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Research Article

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Flat flow solution to the mean curvature flow with volume constraint

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Abstract: In this paper I will revisit the construction of a global weak solution to the volume preserving mean curvature flow via discrete minimizing movement scheme by Mugnai, Seis and Spadaro [L. Mugnai, C. Seis and E. Spadaro, Global solutions to the volume-preserving mean-curvature flow, *Calc. Var. Partial Differential Equations* **55** (2016), no. 1, Article ID 18]. This method is based on the gradient flow approach due to Almgren, Taylor and Wang [F. Almgren, J. E. Taylor and L. Wang, Curvature-driven flows: a variational approach, *SIAM J. Control Optim.* **31** (1993), no. 2, 387–438] and Luckhaus and Sturzenhecker [S. Luckhaus and T. Sturzenhecker, Implicit time discretization for the mean curvature flow equation, *Calc. Var. Partial Differential Equations* **3** (1995), no. 2, 253–271] and my aim is to replace the volume penalization with the volume constraint directly in the discrete scheme, which from practical point of view is perhaps more natural. A technical novelty is the proof of the density estimate which is based on second variation argument.

Keywords: Mean-curvature flow, volume constraint, gradient flow, time discretization

MSC 2020: 35K93, 53C42

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1 Introduction

A smooth family of set $(E_t)_{t\geq 0}$ is said to evolve according to volume preserving mean curvature flow if the normal velocity V_t is proportional to the mean curvature H_{E_t} as

$$V_t = -(H_{E_t} - \bar{H}_{E_t}) \quad \text{on } \partial E_t, \tag{1.1}$$

where $\bar{H}_{E_t} = \fint_{\partial E_t} H_{E_t}\, d\mathcal{H}^n$. Such a geometric equation has been proposed in the physical literature to model coarsening phenomena, where the system consisting of several subdomains evolves such that it decreases the interfacial area while keeping the total volume unchanged [7, 18]. From purely mathematical point of view the equation (1.1) can be seen as the L^2-gradient flow of the surface area under the volume constraint [18]. One has to be careful in this interpretation as the Riemannian distance between two sets is in general degenerate [16]. In order to overcome this one may use the idea due to Almgren, Taylor and Wang [2] and Luckhaus and Sturzenhecker [14] and to view (1.1) as the gradient flow of the surface area with respect to a different, non-degenerate, distance. Using the gradient flow structure, one may then construct a discrete-in-time approximation to the solution of (1.1) via the Euler implicit method, also known as the minimizing movements scheme. By letting the time step to zero, one then obtains a candidate for a weak solution of (1.1) called *flat flow*, as the convergence is measured in terms of the "flat norm". This method is implemented to the volume preserving setting in [19].

In [19] the authors observe that from technical point of view it is easier to replace the volume constraint of the problem with volume penalization, as this simplifies certain regularity issues at the level of the discrete approximation. My aim here is to show that one may construct the flat flow solution to (1.1) by implementing the volume constraint in the minimizing movements scheme directly and thus avoid the volume penalization.

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Let me quickly recall the discrete minimizing movements scheme for (1.1). One defines a sequence of sets $(E_k^h)_k$, with fixed time step $h > 0$, iteratively such that $E_0^h = E_0$, where E_0 is the given initial set, and E_{k+1}^h is a minimizer of the functional

$$P(E) + \frac{1}{h}\int_E \bar{d}_{E_k^h}\,dx \quad \text{under the constraint } |E| = |E_k^h|.$$

Here $P(E)$ denotes the perimeter (generalized surface area) of the set E and $\bar{d}_F$ is the signed distance function of the set F (see next section). One then defines an approximative flat flow solution to (1.1) $(E_t^h)_{t\geq 0}$ from the previous sequence by $E_t^h = E_k^h$ for $t \in [kh,(k+1)h)$. Any cluster point of $(E_t^h)_{t\geq 0}$ is then defined as flat flow solution to (1.1). The advantage is that such a solution is defined for all times and for rough initial data. The main result in the paper is the existence of a flat flow solution.

Theorem 1. *Assume that $E_0 \subset \mathbb{R}^{n+1}$ is an open and bounded set with finite perimeter and let $(E_t^h)_{t\geq 0}$ be an approximative flat flow solution to* (1.1) *stating from E_0 (see Definition 2.1). Then there exists a family of bounded sets of finite perimeter $(E_t)_{t\geq 0}$ and a subsequence $h_k \to 0$ such that*

$$\lim_{h_k\to 0} |E_t^{h_k} \,\Delta\, E_t| = 0 \quad \text{for a.e. } t \geq 0$$

and for every $0 < t < s$ it holds $|E_t| = |E_0|$, $P(E_t) \leq P(E_0)$ and

$$|E_t \,\Delta\, E_s| \leq C\sqrt{s-t},$$

where C depends on the dimension and on E_0. Moreover, if the initial set E_0 is $C^{1,1}$-regular, then any such limit flow $(E_t)_{t\geq 0}$ agrees with the unique classical solution of (1.1) *as long as the latter exists.*

The above theorem thus provides the existence of a flat flow solution and guarantees that this notion is consistent with the classical solution when the initial set is regular enough. The disadvantage of the flat flow is that it is not clear if it provides a solution to the original equation (1.1) in any weak sense after the first singular time. However, the conditional result in the spirit of Luckhaus and Sturzenhecker [14] holds also in this case.

Theorem 2. *Let $(E_t^h)_{t\geq 0}$ be an approximative flat flow solution to* (1.1) *and let $(E_t^{h_k})_{t\geq 0}$ be the converging subsequence in Theorem 1. Assume further that it holds*

$$\lim_{h_k\to 0} P(E_t^{h_k}) = P(E_t) \quad \text{for a.e. } t \geq 0.$$

Then for $n \leq 6$ the flat flow $(E_t)_{t\geq 0}$ is a distributional solution to (1.1) *(see Definition 4.5).*

One may also try to view equation (1.1) as a mean curvature flow with forcing, where the forcing term depends on the flow itself. In this way one may try to use different methods to construct a solution to the equation see, e.g., [5, 6]. I also refer the recent work [13] for a weak-strong uniqueness result related to (1.1).

As I already mentioned, the flat flow is defined for all times and one may study its asymptotical behavior. Indeed, by using the methods from [10, 11] one may deduce the convergence of the flow in low dimensions. I will state this merely as a remark as it follows from the above methods without any modifications.

Remark 1.1. Assume that $E_0 \subset \mathbb{R}^{n+1}$, with $n \leq 2$, is as in Theorem 1 and let $(E_t)_{t\geq 0}$ be a limit flat flow. When $n = 1$, the flow E_t converges to a union of disjoint balls exponentially fast and when $n = 2$ the flow converges to a union of disjoint balls up to a possible translation of the components.

The main technical challenge in proving Theorem 1 is to obtain the sharp density estimate for the discrete flow. This is also the main technical novelty of this paper. There are several techniques to deal with the volume constraint in variational problems, e.g., by using the argument from [3] (see also [15, Lemma 17.21]) or from [8] (see also [4, 9]). However, due to the presence of the dissipation term in the energy it is not obvious how to apply these arguments in order to obtain sharp density estimates in terms of the time step h. I will use an argument which is based on the second variation condition of the energy to prove the density estimate in Proposition 3.1. After this the proof of Theorem 1 follows exactly as in [14, 19] and the consistency follows almost directly using the argument in [12]. The proof also provides the dissipation inequality and therefore the results in [10, 11] hold and one obtains the result stated in Remark 1.1. Finally, I would like to point out that this article is not self-consistent as I will take several well-known arguments for granted, in particular, in Section 4.

2 Preliminaries

In this section I will briefly introduce the notation, the definition of the flat flow solution and recall some of its basic properties.

Given a set $E \subset \mathbb{R}^{n+1}$, the distance function $\operatorname{dist}(\cdot, E) : \mathbb{R}^{n+1} \to [0, \infty)$ is defined, as usual, as

$$\operatorname{dist}(x, E) := \inf_{y \in E} |x - y|$$

and denote the signed distance function by $\bar{d}_E : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$\bar{d}_E(x) := \begin{cases} -\operatorname{dist}(x, \partial E) & \text{for } x \in E, \\ \operatorname{dist}(x, \partial E) & \text{for } x \in \mathbb{R}^{n+1} \setminus E. \end{cases}$$

Then clearly it holds $\operatorname{dist}(\cdot, \partial E) = |\bar{d}_E|$. I denote the ball with radius r centered at x by $B_r(x)$ and by B_r if it is centered at the origin.

For a measurable set $E \subset \mathbb{R}^{n+1}$ the perimeter in an open set $U \subset \mathbb{R}^{n+1}$ is defined as

$$P(E, U) := \sup\Big\{ \int_E \operatorname{div} X \, dx : X \in C_0^1(U, \mathbb{R}^{n+1}),\ \|X\|_{L^\infty} \le 1 \Big\}$$

and write $P(E) = P(E, \mathbb{R}^{n+1})$. If $P(E) < \infty$, then E is called a set of finite perimeter. For an introduction to the topic I refer to [15]. The reduced boundary of a set of finite perimeter E is denoted by $\partial^* E$ and the generalized unit outer normal by ν_E. Note that it holds $P(E, U) = \mathcal{H}^n(\partial^* E \cap U)$ for open sets U. Recall also that if E is regular enough, say with Lipschitz boundary, then $P(E) = \mathcal{H}^n(\partial E)$. For a given vector field $X \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ and a set of finite perimeter E denote the tangential divergence on ∂E^* as $\operatorname{div}_\tau X = \operatorname{div} X - \langle DX\nu_E, \nu_E\rangle$. The distributional mean curvature $H_E \in L^1(\partial^* E, \mathbb{R})$ is defined via the divergence theorem such that for every test vector field $X \in C_0^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ it holds

$$\int_{\partial^* E} \operatorname{div}_\tau X \, d\mathcal{H}^n = \int_{\partial^* E} H_E \langle X, \nu_E \rangle \, d\mathcal{H}^n.$$

I will consider a flat flow solution to (1.1) in the spirit of Almgren, Taylor and Wang [2] and Luckhaus and Sturzenhecker [14]. To this end, for a fixed $h \in (0, 1)$ and a given (open) set $F \subset \mathbb{R}^{n+1}$, I define the functional

$$\mathcal{F}_h(E, F) = P(E) + \frac{1}{h} \int_E \bar{d}_F \, dx. \tag{2.1}$$

The flat flow solution is defined analogously as in [19].

Definition 2.1. Let $E_0 \subset \mathbb{R}^{n+1}$ be an open and bounded set of finite perimeter and fix $h \in (0, 1)$. Define the sequence of sets $(E_k^h)_{k=0}^\infty$ iteratively as $E_0^h = E_0$ and E_{k+1}^h is a minimizer of the problem

$$\min\{\mathcal{F}_h(E, E_k^h) : |E| = |E_0|\}.$$

Moreover, define an approximative flat flow $(E_t^h)_{t \ge 0}$ for (1.1) starting from E_0 as

$$E_t^h = E_k^h \quad \text{for } t \in [kh, (k+1)h).$$

One has to be careful in the definition of the functional (2.1) if the set F is merely a set of finite perimeter as its value depends on the choice of the representative of F. One may overcome this by choosing a proper representative of the set F. However, this is not necessary as the regularity theorem below implies that one may in fact assume the sets E_k^h to be open. The difference in Definition 2.1 to the scheme in [19] is that here the minimizing problem is under volume constraint. On one hand this makes the minimization problem more natural, but on the other hand, it makes the quantitative density estimates more difficult to prove.

For a given open and bounded set $F \subset \mathbb{R}^{n+1}$ consider the minimization problem

$$\min\{\mathcal{F}_h(E,F) : |E| = |F|\}, \tag{2.2}$$

where $\mathcal{F}_h(\cdot\,,F)$ is defined in (2.1). One may use an argument similar to [9] or [15, Lemma 17.21] to remove the volume constraint in (2.2) and deduce that a minimizer of (2.2) is a minimizer also for

$$\min\{\mathcal{F}_h(E,F) + \tilde{\Lambda}||E| - |F||\}, \tag{2.3}$$

when $\tilde{\Lambda}$ is chosen large. Note that the constant $\tilde{\Lambda}$ may have nonoptimal dependence on E and on h. However, the property (2.3) is enough to deduce qualitative regularity properties since it implies that the minimizer inherits the regularity from the theory of the perimeter minimizers [15]. One may also write the Euler–Lagrange equation and by standard calculations (see, e.g., [1]) we have the second variation condition. We state this in the following proposition.

Proposition 2.2. *Let $F \subset \mathbb{R}^{n+1}$ be an open and bounded set, fix $h \in (0,1)$ and let E be a minimizer of* (2.2). *Then E can be chosen to be open, which topological boundary is $C^{2,\alpha}$-regular up to a relatively closed singular set which Hausdorff dimension is at most $n-7$. The regular part is exactly the reduced boundary $\partial^* E$.*

The Euler–Lagrange equation

$$\frac{d_F}{h} = -H_E + \lambda, \tag{2.4}$$

where $\lambda \in \mathbb{R}$ is the Lagrange-multiplier, holds point-wise on $\partial^* E$ and in a distributional sense on ∂E. The quadratic form associated with the second variation of the energy is non-negative, i.e., for all $\varphi \in H^1(\partial^* E)$ with $\int_{\partial^* E} \varphi \, d\mathcal{H}^n = 0$ it holds

$$\int_{\partial^* E} |\nabla_\tau \varphi|^2 - |B_E|^2 \varphi^2 \, d\mathcal{H}^n + \frac{1}{h} \int_{\partial^* E} \langle \nabla \bar{d}_F, \nu_E \rangle \varphi^2 \, d\mathcal{H}^n \geq 0, \tag{2.5}$$

where $B_E(x)$ denotes the second fundamental form of E at $x \in \partial^* E$.

Proof. Since the argument is standard, I will only give the outline. As I already mentioned, the minimizer E is also a minimizer of problem (2.3) for some large constant $\tilde{\Lambda}$, which depends on h and on E itself. This implies that the set E is a Λ-minimizer of the perimeter and thus the reduced boundary $\partial^* E$ is relatively open, $C^{1,\alpha}$-regular hypersurface and the singular set $\partial E \setminus \partial^* E$ has dimension at most $n-7$ (see [15]). The $C^{2,\alpha}$-regularity then follows from the Euler–Lagrange equation and from standard Schauder-estimates for elliptic PDEs.

One may obtain the second variation condition (2.5) by using the argument from [1]. Indeed, given a function $\varphi \in C_0^2(\partial^* E)$ with $\int_{\partial^* E} \varphi \, d\mathcal{H}^n = 0$, we may construct a family of diffeomorphisms Φ_t such that $\Phi_0 = \mathrm{id}$, $|\Phi_t(E)| = |E|$ and $\frac{\partial}{\partial t}\big|_{t=0} \Phi_t(x) \cdot \nu_E = \varphi$. Then the inequality follows from the minimality of E as

$$\frac{\partial^2}{\partial t^2}\bigg|_{t=0} \mathcal{F}_h(\Phi_t(E), F) \geq 0$$

and following the standard calculation of the second variation (see, e.g., [1]). Finally, one obtains (2.5) for all $\varphi \in H^1(\partial^* E)$ by approximation argument and by the fact that the singular set has zero capacity. □

3 Density estimates

This section is the theoretical core of the paper. The aim is to prove the following density estimate.

Proposition 3.1. *Let $F \subset \mathbb{R}^{n+1}$ be an open and bounded set of finite perimeter, fix $h \in (0,1)$ and let E be a minimizer of* (2.2). *Then there is a constant $c > 0$, which depends on the dimension n, $|F|$ and on $P(F)$ such that for all $r \leq \sqrt{h}$ and all $x \in \partial E$ it holds*

$$\min\{|E \cap B_r(x)|, |B_r(x) \setminus E|\} \geq c r^{n+1}$$

and for all $r \le C_0\sqrt{h}$, *where* $C_0 \ge 1$, *it holds*

$$cr^n \le P(E, B_r(x)) \le C_1 r^n,$$

where C_1 *depends also on* C_0. *Moreover, the following estimates hold*

$$\|H_E\|_{L^\infty(\partial^* E)} \le \frac{1}{c\sqrt{h}} \quad \text{and} \quad \|\bar{d}_F\|_{L^\infty(\partial E)} \le c^{-1}\sqrt{h}.$$

It is interesting that in [19, Corollary 3.3] the authors obtain similar result for their scheme for a constant which is independent of $P(F)$.

I need several lemmas in order to prove Proposition 3.1 and therefore I postpone its proof to the end of the section. Before proceeding to technical details, I state a useful consequence of Proposition 3.1.

Proposition 3.2. *Let* $F, E \subset \mathbb{R}^{n+1}$ *be as in Proposition 3.1. Then there are constants* $C \ge 1$, $c > 0$ *and* $h_0 > 0$, *depending on the dimension,* $|F|$ *and* $P(F)$ *such that* E *is the* (Λ, r)*-minimizer of the perimeter for* $\Lambda = \frac{C}{\sqrt{h}}$, $r = c\sqrt{h}$ *and for* $h < h_0$. *To be more precise, for sets* $G \subset \mathbb{R}^{n+1}$ *with* $E \,\Delta\, G \subset B_{c\sqrt{h}}(x_0)$ *it holds*

$$P(E) \le P(G) + \frac{C}{\sqrt{h}}|E \,\Delta\, G|.$$

Proof. The argument is standard but I recall it for the reader's convenience. Let me first show that there is $x \in E$ and $\tilde{c} > 0$ such that for $\rho = \tilde{c}\sqrt{h}$ it holds $B_\rho(x) \subset E$. Fix ρ and apply the Besicovitch covering theorem to find disjoint balls $\{B_\rho(x_i)\}_{i=1}^N$ such that $x_i \in E$ and

$$\sum_{i=1}^N |B_\rho(x_i)| = N|B_1|\rho^{n+1} \ge c|E|. \tag{3.1}$$

I claim that for some $i = 1, 2, \dots, N$ it holds $B_{\frac{\rho}{2}}(x_i) \subset E$. Indeed, if this is not the case, then Proposition 3.1 implies

$$P(E, B_\rho(x_i)) \ge c\rho^n$$

for all i. Since the balls are disjoint, one has by the above and by (3.1) that

$$P(E) \ge \sum_{i=1}^N P(E, B_\rho(x_i)) \ge cN\rho^n \ge c\frac{|E|}{\rho} \ge \frac{c}{\sqrt{h}}.$$

This is a contradiction when h is small enough.

Fix x_0 and G as in the claim. Note that in general the set G does not have the same measure as E and one needs to modify it to $\tilde{G}$ with $|\tilde{G}| = |E|$, e.g., by using the argument from [9] as follows. Assume that $|G| < |E|$ (the case $|G| > |E|$ follows from similar argument). Since $B_\rho(x) \subset E$, by decreasing ρ and r if needed, it holds $B_\rho(x) \subset G$. By continuity there is $z \in \mathbb{R}^{n+1}$ such that $|z - x_0| \ge 2\rho$ and $|G \cup B_\rho(z)| = |E|$. Define $\tilde{G} = G \cup B_\rho(z)$. Then by the minimality of E and Proposition 3.1 it holds

$$P(E) \le P(\tilde{G}) + \frac{C}{\sqrt{h}}|\tilde{G} \,\Delta\, E|.$$

Arguing as in [9], one then deduces

$$\begin{aligned} P(\tilde{G}) - P(G) &\le \mathcal{H}^n(\partial B_\rho(z) \setminus G) - \mathcal{H}^n(\partial G \cap B_\rho(z)) \\ &\le \frac{C}{\rho}|B_\rho(z) \setminus G| \le \frac{C}{\sqrt{h}}|\tilde{G} \,\Delta\, E| \end{aligned}$$

and the claim follows as $|\tilde{G} \,\Delta\, E| \le 2|G \,\Delta\, E|$. □

The first technical result which I need is the classical density estimate which can be found, e.g., in [20].

Lemma 3.3. *Assume* $E \subset \mathbb{R}^{n+1}$ *is a set of finite perimeter with distributional mean curvature* H_E *which satisfies* $\|H_E\|_{L^\infty(B_{2R}(x_0))} \le \Lambda$. *Then for all* $x \in B_R(x_0)$ *which are on the boundary of* E *and* $r \le \min\{R, \Lambda^{-1}\}$ *it holds*

$$P(E, B_r(x)) \ge c_n r^n$$

for a dimensional constant $c_n > 0$.

For a minimizer of (2.2) it holds the inverse of the isoperimetric inequality.

Lemma 3.4. *Let $F \subset \mathbb{R}^{n+1}$ be an open and bounded set of finite perimeter, fix $h \in (0,1)$ and let E be a minimizer of (2.2). Then for all $x \in \partial E$ and $r \leq C_0\sqrt{h}$ it holds*

$$P(E, B_r(x)) \leq \frac{C}{r} \min\{|E \cap B_{2r}(x)|, |B_{2r}(x) \setminus E|\}$$

for a constant which depends on the dimension and on $C_0 > 0$. In, particular it holds $P(E, B_r(x)) \leq Cr^n$.

Proof. Fix $h \in (0,1)$, x and $r > 0$ as in the claim and without loss of generality assume that $x = 0$. One may also consider only the case $|E \cap B_{2r}| \leq |B_{2r} \setminus E|$ as the other case is similar. In particular, it holds $|E \cap B_{2r}| \leq \frac{1}{2}|B_{2r}|$. Since

$$\int_0^{2r} \mathcal{H}^n(\partial B_\rho \cap E) = |E \cap B_{2r}|,$$

there is $\rho \in (r, 2r)$ such that

$$\mathcal{H}^n(\partial B_\rho \cap E) \leq C_n \frac{|E \cap B_{2r}|}{r} \quad \text{and} \quad |B_\rho| \geq \frac{2}{3}|B_{2r}|. \tag{3.2}$$

Consider first the set $E_1 = E \setminus \bar{B}_\rho$. In order to have a competing set with the volume of E, define $\tilde{\rho} \leq \rho$ to be a radius such that $|B_{\tilde{\rho}}| = |E \cap B_\rho|$ and define $E_2 = E_1 \cup B_{\tilde{\rho}}$. Then it holds by construction that $|E_2| = |E|$, $\tilde{\rho} < \rho$ and

$$\mathcal{H}^n(\partial B_{\tilde{\rho}}) = c_n|B_{\tilde{\rho}}|^{\frac{n}{n+1}} = c_n|E \cap B_\rho|^{\frac{n}{n+1}} \leq c_n|E \cap B_{2r}|^{\frac{n}{n+1}} \leq C_n \frac{|E \cap B_{2r}|}{r}. \tag{3.3}$$

By the minimality of E we have

$$P(E) + \frac{1}{h}\int_E \bar{d}_F \, dx \leq P(E_2) + \frac{1}{h}\int_{E_2} \bar{d}_F \, dx.$$

Estimate the perimeter of E_2 using (3.2) and (3.3) as

$$\begin{aligned} P(E_2) &\leq P(E, \mathbb{R}^{n+1} \setminus B_\rho) + \mathcal{H}^n(\partial B_\rho \cap E) + \mathcal{H}^n(\partial B_{\tilde{\rho}}) \\ &\leq P(E, \mathbb{R}^{n+1} \setminus B_\rho) + C_n \frac{|E \cap B_{2r}|}{r}. \end{aligned}$$

Use then $E \,\Delta\, E_2 \subset B_{2r}$, $|E| = |E_2|$ and the fact that the signed distance function is 1-Lipschitz to estimate

$$\left| \int_E \bar{d}_F \, dx - \int_{E_2} \bar{d}_F \right| \leq 4r|E \cap B_\rho| \leq 4r|E \cap B_{2r}|.$$

Therefore one obtains by combining the three above inequalities and $r \leq C_0\sqrt{h}$

$$P(E, B_r) \leq P(E, B_\rho) \leq C_n \frac{|E \cap B_{2r}|}{r} + \frac{4r}{h}|E \cap B_{2r}| \leq C\frac{|E \cap B_{2r}|}{r}. \qquad \square$$

By Lemma 3.3 and Lemma 3.4 it is clear that for Proposition 3.1 it is crucial to prove the curvature estimate $\|H_E\|_{L^\infty} \leq \frac{C}{\sqrt{h}}$. The next lemma is a step towards this.

Lemma 3.5. *Let F, E and h be as in Proposition 3.1. Then it holds*

$$\|H_E\|_{L^2(\partial^* E)} \leq \frac{C_1}{\sqrt{h}},$$

where the constant C_1 depends on the dimension and on $|F|$ and $P(F)$.

Proof. The proof relies on the second variation inequality in Proposition 2.2. I would like to point out that in the case of the mean curvature flow, when there is no volume constraint, the proof is considerable easier as one could choose constant function in (2.5). In the volume preserving case I will choose a cut-off function for a test function.

To this end, use first [11, Proposition 2.3] (see also [17, Lemma 2.1]) to find a point $x_0 \in \mathbb{R}^{n+1}$ and a radius $r \in (c, 1)$, where $c = c(n, |F|, P(F))$, such that

$$|E \cap B_r(x_0)| = \frac{1}{2}|B_r|.$$

Note that the minimality of E yields $P(E) \le P(F) \le C$. Moreover, by the isoperimetric inequality it holds $P(E) \ge c_n |E|^{\frac{n}{n+1}} = c_n |F|^{\frac{n}{n+1}} \ge c$. These estimates are used repeatedly from now on without mentioning. Without loss of generality assume that $x_0 = 0$. Choose $\rho < r$ such that $|B_r \setminus B_\rho| = \frac{1}{4}|B_r|$. Note that then $r - \rho \ge c_n > 0$ and

$$\frac{3}{4}|B_\rho| \ge |E \cap B_\rho| \ge \frac{1}{4}|B_\rho|. \tag{3.4}$$

Define first a cut-off function $\zeta \in C_0^1(\mathbb{R}^{n+1})$ such that $0 \le \zeta \le 1$, $\zeta = 1$ in B_ρ, $\zeta = 0$ outside B_r and $|\nabla \zeta| \le C_n$. Choose then $\varphi = \zeta - \bar{\zeta}$, where $\bar{\zeta} = \fint_{\partial^* E} \zeta \, d\mathcal{H}^n$, as a test function in (2.5), use $|\langle \nabla \bar{d}_F, \nu_E \rangle| \le 1$ and $|\varphi| \le 1$, and obtain

$$\int_{\partial^* E} |B_E|^2 (\zeta - \bar{\zeta})^2 \, d\mathcal{H}^n \le \int_{\partial^* E} |\nabla_\tau \zeta|^2 \, d\mathcal{H}^n + \frac{P(E)}{h}. \tag{3.5}$$

Since $H_E = \mathrm{Trace}(B_E)$, it holds point-wise on $\partial^* E$

$$|B_E|^2 \ge \frac{H_E^2}{n}. \tag{3.6}$$

Recall that $0 \le \zeta \le 1$. Moreover, by the isoperimetric inequality and by (3.4) it holds $P(E, B_\rho) \ge c_n |E \cap B_\rho|^{\frac{n}{n+1}} \ge c$. Therefore $\bar{\zeta} \ge c$ for $c = c(n, |F|, P(F))$. In particular, it holds $|\zeta(x) - \bar{\zeta}| \ge c$ for $x \in \partial^* E \setminus B_r$. Hence, we have by (3.5),

$$\int_{\partial^* E \setminus B_r} |H_E|^2 \, d\mathcal{H}^n \le \frac{C}{h} P(F).$$

We repeat the same argument by defining a cut-off function $\zeta \in C^1(\mathbb{R}^{n+1})$ as $\zeta = 0$ in B_r, $\zeta = 1$ outside B_R and $|\nabla \zeta| \le C_n$, where $R > r$ is such that $|B_R \setminus B_r| = \frac{1}{4}|B_r|$. Using $\varphi = \zeta - \bar{\zeta}$ in (2.5) and arguing as above yields

$$\int_{\partial^* E \cap B_r} |H_E|^2 \, d\mathcal{H}^n \le \frac{C}{h} P(F)$$

and the claim follows. □

The last lemma I need is a bound on the Lagrange multiplier in the Euler–Lagrange equation (2.4).

Lemma 3.6. *Let F, E and h be as in Proposition 3.1. Then for the Lagrange multiplier in* (2.4)*, i.e.,*

$$\frac{\bar{d}_F}{h} = -H_E + \lambda \quad \text{on } \partial^* E$$

it holds

$$|\lambda| \le \frac{C_2}{\sqrt{h}},$$

where the constant C_2 depends on the dimension, on $|F|$ and on $P(F)$.

Proof. Let $\Lambda \ge 0$ be such that $|\lambda| = \frac{\Lambda}{\sqrt{h}}$. Below all the constants depend on n, $|F|$ and $P(F)$. I only treat the case when λ is positive as in the negative case the proof is similar. Define the set

$$\Sigma = \left\{ x \in \partial^* E : |H_E(x)| < \frac{\hat{C}}{\sqrt{h}} \right\}.$$

I claim that we may choose $\hat{C} > 2$ such that it depends on n, $|F|$, $P(F)$ and on C_1 from Lemma 3.5 and it holds

$$\mathcal{H}^n(\Sigma) \ge \frac{P(E)}{2}. \tag{3.7}$$

Indeed, by Lemma 3.5 and by $\hat{C} > 2$ it holds

$$\frac{\hat{C}^2}{h} \mathcal{H}^n(\partial^* E \setminus \Sigma) \le \int_{\partial^* E \setminus \Sigma} H_E^2 \, d\mathcal{H}^n \le \int_{\partial^* E} H_E^2 \, d\mathcal{H}^n \le \frac{C_1^2}{h}.$$

By choosing $\hat{C}$ large enough one then obtains $\mathcal{H}^n(\partial^* E \setminus \Sigma) < \frac{P(E)}{2}$ and (3.7) follows.

By the Besicovitch covering theorem one finds disjoint balls of radius $\sqrt{h}$, denote them by $\{B_{\sqrt{h}}(x_i)\}_{i=1}^N$, with $x_i \in \Sigma$ such that

$$\sum_{i=1}^N P(E, B_{\sqrt{h}}(x_i)) \geq c_n \mathcal{H}^n(\Sigma) \geq cP(E). \tag{3.8}$$

By the Euler–Lagrange equation (2.4) and by the definition of the set Σ it holds for all $x \in \partial^* E \cap B_{\sqrt{h}}(x_i)$, with $x_i \in \Sigma$, that

$$|H_E(x)| \leq \left|\frac{\bar{d}_F(x)}{h} - \lambda\right| \leq \frac{|\bar{d}_F(x) - \bar{d}_F(x_i)|}{h} + \left|\frac{\bar{d}_F(x_i)}{h} - \lambda\right| \leq \frac{1}{\sqrt{h}} + |H_E(x_i)| \leq \frac{2\hat{C}}{\sqrt{h}}. \tag{3.9}$$

Therefore by Lemma 3.3 it holds

$$P(E, B_{\frac{\sqrt{h}}{2}}(x_i)) \geq P(E, B_{\frac{\sqrt{h}}{2\hat{C}}}(x_i)) \geq ch^{\frac{n}{2}}.$$

On the other hand, applying Lemma 3.4 first with $r = \frac{\sqrt{h}}{2}$ yields

$$|E \cap B_{\sqrt{h}}(x_i)| \geq c\sqrt{h}P(E, B_{\frac{\sqrt{h}}{2}}(x_i))$$

and then with $r = \sqrt{h}$ yields $h^{\frac{n}{2}} \geq cP(E, B_{\sqrt{h}}(x_i))$. In conclusion, it holds

$$|E \cap B_{\sqrt{h}}(x_i)| \geq c\sqrt{h}P(E, B_{\sqrt{h}}(x_i)) \tag{3.10}$$

for all balls in the cover.

The minimality of E implies

$$\frac{1}{h}\int_{E\setminus F} \bar{d}_F \,dx \leq P(E) + \frac{1}{h}\int_{E\Delta F} |\bar{d}_F| \,dx \leq P(F). \tag{3.11}$$

Note that by (3.9), by $\lambda = \frac{\Lambda}{\sqrt{h}}$ and by the Euler–Lagrange equation (2.4) we have for all $x \in B_{\sqrt{h}}(x_i)$ that

$$\bar{d}_F(x) \geq \lambda h - |H(x)|h \geq (\Lambda - 2\hat{C})\sqrt{h}.$$

Therefore either $\Lambda \leq 4\hat{C}$, in which case the claim follows trivially, or

$$\bar{d}_F(x) \geq \frac{\Lambda}{2}\sqrt{h}.$$

I assume the latter and show that also in this case Λ is bounded. Indeed, by the above discussion the balls $B_{\sqrt{h}}(x_i)$ are in the exterior of F. Therefore we estimate by (3.10) and by (3.8) that

$$\begin{aligned}
\frac{1}{h}\int_{E\setminus F} \bar{d}_F \,dx &\geq \frac{1}{h}\sum_{i=1}^N \int_{E\cap B_{\sqrt{h}}(x_i)} \bar{d}_F \,dx \\
&\geq \frac{1}{h}\sum_{i=1}^N \Big(\frac{\Lambda}{2}\sqrt{h}|E \cap B_{\sqrt{h}}(x_i)|\Big) \\
&\geq c\Lambda \sum_{i=1}^N P(E, B_{\sqrt{h}}(x_i)) \\
&\geq c\Lambda \mathcal{H}^n(\Sigma) \geq c\Lambda P(E).
\end{aligned}$$

Since $P(E) \geq c_n|E|^{\frac{n}{n+1}} = c_n|F|^{\frac{n}{n+1}}$, the above and (3.11) gives a bound for Λ and the claim follows. □

Here is the proof of the density estimate.

Proof of Proposition 3.1. By Lemma 3.3, Lemma 3.4, Lemma 3.6 and by the Euler–Lagrange equation (2.4) it is enough to prove

$$\|\bar{d}_F\|_{L^\infty(\partial^* E)} \leq C\sqrt{h}. \tag{3.12}$$

Argue by contradiction and assume that there is $x_0 \in \partial^* E$ such that

$$|\bar{d}_F(x_0)| = \Lambda\sqrt{h}$$

for large $\Lambda \gg 1$. Without loss of generality assume that $x_0 = 0$ and consider only the case $\bar{d}_F(x_0) > 0$ as the case $\bar{d}_F(x_0) < 0$ is similar.

By the Euler–Lagrange equation (2.4), by Lemma 3.3 and Lemma 3.6 it holds for $r_0 = \frac{\sqrt{h}}{2\Lambda}$ that

$$P(E, B_{r_0}) \geq c r_0^n. \tag{3.13}$$

Define radii $r_k = k\sqrt{h} + r_0$ for $k = 0, 1, \dots, n+1$, where $n+1$ is the dimension of the ambient space. For every $k = 0, 1, \dots, n+1$ choose a cut-off function $\zeta_k \in C_0^1(\mathbb{R}^{n+1})$ such that $0 \leq \zeta \leq 1$, $\zeta_k = 1$ in B_{r_k}, $\zeta_{k+1} = 0$ in $\mathbb{R}^{n+1} \setminus B_{r_{k+1}}$ and $|\nabla \zeta_k| \leq \frac{2}{\sqrt{h}}$. Choose

$$\varphi = \zeta_k - \bar{\zeta}_k,$$

where $\bar{\zeta}_k = \fint_{\partial^* E} \zeta_k \, d\mathcal{H}^n$, as a test function in the second variation condition (2.5), use $|\langle \nabla \bar{d}_F, \nu_E \rangle| \leq 1$ and (3.6), and obtain

$$\frac{1}{n} \int_{\partial^* E} |H_E|^2 (\zeta_k - \bar{\zeta}_k)^2 \, d\mathcal{H}^n \leq \frac{1}{h} \int_{\partial^* E} (\zeta_k - \bar{\zeta}_k)^2 \, d\mathcal{H}^n + \int_{\partial^* E} |\nabla_\tau \zeta_k|^2 \, d\mathcal{H}^n. \tag{3.14}$$

Since $\zeta_k = 0$ outside $B_{r_{k+1}}$, one may estimate

$$\int_{\partial^* E} |\nabla_\tau \zeta_k|^2 \, d\mathcal{H}^n \leq \frac{4}{h} P(E, B_{r_{k+1}}).$$

Moreover, since $0 \leq \zeta_k \leq 1$, it holds

$$\int_{\partial^* E} (\zeta_k - \bar{\zeta}_k)^2 \, d\mathcal{H}^n = \int_{\partial^* E} \zeta_k^2 - \bar{\zeta}_k^2 \, d\mathcal{H}^n \leq \int_{\partial^* E} \zeta_k^2 \, d\mathcal{H}^n \leq P(E, B_{r_{k+1}}).$$

When Λ is large enough, for all $x \in \partial^* E \cap B_{r_{k+1}}$ and all $k \leq n+1$ it holds $\bar{d}_F(x) \geq \frac{\Lambda}{2}\sqrt{h}$. By the Euler–Lagrange equation (2.4), by $\bar{d}_F(x) \geq \frac{\Lambda}{2}\sqrt{h}$, and by Lemma 3.6 it holds

$$|H_E| \geq \frac{\Lambda}{4\sqrt{h}} \quad \text{on } \partial^* E \cap B_{r_k}$$

when Λ is large. Therefore, by using the notation from Lemma 3.6, it holds

$$\partial^* E \cap B_{r_{k+1}} \subset \left\{ x \in \partial^* E : |H_E(x)| \geq \frac{\hat{C}}{\sqrt{h}} \right\} = \partial^* E \setminus \Sigma$$

when Λ is large. Then one may deduce from (3.7) that $P(E, B_{r_{k+1}}) \leq \frac{1}{2} P(E)$. This yields

$$0 \leq \bar{\zeta}_k \leq \frac{1}{2}.$$

Therefore it holds

$$\begin{aligned} \frac{1}{n} \int_{\partial^* E} |H_E|^2 (\zeta_k - \bar{\zeta}_k)^2 \, d\mathcal{H}^n &\geq \frac{1}{n} \int_{\partial^* E \cap B_{r_k}} |H_E|^2 (1 - \bar{\zeta}_k)^2 \, d\mathcal{H}^n \\ &\geq c_n \frac{\Lambda^2}{h} P(E, B_{r_k}). \end{aligned}$$

Combining the three above estimates with (3.14) yields

$$c_n \frac{\Lambda^2}{h} P(E, B_{r_k}) \leq \frac{5}{h} P(E, B_{r_{k+1}}).$$

For Λ large enough this implies

$$\Lambda P(E, B_{r_k}) \leq P(E, B_{r_{k+1}}). \tag{3.15}$$

Use (3.15) $(n+1)$-times from $k = 0$ to $k = n$, use then (3.13) and recall that $r_0 = \frac{\sqrt{h}}{2\Lambda}$ and obtain finally that

$$P(E, B_{r_{n+1}}) \geq \Lambda^{n+1} P(E, B_{r_0}) \geq c\Lambda^{n+1} r_0^n = c\Lambda^{n+1} \left(\frac{\sqrt{h}}{2\Lambda} \right)^n = c\Lambda h^{\frac{n}{2}}. \tag{3.16}$$

But now since $r_{n+1} = (n+1)\sqrt{h} + r_0 \leq 2(n+1)\sqrt{h}$, one obtains from Lemma 3.4 with $r = 2(n+1)\sqrt{h}$ that

$$P(E, B_{r_{n+1}}) \leq P(E, B_r) \leq Ch^{\frac{n}{2}},$$

which contradicts (3.16) when Λ is large. □

4 Existence of the flat flow

Now that the density estimates are proven, the proof of Theorem 1 follows from the arguments from [14, 19] without major changes. In this section I consider the approximative flat flow $(E_t^h)_{t\geq 0}$ and the associated sequence $(E_k^h)_{k\geq 0}$ as in Definition 2.1, starting from an open and bounded set of finite perimeter E_0. The proof for the following "interpolation" result can be found in [14, Lemma 1.5].

Lemma 4.1. *Let $(E_t^h)_{t\geq 0}$ be an approximative flat flow starting from E_0 and fix $h \in (0,1)$ and $t > h$. Then for all $l \leq \sqrt{h}$ it holds*

$$|E_t^h \,\Delta\, E_{t-h}^h| \leq C\Big(lP(E_{t-h}^h) + \frac{1}{l} \int\limits_{E_t^h \Delta E_{t-h}^h} |\bar{d}_{E_{t-h}^h}| \,\mathrm{d}x \Big).$$

By the regularity result stated in Proposition 2.2, the Euler–Lagrange equation

$$\frac{\bar{d}_{E_{t-h}^h}}{h} = -H_{E_t^h} + \lambda_{t,h} \tag{4.1}$$

holds point-wise on $\partial^* E_t^h$ and in a distributional sense on ∂E_t^h. Here $\lambda_{t,h}$ is the Lagrange multiplier. Using the minimality of E_t^h against the previous set E_{t-h}^h, one obtains the important inequality

$$P(E_t^h) + \frac{1}{h} \int\limits_{E_t^h \Delta E_{t-h}^h} |\bar{d}_{E_{t-h}^h}| \,\mathrm{d}x \leq P(E_{t-h}^h). \tag{4.2}$$

Using (4.2) and the argument [14, Lemma 2.1] or [19, Lemma 3.6], one obtains the following dissipation inequality.

Lemma 4.2. *Let $(E_t^h)_{t\geq 0}$ be an approximative flat flow starting from E_0 and fix $h \in (0,1)$. Then for all $T_2 > T_1 \geq h$ it holds*

$$\int_{T_1}^{T_2} \|H_{E_t^h} - \lambda_{t,h}\|^2_{L^2(\partial^* E_t^h)} \,\mathrm{d}t \leq C(P(E_{T_1-h}) - P(E_{T_2})).$$

Moreover, it holds

$$\int_{T_1}^{T_2} (\|H_{E_t^h}\|^2_{L^2(\partial^* E_t^h)} + \lambda_{t,h}^2) \,\mathrm{d}t \leq C(1 + T_2 - T_1).$$

The constant depends on the dimension, $|E_0|$ and $P(E_0)$.

Proof. I will only sketch the proof. Let (E_k^h) be the sequence of sets associated with $(E_t^h)_{t\geq 0}$. For $l \in \mathbb{Z}$ with $2^l \leq 2Ch^{-\frac{1}{2}}$ set

$$K(l) = \{x \in \mathbb{R}^{n+1} : 2^l h < |\bar{d}_{E_{t-h}^h}(x)| \leq 2^{l+1} h\}.$$

Here C is such that $|\bar{d}_{E_{t-h}^h}| \leq C\sqrt{h}$ on ∂E_t^h. Proposition 3.1 yields that for every $x \in \partial E_t^h$,

$$|E_t^h \cap B_{2^l h}(x)| \geq c(2^l h)^{n+1} \quad \text{and} \quad \mathcal{H}^n(\partial E_t^h \cap B_{2^l h}(x)) \leq C(2^l h)^n.$$

Therefore for all $x \in \partial E_t^h \cap K(l)$ it holds

$$\int\limits_{B_{2^l h}(x) \cap E_t^h \Delta E_{t-h}^h} |\bar{d}_{E_{t-h}^h}| \,\mathrm{d}x \geq c(2^l h)^{n+2} \quad \text{and} \quad \int\limits_{B_{2^l h}(x) \cap \partial E_t^h} \bar{d}^2_{E_{t-h}^h} \,\mathrm{d}\mathcal{H}^n \leq C(2^l h)^{n+2}.$$

Combining these two yields

$$\int\limits_{B_{2^l h}(x) \cap \partial E_t^h} \bar{d}^2_{E_{t-h}^h} \,\mathrm{d}\mathcal{H}^n \leq C \int\limits_{B_{2^l h}(x) \cap E_t^h \Delta E_{t-h}^h} |\bar{d}_{E_{t-h}^h}| \,\mathrm{d}x.$$

By applying the Besicovitch covering theorem and summing over $l \in \mathbb{Z}$ (see [19, Lemma 3.6] for details) yields

$$\int\limits_{\partial E_t^h} \bar{d}^2_{E_{t-h}^h} \,\mathrm{d}\mathcal{H}^n \leq \int\limits_{E_t^h \Delta E_{t-h}^h} |\bar{d}_{E_{t-h}^h}| \,\mathrm{d}x.$$

This together with (4.1) and (4.2) implies

$$h \int_{\partial E_t^h} (H_{E_t^h} - \lambda_{t,h})^2 \,\mathrm{d}\mathcal{H}^n \le C(P(E_{t-h}^h) - P(E_t^h)).$$

The first inequality then follows by iterating the above.

By [11, Lemma 2.4] it holds

$$|\lambda_{t,h}| \le C(1 + \|H_{E_t^h} - \lambda_{t,h}\|_{L^1(\partial^* E_t^h)})$$

for a constant that depends on the dimension, on $|E_0|$ and $P(E_0)$. Note that then

$$\|H_{E_t^h}\|_{L^2}^2 + \lambda_{t,h}^2 \le C(1 + \|H_{E_t^h} - \lambda_{t,h}\|_{L^2}^2).$$

Therefore by the first inequality one obtains

$$\int_{T_1}^{T_2} (\|H_{E_t^h}\|_{L^2}^2 + \lambda_{t,h}^2)\,\mathrm{d}t \le C \int_{T_1}^{T_2} (1 + \|H_{E_t^h} - \lambda_{t,h}\|_{L^2}^2)\,\mathrm{d}t \le C(1 + T_2 - T_1). \qquad \square$$

The third lemma we need is a quantitative bound on the diameter of the sets (E_t^h), which is essentially the same as [19, Lemma 3.8].

Lemma 4.3. *Let $(E_t^h)_{t\ge 0}$ be an approximative flat flow starting from E_0 for $h \in (0,1)$. Then for all $T > 0$ there is R_T, which depends on T, on the dimension and on the diameter of the initial set E_0, such that $E_t^h \subset B_{R_T}$ for all $t \le T$.*

Proof. As in [19, Lemma 3.8] define r_t for all $t \le T$ as

$$r_t := \inf\{r > 0 : E_t^h \subset B_r\}.$$

Arguing as in [19, Lemma 3.8], one deduces that at the point $y \in \partial B_{r_t} \cap \partial E_t^h$ it holds $H_{E_t^h}(y) \ge 0$ and therefore by (4.1)

$$r_t \le r_{t-h} + h|\lambda_{t,h}|.$$

Iterating this and using Lemma 4.2 yields

$$R_T - R_0 \le \int_0^T |\lambda_{t,h}|\,\mathrm{d}t \le \int_0^T (1 + \lambda_{t,h}^2)\,\mathrm{d}t \le C(1 + T). \qquad \square$$

Proof of Theorem 1. Let $(E_t^h)_{t\ge 0}$ be an approximative flat flow starting from E_0 for $h \in (0,1)$ and fix $T \ge 1$. Then by (4.2) it holds $P(E_t^h) \le P(E_0)$ and by Lemma 4.3 it holds $E_t^h \subset B_{R_T}$ for all $t \le T$. I claim that for $0 < t < s$ with $s - t \ge h$ it holds

$$|E_t^h \,\Delta\, E_s^h| \le C\sqrt{t-s}. \tag{4.3}$$

Once (4.3) is obtained, then the convergence of a subsequence $E_t^{h_k} \to E_t$ in measure follows as in [14, 19].

Let j, k be such that $s \in [jh, (j+1)h)$ and $t \in [(j+k)h, (j+k+1)h)$. Then by applying Lemma 4.1 for $l = \frac{h}{\sqrt{s-t}}$ and by (4.2) one obtains

$$\begin{aligned}
|E_t^h \,\Delta\, E_s^h| &\le \sum_{m=j}^{j+k} |E_{mh}^h \,\Delta\, E_{(m+1)h}^h| \\
&\le C \sum_{m=j}^{j+k} \left(\frac{h}{\sqrt{s-t}} P(E_{mh}^h) + \frac{\sqrt{s-t}}{h} \int_{E_{(m+1)h}^h \Delta E_{mh}^h} |\bar{d}_{E_{mh}^h}|\,\mathrm{d}x \right) \\
&\le C \sum_{m=j}^{j+k} \frac{h}{\sqrt{s-t}} P(E_0) + C\sqrt{s-t} \sum_{m=j}^{j+k} (P(E_{mh}^h) - P(E_{(m+1)h}^h)) \\
&\le C \frac{kh}{\sqrt{s-t}} P(E_0) + C\sqrt{s-t}P(E_0).
\end{aligned}$$

Since $kh \le 2(s-t)$, one obtains (4.3).

The proof of the consistency principle for $C^{1,1}$-regular initial sets follows by using the arguments in [12]. The volume penalization is used only in [12, Lemma 3.2], but one may overcome this by using the lemma below. □

Lemma 4.4. *Let $F \subset \mathbb{R}^{n+1}$ be an open and bounded set which satisfies interior and exterior ball condition with radius $r_0 > 0$ and let E be a minimizer of (2.2). There are ρ_0 and h_0 with the property that if G is a set of finite perimeter such that*

$$G \,\Delta\, E \subset B_\rho(x) \cap \mathcal{N}_{C_0 h}(\partial F)$$

for $\rho \le \rho_0$ and $h \le h_0$, where $\mathcal{N}_{C_0 h}(\partial F) = \{x : \operatorname{dist}(x, \partial F) < C_0 h\}$, then it holds

$$P(E) \le P(G) + C\rho^{n+1}.$$

Above the constant depends on the dimension, on r_0, C_0, $|F|$ and $P(F)$.

Proof. By approximation one may assume G to be smooth. Since F satisfies interior and exterior ball condition, by [12, Lemma 3.1] it holds

$$\max_{x \in E \Delta F} \operatorname{dist}(x, \partial F) \le Ch \tag{4.4}$$

when $h \le h_0$. As in the proof of Proposition 3.2 the set G may not have the same measure as E and one has to modify it to $\tilde{G}$ with $|\tilde{G}| = |E|$. Assume that $|G| < |E|$. Since F satisfies interior ball condition with radius r_0, there is $y \in G$ such that $B_{\frac{r_0}{2}}(y) \subset G$. By continuity there is $z \in \mathbb{R}^{n+1}$ such that $|z - x| \ge 2\rho_0$ and $|G \cup B_{\frac{r_0}{2}}(z)| = |E|$ when ρ_0 is small. Define $\tilde{G} = G \cup B_{\frac{r_0}{2}}(z)$. By the minimality of E, by (4.4) and by the assumption $G \,\Delta\, E \subset B_\rho(x) \cap \mathcal{N}_{C_0 h}(\partial F)$ it holds

$$P(E) \le P(\tilde{G}) + C|\tilde{G} \,\Delta\, E| \le P(\tilde{G}) + C\rho^{n+1}.$$

Arguing as in [9], one then deduces

$$\begin{aligned} P(\tilde{G}) - P(G) &\le \mathcal{H}^n(\partial B_{\frac{r_0}{2}}(z) \setminus G) - \mathcal{H}^n(\partial G \cap B_{\frac{r_0}{2}}(z)) \\ &\le \frac{2(n+1)|B_1|}{r_0} |B_{\frac{r_0}{2}}(z) \setminus G| \\ &\le C|\tilde{G} \,\Delta\, E| \le C\rho^{n+1} \end{aligned}$$

and the claim follows. □

The paper concludes with Theorem 2. To this end, I recall the definition of a distributional solution of (1.1) from [14].

Definition 4.5. A family of sets of finite perimeter $(E_t)_{t \ge 0}$ is a distributional solution to (1.1) starting from $E_0 \subset \mathbb{R}^{n+1}$ if the following holds:

(1) For almost every $t > 0$ the set E_t has mean curvature $H_{E_t} \in L^2(\partial^* E_t)$ in a distributional sense and for every $T > 0$,

$$\int_0^T \|H_{E_t}\|^2_{L^2(\partial^* E_t)} \, \mathrm{d}t < \infty.$$

(2) There exists $v : \mathbb{R}^{n+1} \times (0, \infty) \to \mathbb{R}$ with $v \in L^2(0, T; L^2(\partial^* E_t))$ such that for every $\phi \in C_0^1(\mathbb{R}^{n+1} \times [0, \infty))$ it holds

$$-\int_0^T \int_{\partial^* E_t} v\phi \, \mathrm{d}\mathcal{H}^n \, \mathrm{d}t = \int_0^T \int_{\partial^* E_t} (H_{E_t} - \bar{H}_{E_t})\phi \, \mathrm{d}\mathcal{H}^n \, \mathrm{d}t,$$

$$\int_0^T \int_{E_t} \partial_t \phi \, \mathrm{d}x \mathrm{d}t + \int_{E_0} \phi(\,\cdot\,, 0) \, \mathrm{d}x = -\int_0^T \int_{\partial^* E_t} v\phi \, \mathrm{d}\mathcal{H}^n \, \mathrm{d}t.$$

Proof of Theorem 2. The proof is exactly the same as [19, Theorem 2.3]. Note that Proposition 3.2 implies that the sets E_t^h are $(Ch^{-\frac{1}{2}}, c\sqrt{h})$-minimizers of the perimeter, i.e., for every F with $F \,\Delta\, E_t^h \subset B_{c\sqrt{h}}(x_0)$ it holds

$$P(E_t^h) \le P(F) + \frac{C}{\sqrt{h}} |E_t^h \,\Delta\, F|.$$

□

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References

[1] E. Acerbi, N. Fusco and M. Morini, Minimality via second variation for a nonlocal isoperimetric problem, *Comm. Math. Phys.* **322** (2013), no. 2, 515–557.

[2] F. Almgren, J. E. Taylor and L. Wang, Curvature-driven flows: a variational approach, *SIAM J. Control Optim.* **31** (1993), no. 2, 387–438.

[3] F. J. Almgren, Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, *Mem. Amer. Math. Soc.* **4** (1976), no. 165, 1–199.

[4] G. Antonelli, E. Pasqualetto and M. Pozzetta, Isoperimetric sets in spaces with lower bounds on the Ricci curvature, *Nonlinear Anal.* **220** (2022), Article ID 112839.

[5] G. Bellettini, V. Caselles, A. Chambolle and M. Novaga, The volume preserving crystalline mean curvature flow of convex sets in $\mathbb{R}^N$, *J. Math. Pures Appl. (9)* **92** (2009), no. 5, 499–527.

[6] L. Bronsard and B. Stoth, Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg–Landau equation, *SIAM J. Math. Anal.* **28** (1997), no. 4, 769–807.

[7] W. Carter, A. Roosen, J. Cahn and J. Taylor, Shape evolution by surface diffusion and surface attachment limited kinetics on completely faceted surfaces., *Acta Metall. Materialia* **43** (1995), 4309–4323.

[8] E. Gonzalez, U. Massari and I. Tamanini, On the regularity of boundaries of sets minimizing perimeter with a volume constraint, *Indiana Univ. Math. J.* **32** (1983), no. 1, 25–37.

[9] M. Grüter, Boundary regularity for solutions of a partitioning problem, *Arch. Ration. Mech. Anal.* **97** (1987), no. 3, 261–270.

[10] V. Julin, M. Morini, M. Ponsiglione and E. Spadaro, The asymptotics of the area-preserving mean curvature and the Mullins–Sekerka flow in two dimensions, *Math. Ann.* **387** (2023), no. 3–4, 1969–1999.

[11] V. Julin and J. Niinikoski, Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow, *Anal. PDE* **16** (2023), no. 3, 679–710.

[12] V. Julin and J. Niinikoski, Consistency of the flat flow solution to the volume preserving mean curvature flow, *Arch. Ration. Mech. Anal.* **248** (2024), no. 1, Paper No. 1.

[13] T. Laux, Weak-strong uniqueness for volume-preserving mean curvature flow, *Rev. Mat. Iberoam.* **40** (2024), no. 1, 93–110.

[14] S. Luckhaus and T. Sturzenhecker, Implicit time discretization for the mean curvature flow equation, *Calc. Var. Partial Differential Equations* **3** (1995), no. 2, 253–271.

[15] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems. An Introduction to Geometric Measure Theory*, Cambridge Stud. Adv. Math. 135, Cambridge University, Cambridge, 2012.

[16] P. W. Michor and D. Mumford, Riemannian geometries on spaces of plane curves, *J. Eur. Math. Soc. (JEMS)* **8** (2006), no. 1, 1–48.

[17] M. Morini, M. Ponsiglione and E. Spadaro, Long time behavior of discrete volume preserving mean curvature flows, *J. Reine Angew. Math.* **784** (2022), 27–51.

[18] L. Mugnai and C. Seis, On the coarsening rates for attachment-limited kinetics, *SIAM J. Math. Anal.* **45** (2013), no. 1, 324–344.

[19] L. Mugnai, C. Seis and E. Spadaro, Global solutions to the volume-preserving mean-curvature flow, *Calc. Var. Partial Differential Equations* **55** (2016), no. 1, Article ID 18.

[20] L. Simon, Introduction to geometric measure theory, Tsinghua Lectures, 2014.