial_{x_n}^k c_2(x',0), \quad x' \in \Omega, \text{ for all } \quad k = 0,1,2,\dots,N \in \mathbb{N}, N > 3.$$

Let us do a subinduction to prove

$$\partial_{l_1\dots l_k}^k u_1(x',0) = \partial_{l_1\dots l_k}^k u_2(x',0), \quad x' \in \Omega,$$

for all $k = 1,\dots,N$, where $\partial_{l_1\dots l_k}^k u_j(x',0) = \frac{\partial^k u_j(x',0)}{\partial_{\varepsilon_{l_1}}\dots\partial_{\varepsilon_{l_k}}}$. Above we have shown this for $k = 1,2$. Assume that it holds for $k \le K < N$. Then the linearization of order $K+1$ is, when evaluated at $\varepsilon_1 = \dots = \varepsilon_{K+1} = 0$,

$$\begin{aligned}
&\mathrm{div}\left(c_j(x',0)^{\frac{n-1}{2}} \nabla \left(\partial_{l_1\dots l_{K+1}}^{K+1} u_j(x',0)\right)\right) + R_K(u_j, c_j(x',0), 0) \\
&+ C c_j(x',0)^{\frac{n-1}{2}-1} \partial_{x_n}^{K+2} c_j(x',0) \left(\prod_{k=1}^{K+1} v^{(l_k)}\right) = 0,
\end{aligned} \tag{5.6}$$

$x' \in \Omega$, where $C \neq 0$. Actually $C = \frac{n-1}{2}$, since it comes from the u derivatives of F and is the constant $\frac{n-1}{2}$ appearing in front of the second term of F. Also, here R is a polynomial of $\partial_{x_n}^k c_j(x',0), \partial_{l_1\dots l_k}^k u_1(x',0)$ and the components of $\nabla_{x'}\left(\partial_{x_n}^k c_j(x',0)\right)$. Now an integration by parts argument similar to the case of the second linearization and together with Proposition 5.1 (choosing $v^3 = \cdots = v^{K+1} = 1$) gives $\partial_{x_n}^{K+2} c_1(x',0) = \lambda \partial_{x_n}^{K+2} c_2(x',0)$.

Subtracting Eqs. (5.6) (similarly as for Eqs. (5.3) and (5.4)) for $j = 1,2$ we get

$$\begin{cases} \operatorname{div}\left(c_1(x',0)^{\frac{n-1}{2}} \nabla \left(\partial_{l_1\dots l_{K+1}}^{K+1} u_1(x',0) - \partial_{l_1\dots l_{K+1}}^{K+1} u_2(x',0)\right)\right) = 0, & \text{in } \Omega \\ \partial_{l_1\dots l_{K+1}}^{K+1} u_1(x',0) - \partial_{l_1\dots l_{K+1}}^{K+1} u_2(x',0) = 0, & \text{on } \partial\Omega. \end{cases}$$

This is true, since by induction assumptions for all $x' \in \Omega$ we have $\nabla_{x'}\left(\partial_{x_n}^k c_1(x',0) - \partial_{x_n}^k c_2(x',0)\right) = 0$ and the other terms agree for $j = 1,2$, $k \leq K$. Again, by the uniqueness of solutions, $\partial_{l_1\dots l_{K+1}}^{K+1} u_1(x',0) = \partial_{l_1\dots l_{K+1}}^{K+1} u_2(x',0)$, $x' \in \Omega$, which ends the subinduction.

Returning to the original induction, the linearization of order $N+1$ at $\varepsilon_1 = \cdots = \varepsilon_{N+1} = 0$ is

$$\begin{aligned} &\operatorname{div}\left(c_j(x',0)^{\frac{n-1}{2}} \nabla \left(\partial_{l_1\dots l_{N+1}}^{N+1} u_j(x',0)\right)\right) \\ &+ R_{N+1}(u_j, c_j(x',0), 0) + C c_j(x',0)^{\frac{n-1}{2}-1} \partial_{x_n}^{N+2} c_j(x',0) \left(\prod_{k=1}^{N+1} v^{(l_k)}\right) = 0, \end{aligned}$$

$x' \in \Omega$. By the subinduction, the terms $R_N(u_j, c_j(x',0), 0)$ agree for $j = 1,2$. Thus by subtracting, using integration by parts and that $\partial_\nu \partial_{l_1\dots l_{N+1}}^{N+1} u_1(x',0)|_{\partial\Omega} = \partial_\nu \partial_{l_1\dots l_{N+1}}^{N+1} u_2(x',0)|_{\partial\Omega}$ we get

$$\int_\Omega c_j(x',0)^{\frac{n-1}{2}-1} \left(\partial_{x_n}^{N+2} c_1(x',0) - \lambda \partial_{x_n}^{N+2} c_2(x',0)\right) \prod_{k=1}^{N+1} v^{l_k}\, dx' = 0.$$

Choosing all but two of the functions v^{l_k} to be equal to 1, we have by the completeness of such solutions (Proposition 5.1) that

$$\partial_{x_n}^{N+2} c_1(x',0) = \lambda \partial_{x_n}^{N+2} c_2(x',0), \quad x' \in \Omega,$$

which ends the proof for case (1).

Case (2): Now we assume that the partial DN maps coincide for $f \in U_\delta$, $\operatorname{spt}(f) \subset \Gamma$. Then, as in the previous case, from the first linearization we get $c_1(x',0) = \lambda c_2(x',0)$ in Ω (using first the boundary determination from [2], then [10] for $n = 3$ and [12] for $n > 3$). Now define $v^l := v_1^l = v_2^l$ again by uniqueness of solutions.

Moving to the second order linearizations and recovering higher order derivatives produces some extra work since we only have partial data. From the assumption that the DN maps coincide we get

$$\partial_\nu w_1^{al}\big|_\Gamma = \partial_\nu w_2^{al}\big|_\Gamma. \tag{5.7}$$

If we would now integrate the difference of (5.3), for $j = 1$, and (5.4) against $v \equiv 1$ and integrate by parts, some terms would not cancel out. Let us instead introduce the function $v^{(0)}$ which is a solution to

$$\begin{cases} \operatorname{div}\left(c_1(x',0)^{\frac{n-1}{2}} \nabla v^{(0)}\right) = 0, & \text{in } \Omega \\ v^{(0)} = 0, & \text{on } \partial\Omega \setminus \Gamma \\ v^{(0)} = g, & \text{on } \Gamma, \end{cases} \tag{5.8}$$

where $g \in C_c^\infty(\Gamma)$ such that $g \geq 0$ and $g \not\equiv 0$. Then by the maximum principle $v^{(0)} > 0$ in Ω. Now we integrate against this and use integration by parts to have

$$\begin{aligned}
&\int_\Omega \frac{n-1}{2} c_1(x',0)^{\frac{n-1}{2}-1} \left(\partial^3_{x_n} c_1(x',0) - \lambda \partial^3_{x_n} c_2(x',0)\right) v^{(0)} v^l v^a \, dx' \\
&= \int_\Omega \operatorname{div}\left(c_1(x',0)^{\frac{n-1}{2}} \nabla \left(w_1^{(al)} - w_2^{(al)}\right)\right) v^{(0)} \, dx' \\
&= \int_\Omega (w_1^{al} - w_2^{al}) \operatorname{div}\left(c_1(x',0)^{\frac{n-1}{2}} \nabla v^{(0)}\right) dx' \\
&+ \int_\Gamma c_1(x',0)^{\frac{n-1}{2}} \left(\partial_\nu (w_1^{al} - w_2^{al}) v^{(0)} - (w_1^{al} - w_2^{al}) \partial_\nu v^{(0)}\right) dS \\
&+ \int_{\partial\Omega \setminus \Gamma} c_1(x',0)^{\frac{n-1}{2}} \left(\partial_\nu (w_1^{al} - w_2^{al}) v^{(0)} - (w_1^{al} - w_2^{al}) \partial_\nu v^{(0)}\right) dS \\
&= 0.
\end{aligned}$$

In the last inequality we used Eq. (5.7), the fact that $v^{(0)}$ solves (5.8) and that $w_j^{al} = 0$ on $\partial\Omega$ for $j = 1, 2$. Then using Proposition 5.1 and the positivity of $v^{(0)}$ we can conclude

$$\partial^3_{x_n} c_1(x',0) = \lambda \partial^3_{x_n} c_2(x',0), \quad x' \in \Omega.$$

As in the previous case we use induction to show $\partial^k_{x_n} c_1(x',0) = \lambda \partial^k_{x_n} c_2(x',0)$ for all $k \in \mathbb{N}$. By the above this already holds for $k = 0, 1, 2, 3$. Our assumption now is

$$\partial^k_{x_n} c_1(x',0) = \lambda \partial^k_{x_n} c_2(x',0), \quad x' \in \Omega, \text{ for all } \quad k = 0, 1, 2, \ldots, N \in \mathbb{N}, N > 3.$$

By a subinduction we can show that

$$\partial^k_{l_1 \cdots l_k} u_1(x',0) = \partial^k_{l_1 \cdots l_k} u_2(x',0), \quad x' \in \Omega,$$

for all $k = 1, \ldots, N$, where $\partial^k_{l_1 \ldots l_k} u_j(x',0) = \frac{\partial^k u_j(x',0)}{\partial \varepsilon_{l_1} \cdots \partial \varepsilon_{l_k}}$. This goes in the same way as in the previous case except the integration by parts argument needs to be done as shown in this case.

Returning to the original induction, the rest of the proof is again the same as in case (2). We only need to modify the integration by parts argument using again the function $v^{(0)}$ and Proposition 5.1 which finishes the proof. □

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(B)

DETERMINING AN UNBOUNDED POTENTIAL FOR AN ELLIPTIC EQUATION WITH A POWER TYPE NONLINEARITY

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In this article we focus on inverse problems for a semilinear elliptic equation. We show that a potential q in $L^{n/2+\varepsilon}$, $\varepsilon > 0$, can be determined from the full and partial Dirichlet-to-Neumann map. This extends the results from [20] where this is shown for Hölder continuous potentials. Also we show that when the Dirichlet-to-Neumann map is restricted to one point on the boundary, it is possible to determine a potential q in $L^{n+\varepsilon}$. The authors of [25] proved this to be true for Hölder continuous potentials.

Contents

1. Introduction

In this paper we consider an inverse problem of determining a potential in $L^{\frac{n}{2}+\varepsilon}$, for positive ε, from the Dirichlet-to-Neumann (DN) map related to the boundary value problem for a semilinear elliptic equation

$$\begin{cases} \Delta u + qu^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $m \geq 2$, $m \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^n$ open and bounded. This boundary value problem is well posed for $q \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ and a certain class of boundary values. In fact we show that there is $\delta > 0$ such that for all (see [21] for Sobolev spaces)

$$f \in U_\delta := \{h \in W^{2-\frac{1}{p},p}(\partial\Omega) \colon ||h||_{W^{2-\frac{1}{p},p}(\partial\Omega)} < \delta\}$$

there exists a unique small solution $u \in W^{2,p}(\Omega)$ with sufficiently small norm. Here and in the rest of this article, we denote $p := \frac{n}{2} + \varepsilon$. Thus the DN map can be defined as

$$\Lambda_q \colon U_\delta \to W^{1-\frac{1}{p},p}(\partial\Omega), \quad f \mapsto \partial_\nu u_f|_{\partial\Omega}.$$

Our first main result shows that we can determine the potential from the knowledge of the DN map.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, $\varepsilon > 0$ and $q_1, q_2 \in L^{\frac{n}{2}+\varepsilon}(\Omega)$. Let Λ_{q_j} be the DN maps associated to the boundary value problems*

$$\begin{cases} \Delta u + q_j u^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

for $j = 1, 2$, and assume that $\Lambda_{q_1} f = \Lambda_{q_2} f$ for all $f \in U_\delta$ with $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω.

This result is a special case of Theorem 1.2 but we give a proof because it is helpful for the other two main theorems of this paper. Also the proof of Theorem 1.1 gives a reconstruction formula for the potential q via the Fourier transform (see Corollary 3.1).

The proof Theorem 1.1 is quite similar as in [19] and it uses the method of higher order linearization first introduced in [18] and further developed in the works [9], [19]. The key ingredient in this proof is the following integral identity which characterizes the m-th order linearization of the DN map $(D^m\Lambda_q)_0$ at 0 [19, Proposition 2.2]:

$$\int_{\partial\Omega} (D^m\Lambda_{q_1} - D^m\Lambda_{q_2})_0(f_1, \ldots, f_m) f_{m+1}\, dS = -(m!) \int_{\Omega} (q_1 - q_2) v_{f_1} \cdots v_{f_{m+1}}\, dx. \tag{1.3}$$

Here v_{f_k} are solutions to $\Delta v_{f_k} = 0$ with boundary values $v_{f_k}|_{\partial\Omega} = f_k$. Using this integral identity together with a result on density of products of solutions eventually gives $q_1 = q_2$ in Ω.

Theorem 1.1 has been proved for Hölder continuous potentials in [9] and [19] but in this article we give a first result for a less regular potential (at least to the best of our knowledge). The difference is in proving that (1.2) is well-posed when the potential is in $L^p(\Omega)$ and defining the DN map as a map from U_δ to $W^{1-\frac{1}{p},p}(\partial\Omega)$.

In the linear case $(\Delta + q)u = 0$, when $n \geq 3$, a similar result for $q \in L^{\frac{n}{2}}(\Omega)$ has been obtained in the works [23], [6] and in a more general Riemannian manifold setting in [8], where they used L^p Carleman estimates in their proof. The case $q \in L^{\frac{n}{2}}(\Omega)$ is considered optimal in the sense of standard well-posedness theory and for the strong unique continuation principle [15]. There are also results when one assumes that $q \in W^{-1,n}(\Omega)$, see for example [11]. When $n = 2$ the lowest regularity for the potential to have uniqueness in the inverse problem, at least to the best of our knowledge, is $L^{\frac{4}{3}}(\Omega)$ [3]. The same result is true on compact Riemannian surfaces with smooth boundary [22]. In dimension two the unique continuation principle holds for potentials in $L^p(\Omega)$ where $p > 1$ (see for example [1], [2]).

In addition to the full data case, we consider some partial data results for the Schrödinger equation with unbounded potentials. In particular, let Γ be an open subset of the boundary $\partial\Omega$. Define the partial Dirichlet-to-Neumann map for $f \in U_\delta$, $\text{spt}(f) \subset \Gamma$, as

$$\Lambda_q^\Gamma f = \partial_\nu u|_\Gamma.$$

Then from the knowledge of this partial DN map it is possible to determine the potential.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected open and bounded set with C^∞ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial\Omega$. Let $\varepsilon > 0$, $q_1, q_2 \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ and $\Lambda_{q_j}^\Gamma$ be the partial DN maps associated to the boundary value problems*

$$\begin{cases} \Delta u + q_j u^m = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \setminus \Gamma \\ u = f, & \text{on } \Gamma \end{cases}$$

for $j = 1, 2$. Assume that

$$\Lambda_{q_1}^\Gamma f = \Lambda_{q_2}^\Gamma f$$

for all $f \in U_\delta$ with $\operatorname{spt}(f) \subset \Gamma$, where $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω.

When the potentials are assumed to be Hölder continuous, then this theorem has been proved in [17] and [20] using the method of higher order linearization, which we will also use. Here again the key ingredients are the integral identity (1.3) and a density result for solutions of the Laplacian [25] (see also [5, Section 4]).

For the linear Schrödinger equation, partial data results with unbounded potentials have been proved only for special cases of partial data. When $n \geq 3$, it is proved in [7] that from the knowledge of the partial DN map in a specific situation it is possible to determine a potential in $L^{\frac{n}{2}}(\Omega)$. The authors use a method involving the construction of a Dirichlet Green's function for the conjugated Laplacian. In a similar situation on a manifold setting, [26] shows that a potential in $L^{\frac{n}{2}}$ can be determined from a particular case of partial data. When $n = 2$ the best known result for the case of an arbitrary open subset of the boundary is for potentials in the Sobolev space $W^{1,p}(\Omega)$, for $p > 2$ [14].

For partial data results, there is still the case when we are restricted to only one point on the boundary. In the situation of $\Delta u + qu^m$ with the potential q in $C^\alpha(\bar{\Omega})$ this has been proved in [25] using the method of higher order linearization. Here we show that the same result holds even if we only assume that $q \in L^{n+\varepsilon}(\Omega)$ for a positive ε.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected open and bounded set with C^∞ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial\Omega$. Suppose that $\mu \not\equiv 0$ is a fixed measure on $\partial\Omega$ and let $\varepsilon > 0$. Assume that $q_1, q_2 \in L^{n+\varepsilon}(\Omega)$ satisfy*

$$\int_{\partial\Omega} \Lambda_{q_1}(f)\, d\mu = \int_{\partial\Omega} \Lambda_{q_2}(f)\, d\mu \tag{1.4}$$

for all $f \in U_\delta$ with $\operatorname{spt}(f) \subset \Gamma$, where $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω. Thus when choosing $\mu = \delta_{x_0}$ for some fixed $x_0 \in \partial\Omega$ the condition

$$\Lambda_{q_1}(f)(x_0) = \Lambda_{q_2}(f)(x_0) \quad \text{for all } f \in U_\delta \text{ with } \operatorname{spt}(f) \subset \Gamma$$

gives $q_1 = q_2$ in Ω.

The proof of this theorem is very similar to the one in [25] and it uses heavily the identity (1.3) and a density result for solutions of the Laplacian [25].

It is an interesting question if in Theorems 1.1 and 1.2 it is enough to assume the potential q to be in $L^{\frac{n}{2}}(\Omega)$ and if in Theorem 1.3 the potential q could be in $L^s(\Omega)$ for $s = n$ or even $s < n$. The argument given for Theorems 1.1 and 1.2 fails when $q \in L^{\frac{n}{2}}(\Omega)$ since the well-posedness (Theorem 2.1) relies on Sobolev embedding theorems that fail for the exponent $\frac{n}{2}$. For Theorem 1.3 the restriction to $s > n$ comes from Lemma 5.1 and that we again use Sobolev embedding theorems that do not work for the exponent n or exponents less than n.

The rest of this paper is organized as follows. In section 2 we prove the well-posedness of the boundary value problem (1.1). In sections 3 to 5 the proofs for Theorems 1.1, 1.2 and 1.3 are given.

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2. Well-posedness

A short reminder for the reader that we denote here and in the rest of this article $p := \frac{n}{2} + \varepsilon$.

Theorem 2.1. *(Well-posedness) Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded open set with C^∞ boundary, $\varepsilon > 0$ and let $q \in L^p(\Omega)$. Then there exist $\delta, C > 0$ such that for any*

$$f \in U_\delta := \{h \in W^{2-\frac{1}{p},p}(\partial\Omega) \colon ||h||_{W^{2-\frac{1}{p},p}(\partial\Omega)} < \delta\},$$

there is a unique small solution u_f in the class $\{v \in W^{2,p}(\Omega) \colon ||w||_{W^{2,p}(\Omega)} \leq C\delta\}$ of the boundary value problem

$$\begin{cases} \Delta u + qu^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $m \in \mathbb{N}$ and $m \geq 2$. Moreover

$$||u||_{W^{2,p}(\Omega)} \leq C||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)},$$

and there are C^∞ maps

$$S \colon U_\delta \to W^{2,p}(\Omega), \quad f \mapsto u_f,$$
$$\Lambda_q \colon U_\delta \to W^{1-\frac{1}{p},p}(\partial\Omega), \quad f \mapsto \partial_\nu u_f|_{\partial\Omega}.$$

The proof uses the implicit function theorem between Banach spaces [24, Theorem 10.6 and Remark 10.5] and is very similar to the one in [19, Proposition 2.1]. The difference here is that we replace Hölder spaces with Sobolev spaces and one needs to be careful with various embeddings for these spaces.

Proof. Let

$$X = W^{2-\frac{1}{p},p}(\partial\Omega), \quad Y = W^{2,p}(\Omega), \quad Z = L^p(\Omega) \times W^{2-\frac{1}{p},p}(\partial\Omega)$$

and $F \colon X \times Y \to Z$,

$$F(f, u) = (Q(u), u|_{\partial\Omega} - f),$$

where $Q(u) = \Delta u + qu^m$. Let us now show that F has the claimed mapping property. Since $u \in W^{2,p}(\Omega)$, this implies that $u|_{\partial\Omega} \in W^{2-\frac{1}{p},p}(\partial\Omega)$ (see [21]) and $\Delta u \in L^p(\Omega)$. Hence we need to show that the term $qu^m \in L^p(\Omega)$. Since $2\left(\frac{n}{2}+\varepsilon\right) > n$, then by the Sobolev embedding theorem [21] $u \in C^{0,\alpha}(\bar{\Omega})$, for $0 < \alpha < 1$, which is a subset of $L^s(\Omega)$ for every $1 \le s \le \infty$. Now this implies

$$||qu^m||_{L^p(\Omega)} \le ||q||_{L^p(\Omega)} ||u^m||_{L^\infty(\Omega)} \le ||q||_{L^p(\Omega)} \left(||u||_{L^\infty(\Omega)}\right)^m < \infty$$

and thus $qu^m \in L^p(\Omega)$. Hence F has the claimed mapping property.

Next we want to show that F is a C^∞ mapping. Since $u \mapsto \Delta u$ is a linear map $W^{2,p}(\Omega) \to L^p(\Omega)$, it is enough to show that $u \mapsto qu^m$ is a C^∞ map $W^{2,p}(\Omega) \to L^p(\Omega)$. This follows since u^m is a polynomial. More precisely, let $u, v \in W^{2,p}(\Omega)$ and use the Taylor formula:

$$\begin{aligned} q(u+v)^m &= \sum_{j=0}^{m} \frac{\partial_u^j (qu^m)}{j!} v^j + \int_0^1 \frac{\partial_u^{m+1}\left(q(u+tv)^m\right)}{m!} v^{m+1}(1-t)\, dt \\ &= \sum_{j=0}^{m} \frac{\partial_u^j (qu^m)}{j!} v^j. \end{aligned}$$

Now for $||v||_{W^{2,p}(\Omega)} \le 1$ the above gives

$$\left|\left| q(u+v)^m - \sum_{j=0}^{m} \frac{\partial_u^j (qu^m)}{j!} v^j \right|\right|_{L^p(\Omega)} = 0 \le ||v||^{k+1}_{W^{2,p}(\Omega)}$$

and thus the map $u \mapsto q(x)u^m$ is C^k (in the sense of [24, Definition 10.2]) for all $k \in \mathbb{N}$. Hence it is a C^∞ map and F is also C^∞.

Our aim is to use the implicit function theorem for Banach spaces to get a unique solution for the boundary value problem (2.1). Firstly, the linearization of F at $(0,0)$ in the second variable is

$$D_u F|_{(0,0)}(v) = (\Delta v, v|_{\partial\Omega}),$$

which is linear and also $F(0,0) = 0$. Secondly, $D_u F|_{(0,0)} \colon Y \to Z$ is a homeomorphism. To see this, let $(\phi, g) \in Z$ and consider the boundary value problem

$$\begin{cases} \Delta v = \phi, & \text{in } \Omega \\ v = g, & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique solution for each pair (ϕ, g) (see for example [10, Theorem 9.15]), and thus $D_u F|_{(0,0)}$ is bijective. We also have the estimate

$$||D_u F|_{(0,0)}(v)||_Z^2 = ||\Delta v||^2_{L^p(\Omega)} + ||v|_{\partial\Omega}||^2_{W^{2-\frac{1}{p},p}(\partial\Omega)} \le M ||v||^2_{W^{2,p}(\Omega)},$$

because the trace operator from $W^{2,p}(\Omega)$ to $W^{2-\frac{1}{p},p}(\partial\Omega)$ is bounded (see [21]). Hence $D_u F|_{(0,0)}$ is also bounded and then the open mapping theorem (see e.g. [24, Theorem 8.33]) tells us that it is also a homeomorphism.

Now by the implicit function theorem [24, Theorem 10.6] there exists $\delta > 0$, a neighborhood $U_\delta = B(0,\delta) \subset X$ and a C^∞ map $S \colon U_\delta \to Y$ such that $F(f, S(f)) = 0$ for $||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)} \le \delta$. Now S is also Lipschitz continuous, $S(0) = 0, S(f) = u$ and thus we have

$$||u||_{W^{2,p}(\Omega)} \leq C||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)}$$

for $C > 0$. By redefining δ if necessary we have the estimates $||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)} \leq \delta$, $||u||_{W^{2,p}(\Omega)} \leq C\delta$ and the implicit function theorem gives that u is the unique small solution of $F(f,u) = 0$. Also the solution operator $S\colon U_\delta \to W^{2,p}(\Omega)$ is a C^∞ map. Because $u \in W^{2,p}(\Omega)$, then $\nabla u \in W^{1,p}(\Omega)$. The trace operator is a bounded linear map from $W^{1,p}(\Omega)$ to $W^{1-\frac{1}{p},p}(\partial\Omega)$ (see [21]) and thus $\partial_\nu u \in W^{1-\frac{1}{p},p}(\partial\Omega)$ is defined almost everywhere on $\partial\Omega$. Hence Λ_q is a well defined C^∞ map between U_δ and $W^{1-\frac{1}{p},p}(\partial\Omega)$. □

Remark 2.2. In the previous proof, we showed that the mapping $D_uF|_{(0,0)}$ is bijective and bounded and deduced that it is a homeomorphism. An alternative way to see this is to look at the inverse map $(D_uF|_{(0,0)})^{-1}\colon Z \to Y$ and show that it is bijective and bounded. In order to do this, one needs to prove the following estimate:

$$||v||_{W^{2,p}(\Omega)} \leq C\left(||\phi||_{L^p(\Omega)} + ||g||_{W^{2-\frac{1}{p},p}(\partial\Omega)}\right),$$

where $C > 0$ does not depend on v, ϕ and g. This can be done for example by combining the estimate

$$||v||_{W^{2,p}(\Omega)} \leq C\left(||\phi||_{L^p(\Omega)} + ||g||_{W^{2-\frac{1}{p},p}(\partial\Omega)} + ||v||_{L^p(\Omega)}\right)$$

from [27, Theorem 9.1.3] with the assumption that 0 is not a Dirichlet eigenvalue and using a compactness argument.

3. Proof of Theorem 1.1

Using the method of higher order linearization we prove that it is possible to determine a potential in $L^p(\Omega)$ from the knowledge of full DN map.

Proof of Theorem 1.1. Let $\lambda_1, \ldots, \lambda_m$ be sufficiently small numbers, $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $f_1, \ldots, f_m \in W^{2-\frac{1}{p},p}(\partial\Omega)$. Let $u_j(x,\lambda) \in W^{2,p}(\Omega)$ be the unique small solution to

$$\begin{cases} \Delta u_j + q_j u_j^m = 0, & \text{in } \Omega \\ u_j = \sum_{k=1}^m \lambda_k f_k, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Differentiating this with respect to $\lambda_l, l \in \{1, \ldots, m\}$ (possible by Theorem 2.1 which shows that S is a C^∞ map) and setting $\lambda = 0$ gives that $v_j^l := \partial_{\lambda_l} u_j(x,\lambda)|_{\lambda=0}$ satisfies

$$\begin{cases} \Delta v_j^l = 0, & \text{in } \Omega \\ v_j^l = f_l, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

This has a unique solution in $W^{2,p}(\Omega)$ (see for example [10, Theorem 9.15]) and thus we can define $v^l := v_1^l = v_2^l$. Also the first linearizations of the DN maps Λ_{q_j} are the DN maps of the Laplace equation.

Let $1 < a \leq m-1$ be an integer and $l_1, \ldots, l_a \in \{1, \ldots, m\}$. Then the a-th order linearization of (3.1) is

$$\begin{cases} \Delta(\partial_{\lambda_{l_1}} \cdots \partial_{\lambda_{l_a}} u_j(x,\lambda)|_{\lambda=0}) = 0, & \text{in } \Omega \\ \partial_{\lambda_{l_1}} \cdots \partial_{\lambda_{l_a}} u_j(x,\lambda)|_{\lambda=0} = 0, & \text{on } \partial\Omega, \end{cases}$$

and uniqueness of solutions for the Laplace equation gives that 0 is the only solution. Thus the a-th order linearizations of the DN maps Λ_{q_j} are equal to 0.

Moving to the m-th order linearization, we apply $\partial_{\lambda_1}\cdots\partial_{\lambda_m}|_{\lambda=0}$ to (3.1) which results in the boundary value problem

$$\begin{cases} \Delta w_j = -m! q_j \prod_{k=1}^m v^k, & \text{in } \Omega \\ w_j = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Here $w_j = \partial_{\lambda_1}\cdots\partial_{\lambda_m} u_j(x,\lambda)|_{\lambda=0}$ and the functions $v^k, k \in \{1,\dots,m\}$, are solutions to equation (3.2) with corresponding boundary values f_k. On the left hand side of (3.3) we are only left with a product of functions v^k, since after differentiating (3.1) m times with respect to ε, all other terms involve a positive power of u_j. Proposition 2.1 says that the solution u_j depends smoothly on ε and thus when evaluating at $\varepsilon = 0$, the function u_j vanishes.

By our assumptions we have that $\Lambda_{q_1}(\sum_{k=1}^m \lambda_k f_k) = \Lambda_{q_2}(\sum_{k=1}^m \lambda_k f_k)$ and thus $\partial_\nu u_1|_{\partial\Omega} = \partial_\nu u_2|_{\partial\Omega}$. Applying $\partial_{\lambda_1}\cdots\partial_{\lambda_m}|_{\lambda=0}$ to this gives $\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}$. Subtracting (3.3) for $j = 1, 2$ and integrating against $v \equiv 1$ (a solution of (3.2)) over Ω implies

$$\int_\Omega m!(q_1 - q_2) \prod_{k=1}^m v^k \, dx = -\int_\Omega \Delta(w_1 - w_2)\, dx = -\int_{\partial\Omega} \partial_\nu (w_1 - w_2)\, dS = 0. \tag{3.4}$$

Let us now choose v^1, v^2 to be the Calderón's exponential solutions [4]

$$v^1(x) := e^{(\eta + i\xi)\cdot x}, \quad v^2(x) := e^{(-\eta + i\xi)\cdot x}, \tag{3.5}$$

where $\eta, \xi \in \mathbb{R}^n$, $\eta \perp \xi$ and $|\eta| = |\xi|$, and $v^k \equiv 1$ for $k = 3, \dots, m$. Then we get that the Fourier transform of the difference $q_1 - q_2$ at -2ξ vanishes. Thus $q_1 = q_2$ since ξ was arbitrary. □

Notice that this proof gives a reconstruction formula for the potential. In particular, inspecting the last lines after equation (3.4) we have the following result which reconstructs the potential q via its Fourier transform.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, $\varepsilon > 0$ and $q \in L^p(\Omega)$. Let Λ_q be the DN map associated to the boundary value problem*

$$\begin{cases} \Delta u + q u^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases}$$

Then, denoting $\lambda = (\lambda_1, \dots, \lambda_m)$,

$$\hat{q}(-2\xi) = -\frac{1}{m!} \int_{\partial\Omega} \frac{\partial^m}{\partial\lambda_1 \cdots \partial\lambda_m}|_{\lambda=0} \Lambda_q \left(\sum_{k=1}^m \lambda_k f_k \right) dS,$$

where f_1, f_2 are the boundary values of Calderón's exponential solutions (3.5), $f_k \equiv 1$ for $3 \leq k \leq m$ and $\hat{q}$ is the Fourier transform of q.

4. Proof of Theorem 1.2

We prove the partial data result for determining a potential in $L^p(\Omega)$ by using higher order linearization. The proof uses similar techniques as in [17] and [20].

Proof of Theorem 1.2. Let $\lambda_1, \dots, \lambda_m$ be sufficiently small numbers, $\lambda = (\lambda_1, \dots, \lambda_m)$ and $f_1, \dots, f_m \in W^{2-\frac{1}{p},p}(\partial\Omega)$ with $\mathrm{spt}(f) \subset \Gamma$. Let $u_j(x, \lambda) \in W^{2,p}(\Omega)$ be the unique small solution to

$$\begin{cases} \Delta u_j + q_j u_j^m = 0, & \text{in } \Omega \\ u_j = \sum_{k=1}^m \lambda_k f_k, & \text{on } \partial\Omega. \end{cases}$$

The first and m-th order linearizations are the same as in the proof of Theorem 1.1, with corresponding boundary values. We also define $v^l := v_1^l = v_2^l$ by uniqueness of solutions to (3.2). Let $v^{(0)}$ be the solution to

$$\begin{cases} \Delta v^{(0)} = 0, & \text{in } \Omega \\ v^{(0)} = 0, & \text{on } \partial\Omega \setminus \Gamma \\ v^{(0)} = g, & \text{on } \Gamma, \end{cases}$$

where $g \in C_c^\infty(\Gamma)$ with g non-negative and not identically zero. By the maximum principle, $v^{(0)} > 0$ in Ω. Then subtracting (3.3) for $j = 1, 2$ and integrating against $v^{(0)}$ gives the following integral identity (compare to (3.4))

$$\begin{aligned} -\int_\Omega m!(q_1 - q_2) v^{(0)} \prod_{k=1}^m v^k \, dx &= \int_\Omega \Delta(w_1 - w_2) v^{(0)} \, dx \\ &= \int_\Omega (w_1 - w_2) \Delta v^{(0)} \, dx \\ &\quad + \int_{\partial\Omega} v^{(0)} \partial_\nu (w_1 - w_2) - (w_1 - w_2) \partial_\nu v^{(0)} \, dS \\ &= \int_{\partial\Omega} v^{(0)} \partial_\nu (w_1 - w_2) - (w_1 - w_2) \partial_\nu v^{(0)} \, dS \end{aligned} \tag{4.1}$$

Here Green's formula and the fact that $\Delta v^{(0)} = 0$ in Ω were used. Now our assumption on the DN maps coinciding gives $\partial_\nu u_1|_\Gamma = \partial_\nu u_2|_\Gamma$ and when applying $\partial_{\lambda_1} \cdots \partial_{\lambda_m}|_{\lambda=0}$ to this, we have $\partial_\nu w_1|_\Gamma = \partial_\nu w_2|_\Gamma$. Also $w_1 - w_2 = 0$ on $\partial\Omega$ by (3.3) and $v^{(0)} = 0$ on $\partial\Omega \setminus \Gamma$. Using these (4.1) becomes

$$\begin{aligned} -\int_\Omega m!(q_1 - q_2) v^{(0)} \prod_{k=1}^m v^k \, dx &= \int_{\partial\Omega} v^{(0)} \partial_\nu (w_1 - w_2) - (w_1 - w_2) \partial_\nu v^{(0)} \, dS \\ &= \int_{\partial\Omega\setminus\Gamma} v^{(0)} \partial_\nu (w_1 - w_2) \, dS + \int_\Gamma v^{(0)} \partial_\nu (w_1 - w_2) \, dS \\ &= 0. \end{aligned} \tag{4.2}$$

Now we can apply Theorem 1.3 in [25] (see also [5, Section 4]) which says that the set of products of two harmonic functions that vanish on $\partial\Omega \setminus \Gamma$ is dense in $L^1(\Omega)$. Thus we can conclude from (4.2) that

$$m!(q_1 - q_2) v^{(0)} \prod_{k=3}^m v^k = 0 \quad \text{in } \Omega.$$

Let $f_k \in C_c^\infty(\Gamma)$, f_k non-negative and $f_k > 0$ somewhere for $k = 3, \dots, m$. Then again the maximum principle gives that $v^k > 0$ in Ω. Combining this with $v^{(0)} > 0$ in Ω then implies $q_1 = q_2 \quad \text{in } \Omega$. □

5. Proof of Theorem 1.3

As in [25], we need a lemma stating that the solution to the boundary value problem with a finite Borel measure μ as boundary value is in $L^r(\Omega)$ for $1 \leq r < \frac{n}{n-1}$. For the lemma, denote by r' the dual exponent of $1 \leq r \leq \infty$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded open set with C^∞ boundary and μ a finite complex Borel measure on $\partial\Omega$. Then for the function*

$$\Psi(x) = \int_{\partial\Omega} P(x,y)\, d\mu(y), \quad x \in \Omega, \tag{5.1}$$

where $P(x,y)$ is the Poisson kernel for Δ in Ω, we have $\Psi \in L^r(\Omega)$, $1 \leq r < \frac{n}{n-1}$. Additionally Ψ solves the boundary value problem

$$\begin{cases} \Delta\Psi = 0, & \text{in } \Omega \\ \Psi = \mu, & \text{on } \partial\Omega, \end{cases}$$

where $\Psi = \mu$ on $\partial\Omega$ means that for any $w \in W^{2,r'}(\Omega)$ with $w|_{\partial\Omega} = 0$, in trace sense, one has

$$\int_{\partial\Omega} \partial_\nu w\, d\mu = \int_{\Omega} (\Delta w)\Psi\, dx. \tag{5.2}$$

Notice that the left hand side of relation (5.2) is well defined since $\partial_\nu w$ is continuous by the Sobolev embedding theorem (see for example [21]): The assumption $w \in W^{2,r'}(\Omega)$ says that $\nabla w \in W^{1,r'}(\Omega)$. This space embeds to $C^{0,1-\frac{n}{r'}}(\bar{\Omega})$ if $r' > n$. Notice that $r' > n$ is equivalent with the assumption that $1 \leq r < \frac{n}{n-1}$. Also the right hand side of (5.2) is well defined by the fact that $\Delta w \in L^{r'}(\Omega), \Psi \in L^r(\Omega)$ implies $(\Delta w)\Psi \in L^1(\Omega)$.

The proof of this lemma is the same as in [25, Lemma 2.1.]. The only difference when compared to the statement in [25], is that we assume $w \in W^{2,r'}(\Omega)$ instead of $w \in C^2(\bar{\Omega})$.

Proof of Theorem 1.3. As before, we use the method of higher order linearization. Let $\lambda_1, \ldots, \lambda_m$ be sufficiently small numbers, $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $f_1, \ldots, f_m \in W^{2-\frac{1}{p},p}(\partial\Omega)$ with $\mathrm{spt}(f) \subset \Gamma$. Let $u_j(x,\lambda) \in W^{2,p}(\Omega)$ be the unique small solution to

$$\begin{cases} \Delta u_j + q_j u_j^m = 0, & \text{in } \Omega \\ u_j = \sum_{k=1}^m \lambda_k f_k, & \text{on } \partial\Omega. \end{cases}$$

The first and m-th order linearizations are the same as in the proof of Theorem 1.1, with corresponding boundary values. We also define $v^l := v_1^l = v_2^l$ by uniqueness of solutions to (3.2).

Let $\varepsilon > 0$ and $q_1, q_2 \in L^{n+\varepsilon}(\Omega)$ be such that (1.4) holds for all $f \in U_\delta$, $\mathrm{spt}(f) \subset \Gamma$ with sufficiently small δ. From $\partial_{\lambda_1} \cdots \partial_{\lambda_m} \Lambda_{q_j}(f) = \partial_{\lambda_1} \cdots \partial_{\lambda_m} \partial_\nu u_j|_{\partial\Omega} = \partial_\nu w_j|_{\partial\Omega}$, where w_j is the solution to (3.3), and equation (1.4) we get that

$$\int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, d\mu = 0.$$

Let $\Psi \in L^{(n+\varepsilon)'}(\Omega)$ be the function given by (5.1) which is a solution to

$$\begin{cases} \Delta\Psi = 0, & \text{in } \Omega \\ \Psi = \mu, & \text{on } \partial\Omega \end{cases}$$

in the sense of Lemma 5.1. Notice that $(n+\varepsilon)' < \frac{n}{n-1}$ and $w_j \in W^{2,n+\varepsilon}(\Omega)$ because $-m!q_j \prod_{k=1}^m v^k \in L^{n+\varepsilon}(\Omega)$ (see for example [10, Theorem 9.15]). Thus combining (5.2) and (3.3) gives

$$0 = \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, d\mu = \int_{\Omega} \Delta(w_1 - w_2)\Psi\, dx = -\int_{\Omega} m!(q_1 - q_2) \prod_{k=1}^{m} v^k \Psi\, dx,$$

where each v^k is a solution to the Laplace equation with corresponding boundary value f_k. Let $f_3, \dots, f_m \in C^\infty(\partial\Omega)$ be such that $\operatorname{spt}(f_k) \subset \Gamma$, $f_k \geq 0$ and $f_k > 0$ somewhere, then by the maximum principle $v^k > 0$ in Ω. Choosing the boundary values $f_1, f_2 \in C^\infty(\partial\Omega)$, $\operatorname{spt}(f_1), \operatorname{spt}(f_2) \subset \Gamma$, we get by elliptic regularity that v^1, v^2 are smooth and thus we may apply Theorem 1.3 from [25] (see also [5, Section 4]) to get

$$m!(q_1 - q_2)v_3 \cdots v_m \Psi = 0 \quad \text{a.e. in } \Omega.$$

The positivity of $v_3, \dots, v_m$ implies that $(q_1 - q_2)\Psi = 0$ a.e. in Ω. Now we claim that Ψ cannot vanish in any set $E \subset \Omega$ of positive measure. This can be seen as follows: We argue by contradiction and assume that $\Psi = 0$ in $E \subset \Omega$ where E has positive measure. Then by a unique continuation principle (see for example [12], $n > 2$, and for $n = 2$ [13]) $\Psi = 0$ in Ω. From [16] there is a constant $c > 0$ such that for all $(x, y) \in \Omega \times \partial\Omega$

$$c \cdot \frac{\operatorname{dist}(x, \partial\Omega)}{|x - y|^n} \leq P(x, y).$$

In view of the definition of Ψ in (5.1) this would imply that $\mu \equiv 0$ which is a contradiction. Hence we must have that $q_1 = q_2$ a.e. in Ω. □

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AN INVERSE PROBLEM FOR THE MINIMAL SURFACE EQUATION IN THE PRESENCE OF A RIEMANNIAN METRIC

by

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AN INVERSE PROBLEM FOR THE MINIMAL SURFACE EQUATION IN THE PRESENCE OF A RIEMANNIAN METRIC

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ABSTRACT. In this work we study an inverse problem for the minimal surface equation on a Riemannian manifold $(\mathbb{R}^n, g)$ where the metric is of the form $g(x) = c(x)(\hat{g} \oplus e)$. Here $\hat{g}$ is a simple Riemannian metric on $\mathbb{R}^{n-1}$, e is the Euclidean metric on $\mathbb{R}$ and c a smooth positive function. We show that if we know the associated Dirichlet-to-Neumann maps corresponding to metrics g and $\tilde{c}g$, then the Taylor series of the conformal factor $\tilde{c}$ at $x_n = 0$ is equal to a positive constant. We also show a partial data result when $n = 3$.

Keywords. Inverse problem, higher order linearization, quasilinear elliptic equation, minimal surface equation

CONTENTS

1. INTRODUCTION

This article focuses on an inverse problem for the minimal surface equation (MSE), which is a quasilinear elliptic PDE. In particular we consider MSE on a manifold $(\mathbb{R}^n, g), n \geq 3$, where

$$g(x', x_n) = c(x', x_n) \begin{pmatrix} \hat{g}(x') & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.1}$$

Here $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $\hat{g}$ is a Riemannian metric on $\mathbb{R}^{n-1}$ and $c \in C^\infty(\mathbb{R}^n)$, $c(x) > 0$ for all $x \in \mathbb{R}^n$. Inverse problems for the MSE were considered in [Nur23a] where $\hat{g}$ was Euclidean and in [CLLO22] where $c = 1$. Now we consider metrics of the form (1.1) which includes both of these cases. Metrics with local coordinates of the form (1.1) were introduced for example in [DKSU09], [DKLS16] where the Calderón problem was studied.

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Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain. In order to define a minimal surface, we introduce notation for Riemannian Hessian and norm of the gradient

$$\nabla_g^2 u = \left(\partial^2_{x_i x_j} u - \Gamma^m{}_{ij}\partial_{x_m} u\right)_{i,j=1}^n, \quad |\nabla_g u|_g^2 = g^{ij}\partial_{x_i} u \partial_{x_j} u.$$

We use the Einstein summation convention where there is no chance to misinterpret. Also here $\Gamma^m{}_{ij}$ are the standard Christoffel symbols. Now define the Laplace-Beltrami operator to be the trace of the Hessian

$$\Delta_g u = \mathrm{Tr}(\nabla_g^2 u) = g^{ij}\left(\partial^2_{x_i x_j} u - \Gamma^m{}_{ij}\partial_{x_m} u\right).$$

Then for the metric (1.1) the MSE has the form

(1.2)

$$\mathrm{div}_g\left(\frac{\nabla_g u}{(1+|\nabla_g u|_g^2)^{\frac{1}{2}}}\right) + \frac{\Delta_g u(1-c^{-1}) + \frac{1}{2c}(\nabla_g u)^j \partial_{x_j} c(1+|\nabla_g u|_g^2 - c^{-2}) + \frac{n-1}{2c^3}\partial_{x_n} c(1+|\nabla_{\hat{g}} u|_{\hat{g}}^2)}{(1+|\nabla_g u|_g^2)^{\frac{3}{2}}} = 0.$$

For the derivation of this equation, see Section 2. When $c = 1$, one gets the MSE from [CLLO22]. The graph of a function $u\colon \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is called a *minimal surface* if it satisfies (1.2) for all $x' \in \Omega$.

We will look at the boundary value problem

$$\begin{cases} F(x', u, \nabla_g u, \nabla_g^2 u) = 0 & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where $F\colon \mathbb{R}^{n^2} \to \mathbb{R}$,

$$F(x', u, \nabla_g u, \nabla_g^2 u) := \left(-\Delta_{\hat{g}} u + \frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r} c \partial_{x_m} u + \frac{n-1}{2c}\partial_{x_n} c\right)(1+|\nabla_{\hat{g}} u|_{\hat{g}}^2) + \nabla_{\hat{g}}^2 u(\nabla_{\hat{g}} u, \nabla_{\hat{g}} u)$$

(this is an equivalent formulation of (1.2), see Section 2). One can show that (1.3) is well-posed for small enough boundary data f for example by following the arguments in [LLLS21a]. In particular one can show that there exist $C, \delta > 0$ and $s > 3$, $s \notin \mathbb{N}$, such that for all

$$f \in U_\delta := \{h \in C^s(\partial\Omega) : ||h||_{C^s(\partial\Omega)} < \delta\}$$

there is a unique small solution u_f in $\{v \in C^s(\bar{\Omega}) : ||v||_{C^s(\bar{\Omega})} < C\delta\}$. In addition to full data we will be considering partial data and for this let $\Gamma \subset \partial\Omega$ be a nonempty open subset of the boundary. We can now define a (partial) Dirichlet-to-Neumann map (DN map) $\Lambda_g^\Gamma\colon C_c^s(\Gamma) \to C^{s-1}(\Gamma)$,

$$\Lambda_g^\Gamma f \mapsto \partial_\nu u_f|_\Gamma \tag{1.4}$$

where u_f is the unique small solution corresponding to the boundary value f, $\partial_\nu u = \hat{g}^{ij}\partial_{x_i} u \nu_j$ and ν is the unit normal of $\partial\Omega$. When $\Gamma = \partial\Omega$, we will denote $\Lambda_g = \Lambda_g^{\partial\Omega}$.

The inverse problem we study in this work is the following: Given the knowledge of the partial DN map for two metrics in the same conformal class, does it hold that the metrics are the same? We note that when trying to recover the metric from boundary measurements one expects an obstruction to uniqueness by a boundary fixing diffeomorphism. When the metrics are in the same conformal class such an obstruction is not present, since the only diffeomorphism that leaves a conformal class invariant is the identity. In this article we will give some partial answers to this inverse problem.

Before stating the results, we will introduce the class of *admissible metrics* [DKSU09].

Definition 1.1. *A Riemannian metric g is called admissible if on $\mathbb{R}^n$ it is of the form* (1.1) *and $(\Omega, \hat{g})$ is a simple manifold.*

Definition 1.2. *A compact manifold $(\hat{M}, \hat{g})$ with boundary is called simple if for any $p \in \hat{M}$ the exponential map $\exp_p$ with its maximal domain of definition in $T_p\hat{M}$ is a diffeomorphism onto $\hat{M}$, and if $\partial\hat{M}$ is strictly convex in the sense that its second fundamental form is positive definite.*

The first result is for the full data case.

Theorem 1.3. *Let $(\mathbb{R}^n, g)$, $n \geq 3$, be a Riemannian manifold where g is as in* (1.1) *with $\partial_{x_n} c(x', 0) = \partial^2_{x_n} c(x', 0) = 0$ and let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain with C^∞ boundary. When $n = 3$ assume that Ω is simply connected and when $n > 3$ assume that $\hat{g}$ is admissible. Let $\tilde{c} \in C^\infty(\mathbb{R}^n)$ be such that $\partial_{x_n}\tilde{c}(x', 0) = \partial^2_{x_n}\tilde{c}(x', 0) = 0$ and let $\partial^k_{x_n}\tilde{c}(x', 0)|_{\partial\Omega} = 0$ for $k \geq 3$. Assume that $\Lambda_g(f) = \Lambda_{\tilde{c}g}(f)$ for all $f \in U_\delta$, where $\delta > 0$ is sufficiently small. Then*

$$\tilde{c}(x', 0) = \lambda, \quad \partial^k_{x_n}\tilde{c}(x', 0) = 0$$

for some $\lambda > 0$ and for all $k > 2$.

Thus, if $\tilde{c}$ is real analytic with respect to x_n, then $\tilde{c}(x) = \lambda$ for all $x \in \Omega \times \mathbb{R}$.

For the partial data we will only consider the case $n = 3$. For $n > 3$ we would be able to get some results, but the partial data results for the magnetic Schrödinger equation (see [SY23]) are more restrictive and hence we do not record these results here.

Theorem 1.4. *Let $(\mathbb{R}^3, g)$, $\tilde{c}$ and Ω be as in Theorem 1.3 and let $\Gamma \subset \partial\Omega$, $\Gamma \neq \emptyset$, be open. Assume that $\Lambda^\Gamma_g(f) = \Lambda^\Gamma_{\tilde{c}g}(f)$ for all $f \in U_\delta$, where $\delta > 0$ is sufficiently small. Then*

$$\tilde{c}(x', 0) = \lambda, \quad \partial^k_{x_3}\tilde{c}(x', 0) = 0$$

for some $\lambda > 0$ and for all $k > 2$.

Thus, if $\tilde{c}$ is real analytic with respect to x_3, then $\tilde{c}(x) = \lambda$ for all $x \in \Omega \times \mathbb{R}$.

These results extend the results from [Nur23a].

We make some comments about the assumptions made in Theorems 1.3 and 1.4. First of all, the assumption $\partial_{x_n} c(x', 0) = \partial_{x_n}\tilde{c}(x', 0) = 0$ is needed for the well-posedness of the boundary value problem (1.3) for small data, that is we want it to be well-posed for both the metric g and the metric $\tilde{c}g$. The assumption $\partial^2_{x_n} c(x', 0) = \partial^2_{x_n}\tilde{c}(x', 0) = 0$ is needed in order for the method used in the proof to work and it is not yet known if this could be removed. Lastly, there is a small gauge invariance in (1.2) and also in the DN map. To be more precise, if one replaces the conformal factor c in (1.1) by μc, $\mu \neq 0$, then both the equation (1.2) and the DN map remain the same. Thus we cannot have $\lambda = 1$ in Theorem 1.3.

There are also some assumptions made when $n = 3$. The assumption that Ω is simply connected is needed to use the Poincaré Lemma. This is something that one might be able relax with some amount of work by looking at the so called gauge isomorphism mentioned in [GT11] that relates two connection 1-forms. The other assumption that $\partial_{x_n}^k \tilde{c}(x', 0)|_{\partial\Omega} = 0$ for $k \geq 3$ could possibly be removed by doing a boundary determination result.

The proof will use the method of higher order linearization. We will begin by looking at the first linearization of (1.3) and the DN map for that. The first linearization of the DN map corresponds to an advection diffusion type equation. From this it is possible to determine the advection term ([GT11] for $n = 3$, [KU18] for $n > 3$) and this will then imply that $\tilde{c}(x', 0) = \lambda$ for some $\lambda \in (0, \infty)$. Now fix $x_0 \in \Omega$ and construct a solution $v^{(0)}$ to the adjoint of the linearized equation so that $v^{(0)}(x_0) \neq 0$. Then using the higher order linearizations one can obtain an integral identity

$$\int_\Omega \tilde{c}(x', 0)^{-1} \partial_{x_n}^{N+2} \tilde{c}(x', 0) v^{(0)} \prod_{k=1}^{N+1} v^{l_k} \, dx' = 0,$$

where $N+1$ is the order of linearization and v^{l_k} are solutions to the linearized equation. Then choosing $v^{l_k} \equiv 1$ for $k > 2$ and using that a product of these solutions form a complete set in L^1 (see Lemma 4.1) we can conclude that

$$\tilde{c}(x', 0)^{-1} \partial_{x_n}^{N+2} \tilde{c}(x', 0) v^{(0)} = 0.$$

From this, using $v^{(0)}(x_0) \neq 0$ and that x_0 was arbitrary, we get $\partial_{x_n}^{N+2} \tilde{c}(x', 0) = 0$. Then one proceeds by induction.

For the partial data, the proof is very similar and has only a few modifications. One of the more significant changes is that instead of using the results of [GT11] to determine the advection term, we use the results for partial data in [Tzo17].

As mentioned above, the main technique in this work is the method of higher order linearization. This method, which uses the nonlinearity of the partial differential equation as a tool, was first introduced in [KLU18] in the case of a nonlinear wave equation and was further developed in [LLLS21a], [FO20] for nonlinear elliptic equations. In these works the equation $\Delta u + a(x, u) = 0$ was considered and further results for example in partial data and obstacle problems were obtained in [KU20a], [LLLS21b]. Also there has been multiple works when the nonlinearity is of power type (e.g. [LLST22], [ST22] and [Nur23b]).

At the same time there have been multiple articles concerning inverse problems for other nonlinear elliptic PDEs. For example different cases of nonlinear conductivity equations have been considered in [CFKKU21], [KKU20] and in the latter they also consider partial data. For the nonlinear magnetic Schrödinger equation the full data case has been treated on conformally transversally anisotropic manifolds in [KU20b] and in the Euclidean setting the partial data case is treated in [LZ20].

Linearization has been used before these works mentioned above. It was used in the parabolic case in [Isa93] where it was shown that the first linearization of the DN map is the DN map for a linear equation. Other nonlinear elliptic cases have been treated, for example in [IS94], [SU97].

The present work focuses on an inverse problem for the minimal surface equation and for this equation there have been some works previously in the literature. The

Euclidean case has been treated in [MU20], in the sense that they deal with a quasilinear conductivity depending on a function u and its gradient ∇u. After that there have been two works on the minimal surface equation in a Riemannian manifold setting, one by the author of this article [Nur23a] and another one in [CLLO22]. These were simultaneously and independently done. The work of this article somehow brings these two articles together in the following sense: In [Nur23a] the metric was conformally Euclidean and thus in this article the setting is more general. In [CLLO22] the authors consider a two dimensional Riemannian manifold (Σ, g) and look at a minimal surface $Y \subset \Sigma \times \mathbb{R}$ given as a graph of a function. Thus if $\Sigma \subset \mathbb{R}^2$ and in this article we would have $c \equiv 1$, the settings in these two articles would be the same.

There is also the work [ABN20] that is related to minimal surfaces and inverse problems. In that article the authors consider a generalization of the boundary rigidity problem in the sense that instead of measuring distances of boundary points they use measurements related to areas of minimal surfaces.

The rest of this article is organized as follows. In Section 2 we derive the minimal surface equation in the setting mentioned above. Section 3 is dedicated to calculating the first and second order linearizations of (1.3) and addressing shortly the well-posedness of (1.3). Finally in Section 4 we prove the main results and two Lemmas to help in the proof.

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2. Deriving the minimal surface equation

In this section we derive the equation (1.2). This is done similarly as in [Nur23a, Section 3]. Let (M, g), $M = \mathbb{R}^n$, $n \geq 3$, be a Riemannian manifold with the metric

$$g(x', x_n) = c(x', x_n) \begin{pmatrix} \hat{g}(x') & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

where

$$(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R},\ c \in C^\infty(\mathbb{R}^n),\ c(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}^n$$

and $\hat{g}$ is a Riemannian metric on $\mathbb{R}^{n-1}$. We use the standard notations g_{ij} for the matrix g and g^{ij} for the inverse g^{-1}. These assumptions are valid for the rest of the article, unless otherwise stated.

Let $u\colon \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$, $u \in C^2(\bar{\Omega})$, and consider the graph of the function u

$$\text{Graph}_u = \{(x', u(x'))\colon x' \in \Omega\} \subset M.$$

This graph is a minimal surface if and only if its mean curvature H is equal to zero at all points on the graph. By defining

$$f\colon \Omega \times \mathbb{R} \to \mathbb{R}, \quad f(x', x_n) = x_n - u(x'),$$

the graph of u is the surface

$$\Sigma := \{(x', x_n) \in \Omega \times \mathbb{R} : f(x', x_n) = 0\}.$$

The mean curvature of Σ at $x \in \Sigma$ is the sum of principal curvatures. We omit the normalizing factor $\frac{1}{n-1}$ when calculating the mean curvature and we use the Einstein summation convention when it does not cause confusion. In order to calculate the principal curvatures, we introduce the Riemannian gradient and Hessian of a function $f\colon M \to \mathbb{R}$:

$$\nabla_g f = g^{ij}\partial_{x_i} f \partial_{x_j}, \quad \nabla_g^2 f = \left(\partial^2_{x_i x_j} f - \Gamma^m{}_{ij}\partial_{x_m} f\right)_{i,j=1}^n,$$

where g^{ij} is the inverse of g_{ij} and $\Gamma^m{}_{ij} = \frac{1}{2} g^{ml}(\partial_{x_i} g_{jl} + \partial_{x_j} g_{il} - \partial_{x_l} g_{ij})$ is the Christoffel symbol related to the metric g. Define also the Laplace-Beltrami operator, which is the trace of the Hessian (this is one way of defining it), and the norm of the gradient:

$$\Delta_g f = \mathrm{Tr}(\nabla_g^2 f) = g^{ij}\left(\partial^2_{x_i x_j} f - \Gamma^m{}_{ij}\partial_{x_m} f\right), \quad |\nabla_g f|_g^2 = g^{ij}\partial_{x_i} f \partial_{x_j} f.$$

Now the principal curvatures of Σ at $x \in \Sigma$ are the eigenvalues of $\nabla_g^2 f(x)$ restricted to the tangent space $T_x\Sigma$ at x. Since $\frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}$ is a normal to Σ at the point x, we have $T_x\Sigma = \{\nabla_g f(x)\}^\perp$, or in other words, the tangent space $T_x\Sigma$ is the orthogonal complement of the vector $\nabla_g f(x)$.

Let $\{E_1, \ldots, E_{n-1}\}$ be an g-orthonormal basis of $T_x\Sigma$. Then $\left\{E_1, \ldots, E_{n-1}, \frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}\right\}$ is an orthonormal basis of $\mathbb{R}^n$. Now the mean curvature of Σ at $x \in \Sigma$ is the trace of $\nabla_g^2 f(x)|_{\{\nabla_g f(x)\}^\perp}$:

$$\begin{aligned}
H(x) &= \sum_{i=1}^{n-1} \langle \nabla_g^2 f(x) E_i, E_i\rangle \\
&= \sum_{i=1}^{n-1} \langle \nabla_g^2 f(x) E_i, E_i\rangle + \left(\nabla_g^2 f(x)\right)\left(\frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}, \frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}\right) \\
&\quad - \left(\nabla_g^2 f(x)\right)\left(\frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}, \frac{\nabla_g f(x)}{|\nabla_g f(x)|_g}\right) \\
&= \mathrm{Tr}(\nabla_g^2 f(x)) - |\nabla_g f(x)|_g^{-2}\left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x)) \\
&= \Delta_g f(x) - |\nabla_g f(x)|_g^{-2}\left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x)).
\end{aligned}$$

Thus Graph_u is a minimal surface if and only if

$$|\nabla_g f(x)|_g^2 \Delta_g f(x) - \left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x)) = 0 \quad \text{for all } x \in \mathrm{Graph}_u. \tag{2.2}$$

Next we will calculate the minimal surface equation more explicitly using the metric (2.1). Now

$$g^{-1} = c^{-1}\begin{pmatrix} \hat{g}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

is the inverse matrix of (2.1). Let us calculate the first term of (2.2):

$$\begin{aligned}
&c^2|\nabla_g f(x)|_g^2 \Delta_g f(x) \\
&\quad = c^2 g^{ij}(\delta_{in} - \partial_{x_i} u)(\delta_{jn} - \partial_{x_j} u) g^{kl}\left(-\partial^2_{x_k x_l} u - \Gamma^m{}_{kl}(\delta_{mn} - \partial_{x_m} u)\right) \\
&\quad = c^2 g^{ij}(\delta_{in}\delta_{jn} - \delta_{in}\partial_{x_j} u - \delta_{jn}\partial_{x_i} u + \partial_{x_i} u \partial_{x_j} u) g^{kl}(-\partial^2_{x_k x_l} u - \Gamma^n{}_{kl} + \Gamma^m{}_{kl}\partial_{x_m} u) \\
&\quad = c\left(1 + |\nabla_{\hat{g}} u|_{\hat{g}}^2\right)\left(-\Delta_g u - g^{kl}\Gamma^n{}_{kl}\right).
\end{aligned} \tag{2.3}$$

The second term of (2.2) becomes

(2.4)

$$
\begin{aligned}
&c^2\left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x))\\
&= c^2(\partial^2_{x_ix_j} f - \Gamma^m{}_{ij}\partial_{x_m} f) g^{ai}\partial_{x_a} f g^{bj}\partial_{x_b} f\\
&= c^2(-\partial^2_{x_ix_j} u - \Gamma^n{}_{ij} + \Gamma^m{}_{ij}\partial_{x_m} u) g^{ai}g^{bj}(\delta_{an}\delta_{bn} - \delta_{an}\partial_{x_b}u - \delta_{bn}\partial_{x_a}u + \partial_{x_a}u\partial_{x_b}u)\\
&= c^2(-\partial^2_{x_ix_j} u - \Gamma^n{}_{ij} + \Gamma^m{}_{ij}\partial_{x_m} u)(g^{ni}g^{nj} - g^{ni}g^{bj}\partial_{x_b}u - g^{ai}g^{nj}\partial_{x_a}u + g^{ai}g^{bj}\partial_{x_a}u\partial_{x_b}u)\\
&= \Gamma^m{}_{nn}\partial_{x_m}u - \Gamma^n{}_{nn} - 2(\Gamma^m{}_{nj}\partial_{x_m}u - \Gamma^n{}_{nj})cg^{bj}\partial_{x_b}u\\
&+ c^2(-\partial^2_{x_ix_j} u - \Gamma^n{}_{ij} + \Gamma^m{}_{ij}\partial_{x_m} u)g^{ai}g^{bj}\partial_{x_a}u\partial_{x_b}u\\
&= \Gamma^m{}_{nn}\partial_{x_m}u - \Gamma^n{}_{nn} - 2(\Gamma^m{}_{nj}\partial_{x_m}u - \Gamma^n{}_{nj})\left(\nabla_{\hat g}u\right)^j\\
&- \nabla_g^2 u(\nabla_{\hat g}u, \nabla_{\hat g}u) - \Gamma^n{}_{ij}\left(\nabla_{\hat g}u\right)^i\left(\nabla_{\hat g}u\right)^j.
\end{aligned}
$$

In order to simplify further we will calculate the Christoffel symbols more explicitly:

$$
\begin{aligned}
2\Gamma^m{}_{ij} &= g^{mr}(\partial_{x_j}g_{ir} + \partial_{x_i}g_{jr} - \partial_{x_r}g_{ij}),\\
2\Gamma^m{}_{nn} &= g^{mr}(2\partial_{x_n}g_{nr} - \partial_{x_r}g_{nn}) = 2g^{mn}\partial_{x_n}g_{nn} - g^{mr}\partial_{x_r}g_{nn}\\
&= 2g^{mn}\partial_{x_n}c - g^{mr}\partial_{x_r}c,\\
2\Gamma^n{}_{ij} &= g^{nr}(\partial_{x_j}g_{ir} + \partial_{x_i}g_{jr} - \partial_{x_r}g_{ij}) = g^{nn}(\partial_{x_j}g_{in} + \partial_{x_i}g_{jn} - \partial_{x_n}g_{ij})\\
&= c^{-1}(\partial_{x_j}g_{in} + \partial_{x_i}g_{jn} - \partial_{x_n}g_{ij}),\\
2\Gamma^m{}_{nj} &= g^{mr}(\partial_{x_j}g_{nr} + \partial_{x_n}g_{jr} - \partial_{x_r}g_{nj}),\\
2\Gamma^n{}_{nj} &= g^{nr}(\partial_{x_j}g_{nr} + \partial_{x_n}g_{jr} - \partial_{x_r}g_{nj}) = g^{nn}(\partial_{x_j}g_{nn} + \partial_{x_n}g_{jn} - \partial_{x_n}g_{nj})\\
&= c^{-1}\partial_{x_j}c.
\end{aligned}
$$

Inserting these into (2.3) we have

$$
\begin{aligned}
&c^2|\nabla_g f(x)|_g^2\Delta_g f(x)\\
&= c\left(1 + |\nabla_{\hat g}u|^2_{\hat g}\right)(-\Delta_g u - \frac{1}{2}g^{kl}c^{-1}(\partial_{x_l}g_{kn} + \partial_{x_k}g_{ln} - \partial_{x_n}g_{kl}))\\
&= \left(1 + |\nabla_{\hat g}u|^2_{\hat g}\right)(-c\Delta_g u - \frac{1}{2}(g^{nl}\partial_{x_l}c + g^{kn}\partial_{x_k}c - g^{kl}\partial_{x_n}g_{kl}))\\
&= \left(1 + |\nabla_{\hat g}u|^2_{\hat g}\right)(-c\Delta_g u - \frac{1}{2}c^{-1}(2-n)\partial_{x_n}c).
\end{aligned}
$$

For calculating the linearizations in Section 3 it is useful to transform $\Delta_g u$ into $\Delta_{\hat g}u$:

$$
\begin{aligned}
\Delta_g u &= g^{ij}\left(\partial^2_{x_ix_j}u - \frac{1}{2}g^{mr}(\partial_{x_j}g_{ir} + \partial_{x_i}g_{jr} - \partial_{x_r}g_{ij})\partial_{x_m}u\right)\\
&= c^{-1}\hat g^{ij}\partial^2_{x_ix_j}u - \frac{1}{2c}\sum_{m,r=1}^{n-1}\hat g^{mr}g^{ij}(\partial_{x_j}g_{ir} + \partial_{x_i}g_{jr} - \partial_{x_r}g_{ij})\partial_{x_m}u\\
&= c^{-1}\hat g^{ij}\partial^2_{x_ix_j}u - \frac{1}{2c}\sum_{m,r=1}^{n-1}\hat g^{mr}\left(\hat g^{ij}((\partial_{x_j}\hat g_{ir} + \partial_{x_i}\hat g_{jr} - \partial_{x_r}\hat g_{ij})) + \frac{2-n}{c}\partial_{x_r}c\right)\partial_{x_m}u\\
&= c^{-1}\Delta_{\hat g}u - \frac{2-n}{2c^2}\hat g^{mr}\partial_{x_r}c\partial_{x_m}u.
\end{aligned}
$$

Thus

$$\begin{aligned}
&c^2|\nabla_g f(x)|_g^2 \Delta_g f(x)\\
&= \left(1+|\nabla_{\hat g} u|_{\hat g}^2\right)\left(-\Delta_{\hat g} u + \frac{2-n}{2c}\hat g^{mr}\partial_{x_r} c\partial_{x_m} u - \frac{1}{2}c^{-1}(2-n)\partial_{x_n} c\right).
\end{aligned}$$

Now the second part (2.4) becomes

$$\begin{aligned}
&c^2\left(\nabla_g^2 f(x)\right)(\nabla_g f(x),\nabla_g f(x))\\
&= -\frac{1}{2}g^{mr}\partial_{x_r} c\partial_{x_m} u - \frac{1}{2}g^{nn}\partial_{x_n} c\\
&- (g^{mr}(\partial_{x_j} g_{nr} + \partial_{x_n} g_{jr} - \partial_{x_r} g_{nj})\partial_{x_m} u - c^{-1}\partial_{x_j} c)(\nabla_{\hat g} u)^j\\
&- \nabla_g^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u) - \frac{1}{2}c^{-1}(\partial_{x_j} g_{in} + \partial_{x_i} g_{jn} - \partial_{x_n} g_{ij})(\nabla_{\hat g} u)^i(\nabla_{\hat g} u)^j\\
&= -\frac{1}{2}c^{-1}\sum_{m,r=1}^{n-1}\hat g^{mr}\partial_{x_r} c\partial_{x_m} u - \frac{1}{2}c^{-1}\partial_{x_n} c\\
&- \sum_{m,r=1}^{n-1} c^{-1}\hat g^{mr}\hat g_{jr}\partial_{x_n} c\partial_{x_m} u(\nabla_{\hat g} u)^j + c^{-1}\partial_{x_j} c(\nabla_{\hat g} u)^j\\
&- \nabla_g^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u) + \frac{1}{2}c^{-1}\partial_{x_n} c\hat g_{ij}(\nabla_{\hat g} u)^i(\nabla_{\hat g} u)^j\\
&= \frac{1}{2}c^{-1}\sum_{m,r=1}^{n-1}\hat g^{mr}\partial_{x_r} c\partial_{x_m} u - \frac{1}{2}c^{-1}\partial_{x_n} c(1+|\nabla_{\hat g} u|_{\hat g}^2) - \nabla_g^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u).
\end{aligned}$$

As before we would like to modify $\nabla_g^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u)$ to have $\nabla_{\hat g}^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u)$:

$$\begin{aligned}
&\nabla_g^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u)\\
&= (\partial^2_{x_i x_j} u - \frac{1}{2}g^{mr}(\partial_{x_i} g_{rj} + \partial_{x_j} g_{ri} - \partial_{x_r} g_{ji})\partial_{x_m} u)\hat g^{ia}\partial_{x_a} u\hat g^{jb}\partial_{x_b} u.
\end{aligned}$$

Since $\partial_{x_n} u = 0$, all the indices run from 1 to $n-1$. Hence

$$\begin{aligned}
&\nabla_g^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u)\\
&= (\partial^2_{x_i x_j} u - \frac{1}{2c}\hat g^{mr}(\partial_{x_i} c\hat g_{rj} + c\partial_{x_i}\hat g_{rj} + \partial_{x_j} c\hat g_{ri} + c\partial_{x_j}\hat g_{ri}\\
&- \partial_{x_r} c\hat g_{ij} - c\partial_{x_r}\hat g_{ij})\partial_{x_m} u)\hat g^{ia}\partial_{x_a} u\hat g^{jb}\partial_{x_b} u\\
&= (\partial^2_{x_i x_j} u - \hat\Gamma^m{}_{ij}\partial_{x_m} u - \frac{1}{2c}(\delta_j^m\partial_{x_i} c + \delta_i^m\partial_{x_j} c - \hat g^{mr}\hat g_{ij}\partial_{x_r} c)\partial_{x_m} u)\hat g^{ia}\partial_{x_a} u\hat g^{jb}\partial_{x_b} u\\
&= \nabla_{\hat g}^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u) - \frac{1}{2c}(\hat g^{ia}\partial_{x_i} c\partial_{x_a} u\hat g^{mb}\partial_{x_m} u\partial_{x_b} u + \hat g^{jb}\partial_{x_j} c\partial_{x_b} u\hat g^{mb}\partial_{x_m} u\partial_{x_a} u\\
&- \hat g^{mr}\partial_{x_r} c\partial_{x_m} u\hat g^{jb}\partial_{x_j} u\partial_{x_b} u)\\
&= \nabla_{\hat g}^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u) - \frac{1}{2c}\hat g^{ia}\partial_{x_i} c\partial_{x_a} u|\nabla_{\hat g} u|_{\hat g}^2,
\end{aligned}$$

where $2\hat\Gamma^m{}_{ij} = \hat g^{mr}(\partial_{x_j}\hat g_{ir} + \partial_{x_i}\hat g_{jr} - \partial_{x_r}\hat g_{ij})$.

Putting these together gives

$$
\begin{aligned}
0 &= c^2|\nabla_g f(x)|_g^2\Delta_g f(x) - c^2\left(\nabla_g^2 f(x)\right)(\nabla_g f(x), \nabla_g f(x)) \\
&= \left(-\Delta_{\hat g} u + \frac{1-n}{2c}\hat g^{mr}\partial_{x_r} c\partial_{x_m} u + \frac{n-1}{2c}\partial_{x_n} c\right)(1+|\nabla_{\hat g} u|_{\hat g}^2) \\
&+ \nabla_{\hat g}^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u).
\end{aligned}
$$

Thus Graph_u is a minimal surface if and only if the function u satisfies the following minimal surface equation

(2.5)

$$
\begin{aligned}
&\left(-\Delta_{\hat g} u + \frac{1-n}{2c(x',u(x'))}\hat g^{mr}(x')\partial_{x_r} c(x',u(x'))\partial_{x_m} u + \frac{n-1}{2c(x',u(x'))}\partial_{x_n} c(x',u(x'))\right)(1+|\nabla_{\hat g} u|_{\hat g}^2) \\
&+ \nabla_{\hat g}^2 u(\nabla_{\hat g} u, \nabla_{\hat g} u) = 0
\end{aligned}
$$

for all $x' \in \Omega$.

We will modify this a bit in order to see the more familiar Euclidean version of the equation directly. For this, we start with the divergence of a vector field defined in local coordinates as

$$
\mathrm{div}_g\, a = \sum_{i=1}^n \left(\partial_{x_i} a^i + \sum_{j=1}^n a^j {\Gamma^i}_{ij}\right).
$$

Let us denote $\eta := \left(1+|\nabla_g u|_g^2\right)^{\frac{1}{2}}$ and expand the following for a CTA metric and a function $u\colon \Omega \to \mathbb{R}$ as before:

$$
\begin{aligned}
&\mathrm{div}_g\left(\frac{\nabla_g u}{\eta}\right) \\
&= \sum_{i=1}^n\left(\partial_{x_i}\left(\frac{(\nabla_g u)^i}{\eta}\right) + \sum_{j=1}^n \frac{(\nabla_g u)^j}{\eta}{\Gamma^i}_{ij}\right) \\
&= \sum_{i=1}^{n-1} \frac{\partial_{x_i} g^{ik}\partial_{x_k} u + g^{ik}\partial^2_{x_i x_k} u}{\eta} \\
&- \frac{1}{2}\frac{(\nabla_g u)^i}{\eta^3}\left(\partial_{x_i} g^{ab}\partial_{x_a} u\partial_{x_b} u + g^{ab}\left(\partial^2_{x_i x_a} u\partial_{x_b} u + \partial_{x_a} u\partial^2_{x_i x_b} u\right)\right) \\
&+ \sum_{j=1}^{n-1}\frac{(\nabla_g u)^j}{\eta}\left({\Gamma^i}_{ij} + {\Gamma^n}_{nj}\right).
\end{aligned}
$$

Now we will use that $\partial_{x_l} g^{ik} = -({\Gamma^i}_{ml} g^{mk} + {\Gamma^k}_{ml} g^{im})$ to have

$$
\begin{aligned}
&\operatorname{div}_g\left(\frac{\nabla_g u}{\eta}\right)\\
&= \sum_{i=1}^{n-1} \frac{-({\Gamma^i}_{il} g^{lk} + {\Gamma^k}_{il} g^{il})\partial_{x_k} u + g^{ik}\partial^2_{x_i x_k} u}{\eta}\\
&- \frac{1}{2}\frac{(\nabla_g u)^i}{\eta^3}\left(-({\Gamma^a}_{il} g^{lb} + {\Gamma^b}_{il} g^{al})\partial_{x_a} u \partial_{x_b} u + g^{ab}\left(\partial^2_{x_i x_a} u \partial_{x_b} u + \partial_{x_a} u \partial^2_{x_i x_b} u\right)\right)\\
&+ \sum_{j=1}^{n-1} \frac{(\nabla_g u)^j}{\eta}\left({\Gamma^i}_{ij} + {\Gamma^n}_{nj}\right)\\
&= \frac{\Delta_g u}{\eta} - \sum_{i,l=1}^{n-1} \frac{(\nabla_g u)^l}{\eta} {\Gamma^i}_{il}\\
&- \frac{1}{2}\frac{(\nabla_g u)^i}{\eta^3}\left(-(\nabla_g u)^l {\Gamma^a}_{il}\partial_{x_a} u - (\nabla_g u)^l {\Gamma^b}_{il}\partial_{x_b} u + (\nabla_g u)^a \partial^2_{x_i x_a} u + (\nabla_g u)^b \partial^2_{x_i x_b} u\right)\\
&+ \sum_{j=1}^{n-1} \frac{(\nabla_g u)^j}{\eta}\left({\Gamma^i}_{ij} + {\Gamma^n}_{nj}\right).
\end{aligned}
$$

After renaming some of the indices, using the definition of the Riemannian Hessian of a function and ${\Gamma^n}_{nj} = \frac{1}{2}c^{-1}\partial_{x_j} c$ we get

$$
\operatorname{div}_g\left(\frac{\nabla_g u}{\eta}\right) = \frac{\Delta_g u}{\eta} - \frac{\nabla^2_g u(\nabla_g u, \nabla_g u)}{\eta^3} + \sum_{j=1}^{n-1} \frac{(\nabla_g u)^j}{2c\eta}\partial_{x_j} c. \tag{2.6}
$$

Let us now go back to (2.5) and modify it to have (2.6) visible. Now

$$
\begin{aligned}
&\left(-\Delta_{\hat g} u + \frac{1-n}{2c}\hat g^{mr}\partial_{x_r} c \partial_{x_m} u + \frac{n-1}{2c}\partial_{x_n} c\right)(1 + |\nabla_{\hat g} u|^2_{\hat g})\\
&+ \nabla^2_{\hat g} u(\nabla_{\hat g} u, \nabla_{\hat g} u)\\
&= \left(-\Delta_{\hat g} u + \frac{2-n}{2c}\hat g^{mr}\partial_{x_r} c \partial_{x_m} u + \frac{n-1}{2c}\partial_{x_n} c\right)(1 + |\nabla_{\hat g} u|^2_{\hat g})\\
&- \frac{1}{2}\hat g^{mr}\partial_{x_r} c \partial_{x_m} u(1 + |\nabla_{\hat g} u|^2_{\hat g}) + \nabla^2_{\hat g} u(\nabla_{\hat g} u, \nabla_{\hat g} u)\\
&= \left(-c\Delta_g u + \frac{n-1}{2c}\partial_{x_n} c\right)(1 + |\nabla_{\hat g} u|^2_{\hat g}) - \frac{1}{2}\hat g^{mr}\partial_{x_r} c \partial_{x_m} u + \nabla^2_g u(\nabla_{\hat g} u, \nabla_{\hat g} u).
\end{aligned}
$$

Then, we will multiply both sides of (2.5) by c^{-2} to get

$$\begin{aligned}
0 &= \left(-\Delta_g u + \frac{n-1}{2c^2}\partial_{x_n} c\right)(c^{-1} + |\nabla_g u|_g^2) - \frac{1}{2}c^{-3}\hat{g}^{mr}\partial_{x_r} c\partial_{x_m} u + \nabla_g^2 u(\nabla_g u, \nabla_g u) \\
&= -\Delta_g u|\nabla_g u|_g^2 - c^{-1}\Delta_g u + \Delta_g u - \Delta_g u + \frac{(\nabla_g u)^j}{2c}\partial_{x_j} c\eta^2 - \frac{(\nabla_g u)^j}{2c}\partial_{x_j} c\eta^2 \\
&+ \frac{n-1}{2c^3}\partial_{x_n} c(1 + |\nabla_{\hat{g}} u|_{\hat{g}}^2) - \frac{1}{2}c^{-3}(\nabla_g u)^r\partial_{x_r} c + \nabla_g^2 u(\nabla_g u, \nabla_g u) \\
&= -\Delta_g u\eta^2 + \nabla_g^2 u(\nabla_g u, \nabla_g u) - \frac{(\nabla_g u)^j}{2c}\partial_{x_j} c\eta^2 + \Delta_g u(1 - c^{-1}) \\
&+ \frac{n-1}{2c^3}\partial_{x_n} c(1 + |\nabla_{\hat{g}} u|_{\hat{g}}^2) + \frac{1}{2c}(\nabla_g u)^j\partial_{x_j} c(\eta^2 - c^{-2}).
\end{aligned}$$

Dividing both sides by $\eta^3 = \left(1 + |\nabla_g u|_g^2\right)^{\frac{3}{2}}$ gives

$$\begin{aligned}
&\operatorname{div}_g\left(\frac{\nabla_g u}{\left(1 + |\nabla_g u|_g^2\right)^{\frac{1}{2}}}\right) \\
&+ \frac{\Delta_g u(1 - c^{-1}) + \frac{1}{2c}(\nabla_g u)^j\partial_{x_j} c(1 + |\nabla_g u|_g^2 - c^{-2}) + \frac{n-1}{2c^3}\partial_{x_n} c(1 + |\nabla_{\hat{g}} u|_{\hat{g}}^2)}{\left(1 + |\nabla_g u|_g^2\right)^{\frac{3}{2}}} = 0.
\end{aligned}$$

When g is the Euclidean metric, we will have the familiar Euclidean minimal surface equation. Moreover, when $c = 1$ we get the minimal surface equation in [CLLO22]:

$$\operatorname{div}_{\hat{g}}\left(\frac{\nabla_{\hat{g}} u}{\left(1 + |\nabla_{\hat{g}} u|_{\hat{g}}^2\right)^{\frac{1}{2}}}\right) = 0.$$

3. First and second order linearizations

Let g be as in (2.1) and define $F\colon \mathbb{R}^{n^2} \to \mathbb{R}$,

(3.1)
$$F(x', u, \nabla_g u, \nabla_g^2 u) := \left(-\Delta_{\hat{g}} u + \frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r} c\partial_{x_m} u + \frac{n-1}{2c}\partial_{x_n} c\right)(1 + |\nabla_{\hat{g}} u|_{\hat{g}}^2)$$

(3.2)
$$+ \nabla_{\hat{g}}^2 u(\nabla_{\hat{g}} u, \nabla_{\hat{g}} u).$$

We want to use the well-posedness result from [Nur23a] to say that the boundary value problem

$$\left\{\begin{array}{ll} F(x', u, \nabla_g u, \nabla_g^2 u) = 0 & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{array}\right. \tag{3.3}$$

has a unique small solution. For this we need that $F(x', 0, 0, 0) = 0$ which guarantees that $u \equiv 0$ is a solution to (3.3) with $f = 0$. From (3.1) we can see that this happens if and only if $\partial_{x_n} c(x', 0) = 0$ for all $x' \in \Omega$.

To use the above mentioned well-posedness, we will need to calculate the first linearization of the equation $F(x', u, \nabla_g u, \nabla_g^2 u) = 0$. Let us do a formal calculation to get the first linearization. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ where $\varepsilon_k \in \mathbb{R}$ small and assume that $u_\varepsilon := u(x, \varepsilon)$ depends smoothly on ε and solves $F(x', u_\varepsilon, \nabla_g u_\varepsilon, \nabla_g^2 u_\varepsilon) = 0$. We will differentiate this in three parts with respect to ε_l and evaluate at $\varepsilon = 0$ (We could have calculated the first linearization as in [Nur23a], but it would be a longer calculation). Denote by $u_\varepsilon^{(l)} := \partial_{\varepsilon_l} u_\varepsilon$. Just to make the calculations a bit nicer we will start with

$$\begin{aligned}\partial_{\varepsilon_l}\left(\frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r}c\partial_{x_m}u_\varepsilon\right) \\ &= -\frac{1-n}{2c^2}\partial_{x_n}cu_\varepsilon^{(l)}\hat{g}^{mr}\partial_{x_r}c\partial_{x_m}u_\varepsilon + \frac{1-n}{2c}\hat{g}^{mr}(\partial^2_{x_rx_n}cu_\varepsilon^{(l)}\partial_{x_m}u_\varepsilon + \partial_{x_r}c\partial_{x_m}u_\varepsilon^{(l)}) \\ &:= Q_l.\end{aligned} \tag{3.4}$$

Now the first term of (3.1) becomes

$$\begin{aligned}&\partial_{\varepsilon_l}\left(\left(-\Delta_{\hat{g}}u_\varepsilon + \frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r}c\partial_{x_m}u_\varepsilon + \frac{n-1}{2c}\partial_{x_n}c\right)(1+|\nabla_{\hat{g}}u_\varepsilon|^2_{\hat{g}})\right) \\ &= \left(-\Delta_{\hat{g}}u_\varepsilon^{(l)} - \frac{n-1}{2c^{-2}}(\partial_{x_n}c)^2u_\varepsilon^{(l)} + \frac{n-1}{2c}\partial^2_{x_n}cu_\varepsilon^{(l)} + Q_l\right)(1+|\nabla_{\hat{g}}u_\varepsilon|^2_{\hat{g}}) \\ &+ \left(-\Delta_{\hat{g}}u_\varepsilon + \frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r}c\partial_{x_m}u_\varepsilon + \frac{n-1}{2c}\partial_{x_n}c\right)\hat{g}^{ij}(\partial_{x_i}u_\varepsilon^{(l)}\partial_{x_j}u_\varepsilon + \partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon^{(l)}).\end{aligned} \tag{3.5}$$

The second term is

$$\begin{aligned}&\partial_{\varepsilon_l}\left(\nabla^2_{\hat{g}}u_\varepsilon(\nabla_{\hat{g}}u_\varepsilon, \nabla_{\hat{g}}u_\varepsilon)\right) \\ &= \partial_{\varepsilon_l}\left((\partial^2_{x_ix_j}u_\varepsilon - \hat{\Gamma}^m{}_{ij}\partial_{x_m}u_\varepsilon)\hat{g}^{ia}\partial_{x_a}u_\varepsilon\hat{g}^{jb}\partial_{x_b}u_\varepsilon\right) \\ &= (\partial^2_{x_ix_j}u_\varepsilon^{(l)} - \hat{\Gamma}^m{}_{ij}\partial_{x_m}u_\varepsilon^{(l)})\hat{g}^{ia}\partial_{x_a}u_\varepsilon\hat{g}^{jb}\partial_{x_b}u_\varepsilon \\ &+ (\partial^2_{x_ix_j}u_\varepsilon - \hat{\Gamma}^m{}_{ij}\partial_{x_m}u_\varepsilon)\hat{g}^{ia}\hat{g}^{jb}(\partial_{x_a}u_\varepsilon^{(l)}\partial_{x_b}u_\varepsilon + \partial_{x_a}u_\varepsilon\partial_{x_b}u_\varepsilon^{(l)}) \\ &:= P_1 + P_2.\end{aligned} \tag{3.6}$$

where P_1 and P_2 are for future reference. When evaluating at $\varepsilon = 0$ we have $u_0 = 0$, also $\nabla_g u_0, \nabla^2_g u_0$ vanish, and thus when combining the above, denoting $v^l := u_\varepsilon^{(l)}|_{\varepsilon=0}$,

$$\begin{aligned}&\partial_{\varepsilon_l}F(x', u_\varepsilon, \nabla_g u_\varepsilon, \nabla^2_g u_\varepsilon)|_{\varepsilon=0} \\ &= -\Delta_{\hat{g}}v^l + \frac{n-1}{2c(x',0)}\partial^2_{x_n}c(x',0)v^l + \frac{1-n}{2c(x',0)}\sum_{i,j=1}^{n-1}\hat{g}^{ij}\partial_{x_i}c(x',0)\partial_{x_j}v^l = 0.\end{aligned}$$

Now this first linearization satisfies the assumptions of [Nur23a, Proposition 2.1] and thus we have the existence and uniqueness of small solutions to (3.3).

Next we will calculate the second linearization for the equation $F(x', u, \nabla_g u, \nabla^2_g u) = 0$. Denote $u_\varepsilon^{(kl)} := \partial^2_{\varepsilon_k\varepsilon_l}u_\varepsilon$. We will differentiate (3.4)-(3.6) with respect to ε_k, $k \neq l$, starting with (3.4):

$$\begin{aligned}
&\partial_{\varepsilon_k}\partial_{\varepsilon_l}\left(\frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r}c\partial_{x_m}u_\varepsilon\right)\\
&=\frac{1-n}{c^3}(\partial_{x_n}c)^2u_\varepsilon^{(k)}u_\varepsilon^{(l)}\hat{g}\partial_{x_r}c\partial_{x_m}u_\varepsilon-\frac{1-n}{2c^2}\partial_{x_n}^2cu_\varepsilon^{(k)}u_\varepsilon^{(l)}\hat{g}\partial_{x_r}c\partial_{x_m}u_\varepsilon\\
&-\frac{1-n}{2c^2}\partial_{x_n}cu_\varepsilon^{(kl)}\hat{g}\partial_{x_r}c\partial_{x_m}u_\varepsilon\\
&-\frac{1-n}{2c^2}\partial_{x_n}cu_\varepsilon^{(l)}\hat{g}^{mr}(\partial_{x_rx_n}^2cu_\varepsilon^{(k)}\partial_{x_m}u_\varepsilon+\partial_{x_r}c\partial_{x_m}u_\varepsilon^{(k)})\\
&-\frac{1-n}{2c^2}\partial_{x_n}c\hat{g}^{mr}(\partial_{x_rx_n}^2cu_\varepsilon^{(l)}\partial_{x_m}u_\varepsilon+\partial_{x_r}c\partial_{x_m}u_\varepsilon^{(l)})\\
&+\frac{1-n}{2c}\hat{g}^{mr}\big(\partial_{x_r}\partial_{x_n}^2cu_\varepsilon^{(k)}u_\varepsilon^{(l)}\partial_{x_m}u_\varepsilon+\partial_{x_rx_n}^2c(u_\varepsilon^{(kl)}\partial_{x_m}u_\varepsilon+u_\varepsilon^{(l)}\partial_{x_m}u_\varepsilon^{k})\\
&+\partial_{x_rx_n}cu_\varepsilon^{(k)}\partial_{x_m}u_\varepsilon^{(l)}+\partial_{x_r}c\partial_{x_m}u_\varepsilon^{(kl)}\big).
\end{aligned}$$

Then we differentiate (3.5) to have

$$\begin{aligned}
&\partial_{\varepsilon_k}\partial_{\varepsilon_l}\left(\left(-\Delta_{\hat{g}}u_\varepsilon+\frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r}c\partial_{x_m}u_\varepsilon+\frac{n-1}{2c}\partial_{x_n}c\right)(1+|\nabla_{\hat{g}}u_\varepsilon|_{\hat{g}}^2)\right)\\
&=\Bigg(-\Delta_{\hat{g}}u_\varepsilon^{(kl)}+\frac{n-1}{c^{-3}}(\partial_{x_n}c)^3u_\varepsilon^{(k)}u_\varepsilon^{(l)}-\frac{n-1}{2c^{-2}}\left(2\partial_{x_n}c\partial_{x_n}^2cu_\varepsilon^{(k)}u_\varepsilon^{(l)}+(\partial_{x_n}c)^2u_\varepsilon^{(kl)}\right)\\
&-\frac{n-1}{2c^{-2}}\partial_{x_n}c\partial_{x_n}^2cu_\varepsilon^{(k)}u_\varepsilon^{(l)}+\frac{n-1}{2c}\left(\partial_{x_n}^3cu_\varepsilon^{(k)}u_\varepsilon^{(l)}+\partial_{x_n}^2cu_\varepsilon^{(kl)}\right)+\partial_{\varepsilon_k}Q_l\Bigg)(1+|\nabla_{\hat{g}}u_\varepsilon|_{\hat{g}}^2)\\
&+\left(-\Delta_{\hat{g}}u_\varepsilon^{(l)}-\frac{n-1}{2c^{-2}}(\partial_{x_n}c)^2u_\varepsilon^{(l)}+\frac{n-1}{2c}\partial_{x_n}^2cu_\varepsilon^{(l)}+Q_l\right)\hat{g}^{ij}(\partial_{x_i}u_\varepsilon^{(k)}\partial_{x_j}u_\varepsilon+\partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon^{(k)})\\
&+\left(-\Delta_{\hat{g}}u_\varepsilon^{(k)}-\frac{n-1}{2c^{-2}}(\partial_{x_n}c)^2u_\varepsilon^{(k)}+\frac{n-1}{2c}\partial_{x_n}^2cu_\varepsilon^{(k)}+Q_k\right)\hat{g}^{ij}(\partial_{x_i}u_\varepsilon^{(l)}\partial_{x_j}u_\varepsilon+\partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon^{(l)})\\
&+\left(-\Delta_{\hat{g}}u_\varepsilon+\frac{1-n}{2c}\hat{g}^{mr}\partial_{x_r}c\partial_{x_m}u_\varepsilon+\frac{n-1}{2c}\partial_{x_n}c\right)\hat{g}^{ij}(\partial_{x_i}u_\varepsilon^{(kl)}\partial_{x_j}u_\varepsilon+\partial_{x_i}u_\varepsilon^{(l)}\partial_{x_j}u_\varepsilon^{(k)}\\
&+\partial_{x_i}u_\varepsilon^{(k)}\partial_{x_j}u_\varepsilon^{(l)}+\partial_{x_i}u_\varepsilon\partial_{x_j}u_\varepsilon^{(kl)}).
\end{aligned}$$

Now we calculate the derivative of (3.6) in two parts, starting with P_1:

$$\begin{aligned}
&\partial_{\varepsilon_k}P_1\\
&=(\partial_{x_ix_j}^2u_\varepsilon^{(kl)}-\hat{\Gamma}^m{}_{ij}\partial_{x_m}u_\varepsilon^{(kl)})\hat{g}^{ia}\partial_{x_a}u_\varepsilon\hat{g}^{ja}\partial_{x_b}u_\varepsilon\\
&+(\partial_{x_ix_j}^2u_\varepsilon^{(l)}-\hat{\Gamma}^m{}_{ij}\partial_{x_m}u_\varepsilon^{(l)})\hat{g}^{ia}\hat{g}^{jb}(\partial_{x_a}u_\varepsilon^{(k)}\partial_{x_b}u_\varepsilon+\partial_{x_a}u_\varepsilon\partial_{x_b}u_\varepsilon^{(k)}).
\end{aligned}$$

For P_2 we get

$$\begin{aligned}
&\partial_{\varepsilon_k}P_2\\
&=(\partial_{x_ix_j}^2u_\varepsilon^{(k)}-\hat{\Gamma}^m{}_{ij}\partial_{x_m}u_\varepsilon^{(k)})\hat{g}^{ia}\hat{g}^{jb}(\partial_{x_a}u_\varepsilon^{(l)}\partial_{x_b}u_\varepsilon+\partial_{x_a}u_\varepsilon\partial_{x_b}u_\varepsilon^{(l)})\\
&+(\partial_{x_ix_j}u_\varepsilon-\hat{\Gamma}^m{}_{ij}\partial_{x_m}u_\varepsilon)\hat{g}^{ia}\hat{g}^{jb}(\partial_{x_a}u_\varepsilon^{(kl)}\partial_{x_b}u_\varepsilon+\partial_{x_a}u_\varepsilon^{(l)}\partial_{x_b}u_\varepsilon^{(k)}\\
&+\partial_{x_a}u_\varepsilon^{(k)}\partial_{x_b}u_\varepsilon^{(l)}+\partial_{x_a}u_\varepsilon\partial_{x_b}u_\varepsilon^{(kl)}).
\end{aligned}$$

Evaluating the above second derivatives at $\varepsilon = 0$, denoting $w^{kl} := u_\varepsilon^{(kl)}|_{\varepsilon=0}$, gives the second linearization of the equation $F(x', u, \nabla_g u, \nabla_g^2 u) = 0$:

$$\begin{aligned}
&\partial^2_{\varepsilon_k\varepsilon_l} F(x', u_\varepsilon, \nabla_g u_\varepsilon, \nabla_g^2 u_\varepsilon)|_{\varepsilon=0} \\
&= -\Delta_{\hat{g}} w^{kl} + \frac{n-1}{2c(x',0)}\left(\partial^3_{x_n} c(x',0) v^k v^l + \partial^2_{x_n} c(x',0) w^{kl}\right) \\
&+ \frac{1-n}{2c(x',0)} \sum_{i,j=1}^{n-1} \hat{g}^{ij} \partial_{x_i} c(x',0) \partial_{x_j} w^{kl} = 0.
\end{aligned}$$

Here we used that $u_0 = 0$, $\nabla_g u_0$, $\nabla_g^2 u_0$ vanish, $\partial_{x_n} c(x',0) = 0$ and that $\partial_{x_k x_n} c(x',0) = 0$ for $k = 1, \dots, n-1$.

4. Proofs of Theorems 1.3 and 1.4

Before going to the proof of Theorem 1.3, we state two lemmas that will be used also in the proof of Theorem 1.4. The first lemma says that the products of two solutions to an advection diffusion equation form a complete set in L^1. For this we note that equations of the form $\Delta_g u + Xu = 0$, for a smooth real valued vector field X, can be written in the form of a magnetic Schrödinger equation. In local coordinates we have

$$L_{g,A,q} u = -|g|^{-\frac{1}{2}} \left(\partial_{x_j} + iA_j\right)\left(|g|^{\frac{1}{2}} g^{jk}\left(\partial_{x_k} + iA_k\right) u\right) + qu = 0$$

for $A = \frac{iX}{2}$ and $q = \frac{1}{4} g(X,X) - \frac{1}{2}\operatorname{div}_g(X)$.

Lemma 4.1. *Let (Ω, g), $\Omega \subset \mathbb{R}^n$, be a smooth Riemannian manifold with boundary. We have two cases:*

(1) When $n = 2$ let $\Gamma \subset \partial\Omega$ be a nonempty open set, let $f \in C^\infty(\bar{\Omega})$ be such that $f|_\Gamma = 0$. Assume that

$$\int_\Omega f u_1 u_2 \, dV_g = 0 \tag{4.1}$$

for all u_j solving $L_{g,A,q} u_j = 0$ in Ω with $u_j|_{\partial\Omega\setminus\Gamma} = 0$. Then $f \equiv 0$ in Ω.

(2) When $n > 2$ let g be admissible, $f \in C^\infty(\bar{\Omega})$ be such that $f|_{\partial\Omega} = 0$. Assume that (4.1) holds for all u_j solving $L_{g,A,q} u_j = 0$ in Ω. Then $f \equiv 0$ in Ω.

Proof. $n = 2$: The proof is written in [Tzo17, Section 7.3] where the author considers identifying a zeroth order term.

$n > 2$: This can be read from the proof of [DKSU09, Theorem 1.7], more precisely from the part where they prove that the two potentials q_1 and q_2 agree in M (using the notation of the referred article). □

The second lemma states that if we know the partial DN maps corresponding to $\Delta_g u + X_j u = 0$ for $j = 1, 2$ and $X_1 = X_2$ on $\Gamma \subset \partial\Omega$, $\Gamma \neq \emptyset$, then $X_1 = X_2$ in Ω. The DN map in question is defined as $\Lambda_X^\Gamma \colon C^{k,\alpha}(\Gamma) \to C^{k-1,\alpha}(\Gamma)$,

$$\Lambda_X^\Gamma f = \partial_\nu u|_\Gamma.$$

Lemma 4.2. *Let* $(\mathbb{R}^2, g)$ *be a Riemannian manifold,* $\Omega \subset \mathbb{R}^2$ *be simply connected with smooth boundary and* $\Gamma \subset \partial\Omega$ *be an open nonempty set. Let* X_j *be smooth vector fields for* $j = 1, 2$ *so that* $X_1|_\Gamma = X_2|_\Gamma$*. If* $\Lambda^\Gamma_{X_1} f = \Lambda^\Gamma_{X_2} f$ *for all* $f \in C^{k,\alpha}(\Gamma)$*, then* $X_1 = X_2$ *in* Ω*.*

Proof. By [Tzo17, Theorem 1.1] we have $iA_1 - iA_2 = \theta^{-1} d\theta$ and $q_1 = q_2$ in Ω, where

$$A_j = \frac{iX_j}{2}, \quad q_j = \frac{1}{4} g(X_j, X_j) - \frac{1}{2}\operatorname{div}_g(X_j)$$

and θ is a nonvanishing function with $\theta|_\Gamma = 1$. Notice that $d(\theta^{-1} d\theta) = d(\theta^{-1}) \wedge d\theta + \theta^{-1} \wedge d^2\theta = 0$ which implies that $dX_1 = dX_2$ and by the Poincaré Lemma we have a function $\varphi \in C^\infty(\bar{\Omega})$ such that

$$X_1 - X_2 = \nabla_g \varphi.$$

Using that $X_1|_\Gamma = X_2|_\Gamma$ gives $0 = (X_1 - X_2) \cdot \nu|_\Gamma = \partial_\nu \varphi|_\Gamma$ and $\varphi|_\Gamma = 0$ (actually we have that $\varphi|_{\partial\Omega}$ is a constant, but we can subtract this constant to obtain $\varphi|_{\partial\Omega} = 0$). Let us see what kind of an equation φ satisfies. Now $X_2 = X_1 - \nabla_g \varphi$ and plugging this into $q_1 = q_2$ gives

$$\begin{aligned}
&\frac{1}{4} g(X_1, X_1) - \frac{1}{2}\operatorname{div}_g(X_1) \\
&= \frac{1}{4} g^{kl} \left(X_{1,k} - (\nabla_g \varphi)_k\right)\left(X_{1,l} - (\nabla_g \varphi)_l\right) - \frac{1}{2}\operatorname{div}_g(X_1) + \frac{1}{2}\operatorname{div}_g(\nabla_g \varphi) \\
&= \frac{1}{4} g(X_1, X_1) - \frac{1}{2} g(X_1, \nabla_g \varphi) + \frac{1}{4} g(\nabla_g \varphi, \nabla_g \varphi) - \frac{1}{2}\operatorname{div}_g(X_1) + \frac{1}{2}\Delta_g \varphi.
\end{aligned}$$

Thus φ solves

$$\begin{cases} \Delta_g \varphi - g(X_1, \nabla_g \varphi) + \frac{1}{2}|\nabla_g \varphi|_g^2 = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma \\ \partial_\nu \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{4.2}$$

From this we get that

$$|\Delta_g \varphi| = |g(X_1, \nabla_g \varphi) - \frac{1}{2}|\nabla_g \varphi|_g^2| \le |g(X_1, \nabla_g \varphi)| + |\frac{1}{2}|\nabla_g \varphi|_g^2| \le C|\nabla_g \varphi|_g \le C(|\nabla_g \varphi|_g + |\varphi|)$$

since $|\nabla_g \varphi|_g \le M$ for some $M > 0$. Then by the unique continuation principle for local Cauchy data $\varphi \equiv 0$ in Ω (see [KSU11, Theorem B.1]) and hence $X_1 = X_2$. □

Proof Theorem 1.3. The assumptions of [Nur23a, Proposition 2.1] are satisfied and thus (3.3) has a unique small solution $u_f \in C^s(\bar{\Omega})$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ where $\varepsilon_k \in \mathbb{R}$ small enough so that (3.3) has a unique small solution for $f_\varepsilon := \sum_{k=1}^N f_k$, where $f_k \in C^s(\partial\Omega)$ with $\|f_k\|_{C^s(\partial\Omega)} < \delta$, $\delta > 0$. Denote by $u := u(x, \varepsilon)$ and $\tilde{u} := \tilde{u}(x, \varepsilon)$ the unique small solutions to (3.3) with the mterics g and $\tilde{c}g$ respectively. Then by [Nur23a, Proposition 2.1] the solutions u and $\tilde{u}$ depend smoothly on ε. Hence we can calculate the first and second order linearizations of (3.3) and the linearizations of the DN map (1.4).

In section 3 we already calculated the first linearization of $F(x', u, \nabla_g u, \nabla_g^2 u) = 0$ and since we also assume that $\partial_{x_n}^2 c(x', 0) = 0$ we have that the first linearization of

(3.3) is

$$\text{(4.3)}\qquad \begin{cases} -\Delta_{\hat g} v^l + \frac{1-n}{2c(x',0)}\sum_{i,j=1}^{n-1} \hat g^{ij}\partial_{x_i} c(x',0)\partial_{x_j} v^l = 0 & \text{in } \Omega \\ v^l = f_l, & \text{on } \partial\Omega, \end{cases}$$

where $v^l = \partial_{\varepsilon_l} u|_{\varepsilon=0}$. Also, the first linearization of the DN map is $(D\Lambda_g)_0 \colon C^s(\partial\Omega) \to C^{s-1}(\partial\Omega)$, $f \mapsto \partial_\nu v^l|_{\partial\Omega}$. Now (4.3) can be written in the form of an advection diffusion equation $-(\Delta_{\hat g} - X)v^l = 0$ for the real vector field X such that

$$Xh = \frac{1-n}{2c(x',0)} \hat g^{ij}\partial_{x_i} c(x',0)\partial_{x_j} h.$$

For the metric $\tilde c g$ the first linearization of $F(x', \tilde u, \nabla_{\tilde c g}\tilde u, \nabla^2_{\tilde c g}\tilde u) = 0$ is

$$\text{(4.4)}\qquad \begin{cases} -\Delta_{\hat g} \tilde v^l + \frac{1-n}{2c(x',0)}\sum_{i,j=1}^{n-1} \hat g^{ij}\partial_{x_i} c(x',0)\partial_{x_j} \tilde v^l \\ +\frac{1-n}{2\tilde c(x',0)}\sum_{i,j=1}^{n-1} \hat g^{ij}\partial_{x_i} \tilde c(x',0)\partial_{x_j} \tilde v^l = 0 & \text{in } \Omega \\ \tilde v^l = f_l, & \text{on } \partial\Omega, \end{cases}$$

where $\tilde v^l = \partial_{\varepsilon_l} \tilde u|_{\varepsilon=0}$. Similarly this can be written as an advection diffusion equation for the real vector field $\tilde X$ such that $\tilde X h = \frac{1-n}{2}\hat g^{ij}\left(\frac{\partial_{x_i} c(x',0)}{c(x',0)} + \frac{\partial_{x_i}\tilde c(x',0)}{\tilde c(x',0)}\right)\partial_{x_j} h$. Since we know that $\Lambda_g(f) = \Lambda_{\tilde c g}(f)$ for all $f \in U_\delta$, where $\delta > 0$ is sufficiently small, we can apply $\partial_{\varepsilon_l}|_{\varepsilon=0}$ to this, which implies

$$(D\Lambda_g)_0 = (D\Lambda_{cg})_0.$$

When $n = 3$ we use lemma 4.2 together with a boundary determination [GT11, Proposition 4.1.] and when $n > 3$ we use [KU18, Theorem 1.4] (we assume that the metric $\hat g$ is admissible) to deduce that $X = \tilde X$ and hence for all $i = 1, \ldots, n-1$

$$\frac{\partial_{x_i}\tilde c(x',0)}{\tilde c(x',0)} = 0, \text{ for } x' \in \Omega.$$

This then implies two things. Firstly, $\tilde c(x',0) = \lambda \in (0,\infty)$. Secondly, since solutions to an advection diffusion equation are unique, we have that $v^l = \tilde v^l$.

Now the second order linearization for (3.3) is

$$\text{(4.5)}\qquad \begin{cases} -\Delta_{\hat g} w^{kl} + \frac{1-n}{2c(x',0)}\sum_{i,j=1}^{n-1} \hat g^{ij}\partial_{x_i} c(x',0)\partial_{x_j} w^{kl} + \frac{n-1}{2c(x',0)}\partial^3_{x_n} c(x',0) v^k v^l = 0 & \text{in } \Omega \\ w^{kl} = 0, & \text{on } \partial\Omega. \end{cases}$$

For the metric $\tilde c g$ the second linearization is

$$\text{(4.6)}\qquad \begin{cases} -\Delta_{\hat g} \tilde w^{kl} + \frac{1-n}{2c(x',0)}\sum_{i,j=1}^{n-1} \hat g^{ij}\partial_{x_i} c(x',0)\partial_{x_j} \tilde w^{kl} + \frac{n-1}{2c(x',0)}\partial^3_{x_n} c(x',0) v^k v^l \\ +\frac{n-1}{2\tilde c(x',0)}\partial^3_{x_n}\tilde c(x',0) v^k v^l = 0 & \text{in } \Omega \\ \tilde w^{kl} = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $w^{kl} = \partial^2_{\varepsilon_k\varepsilon_l} u|_{\varepsilon=0}$ and $\tilde w^{kl}\partial^2_{\varepsilon_k\varepsilon_l}\tilde u|_{\varepsilon=0}$. Fix $x'_0 \in \Omega$. Let now $v^{(0)}$ be a solution to

$$\text{(4.7)}\qquad \Delta_{\hat g} v^{(0)} + X v^{(0)} + q v^{(0)} = 0 \text{ in } \Omega,$$

where $q := \frac{1-n}{2c(x',0)}\left(-\frac{1}{c(x',0)}|\nabla_{\hat g} c(x',0)|_{\hat g}^2 + \partial_{x_j}\hat g^{ij}\partial_{x_i}c(x',0) + \hat g^{ij}\partial_{x_i}c(x',0){\Gamma^k}_{jk}\right)$, so that $v^{(0)}(x_0') \neq 0$. The existence of such a solution can be shown using Runge approximation (see for example [LLS20, Proposition A.2]). More precisely, there exists a solution $\tilde v^{(0)}$ of (4.7) in a small neighborhood $U \subset \Omega$ of x_0' such that $\tilde v^{(0)}(x_0') \neq 0$ [BJS64, Theorem II5.4.1]. By Runge approximation there exists a solution $v^{(0)}$ with the desired properties. Now subtracting equations (4.5), (4.6) and integrating against $v^{(0)}$ gives

(4.8)

$$\begin{aligned}
0 &= \int_\Omega \left(-\Delta_{\hat g}(w^{kl} - \tilde w^{kl}) + \frac{1-n}{2c(x',0)}\sum_{i,j=1}^{n-1}\hat g^{ij}\partial_{x_i}c(x',0)\partial_{x_j}(w^{kl}-\tilde w^{kl})\right)v^{(0)} \\
&+ \frac{n-1}{2\tilde c(x',0)}\partial_{x_n}^3\tilde c(x',0)v^kv^lv^{(0)}\,dV_{\hat g} \\
&= \int_\Omega -(w^{kl}-\tilde w^{kl})\Delta_{\hat g}v^{(0)} + \frac{1-n}{2c(x',0)}\sum_{i,j=1}^{n-1}\hat g^{ij}\partial_{x_i}c(x',0)\partial_{x_j}(w^{kl}-\tilde w^{kl})v^{(0)} \\
&+ \frac{n-1}{2\tilde c(x',0)}\partial_{x_n}^3\tilde c(x',0)v^kv^lv^{(0)}\,dV_{\hat g} + \int_{\partial\Omega}(w^{kl}-\tilde w^{kl})\partial_\nu v^{(0)} - v^{(0)}\partial_\nu(w^{kl}-\tilde w^{kl})\,dS.
\end{aligned}$$

Since the volume form is $dV_{\hat g} = |\hat g|^{\frac12}\,dx$, where $|\hat g|$ is the determinant of $\hat g$, we use integration by parts for the second term in the last equality to have

$$\begin{aligned}
&+ \int_\Omega \frac{1-n}{2c(x',0)}\hat g^{ij}\partial_{x_i}c(x',0)\partial_{x_j}(w^{kl}-\tilde w^{kl})v^{(0)}|\hat g|^{\frac12}\,dx \\
&= -\int_\Omega\left(\frac{1-n}{2c(x',0)}\hat g^{ij}\partial_{x_i}c(x',0)\partial_{x_j}v^{(0)}|\hat g|^{\frac12}\right. \\
&\left.+ \partial_{x_j}\left(\frac{1-n}{2c(x',0)}\hat g^{ij}\partial_{x_i}c(x',0)|\hat g|^{\frac12}\right)v^{(0)}\right)(w^{kl}-\tilde w^{kl})\,dx \\
&+ \int_{\partial\Omega}(w^{kl}-\tilde w^{kl})\frac{1-n}{2c(x',0)}\hat g^{ij}\partial_{x_i}c(x',0)|\hat g|^{\frac12}v^{(0)}\nu_j\,dS \\
&= -\int_\Omega(w^{kl}-\tilde w^{kl})(Xv^{(0)} + qv^{(0)})\,dV_{\hat g}.
\end{aligned}$$

Here we used that $w^{kl} = \tilde w^{kl} = 0$ on $\partial\Omega$. Combining this with (4.8) gives

$$\begin{aligned}
&\int_\Omega \frac{n-1}{2\tilde c(x',0)}\partial_{x_n}^3\tilde c(x',0)v^kv^lv^{(0)}\,dV_{\hat g} \\
&= \int_\Omega (w^{kl}-\tilde w^{kl})(\Delta_{\hat g}v^{(0)} + Xv^{(0)} + qv^{(0)})\,dV_{\hat g} \\
&- \int_{\partial\Omega}(w^{kl}-\tilde w^{kl})\partial_\nu v^{(0)} - v^{(0)}\partial_\nu(w^{kl}-\tilde w^{kl})\,dS \\
&= 0.
\end{aligned}$$

In the last equality we used $w^{kl} = \tilde w^{kl} = 0$ on $\partial\Omega$ and that applying $\partial^2_{\varepsilon_k\varepsilon_l}|_{\varepsilon=0}$ to $\partial_\nu u|_{\partial\Omega} = \partial_\nu\tilde u|_{\partial\Omega}$ implies

$$\partial_\nu w^{kl}|_{\partial\Omega} = \partial_\nu\tilde w^{kl}|_{\partial\Omega}.$$

Then by Lemma 4.1 we have

$$\frac{n-1}{2\tilde{c}(x',0)}\partial^3_{x_n}\tilde{c}(x',0)v^{(0)} = 0 \text{ for } x' \in \Omega.$$

In particular for $x' = x'_0$ this implies $\partial^3_{x_n}\tilde{c}(x'_0,0) = 0$ but since x'_0 was arbitrary we get

$$\partial^3_{x_n}\tilde{c}(x',0) = 0 \text{ for } x' \in \Omega.$$

Now the boundary value problems (4.5) and (4.6) are the same and thus by uniqueness of solutions $w^{kl} = \tilde{w}^{kl}$ in Ω.

The rest of the proof is done very similarly as in [Nur23a] but we record the proof here for completeness. Next we use induction to show $\partial^k_{x_n}\tilde{c}(x',0) = 0$ for all $k \geq 3$. By the above this already holds for $k = 3$. Our assumption now is

$$\partial^k_{x_n}\tilde{c}(x',0) = 0, \quad x' \in \Omega, \ \text{ for all } \quad k = 3, \dots, N \in \mathbb{N}, N > 3.$$

Let us do a subinduction to prove

$$\partial^k_{l_1 \dots l_k} u(x',0) = \partial^k_{l_1 \dots l_k}\tilde{u}(x',0), \quad x' \in \ \Omega,$$

for all $k = 1, \dots, N$, where $\partial^k_{l_1 \dots l_k} = \frac{\partial^k}{\partial_{\varepsilon_{l_1}} \dots \partial_{\varepsilon_{l_k}}}$. Above we have shown this for $k = 1, 2$. Assume that it holds for $k \leq K < N$. Then the linearization of order $K+1$ for the metric g is, when evaluated at $\varepsilon_1 = \cdots = \varepsilon_{K+1} = 0$,

$$\begin{aligned} &- \Delta_{\hat{g}}\partial^{K+1}_{l_1 \dots l_{K+1}} u(x',0) - X\partial^{K+1}_{l_1 \dots l_{K+1}} u(x',0) + R_K(u, g(x',0), 0) \\ &+ \frac{n-1}{2c(x',0)}\partial^{K+2}_{x_n} c(x',0)\left(\prod_{k=1}^{K+1} v^{(l_k)}\right) = 0, \end{aligned} \tag{4.9}$$

$x' \in \Omega$. Here R is a polynomial of components of g, the derivatives of g and $\partial^k_{l_1 \dots l_k} u_1(x',0)$. Also the linearization of order $K+1$ for the metric $\tilde{c}g$ is

$$\begin{aligned} &- \Delta_{\hat{g}}\partial^{K+1}_{l_1 \dots l_{K+1}} \tilde{u}(x',0) - X\partial^{K+1}_{l_1 \dots l_{K+1}} \tilde{u}(x',0) + R_K(\tilde{u}, g(x',0), 0) \\ &+ \left(\frac{n-1}{2c(x',0)}\partial^{K+2}_{x_n} c(x',0) + \frac{n-1}{2\tilde{c}(x',0)}\partial^{K+2}_{x_n}\tilde{c}(x',0)\right)\left(\prod_{k=1}^{K+1} v^{(l_k)}\right) = 0. \end{aligned} \tag{4.10}$$

Here R_K could also have terms with $\partial^k_{x_n}\tilde{c}(x',0)$ and the components of $\nabla_{x'}\left(\partial^k_{x_n}\tilde{c}(x',0)\right)$ but terms containing these are zero by the induction assumption. Now an integration by parts argument similar to the case of the second linearization and together with Lemma 4.1 (choosing $v^3 = \ldots = v^{K+1} = 1$) gives $\partial^{K+2}_{x_n}\tilde{c}(x',0) = 0$.

Subtracting the equations (4.9) and (4.11) we get

$$\begin{cases} -\Delta_{\hat{g}}\left(\partial^{K+1}_{l_1 \dots l_{K+1}} u(x',0) - \partial^{K+1}_{l_1 \dots l_{K+1}}\tilde{u}(x',0)\right) & \\ -X\left(\partial^{K+1}_{l_1 \dots l_{K+1}} u(x',0) - \partial^{K+1}_{l_1 \dots l_{K+1}}\tilde{u}(x',0)\right) = 0, & \text{in } \Omega \\ \partial^{K+1}_{l_1 \dots l_{K+1}} u(x',0) - \partial^{K+1}_{l_1 \dots l_{K+1}}\tilde{u}(x',0) = 0, & \text{on } \partial\Omega. \end{cases}$$

This is true, since by induction assumptions for all $x' \in \Omega$ the other terms agree for $k \leq K$. Again, by the uniqueness of solutions, $\partial^{K+1}_{l_1 \dots l_{K+1}} u(x',0) = \partial^{K+1}_{l_1 \dots l_{K+1}}\tilde{u}(x',0)$, $x' \in \Omega$, which ends the subinduction.

Returning to the original induction, the linearization of order $N+1$ at $\varepsilon_1 = \cdots = \varepsilon_{N+1} = 0$ for the metric g is

$$-\Delta_{\hat{g}}\partial^{N+1}_{l_1\ldots l_{N+1}}u(x',0) - X\partial^{N+1}_{l_1\ldots l_{N+1}}u(x',0) + R_N(u,g(x',0),0) + \frac{n-1}{2c(x',0)}\partial^{N+2}_{x_n}c(x',0)\left(\prod_{k=1}^{N+1} v^{(l_k)}\right) = 0,$$

$x' \in\ \Omega$ and for the metric $\tilde{c}g$ we have

$$-\Delta_{\hat{g}}\partial^{N+1}_{l_1\ldots l_{N+1}}\tilde{u}(x',0) - X\partial^{N+1}_{l_1\ldots l_{N+1}}\tilde{u}(x',0) + R_N(\tilde{u},g(x',0),0) + \left(\frac{n-1}{2c(x',0)}\partial^{N+2}_{x_n}c(x',0) + \frac{n-1}{2\tilde{c}(x',0)}\partial^{N+2}_{x_n}\tilde{c}(x',0)\right)\left(\prod_{k=1}^{N+1} v^{(l_k)}\right) = 0. \tag{4.11}$$

Now the subinduction implies that $R_N(u,g(x',0),0) = R_N(\tilde{u},g(x',0),0)$. Thus by subtracting, integrating against $v^{(0)}$ (the solution of (4.7)), using integration by parts and that $\partial_\nu\partial^{N+1}_{l_1\ldots l_{N+1}}u(x',0)|_{\partial\Omega} = \partial_\nu\partial^{N+1}_{l_1\ldots l_{N+1}}\tilde{u}(x',0)|_{\partial\Omega}$ we get

$$\int_\Omega \tilde{c}(x',0)^{-1}\partial^{N+2}_{x_n}\tilde{c}(x',0)v^{(0)}\prod_{k=1}^{N+1} v^{l_k}\,dx' = 0.$$

Choosing all but two of the functions v^{l_k} to be equal to 1, then by the completeness of such solutions (Lemma 4.1) implies that

$$\partial^{N+2}_{x_n}\tilde{c}(x'_0,0) = 0.$$

Since x'_0 was arbitrary, we have $\partial^{N+2}_{x_n}\tilde{c}(x',0) = 0$ for all $x' \in \Omega$. □

We will next prove Theorem 1.4. The proof will be very similar to the one above and we will only point out the differences.

Proof of Theorem 1.4. Let ε be, as in the previous proof, small enough so that (3.3) has a unique small solution for $f_\varepsilon := \sum_{k=1}^N f_k$, where $f_k \in C^s(\partial\Omega)$ with $\mathrm{spt}(f) \subset \Gamma$ and $\|f_k\|_{C^s(\partial\Omega)} < \delta$, $\delta > 0$.

The first linearizations are (4.3), (4.4) and for the corresponding partial DN maps we have

$$(D\Lambda^\Gamma_g)_0 = (D\Lambda^\Gamma_{cg})_0. \tag{4.12}$$

Now using Lemma 4.2 together with a boundary determination [Tzo17, Proposition 4.1] we get from the first linearization that for all $i = 1,\ldots,n-1$

$$\frac{\partial_{x_i}\tilde{c}(x',0)}{\tilde{c}(x',0)} = 0,\ \text{ for } x' \in \Omega.$$

This then implies two things. Firstly, $\tilde{c}(x',0) = \lambda \in (0,\infty)$. Secondly, since solutions to an advection diffusion equation are unique, we have that $v^l = \tilde{v}^l$.

Let us move to the second order linearization and let $x'_0 \in \Omega$. We will subtract equations (4.5), (4.6) and integrate against the function $v^{(0)}$ that solves

$$\begin{cases} \Delta_{\hat{g}}v^{(0)} - Xv^{(0)} - qv^{(0)} = 0 & \text{in } \Omega \\ v^{(0)} = g, & \text{on } \Gamma \\ v^{(0)} = 0, & \text{on } \partial\Omega\setminus\Gamma. \end{cases} \tag{4.13}$$

for $g \in C_c^\infty(\Gamma)$, $g \geq 0$, $g \not\equiv 0$ and that $v^{(0)}(x_0') \neq 0$. This function can be constructed exactly as before. We now do the integration, where the difference will be on how to deal with the boundary terms:

$$\begin{aligned}
&\int_\Omega \frac{n-1}{2\tilde{c}(x',0)} \partial_{x_n}^3 \tilde{c}(x',0) v^k v^l v^{(0)} \, dV_{\hat{g}} \\
&= \int_\Omega -(w^{kl} - \tilde{w}^{kl})(-\Delta_{\hat{g}} v^{(0)} + X v^{(0)} + q v^{(0)}) \, dV_{\hat{g}} \\
&- \int_{\partial\Omega} (w^{kl} - \tilde{w}^{kl}) \partial_\nu v^{(0)} - v^{(0)} \partial_\nu (w^{kl} - \tilde{w}^{kl}) \, dS \\
&+ \int_{\partial\Omega} (w^{kl} - \tilde{w}^{kl}) \frac{1-n}{2c(x',0)} \hat{g}^{ij} \partial_{x_i} c(x',0) |\hat{g}|^{\frac{1}{2}} v^{(0)} \nu_j \, dS.
\end{aligned}$$

Since $v^{(0)}$ solves (4.13) and $(w^{kl} - \tilde{w}^{kl})|_{\partial\Omega} = 0$ the first and last integral vanish and also the first term in the second integral vanishes. For the remaining part we divide the integral in two parts

$$\int_{\partial\Omega} v^{(0)} \partial_\nu (w^{kl} - \tilde{w}^{kl}) \, dS = \int_\Gamma v^{(0)} \partial_\nu (w^{kl} - \tilde{w}^{kl}) \, dS + \int_{\partial\Omega\setminus\Gamma} v^{(0)} \partial_\nu (w^{kl} - \tilde{w}^{kl}) \, dS.$$

This is zero since $v^{(0)}|_{\partial\Omega\setminus\Gamma} = 0$ and from (4.12) we have that $\partial_\nu (w^{kl} - \tilde{w}^{kl})|_\Gamma = 0$. Hence

$$\int_\Omega \frac{n-1}{2\tilde{c}(x',0)} \partial_{x_n}^3 \tilde{c}(x',0) v^k v^l v^{(0)} \, dV_{\hat{g}} = 0$$

for all v_k, v^l solving (4.3). Continuing as in the proof of Theorem 1.3 we get

$$\partial_{x_n}^3 \tilde{c}(x',0) = 0 \text{ for } x' \in \Omega.$$

and this then implies that $w^{kl} = \tilde{w}^{kl}$ in Ω.

Proceeding with induction as before and taking further care about the boundary terms in the integrals will conclude the proof. ☐

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INVERSE PROBLEMS FOR SEMILINEAR ELLIPTIC PDE WITH A GENERAL NONLINEARITY *a (x,u)*

by

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INVERSE PROBLEMS FOR SEMILINEAR ELLIPTIC PDE WITH A GENERAL NONLINEARITY $a(x, u)$

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Abstract. This article studies the inverse problem of recovering a nonlinearity in an elliptic equation $\Delta u + a(x, u) = 0$ from boundary measurements of solutions. Previous results based on first order linearization achieve this under a sign condition on $\partial_u a(x, u)$, and results based on higher order linearization recover the Taylor series of $a(x, u)$ with respect to u. We improve these results and show that a general nonlinearity, and not just its Taylor series, is uniquely determined up to gauge near a fixed solution. Our method is based on constructing a good solution map that locally parametrizes solutions of the nonlinear equation by solutions of the linearized equation.

1. Introduction

Motivation. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded domain whose boundary is assumed to be C^∞ for simplicity, and let $a \in C^k(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$ where $k \geq 3$ and $0 < \alpha < 1$. We write $a = a(x, z)$ for $x \in \overline{\Omega}$ and $z \in \mathbb{R}$, and consider equations of the form

$$\Delta u(x) + a(x, u(x)) = 0 \text{ in } \Omega. \tag{1.1}$$

In this article we study the inverse problem of identifying the function $a(x, z)$ from certain boundary measurements of solutions of (1.1). For example, the boundary measurements could be encoded by a Dirichlet-to-Neumann (DN) map if the equation is well-posed, or more generally one could use the (full) Cauchy data set

$$C_a := \{(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) : u \in C^{2,\alpha}(\overline{\Omega}) \text{ solves } \Delta u + a(x, u) = 0\}.$$

That is, we wish to answer the question:

Does C_a determine $a(x, z)$?

For linear equations $\Delta u + q(x)u = 0$, the question above is a version of Calderón's inverse problem and there is large literature (see e.g. the survey [Uhl14]). There are also many results for nonlinear equations. The first generation of such results was based on *first order linearization*, i.e. on studying the (first) Fréchet derivative of the nonlinear DN map and using existing results for linear equations. This method was introduced in [Isa93], and further results for determining a nonlinearity $a(x, u)$ as in (1.1) were given in [IS94; IN95; IY13]. These results typically require assumptions such as

$$a(x, 0) = 0, \tag{1.2}$$

$$\partial_u a(x, u) \leq 0, \tag{1.3}$$

which ensure well-posedness and a maximum principle. The assumption (1.3) was weakened in [IN95], and [Sun10] gave a result without assuming (1.2). The results show that one can recover $a(x, u)$ in (some subset of) the *reachable set*

$$E_a := \{(x, z) : x \in \overline{\Omega}, \; z = u(x) \text{ for some solution } u \text{ of } \Delta u + a(x, u) = 0\}.$$

There are many related works for quasilinear and conductivity type equations. References may be found in the survey articles [Sun05; Uhl09].

The works [FO20; LLLS21a] introduced a higher order linearization method in inverse problems for nonlinear elliptic equations, motivated by the earlier work [KLU18] for hyperbolic equations. This method applies to inverse problems for equations like (1.1) without any positivity assumptions as in (1.3). Moreover, unlike in the first order linearization method that reduced matters to known results for linear equations, in higher order linearization the nonlinearity is used as a tool that helps in solving inverse problems. In this way, one can obtain results in partial data problems [LLLS21b; KU20; ST23] or anisotropic problems [FO20; LLLS21a; CFO23; FKO23] that are stronger than the known results for corresponding linear equations.

However, the higher order linearization results for (1.1) start with the assumptions that

$$a(x,0) = 0, \tag{1.4}$$

$$0 \text{ is not a Dirichlet eigenvalue for } \Delta + \partial_u a(x,0) \text{ in } \Omega. \tag{1.5}$$

The first assumption ensures that $u \equiv 0$ is a solution of (1.1). The second assumption ensures that the linearized equation is well-posed for small Dirichlet data, and hence there is a nonlinear DN map Λ_a for (1.1) defined for small Dirichlet data. The additional assumption $\partial_u a(x,0) = 0$ also appears in many results. The works [FO20; LLLS21a] then show that Λ_a (defined for small Dirichlet data) determines $\partial_u^j a(\,\cdot\,,0)$ for many $j \geq 0$. If one additionally assumes that $a(x,z)$ is real-analytic in z, then this is sufficient for determining $a(x,z)$ completely.

Results. Our aim is to consider inverse problems for (1.1) for general functions $a \in C^k(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$. In particular, we wish to remove the assumptions (1.2)–(1.3) in the first order linearization method and (1.4)–(1.5) in the higher order linearization method. This requires certain changes in the problem setup. First of all, the results in [FO20; LLLS21a] are based on looking at solutions of (1.1) close to $u \equiv 0$ and on well-posedness for small data. Moreover, the linearized equation might not be well-posed in general, but by Fredholm theory it is still well-posed for most Dirichlet data (i.e. data that are L^2-orthogonal to a finite dimensional space). It follows that there may not be a Dirichlet-to-Neumann map to work with.

For these reasons, in the general case we consider an arbitrary but fixed solution $w \in C^{2,\alpha}(\overline{\Omega})$ of $\Delta w + a(x,w) = 0$ and for $\delta > 0$ we define the local Cauchy data set

$$C_a^{w,\delta} := \{(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) \,:\, u \in C^{2,\alpha}(\overline{\Omega}) \text{ solves } \Delta u + a(x,u) = 0 \text{ and } \|w-u\|_{C^{2,\alpha}(\overline{\Omega})} \leq \delta\}.$$

If $w \equiv 0$, this would be analogous to small Dirichlet data. By the Fredholm theory fact mentioned above one expects that there are many solutions close to w, and we will prove a precise version of such a result.

Our first main theorem shows that if two nonlinearities a_1 and a_2 admit a common solution w and if their local boundary measurements satisfy the inclusion $C_{a_1}^{w,\delta} \subseteq C_{a_2}^{0,C}$, then $a_1 = a_2$ near the common solution w.

Theorem 1.1. *Let $a_1, a_2 \in C^3(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$, and let $w \in C^{2,\alpha}(\overline{\Omega})$ solve $\Delta w + a_1(x,w) = 0$ and $\Delta w + a_2(x,w) = 0$ in Ω. If for some $\delta, C > 0$ one has*

$$C_{a_1}^{w,\delta} \subseteq C_{a_2}^{0,C},$$

then there is $\varepsilon > 0$ such that

$$a_1(x, w(x) + \lambda) = a_2(x, w(x) + \lambda), \qquad x \in \overline{\Omega}, \quad |\lambda| \leq \varepsilon.$$

This theorem is based on using first order linearization and it is valid for general nonlinearities. In particular, the sign condition (1.3) in the earlier results mentioned above is not needed. We note that the higher order linearization method does not require the sign condition either, but the result in [FO20; LLLS21a] for the case $w \equiv 0$ was $\partial_u^j a_1(\,\cdot\,,0) = \partial_u^j a_2(\,\cdot\,,0)$ for many $j \geq 0$ which is clearly weaker than the conclusion in Theorem 1.1. Moreover, if the linearizations of $\Delta u + a_j(x,u) = 0$ (linearized at the solution w) happen to be well-posed, then by the arguments in Section 2 there are Dirichlet-to-Neumann maps Λ_{a_j} defined for Dirichlet data close to $w|_{\partial\Omega}$, and then by Theorem 1.1 one obtains $a_1 = a_2$ near points $(x, w(x))$ whenever $\Lambda_{a_1} = \Lambda_{a_2}$.

Remark 1.2. It is in order to explain the assumption $C_{a_1}^{w,\delta} \subseteq C_{a_2}^{0,C}$ in the theorem. A more typical way of stating a uniqueness result would be to say that $C_{a_1} = C_{a_2}$ (i.e. the full Cauchy data sets of a_1 and a_2 agree) implies $a_1 = a_2$ somewhere. However, $C_{a_1} = C_{a_2}$ implies our assumption $C_{a_1}^{w,\delta} \subseteq C_{a_2}^{0,C}$ in many cases, e.g. when the equation $\Delta u + a_2(x,u) = 0$ is well-posed for Dirichlet data near $w|_{\partial\Omega}$ or when $\|\partial_u a_2\|_{L^\infty(\Omega\times\mathbb{R})} < \infty$ (the latter fact follows from the Cauchy data estimates in Section 3).

More precisely, the assumption $C_{a_1}^{w,\delta} \subseteq C_{a_2}^{0,C}$ means that if u_1 solves $\Delta u_1 + a_1(x,u_1) = 0$ with $\|u_1 - w\|_{C^{2,\alpha}(\overline{\Omega})} \leq \delta$, then there is u_2 solving $\Delta u_2 + a_2(x,u_2) = 0$, having the same Cauchy data as u_1, and satisfying $\|u_2\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$. The existence of such a constant C is required in the proof to make sure that if u_1 is very close to w, then u_2 will also be close to w and we can use a uniqueness result to guarantee that u_2 can be differentiated with respect to some parameters if the same is true for u_1.

If the nonlinearities a_1 and a_2 do not admit a common solution, then this inverse problem has a gauge invariance as observed in [Sun10]. If $a \in C^k(\mathbb{R}, C^\alpha(\overline{\Omega}))$ is a nonlinearity and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ is any function satisfying $\varphi|_{\partial\Omega} = \partial_\nu\varphi|_{\partial\Omega} = 0$, we define

$$T_\varphi a(x,u) := \Delta\varphi(x) + a(x, u + \varphi(x)). \tag{1.6}$$

Then u solves $\Delta u + a(x,u(x)) = 0$ if and only if $v = u - \varphi$ solves $\Delta v + T_\varphi a(x,v(x)) = 0$. It follows that the solutions of these two equations have the same Cauchy data. Hence, if the Cauchy data sets for a_1 and a_2 agree, one can only expect that $a_2(x,u) = T_\varphi a_1(x,u)$ for (x,u) in the reachable set. There are a number of related works based on the first linearization, see e.g. [IS94; IN95; Sun10]. Recent works that involve a similar gauge invariance are given in [LL23; KLL23]. We also mention the examples in [IS94; FKO23] showing that in general the reachable set is not all of $\overline{\Omega} \times \mathbb{R}$.

Our next result shows that if one knows the Cauchy data for a nonlinearity a and for solutions close to a given solution w, then one can recover a near points $(x, w(x))$ precisely up to the gauge mentioned above.

Theorem 1.3. *Let $a_1, a_2 \in C^3(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$, and let $w_1 \in C^{2,\alpha}(\overline{\Omega})$ solve $\Delta w_1 + a_1(x,w_1) = 0$ in Ω. If*

$$C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$$

for some $\delta, C > 0$, then there is $\varepsilon > 0$ such that

$$a_1(x, w_1(x) + \lambda) = T_\varphi a_2(x, w_1(x) + \lambda)$$

whenever $x \in \overline{\Omega}$ and $|\lambda| \leq \varepsilon$. Here $\varphi := w_2 - w_1$ where $w_2 \in C^{2,\alpha}(\overline{\Omega})$ is the unique solution of $\Delta w_2 + a_2(x,w_2) = 0$ in Ω with $w_1|_{\partial\Omega} = w_2|_{\partial\Omega}$ and $\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}$.

Again, Theorem 1.3 is valid for general nonlinearities. Note that Theorem 1.1 is a corollary of Theorem 1.3 since $w_2 = w_1$ in that case.

Both Theorem 1.1 and 1.3 are based on first order linearization and they rely on the solution of an inverse problem for the linearized equation. In contrast, many of the results based on higher order linearization do not rely directly on the inverse problem for the linearized equation. In fact in these results the equation often has a form where the unknown quantities only appear in higher linearizations and not in the first linearization. For such equations, nonlinearity often helps and one can obtain improved results in the presence of nonlinearity.

In the case of the higher order linearization method, we can remove the assumptions (1.4)–(1.5) that were present in most of the earlier results. The following result is an example of what one can prove.

Theorem 1.4. *Let $a_1, a_2 \in C^{k+1}(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$ with $k \geq 2$, let $w_1 \in C^{2,\alpha}(\overline{\Omega})$ solve $\Delta w_1 + a_1(x, w_1) = 0$ in Ω, and suppose that*

$$C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$$

for some $\delta, C > 0$. Let $w_2 \in C^{2,\alpha}(\overline{\Omega})$ be the unique solution of $\Delta w_2 + a_2(x, w_2) = 0$ in Ω with $w_1|_{\partial\Omega} = w_2|_{\partial\Omega}$ and $\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}$. Assume further that

$$\partial_u^l a_1(x, w_1) = \partial_u^l a_2(x, w_2), \qquad 1 \leq l \leq k-1. \tag{1.7}$$

Then

$$\int_\Omega (\partial_u^k a_1(x, w_1) - \partial_u^k a_2(x, w_2)) v_1 \ldots v_{k+1} \, dx = 0$$

for any v_j solving the linear equation $\Delta v_j + \partial_u a_1(x, w_1) v_j = 0$ in Ω.

In other words, if $C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$ and (1.7) holds, then $\partial_u^k a_1(x, w_1) - \partial_u^k a_2(x, w_2)$ is L^2-orthogonal to products of $k+1$ solutions of the same linear equation. This is a typical conclusion in the higher order linearization method. Under our current assumptions, Theorem 1.3, which is based on solving the inverse problem for the linearized equation, already implies that $\partial_u^k a_1(x, w_1) = \partial_u^k a_2(x, w_2)$. The point is that as long as (1.7) holds (this is true e.g. for polynomial nonlinearities $a_j(x, u) = q_j(x) u^k$ and $w_j \equiv 0$), Theorem 1.4 does not rely on solving the inverse problem for the linearized equation and one can prove this without assuming (1.4)–(1.5). Theorem 1.4 remains valid for more general equations such as $\Delta_g u + a(x, u) = 0$ with a smooth Riemannian metric g, for which the linearized case is not fully understood. For these more general equations one might be able to obtain improved results in the nonlinear case as was done in [FO20; LLLS21a] and subsequent works.

Methods. Let us next describe the methods for proving the above results. The first objective is to show that near a solution w of $\Delta w + a(x, w) = 0$, there are many solutions $u_v = w + v + O(\|v\|^2)$ of the same equation that are parametrized by small solutions v of the linearized equation

$$\Delta v + \partial_u a(x, w) v = 0.$$

It will also be important that u_v depends smoothly on v. We will prove this by a standard argument using the implicit function theorem. This is slightly delicate since the linearized equation may not be well-posed. In order to have a solution u_v depending smoothly on v that is unique in a suitable sense, one needs to use a solution operator for the linearized equation that takes into account the finite dimensional obstructions to solvability coming from Fredholm theory.

One can recast the previous result in a different language. If $q(x) = \partial_u a(x, w(x))$ and

$$V_q = \{v \in C^{2,\alpha}(\overline{\Omega}) \,:\, \Delta v + qv = 0 \text{ in } \Omega\},$$
$$W_a = \{u \in C^{2,\alpha}(\overline{\Omega}) \,:\, \Delta u + a(x, u) = 0 \text{ in } \Omega\},$$

then V_q is the solution space of $\Delta+q$ and W_a is a Banach manifold in $C^{2,\alpha}(\overline{\Omega})$ consisting of solutions of the nonlinear equation. Then V_q is the tangent space of W_a at w, and our result shows that

$$S_{a,w} : v \mapsto u_v$$

is a bijective smooth map from a neighborhood of 0 in V_q onto a neighborhood of w in W_a with $DS_{a,w}(0) = \mathrm{Id}$. Similar ideas appear e.g. in [Eel66; Pal68; Sun75].

Next, starting from the assumption $C^{w,\delta}_{a_1} \subseteq C^{0,C}_{a_2}$, we construct a solution $u_{1,v}$ as above for a_1, and use the inclusion of Cauchy data sets to conclude that there is a solution $u_{2,v}$ for a_2 having the same Cauchy data as $u_{1,v}$. We know that $u_{1,v}$ depends smoothly on v, but this is not known for $u_{2,v}$. In order to show that also $u_{2,v}$ depends smoothly on v, we prove quantitative estimates showing that solutions of both the linearized and nonlinear equations depend continuously on their Cauchy data. For one of these results we invoke a standard Carleman estimate. Since $u_{2,v}$ depends continuously on its Cauchy data, and since the Cauchy data of $u_{2,v}$ is the same as for $u_{1,v}$ and the latter depends smoothly on v, we are able to show that also $u_{2,v}$ depends smoothly on v. This also uses certain functional analytic arguments following [OSSU20].

The first order linearization result, Theorem 1.1, is proved as follows. If $C^{w,\delta}_{a_1} \subseteq C^{0,C}_{a_2}$ and if u_{1,v_t} and u_{2,v_t} are as described above with $v_t = v + th$, we differentiate the equations $\Delta u_{j,v_t} + a_j(x, u_{j,v_t}) = 0$ with respect to t, subtract the resulting equations, and integrate against a solution of $\Delta \tilde{v}_2 + \partial_u a_2(x, u_{2,v})\tilde{v}_2 = 0$ to obtain

$$\int_\Omega (\partial_u a_1(x, u_{1,v}) - \partial_u a_2(x, u_{2,v}))\tilde{v}_1 \tilde{v}_2 \, dx = 0$$

where $\tilde{v}_1 = DS_{a_1,w}(v)h$. Then we use the bijectivity of $S_{a,w}$ above to conclude that any solution $\tilde{v}_1$ of $\Delta\tilde{v}_1 + \partial_u a_1(x, u_{1,v})\tilde{v}_1 = 0$ can be written as $DS_{a_1,w}(v)h$ for some h. Density of products of solutions as in the standard Calderón problem [SU87; Buk08] implies that

$$\partial_u a_1(x, u_{1,v}) = \partial_u a_2(x, u_{2,v}) \text{ for any small } v \in V_q.$$

Next we show that $\varphi_v := u_{2,v} - u_{1,v}$ is independent of v, by observing that the derivative of φ_v with respect to v is identically 0, because it solves a linear elliptic equation and has zero Cauchy data. Since $\varphi_0 = w - w = 0$, we obtain

$$\partial_u a_1(x, u_{1,v}) = \partial_u a_2(x, u_{1,v}) \text{ for any small } v \in V_q.$$

(In the setting of Theorem 1.3 one has $u_{2,v} = u_{1,v} + \varphi$ instead.) It then remains to show that for any fixed x_0, the values $u_{1,v}(x_0)$ generate an interval $[w(x_0) - \varepsilon, w(x_0) + \varepsilon]$ by varying v. This follows since $u_{1,v} = w + v + O(\|v\|^2)$ and since one can generate linear solutions v with $v(x_0) \neq 0$ by using Runge approximation. This concludes the outline of proof of Theorem 1.1. The proof of Theorem 1.3 is analogous, except that φ will be a nonzero function that is independent of v.

Finally, we use the higher order linearization method and differentiate k times the equations $\Delta u_{j,v} + a_j(x, u_{j,v}) = 0$ with respect to v. Subtracting the resulting equations, using the assumption (1.7) and integrating against a solution v_{k+1}, we arrive at Theorem 1.4.

The article is organized as follows. In Section 2 we construct a solution map for equation (1.1). Section 3 is dedicated to quantitative uniqueness results for (1.1) and its linearization. The linearization methods require two smooth solution maps and the second one is constructed in Section 4. In Section 5 we use first order linearization to prove Theorems 1.1 and 1.3. Finally, in Section 6 we give the proof of Theorem 1.4. At the end we have an Appendix where we give a Runge approximation result for the first linearization of (1.1).

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2. A smooth solution operator

The linearization methods used in this work are based on constructing solutions $u = u_v$ of

$$\Delta u + a(x, u) = 0 \text{ in } \Omega, \tag{2.1}$$

so that u_v is close to a fixed solution w of (2.1) and depends smoothly on a small solution v of the linearized equation

$$\Delta v + \partial_u a(x, w)v = 0 \text{ in } \Omega. \tag{2.2}$$

In order to parametrize solutions of (2.1) on solutions of (2.2), we need to be able to single out suitable solutions of the linearized equation which may not be well-posed. This is done in Lemma 2.1 by using the Fredholm alternative. We then construct solutions u_v of (2.1) by solving a nonlinear fixed point equation. This fixed point equation is solved in Lemma 2.3. The construction of the smooth solution map $v \to u_v$ for the equation (2.1) is completed in Lemma 2.4. Finally in Lemma 2.5 we show that the first Fréchet derivative of the solution operator is an isomorphism between spaces of solutions to the linearized equation.

Before we proceed to the results of this section, let us define some spaces and mappings that are used throughout. Let $\Omega \subseteq \mathbb{R}^n$ with $n \geq 2$ be a bounded open set with C^∞ boundary, and let $q \in C^{1,\alpha}(\overline{\Omega})$ be real valued where $0 < \alpha < 1$. First we have the kernel N_q of the operator $\Delta + q : H_0^1(\Omega) \to H^{-1}(\Omega)$ and the space of Neumann data $\partial_\nu N_q$ of the functions in N_q, i.e.

$$N_q = \{\psi \in H_0^1(\Omega) : (\Delta + q)\psi = 0\},$$
$$\partial_\nu N_q = \{\partial_\nu \psi|_{\partial\Omega} : \psi \in N_q\}.$$

These spaces appear due to the use of the Fredholm alternative. These are finite dimensional spaces, and since $q \in C^{1,\alpha}(\overline{\Omega})$, elliptic regularity [Bro62, Theorem 2.2] ensures that $N_q \subseteq C^{3,\alpha}(\overline{\Omega})$ and $\partial_\nu N_q \subseteq C^{2,\alpha}(\partial\Omega)$. The last fact is the only reason why we assume $q \in C^{1,\alpha}(\overline{\Omega})$ (otherwise $q \in C^{\alpha}(\overline{\Omega})$ would have been sufficient). We let $\{\partial_\nu\psi_1, \ldots, \partial_\nu\psi_m\}$ be an orthonormal basis of $\partial_\nu N_q$ with respect to the $L^2(\partial\Omega)$-inner product.

We now show that even in the case when 0 is a Dirichlet eigenvalue of $\Delta + q$ in Ω, the equation $(\Delta + q)u = F$ has a solution u for any F and one can prescribe the Dirichlet data of u in the $L^2(\partial\Omega)$-orthocomplement of the finite dimensional space $\partial_\nu N_q$. Below, the notation $\perp$ will always mean L^2-orthogonality.

Lemma 2.1. *Let $q \in C^{1,\alpha}(\overline{\Omega})$. For any $F \in C^{\alpha}(\overline{\Omega})$ and $f \in C^{2,\alpha}(\partial\Omega)$, there is a unique function $\Phi = \Phi(F, f) \in \partial_\nu N_q$ such that the problem*

$$\begin{cases} \Delta u + qu = F & \text{in } \Omega, \\ u = f - \Phi & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

admits a solution $u \in C^{2,\alpha}(\overline{\Omega})$. The function Φ is given by

$$\Phi(F, f) = \sum_{j=1}^{m} \left(\int_\Omega F\psi_j \, dx + \int_{\partial\Omega} f \partial_\nu \psi_j \, dS \right) \partial_\nu \psi_j. \tag{2.4}$$

Moreover, there is unique solution $u_{F,f} = G_q(F, f)$ such that $u_{F,f} \perp N_q$. The solution $u_{F,f}$ depends linearly on F and f and satisfies

$$\|u_{F,f}\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|F\|_{C^{\alpha}(\overline{\Omega})} + \|f\|_{C^{2,\alpha}(\partial\Omega)}), \tag{2.5}$$

where C is independent of F and f.

Proof. We first consider the case of solvability in $H^2(\Omega)$ with zero Dirichlet data. If $X = H^2(\Omega) \cap H^1_0(\Omega)$ equipped with the $H^2(\Omega)$ norm, then by Fredholm theory [Eva10, Theorem 4 in Section 6.2.3] and elliptic regularity [Eva10, Theorem 4 in Section 6.3.2] the map

$$T : X \to L^2(\Omega), \ Tv = (\Delta + q)v$$

is Fredholm, i.e. it has finite dimensional kernel N_q and its range $\mathrm{Ran}(T) = \{F \in L^2(\Omega) : F \perp N_q\}$ has finite codimension. It follows that the induced map

$$T_1 : X/N_q \to \mathrm{Ran}(T)$$

is bounded and bijective, hence invertible by the open mapping theorem. The space X/N_q can be identified with $Y = \{u \in X : u \perp N_q\}$, and T becomes an isomorphism from Y onto $\mathrm{Ran}(T)$. (To see this, let $E : X \to L^2(\Omega)$ be the restriction to X of the L^2-orthogonal projection onto N_q. Then $X = \mathrm{Ran}(E) \oplus \mathrm{Ker}(E) = N_q \oplus Y$ [Con90, Theorem 13.2 b)], and the map $Y \to X/N_q$, $u \mapsto [u]$ identifies Y with X/N_q.) It follows that for any $F \in \mathrm{Ran}(T)$ there is a unique $v_F \in X$ with $v_F \perp N_q$ such that $T(v_F) = F$. In other words, for any $F \in L^2(\Omega)$ with $F \perp N_q$ there is a unique $v_F \in H^2(\Omega) \cap H^1_0(\Omega)$ with $v_F \perp N_q$ that depends linearly on F and solves

$$(\Delta + q)v = F \text{ in } \Omega, \qquad v|_{\partial\Omega} = 0,$$

and one has

$$\|v_F\|_{H^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}. \tag{2.6}$$

We can obtain a similar statement in Hölder spaces. Let $F \in C^{\alpha}(\overline{\Omega})$ with $F \perp N_q$ and let v_F be as above. By elliptic regularity, $v_F \in C^{2,\alpha}(\overline{\Omega})$ and

$$\|v_F\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|F\|_{C^{\alpha}(\overline{\Omega})}. \tag{2.7}$$

(More precisely, from [Bro62, Theorem 2.2] and [GT01, Lemma 6.18 and Problem 6.2] we obtain $v_F \in C^{2,\alpha}(\overline{\Omega})$ and

$$\|v_F\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|v_F\|_{C(\overline{\Omega})} + \|F\|_{C^{\alpha}(\overline{\Omega})}). \tag{2.8}$$

From Theorem 8.15 and the remark around equation 8.38 in [GT01] it follows that $\|v_F\|_{C(\overline{\Omega})} \leq C\|v_F\|_{L^2(\Omega)}$. Using this with (2.6) and (2.8) yields (2.7).)

We now consider (2.3). To study the uniqueness of u, we fix a bounded extension operator

$$E_q : C^{2,\alpha}(\partial\Omega) \to C^{2,\alpha}(\overline{\Omega}) \text{ with } E_q h|_{\partial\Omega} = h \text{ and } E_q h \perp N_q \text{ for all } h \in C^{2,\alpha}(\partial\Omega).$$

In fact it is enough to take $E_q = (\mathrm{Id} - P_{N_q})E$, where E is any bounded extension operator [Tri83, Theorem 3.3.3 and eq. 1) on p. 51] and P_{N_q} is the $L^2(\Omega)$-orthogonal projection to N_q. We see that u solves (2.3) iff $u = E_q(f - \Phi) + v$, where v solves

$$\begin{cases} \Delta v + qv = \tilde{F} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

where we wrote $\tilde{F} = F - (\Delta + q)E_q(f - \Phi)$. We wish to find a function $\Phi \in \partial_\nu N_q$ such that $\tilde{F} \perp L^2(\Omega)$. If $\psi \in N_q$, integrating by parts gives

$$\int_\Omega \tilde{F}\psi \, dx = \int_\Omega F\psi \, dx + \int_{\partial\Omega} (f - \Phi)\partial_\nu \psi \, dS.$$

Thus $\tilde{F} \perp N_q$ iff Φ satisfies for all $\psi \in N_q$ the condition

$$\int_{\partial\Omega} \Phi \partial_\nu \psi \, dS = \int_\Omega F\psi \, dx + \int_{\partial\Omega} f\partial_\nu \psi \, dS.$$

This holds for Φ iff $\Phi = \Phi(F, f)$ is given by (2.4). For this Φ, we let $v_{\tilde{F}} \perp N_q$ be the solution of (2.9) satisfying (2.6). Then $u_{F,f} := E_q(f - \Phi(F, f)) + v_{\tilde{F}} \perp N_q$ satisfies the required estimate (2.5). □

Next we prove an auxiliary lemma which is used in several places in the remainder of the article.

Lemma 2.2. *Let $a \in C^k(\mathbb{R}, C^\alpha(\overline{\Omega}))$ and $f \in C(\overline{\Omega})$ and let $l \leq k$. Then*

$$\|\partial_u^l a(x, f(x))\|_{C^\alpha(\overline{\Omega})} \leq \|a\|_{C^k([-M,M],C^\alpha(\overline{\Omega}))} \tag{2.10}$$

where $M = \|f\|_{C(\overline{\Omega})}$.

Proof. By manipulating the supremum in the definition of the norm we have

$$\begin{aligned}
\|\partial_u^l a(x, f)\|_{C^\alpha(\overline{\Omega})} &= \sup_{x\in\overline{\Omega}} |\partial_u^l a(x, f(x))| + \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|\partial_u^l a(x, f(x)) - \partial_u^l a(y, f(y))|}{|x - y|^\alpha} \\
&\leq \sup_{x\in\overline{\Omega}} \|\partial_u^l a(x)\|_{C([-M,M])} + \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \sup_{\eta,\theta\in[-M,M]} \frac{|\partial_u^l a(x, \eta) - \partial_u^l a(y, \theta)|}{|x - y|^\alpha} \\
&\leq \sup_{x\in\overline{\Omega}} \|a(x)\|_{C^l([-M,M])} + \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{\|a(x) - a(y)\|_{C^l([-M,M])}}{|x - y|^\alpha} \\
&= \|a\|_{C^k([-M,M],C^\alpha(\overline{\Omega}))}.
\end{aligned}$$

□

Next we study a fixed point equation related to the linearized equation. Below, we let

$$B_\delta = \{u \in C^{2,\alpha}(\overline{\Omega}) \, : \, \|u\|_{C^{2,\alpha}(\overline{\Omega})} < \delta\}. \tag{2.11}$$

Lemma 2.3. *Let $a \in C^3(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$ and $w \in C^{2,\alpha}(\overline{\Omega})$ be a solution of $\Delta w + a(x, w) = 0$ in Ω. Let $q = \partial_u a(x, w)$ and let $G_q(F, f)$ be the solution operator of*

$$\begin{cases} \Delta u + qu = F & \text{in } \Omega \\ u = f - \Phi(F, f) & \text{on } \partial\Omega \end{cases}$$

provided by Lemma 2.1. Define $R_v(r) = R(v + r)$ where $R : C^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ is given by

$$R(h)(x) := \int_0^1 [\partial_u a(x, w(x) + th(x)) - \partial_u a(x, w(x))]h(x) \, dt. \tag{2.12}$$

For fixed $v \in C^{2,\alpha}(\overline{\Omega})$ define $T_v : C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ by $T_v(r) = -G_q(R_v(r), 0)$.

Under the above assumptions, there exists $\delta > 0$ such that $T_v|_{B_\delta} : B_\delta \to B_\delta$ is a contraction. Furthermore,

$$\|T_v(h)\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|v + h\|^2_{C^{2,\alpha}(\overline{\Omega})}, \qquad h \in B_\delta. \tag{2.13}$$

Consequently there exists a unique $r \in B_\delta$ solving the fixed point equation $r = T_v(r)$. The function r is also the unique solution with $r \perp N_q$ of

$$\begin{cases} \Delta r + \partial_u a(x,w) r = -R(v+r) & \text{in } \Omega \\ r|_{\partial\Omega} \in \partial_\nu N_q & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

and necessarily $r|_{\partial\Omega} = \Phi(R(v+r), 0)$.

Proof. We first show that T_v maps B_δ into itself when δ is small enough. Let $v, r \in B_\delta$ where initially $\delta \leq 1$. By the mapping properties for G in Lemma 2.1 and the fundamental theorem of calculus, we have

$$\begin{aligned} \|T_v(r)\|_{C^{2,\alpha}(\overline{\Omega})} &\leq C\|R(v+r)\|_{C^\alpha(\overline{\Omega})} \\ &= C\left\| \int_0^1 [\partial_u a(x, w + t(v+r)) - \partial_u a(x,w)](v+r)\, dt \right\|_{C^\alpha(\overline{\Omega})} \\ &= C\left\| \int_0^1 \int_0^1 \partial_u^2 a(x, w + st(v+r)) t (v+r)^2\, ds\, dt \right\|_{C^\alpha(\overline{\Omega})} \end{aligned} \tag{2.15}$$

From Lemma 2.2 we get for $s, t \in [0,1]$ that

$$\|\partial_u^2 a(x, w + st(v+r))\|_{C^\alpha(\overline{\Omega})} \leq C_{w,a}. \tag{2.16}$$

Since $v \in B_\delta$ we have, by using (2.16) in (2.15), that

$$\|T_v(r)\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|v+r\|^2_{C^\alpha(\overline{\Omega})} \leq C\|v+r\|^2_{C^{2,\alpha}(\overline{\Omega})} \leq C\delta^2.$$

The second inequality above proves (2.13). For δ small enough we get

$$\|T_v(r)\|_{C^{k,\alpha}(\overline{\Omega})} \leq \delta$$

and conclude that T_v indeed maps B_δ into itself.

Next we show the contraction property of T_v. Let $r_1, r_2 \in B_\delta$. Then, as in (2.15), we have

$$\|T_v(r_1) - T_v(r_2)\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|R_v(r_1) - R_v(r_2)\|_{C^\alpha(\overline{\Omega})}.$$

Denote $u_i = v + r_i$, $i = 1, 2$. Then

$$\begin{aligned} R_v(r_1) - R_v(r_2) &= \int_0^1 (\partial_u a(x, w + tu_1) - \partial_u a(x,w))\, u_1 - (\partial_u a(x, w+tu_2) - \partial_u a(x,w))\, u_2\, dt \\ &= \int_0^1 (\partial_u a(x, w+tu_1) - \partial_u a(x,w) + \partial_u a(x, w+tu_2) - \partial_u a(x,w))(u_1 - u_2) \\ &\quad - (\partial_u a(x, w+tu_2) - \partial_u a(x,w)) u_1 + (\partial_u a(x, w+tu_1) - \partial_u a(x,w)) u_2\, dt \\ &= \int_0^1 (u_1 - u_2)\left(tu_1 \int_0^1 \partial_u^2 a(x, w+stu_1)\, ds + tu_2 \int_0^1 \partial_u^2 a(x, w+stu_2)\, ds \right) \\ &\quad - tu_1u_2 \int_0^1 \partial_u^2 a(x, w+stu_2)\, ds + tu_1u_2 \int_0^1 \partial_u^2 a(x, w+stu_1)\, ds\, dt \\ &= \int_0^1 (u_1 - u_2)\Bigg(tu_1 \int_0^1 \partial_u^2 a(x, w+stu_1)\, ds + tu_2 \int_0^1 \partial_u^2 a(x, w+stu_2)\, ds \\ &\quad + t^2 u_1 u_2 \int_0^1 \int_0^1 s\partial_u^3 a(x, w + ystu_1 + (1-y)stu_2)\, dy\, ds \Bigg)\, dt. \end{aligned}$$

Let us estimate the norm of the last expression term by term. Since $v, r_i \in B_\delta$ then $u_i \in B_{2\delta}$ for $i \in \{1,2\}$. Using this and (2.16) we have that

$$\begin{aligned}\left\| tu_i \int_0^1 \partial_u^2 a(x, w+stu_i)\, ds \right\|_{C^\alpha(\overline{\Omega})} &\le t\|u_i\|_{C^\alpha(\overline{\Omega})} \int_0^1 \|\partial_u^2 a(x,w+stu_i)\|_{C^\alpha(\overline{\Omega})}\, ds \\ &\le t\delta C.\end{aligned}$$

Just as in (2.16) we get $\|\partial_u^3 a(x, w+h)\|_{C^\alpha(\overline{\Omega})} \le C_{w,a}$. Using this we estimate

$$\begin{aligned}&\left\| t^2 u_1 u_2 \int_0^1 \int_0^1 s\partial_u^3 a(x, w+ystu_1+(1-y)stu_2)\, dy\, ds \right\|_{C^\alpha(\overline{\Omega})} \\ &\le t^2 \|u_1u_2\|_{C^\alpha(\overline{\Omega})} \int_0^1 \int_0^1 s\|\partial_u^3 a(x,w+ystu_1+(1-y)stu_2)\|_{C^\alpha(\overline{\Omega})}\, dy\, ds \\ &\le t^2\delta^2 C\end{aligned}$$

Finally, for small enough $\delta > 0$ we have

$$\begin{aligned}\|T_v(r_1) - T_v(r_2)\|_{C^{2,\alpha}(\Omega)} &\le C_a \int_0^1 \|u_1-u_2\|_{C^\alpha(\overline{\Omega})}(2t\delta C + t^2\delta^2 C)\, dt \\ &\le \frac{1}{2}\|u_1-u_2\|_{C^{2,\alpha}(\overline{\Omega})} \\ &= \frac{1}{2}\|r_1-r_2\|_{C^{2,\alpha}(\overline{\Omega})}.\end{aligned}$$

Thus T_v is a contraction and the Banach fixed point theorem ensures existence and uniqueness of solution to the equation $r = T_v(r)$ in B_δ. The definition of T_v ensures that r also solves (2.14). $\square$

We now construct the smooth solution map $S_{a,w}$, which maps small solutions v of the linearized equation $\Delta v + \partial_u a(x,w)v = 0$ to solutions u of the nonlinear equation $\Delta u + a(x,u) = 0$ that are close to some fixed solution w. Below, if $F : U \to Y$ is a C^1 map where X and Y are Banach spaces and $U \subseteq X$ is open, we will denote its Fréchet derivative by

$$DF(x) = F'(x).$$

Recall that B_δ is given by (2.11).

Lemma 2.4. *Let $a \in C^k(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$, $k \ge 3$. Let $w \in C^{2,\alpha}(\overline{\Omega})$ be a solution of $\Delta w + a(x,w) = 0$. Let $q(x) = \partial_u a(x, w(x))$. Then there exist $\delta, C > 0$ and a C^{k-1} map $Q = Q_{a,w} : B_\delta \to B_\delta$ satisfying*

$$\begin{aligned}Q(B_\delta) &\subseteq N_q^\perp, \\ Q(B_\delta)|_{\partial\Omega} &\subseteq \partial_\nu N_q, \\ Q(0) &= DQ(0) = 0\end{aligned}$$

and

$$\|Q(v)\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|v\|^2_{C^{2,\alpha}(\overline{\Omega})}, \tag{2.17}$$

such that $S_a \colon B_\delta \to C^{2,\alpha}(\overline{\Omega})$ defined by $u = S_{a,w}(v) = w + v + Q(v)$ is a C^{k-1} map satisfying

$$\Delta u + a(x,u) = \Delta v + qv \quad \text{in } \Omega \tag{2.18}$$

with $S'_{a,w}(0)v = v$. In particular, if v solves $\Delta v + qv = 0$, then $u = S_{a,w}(v)$ solves $\Delta u + a(x,u) = 0$.

Conversely, if δ is small enough, then given any solution $u \in C^{2,\alpha}(\overline{\Omega})$ of $\Delta u + a(x,u) = 0$ with $\|u-w\|_{C^{2,\alpha}(\overline{\Omega})} \le \delta$ there exists a unique solution $v \in C^{2,\alpha}(\overline{\Omega})$ of $\Delta v + qv = 0$ such that $u = S_{a,w}(v)$. The function v is explicitly given by

$$v = P_{N_q}(u-w) + G_q(0, (u-w)|_{\partial\Omega}), \tag{2.19}$$

and one has $\|v\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|u-w\|_{C^{2,\alpha}(\overline{\Omega})}$.

Proof. We first construct the map Q. Let $v \in B_\delta$. We look for a solution u of (2.18) having the form $u = w + v + r$ and formulate a fixed point equation for r. Taylor expansion gives

$$\begin{aligned}\Delta u + a(x,u) &= \Delta(w+v+r) + a(x, w+v+r)\\ &= \Delta w + \Delta(v+r) + a(x,w) + \partial_u a(x,w)(v+r) + R_v(r)\end{aligned}$$

where $R_v(r)(x) = \int_0^1 [\partial_u a(x, w(x) + t[v(x)+r(x)]) - \partial_u a(x, w(x))][v(x)+r(x)]\,dt$. Since w is a solution of $\Delta w + a(x,w) = 0$, we see that u solves $\Delta u + a(x,u) = \Delta v + qv$ if r satisfies

$$\Delta r + \partial_u a(x,w) r + R_v(r) = 0.$$

For each $v \in B_\delta$, Lemma 2.3 ensures existence and uniqueness of a solution $r = r_v$ in B_δ with $r \perp N_q$ and $r|_{\partial\Omega} \in \partial_\nu N_q$. Hence the mapping $v \mapsto r_v$ is well-defined for $v \in B_\delta$. Next we use the implicit function theorem to show that this mapping is C^{k-1}.

Let $F : C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ be defined by

$$F(v,r) = r - T_v(r) = r + G_q(R_v(r), 0).$$

From the definition of R_v and G_q it follows that $F(0,0) = 0$. Next, R_v is C^{k-1} since $\partial_u a \in C^{k-1,\alpha}(\mathbb{R}; C^{1,\alpha}(\overline{\Omega}))$. Consequently, F is C^{k-1} since G_q is linear. Moreover

$$\begin{aligned}&D_r F|_{(0,0)}(h)\\ &= h + G_q\left(\int_0^1 \partial_u^2 a(x, w + t(v+r))(v+r) + \partial_u a(x, w+t(v+r)) - \partial_u a(x,w)\,dt|_{(v,r)=(0,0)}, 0\right)\\ &= h\end{aligned}$$

and this is a linear homeomorphism from $C^{2,\alpha}(\overline{\Omega})$ to itself. Thus the implicit function theorem [HG27, Theorem 4] ensures the existence of open balls $B_{\delta_1}, B_{\delta_2}$ and a C^{k-1} map $Q\colon B_{\delta_1} \to B_{\delta_2}$ such that

$$F(v, Q(v)) = 0.$$

Since r_v found by Lemma 2.3 above is the unique solution of $F(v,\cdot) = 0$ in B_δ for $v \in B_\delta$, we conclude that for $\delta < \min\{\delta_1, \delta_2\}$, r_v belongs to B_{δ_2} and hence $Q(v) = r_v$. We have now shown that for each $v \in B_\delta$ there is a unique $r_v \in B_\delta$ with $r \perp N_q$ and $r|_{\partial\Omega} \in \partial_\nu N_q$ such that $u_v = w + v + r_v$ is a solution of the equation $\Delta u_v + a(x, u_v) = \Delta v + qv$. Moreover, the map $v \mapsto u_v = S_{a,w}(v)$ is C^{k-1}.

Next we show that Q satisfies the other properties in the statement. The estimate (2.13) implies

$$\|r\|_{C^{2,\alpha}(\overline{\Omega})} = \|T_v(r)\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|v\|^2_{C^{2,\alpha}(\overline{\Omega})} + C\|r\|^2_{C^{2,\alpha}(\overline{\Omega})}$$

and from this we get that

$$\|v\|^2_{C^{2,\alpha}(\overline{\Omega})} \ge \|r\|_{C^{2,\alpha}(\overline{\Omega})} - C\|r\|^2_{C^{2,\alpha}(\overline{\Omega})} \ge \|r\|_{C^{2,\alpha}(\overline{\Omega})}(1 - C\delta).$$

For δ small enough, and since $Q(v) = r$, we have

$$\|Q(v)\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|v\|^2_{C^{2,\alpha}(\overline{\Omega})}.$$

This proves (2.17) and shows that $Q(0) = 0$. Using (2.17) together with $Q(0) = 0$ implies that $DQ(0) = 0$. Since $Q(v) = r_v$, we have $Q(v) \in N_q^\perp$ and $Q(v)|_{\partial\Omega} \in \partial_\nu N_q$.

We now prove the converse statement. Suppose that $u \in C^{2,\alpha}(\overline{\Omega})$ solves $\Delta u + a(x, u) = 0$ in Ω and $\|u - w\|_{C^{2,\alpha}(\overline{\Omega})} \leq \delta$. We write $\tilde{u} = u - w$ and want to construct v solving $\Delta v + qv = 0$ such that $\tilde{u} = v + Q(v)$. Denote by P_{N_q} and $P_{\partial_\nu N_q}$ the L^2-orthogonal projections to the finite dimensional spaces N_q and $\partial_\nu N_q$, respectively. Motivated by the conditions $Q(B_\delta) \subseteq N_q^\perp$ and $Q(B_\delta)|_{\partial\Omega} \subseteq \partial_\nu N_q$ we define

$$\psi = P_{N_q} \tilde{u},$$

and let φ to be the unique solution given by Lemma 2.1 of the problem

$$\Delta\varphi + q\varphi = 0 \text{ in } \Omega, \qquad \varphi \perp N_q, \qquad \varphi|_{\partial\Omega} = (\mathrm{Id} - P_{\partial_\nu N_q})(\tilde{u}|_{\partial\Omega}).$$

Let $v = \varphi + \psi$, which means that v is given by (2.19). It follows that $\Delta v + qv = 0$ and $v|_{\partial\Omega} = (\mathrm{Id} - P_{\partial_\nu N_q})(\tilde{u}|_{\partial\Omega})$. We also have

$$\|v\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|\tilde{u}\|_{C^{2,\alpha}(\overline{\Omega})} = C\|u - w\|_{C^{2,\alpha}(\overline{\Omega})}.$$

It remains to show that $r = \tilde{u} - v = u - w - v$ satisfies $r = Q(v)$. By the above conditions we have $r \perp N_q$ and $r|_{\partial\Omega} \in \partial_\nu N_q$, and r satisfies

$$\begin{aligned}(\Delta + q)r = (\Delta + q)(u - w) &= -(a(x, u) - a(x, w) - q(u - w)) \\ &= -(a(x, w + v + r) - a(x, w) - q(v + r)) = -R_v(r).\end{aligned}$$

The first part of the proof implies that $r = Q(v)$ if δ is chosen small enough. This proves that $u = S_{a,w}(v)$. To show that v is unique suppose that $u = S_{a,w}(\tilde{v})$ for another solution $\tilde{v}$. Then the definition of $S_{a,w}$ gives

$$v - \tilde{v} = Q(\tilde{v}) - Q(v).$$

Thus $v - \tilde{v} \perp N_q$ and $v - \tilde{v}|_{\partial\Omega} \in \partial_\nu N_q$. Since $(\Delta + q)(v - \tilde{v}) = 0$, Lemma 2.1 implies $v = \tilde{v}$ showing that v is unique. □

Lemma 2.5. *In the setting of Lemma 2.4, if v is small and solves $\Delta v + qv = 0$ for $q = \partial_u a(x, w)$, define*

$$\begin{aligned} q_v &= \partial_u a(x, S_{a,w}(v)), \\ V_{\tilde{q}} &= \{h \in C^{2,\alpha}(\overline{\Omega}) \,:\, \Delta h + \tilde{q}h = 0\} \end{aligned}$$

If $v \in V_q$ is small, the map $DS_{a,w}(v)$ is an isomorphism from V_q onto V_{q_v}.

Proof. Let v be a small solution of $\Delta v + qv = 0$ and $v_t = v + th$ where $h \in V_q$. Then $u_t = S_{a,w}(v_t)$ solves

$$\Delta u_t + a(x, u_t) = 0.$$

Since u_t is C^1 in t, the function $\dot{u}_0 = \partial_t u_t|_{t=0} = DS_{a,w}(v)h$ satisfies

$$\Delta\dot{u}_0 + \partial_u a(x, S_{a,w}(v))\dot{u}_0 = 0.$$

Thus $DS_{a,w}(v)$ maps V_q into V_{q_v}.

Now suppose that $v \in V_q$ is small and $\tilde{h} \in V_{q_v}$. For t small define $u_t = S_{a,S_{a,w}(v)}(t\tilde{h})$. By the converse part of Lemma 2.4, if v and t are small enough one has $u_t = S_{a,w}(v_t)$ for a unique small solution $v_t \in V_q$, and v_t is given by

$$v_t = P_{N_q}(u_t - w) + G_q(0, (u_t - w)|_{\partial\Omega}).$$

In particular, v_t is C^1 in t, and since $S_{a,w}(v_0) = u_0 = S_{a,S_{a,w}(v)}(0) = S_{a,w}(v)$ uniqueness gives $v_0 = v$. Differentiating the identities $u_t = S_{a,w}(v_t)$ and $u_t = S_{a,S_{a,w}(v)}(t\tilde{h})$ and using $DS(0) = \mathrm{Id}$ gives

$$DS_{a,w}(v)\dot{v}_0 = \dot{u}_0 = DS_{a,S_{a,w}(v)}(0)\tilde{h} = \tilde{h}.$$

This shows that $DS_{a,w}(v) : V_q \to V_{q_v}$ is surjective.

Finally, suppose that $h \in V_q$ satisfies $DS_{a,w}(v)h = 0$. Since $S_{a,w}(v) = w + v + Q_{a,w}(v)$, we have

$$h + DQ_{a,w}(v)h = 0.$$

But $DQ_{a,w}(0) = 0$, which implies that $\|DQ_{a,w}(v)\| \leq 1/2$ when v is sufficiently small. Here we used that Q is a C^{k-1} map where $k \geq 3$. This implies that $\|h\| \leq \frac{1}{2}\|h\|$, showing that $h = 0$. Thus $DS_{a,w}(v) : V_q \to V_{q_v}$ is bijective and bounded, and by the open mapping theorem it is an isomorphism. $\square$

3. Estimates for solutions in terms of their Cauchy data

In this section we prove estimates for functions in terms of their Cauchy data and in particular for solutions of the nonlinear equation

$$\Delta u + a(x, u) = 0 \text{ in } \Omega. \tag{3.1}$$

The estimate for (3.1) is used in section 4 when constructing the second solution map required for the linearization methods.

First we obtain an auxiliary regularity estimate that is then used to prove the quantitative results.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^∞ boundary and let $q \in C^\alpha(\overline{\Omega})$. There is $C > 0$ such that for any $u \in C^{2,\alpha}(\overline{\Omega})$ we have*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|u\|_{C^{2,\alpha}(\partial\Omega)} + \|(\Delta + q)u\|_{C^\alpha(\overline{\Omega})} + \|u\|_{H^1(\Omega)}).$$

Proof. Consider the Banach space $X = C^{2,\alpha}(\partial\Omega) \times C^\alpha(\overline{\Omega}) \times H^1(\Omega)$ with norm

$$\|(f, F, v)\|_X = \|f\|_{C^{2,\alpha}(\partial\Omega)} + \|F\|_{C^\alpha(\overline{\Omega})} + \|v\|_{H^1(\Omega)}.$$

We define the map

$$T : C^{2,\alpha}(\overline{\Omega}) \to X, \quad T(u) = (u|_{\partial\Omega}, (\Delta + q)u, j(u)),$$

where j is the inclusion $C^{2,\alpha}(\overline{\Omega}) \to H^1(\Omega)$. Then T is bounded, linear and injective. We claim that T has closed range. To see this, suppose that $u_j \in C^{2,\alpha}(\overline{\Omega})$ and $T(u_j) \to (f, F, v)$ in X. Then $u_j \to v$ in $H^1(\Omega)$, $u_j|_{\partial\Omega} \to f$ in $C^{2,\alpha}(\partial\Omega)$ and $(\Delta + q)u_j \to F$ in $C^\alpha(\overline{\Omega})$. On the other hand $(\Delta + q)u_j \to (\Delta + q)v$ in $H^{-1}(\Omega)$ and $u_j|_{\partial\Omega} \to v|_{\partial\Omega}$ in $H^{1/2}(\partial\Omega)$, and by uniqueness of limits one has $(\Delta + q)v = F$ and $v|_{\partial\Omega} = f$. By elliptic regularity, the weak solution v satisfies $v \in C^{2,\alpha}(\overline{\Omega})$. Thus $(f, F, v) = T(v)$ and $\mathrm{Ran}(T)$ is closed.

We have proved that $T : C^{2,\alpha}(\overline{\Omega}) \to \mathrm{Ran}(T)$ is a bounded linear bijection between Banach spaces. By the open mapping theorem it has a bounded inverse $S : \mathrm{Ran}(T) \to C^{2,\alpha}(\overline{\Omega})$, and thus for any $u \in C^{2,\alpha}(\overline{\Omega})$ one has

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} = \|STu\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|Tu\|_X.$$

This proves the claim. $\square$

Next we show a quantitative uniqueness result that follows by combining Lemma 3.1 with the unique continuation principle. This is the used in Section 4 related to the first linearization of (3.1).

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^∞ boundary and let $q \in C^\alpha(\overline{\Omega})$. There is $C > 0$ such that for any $u \in C^{2,\alpha}(\overline{\Omega})$ we have*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|u\|_{C^{2,\alpha}(\partial\Omega)} + \|\partial_\nu u\|_{C^{1,\alpha}(\partial\Omega)} + \|\Delta u + qu\|_{C^\alpha(\overline{\Omega})}).$$

Proof. We argue by contradiction and assume that for any m there is u_m such that

$$\|u_m\|_{C^{2,\alpha}(\overline{\Omega})} > m(\|u_m\|_{C^{2,\alpha}(\partial\Omega)} + \|\partial_\nu u_m\|_{C^{1,\alpha}(\partial\Omega)} + \|(\Delta + q)u_m\|_{C^\alpha(\overline{\Omega})}). \tag{3.2}$$

On the other hand, Lemma 3.1 implies that

$$\|u_m\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|u_m\|_{C^{2,\alpha}(\partial\Omega)} + \|(\Delta + q)u_m\|_{C^\alpha(\overline{\Omega})} + \|u_m\|_{H^1(\Omega)}).$$

Normalize u_m so that $\|u_m\|_{H^1(\Omega)} = 1$. Then using (3.2) yields

$$\|u_m\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\frac{1}{m}\|u_m\|_{C^{2,\alpha}(\overline{\Omega})} + 1)$$

Then $\|u_m\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$ uniformly when m is sufficiently large.

By Theorem 1.34 in [AF03] the embedding $C^{2,\alpha}(\overline{\Omega}) \to C^2(\overline{\Omega})$ is compact. Hence there is a subsequence, still denoted u_m, that converges in $C^2(\overline{\Omega})$ to some $u \in C^2(\overline{\Omega})$. On the other hand, from (3.2) and the bound $\|u_m\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$ we see that

$$u_m|_{\partial\Omega} \to 0, \qquad \partial_\nu u_m|_{\partial\Omega} \to 0, \qquad (\Delta + q)u_m \to 0$$

in the respective spaces. By uniqueness of limits we have $u|_{\partial\Omega} = 0$, $\partial_\nu u|_{\partial\Omega} = 0$, and $(\Delta + q)u = 0$. Consequently, $u \equiv 0$ by unique continuation, which contradicts $\|u\|_{H^1(\Omega)} = \lim\|u_m\|_{H^1(\Omega)} = 1$. ☐

Finally, we invoke a Carleman estimate to show that solutions of semilinear equations of the form (3.1) are uniquely and stably determined by their Cauchy data.

Lemma 3.3. *Let $a \in C^2(\mathbb{R}, C^\alpha(\overline{\Omega}))$, and let $u_0 \in C^{2,\alpha}(\overline{\Omega})$ solve $\Delta u_0 + a(x, u_0) = 0$ in Ω. If $u \in C^{2,\alpha}(\overline{\Omega})$ is any other solution of $\Delta u + a(x, u) = 0$ in Ω and $\|u\|_{C^{2,\alpha}(\overline{\Omega})}, \|u_0\|_{C^{2,\alpha}(\overline{\Omega})} \leq M$, then*

$$\|u - u_0\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(M, a)(\|u - u_0\|_{C^{2,\alpha}(\partial\Omega)} + \|\partial_\nu(u - u_0)\|_{C^{1,\alpha}(\partial\Omega)}). \tag{3.3}$$

Proof. We use a standard Carleman estimate (see e.g. [Cho21, Theorem 4.1]): there are $C, \tau_0 > 0$ and $\varphi \in C^\infty(\overline{\Omega})$ such that when $\tau \geq \tau_0$, one has

$$\|e^{\tau\varphi}v\|_{L^2(\Omega)} + \frac{1}{\tau}\|e^{\tau\varphi}\nabla v\|_{L^2(\Omega)} \leq \frac{C}{\tau^{3/2}}\|e^{\tau\varphi}\Delta v\|_{L^2(\Omega)} + C\|e^{\tau\varphi}v\|_{L^2(\partial\Omega)} + \frac{C}{\tau}\|e^{\tau\varphi}\nabla v\|_{L^2(\partial\Omega)}$$

for any $v \in C^2(\overline{\Omega})$. We apply this with $v = u - u_0$ and use the fact that

$$-\Delta v = a(x, u) - a(x, u_0) = \left[\int_0^1 \partial_u a(x, (1 - t)u_0 + tu)\, dt\right] v. \tag{3.4}$$

Since $|u|, |u_0| \leq M$, we get from Lemma 2.2 that

$$|\Delta v(x)| \leq C(M, a)|v(x)|. \tag{3.5}$$

Thus, choosing $\tau = \tau(M, a)$ large but fixed, we get

$$\frac{1}{2}\|e^{\tau\varphi}v\|_{L^2(\Omega)} + \frac{1}{\tau}\|e^{\tau\varphi}\nabla v\|_{L^2(\Omega)} \leq C(\|e^{\tau\varphi}v\|_{L^2(\partial\Omega)} + \|e^{\tau\varphi}\nabla v\|_{L^2(\partial\Omega)}).$$

Since $c(M, a) \leq e^{\tau\varphi} \leq C(M, a)$, we have

$$\|v\|_{H^1(\Omega)} \leq C(M, a)(\|v\|_{H^1(\partial\Omega)} + \|\partial_\nu v\|_{L^2(\partial\Omega)}). \tag{3.6}$$

We still need to estimate $\|v\|_{C^{2,\alpha}(\overline{\Omega})}$. First, Lemma 3.1 gives

$$\|v\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|v\|_{C^{2,\alpha}(\partial\Omega)} + \|\Delta v\|_{C^{\alpha}(\overline{\Omega})} + \|v\|_{H^1(\Omega)}\right).$$

From (3.4) we observe that

$$\|\Delta v\|_{C^{\alpha}(\overline{\Omega})} \leq C\left[\int_0^1 \|\partial_u a(\,\cdot\,,(1-t)u_0(\,\cdot\,)+tu(\,\cdot\,))\|_{C^{\alpha}(\overline{\Omega})}\,dt\right]\|v\|_{C^{\alpha}(\overline{\Omega})}.$$

By using Lemma 2.2 to estimate the integral from above by a constant depending on a, u and u_0 we have

$$\|\Delta v\|_{C^{\alpha}(\overline{\Omega})} \leq C\|v\|_{C^{\alpha}(\overline{\Omega})}.$$

Thus we get

$$\|v\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|v\|_{C^{2,\alpha}(\partial\Omega)} + \|v\|_{C^{\alpha}(\overline{\Omega})} + \|v\|_{H^1(\Omega)}\right). \tag{3.7}$$

Next, we have by the Sobolev embedding [AF03, Theorem 4.12 Part 2] that $W^{1,s} \subseteq C^{\alpha}$ where $s = \frac{n}{1-\alpha}$. Using this and [AF03, Theorem 5.2 (3)] we obtain that

$$\|v\|_{C^{\alpha}(\overline{\Omega})} \leq C\|v\|_{W^{1,s}(\Omega)} \leq C\|v\|^{1/2}_{W^{2,s}(\Omega)}\|v\|^{1/2}_{L^s(\Omega)}.$$

Then we use interpolation of L^p-spaces (see for example [Eva10, Appendix B]) to get

$$\|v\|_{L^s(\Omega)} \leq C\|v\|^{\lambda}_{L^2(\Omega)}\|v\|^{1-\lambda}_{L^r(\Omega)}$$

for some $r > s$. Estimating the L^r- and $W^{2,s}$-norms by the $C^{2,\alpha}$-norm we have

$$\|v\|_{C^{\alpha}(\overline{\Omega})} \leq C\|v\|^{(2-\lambda)/2}_{C^{2,\alpha}(\overline{\Omega})}\|v\|^{\lambda/2}_{L^2(\Omega)}.$$

Using Young's inequality with ε for $p = 2/\lambda$ and $q = p/(p-1)$ gives

$$\|v\|_{C^{\alpha}(\overline{\Omega})} \leq C(\varepsilon\|v\|^{q(2-\lambda)/2}_{C^{2,\alpha}(\overline{\Omega})} + C_\varepsilon\|v\|_{L^2(\Omega)}) = (\varepsilon\|v\|_{C^{2,\alpha}(\overline{\Omega})} + C_\varepsilon\|v\|_{L^2(\Omega)})$$

since $q = \frac{2}{2-\lambda}$. Using this in (3.7) and choosing $\varepsilon > 0$ sufficiently small finally gives

$$\|v\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|v\|_{C^{2,\alpha}(\partial\Omega)} + \|v\|_{H^1(\Omega)}\right).$$

Combining the last estimate with (3.6) proves the result. □

4. A smooth solution map with prescribed Cauchy data

As mentioned previously, in order to prove the main results, we need to construct two smooth solution maps for the nonlinear equations

$$\Delta u + a_i(x,u) = 0 \text{ in } \Omega \tag{4.1}$$

for $i = 1, 2$. In Section 2 we constructed the first one. One reason why we cannot use the solution map S_{a_i,w_i} for both $i \in \{1,2\}$ is that we need to control the Cauchy data. If $u_1 = S_{a_1,w_1}(v_1)$ then we would need to find a solution $u_2 = S_{a_2,w_2}(v_2)$ such that u_1, u_2 have the same Cauchy data. But the solution maps S_{a_i,w_i} don't provide enough control of the Neumann data to guarantee that this is possible. Another issue, in particular when identifying the first derivatives $\partial_u a_i(x,w_i)$, is that the method of linearization relies on differentiating both solution maps S_{a_1,w_1}, S_{a_2,w_2} in the same direction v. But in order to use the same parameter v for both operators, v needs to solve both linearized equations

$$\Delta v + \partial_u a_i(x,w_i)v = 0 \text{ in } \Omega \tag{4.2}$$

for $i \in \{1,2\}$. However, before having identified the first derivatives, $\partial_u a_1(x,w_1) = \partial_u a_2(x,w_2)$, we don't know that such functions v exist. So the goal of this section is to construct a new solution map T_{a_i,w_i} that resolves these two issues. That is, we aim to construct a smooth solution map T_{a_2,w_2} for

$$\Delta u + a_2(x,u) = 0 \text{ in } \Omega \tag{4.3}$$

parametrized on solutions v of

$$\Delta v + \partial_u a_1(x,w_1)v = 0 \text{ in } \Omega$$

such that $T_{a_2,w_2}(v)$ and $S_{a_1,w_1}(v)$ have the same Cauchy data.

Before constructing T_{a_i,w_i}, we establish some preliminary results. The construction of T_{a_i,w_i} is based on the implicit function theorem. In order to properly define the function to which the implicit function theorem is applied, we require the existence of a certain projection mapping and existence of a bounded inverse of the Schrödinger operator $\Delta + q$. We first establish these two results and then proceed to construct T_{a_i,w_i}.

Lemma 4.1. *Let $q \in C^\alpha(\overline{\Omega})$. Then the spaces*

$$Y = \{u \in C^{2,\alpha}(\overline{\Omega}) \colon u|_{\partial\Omega} = \partial_\nu u|_{\partial\Omega} = 0\}$$
$$Z = \{(\Delta + q)u \colon u \in Y\}$$

are Banach spaces.

Proof. It follows from the continuity of the mappings $C^{2,\alpha}(\overline{\Omega}) \ni u \mapsto u|_{\partial\Omega} \in C^{2,\alpha}(\partial\Omega)$ and $C^{2,\alpha}(\overline{\Omega}) \ni u \mapsto \partial_\nu u|_{\partial\Omega} \in C^{1,\alpha}(\partial\Omega)$ that Y is a Banach space. To see that Z is a Banach space, let $v_n = \Delta w_n + q w_n \in Z$ be a sequence converging to some $v \in C^\alpha(\overline{\Omega})$. Using Lemma 3.2, we have

$$\|w_n - w_m\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|\Delta w_n + q w_n - \Delta w_m - q w_m\|_{C^\alpha(\overline{\Omega})},$$

so that w_n is a Cauchy sequence in Y. Hence there is some $w \in Y$ with $w_n \to w$ in $C^{2,\alpha}(\overline{\Omega})$. Next,

$$\|\Delta w_n + q w_n - \Delta w - q w\|_{C^\alpha(\overline{\Omega})} \le C\|w_n - w\|_{C^{2,\alpha}(\overline{\Omega})}$$

So for $v = \Delta w + q w$, we have $v_n \to v$ in $C^\alpha(\overline{\Omega})$ and Z is a Banach space. □

The following result shows that there is a bounded projection $P : C^{2,\alpha}(\overline{\Omega}) \to Z$. If $C^{2,\alpha}(\overline{\Omega})$ were a Hilbert space, the existence of a projection would follow from an orthogonal decomposition $C^{2,\alpha}(\overline{\Omega}) = Z \oplus W$. Since Z is the image of $\Delta + q$ acting on functions whose Cauchy data vanishes, the orthocomplement W would be the set of suitable functions w with $(\Delta + q)w = 0$. Thus any $u \in C^{2,\alpha}(\overline{\Omega})$ could be written as $u = (\Delta + q)y + w$, where $y \in Y$ and $(\Delta + q)w = 0$. This shows that y needs to satisfy $(\Delta + q)^2 y = (\Delta + q)u$. This formal argument turns out to work also in our case.

Lemma 4.2. *Let $q \in C^\alpha(\overline{\Omega})$ and let Y and Z be as in Lemma 4.1. Then there exists a bounded projection $P \colon C^{2,\alpha}(\overline{\Omega}) \to Z$ such that $P(u) = (\Delta + q)y$ where $y \in C^{4,\alpha}(\overline{\Omega})$ is the unique solution of*

$$\begin{cases} (\Delta + q)^2 y = (\Delta + q)u & \text{in } \Omega \\ y = \partial_\nu y = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. We first show that there is a unique solution $y \in C^{4,\alpha}(\overline{\Omega})$. If y and $\tilde{y}$ are solutions, then $(\Delta + q)^2(y - \tilde{y}) = 0$, and integrating this equation against $y - \tilde{y}$ gives $(\Delta + q)(y - \tilde{y}) = 0$. Since $y - \tilde{y}$ has vanising Cauchy data, we see that $y = \tilde{y}$ and solutions are unique. Existence of weak solutions $y \in H_0^2(\Omega)$ for the equation $(\Delta + q)^2 y + \gamma y = F \in H^{-2}(\Omega)$, where $\gamma > 0$ is a constant chosen sufficiently large depending on q, follows by using the Riesz representation theorem with the coercive bilinear form $B(y,w) = ((\Delta+q)y, (\Delta+q)w)_{L^2(\Omega)} + \gamma(y,w)_{L^2(\Omega)}$ for $y, w \in Y$. Fredholm theory shows that there is a countable set of eigenvalues where unique solvability could fail, but our uniqueness argument above shows that one has solvability for $(\Delta+q)^2 y = F$. Elliptic regularity shows that for $u \in C^{2,\alpha}(\overline{\Omega})$, one has $y \in C^{4,\alpha}(\overline{\Omega})$.

Now that we know that the equation is uniquely solvable, let $u \in C^{2,\alpha}(\overline{\Omega})$, and let $y \in C^{4,\alpha}(\overline{\Omega})$ be the solution. Then $P(u) = (\Delta+q)y$. Then $P(P(u)) = P((\Delta+q)y) = (\Delta+q)v$ for the unique solution v of

$$\begin{cases} (\Delta+q)^2 v = (\Delta+q)^2 y & \text{in } \Omega \\ v = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since y has 0 Cauchy data, $w = v - y$ satisfies

$$\begin{cases} (\Delta+q)^2 w = 0 & \text{in } \Omega \\ w = \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases}$$

The unique solution to this equation is $w = 0$. It follows that $v = y$. Thus $P(P(u)) = P(u)$ and P is indeed a projection. ☐

Lemma 4.3. *Let $q \in C^\alpha(\overline{\Omega})$ and let Y and Z be as in Lemma 4.1. Then $\Delta + q\colon Y \to Z$ is bounded and bijective and has a bounded inverse $G\colon Z \to Y$.*

Proof. By definition of Z, $\Delta+q$ is surjective. To see injectivity, suppose $u, v \in Y$ and $(\Delta+q)u = (\Delta+q)v$. Then $w = v - u$ satisfies

$$\begin{cases} (\Delta+q)w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \\ \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows by the unique continuation principle that $w = 0$. Hence $\Delta + q$ is injective. Lastly, we have

$$\|(\Delta+q)u\|_{C^\alpha(\overline{\Omega})} \le \|\Delta u\|_{C^\alpha(\overline{\Omega})} + \|qu\|_{C^\alpha(\overline{\Omega})} \le C\|u\|_{C^{2,\alpha}(\overline{\Omega})}$$

so that $\Delta + q$ is bounded. Now it follows from the open mapping theorem that there exists a bounded inverse G of $\Delta + q$. ☐

Below we will use the ball $V_{q,\delta}$ in the space of solutions,

$$V_{q,\delta} = \{v \in C^{2,\alpha}(\overline{\Omega}) \,:\, \Delta v + qv = 0 \text{ and } \|v\|_{C^{2,\alpha}(\overline{\Omega})} < \delta\}.$$

Lemma 4.4. *Let $a_1, a_2 \in C^{k+1}(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$ with $k \ge 2$ and let w_1, w_2 have the same Cauchy data and solve $\Delta w_i + a_i(x, w_i) = 0$ in Ω. Write $q_i = \partial_u a_i(x, w_i)$. Let $S_{a_1} : V_{q_1,\delta_1} \to C^{2,\alpha}(\overline{\Omega})$ be the solution map from Lemma 2.4, for some $\delta_1 > 0$. Suppose $u_{1,v} = S_{a_1}(v)$ and that $C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$. Then there exists a $\delta_2 > 0$ and a C^k map $T_{a_2}\colon V_{q_1,\delta_2} \to C^{2,\alpha}(\overline{\Omega})$, $T_{a_2}(v) = u_{2,v}$, where $u_{2,v}$ has the same Cauchy data as $u_{1,v}$ and solves $\Delta u_{2,v} + a_2(x, u_{2,v}) = 0$. Moreover, when $\partial_u a_1(x, w_1) = \partial_u a_2(x, w_2)$ then $T'_{a_2}(0)v = v$.*

Proof. First we use $C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$ to find, for any $v \in V_{q_1,\delta_1}$, a function $u_{2,v}$ with the same Cauchy data as $u_{1,v}$ and solving $\Delta u_{2,v} + a_2(x, u_{2,v}) = 0$. Note that $u_{1,0} = w_1$. Moreover, both $u_{2,0}$ and w_2 solve the equation $\Delta u + a_2(x, u) = 0$ and they have the same Cauchy data, so by Lemma 3.3 one has $u_{2,0} = w_2$. By (2.17) we have

$$\|u_{1,v} - w_1\|_{C^{2,\alpha}(\overline{\Omega})} \le C\|v\|_{C^{2,\alpha}(\overline{\Omega})}, \tag{4.4}$$

Using this, Lemma 3.3, $(u_{2,v} - w_2)|_{\partial\Omega} = (u_{1,v} - w_1)|_{\partial\Omega}$, $\partial_\nu(u_{2,v} - w_2)|_{\partial\Omega} = \partial_\nu(u_{1,v} - w_1)|_{\partial\Omega}$ and the fact that $\|u_{2,v}\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$, we have

$$\|u_{2,v} - w_2\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|v\|_{C^{2,\alpha}(\overline{\Omega})}. \tag{4.5}$$

Let $r_v = u_{1,v} - u_{2,v}$. Then r_v satisfies

$$(\Delta + q_2)r_v = q_2 r_v + a_2(x, u_{2,v}) - a_1(x, u_{1,v}) = q_2 r_v + a_2(x, u_{1,v} - r_v) - a_1(x, u_{1,v}).$$

Let G be the inverse of $\Delta + q_2 : Y \to Z$ provided by Lemma 4.3. Then r_v solves the fixed point equation

$$r_v = G(q_2 r_v + a_2(x, u_{1,v} - r_v) - a_1(x, u_{1,v})). \tag{4.6}$$

We would like to show that r_v depends smoothly on v by applying the implicit function theorem to (4.6). However, for a general function r the expression $q_2 r + a_2(x, u_{1,v} - r) - a_1(x, u_{1,v})$ might not be in the domain of G. For this reason we introduce the projection $P\colon C^{2,\alpha}(\overline{\Omega}) \to Z$ from Lemma 4.2. Now define the map $F\colon V_{q_1,\delta_1} \times C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ by

$$F(v, r) = r - GP(q_2 r + a_2(x, u_{1,v} - r) - a_1(x, u_{1,v})).$$

Next we compute $F(0, w_1 - w_2)$ and $D_r F(0, w_1 - w_2; h)$ and find

$$\begin{aligned} F(0, w_1 - w_2) &= w_1 - w_2 - GP(q_2(w_1 - w_2) + a_2(x, w_1 - (w_1 - w_2)) - a_1(x, w_1)) \\ &= w_1 - w_2 - GP(q_2(w_1 - w_2) + \Delta(-w_2 + w_1)) \\ &= w_1 - w_2 - GP((\Delta + q_2)(w_1 - w_2)) \\ &= w_1 - w_2 - G((\Delta + q_2)(w_1 - w_2)) = 0 \end{aligned}$$

and

$$D_r F(0, w_1 - w_2; h) = h - GP(q_2 h - \partial_u a_2(x, w_2)h) = h.$$

Since $h \mapsto D_r F(0, w_1 - w_2; h)$ is bijective, it follows from the implicit function theorem [HG27, Theorem 4] that there exists a δ_2 with $0 < \delta_2 \leq \delta_1$ and a C^k map $R\colon V_{q_1,\delta_2} \to C^{2,\alpha}(\overline{\Omega})$ such that $\tilde{r} = R(v)$ is the unique solution to

$$\tilde{r} = GP(q_2\tilde{r} + a_2(x, u_{1,v} - \tilde{r}) - a_1(x, u_{1,v})). \tag{4.7}$$

for $\tilde{r}$ close to $w_1 - w_2$. Choosing $v \in V_{q_1,\delta_2}$ in $u_{1,v} = S_{a_1}(v)$, we find that r_v is in the range of R and that r_v satisfies (4.7). Moreover, by (4.4) and (4.5) we also have

$$\|r_v - (w_1 - w_2)\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|v\|_{C^{2,\alpha}(\overline{\Omega})}.$$

By the uniqueness of $\tilde{r} = R(v)$ near $w_1 - w_2$ we have $r_v = R(v)$ for $v \in B_{\delta_2}$. Thus the map $v \mapsto r_v$ is indeed C^k.

Since $r_v = u_{1,v} - u_{2,v}$ we can define the C^k map

$$T_{a_2}(v) := S_{a_1}(v) - R(v).$$

It remains to show that $T'_{a_2}(0)v = v$, provided $\partial_u a_1(x, w_1) = \partial_u a_2(x, w_2)$. To do this, we use the implicit function theorem to compute $R'(0)$,

$$R'(0)v = -[D_r F(0, R(0))]^{-1} D_v F(0, R(0))v.$$

Since $D_r F(0, R(0))v = v$ and $D_v F(0, R(0))v = 0$ it follows that $R'(0)v = 0$. Now we have

$$T'_{a_2}(0)v = S'_{a_1}(0)v + R'(0)v = S'_{a_1}(0)v = v. \qquad \square$$

5. First linearization

Throughout this section, we let $a_1, a_2 \in C^{3,\alpha}(\mathbb{R}, C^{1,\alpha}(\overline{\Omega}))$ and let $w \in C^{2,\alpha}(\overline{\Omega})$ be a fixed solution of $\Delta w + a_1(x, w) = 0$ in Ω. Write $q = \partial_u a_1(x, w)$ and consider the sets

$$V_q = \{v \in C^{2,\alpha}(\overline{\Omega}) \,:\, \Delta v + qv = 0 \text{ in } \Omega\},$$
$$V_{q,\delta} = \{v \in V_q \,:\, \|v\|_{C^{2,\alpha}(\overline{\Omega})} < \delta\}.$$

Assume $C_{a_1}^{w,\delta} \subseteq C_{a_2}^{0,C}$. For any $v \in V_{q,\delta}$ with δ small, we let $u_{1,v} = S_{a_1,w}(v)$ and $u_{2,v} = T_{a_2,w}(v)$ be the solutions of $\Delta u_{j,v} + a_j(x, u_{j,v}) = 0$ given by Lemmas 2.4 and 4.4.

Lemma 5.1. *Suppose that $C_{a_1}^{w,\delta} \subseteq C_{a_2}^{0,C}$. There is $\delta_1 > 0$ such that for any $v \in V_{q,\delta_1}$ one has*

$$\partial_u a_1(x, u_{1,v}(x)) = \partial_u a_2(x, u_{2,v}(x)), \qquad x \in \overline{\Omega}.$$

Proof. Let $v \in V_{q,\delta}$ and let $v_t = v + th$ where h solves $\Delta h + qh = 0$ and t is small. Consider the solutions $u_{1,v_t} = S_{a_1,w}(v_t)$ and $u_{2,v_t} = T_{a_2,w}(v_t)$ of

$$\Delta u_{j,v_t} + a_j(x, u_{j,v_t}) = 0.$$

The solutions u_{j,v_t} are C^2 with respect to t and have the same Cauchy data. Differentiating the above equation in t and writing $\dot{u}_j = \partial_t u_{j,v_t}|_{t=0}$, we obtain

$$\Delta \dot{u}_j + \partial_u a_j(x, u_{j,v}) \dot{u}_j = 0.$$

Subtracting the equations for $j = 1, 2$ and rewriting yields

$$(\Delta + \partial_u a_2(x, u_{2,v}))(\dot{u}_1 - \dot{u}_2) + (\partial_u a_1(x, u_{1,v}) - \partial_u a_2(x, u_{2,v}))\dot{u}_1 = 0. \tag{5.1}$$

Suppose that $\tilde{v}_2$ solves $(\Delta + \partial_u a_2(x, u_{2,v}))\tilde{v}_2 = 0$. Integrating (5.1) against $\tilde{v}_2$ and using that $\dot{u}_1 - \dot{u}_2$ has zero Cauchy data gives

$$\int_\Omega (\partial_u a_1(x, u_{1,v}) - \partial_u a_2(x, u_{2,v}))\dot{u}_1 \tilde{v}_2 \, dx = 0.$$

It remains to study $\dot{u}_1 = DS_{a_1,w}(v)h$. By Lemma 2.5, when $v \in V_q$ is sufficiently small any solution $\tilde{v}_1$ of $(\Delta + \partial_u a_1(x, u_{1,v}))\tilde{v}_1 = 0$ can be written as $DS_{a_1,w}(v)h$ for a suitable h. It follows that

$$\int_\Omega (\partial_u a_1(x, u_{1,v}) - \partial_u a_2(x, u_{2,v}))\tilde{v}_1 \tilde{v}_2 \, dx = 0$$

for any solutions $\tilde{v}_j$ of $(\Delta + \partial_u a_j(x, u_{j,v}))\tilde{v}_j = 0$. Now it follows from the density of products of solutions as in the standard Calderón problem (see [SU87] for $n \geq 3$ and [Buk08; BTW20] for $n = 2$) that $\partial_u a_1(x, u_{1,v}) = \partial_u a_2(x, u_{2,v})$. ☐

Lemma 5.2. *In the setting of Lemma 5.1, the function*

$$\varphi_v = u_{2,v} - u_{1,v}$$

is independent of $v \in V_{q,\delta_1}$.

Proof. Write $\psi_t = \varphi_{tv}$. The function ψ_t is C^2 in t, has zero Cauchy data on $\partial\Omega$, and satisfies

$$\Delta \psi_t = a_1(x, u_{1,tv}) - a_2(x, u_{2,tv}).$$

Thus the derivative $z_t = \partial_t \psi_t$ satisfies

$$\Delta z_t = \partial_u a_1(x, u_{1,tv}) \partial_t u_{1,tv} - \partial_u a_2(x, u_{2,tv}) \partial_t u_{2,tv}.$$

Combining this with Lemma 5.1 yields

$$\Delta z_t = -\partial_u a_1(x, u_{1,tv}) z_t.$$

Since z_t has zero Cauchy data, it follows that $z_t = 0$ and consequently ψ_t is independent of t. In particular, $\varphi_v = \varphi_0$. □

We can now give the proofs of Theorem 1.1 and 1.3.

Proof of Theorem 1.3. Let w_1 solve $\Delta w_1 + a_1(x, w_1) = 0$ and assume that $C_{a_1}^{w_1,\delta} \subseteq C_{a_2}^{0,C}$. Using Lemma 5.2, we have

$$\begin{aligned}\Delta\varphi = \Delta u_{2,v} - \Delta u_{1,v} &= a_1(x, u_{1,v}) - a_2(x, u_{2,v}) \\ &= a_1(x, u_{1,v}) - a_2(x, u_{1,v} + \varphi).\end{aligned}$$

This can be rewritten as

$$a_1(x, u_{1,v}(x)) = T_\varphi a_2(x, u_{1,v}(x)).$$

It is enough to show that there is $\varepsilon > 0$ such that for any $\bar{x} \in \overline{\Omega}$ and $\lambda \in [-\varepsilon, \varepsilon]$, one can find a small solution v such that

$$u_{1,v}(\bar{x}) = w_1(\bar{x}) + \lambda. \tag{5.2}$$

Fix $x_0 \in \overline{\Omega}$, and use Runge approximation (Lemma A.1) to generate a solution $v = v_{x_0}$ of $\Delta v + \partial_u a_1(x, w_1)v = 0$ with $v(x_0) = 4$. Let U_{x_0} be a neighborhood of x_0 so that $v(x) \geq 2$ for $x \in \overline{U}_{x_0} \cap \overline{\Omega}$. In the notation of Lemma 2.4 one has

$$u_{1,tv} = w_1 + tv + Q_{a_1,w_1}(tv)$$

where

$$\|Q_{a_1,w_1}(tv)\| \leq C_{a_1,w_1} t^2 \|v\|^2_{C^{2,\alpha}(\overline{\Omega})}.$$

Thus for $x \in \overline{U}_{x_0} \cap \overline{\Omega}$ one has

$$|u_{1,tv}(x) - w_1(x)| \geq 2|t| - C_{a_1,w_1} t^2 \|v\|^2_{C^{2,\alpha}(\overline{\Omega})}.$$

Set $\varepsilon_{x_0} = 1/(C_{a_1,w_1}\|v\|^2_{C^{2,\alpha}(\overline{\Omega})})$. Then for $|t| \leq \varepsilon_{x_0}$

$$|u_{1,tv}(x) - w_1(x)| \geq |t|.$$

The next step is to use compactness to find a finite cover $\{U_{x_1}, \ldots, U_{x_N}\}$ of $\overline{\Omega}$ and to set

$$\varepsilon = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_N}, \delta_0\}.$$

Here δ_0 is chosen so that $\|tv_{x_j}\|_{C^{2,\alpha}} \leq \delta$ whenever $|t| \leq \delta_0$ and $1 \leq j \leq N$.

Now fix any $\bar{x} \in \overline{\Omega}$ and $\lambda \in [-\varepsilon, \varepsilon]$, and choose j so that $\bar{x} \in U_j$. Define

$$\eta(t) = u_{1,tv_{x_j}}(\bar{x}) - w_1(\bar{x}).$$

Then $\eta : [-\varepsilon, \varepsilon] \to \mathbb{R}$ is continuous with $\eta(\varepsilon) \geq \varepsilon$ and $\eta(-\varepsilon) \leq -\varepsilon$. By continuity, there is $\bar{t} \in [-\varepsilon, \varepsilon]$ such that $\eta(\bar{t}) = \lambda$. This proves that one has (5.2) for some choice of v, which proves the theorem. □

Proof of Theorem 1.1. Since w is a common solution for nonlinearities a_1 and a_2, we have $w_1 = w_2 = w$ in Theorem 1.3. Consequently $\varphi = 0$ and $T_\varphi a_2 = a_2$. The result now follows from Theorem 1.3. □

6. Higher order linearization

In this section we prove theorem 1.4. We use the higher order linearization method with the smooth solution maps from Sections 2 and 4. Essentially the method is to show that the derivatives of order k of the solution maps satisfy a certain partial differential equation and have the same Cauchy data, provided $\partial_u^l a_1(x, w_1) = \partial_u^l a_2(x, w_2)$ for $l \le k-1$. Then Theorem 1.4 follows from an integration by parts argument. We start by proving that the derivatives of order k of the solution maps satisfy a certain differential equation in the following Lemma.

Lemma 6.1. *Let $a_1, a_2 \in C^{k+2,\alpha}(\mathbb{R}; C^{1,\alpha}(\overline{\Omega}))$ with $k \ge 1$. Let S_{a_1}, T_{a_2} be the solution operators $\Delta u + a_i(x, u) = 0$ from Lemma 2.4 and Lemma 4.4. Suppose that $\partial_u^l a_1(x, w_1) = \partial_u^l a_2(x, w_2)$ for all $1 \le l \le k$, then $f := D^{k+1} S_{a_1}(0; v_1, \ldots, v_{k+1}) - D^{k+1} T_{a_2}(0; v_1, \ldots, v_{k+1})$ satisfies*

$$\begin{cases} \Delta f + qf = [\partial_u^{k+1} a_2(x, w_2) - \partial_u^{k+1} a_1(x, w_1)] \prod_{i=1}^{k+1} v_i & \textit{in } \Omega \\ f = 0 & \textit{on } \partial\Omega \\ \partial_\nu f = 0 & \textit{on } \partial\Omega \end{cases}$$

for any solutions v_i of $\Delta v + qv = 0$ where $q = \partial_u a_1(x, w_1) = \partial_u a_2(x, w_2)$.

Proof. We start with the boundary conditions. We have by construction that $S_{a_1}(v)|_{\partial\Omega} = T_{a_2}(v)|_{\partial\Omega}$. Let $\iota \colon C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\partial\Omega)$ be the natural injection. Then the operator $v \mapsto \iota(S_{a_1}(v) - T_{a_2}(v))$ is identically equal to 0. Hence its derivatives are also 0, $D^l S_{a_1}(0; v_1, v_2)|_{\partial\Omega} = D^l T_{a_2}(0; v_1, v_2)|_{\partial\Omega}$ for all $l \in \mathbb{N}$, or equivalently $f|_{\partial\Omega} = 0$. Similarly, $\partial_\nu f|_{\partial\Omega} = 0$

The proof for the differential equation is by induction on k and we start with the base case $k = 1$, where we assume that $\partial_u a_1(x, w_1) = \partial_u a_2(x, w_2)$. Since $S_{a_1}(v)$ and $T_{a_2}(v)$ are solution maps for $\Delta u + a_1(x, u) = 0$ and $\Delta u + a_2(x, u) = 0$, respectively, the operators $v \mapsto \Delta S_{a_1}(v) + a_1(x, S_{a_1}(v))$ and $v \mapsto \Delta T_{a_2}(v) + a_2(x, T_{a_2}(v))$ are identically equal to 0. Hence their derivatives are also 0. The second derivatives being 0 can be rewritten as

$$\Delta D^2 S_{a_1}(0; v_1, v_2) + q D^2 S_{a_1}(0; v_1, v_2) = -\partial_u^2 a_1(x, S_{a_2}(0)) D S_{a_1}(0; v_1) D S_{a_1}(0; v_2) \quad \text{in } \Omega$$

and

$$\Delta D^2 T_{a_2}(0; v_1, v_2) + q D^2 T_{a_2}(0; v_1, v_2) = -\partial_u^2 a_2(x, T_{a_2}(0)) D T_{a_2}(0; v_1) D T_{a_2}(0; v_2) \quad \text{in } \Omega$$

From $\partial_u a_1(x, w_1) = \partial_u a_2(x, w_2)$ we get $D S_{a_1}(0; v) = D T_{a_2}(0; v) = v$. Using this and $S_{a_1}(0) = w_1$, $T_{a_2}(0) = w_2$ we get, by subtracting the equations, that

$$\Delta f + qf = [\partial_u^2 a_2(x, w_1) - \partial_u^2 a_2(x, w_2)] v_1 v_2 \quad \text{in } \Omega$$

where $f := D^2 S_{a_1}(0; v_1, v_2) - D^2 T_{a_2}(0; v_1, v_2)$. Now suppose that, for some $m \ge 2$, the statement holds for $k = m - 1$. That is, suppose for $l \le m - 1$ that $\partial_u^l a_1(x, w_1) = \partial_u^l a_2(x, w_2)$ and that $f := D^m S_{a_1}(0; v_1, \ldots, v_m) - D^m T_{a_2}(0; v_1, \ldots, v_m)$ solves

$$\begin{cases} \Delta f + qf = [\partial_u^m a_2(x, w_2) - \partial_u^m a_1(x, w_1)] \prod_{i=1}^m v_i & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega \\ \partial_\nu f = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

To show that it holds also for $k = m$ assume additionally that $\partial_u^m a_1(x, w_1) = \partial_u^m a_2(x, w_2)$. Then (6.1) simplifies to

$$\begin{cases} \Delta f + qf = 0 & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega \\ \partial_\nu f = 0 & \text{on } \partial\Omega \end{cases}$$

and we conclude that $f = 0$. Let $F(v) := S_{a_1}(v) - T_{a_2}(v)$, so that $D^m F(0; v_1, \ldots, v_m) = f = 0$ and let $g = D^{m+1}F(0; v_1, \ldots, v_{m+1})$. Let δ_{ij} denote the Kronecker delta. Then we have by differentiating (6.1) in direction v_{m+1} that

$$\begin{aligned}\Delta g + qg =& -\partial_u^2 a_1(x, w_1) \underbrace{D^m F(0; v_1, \ldots, v_m)}_{=0} v_{m+1} \\ &+ \underbrace{[\partial_u^m a_2(x, w_2) - \partial_u^m a_1(x, w_1)]}_{=0} \sum_{i=1}^m D^2 S_{a_1}(0; v_i, v_{m+1}) \prod_{j=1}^m (1 - \delta_{ij}) v_j \\ &+ [\partial_u^{m+1} a_2(x, w_2) - \partial_u^{m+1} a_1(x, w_1)] \prod_{i=1}^{m+1} v_i \\ =& [\partial_u^{m+1} a_2(x, w_2) - \partial_u^{m+1} a_1(x, w_1)] \prod_{i=1}^{m+1} v_i\end{aligned}$$

and this proves the lemma. □

Proof of Theorem 1.4. Let $l \geq 2$ be an arbitrary integer and suppose that $\partial_u^j a_1(x, w_1) = \partial_u^j a_2(x, w_2)$ for $1 \leq j \leq l - 1$. Then we have from Lemma 6.1 that

$$(6.2) \qquad \begin{cases} \Delta f + qf = [\partial_u^l a_2(x, w_2) - \partial_u^l a_1(x, w_1)] \prod_{i=1}^l v_i & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega \\ \partial_\nu f = 0 & \text{on } \partial\Omega \end{cases}$$

where v_j solve $\Delta v_j + \partial_u a_1(x, w_1) v_j = 0$ for $j \in \{1, \ldots, l\}$. Let v_{l+1} solve $\Delta v_{l+1} + \partial_u a_1(x, w_1) v_{l+1} = 0$. Multiplying the differential equation (6.2) by v_{l+1} and integrating by parts twice gives

$$\int_\Omega [\partial_u^l a_2(x, w_2) - \partial_u^l a_1(x, w_1)] \prod_{i=1}^{l+1} v_i \, dx = 0. \qquad □$$

Appendix A. Runge approximation

In the proof of Theorem 1.3 we need to find a solution of the linearized equation which is nonzero in some fixed but arbitrary point of the domain. A few ways to achieve this are described in [LLLS21b, Remark 2.2]. For the sake of completeness, we give a proof based on Runge approximation that is valid in our situation following [LLS20].

Lemma A.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $q \in C^\alpha(\overline{\Omega})$. For any $x_0 \in \overline{\Omega}$, there is $u \in C^{2,\alpha}(\overline{\Omega})$ solving $(-\Delta + q)u = 0$ in Ω with $u(x_0) \neq 0$.*

Proof. Let Ω_2 be a large ball with $\overline{\Omega} \subseteq \Omega_2$, and extend q as a function in $C_c^\alpha(\Omega_2)$. We may choose Ω_2 in such a way that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω_2 (see e.g. [Ste90, Lemma 3.2]). Now by [BJS64, Theorem 1 in Section 5.4], there is a small ball Ω_1 centered at x_0 and a function $u_0 \in C^{2,\alpha}(\overline{\Omega}_1)$ solving $(-\Delta + q)u_0 = 0$ in Ω_1 with $u_0(x_0) = 1$. By Runge approximation (see Lemma A.3 below), there is $u \in C^{2,\alpha}(\overline{\Omega}_2)$ solving $(-\Delta + q)u = 0$ in Ω_2 with $u(x_0)$ arbitrarily close to $u_0(x_0) = 1$. This concludes the proof. □

It remains to prove the Runge approximation result. Since the approximation is in the $C(\overline{\Omega}_1)$ norm, we need a notion of suitable weak solutions with measure data in the duality argument. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary, let $q \in L^\infty(\Omega)$, and assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. If μ is a bounded linear functional on $C(\overline{\Omega})$ (i.e. μ is a measure), we say that $u \in L^1(\Omega)$ is a *very weak solution* of

$$(-\Delta + q)u = \mu \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{A.1}$$

if

$$\int_\Omega u(-\Delta + q)\varphi \, dx = \mu(\varphi)$$

for any $\varphi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$.

Proposition A.2. *For any $p < \frac{n}{n-1}$ there is $C > 0$ such that for any bounded linear functional μ on $C(\overline{\Omega})$, there is a unique very weak solution $u \in W^{1,p}(\Omega)$ of (A.1) satisfying*

$$\|u\|_{W^{1,p}(\Omega)} \leq C\|\mu\|,$$

where $\|\mu\| = \sup_{\|\varphi\|_{C(\overline{\Omega})}=1} |\mu(\varphi)|$.

Proof. If $q \geq 0$ this follows from [Sta65, Theorem 9.1]. In general we may replace q by $q + \gamma$ where $\gamma > 0$ is a large constant, and use the part of [Sta65, Theorem 9.1] where λ is away from the spectrum. □

We can now prove the Runge approximation result.

Lemma A.3. *Let Ω_1, $\Omega \subseteq \mathbb{R}^n$ be bounded open sets so that $\overline{\Omega}_1 \subseteq \Omega$, $\Omega \setminus \overline{\Omega}_1$ is connected, and Ω has smooth boundary. Suppose that $q \in C_c^\alpha(\Omega)$ and that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω. Consider the sets*

$$S_1 = \{u \in C^{2,\alpha}(\overline{\Omega}_1),\ (-\Delta + q)u = 0 \text{ in } \Omega_1\},$$
$$S = \{u \in C^{2,\alpha}(\overline{\Omega}),\ (-\Delta + q)u = 0 \text{ in } \Omega\}.$$

For any $u \in S_1$ and any $\varepsilon > 0$, there is $v \in S$ with $\|u - v|_{\Omega_1}\|_{C(\overline{\Omega}_1)} \leq \varepsilon$.

Proof. By the Hahn-Banach theorem [Con90, Corollary 3.15], it is enough to show that any continuous linear functional on $C(\overline{\Omega}_1)$ that vanishes on $S|_{\Omega_1}$ must also vanish on S_1. Thus, let μ be a continuous linear functional on $C(\overline{\Omega}_1)$ that satisfies

$$\mu(v|_{\overline{\Omega}_1}) = 0 \text{ for all } v \in S. \tag{A.2}$$

We consider the extension defined by

$$\bar{\mu} : C(\overline{\Omega}) \to \mathbb{R}, \quad \bar{\mu}(u) = \mu(u|_{\overline{\Omega}_1}).$$

By the Riesz representation theorem, $\bar{\mu}$ is a measure in $\overline{\Omega}$ with $\mathrm{supp}(\bar{\mu}) \subseteq \overline{\Omega}_1$.

We use Proposition A.2 to find a very weak solution $w \in W^{1,p}(\Omega)$ of the problem

$$(-\Delta + q)w = \bar{\mu} \text{ in } \Omega, \qquad w|_{\partial\Omega} = 0. \tag{A.3}$$

We use the assumption (A.2) and the unique continuation principle to prove that

$$w = 0 \text{ in } \Omega \setminus \overline{\Omega}_1. \tag{A.4}$$

Assuming (A.4), the proof can be concluded as follows. Since $\mathrm{supp}(w) \subseteq \overline{\Omega}_1$, there exist $w_j \in C_c^\infty(\Omega_1)$ with $w_j \to w$ in $W^{1,p}(\Omega)$. Given any $u \in S_1$, we let $\bar{u}$ be some function in $C_c^{2,\alpha}(\Omega)$ with $\bar{u}|_{\overline{\Omega}_1} = u$ and compute

$$\mu(u) = \bar{\mu}(\bar{u}) = \int_\Omega w(-\Delta + q)\bar{u} \, dx = \lim \int_\Omega w_j(-\Delta + q)\bar{u} \, dx = \lim \int_{\Omega_1} w_j(-\Delta + q)u \, dx = 0.$$

Thus $\mu|_{S_1} = 0$ as required.

It remains to prove (A.4). We begin by studying the regularity of w near $\partial\Omega$. Choose a ball Ω_2 with $\overline{\Omega} \subseteq \Omega_2$ so that 0 is not a Dirichlet eigenvalue, and a very weak solution $\tilde{w}$ of

$$(-\Delta + q)\tilde{w} = \tilde{\mu} \text{ in } \Omega_2, \qquad \tilde{w}|_{\partial\Omega_2} = 0,$$

where $\tilde{\mu}$ is the extension of $\bar{\mu}$ by zero to $\overline{\Omega}_2$. Using the definition of very weak solutions and the facts that $q \in C_c^\alpha(\Omega)$ and $\operatorname{supp}(\bar{\mu}) \subseteq \overline{\Omega}_1$, we see that $\Delta\tilde{w} = 0$ near $\partial\Omega$ in the sense of distributions. Hence $\tilde{w}$ is C^∞ near $\partial\Omega$. Let $g \in C^{2,\alpha}(\overline{\Omega})$ be the solution of

$$(-\Delta + q)g = 0 \text{ in } \Omega, \qquad g|_{\partial\Omega} = -\tilde{w}|_{\partial\Omega}.$$

Then both w and $\tilde{w}|_\Omega + g$ are very weak solutions of (A.3), and by uniqueness one has

$$w = \tilde{w}|_\Omega + g.$$

It follows that w is $C^{2,\alpha}$ near $\partial\Omega$. Moreover, since w is a $W^{1,p}$ solution of $(-\Delta + q)w = 0$ in $\Omega \setminus \overline{\Omega}_1$, it follows from [HR72, see section 4. Concluding remarks] that w is $W^{1,2}$ and consequently $C^{2,\alpha}$ in $\Omega \setminus \overline{\Omega}_1$.

We now let $v \in S$ and choose $\chi \in C_c^\infty(\Omega)$ such that $\chi = 1$ near $\overline{\Omega}_1$ and w is $C^{2,\alpha}$ in $\operatorname{supp}(1-\chi) \cap \overline{\Omega}$. Then

$$\int_\Omega w(-\Delta + q)v\,dx = \int_\Omega w(-\Delta + q)(\chi v)\,dx + \int_\Omega w(-\Delta + q)((1-\chi)v)\,dx.$$

We use the definition of very weak solutions in the first term, and since w is regular in $\operatorname{supp}(1-\chi)$ we may integrate by parts in the second term. This yields

$$\int_\Omega w(-\Delta + q)v\,dx = \bar{\mu}(\chi v) + \int_{\partial\Omega} (\partial_\nu w)v\,dS = \mu(v|_{\overline{\Omega}_1}) + \int_{\partial\Omega} (\partial_\nu w)v\,dS.$$

Since $v \in S$, we have $\mu(v|_{\overline{\Omega}_1}) = 0$ by the assumption (A.2). Since we can vary the Dirichlet data of $v \in S$, it follows that $\partial_\nu w|_{\partial\Omega} = 0$. Thus w in particular satisfies

$$(-\Delta + q)w = 0 \text{ in } \Omega \setminus \overline{\Omega}_1, \qquad w|_{\partial\Omega} = \partial_\nu w|_{\partial\Omega} = 0.$$

Since w is $C^{2,\alpha}$ in $\Omega \setminus \overline{\Omega}_1$ and this set is connected, the unique continuation principle yields (A.4). This finishes the proof. □

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