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DISSERTATIONES

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ESKO NIEMINEN

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Esko Nieminen

Contents

1. Introduction

In the early 1970s Muckenhoupt discovered his famous concept of A_p weights in connection with weighted norm inequalities for the Hardy–Littlewood maximal function. In general, weighted norm inequalities have the form

$$\int |Tf(y)|^p w(y)\,dy \leq c \int |f(y)|^p w(y)\,dy$$

where, for example, T is a singular integral operator or a maximal function operator, and the constant c depends only on n, p, and the weight w. Such inequalities arise naturally in many areas of harmonic analysis in $\mathbb{R}^n$. During the past two decades a number of papers have appeared concerning different types of integral transforms, in particular, singular integral operators (see [GR] and [Tor]), weighted nonlinear potential theory (see [Ad] and [HKM]), weighted Sobolev spaces (see [Ch], [FKS], and [Ku]), and weighted Beppo Levi spaces (see [Ai]).

In the first part of the paper we investigate weighted Hausdorff measures, weighted capacity densities, and weighted content densities. It turns out that in many cases the weighted Hausdorff dimension can be estimated from below in terms of the ordinary Hausdorff dimension. Weighted capacity densities and weighted content densities are studied by making a comparison between them. A connection between the weighted Hausdorff dimension and the weighted capacity density is given in terms of weighted content density closely related to weighted capacity density and a linearly increasing gauge. This leads to upper and lower bounds for the weighted Hausdorff dimension of a set on condition that the weighted capacity density of the set is zero everywhere. Moreover, we produce upper bounds for the ordinary Hausdorff dimension of a set of zero weighted capacity density by means of the weighted Hausdorff dimension. For earlier results concerning the subject see [Ne], [Res], [Fe], and [Ma].

In the second part we characterize weighted Sobolev spaces as weighted Bessel potential spaces. This is a generalization of a well known result in the unweighted case, see [AMS] or [St]. Roughly speaking, every function in a weighted Sobolev space of order k has a representation by means of Bessel kernels and Riesz transforms. As a byproduct, we obtain the fact that weighted Sobolev space and the space of integrable functions up to order p, $1 < p < \infty$, with respect to the measure induced by a given weight are quasi isometrical. Furthermore, each function in the latter space can be represented in terms of Bessel kernels and Riesz transforms.

List of Notations. The following notation will be used thoroughout this paper.

$x = (x_1, x_2, \ldots, x_n)$ a point in the Euclidean n-space $\mathbb{R}^n$,
$x \cdot y = \sum_{j=1}^{n} x_j \cdot y_j$ the inner product of x and y,

$|x| = (x \cdot x)^{\frac{1}{2}}$ the norm of x in $\mathbb{R}^n$,
$\overline{S}$ the closure of a set S in $\mathbb{R}^n$,
$B(x,r) = \{y : |x-y| < r\}$ an open ball with center x and radius r,
$Q(x,r) = \{y : |x_j - y_j| < \frac{r}{2}, \forall j = 1,\dots,n\}$ an open cube with center x and edge length r,
$\operatorname{spt} u = \overline{\{x : u(x) \neq 0\}}$ the support of u, $u : \Omega \mapsto \mathbb{R}$, and Ω is an open set in $\mathbb{R}^n$,
$\partial_i f$ the i^{th} weak partial derivative of f,
$\nabla g = (\partial_1 g, \partial_2 g, \dots, \partial_n g)$ the gradient of g,
χ_E the characteristic function of a set E,
$|E| = \int_{\mathbb{R}^n} \chi_E \, dy$ the Lebesgue measure of a set E,

$$\fint_{B(x,r)} f \, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} f \, dy$$

is the mean value of a locally integrable function f,
ω the area of the unit sphere in $\mathbb{R}^n$,
c, c_1, c_2, c_3 positive constants, $c = c(n,p,\alpha,\dots,\beta)$ means that c depends on n, p, α, $\dots$, and β only.

Let Ω be an open set in $\mathbb{R}^n$.

$C_0^k(\Omega)$ k-times continuously differentiable functions whose supports belong to Ω, and $C_0^\infty = C_0^\infty(\mathbb{R}^n)$,
$\mathfrak{S}$ the class of Schwartz test functions, see [SW, §3 p. 19].

2. Preliminaries

In this chapter we introduce weights and weighted variational capacities. We also state some auxiliary results concerning basic properties of capacity. That is, we find lower and upper bounds for capacity and the subadditivity of capacity. It is pointed out that all the lemmas in the chapter are well known and therefore some proofs are omitted.

A nonnegative measurable function w defined on $\mathbb{R}^n$ is called a *weight* if $0 < w < \infty$ a.e. (almost everywhere) in $\mathbb{R}^n$ and w is locally integrable (in the Lebesgue sense). We also identify the weight w and the corresponding measure $E \mapsto \int_E w(y)\, dy$.

Since, by the definition of a weight w, the Lebesgue measure and $\int_E w(y)\, dy$ are mutually absolutely continuous, there is no need to specify the measure when using the phrase "almost everywhere". Moreover, we need not identify the measure when speaking about a measurable set or function.

2.1. L^p_w **spaces.** Let $1 \le p < \infty$ and let w be a weight. If $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ is measurable, we write

$$\|f\|_{L^p_w} = \Big(\int_{\mathbb{R}^n} |f|^p \, w \, dy \Big)^{\frac{1}{p}} .$$

The collection of all functions f with $\|f\|_{L^p_w} < \infty$ is denoted by L^p_w.

2.2. Doubling weight. A weight w is called a *doubling weight* if there is a constant, denoted by $C_D = C_D(w)$, such that

$$\int_{B(x,2t)} w\, dy \leq C_D \int_{B(x,t)} w\, dy \tag{2.3}$$

for all balls $B(x,t)$ in $\mathbb{R}^n$. We call (2.3) the *doubling condition* and C_D is said to be the *doubling constant* of w.

2.4. A_p weights. Let $p \in [1,\infty)$. We say that a weight w on $\mathbb{R}^n$ is an A_p *weight* if

$$\Big(\fint_{B(x,t)} w\, dy\Big)\Big(\fint_{B(x,t)} w^{\frac{1}{1-p}}\, dy\Big)^{p-1} \leq C_A \quad \text{if } 1 < p < \infty \tag{2.5}$$

or

$$\fint_{B(x,t)} w\, dy \leq C_A \operatorname*{ess\,inf}_{y \in B(x,t)} w(y) \quad \text{if } p = 1 \tag{2.6}$$

for all balls $B(x,t)$ in $\mathbb{R}^n$ and for a finite constant C_A independent of $B(x,t)$. This is the well known Muckenhoupt A_p *condition* and C_A is the A_p *constant* of w. To express that w is an A_p weight we write $w \in A_p$ and use a phrase "w belongs to the A_p class". Moreover, the A_∞ class is defined by letting

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p\,.$$

2.7. Remarks. (a) Let $1 < p < \infty$. From (2.5) and the Hölder inequality it follows that

$$1 \leq \Big(\fint_{B(x,t)} w\, dy\Big)\Big(\fint_{B(x,t)} w^{\frac{1}{1-p}}\, dy\Big)^{p-1} \leq C_A \tag{2.8}$$

whenever $B(x,t) \subset \mathbb{R}^n$. Repeating the argument, using (2.6), we find that the corresponding double inequality holds for $p = 1$ as well. Hence we have $1 \leq C_A < \infty$ for every A_p weight.

(b) It is due to the geometry of $\mathbb{R}^n$ that a cube sometimes gives us an advantage over a ball in the integral calculus. Thereby, sometimes a ball is replaced by a cube in definitions 2.2 and 2.4 for the sake of convenience in calculus. However, we point out that these changes produce concepts which are similar to the old ones in connection with the doubling weights.

(*c*) A function w defined by $w(x) = |x|^\gamma$ satisfies the Muckenhoupt A_p condition, $1 < p < \infty$, exactly if $-n < \gamma < np - n$, where n is the dimension of the space $\mathbb{R}^n$, see [Tor, Corollary 4.4 p. 237].

(*d*) Let $1 < p < \infty$ and suppose that w_j is measurable and $0 < w_j(x) < \infty$ a.e. in $\mathbb{R}$ for $j = 1, \dots, n$. Then the following two conditions are equivalent:

(i) The functions w_j for $j = 1, \dots, n$ are A_p weights on the real line $\mathbb{R}$.
(ii) The function $w(x) = w_1(x_1) w_2(x_2) \cdots w_n(x_n)$ belongs to the A_p class in $\mathbb{R}^n$.

That (i) implies (ii) is a consequence of the Fubini theorem and (b). It follows from (2.8), (b), and the Fubini theorem that (ii) implies (i).

For $x_j \in \mathbb{R}$ define $w_j(x_j) = |x_j|^{\gamma_j}$ provided that $-1 < \gamma_j < p - 1$ and $j = 1, \dots, n$. Since w_j is in A_p on $\mathbb{R}$, the function

$$w(x) = |x_1|^{\gamma_1} |x_2|^{\gamma_2} \cdots |x_n|^{\gamma_n}$$

satisfies the A_p condition on $\mathbb{R}^n$ whenever $-1 < \gamma_j < p - 1$ for $j = 1, \dots, n$.

2.9. Lemma. *Let* $1 \le p < \infty$ *and* $w \in A_p$. *Then* w *is a doubling weight with* $C_D = 2^{pn} C_A$.

Proof. Since $A_1 \subset A_p$, there is no loss of generality in assuming that $p \in (1, \infty)$. Now (2.8) yields

$$\begin{aligned}\int_{B(x,2t)} w \, dy &\le C_A \, |B(x,2t)|^p \Big(\int_{B(x,2t)} w^{\frac{1}{1-p}} \, dy \Big)^{1-p} \\ &\le 2^{pn} C_A \, |B(x,t)|^p \Big(\int_{B(x,t)} w^{\frac{1}{1-p}} \, dy \Big)^{1-p} \\ &\le 2^{pn} C_A \int_{B(x,t)} w \, dy ;\end{aligned}$$

hence the doubling condition holds with $C_D = 2^{pn} C_A$, as desired. □

For further information on doubling weights and A_p weights, see [GR] and [Tor]. Now we turn our attention to weighted capacities.

2.10. Capacity. Let Ω be an open set in $\mathbb{R}^n$ and let F be a compact subset of Ω. The pair (F, Ω) is said to be a *condenser*. Let

$$\mathcal{A}(F, \Omega) = \{ u \in C_0^1(\Omega) : u \ge 0 \text{ and } u(x) \ge 1 \text{ for all } x \in F \} .$$

We say that $\mathcal{A}(F, \Omega)$ is the *set of admissible functions* for a condenser (F, Ω).

Let $1 < p < \infty$ and let w be a weight. The *variational* (p, w)*-capacity* of a condenser (F, Ω) is the number

$$\operatorname{cap}_{p,w}(F, \Omega) = \inf_{u \in \mathcal{A}(F,\Omega)} \int_{\mathbb{R}^n} |\nabla u|^p \, w \, dy .$$

We close this section with three lemmas.

2.11. Lemma (Poincaré inequality). *Let $p \in (1,\infty)$ and $w \in A_p$. Then there exists a constant $c = c(n,p,C_A)$ such that for all balls $B(x,r)$ and all functions $g \in C_0^1(B(x,r))$ we have*

$$\int_{B(x,r)} |g|^p \, w \, dy \le c \, r^p \int_{B(x,r)} |\nabla g|^p \, w \, dy .$$

Proof. See [FKS, Theorem 1.2 p. 84]. □

The following estimates are well known, see e.g. [HKM].

2.12. Lemma. *Let $1 < p < \infty$, $x \in \mathbb{R}^n$, and $r > 0$.*

(i) *If $w \in A_p$, then*

$$c_1 \, r^{-p} \int_{B(x,r)} w \, dy \le \operatorname{cap}_{p,w}\left(\overline{B}(x,r), B(x,2r)\right),$$

where c_1 is a positive constant depending only on n, p, and C_A.

(ii) *If w is a doubling weight, then*

$$\operatorname{cap}_{p,w}\left(\overline{B}(x,r), B(x,2r)\right) \le C_D \, r^{-p} \int_{B(x,r)} w \, dy .$$

Proof. To prove (i) let $w \in A_p$ and let u be a function in $\mathcal{A}\big(\overline{B}(x,r), B(x,2r)\big)$. Now $u(z) \ge 1$ for all z in $\overline{B}(x,r)$. Thus from the Poincaré inequality we conclude that

$$\int_{B(x,r)} w \, dy \le \int_{B(x,r)} |u|^p \, w \, dy \le \int_{B(x,2r)} |u|^p \, w \, dy \le c(n,p) r^p \int_{B(x,2r)} |\nabla u|^p \, w \, dy$$

and because u was an arbitrary admissible function we have

$$c_1(n,p) r^{-p} \int_{B(x,r)} w \, dy \le \operatorname{cap}_{p,w}\left(\overline{B}(x,r), B(x,2r)\right),$$

which proves (i).

To verify (ii) we may assume that $x = 0$. For a fixed $j = 3, 4, \ldots$ we consider the continuous function f_j defined on $\mathbb{R}$,

$$f_j(t) = \begin{cases} 2^j/r\big(t - r(1+2^{-j})\big), & r(1+2^{-j}) < t < r(1+2^{-j+1}), \\ 1, & r(1+2^{-j+1}) \le t \le r(2-2^{-j+1}), \\ 2^j/r\big(r(2-2^{-j}) - t\big), & r(2-2^{-j+1}) < t < r(2-2^{-j}), \\ 0 & \text{otherwise.} \end{cases}$$

Now $\int_r^{2r} f_j\,dt = r(1-3\cdot 2^{-j})$. Let $z \in \mathbb{R}^n$ and define

$$g_j(z) = \frac{\int_{|z|}^{2r} f_j\,dt}{r(1-3\cdot 2^{-j})} \quad \text{for } j = 3,4,\dots .$$

Clearly $\operatorname{spt} g_j \subset B(0,2r)$, $g_j \in C_0^1\big(B(0,2r)\big)$, and $g_j(z) = 1$ for all $z \in \overline{B}(0,r)$. Hence g_j is an admissible function for the condenser $\big(\overline{B}(0,r), B(0,2r)\big)$. Furthermore, we have

$$\partial_i g_j(z) = -\frac{f_j(|z|)}{r(1-3\cdot 2^{-j})}\frac{z_i}{|z|} \quad \text{for } i = 1,\dots,n$$

and $|f_j(|z|)| \le 1$ for all $z \in \mathbb{R}^n$. Consequently

$$|\nabla g_j(z)| \le \frac{1}{r(1-3\cdot 2^{-j})} \quad \text{for all } z \in \mathbb{R}^n .$$

From this inequality and from the doubling condition we obtain

$$\begin{aligned}\int_{B(0,2r)} |\nabla g_j|^p\, w\,dy &\le r^{-p}(1-3\cdot 2^{-j})^{-p}\int_{B(0,2r)} w\,dy\\ &\le r^{-p}(1-3\cdot 2^{-j})^{-p}\,C_D\int_{B(0,r)} w\,dy .\end{aligned}$$

Letting $j \to \infty$ we arrive at (ii), which completes the proof. □

2.13. Lemma. *Let $1 < p < \infty$ and $w \in A_p$. Assume that (F,Ω) is a condenser and that (F_j,Ω_j) is a sequence of condensers such that $F \subset \bigcup_j F_j$ and $\Omega_j \subset \Omega$ for $j = 1,2,3,\dots$. Then*

$$\operatorname{cap}_{p,w}(F,\Omega) \le \sum_{j=1}^{\infty} \operatorname{cap}_{p,w}(F_j,\Omega_j) .$$

Proof. The desired conclusion follows from [HKM]. □

2.14. Remark. Note that A_p weights satisfy a stronger inequality than the doubling condition (Lemma 2.9), see [Tor, Theorem 2.1 p. 226].

The assumption that $w \in A_p$ for $1 < p < \infty$ in Lemma 2.11 and Lemma 2.13 is not really necessary. Other weights for which the Poincaré inequality holds can be constructed, for example, by the aid of quasiconformal mappings, see [FKS, §3 p. 104 and Property 3 p. 107]. For more general assumptions for Lemma 2.13, see e.g. [HKM].

3. Weighted Hausdorff measures

An α-dimensional weighted spherical Hausdorff type measure is constructed and a weighted Hausdorff dimension of a set in $\mathbb{R}^n$ is introduced. For certain weights we also discuss lower and upper bounds for the weighted Hausdorff dimension of a set. This leads to an upper bound for the ordinary Hausdorff dimension in terms of the weighted Hausdorff dimension. The relationship between weighted Hausdorff measures and content densities is investigated by means of a linearly increasing gauge on a subset of $\mathbb{R}^n$. In Chapter 4 we will continue our discussion on the (weighted) Hausdorff dimension and content densities in connection with capacity densities.

We begin by introducing gauges and contents. Let ρ be a positive measurable function from $(0,\infty)$ to $(0,\infty)$ and let w be a weight. We call the function $h_{\rho,w}$,

$$h_{\rho,w}(x,t) = h_{\rho,w}\big(B(x,t)\big) = \rho(t)\int_{B(x,t)} w\,dy$$

a *gauge*.

3.1. Remark. Because the weight w is locally integrable with respect to the Lebesgue measure, we have

$$h_{\rho,w}(x,t) = h_{\rho,w}\big(B(x,t)\big) = \rho(t)\int_{\overline{B}(x,t)} w\,dy$$

whenever $\overline{B}(x,t)$ is a closed ball in $\mathbb{R}^n$.

Let E be a subset of $\mathbb{R}^n$ and let $h_{\rho,w}$ be a gauge. For $0<\delta\le\infty$ define

$$H^{\delta}_{\rho,w}(E) = \inf\sum_{j=1}^{\infty} h_{\rho,w}(x_j,r_j)\,,$$

where the infimum is over all coverings $\{\overline{B}(x_j,r_j)\}$ of E with $r_j<\delta$ for all $j=1,2,3,\dots$. The quantity $H^{\delta}_{\rho,w}(E)$ is called the *(δ-ρw)-content* of E.

It is easy to see that a content $H^{\delta}_{\rho,w}$ is an outer measure on $\mathbb{R}^n$. If $\rho(t)=t^{\alpha}$, $\alpha\in\mathbb{R}$, we write $h_{\alpha,w}=h_{\rho,w}$ and $H^{\delta}_{\alpha,w}=H^{\delta}_{\rho,w}$.

First we prove a well known lemma of Cartan for the contents $H^{\delta}_{\rho,w}$. See [Ne, Theorem of Cartan p. 146] or [Res, Lemma 3.7 p. 115].

3.2. Lemma. *Suppose that $h_{\rho,w}$ is a gauge and that ν is a finite Borel measure on $\mathbb{R}^n$. Let $c_0>0$ and $\tau>0$. If*

$$E_\tau = \{x\in\mathbb{R}^n : \tau\,\nu\big(\overline{B}(x,t)\big)\le h_{\rho,w}(x,t) \text{ for all } 0<t<c_0\}\,,$$

then there exists a constant c depending on n only such that

$$H^{\delta}_{\rho,w}(\mathbb{R}^n \setminus E_\tau) \le c(n)\,\tau\,\nu(\mathbb{R}^n)$$

whenever $c_0 < \delta \le \infty$.

Proof. For each point x of $\mathbb{R}^n \setminus E_\tau$ there is a radius $r_x \le c_0$ such that

$$h_{\rho,w}(x, r_x) < \tau\,\nu\big(\overline{B}(x, r_x)\big)\,. \tag{3.3}$$

Hence

$$\mathbb{R}^n \setminus E_\tau \subset \bigcup_{x\in\mathbb{R}^n\setminus E_\tau} \overline{B}(x, r_x) \quad\text{where}\quad \sup_{x\in\mathbb{R}^n\setminus E_\tau} r_x \le c_0 < \infty\,.$$

By the Besicovitch covering theorem [Z, Theorem 1.3.5 p. 9] we find a constant $c(n) < \infty$ and a new covering of $\mathbb{R}^n \setminus E_\tau$ such that

$$\sum_{j=1}^{\infty} \chi_{\overline{B}(x_j, r_j)}(x) \le c(n) \quad\text{and}\quad \mathbb{R}^n \setminus E_\tau \subset \bigcup_{j=1}^{\infty} \overline{B}(x_j, r_j)\,,$$

where x_j and $r_j = r_{x_j}$ satisfy (3.3). Hence

$$\begin{aligned}\sum_{j=1}^{\infty} h_{\rho,w}(x_j, r_j) &\le \tau \sum_{j=1}^{\infty} \nu\big(\overline{B}(x_j, r_j)\big)\\ &\le c(n)\,\tau\,\nu(\mathbb{R}^n)\,,\end{aligned}$$

which establishes the result. □

We use the well known Caratheory construction to obtain weighted Hausdorff measures; for this construction see [Fe, §2.10 p. 169]. The construction is applied to the gauge

$$h_{\alpha,w}(x,t) = t^{\alpha} \int_{\overline{B}(x,t)} w\,dy\,,$$

where $\alpha \in \mathbb{R}$ and w is a weight. Let $H^{\delta}_{\alpha,w}$ be the corresponding content and let E be a set in $\mathbb{R}^n$. Define

$$\mathcal{H}_{\alpha,w}(E) = \lim_{\delta\to 0} H^{\delta}_{\alpha,w}(E) = \sup_{\delta>0} H^{\delta}_{\alpha,w}(E)\,. \tag{3.4}$$

The quantity $\mathcal{H}_{\alpha,w}$ is called the *α-dimensional weighted spherical Hausdorff measure* on $\mathbb{R}^n$ or, for short, *the weighted Hausdorff measure.* Because for a fixed $E \subset \mathbb{R}^n$ the content $H^{\delta}_{\alpha,w}(E)$ increases as δ decreases, the limit in (3.4) exists but may be infinite.

3.5. *Remark.* It is easy to check that $\mathcal{H}_{\alpha,w}$ is a metric outer measure and so every open set is measurable. In fact, $\mathcal{H}_{\alpha,w}$ is a Borel regular measure on $\mathbb{R}^n$, see [Fe, §2.10 p. 169].

For a set E in $\mathbb{R}^n$, the number

$$\dim_w(E) = n + \inf\{\alpha \in \mathbb{R} : \mathcal{H}_{\alpha,w}(E) = 0\}$$

is called the *weighted Hausdorff dimension* of E.

3.6. *Remarks.* (*a*) For $E \subset \mathbb{R}^n$, $\dim_w(E)$ is uniquely determined by the properties:

$\mathcal{H}_{\alpha,w}(E) = 0$ if $\dim_w(E) < n + \alpha$,
$\mathcal{H}_{\alpha,w}(E) = \infty$ if $n + \alpha < \dim_w(E)$.

(*b*) For all subsets E of $\mathbb{R}^n$ we have $\dim_w(E) \leq n$ and, in particular, $\dim_w(\mathbb{R}^n) \leq n$. On the other hand, note that $\dim_w(E)$ can be negative, see Theorem 3.10 and Example 3.12.(c) below.

(*c*) In case $w = 1$ the gauge $h_{\alpha,w}$ has a form

$$h_{\alpha,1}(x,t) = t^\alpha \int_{B(x,t)} 1\, dy = \tfrac{\omega}{n} t^{\alpha+n}.$$

Here ω stands for the area of the unit sphere in $\mathbb{R}^n$. The ordinary (spherical) Hausdorff dimension of a set E in $\mathbb{R}^n$ is defined as

$$\dim(E) = \inf\Big\{\beta > 0 : \lim_{\delta\to 0}\big\{\sum t_j^\beta : E \subset \cup\overline{B}(x_j,t_j),\, t_j < \delta\big\} = 0\Big\},$$

see [Fa, §1.2 Hausdorff measure p. 7]. The ordinary Hausdorff dimension of a set coincides with the weighted Hausdorff dimension of the set provided that $w = 1$.

For $0 < d \leq \infty$ we let

$$\mathcal{P}(E,d) = \bigcup_{y\in E} B(y,d)$$

be the *d-inflation* of a set E in $\mathbb{R}^n$.

Next we derive a lower bound for the weighted Hausdorff dimension of a set by means of the ordinary Hausdorff dimension. We point out that the lower bound is always nonpositive. Recall that $\dim_w(\mathbb{R}^n) \leq n$ and $\dim_w(\emptyset) = -\infty$. Our approach to the lower bound is based on the comparison of $\dim_w$ and dim. First we investigate what is the connection between the two different weighted Hausdorff dimensions of a set.

3.7. Theorem. *Let E be a nonempty subset of $\mathbb{R}^n$ and let $1 < q < \infty$. Assume that v is a doubling weight and that w is a function such that wv is a weight with $\int_{\mathcal{P}(E,d_0)} w(y)^{\frac{1}{1-q}} v(y)\,dy < \infty$ for some $0 < d_0 < \infty$. Moreover, suppose that $-\infty < \beta \le \dim_v(E)$, $\lambda \ge 0$, and $\dim_{wv}(E) \le n - \lambda$. Then*

$$\lambda \le q(n-\beta)\,, \ \dim_v(E) \le n - \tfrac{\lambda}{q}\,, \quad \text{and} \quad n + q(\beta - n) \le \dim_{wv}(E)\,.$$

Proof. Let $\alpha > -\lambda$. From the fact that $0 < w(x) < \infty$ a.e. in $\mathbb{R}^n$ and from the Hölder inequality we obtain

(3.8)
$$h_{\frac{\alpha}{q},v}(x,r) = r^{\frac{\alpha}{q}} \int_{\overline{B}(x,r)} v\,dy \le \Big(r^\alpha \int_{\overline{B}(x,r)} wv\,dy \Big)^{\frac{1}{q}} \Big(\int_{\overline{B}(x,r)} w^{\frac{1}{1-q}} v\,dy \Big)^{\frac{q-1}{q}}$$

for all closed balls $\overline{B}(x,r)$ in $\mathcal{P}(E,d_0)$. Let $0 < d < d_0/3$. Since $\dim_w(E) < n + \alpha$, we have $H^d_{\alpha,wv}(E) \le \mathcal{H}_{\alpha,wv}(E) = 0$. Thus we may choose a covering $\{\overline{B}(x_j,r_j)\}_{j\in\mathbb{N}}$ of E such that $r_j < d$ for all $j \in \mathbb{N}$ and

$$\sum_{j\in\mathbb{N}} h_{\alpha,wv}(x_j,r_j) < H^d_{\alpha,wv}(E) + 1 = 1\,. \tag{3.9}$$

We may assume that $\overline{B}(x_j,r_j) \cap E$ is nonempty for all $j \in \mathbb{N}$; thus $\overline{B}(x_j,r_j) \subset \mathcal{P}(E,3d)$.

Employing a standard covering theorem [Z, Theorem 1.3.1 p. 7] we find a subfamily $\{\overline{B}(x_j,r_j)\}_{j\in J}$ of $\{\overline{B}(x_j,r_j)\}_{j\in\mathbb{N}}$ such that

$$\bigcup_{j\in\mathbb{N}} \overline{B}(x_j,r_j) \subset \bigcup_{j\in J} \overline{B}(x_j,5r_j)\,, \quad J \subset \mathbb{N}\,,$$

and that the balls in $\{\overline{B}(x_j,r_j)\}_{j\in J}$ are pairwise disjoint. Hence using the doubling condition, (3.8), the Hölder inequality, and (3.9) we arrive at

$$\begin{aligned}
\sum_{j\in J} h_{\frac{\alpha}{q},v}(x_j,5r_j) &= 5^{\frac{\alpha}{q}} \sum_{j\in J} r_j^{\frac{\alpha}{q}} \int_{\overline{B}(x_j,5r_j)} v\,dy \\
&\le 5^{\frac{\alpha}{q}} C_D(v)^3 \sum_{j\in J} r_j^{\frac{\alpha}{q}} \int_{\overline{B}(x_j,r_j)} v\,dy \\
&\le 5^{\frac{\alpha}{q}} C_D(v)^3 \sum_{j\in J} \Big(r_j^\alpha \int_{\overline{B}(x_j,r_j)} wv\,dy \Big)^{\frac{1}{q}} \Big(\int_{\overline{B}(x_j,r_j)} w^{\frac{1}{1-q}} v\,dy \Big)^{\frac{q-1}{q}} \\
&\le 5^{\frac{\alpha}{q}} C_D(v)^3 \Big(\sum_{j\in J} r_j^\alpha \int_{\overline{B}(x_j,r_j)} wv\,dy \Big)^{\frac{1}{q}} \Big(\sum_{j\in J} \int_{\overline{B}(x_j,r_j)} w^{\frac{1}{1-q}} v\,dy \Big)^{\frac{q-1}{q}} \\
&\le 5^{\frac{\alpha}{q}} C_D(v)^3 \Big(\int_{\mathcal{P}(E,3d)} w^{\frac{1}{1-q}} v\,dy \Big)^{\frac{q-1}{q}}\,.
\end{aligned}$$

Since $E \subset \bigcup_{j\in J} \overline{B}(x_j, 5r_j)$ and $\mathcal{P}(E, 3d) \subset \mathcal{P}(E, d_0)$ whenever $0 < d < d_0/3$, we have

$$H^{5d}_{\frac{\alpha}{q},v}(E) \le 5^{\frac{\alpha}{q}} C_D(v)^3 \left(\int_{\mathcal{P}(E,d_0)} w^{\frac{1}{1-q}} v\, dy \right)^{\frac{q-1}{q}} .$$

By letting $d \to 0$ we find

$$\mathcal{H}_{\frac{\alpha}{q},v}(E) \le 5^{\frac{\alpha}{q}} C_D(v)^3 \left(\int_{\mathcal{P}(E,d_0)} w^{\frac{1}{1-q}} v\, dy \right)^{\frac{q-1}{q}} < \infty .$$

Now it follows that $\dim_v(E) \le n - \frac{\lambda}{q}$ and, in particular, $\lambda \le q(n-\beta)$. Consequently $n + q(\beta - n) \le \dim_{wv}(E)$, which completes the proof of the theorem. □

3.10. Theorem. *Let E be a nonempty subset of $\mathbb{R}^n$ and let $1 < q < \infty$. Suppose that w is a weight with $\int_{\mathcal{P}(E,d_0)} w^{\frac{1}{1-q}}\, dy < \infty$ for some $0 < d_0 < \infty$. If $\lambda \ge 0$ and $\dim_w(E) \le n - \lambda$, then $\lambda \le nq$ and $\dim(E) \le n - \frac{\lambda}{q}$.*

In particular, $n - nq \le \dim_w(E) \le n$.

Proof. The result follows from Theorem 3.7 with $v = 1$ and $\beta = 0$. □

3.11. Corollary. *Let w be in A_∞ and $q = \inf\{t > 1 : w \in A_t\}$. Suppose that E is a nonempty set in $\mathbb{R}^n$. If $\lambda \ge 0$ and $\dim_w(E) \le n - \lambda$, then $\lambda \le nq$ and $\dim(E) \le n - \frac{\lambda}{q}$. Moreover, $n - nq \le \dim_w(E) \le n$.*

Proof. The conclusion is obtained from Theorem 3.10. □

3.12. Examples. (a) Such nontrivial weights will be constructed such that it is possible to calculate weighted Hausdorff dimensions of sets in $\mathbb{R}^n$ in terms of ordinary Hausdorff dimensions.

To this end, let k equal to $1, 2, \ldots, n$, and let $\mathbb{R}^{n-k}$ be a subspace of $\mathbb{R}^n$, $\mathbb{R}^{n-k} = \{x \in \mathbb{R}^n : x_j = 0 \text{ for all } j = n-k+1, \ldots, n\}$. Let λ be in (k, ∞). The weight w is defined by $w(x) = \prod_{j=n-k+1}^{n} |x_j|^{\frac{\lambda}{k}-1}$ where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Now, if E is any subset of $\mathbb{R}^{n-k}$, then $\dim_w(E) = \dim(E) - \lambda + k$.

To prove the formula, recall that w belongs to the A_∞ class since $-1 < \frac{\lambda}{k} - 1 < p - 1$ whenever $p > \frac{\lambda}{k}$, and so $w \in A_p \subset A_\infty$ for some $p > \frac{\lambda}{k}$, see Remarks 2.7.(d). Furthermore, we recall that $\dim_w$ (and dim) does not change if we replace a ball covering by a cubic covering, since w is a doubling weight. So, let us consider a cube $\overline{Q}(x, r) = \{y \in \mathbb{R}^n : |x_j - y_j| \le \frac{r}{2} \text{ for } j = 1, \ldots, n\}$ centered at a point $x = (x_1, x_2, \ldots, x_{n-k}, 0, 0, \ldots, 0) \in \mathbb{R}^{n-k}$ and a gauge $h_{\alpha,w}$

for $\alpha > -\lambda - n + k$. In this case we obtain

$$\begin{aligned} h_{\alpha,w}(x,r) &= r^{\alpha} \int_{\overline{Q}(x,r)} |y_{n-k+1}|^{\frac{\lambda}{k}-1} \cdots |y_{n-1}|^{\frac{\lambda}{k}-1} |y_n|^{\frac{\lambda}{k}-1}\, dy \\ &= r^{\alpha+n-k} \int_{-\frac{r}{2}}^{\frac{r}{2}} |y_{n-k+1}|^{\frac{\lambda}{k}-1}\, dy_{n-k+1} \cdots \int_{-\frac{r}{2}}^{\frac{r}{2}} |y_n|^{\frac{\lambda}{k}-1}\, dy_n \\ &= 2^{k-\lambda}\, \lambda^{-k}\, k^k\, r^{\alpha+n+\lambda-k} \\ &= 2^{k-\lambda}\, \lambda^{-k}\, k^k\, h_{\alpha+\lambda-k,1}(x,r)\,. \end{aligned}$$

That is, if $x \in \mathbb{R}^{n-k}$ and $c(\lambda,k) = 2^{k-\lambda}\, \lambda^{-k}\, k^k$, then

$$h_{\alpha,w}(x,r) = c(\lambda,k)\, h_{\alpha+\lambda-k,1}(x,r)\,. \tag{3.13}$$

Now we are in a position to prove that if E is a subset of $\mathbb{R}^{n-k}$, then

$$\frac{1}{c(\lambda,k)}\, H^d_{\alpha,w}(E) \le H^d_{\alpha+\lambda-k,\,1}(E) \le \frac{C_D}{c(\lambda,k)}\, H^d_{\alpha,w}(E) \tag{3.14}$$

for $\alpha > -\lambda - n + k$. To this purpose, let $\{\overline{Q}(x_j,r_j)\}$ be a covering of E with $r_j < d$ for all $j = 1,2,\ldots$. With no loss of generality we may suppose that $\overline{Q}(x_j,r_j) \cap E$ is nonempty for all $j = 1,2,3,\ldots$. By the projection of the center x_j of $\overline{Q}(x_j,r_j)$ to $\mathbb{R}^{n-k}$ we are able to find a cube $\overline{Q}(x'_j,r_j)$ such that $x'_j \in \mathbb{R}^{n-k}$ and $r_j < d$. In particular, $\overline{Q}(x'_j,r_j) \cap E = \overline{Q}(x_j,r_j) \cap E$. Thus we have a new covering of E obtained from the covering $\{\overline{Q}(x_j,r_j)\}$ such that $E \subset \bigcup_j \overline{Q}(x'_j,r_j)$ and $r_j < d$ for all $j = 1,2,\ldots$.

For $\alpha > -\lambda - n + k$ from (3.13) we infer that

$$\begin{aligned} h_{\alpha,w}(x'_j,r_j) &= c(\lambda,k)\, h_{\alpha+\lambda-k,1}(x'_j,r_j) \\ &= c(\lambda,k)\, r_j^{\alpha+\lambda-k+n} \\ &= c(\lambda,k)\, h_{\alpha+\lambda-k,1}(x_j,r_j)\,. \end{aligned}$$

Hence

$$\sum_{j=1}^{\infty} h_{\alpha,w}(x'_j,r_j) = c(\lambda,k) \sum_{j=1}^{\infty} h_{\alpha+\lambda-k,1}(x_j,r_j)$$

and passing to the infima, the left part of (3.14) follows.

Using again (3.13), the fact that $\overline{Q}(x'_j, r_j) \subset \overline{Q}(x_j, 2r_j)$, and the doubling condition we obtain for $\alpha > -\lambda - n + k$

$$\begin{aligned} h_{\alpha+\lambda-k,1}(x'_j, r_j) &= \frac{1}{c(\lambda, k)}\, h_{\alpha,w}(x'_j, r_j) \\ &= \frac{1}{c(\lambda, k)}\, r_j^\alpha \int_{\overline{Q}(x'_j, r_j)} w\, dy \\ &\leq \frac{1}{c(\lambda, k)}\, r_j^\alpha \int_{\overline{Q}(x_j, 2r_j)} w\, dy \\ &\leq \frac{C_D}{c(\lambda, k)}\, r_j^\alpha \int_{\overline{Q}(x_j, r_j)} w\, dy \\ &= \frac{C_D}{c(\lambda, k)}\, h_{\alpha,w}(x_j, r_j)\,. \end{aligned}$$

Consequently,

$$\sum_{j=1}^{\infty} h_{\alpha+\lambda-k,1}(x'_j, r_j) \leq \frac{C_D}{c(\lambda, k)} \sum_{j=1}^{\infty} h_{\alpha,w}(x_j, r_j)$$

and by taking the infimum over all coverings, we arrive at the right part of (3.14).

Finally, by letting $d \to 0$ in (3.14) we find for $E \subset \mathbb{R}^{n-k}$

$$c_1(\lambda, k)\, \mathcal{H}_{\alpha,w}(E) \leq \mathcal{H}_{\alpha+\lambda-k,1}(E) \leq c_1(\lambda, k)\, C_D \mathcal{H}_{\alpha,w}(E)\,. \tag{3.15}$$

Now, if $\alpha + \lambda - k + n = \dim(E)$, then $\alpha = \dim(E) - \lambda + k - n$. Hence (3.15) yields $\dim_w(E) = \dim(E) - \lambda + k$ for λ in (k, ∞) and $k = 1, 2, \ldots, n$, as desired.

(*b*) We show that in Corollary 3.11 the upper bound for the ordinary Hausdorff dimension of a set is sharp whenever λ is in $(1, \infty)$. Indeed, if k equals to $1, 2, \ldots, n$, then for all $\lambda \in (k, \infty)$ there exist a set E in $\mathbb{R}^n$ and a weight w satisfying the assumptions of Corollary 3.11 such that $\dim_w(E) = n - \lambda$, $0 < \lambda \leq nq$, and $\dim(E) = n - \frac{\lambda}{q}$. Here $q = \frac{\lambda}{k} = \inf\{p > 1 : w \in A_p\}$.

To this end, let $\mathbb{R}^{n-k}$ and the weight w be as in (a) for $k = 1, 2, \ldots, n$ and $\lambda \in (k, \infty)$. Now, by (a) we have that w is in A_∞ and $q = \frac{\lambda}{k}$; clearly λ is in $(0, \frac{n\lambda}{k}]$. Let E be a unit cube in $\mathbb{R}^{n-k}$. It is known that $\dim(E) = n - k$. From (a) it follows that $\dim_w(E) = n - k - \lambda + k = n - \lambda$, and moreover, $\dim(E) = n - k = n - \frac{\lambda}{q}$. To complete the example, choose $k = 1$.

(*c*) It is shown that $\dim_w$ can be strictly negative. Again, let $\mathbb{R}^{n-k}$ and the weight w be as in (a) when $k = 1$. Now $q = \frac{\lambda}{1}$. Furthermore, let E be a cube in $\mathbb{R}^{n-k}$. If λ is in (n, ∞), then λ is always in $(n, n\lambda]$ and $n - n\lambda < n - \lambda = \dim_w(E) < 0$. However, $\dim(E) = n - 1$.

(d) In Corollary 3.11 the lower bound for the weighted Hausdorff dimension of a set is sharp if λ belongs to (n,∞). Of course, the corresponding upper bound for the ordinary Hausdorff dimension of the set is zero, and in particular, the upper bound is also simultaneously sharp. In other words, for each λ in (n,∞) there exist a set E and a weight w in the A_∞ class such that $\dim_w(E) = n - nq = n - \lambda$, $q = \frac{\lambda}{n} = \inf\{p > 1 : w \in A_p\}$, and $\dim(E) = 0$.

Let $\lambda \in (n,\infty)$. Choose the set E to be $\{0\}$ and the weight to be defined by $w(x) = |x_1|^{\frac{\lambda}{n}-1}|x_2|^{\frac{\lambda}{n}-1}\cdots|x_n|^{\frac{\lambda}{n}-1}$ for $x = (x_1,\dots,x_n)$ in $\mathbb{R}^n$. According to (a) we arrive at $\dim_w(E) = \dim(E) - \lambda + n$ with $q = \frac{\lambda}{n}$ whenever λ is in (n,∞). This yields

$$n - nq = n - n\frac{\lambda}{n} = n - \lambda = \dim_w(E),$$

with $\dim(E) = 0$, as desired.

As a consequence, the weighted Hausdorff dimension of the origin can be as small as we please although the ordinary Hausdorff dimension of the origin is zero.

(e) The next counterexample shows that the following conclusion is false: if $0 < \lambda \le nq$ and $\dim(E) \le n - \frac{\lambda}{q}$, then $\dim_w(E) \le n - \lambda$, where E, n, q, and w are as in Corollary 3.11.

Let $w(x) = |x_2|$ for $x \in \mathbb{R}^2$. Hence $q = 2$. Choose a set E to be $\frac{1}{4}$-Cantor set, see [Fa, §1.5 p 14], lying in the coordinate axis $\{x \in \mathbb{R}^2 : x = (x_1, 0),\ x_1 \in \mathbb{R}\}$ with $\dim(E) = \frac{1}{2}$. Let $\lambda = 3$. Now $w \in A_3 \subset A_\infty$ and $\dim(E) = \frac{1}{2} = n - \frac{\lambda}{q} = 2 - \frac{3}{2}$. Moreover, from (a) for $w(x) = |x_2|^{\frac{2}{1}-1}$ we obtain $\dim_w(E) = \dim(E) - 1$. Therefore

$$\dim_w(E) = \tfrac{1}{2} - 1 = -\tfrac{1}{2} > -1 = 2 - 3 = n - \lambda,$$

as required.

Next we are going to compare the content $H^{\delta}_{\rho,w}$ with the content $H^{d}_{\rho,w}$ for $\delta < d \le \infty$. Trivially the latter is less than the former. The question is: when can we obtain a reverse inequality?

Let E be a nonempty subset of $\mathbb{R}^n$. We say that a gauge $h_{\rho,w}$ is *linearly increasing on* E if there exists a pair of constants c_0 and d, $c_0 \ge 1$, $0 < d \le \infty$, such that

$$h_{\rho,w}(x,t_1) \le c_0\, h_{\rho,w}(x,t_2) \tag{3.16}$$

for all $x \in E$ and all $0 < t_1 \le t_2 < d$.

For the sake of brevity, the gauge $h_{\rho,w}$ is called *linearly increasing on* E, or *linearly increasing on* E *with constants* c_0 *and* d, if (3.16) holds for all $x \in E$ and all $0 < t_1 \le t_2 < d$.

Furthermore, we say that the gauge $h_{\rho,w}$ is *linearly increasing* if it is linearly increasing on $\mathbb{R}^n$ with c_0 and ∞.

3.17. Examples. (*a*) Let $w(x) = |x|^\gamma$. We recall that $w \in A_p$ whenever $-n < \gamma < np - n$, see Remarks 2.7.(c). A straightforward calculation shows that $|\cdot|^\gamma$ is a doubling weight with $C_D = c_1(n,\gamma)$ whenever $\gamma > -n$.

We will verify that there exists a constant $c(n,\gamma)$ such that if $\gamma > -n$ and $\alpha \geq \max\{-n, -\gamma - n\}$ then

$$t_1^\alpha \int_{\overline{B}(x,t_1)} |y|^\gamma \, dy \leq c(n,\gamma)\, t_2^\alpha \int_{\overline{B}(x,t_2)} |y|^\gamma \, dy \tag{3.18}$$

for all $x \in \mathbb{R}^n$ and all $0 < t_1 \leq t_2 < \infty$. Consequently, the gauge $h_{\alpha,w}$ is linearly increasing.

To this end, suppose first that $0 < t_1 \leq t_2 \leq |x|/2$ and $-n < \gamma < 0$. Now

$$\begin{aligned} h_{\alpha,w}(x,t_1) &= t_1^\alpha \int_{\overline{B}(x,t_1)} |y|^\gamma \, dy \leq t_1^\alpha \left(\tfrac{|x|}{2}\right)^\gamma \int_{\overline{B}(x,t_1)} dy \\ &= \tfrac{\omega}{n}\, t_1^{\alpha+n} \left(\tfrac{|x|}{2}\right)^\gamma \leq \tfrac{\omega}{n}\, t_2^{\alpha+n} \left(\tfrac{3|x|}{2}\right)^\gamma 3^{-\gamma} \\ &\leq 3^{-\gamma}\, t_2^\alpha \int_{\overline{B}(x,t_2)} |y|^\gamma \, dy = 3^{-\gamma}\, h_{\alpha,w}(x,t_2) \end{aligned}$$

provided that $\alpha + n \geq 0$. Applying a similar method in case $\gamma \geq 0$ we see

$$h_{\alpha,w}(x,t_1) \leq 3^{|\gamma|}\, h_{\alpha,w}(x,t_2) \tag{3.19}$$

where $\alpha + n \geq 0$ and $0 < t_1 \leq t_2 \leq |x|/2$.

Secondly, suppose that $|x|/2 \leq t_1 \leq t_2 < \infty$ and $-n < \gamma < \infty$. Using the doubling condition twice we obtain

$$\begin{aligned} h_{\alpha,w}(x,t_1) &= t_1^\alpha \int_{\overline{B}(x,t_1)} |y|^\gamma \, dy \leq t_1^\alpha \int_{\overline{B}(0,3t_1)} |y|^\gamma \, dy \\ &= 3^{\gamma+n} \tfrac{\omega}{n(\gamma+n)}\, t_1^{\alpha+\gamma+n} \leq 3^{\gamma+n} \tfrac{\omega}{n(\gamma+n)}\, t_2^{\alpha+\gamma+n} \\ &= 3^{\gamma+n}\, t_2^\alpha \int_{\overline{B}(0,t_2)} |y|^\gamma \, dy \leq 3^{\gamma+n}\, t_2^\alpha \int_{\overline{B}(x,3t_2)} |y|^\gamma \, dy \\ &\leq 3^{\gamma+n}\, C_D^2\, t_2^\alpha \int_{\overline{B}(x,t_2)} |y|^\gamma \, dy = c_1(n,\gamma)\, h_{\alpha,w}(x,t_2) \end{aligned} \tag{3.20}$$

whenever $\alpha + \gamma + n \geq 0$.

Finally, suppose that $0 < t_1 \leq |x|/2 \leq t_2 < \infty$ and $-n < \gamma < \infty$. It follows from (3.19) and (3.20) that

$$h_{\alpha,w}(x,t_1) \leq c_2(n,\gamma)\, h_{\alpha,w}(x,t_2) \tag{3.21}$$

for $\alpha \geq \max\{-n, -\gamma - n\}$. Thus invoking (3.19), (3.20), and (3.21) we arrive at (3.18).

(*b*) Choose $\alpha = -n-1$ and $\gamma = -n+1$. Then

$$\alpha + n = -n - 1 + n = -1 < 0 \text{ and}$$
$$\alpha + \gamma + n = -n - 1 - n + 1 + n = -n < 0\,.$$

It follows from (3.19), (3.20), and (3.21) that we can find a constant $c(n)$ such that

$$t_1^{-n-1} \int_{\overline{B}(x,t_1)} |y|^{-n+1}\, dy \geq c(n)\, t_2^{-n-1} \int_{\overline{B}(x,t_2)} |y|^{-n+1}\, dy$$

whenever $0 < t_1 \leq t_2 < \infty$ and $x \in \mathbb{R}^n$. Since for each $\delta < 0$ we have $t_1^\delta > t_2^\delta$ if $t_1 < t_2$, for all $\alpha < -n-1$ the corresponding gauge $h_{\alpha,w}$ is not linearly increasing. In spite of that the weight $|\cdot|^{-n+1}$ is in A_p for every p with $1 \leq p < \infty$. Therefore neither the A_p condition nor the doubling condition implies the linearly increasing property (3.16) of a gauge.

(*c*) It is easy to see that if a gauge $h_{\alpha_1,w}$ is linearly increasing on a set in $\mathbb{R}^n$, then for every $\alpha_2 \geq \alpha_1$ a gauge $h_{\alpha_2,w}$ is also linearly increasing on the set.

3.22. Lemma. *Assume that E is a set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Let w be a doubling weight and suppose that $h_{\rho,w}$ is linearly increasing on E with constants c_0 and d. If $0 < \frac{r}{4} \leq \delta$ and $\delta < d \leq \infty$, then*

$$H^\delta_{\rho,w}\big(E \cap \overline{B}(x,r)\big) \leq c\, H^d_{\rho,w}\big(E \cap \overline{B}(x,r)\big)\,,$$

where $c = 17^n\, {C_D}^6\, c_0$.

Proof. We may suppose that $E \cap \overline{B}(x,r)$ is nonempty. Let $\{\overline{B}(x_j,r_j)\}_j^\infty$ be a covering of $E \cap \overline{B}(x,r)$ with $r_j < d$ for all $j = 1,2,3,\ldots$. If $r_j < \delta$ for all $j = 1,2,3,\ldots$, then there is nothing to prove.

Assume that $\delta \leq r_k < d$ for some k and $\overline{B}(x_k,r_k) \cap E \cap \overline{B}(x,r)$ is not empty. Similarly as in the proof of the Besicovitch covering theorem (cf. [Z, Theorem 1.3.5 p. 9]), we may find points $z_j \in \overline{B}(x,r)$, $j = 1,\ldots,m$, such that the balls $\{\overline{B}(z_j,\frac{r}{8})\}$ cover $E \cap \overline{B}(x,r)$ and the balls $\{\overline{B}(z_j,\frac{r}{16})\}$ are mutually disjoint and that $\overline{B}(z_j,\frac{r}{8}) \cap E \cap \overline{B}(x,r)$ is nonempty for all $j = 1,2,\ldots,m$. Comparing Lebesgue measures of $\cup_{j=1}^m \overline{B}(z_j,\frac{r}{16})$ and $\overline{B}(z_j,\frac{17r}{16})$ we infer that $m \leq 17^n$.

Now fix an integer $j = 1,\ldots,m$ and pick a point z' from $\overline{B}(z_j,\frac{r}{2}) \cap E \cap \overline{B}(x,r)$. From the doubling condition and from (3.16) we infer that

$$\begin{aligned} h_{\rho,w}(z_j,\tfrac{r}{8}) &= \rho(\tfrac{r}{8}) \int_{\overline{B}(z_j,\frac{r}{8})} w\, dy \leq \rho(\tfrac{r}{8}) \int_{\overline{B}(z',\frac{r}{4})} w\, dy \\ &\leq C_D\, \rho(\tfrac{r}{8}) \int_{\overline{B}(z',\frac{r}{8})} w\, dy \leq C_D\, c_0\, \rho(r_k) \int_{\overline{B}(z',r_k)} w\, dy\,. \end{aligned}$$

But $\overline{B}(z', r_k) \subset \overline{B}(x_k, 18r_k)$ and hence by a repeated use of the doubling condition we have

$$\begin{aligned} h_{\rho,w}(z_j, \tfrac{r}{8}) &\le C_D\, c_0\, \rho(r_k) \int_{\overline{B}(x_k,18r_k)} w\, dy \\ &\le {C_D}^6\, c_0\, \rho(r_k) \int_{\overline{B}(x_k,r_k)} w\, dy\,, \end{aligned}$$

in other words,

$${C_D}^6\, c_0\, h_{\rho,w}(x_k, r_k) \ge h_{\rho,w}(z_j, \tfrac{r}{8}) \quad \text{for } j = 1, \dots, m\,.$$

This with $E \cap \overline{B}(x,r) \subset \bigcup_{j=1}^m \overline{B}(z_j, \frac{r}{2})$ for $\frac{r}{8} < \frac{r}{4} \le \delta$ implies

$$\begin{aligned} 17^n\, {C_D}^6\, c_0 \sum_{j=1}^{\infty} h_{\rho,w}(x_j, r_j) &\ge 17^n\, {C_D}^6\, c_0\, h_{\rho,w}(x_k, r_k) \\ &\ge \sum_{j=1}^{m} h_{\rho,w}(z_j, \tfrac{r}{8}) \\ &\ge H^{\delta}_{\rho,w}\big(E \cap \overline{B}(x,r)\big)\,, \end{aligned}$$

which completes the proof. □

Let E be a set in $\mathbb{R}^n$ and let $h_{\alpha,w}$ be a gauge. If ν is an outer measure on $\mathbb{R}^n$, $t > 0$, and $\delta > 0$, then we write

$$R(E, \nu, t, \delta) = \big\{x \in E : \nu\big(E \cap \overline{B}(x,r)\big) \le t\, h_{\alpha,w}(x,r) \ \text{ for all } 0 < r \le \delta\big\}\,.$$

For $0 < \delta < 1$ we use the abbreviation

$$T(E,\delta) = R\big(E, H^{\delta}_{\alpha,w}, (1-\delta)2^{-|\alpha|}\, {C_D}^{-2}, 2\delta\big)\,.$$

3.23. Lemma. *Let $E \subset \mathbb{R}^n$ and $0 < \delta < 1$. If w is a doubling weight and $H^{\delta}_{\alpha,w}\big(T(E,\delta)\big) < \infty$, then $H^{\delta}_{\alpha,w}\big(T(E,\delta)\big) = 0$.*

Proof. Let $\{\overline{B}(x_j, r_j)\}$ be a covering of $T(E,\delta)$ such that $r_j < \delta$ and that a set $T(E,\delta) \cap \overline{B}(x_j, r_j)$ is nonempty for all $j = 1, 2, 3, \dots$. Now for each $j = 1, 2, 3, \dots$ we can pick a point $y_j \in T(E,\delta) \cap \overline{B}(x_j, r_j)$ and therefore $\overline{B}(x_j, r_j) \subset$

$\overline{B}(y_j, 2r_j) \subset \overline{B}(x_j, 4r_j)$. Using the doubling condition we obtain

$$\begin{aligned}
H^{\delta}_{\alpha,w}\big(T(E,\delta)\big) &\le \sum_{j=1}^{\infty} H^{\delta}_{\alpha,w}\big(T(E,\delta)\cap \overline{B}(x_j,r_j)\big)\\
&\le \sum_{j=1}^{\infty} H^{\delta}_{\alpha,w}\big(T(E,\delta)\cap \overline{B}(y_j,2r_j)\big)\\
&\le (1-\delta)\,2^{-|\alpha|}\,C_D{}^{-2}\sum_{j=1}^{\infty}(2r_j)^{\alpha}\int_{\overline{B}(y_j,2r_j)} w\,dy\\
&\le (1-\delta)\,2^{-|\alpha|}\,C_D{}^{-2}\,2^{\alpha}\sum_{j=1}^{\infty} r_j^{\alpha}\int_{\overline{B}(x_j,4r_j)} w\,dy\\
&\le (1-\delta)\,2^{-|\alpha|}\,C_D{}^{-2}\,2^{|\alpha|}\,C_D{}^{2}\sum_{j=1}^{\infty} r_j^{\alpha}\int_{\overline{B}(x_j,r_j)} w\,dy\\
&= (1-\delta)\sum_{j=1}^{\infty} h_{\alpha,w}(x_j,r_j)\,.
\end{aligned}$$

Since $0<\delta<1$, from $H^{\delta}_{\alpha,w}\big(T(E,\delta)\big)<\infty$ it now follows that $H^{\delta}_{\alpha,w}\big(T(E,\delta)\big)=0$, as required. □

Let $\alpha\in\mathbb{R}$ and $0<\delta\le\infty$. The *upper* (δ-α, w)*-content density* of a subset F of $\mathbb{R}^n$ at a point $x\in\mathbb{R}^n$ is

$$\Phi^{\delta *}_{\alpha,w}(x,F)=\limsup_{r\to 0}\frac{H^{\delta}_{\alpha,w}\big(F\cap\overline{B}(x,r)\big)}{h_{\alpha,w}(x,r)}\,.$$

The following theorem generalizes [Fe, §2.10.19(2) p. 181]. Roughly speaking, it says that for a linearly increasing gauge on a set E with constants c_0 and δ the corresponding weighted Hausdorff measure of the set of points x in E with the property $\Phi^{\delta *}_{\alpha,w}(x,E)=0$ is zero.

3.24. Theorem. *Let w be a doubling weight. Suppose that a gauge $h_{\alpha,w}$ is linearly increasing on a subset E of $\mathbb{R}^n$ with constants c_0 and $\mathbf{d}$. If $H^{\delta}_{\alpha,w}(E)<\infty$ for all $0<\delta<1$, then*

$$\mathcal{H}_{\alpha,w}\big(E\cap\{x\in\mathbb{R}^n : 0\le\Phi^{d*}_{\alpha,w}(x,E)<c^{-1}\,2^{-|\alpha|}\,C_D{}^{-2}\}\big)=0\,,$$

where $c=17^n\,C_D{}^6\,c_0$.

Proof. First we verify that

$$F_j=R\big(E,H^{\mathbf{d}}_{\alpha,w},(1-\tfrac{1}{j})\,c^{-1}\,2^{-|\alpha|}\,C_D{}^{-2},\tfrac{2}{j}\big)\subset T(E,\delta) \tag{3.25}$$

for $\delta < \min\{d, \frac{1}{j}\}$, $j = 1, 2, 3, \ldots$. To this end, let

$x \in F_j =$
$\{z \in E : H^d_{\alpha,w}(E \cap \overline{B}(z,r)) \le (1 - \frac{1}{j})\, c^{-1} 2^{-|\alpha|} C_D{}^{-2}\, h_{\alpha,w}(z,r) \text{ for all } 0 < r \le \frac{2}{j}\}$.

By Lemma 3.22 and by the preceding line we see that

$$\begin{aligned} H^\delta_{\alpha,w}(E \cap \overline{B}(x,r)) &\le c\, H^d_{\alpha,w}(E \cap \overline{B}(x,r)) \\ &\le c(1 - \tfrac{1}{j})\, c^{-1}\, 2^{-|\alpha|}\, C_D{}^{-2}\, h_{\alpha,w}(x,r) \\ &\le (1 - \delta)\, 2^{-|\alpha|}\, C_D{}^{-2} h_{\alpha,w}(x,r) \end{aligned}$$

for all $0 < r < 2\delta$ provided that $\delta < \min\{d, \frac{1}{j}\}$. Hence $x \in T(E, \delta)$ and (3.25) follows.

Next we prove that

$$\mathcal{H}_{\alpha,w}(F_j) = 0 \quad \text{for } j = 1, 2, 3, \ldots . \tag{3.26}$$

From (3.25) and from the fact that $H^\delta_{\alpha,w}(E) < \infty$ for all $0 < \delta < 1$ together with Lemma 3.23 it follows for $j = 1, 2, 3, \ldots$ that $H^\delta_{\alpha,w}(F_j) = 0$ whenever $\delta < \min\{d, \frac{1}{j}\}$; consequently

$$\mathcal{H}_{\alpha,w}(F_j) = \sup_{\delta > 0} H^\delta_{\alpha,w}(F_j) = \sup_{\delta > 0} 0 = 0\,,$$

as required.

Finally, write

$$F = E \cap \{x \in \mathbb{R}^n : 0 \le \Phi^{d^*}_{\alpha,w}(x, E) < c^{-1}\, 2^{-|\alpha|}\, C_D{}^{-2}\}\,;$$

now

$$F = \bigcup_{j=1}^{\infty} F_j$$

and hence by (3.26) we have

$$\mathcal{H}_{\alpha,w}(F) \le \sum_{j=1}^{\infty} \mathcal{H}_{\alpha,w}(F_j) = 0\,,$$

which finishes the proof. □

4. Capacity and measure densities

First we consider upper bounds for content densities in terms of capacity densities. Theorem 4.7 shows that if the capacity density of a set is zero at a given point, then the point also has zero content density. After that we prove a result concerning capacity densities and weighted Hausdorff dimensions; it demostrates the relationship between capacity density, linearly increasing gauges, and Hausdorff dimensions with respect to a given set. We close this chapter by proving upper bounds for capacity densities in terms of content densities.

Regarding the first subject of the chapter in the light of earlier results, see [Ne, Chapter V, sections 5 and 6], [Res, Theorem 3.3 p. 118], and [Ma].

For the next lemma, let $w \in A_p$ for some $1 < p < \infty$ and suppose that

(4.1) $g \in L^p_w$ is nonnegative with $\operatorname{spt} g \subset B(0, r_0)$ for some $r_0 > 0$;

(4.2) $f(x) = \int_{\mathbb{R}^n} \frac{g(y)\,dy}{|x-y|^{n-1}}$ for x in $\mathbb{R}^n$.

For $\tau > 0$ write

(4.3) $I(\tau, r_0) = \frac{C_A^{\frac{1}{p}}\,\omega(n-1)}{\tau\, n} \int_0^{r_0} \rho(t)^{\frac{1}{p}}\,dt + r_0^{1-n} \|g\|_{L^p_w} \Big(\int_{B(0,r_0)} w^{\frac{1}{1-p}}\,dy \Big)^{\frac{1}{p'}}$,
where $\frac{1}{p} + \frac{1}{p'} = 1$.

Recall that C_A is the A_p-constant of a weight.

4.4. Lemma. *Let $r_0 < d \le \infty$. Then we have the estimate*

$$H^d_{\rho,w}\big(\{x \in \mathbb{R}^n : f(x) > I(\tau, r_0)\}\big) \le c(n)\Big(\tau\,\|g\|_{L^p_w}\Big)^p .$$

Proof. Let g satisfy (4.1) and let f be as in (4.2). Using the well known formula

$$\int_{\mathbb{R}^n} |f|^p\,d\nu = p \int_0^\infty t^{p-1} \nu(\{|f| > t\})\,dt$$

for a nonnegative measure ν and for $p \ge 1$ with a change of variable we easily arrive at

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^n} \frac{g(y)\,dy}{|x-y|^{n-1}} = (n-1) \int_0^\infty \int_{B(x,t)} g\,dy\,t^{-n}\,dt \\ &= (n-1) \int_0^{r_0} \int_{B(x,t)} g\,dy\,t^{-n}\,dt + (n-1) \int_{r_0}^\infty \int_{B(x,t)} g\,dy\,t^{-n}\,dt . \end{aligned}$$

The Hölder inequality and the A_p condition yield

$$\begin{aligned} &\int_0^{r_0} \Big(\int_{B(x,t)} g\,dy \Big) t^{-n}\,dt \\ &\le \int_0^{r_0} \Big(\int_{B(x,t)} g^p w\,dy \Big)^{\frac{1}{p}} \Big(\int_{B(x,t)} w^{\frac{1}{1-p}}\,dy \Big)^{\frac{1}{p'}} t^{-n}\,dt \qquad (4.5) \\ &\le C_A^{\frac{1}{p}} \frac{\omega}{n} \int_0^{r_0} \Big(\frac{\int_{B(x,t)} g^p w\,dy}{\int_{B(x,t)} w\,dy} \Big)^{\frac{1}{p}} dt . \end{aligned}$$

On the other hand, a simple computation together with the Hölder inequality gives

$$\begin{aligned}(4.6)\qquad \int_{r_0}^{\infty}\int_{B(x,t)} g\,dy\,t^{-n}\,dt &\le \int_{r_0}^{\infty} t^{-n}\,dt \int_{B(0,r_0)} g\,dy \\ &\le \tfrac{{r_0}^{1-n}}{n-1}\|g\|_{L^p_w}\Big(\int_{B(0,r_0)} w^{\frac{1}{1-p}}\,dy\Big)^{\frac{1}{p'}}.\end{aligned}$$

For $\tau > 0$ write

$$E_\tau = \Big\{x \in \mathbb{R}^n : \int_{B(x,t)} g^p w\,dy \le \tau^{-p}\,h_{\rho,w}(x,t) \ \text{ for all } \ 0 < t < r_0\Big\};$$

here $h_{\rho,w}$ is the gauge of $H^d_{\rho,w}$. If $x \in E_\tau$, then (4.5) implies

$$\begin{aligned}\int_0^{r_0}\Big(\int_{B(x,t)} g\,dy\Big)t^{-n}\,dt &\le \frac{C_A^{\frac{1}{p}}\,\omega}{\tau\,n}\int_0^{r_0}\Big(\frac{h_{\rho,w}(x,t)}{\int_{B(x,t)} w\,dy}\Big)^{\frac{1}{p}}\,dt \\ &= \frac{C_A^{\frac{1}{p}}\,\omega}{\tau\,n}\int_0^{r_0}\rho(t)^{\frac{1}{p}}\,dt;\end{aligned}$$

and taking also (4.6) and (4.3) into account we have

$$f(x) \le I(\tau, r_0)$$

and thus

$$\{x \in \mathbb{R}^n : f(x) > I(\tau, r_0)\} \subset \mathbb{R}^n \setminus E_\tau.$$

Now from Lemma 3.2 for $r_0 < d \le \infty$ we obtain

$$H^d_{\rho,w}(\mathbb{R}^n \setminus E_\tau) \le c(n)\,\tau^p\|g\|^p_{L^p_w},$$

and the lemma follows. □

4.7. Theorem. *Let $1 < p < \infty$ and $w \in A_p$. Suppose that F is a closed set in $\mathbb{R}^n$ and that ρ is a positive measurable function on $(0,\infty)$ satisfying*

$$(4.8)\qquad \int_0^{2r}\rho(t)^{\frac{1}{p}}\,dt \le c_1\,\rho(r)^{\frac{1}{p}}\,r \quad \text{for all } r \in (0, c_0).$$

If $2r < d \le \infty$ and $x \in \mathbb{R}^n$, then

$$\frac{H^d_{\rho,w}(F \cap \overline{B}(x,r))}{h_{\rho,w}(x,r)} \le c_2\,\frac{\operatorname{cap}_{p,w}\big(F \cap \overline{B}(x,r), B(x,2r)\big)}{\operatorname{cap}_{p,w}\big(\overline{B}(x,r), B(x,2r)\big)}.$$

for all $r \in (0, c_0)$. *The constant* c_2 *depends only on* n, p, C_A, *and* c_1.

Proof. We may assume without loss of generality that $x = 0$ and that $F \cap \overline{B}(0,r) \neq \emptyset$ for $0 < r < c_0$. Moreover, we may suppose that

$$\frac{\operatorname{cap}_{p,w}\big(F \cap \overline{B}(0,r), B(0,2r)\big)}{r^{-p}\int_{B(0,r)} w\,dy} < K = 2^{-2p}\left(\tfrac{n}{\omega}\right)^p {C_A}^{-1}\omega$$

because of Lemma 2.12 and for $r < d \leq \infty$ it holds

$$\frac{H^d_{\rho,w}\big(F \cap \overline{B}(0,r)\big)}{h_{\rho,w}(0,r)} \leq 1.$$

Fix $\varepsilon > 0$. Choose $u \in \mathcal{A}(F \cap \overline{B}(0,r), B(0,2r))$ such that

$$\int_{\mathbf{R}^n} |\nabla u|^p\, w\,dy \leq \operatorname{cap}_{p,w}\big(F \cap \overline{B}(0,r), B(0,2r)\big) + \varepsilon \quad \text{and} \tag{4.9}$$

$$\int_{\mathbf{R}^n} |\nabla u|^p\, w\,dy < K\, r^{-p} \int_{B(0,r)} w\,dy. \tag{4.10}$$

Next we apply Lemma 4.4 with $r_0 = 2r < d \leq \infty$ and $g = \frac{|\nabla u|}{\omega}$. The function u has a representation (see [GT, Lemma 7.14 p. 161]),

$$\begin{aligned} u(x) &= \tfrac{1}{\omega}\int_{\mathbf{R}^n} \frac{\nabla u(y)\cdot(x-y)\,dy}{|x-y|^n} \\ &\leq \tfrac{1}{\omega}\int_{\mathbf{R}^n} \frac{|\nabla u(y)|\,dy}{|x-y|^{n-1}} = f(x). \end{aligned} \tag{4.11}$$

Moreover, we invoke (4.10) and the A_p condition to obtain the following estimates

$$\begin{aligned} &(2r)^{1-n}\|\tfrac{\nabla u}{\omega}\|_{L^p_w}\Big(\int_{B(0,2r)} w^{\frac{1}{1-p}}\,dy\Big)^{\frac{1}{p'}} \\ &\leq (2r)^{1-n}\Big(\tfrac{K}{\omega}\Big)^{\frac{1}{p}} r^{-1}\Big(\int_{B(0,r)} w\,dy\Big)^{\frac{1}{p}}\Big(\int_{B(0,2r)} w^{\frac{1}{1-p}}\,dy\Big)^{\frac{1}{p'}} \\ &\leq (2r)(2r)^{-n}2^{-2}n\omega^{-1}{C_A}^{-\frac{1}{p}}\omega^{\frac{1}{p}}\omega^{-\frac{1}{p}}r^{-1}{C_A}^{\frac{1}{p}}(2r)^n\omega\, n^{-1} \\ &= \tfrac{1}{2} < \tfrac{5}{6}. \end{aligned} \tag{4.12}$$

Thus we can choose a number $\tau > 0$ such that

$$\begin{aligned} &\frac{C_A^{\frac{1}{p}}\omega(n-1)}{\tau\, n}\int_0^{2r} \rho(t)^{\frac{1}{p}}\,dt + (2r)^{1-n}\|\tfrac{\nabla u}{\omega}\|_{L^p_w}\Big(\int_{B(0,2r)} w^{\frac{1}{1-p}}\,dy\Big)^{\frac{1}{p'}} \\ &= I(\tau, 2r) = \tfrac{5}{6}. \end{aligned}$$

Hence, if $x \in F \cap \overline{B}(0,r)$, then by (4.11) we have

$$I(\tau, 2r) = \tfrac{5}{6} < 1 \leq u(x) \leq f(x),$$

therefore

$$F \cap \overline{B}(0,r) \subset \{x \in \mathbb{R}^n : f(x) > I(\tau, 2r)\}.$$

Now from Lemma 4.4 for $2r < d \leq \infty$ and from (4.12) we infer that

$$\begin{aligned} &H^d_{\rho,w}(F \cap \overline{B}(0,r)) \\ &\leq c(n) \|\tfrac{\nabla u}{\omega}\|^p_{L^p_w} \left[\frac{\frac{C_A^{\frac{1}{p}} \omega(n-1)}{n} \int_0^{2r} \rho(t)^{\frac{1}{p}}\, dt}{\frac{5}{6} - (2r)^{1-n} \|\frac{\nabla u}{\omega}\|_{L^p_w} \left(\int_{B(0,2r)} w^{\frac{1}{1-p}}\, dy \right)^{\frac{1}{p'}}} \right]^p \\ &\leq c(n) \omega^{-p} 3^p \left[C_A^{\frac{1}{p}} \tfrac{\omega}{n}(n-1) \int_0^{2r} \rho(t)^{\frac{1}{p}}\, dt \right]^p \left[\operatorname{cap}_{p,w} \left(F \cap \overline{B}(0,r), B(0,2r) \right) + \varepsilon \right]. \end{aligned}$$

Hence letting $\varepsilon \to 0$ and using (4.8) we obtain for $r \in (0, c_0)$

$$H^d_{\rho,w}(F \cap \overline{B}(0,r)) \leq c(n)\, C_A \left[3 \tfrac{n-1}{n} c_1 \right]^p \rho(r) r^p \operatorname{cap}_{p,w} \left(F \cap \overline{B}(0,r), B(0,2r) \right).$$

Since $h_{\rho,w}(0,r) > 0$ we have

$$\frac{H^d_{\rho,w}(F \cap \overline{B}(0,r))}{h_{\rho,w}(0,r)} \leq c(n,p,C_A,c_1) \frac{\operatorname{cap}_{p,w} \left(F \cap \overline{B}(0,r), B(0,2r) \right)}{r^{-p} \int_{B(0,r)} w\, dy}.$$

Finally, by Lemma 2.12.(ii) we easily conclude the proof. □

Let $\alpha \in \mathbb{R}$ and $0 < \delta \leq \infty$. Recall that the content density of a set F in $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is

$$\Phi^{\delta *}_{\alpha,w}(x,F) = \limsup_{r \to 0} \frac{H^\delta_{\alpha,w}(F \cap \overline{B}(x,r))}{h_{\alpha,w}(x,r)}.$$

Furthermore, we say that F has *zero content density* at x, if

$$\Phi^{\delta *}_{\alpha,w}(x,F) = 0 \quad \text{for all } 0 < \delta \leq \infty.$$

In this case we write $\Phi^*_{\alpha,w}(x,F) = 0$.

Let $1 < p < \infty$ and let w be a weight. Suppose that F is a closed set in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$

$$\Psi^*_{p,w}(x,F) = \limsup_{r \to 0} \frac{\operatorname{cap}_{p,w} \left(F \cap \overline{B}(x,r), B(x,2r) \right)}{\operatorname{cap}_{p,w} \left(\overline{B}(x,r), B(x,2r) \right)}$$

defines the *upper (p,w)-capacity density* of F at x.

We are now in a position to prove a result concerning the relationship between capacity density and content density.

4.13. Theorem. *Let $1 < p < \infty$, $w \in A_p$, and $\alpha > -p$. Suppose that F is a closed set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. If $\Psi^*_{p,w}(x,F) = 0$, then $\Phi^*_{\alpha,w}(x,F) = 0$.*

Proof. Now

$$\int_0^{2r} t^{\frac{\alpha}{p}}\, dt = \tfrac{p}{\alpha+p} 2^{\frac{\alpha+p}{p}}\, r^{\frac{\alpha}{p}}\, r$$

provided that $\alpha > -p$. Hence the function $\rho(t) = t^\alpha$, $\alpha > -p$, satisfies (4.8) with constants $c_1 = \frac{p}{\alpha+p} 2^{\frac{\alpha+p}{p}}$ and $c_0 = \infty$. The desired result follows from Theorem 4.7. □

Next we consider the connection between the weighted Hausdorff dimension of a set and weighted capacity densities using content densities. As a consequence, we obtain a result on weighted capacity density and the ordinary Hausdorff dimension of a set.

Let $1 < q < \infty$ and let E be a nonempty subset of $\mathbb{R}^n$. Define

$$\Lambda(q,\ E) = \inf\{\alpha > -q : \text{ a gauge } h_{\alpha,w} \text{ is linearly increasing on } E\}. \tag{4.14}$$

If the gauge is not linearly increasing on E, we write $\Lambda(q,E) = \infty$.

4.15. Lemma. *Let $1 < p < \infty$, $w \in A_p$, and $\alpha > -p$. Assume that F is a closed set in $\mathbb{R}^n$. Suppose that a gauge $h_{\alpha,w}$ is linearly increasing on F. If $\Psi^*_{p,w}(x,F) = 0$ for all $x \in \mathbb{R}^n$, then $\dim_w(F) \le n + \Lambda(p,F)$.*

Proof. Assume first that F is a compact set with $\Psi^*_{p,w}(x,F) = 0$ for all $x \in \mathbb{R}^n$. By Theorem 4.13 we see $\Phi^*_{\alpha,w}(x,F) = 0$ for all $x \in \mathbb{R}^n$ and for all $\alpha > -p$. Fix $\alpha > -p$ such that $h_{\alpha,w}$ is linearly increasing on F with constants c_0 and d. Now Theorem 3.24 gives $\mathcal{H}_{\alpha,w}(F) = 0$ and so $\dim_w(F) \le n + \alpha$.

In the general case we write F as the union of sets $F_j = F \cap \overline{B}(0,j)$ with $j = 1,2,3,\dots$. Then

$$\mathcal{H}_{\alpha,w}(F) \le \sum_{j=1}^{\infty} \mathcal{H}_{\alpha,w}(F_j) = 0.$$

Thus we have again $\dim_w(F) \le n + \alpha$ and the lemma follows. □

4.16. Theorem. *Let $1 < p < \infty$, $w \in A_p$, and $q = \inf\{t > 1 : w \in A_t\}$. Assume that F is a closed nonempty set in $\mathbb{R}^n$ and that a gauge $h_{\alpha,w}$ is linearly increasing on F for some $\alpha > -p$. If $\Psi^*_{p,w}(x,F) = 0$ for all $x \in \mathbb{R}^n$, then*

$$n - nq \le \dim_w(F) \le n + \Lambda(p,F) \ \text{ and } \ \dim(F) \le n + \frac{\Lambda(p,F)}{q}.$$

Proof. Combine Lemma 4.15 with Corollary 3.11 to obtain the desired result. □

Note that if $\Lambda(p,F) \geq 0$, then the lemma and theorem above are trivial.

4.17. Examples. Let $1 < p < \infty$ and $w \in A_p$. Suppose that F is a closed nonempty set with the property $\Psi^*_{p,w}(x,F) = 0$ for all $x \in \mathbb{R}^n$. Write $q = \inf\{t > 1 : w \in A_t\}$.

(*a*) Take $w = 1$. Now $q = 1$ and the gauge $h_{\alpha,w}(x,r) = \frac{\omega}{n} r^{\alpha+n}$ is linearly increasing on $\mathbb{R}^n$ with constants $c_0 = 1$ and $d = \infty$ if $\alpha \geq -n$. Hence, for $n \geq p > 1$ we have $\Lambda(p,F) = -p$ and so by Theorem 4.16, $0 \leq \dim_w(F) \leq n - p$.

(*b*) Choose $w(x) = |x|^\gamma$, Recall that the weight w belongs to A_p if $-n < \gamma < np - n$ for $p \in (1,\infty)$. Thereby $q = 1$ if $-n < \gamma \leq 0$ and $q = \frac{\gamma}{n} + 1$ if $0 < \gamma < np - n$. Write

$$h_{\alpha,w}(x,r) = r^\alpha \int_{\overline{B}(x,r)} |y|^\gamma \, dy\,. \tag{4.18}$$

Suppose that $\operatorname{dist}(F,0) = 2d > 0$. From (3.19) it follows that $h_{\alpha,w}$ is linearly increasing on F with constants $3^{|\gamma|}$ and d if $\alpha + n \geq 0$. The requirement $\alpha > -p$ gives $\alpha + n > n - p$. Hence if $n \geq p > 1$, we have $\Lambda(p,F) = -p$ whenever $-n < \gamma < np - n$. So by Theorem 4.16 we obtain $0 \leq \dim_w(F) \leq n - p$ if $-n < \gamma \leq 0$, $-\gamma \leq \dim_w(F) \leq n - p$ if $0 < \gamma < np - n$, and for the ordinary Hausdorff dimension, $\dim(F) \leq n - p$ if $-n < \gamma \leq 0$ and $\dim(F) \leq n - \frac{pn}{\gamma+n}$ if $0 < \gamma < np - n$. However, by (a) we know that $\dim(F) \leq n - p$ and in this case is independent of γ.

(*c*) Let $h_{\alpha,w}$ be as in (4.18) and let d be positive. In addition, suppose that the origin belongs to F. According to Examples 3.17.(a) the gauge $h_{\alpha,w}$ is linearly increasing on F with constants $c_0 = c_0(n,\gamma)$ and d whenever

$$\begin{cases} \alpha + n & \geq 0 \\ \alpha + \gamma + n & \geq 0\,. \end{cases}$$

Furthermore, the number $\Lambda(p,F)$, see (4.14), gives a condition $\alpha + n > n - p$. Hence we obtain $\Lambda(p,F) = \max\{-n, -\gamma - n, -p\}$; thereby

$$\Lambda(p,F) = \begin{cases} -n & \text{if } -n < \gamma \leq 0 \text{ and } p \geq n, \\ -n-\gamma & \text{if } -n < \gamma \leq \min\{0, p-n\}, \\ -p & \text{if } p - n \leq \gamma < np - n \text{ and } n > p > 1. \end{cases}$$

The number q is characterized as in (b). Consequently, by Theorem 4.16 we arrive at the following estimates for the weighted Hausdorff dimension of F:

$$0 \leq \dim_w(F) \leq \begin{cases} 0 & \text{if } -n < \gamma \leq 0 \text{ and } p \geq n, \\ -\gamma & \text{if } -n < \gamma \leq \min\{0, p-n\}, \\ n-p & \text{if } p - n \leq \gamma \leq 0 \text{ and } n > p > 1, \text{ and} \end{cases}$$

$$-\gamma \leq \dim_w(F) \leq n - p \ \text{ if } \ 0 < \gamma < np - n \text{ and } n > p > 1.$$

And the corresponding inequalities for the ordinary Hausdorff dimension of F:

$$\dim(F) \le \begin{cases} 0 & \text{if } -n < \gamma \le 0 \text{ and } p \ge n, \\ -\gamma & \text{if } -n < \gamma \le \min\{0, p-n\}, \\ n-p & \text{if } p-n \le \gamma \le 0 \text{ and } n > p > 1, \\ n - \frac{np}{\gamma+n} & \text{if } 0 < \gamma < np-n \text{ and } n > p > 1. \end{cases}$$

We conclude this section by discussing upper bounds for capacity densities. A starting point is to consider particular types of gauges

$$h_{-p,w}(x,t) = t^{-p} \int_{\overline{B}(x,t)} w\, dy\,,$$

where $1 < p < \infty$ and w is a weight. Note that $\rho(t) = t^{-p}$ does not satisfy condition (4.8), see the proof of Theorem 4.13.

We need one auxiliary lemma more to establish the final result of the chapter: if for $x \in \mathbb{R}^n$ we have $\Phi^*_{-p,w}(x,F) = 0$, then $\Psi^*_{p,w}(x,F) = 0$ as well.

4.19. Lemma. *Let $1 < p < \infty$, $w \in A_p$, and let F be a closed set in $\mathbb{R}^n$. Assume that the gauge $h_{-p,w}$ is linearly increasing on F with constants c_0 and d. If $0 < \frac{r}{4} < d \le \infty$, then*

$$\operatorname{cap}_{p,w}\big(F \cap \overline{B}(x,r), B(x,2r)\big) \le c(n,p,C_A,c_0)\, H^d_{-p,w}\big(F \cap \overline{B}(x,r)\big)\,.$$

for $x \in \mathbb{R}^n$.

Proof. Let $\{\overline{B}(x_j,r_j)\}_j^\infty$ be a covering of $F \cap \overline{B}(x,r)$ such that $r_j < \frac{r}{4}$. We may suppose without loss of generality that $\overline{B}(x_j,r_j) \cap F \cap \overline{B}(x,r)$ is nonempty for all $j = 1,2,3,\ldots$. In particular, $B(x_j,2r_j) \subset B(x,2r)$ for all $j = 1,2,3,\ldots$. Now from Lemma 2.13 and from Lemma 2.12.(ii) we infer that

$$\begin{aligned} \operatorname{cap}_{p,w}\big(F \cap \overline{B}(x,r), B(x,2r)\big) &\le \sum_{j=1}^{\infty} \operatorname{cap}_{p,w}\big(F \cap \overline{B}(x_j,r_j), B(x_j,2r_j)\big) \\ &\le C_D \sum_{j=1}^{\infty} r_j^{-p} \int_{\overline{B}(x_j,r_j)} w\, dy\,. \end{aligned}$$

Taking the infimum over all such coverings we arrive at

$$\operatorname{cap}_{p,w}\big(F \cap \overline{B}(x,r), B(x,2r)\big) \le C_D\, H^{\frac{r}{4}}_{-p,w}\big(F \cap \overline{B}(x,r)\big)\,.$$

Hence Lemma 3.22 yields

$$\operatorname{cap}_{p,w}\left(F\cap\overline{B}(x,r),B(x,2r)\right)\le C_D\, c(n,C_D,c_0)\, H^d_{-p,w}\left(F\cap\overline{B}(x,r)\right)$$

whenever $\frac{r}{4}<d\le\infty$, as desired. □

4.20. Theorem. *Let $1<p<\infty$, $w\in A_p$, and let F be a closed set in $\mathbb{R}^n$. Fix a point $x\in\mathbb{R}^n$. If the gauge $h_{-p,w}$ is linearly increasing on F, then $\Phi^*_{-p,w}(x,F)=0$ implies $\Psi^*_{p,w}(x,F)=0$.*

Proof. Let $h_{-p,w}$ be linearly increasing on F with constants c_0 and d. Now Lemma 4.19 yields for all $0<\frac{t}{4}<d$

$$\frac{\operatorname{cap}_{p,w}\left(F\cap\overline{B}(x,t),B(x,2t)\right)}{h_{-p,w}(x,t)}\le c_1(n,p,C_A,c_0)\frac{H^d_{-p,w}\left(F\cap\overline{B}(x,t)\right)}{h_{-p,w}(x,t)}$$

Therefore, using Lemma 2.12.(i) we obtain

$$\frac{\operatorname{cap}_{p,w}\left(F\cap\overline{B}(x,t),B(x,2t)\right)}{\operatorname{cap}_{p,w}\left(\overline{B}(x,t),B(x,2t)\right)}\le c_2(n,p,C_A,c_0)\frac{H^d_{-p,w}\left(F\cap\overline{B}(x,t)\right)}{h_{-p,w}(x,t)}.$$

for all $0<\frac{t}{4}<d$. The theorem follows. □

5. Weighted Sobolev spaces and weighted Bessel spaces

It is well known that the Sobolev spaces can be characterized as the Bessel spaces. Here we prove a similar result for weighted spaces. We assume thoughout this section that $w\in A_p$.

First we introduce some notation and basic concepts of harmonic analysis in $\mathbb{R}^n$.

Let f be a locally integrable function on $\mathbb{R}^n$ in the Lebesgue sense. The Hardy–Littlewood maximal function f^* of f is defined as

$$f^*(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f|\,dy\,.$$

If $f\in\mathfrak{S}$ (also in $L^1_{w=1}$), the Fourier transform of f is the function $\widehat{f}$ defined by letting

$$\widehat{f}(x)=\int_{\mathbb{R}^n}f(y)e^{-2\pi i x\cdot y}\,dy\,.$$

Similary, if ξ is a finite Borel measure on $\mathbb{R}^n$, we define $\widehat{\xi}$ by

$$\widehat{\xi}(x)=\int_{\mathbb{R}^n}e^{-2\pi i x\cdot y}\,d\xi(y)\,.$$

See [SW, §1 p. 2].

Let $\alpha > 0$. If $x \in \mathbb{R}^n$, define

$$G_\alpha(x) = \frac{\pi^{n/2}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-(s+\frac{\pi^2|x|^2}{s})} s^{\frac{\alpha-n}{2}-1}\, ds\,.$$

The function G_α is the Bessel kernel of order α. Our Bessel kernel is up to constants the same as in [St, §3 p. 130]. Here Γ stands for the Euler gamma function, see [Str, Definition 7.60 p. 461]. Let β also be positive. A computation shows that $\int_{\mathbb{R}^n} G_\alpha\, dy = 1$,

$$\widehat{G_\alpha}(x) = \frac{1}{(1+|x|^2)^{\frac{\alpha}{2}}}\,, \quad \text{and} \tag{5.1}$$

$$G_\alpha * G_\beta = G_{\alpha+\beta}\,. \tag{5.2}$$

Let δ_0 be the Dirac delta measure at zero. If $E \subset \mathbb{R}^n$ is measurable and $\alpha > 0$, define

$$\mu_\alpha(E) = \delta_0(E) + \sum_{k=1}^{\infty} b(\alpha,k) \int_E G_{2k}\, dy\,,$$

where $b(\alpha,k) = \frac{1}{k!}(-1)^k \prod_{j=0}^{k-1}(\frac{\alpha}{2}-j)$, $k = 1,2,3,\dots$, see [St, §3.2(34) p. 134] and [Str, Theorem 7.46 p. 437]. For $\alpha = 0$ we set $\mu_0 = \delta_0$. Note that the measure μ_α is a finite signed Borel measure on $\mathbb{R}^n$ whenever $\alpha \geq 0$. Thereby, for $\alpha \geq 0$ we have

$$\widehat{\mu_\alpha}(x) = \frac{|x|^\alpha}{(1+|x|^2)^{\frac{\alpha}{2}}}\,, \tag{5.3}$$

see [St, Lemma 2 p. 133].

Let $1 < p < \infty$ and $w \in A_p$. The Riesz transform R_j, $j = 1,\dots,n$, is defined as

$$R_j(f)(x) = \lim_{\varepsilon\to 0} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \frac{y_j f(x-y)\, dy}{|y|^{n+1}}$$

at a point $x \in \mathbb{R}^n$ for $f \in L^p_w$. According to [SW, §2 p. 224] and [Tor, Theorem 2.2 p. 331] this definition makes sense for every function in L^p_w. Here Γ is the Euler gamma function, as in connection with G_α.

For a multi index $\beta = (\beta_1,\beta_2,\dots,\beta_n) \in \mathbb{N}^n$ we write $|\beta| = \beta_1+\beta_2+\dots+\beta_n$ and, in addition, if $x \in \mathbb{R}^n$, we denote $x^\beta = x_1^{\beta_1}\cdot x_2^{\beta_2}\cdots x_n^{\beta_n}$. The weak partial derivative of f of order $|\beta|$ is denoted by

$$\partial_\beta f = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\cdot \partial x_2^{\beta_2}\cdots \partial x_n^{\beta_n}} f\,.$$

The multi Riesz transform is defined for $f \in L^p_w$, $1 < p < \infty$, and $w \in A_p$, as

$$\begin{aligned} R_\beta(f) &= R_1^{\beta_1} \circ R_2^{\beta_2} \circ \cdots R_n^{\beta_n}(f) \\ &= \underbrace{R_1 \circ \cdots \circ R_1}_{\beta_1 \text{ times}} \circ \underbrace{R_2 \circ \cdots \circ R_2}_{\beta_2 \text{ times}} \circ \ldots \circ \underbrace{R_n \circ \cdots \circ R_n}_{\beta_n \text{ times}}(f), \end{aligned}$$

where $\circ$ stands for a decomposition of functions. If $\beta = (0, 0, \ldots, 0) \in \mathbb{N}^n$, we write $R_\beta(f) = f$. Now, let $f \in \mathfrak{S}$. It is easy to verify that

$$\widehat{R_\beta(f)}(x) = \left(\frac{-ix_1}{|x|}\right)^{\beta_1}\left(\frac{-ix_2}{|x|}\right)^{\beta_2} \cdots \left(\frac{-ix_n}{|x|}\right)^{\beta_n} \widehat{f}(x), \tag{5.4}$$

$$\widehat{R_\beta(\partial_\beta f)}(x) = \left(\frac{-2\pi x_1^2}{|x|}\right)^{\beta_1}\left(\frac{-2\pi x_2^2}{|x|}\right)^{\beta_2} \cdots \left(\frac{-2\pi x_n^2}{|x|}\right)^{\beta_n} \widehat{f}(x), \tag{5.5}$$

$$\widehat{\partial_\beta f}(x) = (-2\pi i)^{|\beta|} x^\beta \widehat{f}(x), \tag{5.6}$$

see [SW, Theorem 1.7 p. 4 and §2.7 p. 224].

Finally, let $k \in \mathbb{N}$. From the Binomial Theorem and by induction we obtain

$$\begin{aligned} (1+|x|^2)^k &= \sum_{l=0}^{k} \binom{k}{l} (|x|^2)^l \\ &= \sum_{l=0}^{k} \binom{k}{l} \sum_{|\beta|=l} \binom{l}{\beta} x_1^{2\beta_1} \cdot x_2^{2\beta_2} \cdots x_n^{2\beta_n} \end{aligned} \tag{5.7}$$

where $\binom{k}{l} = \frac{k!}{l!(k-l)!}$ and $\binom{l}{\beta} = \frac{l!}{\beta_1! \cdot \beta_2! \cdots \beta_n!}$. Recall that $0! = 1$.

We need three different types of convolution operators defined on L^p_w whenever $1 < p < \infty$ and $w \in A_p$. Next we verify that our convolutions by G_α and by μ_α will make sense on L^p_w. It is due to Muckenhoupt that the Riesz transforms are well defined convolution operators on L^p_w. Recall that C_A stands for the A_p constant of a weight.

5.8. Theorem. *Let $1 < p < \infty$ and $w \in A_p$. If $f \in L^p_w$, then*

$$\|R_j(f)\|_{L^p_w} \le c_1(n, p, C_A)\|f^*\|_{L^p_w} \le c_2(n, p, C_A)\|f\|_{L^p_w} \quad \text{for } j = 1, \ldots, n.$$

Proof. See [Tor, Theorem 4.1 p. 233 and Theorem 2.2 p. 331]. □

Let $1 \le p < \infty$, $w \in A_p$, and $\alpha > 0$. For $f \in L^p_w$ define the following functions

$$f * \delta_0 = f,$$

$$G_\alpha * f(x) = \int_{\mathbb{R}^n} G_\alpha(x-y) f(y)\, dy \quad \text{for } x \text{ in } \mathbb{R}^n, \text{ and}$$

$$f * \mu_\alpha = f * \delta_0 + \sum_{k=1}^{\infty} b(\alpha, k)\, G_{2k} * f.$$

Furthermore, if $\alpha = 0$, we write $G_0 * f = f$. Observe that $f * \mu_0 = f$. The function $f * \delta_0$, and $f * \mu_\alpha$, is called the convolution of f with the measure δ_0, and the measure μ_α, respectively. In the same manner the function $G_\alpha * f$ is called the convolution of G_α and f.

The inequality (i) below is a special case of a well known result in connection with the convolution of the Hardy–Littlewood maximal function of a locally Lebesgue integrable function with a proper class of kernels, see [GR, Theorem 4.13 p. 179]. It is easy to see that each function in L^p_w is locally Lebesgue integrable.

5.9. Lemma. *Let $1 \le p < \infty$, $w \in A_p$, and $\alpha \ge 0$. If $f \in L^p_w$, then*

(i) $|G_\alpha * f(x)| \le f^*(x)$ *a.e. in* $\mathbb{R}^n$,
(ii) $\|G_\alpha * f\|_{L^p_w} \le c_1(n,p,C_A)\|f\|_{L^p_w}$,
(iii) $\|f * \mu_\alpha\|_{L^p_w} \le c_2(n,p,\alpha,C_A)\|f\|_{L^p_w}$,
(iv) $f * \mu_\alpha * \mu_\beta = f * \mu_{\alpha+\beta}$ *whenever* $\beta \ge 0$.

Proof. If $\alpha = 0$, then (i), (ii), (iii), and (iv) are trivially true. So let $\alpha > 0$. To prove (i) let $1 \le p < \infty$ and $f \in L^p_w$. We may suppose that $f^*(x) < \infty$ since otherwise there is nothing to prove. We can approximate G_α from below by an increasing sequence of nonnegative radial simple functions $(g_j)_j^\infty$ such that

$$\begin{aligned} g_j(y) &= \sum_{k=1}^{m_j} a_k \chi_{B(0,r_k)}(y), \\ g_j &\le g_{j+1}, \\ g_j &\to G_\alpha \quad \text{a.e. in } \mathbb{R}^n, \text{ and} \\ \int_{\mathbb{R}^n} g_j\, dy &\le \int_{\mathbb{R}^n} G_\alpha\, dy = 1. \end{aligned}$$

Now we have

$$\begin{aligned} |g_j * f(x)| &\le \int_{\mathbb{R}^n} g_j(x-y)|f(y)|\, dy = \sum_{k=1}^{m_j} a_k \int_{B(x,r_k)} |f|\, dy \\ &= \sum_{k=1}^{m_j} a_k |B(x,r_k)| \frac{1}{|B(x,r_k)|} \int_{B(x,r_k)} |f|\, dy \le f^*(x) < \infty \end{aligned}$$

for all $j = 1,2,3,\dots$. Thus the Monotone Convergence Theorem yields (i).

Now (ii) follows immediately from (i) and Theorem 5.8 for $1 < p < \infty$. Let $p = 1$. Since $w \in A_1$, we have $w^*(y) \le C_A\, w(y)$ a.e. in $\mathbb{R}^n$. After the Fubini theorem we apply this and (i) to a locally integrable function w:

$$\begin{aligned}
&\int_{\mathbb{R}^n} |G_\alpha * f(y)| w(y)\, dy \le \int_{\mathbb{R}^n} G_\alpha * |f|(y) w(y)\, dy \\
&= \int_{\mathbb{R}^n} G_\alpha * w(y) |f|(y)\, dy \le \int_{\mathbb{R}^n} w^*(y) |f|(y)\, dy \\
&\le C_A \int_{\mathbb{R}^n} w(y) |f|(y)\, dy\,,
\end{aligned} \tag{5.10}$$

which completes the proof of (ii).

To prove (iii) let $1 < p < \infty$. First we observe that by (i)

$$|\sum_{k=1}^{\infty} b(\alpha,k) G_{2k} * f(x)| \le \Big(\sum_{k=1}^{\infty} |b(\alpha,k)|\Big) f^*(x) < \infty \text{ a.e. in } \mathbb{R}^n$$

since the series on the right hand converges. Hence from this and from Theorem 5.8 we infer that

$$\|\sum_{k=1}^{\infty} b(\alpha,k) G_{2k} * f\|_{L^p_w} \le \Big(\sum_{k=1}^{\infty} |b(\alpha,k)|\Big) c(n,p,C_A) \|f\|_{L^p_w} \tag{5.11}$$

for $1 < p < \infty$. It is easy to see using the same idea as in (5.10) that (5.11) is true for $p = 1$ too. Now the Minkowski inequality and (5.11) with $1 \le p < \infty$ yield

$$\begin{aligned}
\|f * \mu_\alpha\|_{L^p_w} &= \|f * \delta_0 + \sum_{k=1}^{\infty} b(\alpha,k) G_{2k} * f\|_{L^p_w} \\
&\le \|f\|_{L^p_w} + \|\sum_{k=1}^{\infty} b(\alpha,k) G_{2k} * f\|_{L^p_w} \\
&\le \Big(1 + \sum_{k=1}^{\infty} |b(\alpha,k)|\Big) c(n,p,C_A) \|f\|_{L^p_w}\,.
\end{aligned}$$

Hence (iii) follows.

We omit a proof for (iv) since it is an easy application of the Fourier transform of a measure μ_α, see (5.3), and (iii). The lemma is proved. □

We now define the weighted Sobolev and Bessel spaces; the definitions are similar to the classical situation.

Let $1 \le p < \infty$ and $w \in A_p$. For $\alpha \ge 0$ the linear space of functions $\{f : f = G_\alpha * g\,,\ g \in L^p_w\}$ endowed with the norm

$$\|f\|_{B^p_{\alpha w}} = \|g\|_{L^p_w} \text{ where } f = G_\alpha * g \text{ with } g \in L^p_w \tag{5.12}$$

is called the *weighted Bessel (potential) space* and denoted by $B^p_{\alpha w}$.

Let $1 \le p < \infty$, $w \in A_p$, and $k = 0, 1, 2, \dots$. The function $f : \mathbb{R}^n \to \mathbb{R}$ has a distributional (weak) partial derivative of order k, denoted by $\partial_\beta f$, $|\beta| = k$, if

$$\int_{\mathbb{R}^n} f\, \partial_\beta \varphi\, dy = (-1)^{|\beta|} \int_{\mathbb{R}^n} \partial_\beta f\, \varphi\, dy \quad \text{for all } \varphi \in C_0^\infty . \tag{5.13}$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that distributional (weak) partial derivatives exist up to order k and $\partial_\beta f \in L^p_w$ whenever $0 \le |\beta| \le k$ equipped with the norm

$$\|f\|_{S^p_{kw}} = \sum_{0 \le |\beta| \le k} \|\partial_\beta f\|_{L^p_w}$$

is said to be the *weighted Sobolev space* and denoted by S^p_{kw}.

5.14. Remark. Because the mapping $f \to G_\alpha * f$ is a one-to-one mapping from L^p_w into L^p_w, the formula (5.12) really defines a norm. This is an easy consequence of [Z, Exercises 2.2 p. 103], the facts that $C_0^\infty \subset \mathfrak{S}$, G_α is radial, and $f \to G_\alpha * f$ is a one-to-one function from $\mathfrak{S}$ onto $\mathfrak{S}$. For further information see [St, §3.3 p. 135].

Clearly $B^p_{\alpha w}$ is a Banach space. Since our weight w belongs to the A_p class for $1 \le p < \infty$, a function $w^{\frac{1}{1-p}}$ is locally integrable in the Lebesgue sense. Thus weighted Sobolev spaces S^p_{kw} are Banach spaces as well.

To proceed we need a reproducing formula for every function in S^p_{kw}. In the unweighted case this is a classical and well known fact, see [AMS, Chapter II §5 p. 230]. We will show that all the functions in S^p_{kw} can be represented in terms of Bessel kernels and Riesz transforms.

5.15. Lemma. *Let $1 < p < \infty$, $w \in A_p$, and $k = 0, 1, 2, \dots$. If $f \in S^p_{kw}$, then*

$$f = G_k * \sum_{l=0}^{k} \binom{k}{l} G_{k-l} * \mu_l * (-2\pi)^{-l} \sum_{|\beta|=l} \binom{l}{\beta} R_\beta(\partial_\beta f) \quad \text{a.e. in } \mathbb{R}^n .$$

Proof. First let $f \in C_0^\infty$. Combining (5.7), (5.1), (5.3), and (5.5) we obtain

$$\widehat{f}(x) = \frac{\sum_{l=0}^{k} \binom{k}{l} \sum_{|\beta|=l} \binom{l}{\beta} x_1^{2\beta_1} \cdot x_2^{2\beta_2} \cdots x_n^{2\beta_n}}{(1+|x|^2)^{\frac{2k}{2}}} \widehat{f}(x)$$

$$= \frac{1}{(1+|x|^2)^{\frac{k}{2}}} \sum_{l=0}^{k} \binom{k}{l} \frac{1}{(1+|x|^2)^{\frac{k-l}{2}}} \frac{|x|^l}{(1+|x|^2)^{\frac{l}{2}}} (-2\pi)^{-l} \sum_{|\beta|=l} \binom{l}{\beta} \Big(\frac{-2\pi x_1^2}{|x|}\Big)^{\beta_1} \cdots \Big(\frac{-2\pi x_n^2}{|x|}\Big)^{\beta_n} \widehat{f}(x)$$

$$= \widehat{G_k}(x) \sum_{l=0}^{k} \binom{k}{l} \widehat{G}_{k-l}(x)\, \widehat{\mu_l}(x)\, (-2\pi)^{-l} \sum_{|\beta|=l} \binom{l}{\beta} \widehat{R_\beta}(\partial_\beta f)(x) .$$

Hence we have

$$f(x) = G_k * \sum_{l=0}^{k} \binom{k}{l} G_{k-l} * \mu_l * (-2\pi)^{-l} \sum_{|\beta|=l} \binom{l}{\beta} R_\beta(\partial_\beta f)(x)$$

for $f \in C_0^\infty$. Next, let f be in S_{kw}^p. Using a sequence $(f_j)_j^\infty$, $f_j \in C_0^\infty$ and $f_j \to f$ in S_{kw}^p, together with Theorem 5.8 and Lemma 5.9 we obtain the result for f. □

By a slight modification of a reproducing formula we will find out that the Bessel space includes the Sobolev space:

5.16. Lemma. *Let $1 < p < \infty$, $w \in A_p$, and $k = 0, 1, 2, \ldots$. If $f \in S_{kw}^p$, then there is a unique g in L_w^p such that $f = G_k * g$ and $\|g\|_{L_w^p} \le c(n, p, k, C_A) \|f\|_{S_{kw}^p}$.*

Proof. Let $f \in S_{kw}^p$. We choose

$$g = \sum_{l=0}^{k} \binom{k}{l} G_{k-l} * \mu_l * (-2\pi)^{-l} \sum_{|\beta|=l} \binom{l}{\beta} R_\beta(\partial_\beta f).$$

Applying Theorem 5.8 and Lemma 5.9 to this formula we see

$$\|g\|_{L_w^p} \le c(n, p, k, C_A) \|f\|_{S_{kw}^p}.$$

From Lemma 5.15 it follows that $f = G_k * g$ and Remark 5.14 implies the uniqueness of g. The proof is complete. □

Conversely, we ask if the Sobolev space includes the Bessel space. This question has an affirmative answer. For background see [St, Theorem 3 p. 135].

5.17. Lemma. *Let $1 < p < \infty$, $w \in A_p$, and $k = 0, 1, 2, \ldots$. If $g \in L_w^p$, then $G_k * g$ belongs to S_{kw}^p with distributional partial derivatives*

$$\partial_\beta(G_k * g) = (2\pi)^{|\beta|} G_{k-|\beta|} * R_\beta(g) * \mu_{|\beta|} \tag{5.18}$$

and, in addition,

$$\|G_k * g\|_{S_{kw}^p} \le c(n, p, k, C_A) \|g\|_{L_w^p}.$$

Proof. Let $g \in \mathfrak{S}$. Now by the Fourier transform, (5.6), (5.1), (5.3), and (5.4) we see

$$\begin{aligned}
\widehat{\partial_\beta}(G_k * g)(x) &= (-2\pi i)^{|\beta|} x^\beta \widehat{G_k}(x) \widehat{g}(x) \\
&= (2\pi)^{|\beta|} \frac{1}{(1+|x|^2)^{\frac{k-|\beta|}{2}}} \left(\frac{-ix_1}{|x|}\right)^{\beta_1} \cdots \left(\frac{-ix_n}{|x|}\right)^{\beta_n} \widehat{g}(x) \frac{|x|^{|\beta|}}{(1+|x|^2)^{\frac{|\beta|}{2}}} \\
&= (2\pi)^{|\beta|} \widehat{G}_{k-|\beta|}(x) \widehat{R_\beta}(g)(x) \widehat{\mu}_{|\beta|}(x).
\end{aligned}$$

So (5.18) holds whenever $g \in \mathfrak{S}$.

In the general case $g \in L^p_w$ we take a sequence $(g_j)_1^\infty \in \mathfrak{S}$ such that $g_j \to g$ in L^p_w. By the standard tricks together with Theorem 5.8 and Lemma 5.9 we obtain that (5.18) is true a.e. in $\mathbb{R}^n$. Furthermore, since $w^{\frac{1}{1-p}}$ is locally integrable in the Lebesgue sense, we conclude that the limit function $\partial_\beta(G_k * g)$ satisfies (5.13), which finishes the proof. □

Summing up, we have proved the following result:

5.19. Theorem. *Let $1 < p < \infty$ and $w \in A_p$. Then $S^p_{kw} = B^p_{kw}$ for all $k = 0, 1, 2, 3, \dots$, and the norms are equivalent, i.e.*

$$c_1(n,p,k,C_A)\,\|f\|_{B^p_{kw}} \le \|f\|_{S^p_{kw}} \le c_2(n,p,k,C_A)\,\|f\|_{B^p_{kw}} \quad \text{for } f \in S^p_{kw}.$$

Proof. The result follows from Lemma 5.16 and Lemma 5.17. □

5.20. Remarks. (a) The equivalence of the spaces B^p_{kw} and S^p_{kw} already fails in case $w = 1$ when $p = 1$ or $p = \infty$, see [St, §6.6 p. 160] and [Z, Theorem 2.6.1 p. 66].

(b) The formula (5.7) derived from the Binomial Theorem used in the proof of Lemma 5.15 can be interpreted in the following way: let $1 < p < \infty$, $w \in A_p$, and $f \in L^p_w$. Then for $k = 0, 1, 2, \dots$ we find that

$$f = \sum_{l=0}^{k} \binom{k}{l} G_{2(k-l)} * \mu_{2l} * (-1)^{-l} \sum_{|\beta|=l} \binom{l}{\beta} R_\beta R_\beta(f) \quad \text{a.e. in } \mathbb{R}^n.$$

In case $k = 1$ we have $f = G_2 * f + f * \mu_2$ a.e. in $\mathbb{R}^n$ for all $f \in L^p_w$ and $1 < p < \infty$.

Hence every function in L^p_w has a reproducing formula, which differs from that one of Lemma 5.15.

(c) The proof of Lemma 5.16 and Lemma 5.17 inspires us to define two mappings between spaces S^p_{kw} and L^p_w when $1 < p < \infty$, $w \in A_p$, and $k = 0, 1, 2, \dots$.

The mapping J_{-k} from S^p_{kw} to L^p_w is defined by

$$J_{-k}(f) = \sum_{l=0}^{k} \binom{k}{l} G_{k-l} * \mu_l * (-2\pi)^{-l} \sum_{|\beta|=l} \binom{l}{\beta} R_\beta(\partial_\beta f)$$

for $f \in S^p_{kw}$. The mapping J_k from L^p_w to S^p_{kw} is defined by letting

$$J_k(g) = G_k * g \quad \text{for } g \in L^p_w.$$

Using the Fourier transform we verify that J_k is the inverse of J_{-k} and that J_{-k} is the inverse of J_k. Thus the mapping J_k provides a linear bijective quasi isometry between the spaces L^p_w and S^p_{kw}.

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