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MAARIT JÄRVENPÄÄ

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Maarit Järvenpää

Contents

1. Introduction

Throughout this paper, m and n will be integers such that $0 < m < n$. We consider how the upper Minkowski dimension and the packing dimension, which was introduced by Tricot in [T], change under orthogonal projections. Since neither the upper Minkowski dimension, denoted by $\dim_{\mathrm{M}} A$ for all bounded $A \subset \mathbf{R}^n$, nor the packing dimension, denoted by $\dim_{\mathrm{p}} A$ for all $A \subset \mathbf{R}^n$, can increase under Lipschitz maps (see [F, p. 44] for the upper Minkowski dimension and [F, Proposition 3.8 and p. 46] for the packing dimension), we have that

$$\dim_{\mathrm{M}} P_V(A) \leq \dim_{\mathrm{M}} A$$

for all bounded $A \subset \mathbf{R}^n$, and

$$\dim_{\mathrm{p}} P_V(A) \leq \dim_{\mathrm{p}} A$$

for all $A \subset \mathbf{R}^n$, where $P_V \colon \mathbf{R}^n \to V$ denotes the orthogonal projection onto an m-dimensional linear subspace $V \subset \mathbf{R}^n$.

We denote by $G(n,m)$ the Grassmann manifold of all m-dimensional linear subspaces of $\mathbf{R}^n$. Let $\gamma_{n,m}$ be the unique orthogonally invariant Radon probability measure on $G(n,m)$ (see [M3, 3.9]).

We use the notation $\dim_{\mathrm{H}}$ for the Hausdorff dimension (for its definition see [M3, 4.8]). Let $A \subset \mathbf{R}^n$ be a Borel set. The behaviour of the Hausdorff dimension $s = \dim_{\mathrm{H}} A$ of A under orthogonal projections onto m-dimensional linear subspaces $V \subset \mathbf{R}^n$ depends on whether $0 \leq s \leq m$ or $m < s \leq n$. In fact, if $s \leq m$, then $\dim_{\mathrm{H}} P_V(A) = s$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$, and if $s > m$, then $\dim_{\mathrm{H}} P_V(A) = m$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$. In the plane, a geometric proof of these projection theorems for the Hausdorff dimension was first given by Marstrand in [M]. He considered the case where the s-dimensional Hausdorff measure of A (for the definition of the Hausdorff measures see [M3, 4.3]) is positive and finite. However, Davies has shown in [D] that if the s-dimensional Hausdorff measure of any Borel set is infinite, then it has a compact subset with positive and finite s-dimensional Hausdorff measure. Potential theoretical concepts were first used by Kaufman in [K] in order to prove these projection theorems in the plane. In [M1], Mattila obtained extensions of these theorems to higher dimensions.

Let $A \subset \mathbf{R}^n$ be a bounded Borel set. If $\dim_{\mathrm{M}} A = \dim_{\mathrm{H}} A$, then we have by the above mentioned projection theorems for the Hausdorff dimension that $\dim_{\mathrm{M}} P_V(A) = \min\{\dim_{\mathrm{M}} A, m\}$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$, since $\dim_{\mathrm{H}} A \leq \dim_{\mathrm{M}} A$ for all bounded $A \subset \mathbf{R}^n$. One example [M3, Corollary 5.8] of this kind of set is an arbitrary self-similar set generated by similitudes $S_1, \ldots, S_N$ for which there is a non-empty, bounded and open set U such that

$$\bigcup_{i=1}^{N} S_i(U) \subset U \text{ and } S_i(U) \cap S_j(U) = \emptyset \text{ for } i \neq j.$$

Analogously, if $A \subset \mathbf{R}^n$ is a Borel set with $\dim_{\mathrm{p}} A = \dim_{\mathrm{H}} A$, then $\dim_{\mathrm{p}} P_V(A) = \min\{\dim_{\mathrm{p}} A, m\}$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$, since $\dim_{\mathrm{H}} A \leq \dim_{\mathrm{p}} A$ for all $A \subset \mathbf{R}^n$.

In this paper, we construct an example showing that, in general, the behaviour of both the upper Minkowski dimension and the packing dimension under orthogonal projections differs from the corresponding behaviour of the Hausdorff dimension. This construction gives a negative answer to a question asked by Hu and Taylor in [HT, Introduction, p. 528].

We also show that if $(e_1, \ldots, e_n)$ is a basis of $\mathbf{R}^n$, then for all bounded $A \subset \mathbf{R}^n$ there is an m-dimensional linear subspace V spanned by some of the vectors $(e_1, \ldots, e_n)$ such that

$$\dim_{\mathrm{M}} P_V(A) \geq \frac{m}{n} \dim_{\mathrm{M}} A.$$

Using the relation between the upper Minkowski dimension and the packing dimension, we obtain a corresponding result for the packing dimensions of any set in $\mathbf{R}^n$ and its images under the orthogonal projections onto the m-planes spanned by a given basis of $\mathbf{R}^n$. Further, these inequalities are the best possible ones, and the analogue of these results is not true for the Hausdorff dimension.

By means of the same method for both the upper Minkowski dimension and the packing dimension, the above mentioned results can be extended to $\gamma_{n,m}$ almost all $V \in G(n,m)$. In this way we are able to prove that for all bounded $A \subset \mathbf{R}^n$,

$$\dim_{\mathrm{M}} P_V(A) \geq \frac{m}{n} \dim_{\mathrm{M}} A$$

for $\gamma_{n,m}$ almost all $V \in G(n,m)$, and for all Suslin sets $A \subset \mathbf{R}^n$,

$$\dim_{\mathrm{p}} P_V(A) \geq \frac{m}{n} \dim_{\mathrm{p}} A$$

for $\gamma_{n,m}$ almost all $V \in G(n,m)$. The cases $m = 1$ and $m = n-1$ are immediate consequences of the above mentioned results. In fact, if $m = 1$ or $m = n-1$, the assumption that A is a Suslin subset of $\mathbf{R}^n$ is not necessary for the packing dimension (see Remarks 3.6 (d) and (f)). The case $1 < m < n-1$ involves some more work including measurability considerations (see Lemmas 2.2 and 2.3). However, these estimates concerning the decrease of the upper Minkowski dimension and the packing dimension under orthogonal projections are not the best possible ones. After this work was completed, the best possible lower bounds were found by Falconer and Howroyd using different methods (see [FH]). They showed that if $A \subset \mathbf{R}^n$ is bounded, then

$$\dim_{\mathrm{M}} P_V(A) \geq \frac{\dim_{\mathrm{M}} A}{1 + (1/m - 1/n)\dim_{\mathrm{M}} A}$$

for $\gamma_{n,m}$ almost all $V \in G(n,m)$, and if $A \subset \mathbf{R}^n$ is a Suslin set, then

$$\dim_{\mathrm{p}} P_V(A) \geq \frac{\dim_{\mathrm{p}} A}{1 + (1/m - 1/n)\dim_{\mathrm{p}} A}$$

for $\gamma_{n,m}$ almost all $V \in G(n,m)$. Note that $ms/n < s/(1 + (1/m - 1/n)s) < \min\{s, m\}$ for each $0 < s < n$.

2. Preliminaries

We denote by $d(A) := \sup\{|y - z| \mid y, z \in A\}$ the diameter of a non-empty set $A \subset \mathbf{R}^n$ and by $d(x, A) := \inf\{|x - y| \mid y \in A\}$ the distance between $x \in \mathbf{R}^n$ and A. The distance between two non-empty sets $A, B \subset \mathbf{R}^n$ is denoted by $d(A, B) := \inf\{|x - y| \mid x \in A,\ y \in B\}$.

Let $x \in \mathbf{R}^n$ and $0 < r < \infty$. For closed balls we use the notation $B(x, r) := \{y \in \mathbf{R}^n \mid |x - y| \leq r\}$ and for open balls the notation $U(x, r) := \{y \in \mathbf{R}^n \mid |x - y| < r\}$.

When k is an integer with $1 \leq k \leq n$ and $(x_1, \ldots, x_k)$ is a linearly independent sequence in $\mathbf{R}^n$, we use the notation $V(x_1, \ldots, x_k)$ for the k-dimensional linear subspace of $\mathbf{R}^n$ spanned by $\{x_1, \ldots, x_k\}$. Further, if V_1 and V_2 are linear subspaces of $\mathbf{R}^n$, we define $V_1 + V_2 := \{x + y \mid x \in V_1,\ y \in V_2\}$.

Throughout this paper, a measure means a non-negative, monotone, and countably subadditive set function which vanishes for the empty set. If μ is a measure on a set X, we denote by $\mu \llcorner A$ the restriction of the measure μ to a set $A \subset X$, that is,

$$(\mu \llcorner A)(B) = \mu(A \cap B)$$

for all $B \subset X$. Further, we use the notation $f_\sharp\mu$ for the image of the measure μ under a map $f\colon X \to Y$, that is,

$$f_\sharp\mu(A) = \mu(f^{-1}(A))$$

for all $A \subset Y$. The following lemma, which is a modification of [M3, Theorem 1.18], will be used later when proving Theorem 3.7.

2.1. Lemma. *Let X and Y be σ-compact metric spaces. If μ is a Radon measure on X with $\mu(X) < \infty$ and $f\colon X \to Y$ is continuous, then $f_\sharp\mu$ is a Radon measure on Y.*

Proof. Since a measure on a σ-compact metric space is a Radon measure if and only if it is locally finite and Borel regular (see [M3, Corollary 1.11]), it is sufficient to prove that $f_\sharp\mu$ is Borel regular.

Let $A \subset Y$ and (ε_i) be a sequence of positive real numbers such that $\lim_{i\to\infty} \varepsilon_i = 0$. Since μ is a finite Radon measure, there exists for all i a compact set $C_i \subset X$ such that $\mu(X \setminus C_i) < \varepsilon_i$.

For all i, we define

$$\nu_i := \mu \llcorner C_i.$$

Since μ is Borel regular and C_i is a Borel set with $\mu(C_i) < \infty$, the measure ν_i is Borel regular (see [M3, Theorem 1.9]) and finite. Thus ν_i is a Radon measure with compact support for all i, and so $f_\sharp\nu_i$ is a Radon measure (see [M3, Theorem 1.18]); in particular, it is Borel regular. Thus for all i, there exists a Borel set $B_i \subset Y$ with $A \subset B_i$ and $f_\sharp\nu_i(B_i) = f_\sharp\nu_i(A)$.

Define

$$B := \bigcap_{i=1}^{\infty} B_i.$$

Then B is a Borel set with $A \subset B$ and

$$f_\sharp\mu(B) \leq f_\sharp\mu(B_i) \leq f_\sharp\nu_i(B_i) + \varepsilon_i = f_\sharp\nu_i(A) + \varepsilon_i \leq f_\sharp\mu(A) + \varepsilon_i$$

for all i. Thus $f_\sharp\mu(B) - f_\sharp\mu(A)$. □

We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbf{R}^n$. Further, we use the notation $\alpha(n) := \mathcal{L}^n(B(0,1))$.

As mentioned before, we use the notation $G(n,m)$ for the Grassmann manifold consisting of all m-dimensional linear subspaces of $\mathbf{R}^n$. For $V \in G(n,m)$, we denote by $V^\perp \in G(n,n-m)$ the orthogonal complement of V and by $P_V\colon \mathbf{R}^n \to V$ the orthogonal projection onto V. Identifying $V \in G(n,m)$ with $\mathbf{R}^m$, we can determine the m-dimensional area measure for any subset of V by using the m-dimensional Lebesgue measure $\mathcal{L}^m$. By means of this identification, any orthogonal projection onto an m-dimensional linear subspace of $\mathbf{R}^n$ becomes a map onto $\mathbf{R}^m$.

Equipped with the metric

$$\sigma(V,W) := \|P_V - P_W\|$$

for $V, W \in G(n,m)$, where $\|L\| := \sup_{|x|=1} |Lx|$ denotes the usual norm for a linear mapping L, the Grassmann manifold $G(n,m)$ is a compact metric space. As mentioned before, there exists a unique orthogonally invariant Radon probability measure on $G(n,m)$. We denote it by $\gamma_{n,m}$.

We use the notation $\mathcal{N}$ for the set of all infinite sequences of positive integers. Then $\mathcal{N}$ is a cartesian product with all factors equal to the set consisting of all positive integers, and using the discrete topology on each of these factors, we can define the cartesian product topology on $\mathcal{N}$. A Suslin subset of a topological space X is a set of the form $p(C)$ where $p\colon X \times \mathcal{N} \to X$ is the projection and $C \subset X \times \mathcal{N}$ is closed. Every Borel set in $\mathbf{R}^n$ is a Suslin set (see [F1, p. 66]), and every Suslin set in $\mathbf{R}^n$ is Lebesgue measurable (see [F1, Theorem 2.2.12]). Further, if A is a Suslin subset of $\mathbf{R}^n$, then $P_V(A)$ is a Suslin subset of V for all m-dimensional linear subspaces $V \subset \mathbf{R}^n$ (see [F1, p. 65]) .

Let $A \subset \mathbf{R}^n$ be non-empty and bounded. Define for all $0 < \varepsilon < \infty$

$$A(\varepsilon) := \{x \in \mathbf{R}^n \mid d(x, A) \le \varepsilon\}.$$

The upper Minkowski dimension of A is defined by

$$\dim_{\mathrm{M}} A := \inf\{s \mid \mathcal{M}^s(A) = 0\} = \sup\{s \mid \mathcal{M}^s(A) > 0\},$$

where

$$\mathcal{M}^s(A) := \limsup_{\varepsilon \to 0} (2\varepsilon)^{s-n} \mathcal{L}^n(A(\varepsilon))$$

is the s-dimensional upper Minkowski content of A for all real numbers $s \ge 0$. Then $0 \le \dim_{\mathrm{M}} A \le n$.

There are some clearly equivalent ways to define the upper Minkowski dimension (see [F, pp. 38–41]). For example,

$$\dim_{\mathrm{M}} A = \limsup_{\varepsilon \to 0} \frac{\log P(A, \varepsilon)}{-\log \varepsilon},$$

where for all $0 < \varepsilon < \infty$,

$$P(A, \varepsilon) := \max\{k \mid \text{there are } x_1, \dots, x_k \in A \text{ such that } B(x_1, \varepsilon), \dots, B(x_k, \varepsilon) \text{ are disjoint}\},$$

and further,

$$\dim_{\mathrm{M}} A = \limsup_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$

where for all $0 < \varepsilon < \infty$,

$$N(A, \varepsilon) := \min\{k \mid \text{there are } E_1, \dots, E_k \subset \mathbf{R}^n \text{ such that } A \subset \bigcup_{i=1}^{k} E_i \text{ and } d(E_i) \le \varepsilon \text{ for all } 1 \le i \le k\}.$$

The upper Minkowski dimension can also be determined by using dyadic cubes in $\mathbf{R}^n$, that is, cubes of the form

$$\{x \in \mathbf{R}^n \mid k_j 2^{-i} \le x_j < (k_j+1)2^{-i} \text{ for all } 1 \le j \le n\},$$

where k_j and i are integers. We use the notation $N_i(A)$ for the number of the dyadic cubes with side-length 2^{-i} which intersect A. Then

$$\dim_{\mathrm{M}} A = \limsup_{i \to \infty} \frac{\log N_i(A)}{i \log 2}.$$

Finally, we set $\dim_{\mathrm{M}} \emptyset := 0$.

Let $A \subset \mathbf{R}^n$. The packing dimension of A can be defined by means of the packing measures (see [M3, 5.10]), which were introduced by Tricot in [T] and independently in a different form by Sullivan in [S]. Apart from the proof of Corollary 3.10, we only need the following equivalent definition, which is based on the relation between the upper Minkowski dimension and the packing dimension:

$$\dim_{\mathrm{p}} A := \inf\{\sup_i \dim_{\mathrm{M}} E_i \mid E_i \subset \mathbf{R}^n \text{ is bounded for all } i \text{ and } A \subset \bigcup_{i=1}^{\infty} E_i\}.$$

Clearly, $\dim_{\mathrm{p}} A \le \dim_{\mathrm{M}} A$ whenever A is bounded. Further, $\dim_{\mathrm{H}} A \le \dim_{\mathrm{p}} A$ for all $A \subset \mathbf{R}^n$ (see [F, (3.29)]).

2.2. Lemma. *If $A \subset \mathbf{R}^n$ is bounded, then the map $f\colon G(n,m) \to \mathbf{R}$ defined by*

$$f(V) := \dim_{\mathrm{M}} P_V(A)$$

for all $V \in G(n,m)$ is a Borel function.

Proof. We may assume that $A \ne \emptyset$. Since

$$\dim_{\mathrm{M}} P_V(A) = \limsup_{i \to \infty} \frac{\log M(P_V(A), c^i)}{-\log c^i},$$

where i tends to infinity through integer values, $M(P_V(A), r)$ is for all $r > 0$ the smallest number of closed balls with radius r which cover $P_V(A)$, and $0 < c < 1$ (see [F, p. 41]), it is enough to prove that for fixed $r > 0$ the map $g\colon G(n,m) \to \mathbf{R}$ defined by

$$g(V) := M(P_V(A), r)$$

for all $V \in G(n,m)$ is lower semicontinuous, that is,

$$M(P_V(A), r) \le \liminf_{i \to \infty} M(P_{V_i}(A), r),$$

when $V_i \in G(n,m)$ for all i with $\lim_{i\to\infty} \sigma(V_i, V) = 0$. Indeed, since any lower semicontinuous function is a Borel function, this gives the desired result.

Define $k_i := M(P_{V_i}(A), r)$ for all i and $k := \liminf_{i\to\infty} k_i$. Since (k_i) is a bounded sequence of positive integers, there exists a subsequence (k_{i_j}) such that $k_{i_j} = k$ for all j. So we may assume that $M(P_{V_i}(A), r) = k$ for all i.

For all i, we fix $x_1^i, \ldots, x_k^i \in V_i$ such that

$$P_{V_i}(A) \subset \bigcup_{j=1}^{k} B(x_j^i, r). \tag{1}$$

Since (x^i), where $x^i = (x_1^i, \ldots, x_k^i)$ for all i, can be identified with a bounded sequence in $\mathbf{R}^{nk}$, it has a convergent subsequence. By considering this subsequence, we may assume that for all $1 \le j \le k$ there is $x_j \in V$ with $\lim_{i\to\infty} x_j^i = x_j$. Now it suffices to show that

$$P_V(A) \subset \bigcup_{j=1}^{k} B(x_j, r).$$

Consider $x \in A$. Let (ε_l) be a sequence of positive real numbers such that $\lim_{l\to\infty} \varepsilon_l = 0$. For all l, there is an integer $I(l)$ such that

$$|x_j^{I(l)} - x_j| < \varepsilon_l$$

for all $1 \le j \le k$ and

$$|P_{V_{I(l)}}(x) - P_V(x)| < \varepsilon_l.$$

Consider a positive integer l. By (1), there is an integer $1 \le j(l) \le k$ such that $|P_{V_{I(l)}}(x) - x_{j(l)}^{I(l)}| \le r$. Then,

$$|P_V(x) - x_{j(l)}| \le |P_V(x) - P_{V_{I(l)}}(x)| + |P_{V_{I(l)}}(x) - x_{j(l)}^{I(l)}| + |x_{j(l)}^{I(l)} - x_{j(l)}| \le r + 2\varepsilon_l,$$

and so there are a subsequence (ε_{l_p}) of (ε_l) and an integer $1 \le j \le k$ such that

$$|P_V(x) - x_j| \le r + 2\varepsilon_{l_p}$$

for all p. Thus $|P_V(x) - x_j| \le r$. □

2.3. Lemma. *If $A \subset \mathbf{R}^n$ is compact, then the map $\mathfrak{g}\colon G(n,m) \to \mathbf{R}$ defined by*

$$\mathfrak{g}(V) := \dim_{\mathrm{p}} P_V(A)$$

for all $V \in G(n,m)$ is measurable with respect to the σ-algebra consisting of the $\gamma_{n,m}$-measurable sets.

Proof. We may assume that $A \neq \emptyset$. We use the notation $\mathcal{K}(\mathbf{R}^n)$ for the family of the non-empty, compact subsets of $\mathbf{R}^n$ and equip $\mathcal{K}(\mathbf{R}^n)$ with the Hausdorff metric

$$\rho(K,L) := \sup\{d(x,L), d(y,K) \mid x \in K, y \in L\}$$

for $K, L \in \mathcal{K}(\mathbf{R}^n)$. Now we obtain for all real numbers c that

$$\{V \in G(n,m) \mid g(V) \geq c\} = f^{-1}(\{K \in \mathcal{K}(\mathbf{R}^n) \mid \dim_{\mathrm{p}} K \geq c\}),$$

where $f\colon G(n,m) \to \mathcal{K}(\mathbf{R}^n)$ is a continuous function defined by

$$f(V) := P_V(A)$$

for all $V \in G(n,m)$. The measurability of g is an immediate consequence of a result by Mattila and Mauldin (see [MM]) stating that for all real numbers c, the set $\{K \in \mathcal{K}(\mathbf{R}^n) \mid \dim_{\mathrm{p}} K \geq c\}$ is a Suslin subset of $\mathcal{K}(\mathbf{R}^n)$. Then $f^{-1}(\{K \in \mathcal{K}(\mathbf{R}^n) \mid \dim_{\mathrm{p}} K \geq c\})$ is a Suslin subset of $G(n,m)$ (see [F1, p. 66]) and thus $\gamma_{n,m}$-measurable (see [F1, Theorem 2.2.12]). □

2.4. Remark. It is easy to see that Lemma 2.3 holds for σ-compact sets. In fact, if $A = \bigcup_{i=1}^{\infty} A_i$ where A_i is compact for all i, then, by [F, (3.26)], $g = \sup_i g_i$ where the map $g_i : G(n,m) \to \mathbf{R}$ defined by

$$g_i(V) := \dim_{\mathrm{p}} P_V(A_i)$$

for all $V \in G(n,m)$ is by Lemma 2.3 measurable with respect to the σ-algebra consisting of the $\gamma_{n,m}$-measurable sets for all i. Thus g is measurable.

3. Projection results

The behaviour of both the upper Minkowski dimension and the packing dimension under orthogonal projections differs from the projection properties of the Hausdorff dimension, as shown by the following example, which is a modification of a construction from [M2]. In fact, it also gives an example which shows that the upper Minkowski dimension and the packing dimension estimates discussed by Falconer and Howroyd in [FH] are the best possible ones (for other such examples, see [FH]).

3.1. Example. For any $0 < s < n$ and $s/(1+(1/m-1/n)s) < t < \min\{s,m\}$ there is a compact set $E \subset \mathbf{R}^n$ such that $\dim_{\mathrm{M}} E = \dim_{\mathrm{p}} E = s$ and $\mathcal{M}^t(P_V(E)) = 0$ for all $V \in G(n,m)$; in particular, $\dim_{\mathrm{M}} P_V(E) \leq t$ for all $V \in G(n,m)$.

Consider $0 < s < n$. Let $R_1 = 1$. For all integers $k \geq 1$, we choose an integer $l_{k+1} > 1$ and define

$$r_k := \frac{R_k}{l_{k+1}^{(n-s)/s}} \quad \text{and} \quad R_{k+1} := \frac{r_k}{l_{k+1}}. \tag{2}$$

Then, $R_k > r_k > R_{k+1}$ and

$$l_{k+1}^n R_{k+1}^s = R_k^s \tag{3}$$

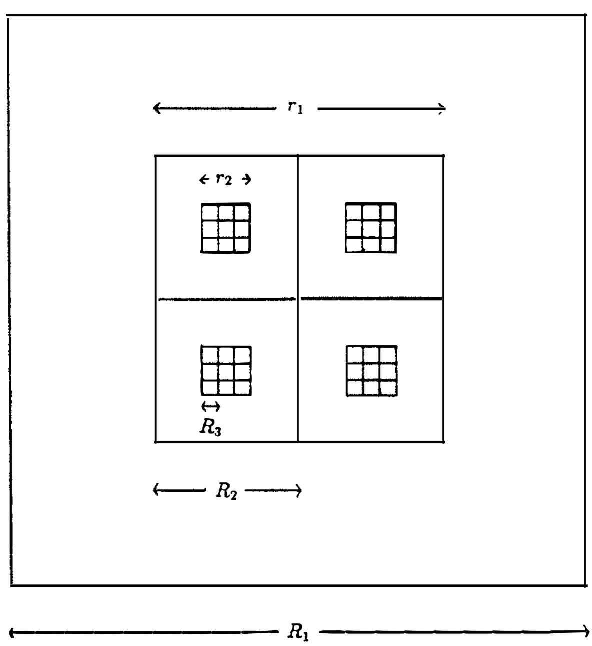
for all $k \geq 1$. Further, we obtain by induction that

$$(l_2 \dots l_k)^n R_k^s = 1 \tag{4}$$

for all $k \geq 2$.

Let $P_{1,1}$ be a closed cube with side-length R_1 and $Q_{1,1} \subset P_{1,1}$ a closed cube with the same centre as $P_{1,1}$ and side-length r_1. We divide $Q_{1,1}$ into l_2^n closed cubes $P_{2,1}, \dots, P_{2,l_2^n}$ with side-length R_2. For all $1 \leq i \leq l_2^n$, let $Q_{2,i} \subset P_{2,i}$ be a closed cube with the same centre as $P_{2,i}$ and side-length r_2. When dividing each cube $Q_{2,i}$ into l_3^n closed cubes with side-length R_3, we obtain $(l_2 l_3)^n$ closed cubes $P_{3,1}, \dots, P_{3,(l_2 l_3)^n}$. We continue this construction and define

$$E := \bigcap_{k=2}^{\infty} \bigcup_{i=1}^{(l_2 \dots l_k)^n} Q_{k,i}.$$



Let $0 < \varepsilon < R_2$. We fix $k \geq 2$ such that $R_{k+1} \leq \varepsilon \leq R_k$.

If $R_{k+1} \leq \varepsilon < r_k$, then for all $1 \leq i \leq (l_2 \dots l_k)^n$, the cube $Q_{k,i}$ can be covered with p^n n-cubes with side-length ε for some $p \leq 2r_k/\varepsilon$. Since E is contained in $(l_2 \dots l_k)^n p^n$ such cubes, we obtain, using (4), (2), and (3), that

$$(5) \quad N(E, \sqrt{n}\varepsilon) \leq (l_2 \dots l_k)^n p^n \leq 2^n R_k^{-s} R_{k+1}^n l_{k+1}^n \varepsilon^{-n} = 2^n R_{k+1}^{n-s} \varepsilon^{-n} \leq 2^n \varepsilon^{-s}.$$

If $r_k \leq \varepsilon \leq R_k$, then by (4),

$$(6) \qquad N(E, \sqrt{n}\varepsilon) \leq N(E, \sqrt{n} r_k) \leq (l_2 \dots l_k)^n = R_k^{-s} \leq \varepsilon^{-s}.$$

Now, (5) and (6) together imply $\dim_{\mathrm{M}} E \leq s$.

Since for all $k \geq 2$,

$$\bigcup_{i=1}^{(l_2 \dots l_k)^n} P_{k,i} \subset E(\sqrt{n} R_k),$$

we have by (4) that

$$\mathcal{L}^n(E(\sqrt{n} R_k)) \geq (l_2 \dots l_k)^n R_k^n = R_k^{n-s},$$

and so $\mathcal{M}^s(E) > 0$, which gives $\dim_{\mathrm{M}} E \geq s$. Thus $\dim_{\mathrm{M}} E = s$.

Since $\dim_{\mathrm{M}}(E \cap Q_{k,i}) = s$ for all k, i, we have that $\dim_{\mathrm{M}}(E \cap U) = s$ for all open $U \subset \mathbf{R}^n$ with $E \cap U \neq \emptyset$, and so [F, Corollary 3.9] implies that $\dim_{\mathrm{p}} E = s$.

Consider $s/(1 + (1/m - 1/n)s) < t < \min\{s, m\}$. We choose l_k and positive real numbers u_k and v_k such that $\lim_{k \to \infty} u_k = \lim_{k \to \infty} v_k = 0$ and

$$(7) \qquad u_k (v_k R_{k+1})^{s(m-t)/t} = (R_{k+1} l_{k+1})^m R_k^{-s}$$

for all k. Since (7) is equivalent to the equation

$$u_k v_k^{s(m-t)/t} = (l_2 \dots l_k)^{nm(s-t)/st} l_{k+1}^{(snm - (s(n-m)+nm)t)/st}$$

by (3) and (4), this is possible by choosing successive l_{k+1} so large that the right-hand side tends to zero as $k \to \infty$.

Consider $V \in G(n, m)$. Let $r_{k+1} \leq \varepsilon < r_k$ with $k \geq 2$. We first assume that $(v_k R_{k+1})^{s/t} \leq \varepsilon$. Since for all $1 \leq i \leq (l_2 \dots l_k)^n$ the projection $P_V(Q_{k,i})$ can be covered with p^m m-cubes with side-length ε for some $p \leq 2nr_k/\varepsilon$, the projection $P_V(E)$ can be covered with $p^m (l_2 \dots l_k)^n$ m-cubes with side-length ε. Since $P_V(E)(\varepsilon) \cap V$ is contained in $(3p)^m (l_2 \dots l_k)^n$ such cubes, we obtain, using (4), (2), and (7), that

$$\begin{aligned} (8) \qquad (2\varepsilon)^{t-m} \mathcal{L}^m(P_V(E)(\varepsilon) \cap V) &\leq 2^{t-m} (3p)^m (l_2 \dots l_k)^n \varepsilon^t \\ &\leq 2^{t-m} (6n)^m r_k^m R_k^{-s} \varepsilon^{t-m} \leq 2^{t-m} (6n)^m (R_{k+1} l_{k+1})^m R_k^{-s} \varepsilon^{t-m} \\ &\leq 2^{t-m} (6n)^m u_k. \end{aligned}$$

We now assume that $\varepsilon \le (v_k R_{k+1})^{s/t}$. Since E can be covered with $(l_2 \dots l_{k+1})^n$ n-cubes with side-length ε, $P_V(E)(\varepsilon) \cap V$ is contained in $(2n)^m (l_2 \dots l_{k+1})^n$ m-cubes with side-length ε, and so we obtain by (4) that

$$(9) \qquad \begin{aligned}(2\varepsilon)^{t-m}\mathcal{L}^m(P_V(E)(\varepsilon) \cap V) &\le 2^{t-m}(2n)^m(l_2 \dots l_{k+1})^n \varepsilon^t \\ &= 2^{t-m}(2n)^m R_{k+1}^{-s}\varepsilon^t \le 2^{t-m}(2n)^m v_k^s.\end{aligned}$$

Now, (8) and (9) together imply $\mathcal{M}^t(P_V(E)) = 0$, which completes the construction. □

We now consider some projection properties of the upper Minkowski dimension and the packing dimension. We begin our consideration by comparing the upper Minkowski dimension of a bounded set in $\mathbf{R}^n$ with the upper Minkowski dimensions of its images under the orthogonal projections onto the m-dimensional linear subspaces spanned by a given basis of $\mathbf{R}^n$. As a corollary, we obtain a corresponding result for the packing dimension. For this purpose, we need the following two lemmas, in which we study the relations between the n-dimensional Lebesgue measure of a Suslin set in $\mathbf{R}^n$ and the m-dimensional Lebesgue measures of its images under the orthogonal projections onto the m-dimensional linear subspaces spanned by a given basis of $\mathbf{R}^n$. The first one of these two lemmas, Lemma 3.2, is used when proving the second one, Lemma 3.3. In Theorem 3.4, the estimates of Lemma 3.3 are applied to compact sets in order to prove the above mentioned result for the upper Minkowski dimension.

3.2. Lemma. *If $(e_1, \dots, e_n)$ is a basis of $\mathbf{R}^n$, then there exists a constant $0 < c < \infty$ such that for all bounded Suslin sets $A \subset \mathbf{R}^n$*

$$\mathcal{L}^{n-1}(P_{V(e_{i_1},\dots,e_{i_{n-1}})}(A)) \ge c\mathcal{L}^n(A)^{(n-1)/n}$$

for some $1 \le i_1 < \dots < i_{n-1} \le n$.

Proof. Let $A \subset \mathbf{R}^n$ be a bounded Suslin set. We may assume that $\mathcal{L}^n(A) > 0$ and $|e_i| = 1$ for all $1 \le i \le n$. We proceed by induction on n. Let $n = 2$. Define a linear map $L\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$L(t_1e_1 + t_2e_2) := (1 - (e_1 \cdot e_2)^2)^{-1}[(t_1 - t_2e_1 \cdot e_2)e_1 + (t_2 - t_1e_1 \cdot e_2)e_2]$$

for real numbers t_1 and t_2, where $e_i \cdot e_j$ denotes the usual inner product. Then, $L(P_{V(e_1)}(x) + P_{V(e_2)}(x)) = x$ for all $x \in \mathbf{R}^2$, and so

$$\begin{aligned}\mathcal{L}^2(A)^{1/2} &\le |\det L|^{1/2}(\mathcal{L}^1(P_{V(e_1)}(A))\mathcal{L}^1(P_{V(e_2)}(A)))^{1/2} \\ &\le \tfrac{1}{2}(1 - (e_1 \cdot e_2)^2)^{-1/2}(\mathcal{L}^1(P_{V(e_1)}(A)) + \mathcal{L}^1(P_{V(e_2)}(A))),\end{aligned}$$

where $\det L$ denotes the determinant of L. Thus, either

$$\mathcal{L}^1(P_{V(e_1)}(A)) \geq (1-(e_1\cdot e_2)^2)^{1/2}\mathcal{L}^2(A)^{1/2}$$

or

$$\mathcal{L}^1(P_{V(e_2)}(A)) \geq (1-(e_1\cdot e_2)^2)^{1/2}\mathcal{L}^2(A)^{1/2}.$$

Now, let $n \geq 3$ be such that the lemma holds with n replaced by $n-1$. We use for all $t \in \mathbf{R}$ the notation

$$A_t^n := \{x \in W \mid x + te_n \in A\},$$

where $W \in G(n,n-1)$ denotes the orthogonal complement of $V(e_n)$. Then A_t^n is a Suslin set since it is the inverse image of the Suslin set A under the continuous function $x \mapsto x + te_n$, where $x \in W$ (see [F1, p. 66]).

We first assume that $\mathcal{L}^{n-1}(A_t^n) \geq \mathcal{L}^n(A)^{(n-1)/n}$ for some $t \in \mathbf{R}$. If $W = V(e_1,\dots,e_{n-1})$, then $\mathcal{L}^{n-1}(P_{V(e_1,\dots,e_{n-1})}(A)) \geq \mathcal{L}^{n-1}(A_t^n) \geq \mathcal{L}^n(A)^{(n-1)/n}$.

If $W \neq V(e_1,\dots,e_{n-1})$, then $\widetilde{W} := W \cap V(e_1,\dots,e_{n-1}) \in G(n,n-2)$. For all $x \in \widetilde{W}$ we define

$$A_x := \{y \in A_t^n \mid P_{\widetilde{W}}(y) = x\}$$

and

$$B_x := \{y \in P_{V(e_1,\dots,e_{n-1})}(A_t^n) \mid P_{\widetilde{W}}(y) = x\}.$$

Then $\mathcal{L}^1(B_x) = \mathcal{L}^1(A_x)\cos\alpha$, where $0 < \alpha < \frac{1}{2}\pi$ is the angle which the line $V(e_n)$ forms with the orthogonal complement of $V(e_1,\dots,e_{n-1})$. Using Fubini's theorem, we obtain that

$$\begin{aligned}\mathcal{L}^n(A)^{(n-1)/n}\cos\alpha \leq \mathcal{L}^{n-1}(A_t^n)\cos\alpha &= \int_{\widetilde{W}} \mathcal{L}^1(A_x)\cos\alpha \, d\mathcal{L}^{n-2}x \\ &= \int_{\widetilde{W}} \mathcal{L}^1(B_x)\, d\mathcal{L}^{n-2}x = \mathcal{L}^{n-1}(P_{V(e_1,\dots,e_{n-1})}(A_t^n)) \\ &\leq \mathcal{L}^{n-1}(P_{V(e_1,\dots,e_{n-1})}(A)),\end{aligned}$$

where the last inequality follows from the translation invariance of the $(n-1)$-dimensional Lebesgue measure $\mathcal{L}^{n-1}$.

We now assume that $\mathcal{L}^{n-1}(A_t^n) \leq \mathcal{L}^n(A)^{(n-1)/n}$ for all $t \in \mathbf{R}$. Using Fubini's theorem, we obtain that

$$\mathcal{L}^n(A)^{(n-1)/n} = \mathcal{L}^n(A)^{-1/n}\int \mathcal{L}^{n-1}(A_t^n)\,dt \leq \int \mathcal{L}^{n-1}(A_t^n)^{(n-2)/(n-1)}\,dt. \tag{10}$$

For all $1 \leq i \leq n-1$, there exists $\tilde{e}_i \in V(e_i, e_n)$ such that $|\tilde{e}_i| = 1$ and $\tilde{e}_i \cdot e_n = 0$, whence $\tilde{e}_i \in V(e_i, e_n) \cap W$. Since $(e_1, \ldots, e_n)$ is a basis of $\mathbf{R}^n$, we see that $W = V(\tilde{e}_1, \ldots, \tilde{e}_{n-1})$. Further, using the fact that $V(e_i, e_n) = V(\tilde{e}_i, e_n)$ for all $1 \leq i \leq n-1$, we see that

$$V(e_{i_1}, \ldots, e_{i_{n-2}}, e_n) = V(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{n-2}}, e_n) \tag{11}$$

for all $1 \leq i_1 < \cdots < i_{n-2} \leq n-1$.

Identifying W with $\mathbf{R}^{n-1}$ and A_t^n with a Suslin subset of $\mathbf{R}^{n-1}$ for all $t \in \mathbf{R}$, we find that there exists by the induction hypothesis a constant $0 < c < \infty$ such that

$$\mathcal{L}^{n-1}(A_t^n)^{(n-2)/(n-1)} \leq \frac{1}{c} \sum_{1 \leq i_1 < \cdots < i_{n-2} \leq n-1} \mathcal{L}^{n-2}(P_{V(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{n-2}})}(A_t^n)) \tag{12}$$

for all $t \in \mathbf{R}$. Using (10), (12), Fubini's theorem, and (11), we obtain that

$$\begin{aligned} \mathcal{L}^n(A)^{(n-1)/n} &\leq \frac{1}{c} \sum_{1 \leq i_1 < \cdots < i_{n-2} \leq n-1} \int \mathcal{L}^{n-2}(P_{V(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{n-2}})}(A_t^n))\, dt \\ &= \frac{1}{c} \sum_{1 \leq i_1 < \cdots < i_{n-2} \leq n-1} \mathcal{L}^{n-1}(P_{V(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{n-2}}, e_n)}(A)) \\ &= \frac{1}{c} \sum_{1 \leq i_1 < \cdots < i_{n-2} \leq n-1} \mathcal{L}^{n-1}(P_{V(e_{i_1}, \ldots, e_{i_{n-2}}, e_n)}(A)). \end{aligned}$$

Thus,

$$\mathcal{L}^{n-1}(P_{V(e_{i_1}, \ldots, e_{i_{n-2}}, e_n)}(A)) \geq \frac{c}{n-1} \mathcal{L}^n(A)^{(n-1)/n}$$

for some $1 \leq i_1 < \cdots < i_{n-2} \leq n-1$. □

Using Lemma 3.2, we obtain:

3.3. Lemma. *If $(e_1, \ldots, e_n)$ is a basis of $\mathbf{R}^n$, then there exists a constant $0 < c < \infty$ such that for all bounded Suslin sets $A \subset \mathbf{R}^n$*

$$\mathcal{L}^{n-m}(P_{V(e_{i_1}, \ldots, e_{i_{n-m}})}(A)) \geq c\mathcal{L}^n(A)^{(n-m)/n}$$

for some $1 \leq i_1 < \cdots < i_{n-m} \leq n$.

Proof. Let $A \subset \mathbf{R}^n$ be a bounded Suslin set. We proceed by induction on m. If $m = 1$, the result follows from Lemma 3.2.

We now assume that there are a constant $0 < c_1 < \infty$ depending only on n, m, and the basis $(e_1, \ldots, e_n)$, and an $(n-(m-1))$-dimensional linear subspace W_1 spanned by $\{e_{i_1}, \ldots, e_{i_{n-(m-1)}}\}$ for some $1 \leq i_1 < \cdots < i_{n-(m-1)} \leq n$ such that

$$\mathcal{L}^{n-(m-1)}(P_{W_1}(A)) \geq c_1 \mathcal{L}^n(A)^{(n-(m-1))/n}.$$

Identifying W_1 with $\mathbf{R}^{n-(m-1)}$ and $P_{W_1}(A)$ with a Suslin subset of $\mathbf{R}^{n-(m-1)}$ and using Lemma 3.2, we find a constant $0 < c_2 < \infty$ depending only on n, m, and the basis $(e_{i_1}, \dots, e_{i_{n-(m-1)}})$, and an $(n-m)$-dimensional subspace $W_2 \subset W_1$ spanned by $\{e_{i_1}, \dots, e_{i_{j-1}}, e_{i_{j+1}}, \dots, e_{i_{n-(m-1)}}\}$ for some j such that

$$\mathcal{L}^{n-m}(P_{W_2} P_{W_1}(A)) \geq c_2 \mathcal{L}^{n-(m-1)}(P_{W_1}(A))^{(n-m)/(n-(m-1))}.$$

Thus

$$\mathcal{L}^{n-m}(P_{W_2}(A)) \geq c_1^{(n-m)/(n-(m-1))} c_2 \mathcal{L}^n(A)^{(n-m)/n}. \quad \square$$

By means of Lemma 3.3 we obtain:

3.4. Theorem. *If* $(e_1, \dots, e_n)$ *is a basis of* $\mathbf{R}^n$, *then for all bounded* $A \subset \mathbf{R}^n$,

$$\dim_{\mathrm{M}} P_{V(e_{i_1}, \dots, e_{i_m})}(A) \geq \frac{m}{n} \dim_{\mathrm{M}} A$$

for some $1 \leq i_1 < \cdots < i_m \leq n$.

Proof. We may assume that $\dim_{\mathrm{M}} A > 0$. Let $V_1, \dots, V_l$ be the m-dimensional linear subspaces spanned by the basis $(e_1, \dots, e_n)$. Assume that $\dim_{\mathrm{M}} P_{V_j}(A) < (m/n) \dim_{\mathrm{M}} A$ for all $1 \leq j \leq l$, and fix a real number t such that $\dim_{\mathrm{M}} P_{V_j}(A) < mt/n < (m/n) \dim_{\mathrm{M}} A$ for all $1 \leq j \leq l$. Since $\dim_{\mathrm{M}} A > t$, there exists a sequence (ε_i) of positive real numbers such that $\lim_{i \to \infty} \varepsilon_i = 0$ and $P(A, \varepsilon_i) > \varepsilon_i^{-t}$ for all i. Further, by Lemma 3.3 there exists a constant $0 < c < \infty$ and a subsequence (ε_{i_k}) of (ε_i) such that for some $1 \leq j \leq l$,

$$\mathcal{L}^m(P_{V_j}(A(\varepsilon_{i_k}))) \geq c \mathcal{L}^n(A(\varepsilon_{i_k}))^{m/n}$$

for all k.

Since $\mathcal{L}^n(A(\varepsilon)) \geq P(A, \varepsilon) a(n) \varepsilon^n$ and $P_{V_j}(A(\varepsilon)) \subset (P_{V_j}(A))(\varepsilon) \cap V_j$ for all $0 < \varepsilon < \infty$, we have that

$$\begin{aligned}(2\varepsilon_{i_k})^{mt/n-m} \mathcal{L}^m((P_{V_j}(A))(\varepsilon_{i_k}) \cap V_j) &\geq c 2^{mt/n-m} a(n)^{m/n} \varepsilon_{i_k}^{mt/n} P(A, \varepsilon_{i_k})^{m/n} \\ &\geq c 2^{mt/n-m} a(n)^{m/n}\end{aligned}$$

for all k, and so $\mathcal{M}^{mt/n}(P_{V_j}(A)) > 0$. Thus $\dim_{\mathrm{M}} P_{V_j}(A) \geq mt/n$, which gives a contradiction by the choice of t. $\square$

3.5. Corollary. *If* $(e_1, \dots, e_n)$ *is a basis of* $\mathbf{R}^n$, *then for all* $A \subset \mathbf{R}^n$,

$$\dim_{\mathrm{p}} P_{V(e_{i_1}, \dots, e_{i_m})}(A) \geq \frac{m}{n} \dim_{\mathrm{p}} A$$

for some $1 \leq i_1 < \cdots < i_m \leq n$.

Proof. We may assume that $\dim_p A > 0$. Let $V_1, \dots, V_l$ be the m-dimensional linear subspaces spanned by the basis $(e_1, \dots, e_n)$. Assume that $\dim_p P_{V_j}(A) < (m/n) \dim_p A$ for all $1 \le j \le l$, and fix a real number t such that $\dim_p P_{V_j}(A) < t < (m/n) \dim_p A$ for all $1 \le j \le l$. Then for all $1 \le j \le l$, there are bounded sets $A_1^j, A_2^j, \dots$ in $\mathbf{R}^n$ such that

$$P_{V_j}(A) \subset \bigcup_{i=1}^{\infty} A_i^j$$

and $\sup_i \dim_M A_i^j < t$. Now,

$$A = \bigcup_{i_1, \dots, i_l \ge 1} E_{i_1, \dots, i_l},$$

where

$$E_{i_1, \dots, i_l} := \bigcap_{j=1}^{l} (P_{V_j}^{-1}(A_{i_j}^j) \cap A)$$

is a bounded set, and thus

$$\dim_p A \le \sup_{i_1, \dots, i_l \ge 1} \dim_M E_{i_1, \dots, i_l}. \tag{13}$$

If $i_1, \dots, i_l \ge 1$, then, by Theorem 3.4, there is j such that $\dim_M P_{V_j}(E_{i_1, \dots, i_l}) \ge (m/n) \dim_M E_{i_1, \dots, i_l}$. Thus

$$\dim_M E_{i_1, \dots, i_l} \le \frac{n}{m} \dim_M P_{V_j}(E_{i_1, \dots, i_l}) \le \frac{n}{m} \dim_M A_{i_j}^j < \frac{n}{m} t.$$

By (13) we obtain that $\dim_p A \le mt/n$, which gives a contradiction by the choice of t. □

3.6. Remarks. (a) We will now construct an example which shows that the inequalities proved in Theorem 3.4 and Corollary 3.5 are the best possible ones. For this purpose, let $(e_1, \dots, e_n)$ be the standard basis of $\mathbf{R}^n$. We will show that for any $0 < s < n$ there is a compact set $E \subset \mathbf{R}^n$ such that $\dim_M E = \dim_p E = s$ and $\dim_M P_{V(e_{i_1}, \dots, e_{i_m})}(E) = \dim_p P_{V(e_{i_1}, \dots, e_{i_m})}(E) = ms/n$ for all $1 \le i_1 < \dots < i_m \le n$.

Consider $0 < s < n$. For all $1 \le i \le n$, we construct a compact set $E_i \subset V(e_i)$ with $\dim_H E_i = \dim_M E_i = \dim_p E_i = s/n$. One example of this kind of set is the Cantor λ-set $C(\lambda)$, where $\lambda := 2^{-n/s}$ (for the construction of $C(\lambda)$ see [M3, 4.10]). The fact that $\dim_H C(\lambda) = s/n$ follows from [M3, 4.10], and the fact that $\dim_M C(\lambda) = s/n$ follows from [M3, Corollary 5.8]. Thus $\dim_p C(\lambda) = s/n$.

We define

$$E := E_1 + \dots + E_n.$$

Then, [F, Product formulas 7.2 and 7.5] imply that $\dim_{\mathrm{M}} E = \dim_{\mathrm{p}} E = s$, $\dim_{\mathrm{M}} P_{V(e_{i_1},\dots,e_{i_m})}(E) = \dim_{\mathrm{M}}(E_{i_1} + \cdots + E_{i_m}) = ms/n$, and analogously, $\dim_{\mathrm{p}} P_{V(e_{i_1},\dots,e_{i_m})}(E) = ms/n$ for all $1 \le i_1 < \cdots < i_m \le n$, which completes the construction.

(b) The fact that the analogue of Theorem 3.4 and Corollary 3.5 does not hold for the Hausdorff dimension follows from [F, Example 7.8].

(c) Let $A \subset \mathbf{R}^n$ be bounded. We define

$$B := \Big\{L \in G(n,1) \mid \dim_{\mathrm{M}} P_L(A) < \frac{1}{n} \dim_{\mathrm{M}} A\Big\}$$

and show that $\gamma_{n,1}(B) = 0$.

Since $G(n,1)$ can be identified with $S^{n-1}/\sim$, where $S^{n-1} := \{x \in \mathbf{R}^n \mid |x| = 1\}$ and $\sim$ is an equivalence relation defined for all $x, y \in S^{n-1}$ by

$$x \sim y \iff x = y \text{ or } x = -y,$$

we have $\gamma_{n,1} = \pi_\sharp \omega_{n-1}$, where $\pi\colon S^{n-1} \to S^{n-1}/\sim$ is the projection and ω_{n-1} is the normalized area measure on S^{n-1}. By Theorem 3.4, there is an $(n-1)$-dimensional linear subspace $W \subset \mathbf{R}^n$ containing each $L \in B$, and so $\pi^{-1}(B)$ lies in $W \cap S^{n-1}$, whence $\gamma_{n,1}(B) = 0$.

(d) In the same way as in the previous remark, we see by Corollary 3.5 that for all $A \subset \mathbf{R}^n$ there is an $(n-1)$-plane containing each $L \in G(n,1)$ for which $\dim_{\mathrm{p}} P_L(A) < (1/n) \dim_{\mathrm{p}} A$; in particular,

$$\dim_{\mathrm{p}} P_L(A) \ge \frac{1}{n} \dim_{\mathrm{p}} A$$

for $\gamma_{n,1}$ almost all $L \in G(n,1)$.

(e) Let $A \subset \mathbf{R}^n$ be bounded. We define

$$B := \Big\{V \in G(n,n-1) \mid \dim_{\mathrm{M}} P_V(A) < \frac{n-1}{n} \dim_{\mathrm{M}} A\Big\}$$

and show that there are no $V_1, \dots, V_n \in B$ with $V_1^\perp + \cdots + V_n^\perp = \mathbf{R}^n$.

Assume that there are $V_1, \dots, V_n \in B$ such that $V_1^\perp + \cdots + V_n^\perp = \mathbf{R}^n$, and fix $x_j \in V_j^\perp$ such that $x_j \ne 0$ for all $1 \le j \le n$.

Then $V_{i_1} \cap \cdots \cap V_{i_{n-1}} = (V_{i_1}^\perp + \cdots + V_{i_{n-1}}^\perp)^\perp$ is a line for all $1 \le i_1 < \cdots < i_{n-1} \le n$.

Further, we will show that the lines which are of the form $V_{i_1} \cap \cdots \cap V_{i_{n-1}}$ for some $1 \le i_1 < \cdots < i_{n-1} \le n$ span $\mathbf{R}^n$.

We fix $y_1 \in V_2 \cap \cdots \cap V_n$, $y_k \in V_1 \cap \cdots \cap V_{k-1} \cap V_{k+1} \cap \cdots \cap V_n$ for $2 \le k \le n-1$, and $y_n \in V_1 \cap \cdots \cap V_{n-1}$ such that $y_j \ne 0$ for all $1 \le j \le n$. Assume that there are an integer $2 \le k \le n-1$ (the cases $k = 1$ and $k = n$ are similar to the case considered here) and real numbers $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n$ such that $y_k = a_1y_1 + \cdots + a_{k-1}y_{k-1} + a_{k+1}y_{k+1} + \cdots + a_ny_n$. Fix real numbers $b_1, \ldots, b_n$ such that $y_k = b_1x_1 + \cdots + b_nx_n$. Since $y_i \cdot x_j = 0$ whenever $i \neq j$, we have that

$$\begin{aligned} y_k \cdot y_k &= y_k \cdot (b_1x_1 + \cdots + b_nx_n) = b_ky_k \cdot x_k \\ &= b_k(a_1y_1 + \cdots + a_{k-1}y_{k-1} + a_{k+1}y_{k+1} + \cdots + a_ny_n) \cdot x_k = 0, \end{aligned}$$

which gives a contradiction.

Thus, the lines which are of the form $V_{i_1} \cap \cdots \cap V_{i_{n-1}}$ for some $1 \leq i_1 < \cdots < i_{n-1} \leq n$ span $\mathbf{R}^n$. Further, for each $1 \leq k \leq n$, the $n-1$ lines $\cap_{i \neq l} V_i$ with $1 \leq l \leq n$, $l \neq k$, span V_k, and so Theorem 3.4 implies that $\dim_{\mathrm{M}} P_{V_j}(A) \geq ((n-1)/n) \dim_{\mathrm{M}} A$ for some $1 \leq j \leq n$, which gives a contradiction by the definition of B.

Since there are no $V_1, \ldots, V_n \in B$ such that $V_1^\perp + \cdots + V_n^\perp = \mathbf{R}^n$, the union of the orthogonal complements of those hyperplanes V for which $\dim_{\mathrm{M}} P_V(A) < ((n-1)/n) \dim_{\mathrm{M}} A$ lies in some $(n-1)$-plane. Since $\gamma_{n,n-1}(B) = \gamma_{n,1}(\{V^\perp \mid V \in B\})$ by the uniqueness of the measure $\gamma_{n,n-1}$, we see in the same way as in Remark (c) that $\gamma_{n,n-1}(B) = 0$.

(f) Let $A \subset \mathbf{R}^n$. Define

$$B := \Big\{ V \in G(n, n-1) \mid \dim_{\mathrm{p}} P_V(A) < \frac{n-1}{n} \dim_{\mathrm{p}} A \Big\}.$$

Similarly to the previous remark, we see by means of Corollary 3.5 that there are no $V_1, \ldots, V_n \in B$ such that $V_1^\perp + \cdots + V_n^\perp = \mathbf{R}^n$; in particular, $\gamma_{n,n-1}(B) = 0$.

(g) For $n = 2$, we will now present another way of proving Theorem 3.4.

Let $A \subset \mathbf{R}^2$ be bounded. We may assume that $\dim_{\mathrm{M}} A > 0$. Assume that there exists $L \in G(2,1)$ such that $\dim_{\mathrm{M}} P_L(A) < \frac{1}{2} \dim_{\mathrm{M}} A$. We will prove that $\dim_{\mathrm{M}} P_{\widetilde{L}}(A) \geq \frac{1}{2} \dim_{\mathrm{M}} A$ for all $\widetilde{L} \in G(2,1)$ with $\widetilde{L} \neq L$. We may assume that L is the x_1-axis.

For all integers i and k, we use the notation $\mathcal{D}_i^k$ for the family of the dyadic squares which meet A and are of the form

$$\{x \in \mathbf{R}^2 \mid (k-1)2^{-i} \leq x_1 < k2^{-i}, \quad (l-1)2^{-i} \leq x_2 < l2^{-i}\},$$

where l is an arbitrary integer. We denote by M_i^k the number of the squares belonging to $\mathcal{D}_i^k$. Let K_i be such an integer that

$$M_i^{K_i} := \max_k M_i^k.$$

We consider real numbers t and u such that $\dim_{\mathrm{M}} P_L(A) < t < \frac{1}{2} \dim_{\mathrm{M}} A$ and $u < t$. Then, $N_i(A) \leq N_i(P_L(A)) M_i^{K_i}$ for all i, $N_i(A) \geq 2^{2ti}$ for arbitrarily large

i, and $N_i(P_L(A)) \le 2^{ti}$ for all sufficiently large i. Hence, $M_i^{K_i} \ge 2^{ti}$ for arbitrarily large i.

Consider an integer i with $M_i^{K_i} \ge 2^{ti}$, and define

$$\alpha_i := 2 \cdot 2^{-(t-u)i}.$$

Let p_i be the smallest integer with $p_i \ge \alpha_i^{-1}$. We may assume that $p_i + 1 \le 2\alpha_i^{-1}$.

We choose the smallest integer l such that the square

$$Q_i^1 := \{x \in \mathbf{R}^2 \mid (K_i - 1)2^{-i} \le x_1 < K_i 2^{-i}, \quad (l-1)2^{-i} \le x_2 < l2^{-i}\}$$

intersects A. Next we select, if it is possible, the square $Q_i^2 \in \mathcal{D}_i^{K_i}$ which is closest to Q_i^1 such that $d(Q_i^1, Q_i^2) \ge p_i 2^{-i}$. Continuing in this way, we select a maximum number n_i of squares $Q_i^1, \ldots, Q_i^{n_i} \in \mathcal{D}_i^{K_i}$ with $d(Q_i^j, Q_i^{j'}) \ge p_i 2^{-i}$ if $j \ne j'$. Since $2^{ti} \le M_i^{K_i} \le n_i(p_i + 1) \le 2n_i\alpha_i^{-1}$, we have that $n_i \ge 2^{ui}$.

Let $L_\theta \in G(2,1)$ be the line forming an angle $0 < \theta < \pi$ with the positive x_1-axis. If the distance between two squares belonging to $\mathcal{D}_i^{K_i}$ is greater than $2^{-i}(\tan\theta)^{-1}$ for $0 < \theta < \frac{1}{2}\pi$ and greater than $2^{-i}(\tan(\pi - \theta))^{-1}$ for $\frac{1}{2}\pi < \theta < \pi$, then they project as disjoint intervals to the line L_θ.

Now if $\alpha_i < \theta < \frac{1}{2}\pi$, then

$$d(Q_i^j, Q_i^{j+1}) \ge 2^{-i}p_i \ge 2^{-i}\alpha_i^{-1} > 2^{-i}(\tan\theta)^{-1}$$

for all $1 \le j \le n_i - 1$, and so the squares $Q_i^1, \ldots, Q_i^{n_i}$ project as disjoint intervals of length greater than 2^{-i} to the line L_θ. Thus $N_i(P_{L_\theta}(A)) \ge \frac{1}{2}n_i \ge \frac{1}{2}2^{ui}$. Similarly, we see that $N_i(P_{L_\theta}(A)) \ge \frac{1}{2}2^{ui}$, if $\frac{1}{2}\pi \le \theta < \pi - \alpha_i$.

Thus $N_i(P_{L_\theta}(A)) \ge \frac{1}{2}2^{ui}$ for arbitrarily large integers i, and so

$$\dim_{\mathrm{M}} P_{L_\theta}(A) \ge u$$

for all u and t with $u < t < \frac{1}{2}\dim_{\mathrm{M}} A$, which gives the desired result.

(h) Let $A \subset \mathbf{R}^{2m}$ be a bounded set such that $\dim_{\mathrm{M}} A > 0$. By generalizing the method used in the previous remark, we can characterize the set of those $V \in G(2m, m)$ for which $\dim_{\mathrm{M}} P_V(A) < \frac{1}{2}\dim_{\mathrm{M}} A$.

We assume that there exists $V \in G(2m, m)$ such that $\dim_{\mathrm{M}} P_V(A) < \frac{1}{2}\dim_{\mathrm{M}} A$. We will prove that $\dim_{\mathrm{M}} P_W(A) \ge \frac{1}{2}\dim_{\mathrm{M}} A$ for all $W \in G(2m, m)$ with $V \cap W = \{0\}$. We may assume that V is the $x_1 \ldots x_m$-plane.

Let $W \in G(2m, m)$ be such that $V \cap W = \{0\}$. We claim that there exists a constant $0 < L \le 1$ such that

$$L|x - y| \le |P_W(x) - P_W(y)| \tag{14}$$

for all $x, y \in V^{\perp}$. In fact, we may assume that $y = 0$ and $|x| = 1$ in (14). Now it suffices to observe that, since $V^{\perp} \cap W^{\perp} = \{0\}$, the continuous function $x \mapsto |P_W(x)|$ on the compact set $\{x \in V^{\perp} \mid |x| = 1\}$ is positive and thus attains a positive minimum.

Let N be a positive integer such that $N > (L^{-2}(1+\sqrt{m})^2 + m)^{1/2}$.

For all integers i and $k_1, \ldots, k_m$, we use the notation $\mathcal{D}_i^{k_1,\ldots,k_m}$ for the family of the dyadic cubes which meet A and are of the form

$$\{x \in \mathbf{R}^{2m} \mid (k_j - 1)2^{-i} \leq x_j < k_j 2^{-i} \text{ for all } 1 \leq j \leq 2m\},$$

where $k_{m+1}, \ldots, k_{2m}$ are arbitrary integers. We denote by $M_i^{k_1,\ldots,k_m}$ the number of the dyadic cubes belonging to $\mathcal{D}_i^{k_1,\ldots,k_m}$. For each i, let $K_1(i), \ldots, K_m(i)$ be the integers with

$$M_i := M_i^{K_1(i),\ldots,K_m(i)} = \max_{k_1,\ldots,k_m} M_i^{k_1,\ldots,k_m}.$$

Consider a real number t with $\dim_{\mathrm{M}} P_V(A) < t < \frac{1}{2}\dim_{\mathrm{M}} A$. In the same way as in the previous remark, we see that $M_i \geq 2^{ti}$ for arbitrarily large i.

Consider an integer i with $M_i \geq 2^{ti}$. Letting the cubes in $\mathcal{D}_i^{K_1(i),\ldots,K_m(i)}$ be Q_i^j with $1 \leq j \leq M_i$, we choose $a_i^j \in A \cap Q_i^j$ for each j. We use the notation

$$B_i^j := P_V(B(a_i^j, N2^{-i})) + P_{V^{\perp}}(B(a_i^j, N2^{-i})).$$

Then, $Q_i^j \subset B_i^j$ for all $1 \leq j \leq M_i$. Further, we define

$$3B_i^j := P_V(B(a_i^j, 3N2^{-i})) + P_{V^{\perp}}(B(a_i^j, 3N2^{-i})).$$

Now, [F1, Corollary 2.8.5] implies that there are $1 \leq j_1 < \cdots < j_{P_i} \leq M_i$ such that the family $\{B_i^{j_k} \mid 1 \leq k \leq P_i\}$ is disjoint and

$$\bigcup_{1\leq j\leq M_i} Q_i^j \subset \bigcup_{1\leq j\leq M_i} B_i^j \subset \bigcup_{1\leq k\leq P_i} 3B_i^{j_k},$$

and so, by comparing the $2m$-dimensional Lebesgue measures, we obtain that $P_i \geq \alpha(m)^{-2}(3N)^{-2m} M_i \geq \alpha(m)^{-2}(3N)^{-2m} 2^{ti}$.

Assume that $1 \leq k < l \leq P_i$. Then,

$$\begin{aligned}(2N2^{-i})^2 < |a_i^{j_k} - a_i^{j_l}|^2 &= |P_V(a_i^{j_k}) - P_V(a_i^{j_l})|^2 + |P_{V^{\perp}}(a_i^{j_k}) - P_{V^{\perp}}(a_i^{j_l})|^2 \\ &\leq m2^{-2i} + |P_{V^{\perp}}(a_i^{j_k}) - P_{V^{\perp}}(a_i^{j_l})|^2,\end{aligned}$$

and so

$$|P_{V^{\perp}}(a_i^{j_k}) - P_{V^{\perp}}(a_i^{j_l})| \geq (4N^2 - m)^{1/2} 2^{-i}. \tag{15}$$

Now we have by (14), (15), and the choice of N that

$$\begin{aligned}|P_W(a_i^{j_k}) - P_W(a_i^{j_l})| &\\ &\geq |P_W(P_{V^\perp}(a_i^{j_k})) - P_W(P_{V^\perp}(a_i^{j_l}))| - |P_W(P_V(a_i^{j_k})) - P_W(P_V(a_i^{j_l}))| \\ &\geq L|P_{V^\perp}(a_i^{j_k}) - P_{V^\perp}(a_i^{j_l})| - |P_V(a_i^{j_k}) - P_V(a_i^{j_l})| \\ &\geq (L(4N^2 - m)^{1/2} - \sqrt{m})2^{-i} > 2^{-i}.\end{aligned}$$

Hence, $P(P_W(A), \frac{1}{2}2^{-i}) \geq P_i \geq \alpha(m)^{-2}(3N)^{-2m}2^{ti}$. Since i can be chosen arbitrarily large, we get $\dim_{\mathrm{M}} P_W(A) \geq t$. Since this holds for all $t < \frac{1}{2}\dim_{\mathrm{M}} A$, we get the desired result.

Thus, if we define

$$B := \{V \in G(2m, m) \mid \dim_{\mathrm{M}} P_V(A) < \tfrac{1}{2}\dim_{\mathrm{M}} A\}$$

and assume that there exists $V \in B$, then we have by the above considerations that

$$B \subset \{W \in G(2m, m) \mid V \cap W \neq \{0\}\},$$

and so, using [M3, Lemma 3.13], we obtain that $\gamma_{2m,m}(B) = 0$.

The method by which Theorem 3.4 and Corollary 3.5 can be extended to $\gamma_{n,m}$ almost all $V \in G(n, m)$ is based on the following theorem.

3.7. Theorem. *Assume that $B \subset G(n, m)$ has the following properties:*

(i) *B is $\gamma_{n,m}$-measurable.*

(ii) *For every basis $(e_1, \ldots, e_n)$ of $\mathbf{R}^n$ there are $1 \leq i_1 < \cdots < i_m \leq n$ such that $V(e_{i_1}, \ldots, e_{i_m}) \notin B$.*

Then $\gamma_{n,m}(B) = 0$.

Proof. We assume that $\gamma_{n,m}(B) > 0$. Then B has a compact subset with positive measure (see [F1, p. 63]); clearly, condition (ii) is also true if B is replaced by this subset. Thus, by considering this subset, we may assume that B is compact.

We use the notation

$$U := \{(x_1, \ldots, x_m) \in U(0,1)^m \mid \text{the sequence } (x_1, \ldots, x_m) \text{ is linearly independent}\}$$

and define a continuous function $f\colon U \to G(n, m)$ by

$$f(x_1, \ldots, x_m) := V(x_1, \ldots, x_m).$$

Then $f^{-1}(B)$ is closed in U.

By Lemma 2.1, the measure $f_\sharp((\mathcal{L}^n \times \cdots \times \mathcal{L}^n) \llcorner U)$, which is orthogonally invariant, is a Radon measure on $G(n,m)$. So the uniqueness of $\gamma_{n,m}$ implies that there exists a constant $0 < c < \infty$ such that

$$\gamma_{n,m} = c f_\sharp((\mathcal{L}^n \times \cdots \times \mathcal{L}^n) \llcorner U).$$

Now $\mathcal{L}^n \times \cdots \times \mathcal{L}^n(f^{-1}(B)) > 0$, and so $f^{-1}(B)$ has a density point $(b_1, \ldots, b_m)$ (see [M3, Corollary 2.14]), that is,

$$\text{(16)} \qquad \lim_{r \to 0} \frac{\mathcal{L}^n \times \cdots \times \mathcal{L}^n((B(b_1,r) \times \cdots \times B(b_m,r)) \cap f^{-1}(B))}{\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B(b_1,r) \times \cdots \times B(b_m,r))} = 1.$$

We now select $b_{m+1}, \ldots, b_n \in V(b_1, \ldots, b_m)$ such that $(b_{i_1}, \ldots, b_{i_m}) \in U$ for all $1 \le i_1 < \cdots < i_m \le n$. Then for all $1 \le k \le n-m$ and $1 \le i_1 < \cdots < i_m \le m+k-1$ there exists $\alpha_j^k(i_1, \ldots, i_m) \in \mathbf{R} \setminus \{0\}$ for all $1 \le j \le m$ such that

$$b_{m+k} = \sum_{j=1}^{m} \alpha_j^k(i_1, \ldots, i_m) b_{i_j}.$$

Define

$$M := \max_{\substack{1 \le k \le n-m \\ 1 \le i_1 < \cdots < i_m \le m+k-1}} \Big\{1, \sum_{j=1}^{m} |\alpha_j^k(i_1, \ldots, i_m)|\Big\}.$$

Since U is open, there exists $R > 0$ such that

$$B_{i_1,\ldots,i_m}(r) := B(b_{i_1}, r) \times \cdots \times B(b_{i_m}, r) \subset U$$

for all $1 \le i_1 < \cdots < i_m \le n$ and $r \le R$.

We will prove that there are constants $0 < C \le 1$ and $1 \le D < \infty$ such that for all $1 \le i_1 < \cdots < i_m \le n$

$$\text{(17)} \quad \mathcal{L}^n \times \cdots \times \mathcal{L}^n(B_{i_1,\ldots,i_m}(r) \cap (U \setminus f^{-1}(B))) \le D\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B_{1,\ldots,m}(r/C) \cap (U \setminus f^{-1}(B)))$$

for all $r \le CR/M$.

In order to prove (17), we will show that for all $0 \le k \le n-m$ there are constants $0 < C_k \le 1$ and $1 \le D_k < \infty$ such that for all $1 \le i_1 < \cdots < i_m \le m+k$

$$\text{(18)} \quad \mathcal{L}^n \times \cdots \times \mathcal{L}^n(B_{i_1,\ldots,i_m}(r) \cap (U \setminus f^{-1}(B))) \le D_k\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B_{1,\ldots,m}(r/C_k) \cap (U \setminus f^{-1}(B)))$$

for all $r \le C_k R/M$. This gives (17) when we choose $C := C_{n-m}$ and $D := D_{n-m}$.

To prove (18) we proceed by induction on k. The case $k = 0$ is trivial.

We now assume that (18) is true for k. Let $1 \le i_1 < \dots < i_m \le m + k + 1$. If $i_m \le m+k$, then by the induction hypothesis, (18) is true for $k+1$ when we choose $C_{k+1} \le C_k$ and $D_{k+1} \ge D_k$. So it suffices to consider the case $i_m = m + k + 1$.

Let $l \ge 1$ be the smallest integer with $l \ne i_j$ for all $1 \le j \le m$ and $1 \le p \le m$ the smallest integer with $l < i_p$. We assume that $p \ge 2$; the case $p = 1$ only involves some slight modifications in the following notation.

We define $j_k := i_k$ for all $1 \le k \le p-1$, $j_p := l$, and $j_k := i_{k-1}$ for all $p+1 \le k \le m$.

Define a Lipschitz map $L_{i_1,\dots,i_m}\colon B_{j_1,\dots,j_m}(r) \to B_{i_1,\dots,i_m}(R)$ for all $r \le R/M$ by

$$L_{i_1,\dots,i_m}(x_1,\dots,x_m) := \Big(x_1,\dots,x_{p-1},x_{p+1},\dots,x_m,\sum_{i=1}^{m}\alpha_i^{k+1}(j_1,\dots,j_m)x_i\Big);$$

recall that here $\alpha_i^{k+1}(j_1,\dots,j_m) \in \mathbf{R} \setminus \{0\}$ for all $1 \le i \le m$ and

$$b_{i_m} = b_{m+k+1} = \sum_{i=1}^{m}\alpha_i^{k+1}(j_1,\dots,j_m)b_{j_i}. \tag{19}$$

Since $B_{i_1,\dots,i_m}(R) \subset U$, we have that

$$V(x_1,\dots,x_m) = V(L_{i_1,\dots,i_m}(x_1,\dots,x_m)) \tag{20}$$

for all $(x_1,\dots,x_m) \in B_{j_1,\dots,j_m}(r)$. Furthermore, we will see that

$$B_{i_1,\dots,i_m}(r) \subset L_{i_1,\dots,i_m}(B_{j_1,\dots,j_m}(r/k(i_1,\dots,i_m))) \tag{21}$$

for all $r \le k(i_1,\dots,i_m)R/M$, where

$$k(i_1,\dots,i_m) := \min\Big\{1, |\alpha_p^{k+1}(j_1,\dots,j_m)|\Big(1+\sum_{i\ne p}|\alpha_i^{k+1}(j_1,\dots,j_m)|\Big)^{-1}\Big\}.$$

Now, (21) follows from the fact that

$$(x_1,\dots,x_m) = L_{i_1,\dots,i_m}(x_1,\dots,x_{p-1},y,x_p,\dots,x_{m-1})$$

for all $(x_1,\dots,x_m) \in B_{i_1,\dots,i_m}(r)$, where

$$\begin{aligned} y = {} & (\alpha_p^{k+1}(j_1,\dots,j_m))^{-1}(x_m - \alpha_1^{k+1}(j_1,\dots,j_m)x_1 - \dots \\ & - \alpha_{p-1}^{k+1}(j_1,\dots,j_m)x_{p-1} - \alpha_{p+1}^{k+1}(j_1,\dots,j_m)x_p - \dots - \alpha_m^{k+1}(j_1,\dots,j_m)x_{m-1}). \end{aligned}$$

Here we have by (19) that

$$\begin{aligned}|y-b_{j_p}| &= (|\alpha_p^{k+1}(j_1,\dots,j_m)|)^{-1}|x_m-\alpha_1^{k+1}(j_1,\dots,j_m)x_1\\ &\quad -\dots-\alpha_{p-1}^{k+1}(j_1,\dots,j_m)x_{p-1}-\alpha_{p+1}^{k+1}(j_1,\dots,j_m)x_p\\ &\quad -\dots-\alpha_m^{k+1}(j_1,\dots,j_m)x_{m-1}-\alpha_p^{k+1}(j_1,\dots,j_m)b_{j_p}|\\ &\le (|\alpha_p^{k+1}(j_1,\dots,j_m)|)^{-1}(|x_m-b_{i_m}|+|b_{i_m}-\alpha_1^{k+1}(j_1,\dots,j_m)x_1\\ &\quad -\dots-\alpha_{p-1}^{k+1}(j_1,\dots,j_m)x_{p-1}-\alpha_{p+1}^{k+1}(j_1,\dots,j_m)x_p\\ &\quad -\dots-\alpha_m^{k+1}(j_1,\dots,j_m)x_{m-1}-\alpha_p^{k+1}(j_1,\dots,j_m)b_{j_p}|)\\ &\le r(|\alpha_p^{k+1}(j_1,\dots,j_m)|)^{-1}\Big(1+\sum_{i\neq p}|\alpha_i^{k+1}(j_1,\dots,j_m)|\Big)\le r/k(i_1,\dots,i_m),\end{aligned}$$

whence $y\in B(b_{j_p},r/k(i_1,\dots,i_m))$.

If we choose

$$c_{k+1}:=\min_{1\le i_1<\dots<i_{m-1}<i_m=m+k+1}k(i_1,\dots,i_m)$$

and

$$d_{k+1}:=\max_{1\le i_1<\dots<i_{m-1}<i_m=m+k+1}\{1,\mathrm{Lip}(L_{i_1,\dots,i_m})^{nm}\},$$

where $\mathrm{Lip}(L_{i_1,\dots,i_m})$ denotes the Lipschitz constant of $L_{i_1,\dots,i_m}$, then we have by (21) and (20) for all $1\le i_1<\dots<i_{m-1}<i_m=m+k+1$ that

$$\begin{aligned}(22)\qquad &\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{i_1,\dots,i_m}(r)\cap(U\setminus f^{-1}(B)))\\ &\le\mathcal{L}^n\times\dots\times\mathcal{L}^n(L_{i_1,\dots,i_m}(B_{j_1,\dots,j_m}(r/k(i_1,\dots,i_m))\cap(U\setminus f^{-1}(B))))\\ &\le d_{k+1}\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{j_1,\dots,j_m}(r/c_{k+1})\cap(U\setminus f^{-1}(B)))\end{aligned}$$

for all $r\le c_{k+1}R/M$. Now, $j_m=i_{m-1}\le m+k$, and so we obtain by the induction hypothesis and (22) for all $1\le i_1<\dots<i_{m-1}<i_m=m+k+1$ that

$$\begin{aligned}&\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{i_1,\dots,i_m}(r)\cap(U\setminus f^{-1}(B)))\\ &\qquad\le d_{k+1}D_k\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{1,\dots,m}(r/(c_{k+1}C_k))\cap(U\setminus f^{-1}(B)))\end{aligned}$$

for all $r\le c_{k+1}C_kR/M$. Thus, if we choose $C_{k+1}:=c_{k+1}C_k$ and $D_{k+1}:=d_{k+1}D_k$, we obtain (18) for $k+1$.

Let $\varepsilon>0$. By (16), there exists $R_\varepsilon>0$ such that

$$\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{1,\dots,m}(r)\cap(U\setminus f^{-1}(B)))\le\varepsilon\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{1,\dots,m}(r))$$

for all $r\le R_\varepsilon$, and so by (17), we have for all $1\le i_1<\dots<i_m\le n$ that

$$\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{i_1,\dots,i_m}(r)\cap(U\setminus f^{-1}(B)))\le DC^{-nm}\varepsilon\mathcal{L}^n\times\dots\times\mathcal{L}^n(B_{1,\dots,m}(r))$$

for all $r \leq \min\{CR_\varepsilon, CR/M\}$. Thus for all $1 \leq i_1 < \cdots < i_m \leq n$ we obtain that

$$\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B_{i_1,\dots,i_m}(r) \cap f^{-1}(B)) \geq (1 - DC^{-nm}\varepsilon)\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B_{i_1,\dots,i_m}(r))$$

for all $r \leq \min\{CR_\varepsilon, CR/M\}$. By the following Lemma 3.8, we can choose a basis $(x_1, \dots, x_n)$ of $\mathbf{R}^n$ such that $(x_{i_1}, \dots, x_{i_m}) \in f^{-1}(B)$ for all $1 \leq i_1 < \cdots < i_m \leq n$. Then $V(x_{i_1}, \dots, x_{i_m}) = f(x_{i_1}, \dots, x_{i_m}) \in B$ for all $1 \leq i_1 < \cdots < i_m \leq n$, which contradicts condition (ii) and so completes the proof. □

3.8. Lemma. *Suppose that* $b_1, \dots, b_n \in U(0,1) \subset \mathbf{R}^n$ *and that* $B \subset U(0,1)^m \subset \mathbf{R}^{nm}$ *is a Borel set such that for all* $0 < \varepsilon < 1$ *there exists* $r > 0$ *such that*

$$\mathcal{L}^n \times \cdots \times \mathcal{L}^n((B(b_{i_1}, r) \times \cdots \times B(b_{i_m}, r)) \cap B) \geq (1 - \varepsilon)\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B(b_{i_1}, r) \times \cdots \times B(b_{i_m}, r))$$

for all $1 \leq i_1 < \cdots < i_m \leq n$. *Then there are* $R > 0$ *and* $x_i \in B(b_i, R)$ *for all* $1 \leq i \leq n$ *such that the sequence* $(x_1, \dots, x_n)$ *is linearly independent and* $(x_{i_1}, \dots, x_{i_m}) \in B$ *for all* $1 \leq i_1 < \cdots < i_m \leq n$.

Proof. We omit the easy proof in the case $m = 1$ and assume that $m \geq 2$. Let $k = n - m$.

We define

$$M_i := \sum_{l=\max\{1,i-k\}}^{\min\{i,m\}} \binom{i-1}{l-1}\binom{n-i}{m-l}$$

for all $1 \leq i \leq n$ and

$$M := \max_{1 \leq i \leq n} M_i.$$

Then there exists $R > 0$ such that

$$\mathcal{L}^n \times \cdots \times \mathcal{L}^n((B(b_{i_1}, R) \times \cdots \times B(b_{i_m}, R)) \cap B) > (1 - M^{-m})\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B(b_{i_1}, R) \times \cdots \times B(b_{i_m}, R))$$

for all $1 \leq i_1 < \cdots < i_m \leq n$.

We use the notation

$$\alpha := \min_{1 \leq i_1 < \cdots < i_m \leq n} \mathcal{L}^n \times \cdots \times \mathcal{L}^n((B(b_{i_1}, R) \times \cdots \times B(b_{i_m}, R)) \cap B)$$

and

$$\beta := \mathcal{L}^n(B(0, R))$$

and fix γ such that $(\beta^m - \alpha)^{1/m} < \gamma < \beta/M$.

If $1 \le l \le m-1$ and $1 \le p_1 < \cdots < p_l < i_1 < \cdots < i_{m-l} \le n$, then we define for all $x_{p_q} \in B(b_{p_q}, R)$, $1 \le q \le l$, a Borel set

$$B^{x_{p_1},\dots,x_{p_l}}_{i_1,\dots,i_{m-l}} := \{(x_{i_1},\dots,x_{i_{m-l}}) \in B(b_{i_1},R) \times \cdots \times B(b_{i_{m-l}},R) \mid (x_{p_1},\dots,x_{p_l},x_{i_1},\dots,x_{i_{m-l}}) \in B\}.$$

If $1 \le j \le k+1$ and $j < i_1 < \cdots < i_{m-1} \le n$, we define a Borel set

$$A_j^{i_1,\dots,i_{m-1}} := \{x \in B(b_j,R) \mid \mathcal{L}^n \times \cdots \times \mathcal{L}^n(B^x_{i_1,\dots,i_{m-1}}) \ge \beta^{m-1} - \gamma^{m-1}\}.$$

Since by Fubini's theorem,

$$\begin{aligned}
\alpha &\le \mathcal{L}^n \times \cdots \times \mathcal{L}^n((B(b_j,R) \times B(b_{i_1},R) \times \cdots \times B(b_{i_{m-1}},R)) \cap B) \\
&= \int_{B(b_j,R)} \mathcal{L}^n \times \cdots \times \mathcal{L}^n(B^x_{i_1,\dots,i_{m-1}})\, d\mathcal{L}^n x \\
&\le \beta^{m-1}\mathcal{L}^n(A_j^{i_1,\dots,i_{m-1}}) + (\beta^{m-1} - \gamma^{m-1})\mathcal{L}^n(B(b_j,R) \setminus A_j^{i_1,\dots,i_{m-1}}) \\
&= \beta^m - \gamma^{m-1}(\beta - \mathcal{L}^n(A_j^{i_1,\dots,i_{m-1}})),
\end{aligned}$$

we have by the choice of γ that

$$\mathcal{L}^n(A_j^{i_1,\dots,i_{m-1}}) \ge \beta - \gamma. \tag{23}$$

Further, if $m \ge 3$, $2 \le l \le m-1$, $l \le j \le k+l$, and $1 \le p_1 < \cdots < p_{l-1} < j < i_1 < \cdots < i_{m-l} \le n$, we define for all $x_{p_q} \in B(b_{p_q}, R)$, $1 \le q \le l-1$, a Borel set

$$A_j^{i_1,\dots,i_{m-l}}(x_{p_1},\dots,x_{p_{l-1}}) := \{x \in B(b_j,R) \mid \mathcal{L}^n \times \cdots \times \mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{l-1}},x}_{i_1,\dots,i_{m-l}}) \ge \beta^{m-l} - \gamma^{m-l}\}.$$

If

$$\mathcal{L}^n \times \cdots \times \mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{l-1}}}_{j,i_1,\dots,i_{m-l}}) \ge \beta^{m-(l-1)} - \gamma^{m-(l-1)},$$

we obtain, using Fubini's theorem, that

$$\begin{aligned}
\beta^{m-(l-1)} - \gamma^{m-(l-1)} &\le \mathcal{L}^n \times \cdots \times \mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{l-1}}}_{j,i_1,\dots,i_{m-l}}) \\
&= \int_{B(b_j,R)} \mathcal{L}^n \times \cdots \times \mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{l-1}},x}_{i_1,\dots,i_{m-l}})\, d\mathcal{L}^n x \\
&\le \beta^{m-l}\mathcal{L}^n(A_j^{i_1,\dots,i_{m-l}}(x_{p_1},\dots,x_{p_{l-1}})) \\
&\quad + (\beta^{m-l} - \gamma^{m-l})\mathcal{L}^n(B(b_j,R) \setminus A_j^{i_1,\dots,i_{m-l}}(x_{p_1},\dots,x_{p_{l-1}})) \\
&= \beta^{m-l+1} - \gamma^{m-l}(\beta - \mathcal{L}^n(A_j^{i_1,\dots,i_{m-l}}(x_{p_1},\dots,x_{p_{l-1}}))),
\end{aligned}$$

and so

$$\mathcal{L}^n(A_j^{i_1,\dots,i_{m-l}}(x_{p_1},\dots,x_{p_{l-1}})) \geq \beta - \gamma. \tag{24}$$

For brevity, we will write $A_j^{i_1,\dots,i_{m-l}}(x_{p_1},\dots,x_{p_{l-1}}) := A_j^{i_1,\dots,i_{m-1}}$ and $B_{i_1,\dots,i_{m-l}}^{x_{p_1},\dots,x_{p_{l-1}},x_j} := B_{i_1,\dots,i_{m-1}}^{x_j}$ in the case $l = 1$, originally defined only in the case $l \geq 2$, if $1 \leq l \leq m-1$, $l \leq j \leq k+l$, $1 \leq p_1 < \dots < p_{l-1} < j < i_1 < \dots < i_{m-l} \leq n$, $x_{p_q} \in B(b_{p_q}, R)$ for all $1 \leq q \leq l-1$, and $x_j \in B(b_j, R)$.

We will first inductively construct $x_i \in B(b_i, R)$ for all $1 \leq i \leq m-1$ such that the sequence $(x_1, \dots, x_{m-1})$ is linearly independent and

$$\mathcal{L}^n \times \dots \times \mathcal{L}^n(B_{i_1,\dots,i_{m-l}}^{x_{p_1},\dots,x_{p_{l-1}},x_i}) \geq \beta^{m-l} - \gamma^{m-l} \tag{25}$$

for all $1 \leq i \leq m-1$, $\max\{1, i-k\} \leq l \leq i$, and $1 \leq p_1 < \dots < p_{l-1} < i < i_1 < \dots < i_{m-l} \leq n$.

We begin by selecting x_1. By (23), we obtain that

$$\mathcal{L}^n(\bigcap_{1<i_1<\dots<i_{m-1}\leq n} A_1^{i_1,\dots,i_{m-1}}) \geq \beta - \binom{n-1}{m-1}\gamma = \beta - M_1\gamma \geq \beta - M\gamma > 0,$$

and so we can choose $x_1 \in B(b_1, R)$ with $x_1 \neq 0$ such that

$$\mathcal{L}^n \times \dots \times \mathcal{L}^n(B_{i_1,\dots,i_{m-1}}^{x_1}) \geq \beta^{m-1} - \gamma^{m-1}$$

for all $1 < i_1 < \dots < i_{m-1} \leq n$.

If $m \geq 3$, we continue as follows. We assume that $1 \leq i \leq m-2$ and that we have chosen $x_j \in B(b_j, R)$ for all $1 \leq j \leq i$ such that the sequence $(x_1, \dots, x_i)$ is linearly independent and

$$\mathcal{L}^n \times \dots \times \mathcal{L}^n(B_{i_1,\dots,i_{m-l}}^{x_{p_1},\dots,x_{p_{l-1}},x_j}) \geq \beta^{m-l} - \gamma^{m-l}$$

for all $1 \leq j \leq i$, $\max\{1, j-k\} \leq l \leq j$, and $1 \leq p_1 < \dots < p_{l-1} < j < i_1 < \dots < i_{m-l} \leq n$. We claim that

$$\mathcal{L}^n(A_{i+1}^{i_1,\dots,i_{m-l}}(x_{p_1},\dots,x_{p_{l-1}})) \geq \beta - \gamma \tag{26}$$

for all $\max\{1, i+1-k\} \leq l \leq i+1$ and $1 \leq p_1 < \dots < p_{l-1} < i+1 < i_1 < \dots < i_{m-l} \leq n$. In fact, inequality (26) follows from (23) if $l = 1$, and from the induction hypothesis and (24) if $l \geq 2$.

Now we obtain, using (26), that

$$\begin{aligned}\mathcal{L}^n(\bigcap_{l=\max\{1,i+1-k\}}^{i+1} \bigcap_{\substack{1\le p_1<\cdots<p_{l-1}<i+1\\ i+1<i_1<\cdots<i_{m-l}\le n}} A_{i+1}^{i_1,\ldots,i_{m-l}}(x_{p_1},\ldots,x_{p_{l-1}}))\\ \ge \beta - \sum_{l=\max\{1,i+1-k\}}^{i+1} \binom{i}{l-1}\binom{n-(i+1)}{m-l}\gamma\\ = \beta - M_{i+1}\gamma \ge \beta - M\gamma > 0,\end{aligned}$$

whence, using the fact that the n-dimensional Lebesgue measure equals zero for any i-dimensional linear subspace of $\mathbf{R}^n$, we can choose $x_{i+1}\in B(b_{i+1},R)$ such that the sequence $(x_1,\ldots,x_{i+1})$ is linearly independent and

$$\mathcal{L}^n\times\cdots\times\mathcal{L}^n(B_{i_1,\ldots,i_{m-l}}^{x_{p_1},\ldots,x_{p_{l-1}},x_{i+1}}) \ge \beta^{m-l}-\gamma^{m-l}$$

for all $\max\{1,i+1-k\}\le l\le i+1$ and $1\le p_1<\cdots<p_{l-1}<i+1<i_1<\cdots<i_{m-l}\le n$. This completes the construction of the sequence $(x_1,\ldots,x_{m-1})$.

We will now inductively construct $x_{m+i}\in B(b_{m+i},R)$ for all $0\le i\le k-1$ such that the sequence $(x_1,\ldots,x_{n-1})$ is linearly independent, $(x_{p_1},\ldots,x_{p_{m-1}},x_{m+i})\in B$ for all $0\le i\le k-1$ and $1\le p_1<\cdots<p_{m-1}<m+i$, and

$$\mathcal{L}^n\times\cdots\times\mathcal{L}^n(B_{i_1,\ldots,i_{m-l}}^{x_{p_1},\ldots,x_{p_{l-1}},x_{m+i}}) \ge \beta^{m-l}-\gamma^{m-l} \tag{27}$$

for all $0\le i\le k-1$, $\max\{1,m+i-k\}\le l\le m-1$, and $1\le p_1<\cdots<p_{l-1}<m+i<i_1<\cdots<i_{m-l}\le n$.

We begin by selecting x_m. By (25), we obtain that

$$\mathcal{L}^n(B_m^{x_1,\ldots,x_{m-1}}) \ge \beta-\gamma. \tag{28}$$

Further, we claim that

$$\mathcal{L}^n(A_m^{i_1,\ldots,i_{m-l}}(x_{p_1},\ldots,x_{p_{l-1}})) \ge \beta-\gamma \tag{29}$$

for all $\max\{1,m-k\}\le l\le m-1$ and $1\le p_1<\cdots<p_{l-1}<m<i_1<\cdots<i_{m-l}\le n$. In fact, inequality (29) follows from (23) if $l=1$, and from (25) and (24) if $l\ge 2$.

Using (28), (29), the inequality $\beta > M_m\gamma$, and the fact that the n-dimensional Lebesgue measure equals zero for any $(m-1)$-dimensional linear subspace of $\mathbf{R}^n$, we can choose, like before, $x_m\in B(b_m,R)$ such that the sequence $(x_1,\ldots,x_m)$ is linearly independent, $(x_1,\ldots,x_m)\in B$, and (27) is valid if $i=0$.

If $k\ge 2$, we continue as follows. We assume that $0\le i\le k-2$ and that we have chosen $x_{m+j}\in B(b_{m+j},R)$ for all $0\le j\le i$ such that the sequence

$(x_1,\dots,x_{m+i})$ is linearly independent, $(x_{p_1},\dots,x_{p_{m-1}},x_{m+j})\in B$ for all $0\le j\le i$ and $1\le p_1<\dots<p_{m-1}<m+j$, and

$$\mathcal{L}^n\times\cdots\times\mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{l-1}},x_{m+j}}_{i_1,\dots,i_{m-l}})\ge\beta^{m-l}-\gamma^{m-l}$$

for all $0\le j\le i$, $\max\{1,m+j-k\}\le l\le m-1$, and $1\le p_1<\dots<p_{l-1}<m+j<i_1<\dots<i_{m-l}\le n$.

We claim that

$$\mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{m-1}}}_{m+i+1})\ge\beta-\gamma \tag{30}$$

for all $1\le p_1<\dots<p_{m-1}<m+i+1$ and

$$\mathcal{L}^n(A^{i_1,\dots,i_{m-l}}_{m+i+1}(x_{p_1},\dots,x_{p_{l-1}}))\ge\beta-\gamma \tag{31}$$

for all $\max\{1,m+i+1-k\}\le l\le m-1$ and $1\le p_1<\dots<p_{l-1}<m+i+1<i_1<\dots<i_{m-l}\le n$. Inequality (30) follows from (25) if $p_{m-1}=m-1$, and from the induction hypothesis if $p_{m-1}\ge m$. Inequality (31) follows from (23) if $l=1$, from (25) and (24) if $l\ge2$ and $p_{l-1}\le m-1$, and finally from the induction hypothesis and (24) if $l\ge2$ and $p_{l-1}\ge m$.

By means of (30), (31), the inequality $\beta>M_{m+i+1}\gamma$, and the fact that the n-dimensional Lebesgue measure equals zero for any $(m+i)$-dimensional linear subspace of $\mathbf{R}^n$, we can choose, like before, $x_{m+i+1}\in B(b_{m+i+1},R)$ such that the sequence $(x_1,\dots,x_{m+i+1})$ is linearly independent, $(x_{p_1},\dots,x_{p_{m-1}},x_{m+i+1})\in B$ for all $1\le p_1<\dots<p_{m-1}<m+i+1$, and

$$\mathcal{L}^n\times\cdots\times\mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{l-1}},x_{m+i+1}}_{i_1,\dots,i_{m-l}})\ge\beta^{m-l}-\gamma^{m-l}$$

for all $\max\{1,m+i+1-k\}\le l\le m-1$ and $1\le p_1<\dots<p_{l-1}<m+i+1<i_1<\dots<i_{m-l}\le n$. This completes the construction of the sequence $(x_1,\dots,x_{n-1})$.

Finally, we obtain by (25) if $p_{m-1}=m-1$, and by (27) if $p_{m-1}\ge m$ that

$$\mathcal{L}^n(B^{x_{p_1},\dots,x_{p_{m-1}}}_{n})\ge\beta-\gamma \tag{32}$$

for all $1\le p_1<\dots<p_{m-1}<n$. Using (32), the inequality $\beta>M_n\gamma$, and the fact that the n-dimensional Lebesgue measure equals zero for any $(n-1)$-dimensional linear subspace of $\mathbf{R}^n$, we can choose, in the same way as before, $x_n\in B(b_n,R)$ such that the sequence $(x_1,\dots,x_n)$ is linearly independent and $(x_{p_1},\dots,x_{p_{m-1}},x_n)\in B$ for all $1\le p_1<\dots<p_{m-1}<n$. This completes the proof of the lemma. □

3.9. Corollary. *If $A\subset\mathbf{R}^n$ is bounded, then*

$$\dim_{\mathrm{M}}P_V(A)\ge\frac{m}{n}\dim_{\mathrm{M}}A$$

for $\gamma_{n,m}$ almost all $V\in G(n,m)$.

Proof. By Lemma 2.2 and Theorem 3.4, the set

$$B := \left\{ V \in G(n,m) \mid \dim_{\mathrm{M}} P_V(A) < \frac{m}{n} \dim_{\mathrm{M}} A \right\}$$

has properties (i) and (ii) assumed in Theorem 3.7, and so we obtain by Theorem 3.7 that $\gamma_{n,m}(B) = 0$. □

3.10. Corollary. *If $A \subset \mathbf{R}^n$ is a Suslin set, then*

$$\dim_{\mathrm{p}} P_V(A) \geq \frac{m}{n} \dim_{\mathrm{p}} A$$

for $\gamma_{n,m}$ almost all $V \in G(n,m)$.

Proof. If A is compact, it suffices to observe that by Lemma 2.3 and Corollary 3.5 the set

$$B := \left\{ V \in G(n,m) \mid \dim_{\mathrm{p}} P_V(A) < \frac{m}{n} \dim_{\mathrm{p}} A \right\}$$

has properties (i) and (ii) assumed in Theorem 3.7, and so we obtain by Theorem 3.7 that $\gamma_{n,m}(B) = 0$.

If A is a Suslin set, then [JP, Theorem 1] implies that for all positive integers i there exists a compact set $K_i \subset A$ such that $\dim_{\mathrm{p}} A = \sup_i \dim_{\mathrm{p}} K_i$. Further, the above consideration implies that for all i there is $B_i \subset G(n,m)$ such that $\gamma_{n,m}(B_i) = 0$ and $\dim_{\mathrm{p}} P_V(K_i) \geq (m/n) \dim_{\mathrm{p}} K_i$ for all $V \in G(n,m) \setminus B_i$. We define $B := \bigcup_{i=1}^{\infty} B_i$. Then $\gamma_{n,m}(B) = 0$, and the monotonicity of the packing dimension implies that

$$\dim_{\mathrm{p}} P_V(A) \geq \sup_i \dim_{\mathrm{p}} P_V(K_i) \geq \frac{m}{n} \sup_i \dim_{\mathrm{p}} K_i = \frac{m}{n} \dim_{\mathrm{p}} A$$

for all $V \in G(n,m) \setminus B$. □

3.11. Remarks. (a) As mentioned in the Introduction, the lower bounds obtained in Corollary 3.9 and Corollary 3.10 were improved to the best possible ones by Falconer and Howroyd in [FH].

(b) If $A \subset \mathbf{R}^n$ is a bounded set with $\dim_{\mathrm{M}} A = n$, then we have by Corollary 3.9 that $\dim_{\mathrm{M}} P_V(A) = m$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$.

(c) If $A \subset \mathbf{R}^n$ is a Suslin set with $\dim_{\mathrm{p}} A = n$, then we have by Corollary 3.10 that $\dim_{\mathrm{p}} P_V(A) = m$ for $\gamma_{n,m}$ almost all $V \in G(n,m)$.

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