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Marta Llorente

Contents

1. Introduction

The birth of the theory of fractal dimensions can be traced back to the introduction of the Hausdorff dimension in 1917. However, since then many other concepts of dimension have risen in fractal geometry. One of the motivations for this development, especially within the framework of dynamical systems, was the difficulty of a straightforward calculation of the Hausdorff dimension. In this theory it is natural to study fractal properties of measures rather than of those sets. This thesis analyzes the concept of the average dimension that is typically meant for measures, although the definition can be extended in the usual way to sets.

The concept of the local average dimension of a measure μ at $x \in \mathbb{R}^n$ was introduced by Zähle in [27] as the supremum of those positive numbers α that have zero lower average density of order α at that point (see (1.2)), that is

$$\dim_A \mu(x) = \sup\{\alpha : \; d_A^\alpha \mu(x) = 0\} = \inf\{\alpha : \; d_A^\alpha \mu(x) = \infty\}. \tag{1.1}$$

Note that this is a local concept that can be naturally extended to its global version. If the lower average density is replaced by the upper average density (1.3) in the above definition, the corresponding exponent agrees with the local Hausdorff dimension (2.3).

The average density or order two density was introduced by Bedford and Fisher in [1] to obtain a limit which often exists in cases where the usual density of a measure does not exist. The *lower* and *upper average densities* of a Radon measure μ at x are given by

$$d_A^\alpha \mu(x) := \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{1}{r} \, dr \tag{1.2}$$

$$D_A^\alpha \mu(x) := \limsup_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{1}{r} \, dr \tag{1.3}$$

and, if $d_A^\alpha\mu(x) = D_A^\alpha\mu(x)$, we call the common value the *average density*. For instance, this density exists for the restriction of $\mathcal{H}^\alpha$ to self-similar sets or, more generally, for quasi-self similar sets (see, for example, [1] or [5]). Here $\mathcal{H}^\alpha$ denotes the α-dimensional Hausdorff measure. In a recent work, Zähle [28] gave the value of the avera e densit of the normalized Hausdorff measure on the fractal set generated by a conformal iterated function system.

This new dimension lies between the corresponding local Hausdorff and packing dimensions. These are defined in a similar fashion to (1.1) but taking the lower and upper densities, that is, the lower and upper limits of $\mu(B(x,r))/r^\alpha$, instead of the lower average density (see (2.3) and (2.4) for a rigorous definition).

The understanding of these dimensions involves the study of their behaviour under orthogonal projections, smooth mappings, products, slices and intersections. In this line, both the Hausdorff and the packing dimension are very well understood thanks to the work of Falconer, Howroyd, Järvenpää, Kaufman, Marstrand and Mattila, among others.

With any reasonable definition of dimension, every orthogonal projection, proj_V, of a Borel set E (or a Borel measure μ) onto an m-dimensional subspace V has dimension at most $\min\{\dim E, m\}$ (analogously, $\min\{\dim\mu, m\}$). In the case of the Hausdorff dimension, we have equality for almost all V (see [19] and [13]). Zähle proved in [27] that this equality is also true for the average dimension. But this is not the case with the packing dimension. For $1 \le m \le n-1$ and all $0 < s < n$ there are Borel subsets of $\mathbb{R}^n$ of packing dimension s but whose projections have packing dimension strictly less than $\min(m,s)$. Examples of such sets were given by Järvenpää in [14], who also showed that $\dim_P \mathrm{proj}_V E \ge (m/n)\dim_P E$ for almost all subspaces V. At about the same time Falconer and Howroyd in [7] gave the best possible bound, that is,

$$\dim_P \mathrm{proj}_V E \ge \frac{\dim_P E}{1 + (1/m - 1/n)\dim_P E}.$$

In relation to the study of the behaviour of the intersections of an s-dimensional set with $(n-m)$-dimensional affine subspaces of $\mathbb{R}^n$, Mattila [20] used differentiation theory to construct the slicing measures. That is, given a Radon measure μ on $\mathbb{R}^n$, he constructed these new measures by slicing μ with $(n-m)$-dimensional affine subspaces W of $\mathbb{R}^n$ obtaining a Radon measure supported by $W \cap \mathrm{spt}\,\mu$ (see Section 2.1 for the definition and the main properties). By spt we denote the support of a measure. These new measures and their integral relations with μ were used in [20] to study the capacities of the intersections of a set with $(n-m)$-planes. Earlier Marstrand had explored the fractional dimensional subsets of the plane $\mathbb{R}^2$. Mattila [19] extended Marstrand's result to higher dimensions. He proved that for all $E \subset \mathbb{R}^n$ and $V \in G_{n,n-m}$

$$\dim_H(E \cap V_a) \le \max\{0, \dim_H E - m\} \tag{1.4}$$

for $\mathcal{H}^m$-almost all $a \in V^\perp$, where $V^\perp$ is the orthogonal complement of V, and V_a is the $(n-m)$-plane $\{v + a : v \in V\}$ for all $a \in V^\perp$. Moreover, if $E \subset \mathbb{R}^n$ is a Borel set, then for $\gamma_{n,n-m}$ almost all $V \in G_{n,n-m}$

$$\text{ess sup}_{a \in V^\perp} \dim_H(E \cap V_a) = \max\{0, \dim_H E - m\}.$$

Analogous results for the Hausdorff dimension of measures have been obtained by Järvenpää and Mattila in [18]. There they proved that if m and n are integers with $0 < m < n$ and μ is a Radon measure in $\mathbb{R}^n$ with compact support, then for $\gamma_{n,n-m}$ almost all $V \in G_{n,n-m}$

$$\text{ess inf}\{\dim_H \mu_{V,a} : \ a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^n) > 0\} = \dim_H \mu - m$$

provided that $\dim_H \mu > m$. Here $\mu_{V,a}$ is the slice of μ with the plane V_a. In particular, they proved that for almost all $(n-m)$-dimensional linear subspaces V

$$\dim_H \mu_{V,a} \geq \dim_H \mu - m \tag{1.5}$$

for $\mathcal{H}^m$ almost all $a \in V^\perp$ with $\mu_{V,a}(\mathbb{R}^n) > 0$.

As Theorems 2.6 and 2.8 will show, the average dimension behaves under sections of measures like the Hausdorff dimension does. We will prove that if $\dim_H \mu > m$, then for almost all $(n-m)$-dimensional linear subspaces V it holds that

$$\dim_A \mu_{V,x}(x) = \dim_A \mu(x) - m$$

for μ almost all $x \in \mathbb{R}^n$.

The geometry of the packing dimension in relation to slices is much less regular. In the case of the packing dimension of sets, Falconer [6] proved that if $E \subset \mathbb{R}^n$, then for any $(n-m)$-dimensional subspace V we have

$$\dim_P(E \cap V_a) \leq \max\{0, \dim_P E - m\}$$

for $\mathcal{H}^m$ almost all $a \in V$. By redefining the packing dimension of sets, Falconer and Järvenpää got a result stronger than (1.4) for packing dimensions, since they were able to obtain an estimate for each individual m-dimensional linear subspace V, see [9].

Although we cannot expect equality for the packing dimensions of projections, Falconer and Howroyd [8] proved that, given an analytic set $E \subset \mathbb{R}^n$, $\dim_P \text{proj}_V(E)$ is almost surely a constant. A recent work of Falconer, Järvenpää and Mattila [10] shows that there is not such a result for plane sections. They showed that there exist a compact set $E \subset \mathbb{R}^n$ and compact subsets A and B of $G(n,m)$ with $\gamma_{n,m}(A) > 0$ and $\gamma_{n,m}(B) > 0$ such that for all $V \in A$

$$\mathscr{H}^m(\operatorname{proj}_{V^\perp}(E)) = 0, \quad \text{that is,} \quad E \cap V_a = \varnothing$$

for $\mathscr{H}^{n-m}$ almost all $a \in V^\perp$, and for all $V \in B$

$$\dim_P(E \cap V_a) = m$$

for points a in a non-empty open subset of $V^\perp$. Using ideas similar to those in [22], they showed the instability of the packing dimension of sections under smooth "bending" diffeomorphisms. They also conjectured that, given a Borel function f from the space of affine m-planes in $\mathbb{R}^n$ into the closed subinterval $[0,m]$, there is a Borel set $E \subset \mathbb{R}^n$ such that $\dim_P(E \cap V) = f(V)$ for almost all affine planes V. This conjecture, in a more general setting, has been solved in the plane by Csörnyei in [2].

Regarding packing dimension of measures, Falconer and Mattila [11] considered the $(n-m)$-dimensional slices of measures on $\mathbb{R}^n$ and proved an analogue of (1.5) for the packing dimension, that is given a probability Radon measure μ on $\mathbb{R}^n$ such that $\dim_H \mu > m$, they proved that for μ almost all $x \in \mathbb{R}^n$ the slices of μ by almost all $(n-m)$-planes V_x through x satisfy

$$\dim_P \mu_{V,x} \geq \frac{(n-m)(\dim_P \mu)(\dim_H \mu - m)}{n \dim_H \mu - m \dim_P \mu}.$$

They also get the following result for projections:

If μ is a probability measure on $\mathbb{R}^n$ such that $\dim_H \mu \leq m$, then

$$\dim_P(\operatorname{proj}_V \mu) \geq \frac{(\dim_P \mu)(1 - (1/n)\dim_H \mu)}{1 + (1/m - 1/n)\dim_P \mu - (1/m)\dim_H \mu}$$

for almost all m-dimensional subspaces V. They gave examples to show that both inequalities are sharp.

In Scction 3 we will be concerned with the problem of studying the average dimension of products of Radon measures. We will prove that, given probability Radon measures μ on $\mathbb{R}^m$ and ν on $\mathbb{R}^n$, both with compact support, it holds that

$$\dim_A \mu(x) + \dim_H \nu(y) \leq \dim_A(\mu \times \nu)(x,y) \leq \dim_A \mu(x) + \dim_P \nu(y)$$

for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. As our examples will prove, these bounds are sharp. We also will give examples showing that the known inequalities for the Hausdorff and packing dimensions of product measures

$$\dim_H \mu + \dim_H \nu \leq \dim_H \mu \times \nu \leq \dim_H \mu + \dim_P \nu,$$
$$\dim_P \mu + \dim_H \nu \leq \dim_P \mu \times \nu \leq \dim_P \mu + \dim_P \nu$$

(cf. [12]) cannot be improved with the average dimension. Our examples will be based on the construction of Cantor-type sets with changing dimensional behaviour. Note that in order to get $\dim_A \mu(x) \leq \alpha$, $x \in \mathbb{R}^n$, it would be enough for the measure μ to satisfy $\mu(B(x,r)) \gtrsim r^\alpha$ with $r \in (\delta, \delta^a)$, $\delta > 0$ as small as we wish and $a \in (0,1)$ independent of δ. Observe that, by definition, if we want that $\dim_H \mu(x) \leq \alpha$, we only need that the above inequality holds for a sequence of r_i's converging to zero. However, for the packing dimension we need that $\mu(B(x,r)) \gtrsim r^\alpha$ for every r small enough. On the other hand, to get $\dim_A \mu(x) \geq \alpha$, where $0 < \alpha < 1$, it is enough that there exists a sequence $\{\delta_i\}_{i=1}^\infty$ converging to zero such that $\mu(B(x,r)) \lesssim r^\alpha$ for $r \in (\delta_i, \frac{1}{|\log \delta_i|})$. In the examples we will combine these ideas to obtain the desired measures.

Section 4 will be devoted to the study of the average dimension of intersection measures $\mu \cap f_\# \nu$ when f runs through the similarities or isometries. These measures, which can be regarded as natural measures on $\operatorname{spt} \mu \cap f(\operatorname{spt} \nu)$, were introduced by Mattila in [21]. There he studied the relation between the Hausdorff dimensions of A, B and $A \cap f(B)$, where A and B are Borel sets and f is as before. Let us recall that if f is a similarity map on $\mathbb{R}^n$, it has a unique decomposition as

$$f = \tau_z \circ g \circ \delta_r, \qquad z \in \mathbb{R}^n,\ g \in O(n),\ r \in \mathbb{R}^+,$$

where $\tau_z : \mathbb{R}^n \to \mathbb{R}^n$ is the translation $\tau_z(x) = x + z$, and $\delta_r : \mathbb{R}^n \to \mathbb{R}^n$ is the homothety $\delta_r(x) = rx$. Note that if f is an isometry, we will have the same decomposition but with δ_r being the identity map. Using the relations between the Hausdorff dimension of sets and measures, Järvenpää proved in [17] the following result for measures. If μ and ν are Radon measures on $\mathbb{R}^n$ with compact supports such that

(a) $\dim_H(\mu \times \nu) = \dim_H \mu + \dim_H \nu > n$ and

(b) the t-energy of ν is finite for all $0 < t < \dim_H \nu < n$,

then for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times (0,\infty)$ we have

$$\begin{aligned} \operatorname{ess\,inf}\{\dim_H \mu \cap (\tau_z \circ g \circ \delta_r)_\# \nu : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\# \nu(\mathbb{R}^n) > 0\} \\ = \dim_H \mu + \dim_H \nu - n. \end{aligned}$$

Changing assumption (b) to the following (b') the t-energy of ν is finite for all $0 < t < \dim_H \nu < n$ and $\dim_H \nu > \frac{1}{2}(n+1)$,

one can see that the above result also holds when considering isometries instead of similarities (cf. [17]).

In Theorems 4.9 and 4.10 we will prove the following analogies for the average dimension and similarities. Let $0 < s < n$, $0 < t < n$ such that $s + t - n \geq 0$. If $0 < r_1 < r_2 < \infty$ and μ and ν are probability Radon measures on $\mathbb{R}^n$ with compact support such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$, then

$$\begin{aligned}\dim_A \mu(x) + \dim_H \nu - n &\leq \dim_A(\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu)(x)\\ &\leq \dim_A \mu(x) + \dim_P \nu(x-z) - n\end{aligned}$$

for $\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu \times \mathcal{L}^n \times \theta_n \times \mathcal{L}^1$ almost all $(x, z, g, r) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n) \times [r_1, r_2]$.

Similar results will be obtained when similarities are replaced by isometries, see Theorem 4.13 and Corollary 4.14. Since intersection measures are defined as the orthogonal projections on $\mathbb{R}^n$ of measures obtained by slicing a certain product measure on $\mathbb{R}^n \times \mathbb{R}^n$ with diagonal n-planes (see Section 4.1), the results concerning these intersections are closely related with those obtained for slicing, products and projections of measures. The methods used for proving the lower bounds for intersections in the last section are similar to those used by Järvenpää in [15] and [16], where the same problems are solved for the packing dimension, and are originally from [11].

2. Average dimension and sections of measures

2.1. Notation and preliminaries. Throughout this section, μ will be a probability Radon measure on $\mathbb{R}^n$ with compact support. We will denote by $B(x, r)$ the closed ball with centre x and radius r in $\mathbb{R}^n$. If μ is a measure on a set X, we will denote by $f_{\#}\mu$ the image of μ under a function $f : X \to Y$, that is,

$$f_{\#}\mu(A) = \mu(f^{-1}(A))$$

for all $A \subset Y$. The restriction of μ to a set $B \subset X$ is denoted by $\mu \mid B$, that is,

$$(\mu \mid B)(A) = \mu(B \cap A)$$

for all $A \subset X$. For $0 < t < n$, the *t-energy* of a Radon measure μ on $\mathbb{R}^n$ is defined by

$$I_t(\mu) = \iint |x - y|^{-t} \, d\mu x \, d\mu y.$$

We will denote by $||\mu||$ the total mass of μ. Here μ will be a probability measure, that is, $||\mu|| = 1$. Let μ and ν be measures in a set X. The measure μ is said to be ***absolutely continuous*** with respect to ν, if $\mu(A) = 0$ for any $A \subset X$ with $\nu(A) = 0$. In this case we write $\mu \ll \nu$. Let m and n be integers with $0 < m < n$. We denote by $G_{n,n-m}$ the Grassmannian manifold consisting of all $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$. The unique rotationally invariant Radon probability measure on $G_{n,n-m}$ is denoted by $\gamma_{n,n-m}$. For any $V \in G_{n,n-m}$, let $V^\perp \in G_{n,m}$ be the orthogonal complement of V and $P_{V^\perp} : \mathbb{R}^n \to V^\perp$ the orthogonal projection onto $V^\perp$.

Now we define the slices of a Radon probability measure μ by $(n-m)$-planes, see [23]. The slice of μ by the plane $V_a = \{v + a : v \in V\}$, $V \in G_{n,n-m}$ and $a \in V^\perp$, is the Radon measure $\mu_{V,a}$ on V_a, which exists for $\mathscr{H}^m$ almost all $a \in V^\perp$, such that

$$\int \phi \, d\mu_{V,a} = \lim_{\delta \to 0} \alpha(m)^{-1} \delta^{-m} \int_{\{y \,:\, d(y,V_a) \le \delta\}} \phi \, d\mu$$

for all non-negative continuous functions ϕ on $\mathbb{R}^n$ with compact support, where $\alpha(m)$ is the volume of the m-dimensional unit ball. Here $\mathscr{H}^m$ denotes the m-dimensional Hausdorff measure normalized so that $\mathscr{H}^m$ in $\mathbb{R}^m$ is the Lebesgue measure $\mathcal{L}^m$. Obviously,

$$\operatorname{spt} \mu_{V,a} \subset \operatorname{spt} \mu \cap V_a, \tag{2.1}$$

where spt is the support of a measure. Further,

$$\int_{V^\perp} \int f \, d\mu_{V,a} \, d\mathscr{H}^m a = \int f \, d\mu, \tag{2.2}$$

for all non-negative Borel functions f on $\mathbb{R}^n$ with $\int f \, d\mu < \infty$, provided that $P_{V^\perp \#}\mu \ll \mathscr{H}^m | V^\perp$. This is the case for $\gamma_{n,n-m}$ almost all $V \in G_{n,n-m}$ if $\dim_H \mu > m$ (see below for the definition).

In order to introduce measures $\mu_{V,x}$ on $(n-m)$-planes $V_x = \{v + x : v \in V\}$ through $x \in \mathbb{R}^n$, we simply set $\mu_{V,x} = \mu_{V,a}$ for any $x \in P_{V^\perp}^{-1}(\{a\})$, whenever $a \in V^\perp$ is such that $\mu_{V,a}$ is defined. This holds for $\mathscr{H}^m$ almost all $a \in V^\perp$.

Definition 2.1. Let μ be a finite Radon measure on $\mathbb{R}^n$.

(a) For $0 \le \alpha < \infty$, let

$$d_H^\alpha \mu(x) := \limsup_{\delta \to 0} \frac{\mu(B(x,\delta))}{\delta^\alpha},$$
$$d_P^\alpha \mu(x) := \liminf_{\delta \to 0} \frac{\mu(B(x,\delta))}{\delta^\alpha}$$

be the *upper* and *lower densities* of order α of μ at x, respectively.

We define the *local Hausdorff dimension* of μ at x to be

$$(2.3) \qquad \dim_H \mu(x) := \sup\{\alpha \geq 0 : d^\alpha_H \mu(x) = 0\} = \inf\{\alpha \geq 0 : d^\alpha_H \mu(x) = \infty\},$$

and the *local packing dimension* of μ at x to be

$$(2.4) \qquad \dim_P \mu(x) := \sup\{\alpha \geq 0 : d^\alpha_P \mu(x) = 0\} = \inf\{\alpha \geq 0 : d^\alpha_P \mu(x) = \infty\}.$$

They can also be defined as

$$\dim_H \mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$
$$\dim_P \mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

(b) We define the *Hausdorff dimension* of μ to be

$$\dim_H \mu = \sup\left\{d \ : \ \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq d \text{ for } \mu - a.a.\ x\right\}$$

and the *packing dimension* of μ to be

$$\dim_P \mu = \sup\left\{d \ : \ \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq d \text{ for } \mu - a.a.\ x\right\}.$$

Remark 2.2. Note that by the above definitions we have that if μ is a Radon probability measure on $\mathbb{R}^n$ with $\dim_H \mu > d$, then for μ almost all $x \in \mathbb{R}^n$ there exists b (depending on x) such that:

$$\mu(B(x,r)) \leq br^d \qquad \text{for all} \quad r > 0.$$

Also, if $x \in \mathbb{R}^n$ is such that $\dim_H \mu(x) > d$, then there exists b (depending on x) such that:

$$\mu(B(x,r)) \leq br^d \qquad \text{for all} \quad r > 0.$$

Next we define the local average dimension of a measure, which was introduced by Zähle in [27].

Definition 2.3. For $0 \le \alpha < \infty$ let

$$d_A^\alpha \mu(x) := \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{1}{r} \, dr$$

be the *lower average density* of order α of μ at x. We define the *local average dimension of* μ *at* x as

$$\dim_A \mu(x) := \sup\{\alpha \ge 0 : d_A^\alpha \mu(x) = 0\} = \inf\{\alpha \ge 0 : d_A^\alpha \mu(x) = \infty\}.$$

The average dimension lies between the corresponding local Hausdorff and packing dimensions, that is,

$$\dim_H \mu(x) \le \dim_A \mu(x) \le \dim_P \mu(x),$$

and these inequalities can be strict.

Note that if the lower average density is replaced by the upper average density, the corresponding exponent agrees with the local Hausdorff dimension.

2.2. Average dimension and plane sections. We first introduce the quantity $J_r(x)$, which was used in [11] to analyze the packing dimensions of slices of measures.

Definition 2.4. Let μ be a probability Radon measure on $\mathbb{R}^n$. For $r > 0$ and $x \in \mathbb{R}^n$ we define

$$J_r(x) = \int \mu_{V,x}(B(x,r)) \, d\gamma_{n,n-m} V$$

provided that the right-hand side is defined.

Lemma 2.5. *Let μ be a probability Radon measure on $\mathbb{R}^n$ such that* $\dim_H \mu > m$. *Then for $r > 0$ and for μ almost all $x \in \mathbb{R}^n$*

$$J_r(x) \le c \int_0^{2r} h^{-m-1} \mu(B(x,h)) \, dh,$$

where c depends only on m and n.

Proof. See Lemma 4.4 in [11]. □

Theorem 2.6. *Let μ be a probability Radon measure on $\mathbb{R}^n$ with compact support. If* $\dim_H \mu > m$, *then*

$$\dim_A \mu_{V,x}(x) \ge \dim_A \mu(x) - m$$

for μ almost all $x \in \mathbb{R}^n$ and $\gamma_{n,n-m}$ almost all $V \in G_{n,n-m}$.

In the proof of this Theorem we will need the following Lemma.

Lemma 2.7. *Let μ be a probability Radon measure on $\mathbb{R}^n$. Suppose that $x \in \mathbb{R}^n$, $\alpha > 0$, $\dim_H \mu(x) > d$ and $0 < \alpha + d < \dim_A \mu(x)$. Then*

$$\liminf_{\delta \to 0} \frac{1}{|\log \delta|} \left[\int_{2\delta}^{1} \frac{\mu(B(x,h))}{h^{\alpha+d+1}}\, dh + \delta^{-\alpha} \int_{0}^{2\delta} \frac{\mu(B(x,h))}{h^{d+1}}\, dh \right] = 0. \tag{2.5}$$

Proof. Let $0 < \alpha + d < s < \dim_A \mu(x)$. Then for all $0 < \lambda < 1$, $C > 0$ and $N \in \mathbb{N}^+$ there exists $0 < \delta < \frac{1}{N}$ such that

$$\frac{1}{|\log \delta|} \int_{\delta}^{1} \frac{\mu(B(x,h))}{h^s} \frac{1}{h}\, dh < \frac{\lambda}{2C} < 1. \tag{2.6}$$

For simplicity and without losing generality, we assume in the following that δ verifies $|\log \delta| > 1$.

Step I.

Let $d < p < \dim_H \mu(x)$, $\alpha + d < t < s$ and $0 < \varepsilon < (p-d)/\alpha < 1$. Then there exists $\delta_0 \in (0, 1/4)$ such that if $0 < \delta < \delta_0$ and δ satisfies (2.6), then $\mu(B(x,h)) \leq h^t$ for all $\delta < h < \delta^\varepsilon$.

To prove this, let $r \in (\delta, 1/2)$ be such that $\mu(B(x,r)) > r^t$. Then,

$$\begin{aligned} 1 &> \frac{1}{|\log \delta|} \int_{\delta}^{1} \frac{\mu(B(x,h))}{h^s} \frac{1}{h}\, dh \geq \frac{1}{|\log \delta|} \int_{r}^{1} \frac{\mu(B(x,h))}{h^s} \frac{1}{h}\, dh \\ &\geq \frac{1}{|\log \delta|} \int_{r}^{1} \frac{r^t}{h^s} \frac{1}{h}\, dh = \frac{r^t}{|\log \delta|} \frac{1}{s}(r^{-s} - 1) \geq \frac{c}{|\log \delta|} r^{t-s}, \end{aligned}$$

where c is some constant depending on s.

Thus, $r \geq \left(\frac{c}{|\log \delta|}\right)^{1/(s-t)}$, and so

$$\mu(B(x,h)) \leq h^t \qquad \text{for} \quad \delta < h < \min\left\{\frac{1}{2}, \left(\frac{c}{|\log \delta|}\right)^{1/(s-t)}\right\}.$$

Choose δ_0 such that $0 < \delta_0 < 1/4$ and

$$\delta^\varepsilon < \left(\frac{c}{|\log \delta|}\right)^{\frac{1}{s-t}} \qquad \text{for} \quad 0 < \delta < \delta_0.$$

Then, if $0 < \delta < \delta_0$ and x and δ satisfy (2.6), we have

$$\mu(B(x,h)) \leq h^t \qquad \text{for} \quad \delta < h < \delta^\varepsilon.$$

We have now proved Step I.

Step II

Let δ_0, δ, p, t and x be as in Step I. Then for $0 < \varepsilon < (p-d)/\alpha$ we have that

$$\int_0^{\delta^\varepsilon} \frac{\mu(B(x,h))}{h^d} \frac{1}{h}\, dh < \mathbf{c}\delta^{\varepsilon\alpha}, \tag{2.7}$$

where $\mathbf{c}$ is some constant $\mathbf{c}= \mathbf{c}(p,d,t,x)$.

Using Remark 2.2 and Step I, we have that

$$\begin{aligned}
\int_0^{\delta^\varepsilon} \frac{\mu(B(x,h))}{h^d} \frac{1}{h}\, dh &= \int_0^{\delta} \frac{\mu(B(x,h))}{h^d} \frac{1}{h}\, dh + \int_\delta^{\delta^\varepsilon} \frac{\mu(B(x,h))}{h^d} \frac{1}{h}\, dh \\
&\leq b \int_0^{\delta} h^{p-d-1}\, dh + \int_\delta^{\delta^\varepsilon} h^{t-d-1}\, dh \\
&= b\frac{\delta^{p-d}}{p-d} + \frac{\delta^{\varepsilon(t-d)} - \delta^{t-d}}{t-d} \leq \mathbf{c}\delta^{\varepsilon\alpha}.
\end{aligned}$$

Note that in the last inequality we have used that $p-d > \varepsilon\alpha$ and $t-d > \alpha$.

Now we are ready to prove our main assertion (2.5).

That is, we have to show that for all $\lambda > 0$ and all $N \in \mathbb{N}^+$, there exists $\widetilde{\delta} > 0$ such that $0 < \widetilde{\delta} < \frac{1}{N}$ and

$$\frac{1}{|\log \widetilde{\delta}|} \left[\int_{2\widetilde{\delta}}^1 \frac{\mu(B(x,h))}{h^{\alpha+d+1}}\, dh + \widetilde{\delta}^{-\alpha} \int_0^{2\widetilde{\delta}} \frac{\mu(B(x,h))}{h^{d+1}}\, dh \right] < \lambda.$$

Let $\lambda > 0$, $N \in \mathbb{N}^+$ and δ be as in Step I. We take $\widetilde{\delta} = \frac{\delta^\varepsilon}{2}$ with $0 < \varepsilon < (p-d)/\alpha$ and δ such that $0 < \delta < \min\left(\delta_0, (2/N)^{1/\varepsilon}\right)$ and verify that $\frac{c_2}{\varepsilon|\log\delta|} \leq \frac{\lambda}{2}$, where δ_0 is as in Step I and c_2 is some constant that will appear in the next estimates. Using Step II, we have that

$$\begin{aligned}
\frac{\widetilde{\delta}^{-\alpha}}{|\log \widetilde{\delta}|} \int_0^{2\widetilde{\delta}} \frac{\mu(B(x,h))}{h^{d+1}}\, dh &\leq c_1 \frac{\delta^{-\varepsilon\alpha}}{|\log \delta^\varepsilon|} \int_0^{\delta^\varepsilon} \frac{\mu(B(x,h))}{h^{d+1}}\, dh \\
&\leq \frac{c_2}{|\log \delta^\varepsilon|} = \frac{c_2}{\varepsilon|\log\delta|} \leq \frac{\lambda}{2}.
\end{aligned}$$

By (2.6)

$$\begin{aligned}\frac{1}{|\log\widetilde{\delta}|}\int_{2\widetilde{\delta}}^{1}\frac{\mu(B(x,h))}{h^{d+\alpha}}\frac{1}{h}\,dh &\le \frac{1}{|\log\widetilde{\delta}|}\int_{2\widetilde{\delta}}^{1}\frac{\mu(B(x,h))}{h^{s}}\frac{1}{h}\,dh\\ &\le \frac{1}{\varepsilon|\log\delta|}\int_{\delta^{\varepsilon}}^{1}\frac{\mu(B(x,h))}{h^{s}}\frac{1}{h}\,dh\\ &\le \frac{C}{|\log\delta|}\int_{\delta}^{1}\frac{\mu(B(x,h))}{h^{s}}\frac{1}{h}\,dh \le \frac{\lambda}{2}\,.\end{aligned}$$

Note that for the first and last inequality in the second estimation we have used respectively that $d+\alpha < s$ and that $\delta^{\varepsilon} > \delta$, since $\varepsilon < 1$. □

Let us now prove Theorem 2.6.

Proof. Let $0 < \alpha < \dim_A \mu(x) - m$. We want to show that for μ almost all $x \in \mathbb{R}^n$

$$d_A^{\alpha}\mu_{V,x}(x) = 0 \quad \text{for } \gamma_{n,n-m} \text{ almost all } V \in G_{n,n-m}.$$

Using Fatou's lemma, Fubini's theorem and Lemma 2.5, we have that

$$\begin{aligned}&\int d_A^{\alpha}\mu_{V,x}(x)\;d\gamma_{n,n-m}V\\ &\le \liminf_{\delta\to 0}\int\frac{1}{|\log\delta|}\int_{\delta}^{1}\frac{\mu_{V,x}(B(x,r))}{r^{\alpha}}\frac{1}{r}\,dr\;d\gamma_{n,n-m}V\\ &= \liminf_{\delta\to 0}\int_{\delta}^{1}\frac{r^{-\alpha-1}}{|\log\delta|}\int\mu_{V,x}(B(x,r))\;d\gamma_{n,n-m}V\;dr\\ &\le c\liminf_{\delta\to 0}\int_{\delta}^{1}\frac{r^{-\alpha-1}}{|\log\delta|}\int_{0}^{2r}h^{-m-1}\mu(B(x,h))\;dh\;dr\end{aligned}$$

for μ almost all $x \in \mathbb{R}^n$. Here c is the constant of Lemma 2.5 and depends only on n and m.

Then, by Fubini's theorem, we will get that for μ almost all $x \in \mathbb{R}^n$

$$\begin{aligned}
&\int d_A^\alpha \mu_{V,x}(x)\, d\gamma_{n,n-m} V \\
&\leq c \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \left[\int_{2\delta}^{2} \int_{\frac{h}{2}}^{1} r^{-\alpha-1} h^{-m-1} \mu(B(x,h))\, dr\, dh \right. \\
(2.8) \qquad &\left. + \int_0^{2\delta} \int_\delta^1 r^{-\alpha-1} h^{-m-1} \mu(B(x,h))\, dr\, dh \right] \\
&\leq \mathbf{C} \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \left[\int_{2\delta}^{1} \frac{\mu(B(x,h))}{h^{\alpha+m+1}}\, dh \right. \\
&\left. + \delta^{-\alpha} \int_0^{2\delta} \frac{\mu(B(x,h))}{h^{m+1}}\, dh \right],
\end{aligned}$$

where $\mathbf{C}$ is some constant depending on $c = c\ (n,m)$ and α, that is $\mathbf{C} = \mathbf{C}(n,m,\alpha)$. Note that in the last inequality we have used that

$$\liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_1^2 \frac{\mu(B(x,h))}{h^{\alpha+m+1}}\, dh = 0$$

Using Lemma 2.7 with "$d = m$," we get that (2.8) equals zero.

□

Next we prove that we have equality in Theorem 2.6.

Theorem 2.8. *Let μ be a probability Radon measure on $\mathbb{R}^n$ with compact support. Provided that $\dim_H \mu > m$, then*

$$\dim_A \mu_{V,x}(x) = \dim_A \mu(x) - m$$

for μ almost all $x \in \mathbb{R}^n$ and $\gamma_{n,n-m}$ almost all $V \in G_{n,n-m}$.

Proof. As the lower bound has been proved in the above theorem, we just have to prove the " $\leq$ " inequality.

Let $x \in \mathbb{R}^n$ be such that $\dim_A \mu(x) < \alpha$. We want to show that for all $\varepsilon > 0$, $\dim_A \mu_{V,x}(x) < \alpha - m + \varepsilon$ for μ almost all $x \in \mathbb{R}^n$ and $\gamma_{n,n-m}$ almost all $V \in G_{n,n-m}$.

Let $\mathcal{D}_k$, $k = 1, 2, \ldots$ be the standard half-open disjoint dyadic cubes Q of side-lengths $l(Q) = 2^{-k}$ (see, for example [26]). Denote by $\mathcal{D} = \bigcup_{k=1}^\infty \mathcal{D}_k$. Let $V \in G_{n,n-m}$ be such that $P_{V^\perp \#}\mu \ll \mathcal{H}^m | V^\perp$.

Let $\varepsilon > 0$ and $\eta > 0$. For each $Q \in \mathcal{D}$ we define

$$A_Q = \{x \in 2Q \ ; \ \mu_{V,x}(2Q) < \eta \ \mu(2Q) \ l(Q)^{\varepsilon - m}\},$$

where $2Q$ stands for the cube centered at the same point as Q and with side-length $l(2Q) = 2l(Q)$. Let us denote $A = \bigcup_{Q\in\mathcal{D}} A_Q$. Then, by (2.2), we have that

$$\begin{aligned}
\mu(A) &\leq \sum_{Q\in\mathcal{D}} \mu(A_Q) = \sum_{Q\in\mathcal{D}} \int_{P_{V^\perp}(A_Q)} \mu_{V,a}(2Q) \ d\mathscr{H}^m a \\
&\leq \sum_{Q\in\mathcal{D}} \eta \ \mu(2Q) \ l(Q)^{\varepsilon-m} \ \mathscr{H}^m(P_{V^\perp}(A_Q)) \leq \eta \ c \sum_{k=1}^{\infty} 2^{-k\varepsilon} \sum_{Q\in\mathcal{D}_k} \mu(2Q) \\
&\leq \eta \ c' \ \mu(R^n) \ \frac{1}{2^\varepsilon - 1} = c'' \ \eta,
\end{aligned}$$

where $c = c(m)$, $c' = c'(m,n)$ and $c'' = c''(m,n,\varepsilon)$ are positive constants.

Let $\delta > 0$. Suppose $x \in \mathbb{R}^n \setminus A$ and $r \in (\delta, 1)$. Then there exist $Q \in \mathcal{D}$ and a positive constant $0 < c < 1$ such that $x \in Q$ and

$$B(x, cr) \subset 2Q \subset B(x,r),$$

where c depends only on n. Hence,

$$\begin{aligned}
\mu_{V,x}(B(x,r)) &\geq \mu_{V,x}(2Q) \geq \eta \ \mu(2Q) \ l(Q)^{\varepsilon-m} \\
&\geq \eta \ \mu(B(x,cr)) \ \mathbf{c} \ r^{\varepsilon-m},
\end{aligned} \tag{2.9}$$

where $\mathbf{c} > 0$ is a constant depending on n, m and ε.

Note that, since $c < 1$ we have

$$\int_{c\delta}^{1} \frac{\mu(B(x,r))}{r^\alpha} \frac{dr}{r} \geq \int_{\delta}^{1} \frac{\mu(B(x,r)}{r^\alpha} \frac{dr}{r} \tag{2.10}$$

By (2.9), (2.10) and change of variable we have

$$\begin{aligned}
&\frac{1}{|\log\delta|} \int_\delta^1 \frac{\mu_{V,x}(B(x,r))}{r^{\alpha-m+\varepsilon}} \frac{dr}{r} \geq \frac{1}{|\log\delta|} \int_\delta^1 \mathbf{c}\eta \ \frac{\mu(B(x,cr)}{r^\alpha} \frac{dr}{r} \\
&= C \ \eta \ \frac{1}{|\log\delta|} \int_{c\delta}^{c} \frac{\mu(B(x,r))}{r^\alpha} \frac{dr}{r} \\
&\geq C \ \eta \left[\frac{1}{|\log\delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{dr}{r} - \frac{1}{|\log\delta|} \int_c^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{dr}{r} \right],
\end{aligned}$$

where $C > 0$ is a constant depending on n, m and ε. Since $\dim_A \mu(x) < \alpha$, we get that

$$\liminf_{\delta\to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu_{V,x}(B(x,r))}{r^{\alpha-m+\varepsilon}} \frac{dr}{r} = \infty$$

for $x \in \mathbb{R}^n \setminus A$. Since $\mu(A) \leq c''\eta$ and we can take η as small as we wish, we get that $d_A^{\alpha-m+\varepsilon}\mu_{V,x}(x) = \infty$ for μ almost all $x \in \mathbb{R}^n$ and for $\gamma_{n,n-m}$ almost all $V \in G_{n,n-m}$.

3. Average dimension of product measures

In this section, we are going to study the average dimension of the product of measures. Since it lies between the Hausdorff and the packing dimension, it is natural to ask whether the known inequalities for the Hausdorff and packing dimensions of the product of measures can be improved with this other one. Unfortunately, the answer is no, as examples 3.3, 3.4 and 3.5 will show.

Theorem 3.1. *Let μ and ν be probability Radon measures on $\mathbb{R}^m$ and $\mathbb{R}^n$ with compact support. Then for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$,*

$$\dim_A(\mu \times \nu)(x,y) \geq \dim_A \mu(x) + \dim_H \nu(y).$$

Proof. Let $x \in \mathbb{R}^m$ and $\alpha > 0$ such that $\dim_A \mu(x) > \alpha$. Then

$$0 = d_A^\alpha \mu(x) = \liminf_{\delta\to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{1}{r}\, dr.$$

Since μ and ν are probability Radon measures on $\mathbb{R}^m$ and $\mathbb{R}^n$ with compact support, we have that for $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$

$$\begin{aligned} \mu \times \nu(B((x,y),r)) &\leq \mu \times \nu(B(x,r) \times B(y,r)) \\ &= \mu(B(x,r)) \cdot \nu(B(y,r)). \end{aligned} \tag{3.1}$$

Let $\beta > 0$ such that $\dim_H \nu(y) > \beta$. We want to prove that $\dim_A(\mu \times \nu)(x,y) \geq \alpha + \beta$, that is, $d_A^{\alpha+\beta}(\mu \times \nu)(x,y) = 0$. Using Remark 2.2 and (3.1), we have that

$$\begin{aligned} d_A^{\alpha+\beta}(\mu \times \nu)(x,y) = &\ \liminf_{\delta\to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu \times \nu(B((x,y),r))}{r^{\alpha+\beta}} \frac{1}{r}\, dr \\ < &\ \liminf_{\delta\to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^{\alpha+1}} \frac{\nu(B(y,r))}{r^\beta}\, dr \\ < &\ b \liminf_{\delta\to 0} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^{\alpha+1}}\, dr = 0. \end{aligned}$$

Theorem 3.2. *Let μ and ν be probability Radon measures on $\mathbb{R}^m$ and $\mathbb{R}^n$ with compact support. Then for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$*

$$\dim_A(\mu \times \nu)(x,y) \leq \dim_A \mu(x) \; + \; \dim_P \nu(y).$$

Proof. Let $\alpha > 0$ and $\beta > 0$ be such that $\dim_A \mu(x) < \alpha$ and $\dim_P \nu(y) < \beta$. We want to prove that for all $M > 0$ there exists $N \in \mathbb{N}^+$ such that for all $\delta \in (0, 1/N)$

$$\frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu \times \nu(B((x,y),r))}{r^{\alpha+\beta}} \frac{1}{r} \, dr > M.$$

We know that

$$B(x,r/2) \times B(y,r/2) \subset B((x,y),r). \tag{3.2}$$

Let $M > 0$. Since $d_P^\beta \nu(y) = \infty$ (recall Definition 2.1), there exists $N_1 \in \mathbb{N}^+$, $N_1 \geq 2$ such that for all $r \in (0, 1/N_1)$ we have

$$\frac{\nu(B(y,r))}{r^\beta} > M. \tag{3.3}$$

On the other hand, we know that $d_A^\alpha \mu(x) = \infty$. So there exists $N_2 \in \mathbb{N}^+$ such that for all $\delta \in (0, 1/N_2)$ we get that

$$\frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^{\alpha+1}} \, dr > M. \tag{3.4}$$

It is also clear that there exists $N_3 \in \mathbb{N}^+$ such that for all $\delta \in (0, 1/N_3)$

$$\frac{1}{|\log \delta|} \int_{1/N_1}^1 \frac{\mu(B(x,r))}{r^{\alpha+1}} \, dr < 1. \tag{3.5}$$

Let $N = \max(N_1, N_2, N_3)$. By a change of variables, (3.2) (3.3), (3.4) and (3.5), we have that for all $\delta \in (0, 1/N)$

$$
\begin{aligned}
&\frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu \times \nu(B((x,y),r))}{r^{\alpha+\beta}} \frac{1}{r}\, dr \\
&\geq \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r/2))\nu(B(y,r/2))}{r^{\alpha+\beta}} \frac{1}{r}\, dr \\
&\geq \frac{c}{|\log \delta|} \int_\delta^{1/2} \frac{\mu(B(x,r))\nu(B(y,r))}{r^{\alpha+\beta}} \frac{1}{r}\, dr \\
&\geq \frac{Mc}{|\log \delta|} \int_\delta^{1/N_1} \frac{\mu(B(x,r))}{r^{\alpha+1}}\, dr \\
&= \frac{Mc}{|\log \delta|} \left[\int_\delta^1 \frac{\mu(B(x,r))}{r^{\alpha+1}}\, dr - \int_{1/N_1}^1 \frac{\mu(B(x,r))}{r^{\alpha+1}}\, dr \right] \\
&\geq Mc(M-1) \geq M,
\end{aligned}
$$

if M is big enough and where c is some constant that depends on n, α and β.

Next we give an example that shows that the inequality

$$\dim_H \mu \times \nu \geq \dim_H \mu \; + \; \dim_H \nu,$$

(cf. [12]) cannot be improved with the average dimension.

All the examples in this section consist of natural measures supported in suitable Cantor sets. The process to associate a probability measure to a Cantor set is standard and explained in [4], pp. 13-15. The idea is as follows. For any basic interval I of the Cantor set

$$\mu(I) \cdot (\#\{\text{Intervals of its generation}\}) = ||\mu|| = 1, \tag{3.6}$$

and for given a Borel set $A \subset \mathbb{R}^n$

$$\mu(A) = \inf\Big\{ \sum_i \mu(I_i) : A \cap C_\mu \subset \bigcup_i I_i \text{ and } I_i \text{ is a basic interval of } C_\mu \Big\}, \tag{3.7}$$

where C_μ denotes the Cantor set associated to the measure μ.

Example 3.3. Let $0 < \alpha < \beta < 1$. We construct a probability Radon measure μ on $\mathbb{R}$ such that

$$\dim_H \mu(x) \leq \alpha < \beta \leq \dim_A \mu(x)$$

for μ almost all $x \in \mathbb{R}$. Then, taking the product of this measure μ with $\nu = \mathcal{L} \mid [0,1]$, we will see that

$$\dim_H(\mu \times \nu)(x,y) < \dim_A \mu(x) \ + \ \dim_H \nu$$

for μ almost all $x \in \mathbb{R}$ and all $y \in [0,1]$.

Moreover,

$$\dim_A \nu(x) \ + \ \dim_H \mu(y) < \dim_A(\mu \times \nu)(x,y) \leq \dim_A \mu(x) \ + \ \dim_A \nu(y)$$

for μ almost all $x \in \mathbb{R}$ and ν almost all $y \in \mathbb{R}$.

For this purpose, we first construct a suitable Cantor set, and the measure μ will be determined by repeated subdivision of its initial mass, $||\mu|| = 1$, between the basic intervals of the Cantor set. By (3.6) μ will be uniformly distributed, and moreover, $\mu(I) = \frac{1}{2^i}$ for any interval I of the i-generation. Then the extension of μ to all subsets of $\mathbb{R}$ given in (3.7), defines μ as a measure supported in our Cantor set (see Proposition 1.7 in [4]).

Let $\varepsilon > 0$ be such that $\beta+\varepsilon < 1$. First, take the interval $[0,1]$ and delete a middle interval of it so that the remaining two intervals have both length r_1, satisfying

$$2r_1^\alpha = 1.$$

Let k be any big natural number. We perform k times this "α-dimensional symmetric Cantor operation" on each of these two subintervals. That is, we delete a middle interval of each of the two subintervals so that the remaining four intervals all have length r_2 such that

$$2r_2^\alpha = r_1^\alpha.$$

Thus, after k steps we will have 2^k subintervals of equal length r_k, satisfying

$$2r_k^\alpha = r_{k-1}^\alpha.$$

Note that, for $i = 1, ..., k$, on the i-step we have 2^i subintervals and, by construction, $r_i^\alpha = 2^{-i}$. Hence, (3.6) implies that for all these intervals we have

$$d(I)^\alpha \leq \mu(I) \leq d(I)^{\alpha-\varepsilon}, \tag{3.8}$$

where $d(I)$ denotes the length of the interval I. Note that at this point we have equality here $(d(I)^\alpha = \mu(I))$, but later on we will only get inequalities.

Next we delete a middle interval of each of the 2^k subintervals of the last result, so that the remaining 2^{k+1} intervals have all length r_{k+1} satisfying

$$2r_{k+1}^{\beta+\varepsilon} = r_k^{\beta+\varepsilon}.$$

Now we perform l times this "$(\beta+\varepsilon)$-dimensional symmetric Cantor operation" in the same way as before, having after l steps 2^{m+l} subintervals of equal length, r_{k+l}, satisfying

$$2^l r_{k+l}^{\beta+\varepsilon} = r_k^{\beta+\varepsilon}.$$

The measure of these new intervals I of the generation $k+i$ is given by

$$\mu(I) = 2^{-k-i}, \qquad i = 1, 2, ..., l.$$

Since $r_k = 2^{-k/\alpha}$ and $r_{k+i} = 2^{-i/(\beta+\varepsilon)-k/\alpha}$ for $i = 1, 2, ..., l$, we will have

$$\mu(I) \geq d(I)^{\beta+\varepsilon} \tag{3.9}$$

for all these new intervals. Note that if I is any interval of any of these new generations, we have that

$$\mu(I) = 2^{(\frac{\beta+\varepsilon}{\alpha}-1)k} d(I)^{\beta+\varepsilon} = 2^{(\frac{\beta+\varepsilon}{\alpha}-1)k} d(I)^{\beta+\varepsilon/2} d(I)^{\varepsilon/2}.$$

Therefore if we make the length of I go to zero, then $2^{(\frac{\beta+\varepsilon}{\alpha}-1)k} d(I)^{\varepsilon/2}$ tends to zero. Hence if $d(I)$ is small enough, then $2^{(\frac{\beta+\varepsilon}{\alpha}-1)k} d(I)^{\varepsilon/2} < 1$. That is, if l is big enough, we will have for the intervals of the (last) l-generation that

$$d(I)^{\beta+\varepsilon/2} \geq \mu(I) \geq d(I)^{\beta+\varepsilon}. \tag{3.10}$$

Let m be any big natural number, $m \geq 2$. Now we continue performing this operation m times. So we will have that (3.10) is satisfied for all these new intervals.

Then we perform the "$(\alpha-\varepsilon)$-dimensional symmetric Cantor operation" during a suitable number of times (until we get the condition (3.8)). Now we start again, performing the "α-dimensional symmetric Cantor operation" as many times as we have to, and we repeat the whole construction infinitely many times. So we get that (3.8), (3.9) and (3.10) will remain true for the new generations, respectively.

Note that if $x \in \operatorname{spt}\mu$, $0 < r < 1$ and r_k is the length of the largest interval of our construction which is contained in $B(x,r)$, then $B(x,r)$ is contained in two construction intervals of length r_{k-1}. Using this, we see that for all $\xi > 0$ there exist constants $0 < c_1 < c_2 < \infty$ and $0 < \rho_1 < \rho_2 < \rho_3 < \rho_4 < \xi$, such that $\frac{\rho_2}{\rho_1}$ is as big as we need and for $x \in$ s t

$$\mu(B(x,r)) \geq c_1 r^{\alpha} \qquad \text{for all} \quad \rho_3 \leq r \leq \rho_4 \tag{3.11}$$

and

$$c_2 r^{\beta+\varepsilon/2} \geq \mu(B(x,r)) \geq c_1 r^{\beta+\varepsilon} \qquad \text{for all} \quad \rho_1 \leq r \leq \rho_2, \tag{3.12}$$

where (3.11) comes from (3.8) and (3.12) comes from (3.10).

It is clear from (3.11) and Definition 2.1 that

$$\dim_H \mu(x) \leq \alpha.$$

Let ξ, ρ_1, ρ_2 be as before. By (3.12) we have that

$$\begin{aligned}
&\frac{1}{|\log \rho_1|} \int_{\rho_1}^{1} \frac{\mu(B(x,r))}{r^{\beta}} \frac{1}{r}\, dr \\
&\quad = \frac{1}{|\log \rho_1|} \left(\int_{\rho_1}^{\rho_2} \frac{\mu(B(x,r))}{r^{\beta}} \frac{1}{r}\, dr + \int_{\rho_2}^{1} \frac{\mu(B(x,r))}{r^{\beta}} \frac{1}{r}\, dr \right) \\
&\quad \leq \frac{c_2}{|\log \rho_1|} \left(\int_{\rho_1}^{\rho_2} r^{\varepsilon/2-1}\, dr + \int_{\rho_2}^{1} r^{-\beta-1}\, dr \right) \\
&\quad \leq \frac{c_2}{|\log \rho_1|} \left(\frac{\rho_2^{\varepsilon/2}}{\varepsilon/2} + \frac{\rho_2^{-\beta}}{\beta} \right).
\end{aligned}$$

Thus, taking ρ_1 much smaller than ρ_2, we get $d_A^{\beta}\mu(x) = 0$, and by Definition 2.3 we have that

$$\dim_A \mu(x) \geq \beta$$

for μ almost all $x \in \mathbb{R}$.

To finish the example, just take $\nu = \mathcal{L} \mid [0,1]$; then

$$\dim_H \mu \times \nu \leq \alpha \,+\, 1 < \beta \,+\, 1 \leq \dim_A \mu(x) \,+\, \dim_H \nu$$

for μ almost all $x \in \mathbb{R}$.

The following example shows that the Hausdorff dimension cannot be replaced with the average dimension in Theorem 3.1, that is, in general,

$$\dim_A(\mu \times \nu)(x,y) \geq \dim_A \mu(x) \ + \ \dim_A \nu(y)$$

will not be true.

Moreover, it will show that the known inequality

$$\dim_P(\mu \times \nu) \geq \dim_P \mu + \dim_H \nu$$

cannot be improved with the average dimension.

Example 3.4. We shall construct two probability Radon measures μ and ν with compact support in $\mathbb{R}$ such that

$$\dim_P(\mu \times \nu)(x,y) < \dim_A \mu(x) \ + \ \dim_A \nu(y)$$

for μ almost all $x \in \mathbb{R}$ and ν almost all $y \in \mathbb{R}$.

In particular, it holds

$$\dim_A(\mu \times \nu)(x,y) < \dim_A \mu(x) \ + \ \dim_A \nu(y)$$

for μ almost all $x \in \mathbb{R}$ and ν almost all $y \in \mathbb{R}$.

Let $0 < \alpha < \beta < 1$. Let $\varepsilon > 0$ be such that $\beta+\varepsilon < 1$. We construct the measures μ and ν in the same way as in the last example. That is, we will construct two Cantor sets and define the measures μ and ν by repeated subdivision of their initial mass, $||\mu|| = ||\nu|| = 1$, among the basic intervals of the Cantor sets, respectively. Then we will extend them to all Borel sets, as before, and we obtain the two measures. Let us denote by C_μ and C_ν the Cantor sets for the measures μ and ν, respectively. Let r_i and s_i denote the length of the intervals of the i-generation of C_μ and C_ν, respectively.

We will construct C_μ and C_ν in such a way that the measures μ and ν will have the property that for all $\xi > 0$ there exist $0 < \rho_1 < \rho_2 < \rho_3 < \rho_4 < \xi$ such that $\frac{\rho_4-\rho_3}{\rho_3}$ and $\frac{\rho_2-\rho_1}{\rho_1}$ are as big as we need and

$$\mu(B(x,r)) \leq 2r^{\beta+\varepsilon/2} \tag{3.13}$$

for all $\rho_1 \leq r \leq \rho_2$ and μ almost all $x \in C_\mu$,

$$\nu(B(y,r)) \leq 2r^{\beta+\varepsilon/2} \tag{3.14}$$

for all $\rho_3 \leq r \leq \rho_4$ and ν almost all $x \in C_\nu$.

Moreover, for all $0 < r < 1$ it holds that

$$\mu(B(x,r))\,\nu(B(y,r)) \geq cr^{\alpha+\beta+\varepsilon} \tag{3.15}$$

$\mu \times \nu$ almost all $(x,y) \in C_\mu \times C_\nu$. Here c is some absolute constant.

These estimates will imply that

$$\begin{aligned}
&\dim_A \mu(x) \geq \beta \text{ for } \mu \text{ almost all } x \in C_\mu,\\
&\dim_A \nu(y) \geq \beta \text{ for } \nu \text{ almost all } x \in C_\nu \text{ and}\\
&\dim_P(\mu \times \nu)(x,y) \leq \alpha + \beta + \varepsilon \;\text{ for } \mu \times \nu \text{ almost all } (x,y) \in C_\mu \times C_\nu \;.
\end{aligned}$$

The construction consists of the following steps.

(i) **Construction of C_ν.** We perform k times the *"α-dimensional symmetric Cantor operation"* on the interval $[0,1]$, k being a big natural number.

Construction of C_μ. We perform k times the *"$(\beta+\varepsilon)$-dimensional symmetric Cantor operation"* on the interval $[0,1]$.

(ii) **Construction of C_μ.** We *shrink* the "C_μ-intervals" of the (last) k-generation making their lengths to be equal to the length of the "C_ν-intervals" of the (last) k-generation, but keeping their centres fixed.

(iii) **Construction of C_ν.** We perform M times the *"$(\beta+\varepsilon)$-dimensional symmetric Cantor operation"* on the "C_ν-intervals" of the (last) k-generation. The natural number M has to be big enough.

Construction of C_μ. We perform M times the *"α-dimensional symmetric Cantor operation"* on the intervals obtained in (ii).

(iv) **Construction of C_ν.** We *shrink* the "C_ν-intervals" of the (last) $(k+M)$-generation making their lengths to be equal to the length of the "C_μ-intervals" of the (last) $(k+M)$-generation, but keeping their centres fixed.

(v) **Construction of C_μ.** We perform N times the *"$(\beta+\varepsilon)$-dimensional symmetric Cantor operation"* on the "C_μ-intervals" of the (last) $(k+M)$-generation. The natural number N has to be big enough.

Construction of C_ν. We perform N times the *"α-dimensional symmetric Cantor operation"* on the intervals obtained in (iv).

Then we iterate this process from (ii) to (v) infinitely many times.

Note that this construction is basically the same as the construction in Example 3.3, but here we have added the steps (ii) and (iv). These new steps will allow us to obtain inequality (3.15), that is, to control from above the packing dimension of the product.

We next show how this construction works in detail.

Let k be any big natural number. For C_ν we perform k times the "α-dimensional symmetric Cantor operation" on the interval $[0,1]$. So we will have 2^k subintervals of length s_k, satisfying

$$2s_k^\alpha = s_{k-1}^\alpha,$$
$$2^k s_k^\alpha = 1.$$

By (3.6), the μ measure of all these "C_ν-intervals" is given by

$$\nu(I) = d(I)^\alpha.$$

For C_μ, we perform k times the "$(\beta+\varepsilon)$-dimensional symmetric Cantor operation" on the interval $[0,1]$. We have, after k steps, 2^k subintervals all of length r_k satisfying

$$2r_k^{\beta+\varepsilon} = r_{k-1}^{\beta+\varepsilon},$$
$$2^k r_k^{\beta+\varepsilon} = 1.$$

The μ measure of all these "C_μ-intervals" is given by

$$\mu(I) = d(I)^{\beta+\varepsilon}.$$

Then we shrink the "C_μ-intervals" of the last generation, making their length equal to the length of the "C_ν-intervals" of the last generation, but keeping their centres fixed. That is, we change their length from r_k to s_k. Note that $r_k^{\beta+\varepsilon} = s_k^\alpha$. Let us perform, on these new shorter "C_μ-intervals", the "α-dimensional symmetric Cantor operation" M times , where M will be determined in the construction of C_ν. The μ measure of all these new "C_μ-intervals" is given by

$$\mu(I) = d(I)^\alpha.$$

Now we continue with the construction of C_ν. On the "C_ν-intervals" of the last generation, perform l times the "$(\beta+\varepsilon)$-dimensional symmetric Cantor operation". Our new 2^{k+l} intervals satisfy

$$2^l s_{k+l}^{\beta+\varepsilon} = s_k^{\beta+\varepsilon}$$

Continue this operation still m times and let $M = l+m$. If l is sufficiently big, we see as in Example 3.3 that for all these new "C_ν-intervals"

$$\nu(I) \geq d(I)^{\beta+\varepsilon}, \tag{3.16}$$

and for all the "C_ν-intervals" of the $(l+i)$-generation, where $i = 1, \ldots, m$, we have that

$$d(I)^{\beta+\varepsilon/2} \geq \nu(I) \geq d(I)^{\beta+\varepsilon}. \tag{3.17}$$

Here m can be as big as we wish. Now we again change the size of our intervals by shrinking the "C_ν-intervals" of the last generation, in the same way as before. That is, we make their length equal to the length of the "C_μ-intervals" of the last generation, keeping their centres fixed. Note that

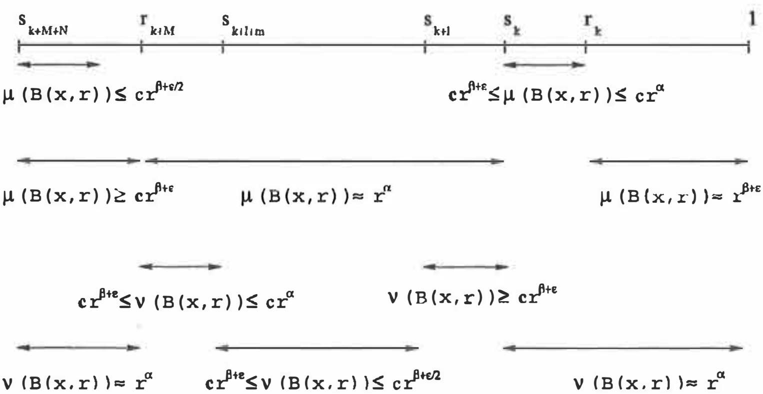
$$\frac{s_k^\alpha}{2^M} = r_{k+M}^\alpha.$$

Hence, if these shorter intervals are denoted by $\widetilde{I}$, it holds that

$$\nu(\widetilde{I}) = d(\widetilde{I})^\alpha.$$

We start again to perform N times the "α-dimensional symmetric Cantor operation" on these new "C_ν-intervals", where N will be determined in the construction of C_μ.

For C_μ, we perform the "$(\beta+\varepsilon)$-dimensional symmetric Cantor operation" N times. Choosing N big enough, we have that the analogues of (3.16) and (3.17) hold for μ.



In this picture we have used the same notation for all constants, but their values might differ from one interval to the other.

The advantage of shrinking the intervals is that now we will have an extra condition that will allow us to bound the product. This means that for any "C_μ-interval" I and "C_ν-interval" I' of the same generation, we have

$$d(I)^{\beta+\varepsilon} \leq \mu(I) \leq d(I)^\alpha, \quad d(I')^{\beta+\varepsilon} \leq \nu(I') \leq d(I')^\alpha$$

and, moreover,

$$\mu(I) \geq d(I)^\alpha \quad \text{or} \quad \nu(I') \geq d(I')^\alpha.$$

Finally, we start the whole process again, and so the scheme of the above picture will be repeated when we approach zero. Our measures will satisfy conditions (3.13), (3.14) and (3.15). In particular, (3.15) follows since at every stage where the constructions are changed, the "C_μ" and "C_ν-intervals" are made to have equal length. So μ and ν are the desired measures.

Next we give another example that shows that in Theorem 3.2 the packing dimension cannot be replaced by the average dimension. Moreover, this example will also show that the well-known inequalities

$$\dim_H(\mu \times \nu) \leq \dim_P \mu \ + \ \dim_H \nu$$

and

$$\dim_P(\mu \times \nu) \leq \dim_P \mu \ + \ \dim_P \nu$$

cannot be improved with the average dimension. That way this example and Remark 3.6 will give us an answer to the following questions:

Questions:
1) Is it possible to get $\dim_H \mu \times \nu \leq \dim_A \mu \ + \ \dim_H \nu$?
2) Is it possible to get $\dim_A \mu \times \nu \leq \dim_A \mu \ + \ \dim_A \nu$?
3) Is it possible to get $\dim_P \mu \times \nu \leq \dim_P \mu \ + \ \dim_A \nu$?
Answers: No, since we have examples where

$$\dim_H(\mu \times \nu)(x,y) > \dim_A \mu(x) \ + \ \dim_A \nu(y) \quad \text{(Example 3.5)}$$

and

$$\dim_P(\mu \times \nu)(x,y) > \dim_P \mu(x) \ + \ \dim_A \nu(y) \quad \text{(Remark 3.6)}$$

for μ almost all $x \in \mathbb{R}$ and for ν almost all $y \in \mathbb{R}$.

Example 3.5. For any $0 < \alpha < \beta < 1$, there are two probability Radon measures, μ and ν, in $\mathbb{R}$ with compact support such that

$$\dim_H(\mu \times \nu)(x,y) \geq \alpha + \beta > \alpha + \alpha \geq \dim_A \mu(x) \ + \ \dim_A \nu(y)$$

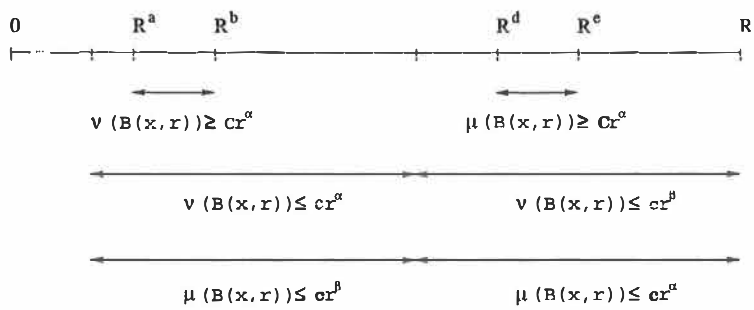
for μ almost all $x \in \mathbb{R}$ and ν almost all $y \in \mathbb{R}$.
In particular, $\dim_A(\mu \times \nu)(x,y) > \dim_A \mu(x) \ + \ \dim_A \nu(y)$.

In the same way as in the last example, to obtain these two measures we will build two Cantor sets, C_μ and C_ν, that will determine the measures μ and ν, respectively, by subdividing their initial mass, $||\mu|| = 1 = ||\nu||$, among their corresponding basic intervals. Their extensions to all Borel sets will give us the required measures.

In order to obtain the lower bound for the Hausdorff dimension, we would need that for all $r \in [0,1]$ and for $\mu \times \nu$ almost all points (x,y) the measure $\mu \times \nu(B((x,y),r))$ remains bounded from above by $r^{\alpha+\beta}$. In addition, to get the desired average dimensions, we will need that there exist sequences of "big enough" intervals of r's which approach the origin and are not too far from each other such that

$$
\begin{aligned}
\mu(B(x,r)) &\gtrsim r^{\alpha} && \text{for } \mu \text{ almost all } x \quad \text{and} \\
\nu(B(x,r)) &\gtrsim r^{\alpha} && \text{for } \nu \text{ almost all } x,
\end{aligned}
$$

for any r on these intervals. A careful analysis of the definition of the average dimension (2.3) shows that it is enough to have bounds for the measure of the balls of radius r with $r \in [\delta^a, \delta^b]$, a and b being independent of δ. Thus, we will build two Cantor sets following the patterns according to the picture below.



We can find constants a, b, c, d and e independent of r such that this structure can be repeated as near zero as we wish.

Hence, we will take our measures μ and ν so that they behave like "α-dimensional" for some radius and like "β-dimensional" for others. As in the last example, we now face the problem of changing the dimensional behaviour of our measures from α to β and conversely. Since $0 < \alpha < \beta < 1$, an adequate contraction of the size of the intervals is enough to change the measure supported on them from "β-dimensional" to "α-dimensional", as we have done in the previous example. The reverse problem is more subtle, since we cannot just expand our intervals without avoiding some undesired overlapping in our Cantor sets. In order to make this change, we now introduce a different method. The idea is to divide every interval I of a given generation into subintervals I_i with their corresponding length l small enough so that their mass is comparable to l^{β}.

Let $0 < \alpha < \beta < 1$. We generate a sequence of positive real numbers $R_{1,1} > R_{2,1} > \ldots > R_{7,1} := R_{1,2} > R_{2,2} > \ldots$ such that

$$\begin{aligned}
&0 < R_{1,1} < 1,\\
&R_{2,1} := R_{1,1}^{\frac{\beta}{\alpha}}, \quad R_{3,1} := R_{2,1}^{2}, \quad bR_{3,1}^{\frac{1-\alpha}{1-\beta}} \le R_{4,1} \le R_{3,1}^{\frac{1-\alpha}{1-\beta}},\\
&b'R_{4,1}^{\frac{\beta}{\alpha}}, \le R_{5,1} := R_{4,1}^{\frac{\beta}{\alpha}}, \quad R_{6,1} := R_{5,1}^{2}, \quad bR_{6,1}^{\frac{1-\alpha}{1-\beta}} \le R_{7,1} \le R_{6,1}^{\frac{1-\alpha}{1-\beta}},\\
&R_{1,2} := R_{7,1}, \ldots
\end{aligned}$$

In general, we will have

$$\begin{aligned}
&R_{1,j} := R_{7,j-1},\\
&R_{2,j} := R_{1,j}^{\frac{\beta}{\alpha}}, \quad R_{3,j} := R_{2,j}^{2}, \quad bR_{3,j}^{\frac{1-\alpha}{1-\beta}} \le R_{4,j} \le R_{3,j}^{\frac{1-\alpha}{1-\beta}},\\
&b'R_{4,j}^{\frac{\beta}{\alpha}}, \le R_{5,j} := R_{4,j}^{\frac{\beta}{\alpha}}, \quad R_{6,j} := R_{5,j}^{2}, \quad bR_{6,j}^{\frac{1-\alpha}{1-\beta}} \le R_{7,j} \le R_{6,j}^{\frac{1-\alpha}{1-\beta}}, \ldots,
\end{aligned}$$

where b and b' are some positive constants independent of j.

Our Cantor sets, C_μ and C_ν, will be defined in such a way that the behaviour of the measures μ and ν on the intervals $[R_{i,j}, R_{i+1,j}]$, $i = 1, \ldots, 7$ and $j \in \mathbb{N}^+$, will be as follows: For every $j \in \mathbb{N}^+$ and for μ almost all $x \in [0,1]$, there exist constants c and c' independent of r such that

$$cr^\beta \le \mu(B(x,r)) \le c'r^\beta \quad \text{for } R_{1,1} \le r \le 1, \tag{3.18}$$

$$r^\beta \le \mu(B(x,r)) \le cr^\alpha \quad \text{for } R_{2,j} \le r \le R_{1,j}, \tag{3.19}$$

$$cr^\alpha \le \mu(B(x,r)) \le c'r^\alpha \quad \text{for } R_{3,j} \le r \le R_{2,j}, \tag{3.20}$$

$$cr^\beta \le \mu(B(x,r)) \le cr^\alpha \quad \text{for } R_{4,j} \le r \le R_{3,j}, \tag{3.21}$$

and

$$cr^\beta \le \mu(B(x,r)) \le c'r^\beta \quad \text{for } R_{7,j} \le r \le R_{4,j}. \tag{3.22}$$

We will construct ν in such a way that for all $r > 0$ with the property $\mu(B(x,r)) \lesssim r^\alpha$, it holds that $\nu(B(y,r)) \lesssim r^\beta$, and for those r's for which $\mu(B(x,r)) \lesssim r^\beta$, we will have that $\nu(B(x,r)) \lesssim r^\alpha$. That is, for ν almost all $x \in [0,1]$ there will exist constants c and c' independent of r such that

$$cr^\beta \leq \nu(B(x,r)) \leq c'r^\beta \quad \text{for } R_{4,j} \leq r \leq R_{7,j-1}, \tag{3.23}$$

where $R_{7,0} := 1$,

$$r^\beta \leq \nu(B(x,r)) \leq cr^\alpha \quad \text{for } R_{5,j} \leq r \leq R_{4,j}, \tag{3.24}$$

$$cr^\alpha \leq \nu(B(x,r)) \leq c'r^\alpha \quad \text{for } R_{6,j} \leq r \leq R_{5,j}, \tag{3.25}$$

and

$$\nu(B(x,r)) \leq cr^\alpha \quad \text{for } R_{7,j} \leq r \leq R_{6,j}. \tag{3.26}$$

Note that the value of the constants c and c' may vary from one line to another but, for simplicity in the notation, we use the same symbol.

In this way, we have that for all $r > 0$ the product measure $\mu \times \nu$ satisfies $\mu \times \nu(B(x,y),r) \leq cr^{\alpha+\beta}$ for $\mu \times \nu$ almost all $(x,y) \in [0,1] \times [0,1]$, with c as before. Hence, recalling the definition of the Hausdorff dimension (2.1), we see that $\dim_H(\mu \times \nu)(x,y) \geq \alpha + \beta$ for $\mu \times \nu$ almost all points in the plane.

On the other hand, by (3.20) and the way the sequence of $R_{i,j}$ has been constructed, it holds for μ almost all $x \in [0,1]$ that if $R_{7,j} \leq \delta \leq R_{3,j}$, there exists a fixed constant η independent of j such that

$$\begin{aligned} \frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{1}{r}\, dr &\geq \frac{1}{|\log R_{7,j}|} \int_{R_{3,j}}^{R_{2,j}} \frac{\mu(B(x,r))}{r^\alpha} \frac{1}{r}\, dr \\ &\geq \frac{c}{|\log R_{7,j}|} \int_{R_{3,j}}^{R_{2,j}} \frac{1}{r}\, dr \geq \frac{c|\log R_{2,j}|}{\eta |\log R_{2,j}|} = c\eta. \end{aligned}$$

In the case when $R_{3,j+1} \leq \delta \leq R_{1,j+1} = R_{7,j}$, we have by the same argument that there exists η' such that for μ almost all $x \in [0,1]$

$$\frac{1}{|\log \delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^\alpha} \frac{1}{r}\, dr \geq c\eta'.$$

Consequently, at μ almost all $x \in [0,1]$, we have $\dim_A \mu(x) \leq \alpha$. By repeating the argument for ν, using (3.26) instead of (3.20), we get that also the average dimension of ν is less than or equal to α almost everywhere.

Construction of C_μ.

Let us start with the unit interval and perform on it k times the "β- dimensional symmetric Cantor operation" (β-S.C.O), like in the last example. In this first step we may assume $k = 1$, but later on the number of steps will be relevant since it has to be related with the behaviour of the measure μ. The lengths of the intervals obtained will satisfy the following relationships

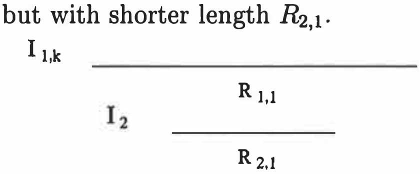
$$\begin{aligned} & r_{1,0} = 1, \\ & 2r_{1,1}^{\beta} = r_{1,0}^{\beta}, \\ & \dots \\ & 2r_{1,k}^{\beta} = r_{1,k-1}^{\beta}. \end{aligned}$$

In a natural way, the measure μ of these basic intervals is given by

$$\begin{aligned} & \mu(I_{1,0}) = r_{1,0} = 1, \\ & \mu(I_{1,1}) = r_{1,1}^{\beta} = 1/2, \\ & \dots \\ & \mu(I_{1,k}) = r_{1,k}^{\beta} = 1/2^k. \end{aligned}$$

Defining $R_{1,1} := r_{1,k}$, it is clear that (3.18) holds for $j = 1$.

Define $R_{2,1} := R_{1,1}^{\frac{\beta}{\alpha}}$. We now change the length of the 2^k intervals obtained before from $R_{1,1}$ to $R_{2,1}$. That is, we take 2^k new intervals I_2 centered at the same point as their "elders" but with shorter length $R_{2,1}$.



The μ measure of these new intervals will be given by

$$\mu(I_2) = \mu(I_{1,k}) = R_{1,1}^{\beta} = R_{2,1}^{\alpha}.$$

Thus (3.19) is satisfied with $j = 1$.

Now, we perform on each of these intervals I_2 the α-S.C.O and stop when the lengths $r_{2,l}$ of the intervals $I_{2,l}$ of the last generation satisfy

$$r_{2,l} \le R_{2,1}^2 := R_{3,1} \le r_{2,l-1}.$$

So we obtain (3.20) for $j=1$.

In order to change the behaviour of μ from "α-dimensional" to "β-dimensional," we divide each of the obtained intervals into $\frac{r_{2,l}}{R_{4,1}}$ and we take the $\left[\frac{r_{2,l}}{R_{4,1}}\right]$ first subintervals I_4 each of length $R_{4,1} := r_{2,l}^{\frac{1-\alpha}{1-\beta}}$. Here $\left[\frac{r_{2,l}}{R_{4,1}}\right]$ is the integer satisfying $\frac{r_{2,l}}{R_{4,1}} - 1 < \left[\frac{r_{2,l}}{R_{4,1}}\right] \leq \frac{r_{2,l}}{R_{4,1}}$.

For these intervals we have

$$\mu(I_4) = \frac{\mu(I_{2,l})}{\left[r_{2,l}/R_{4,1}\right]} \geq \frac{\mu(I_{2,l})}{r_{2,l}/R_{4,1}} = \frac{R_{4,1}}{r_{2,l}} r_{2,l}^{\alpha} = R_{4,1}^{\beta}.$$

Moreover, since we may assume without losing generality that $r_{2,l}^{\frac{\beta-\alpha}{1-\beta}} \leq 1 - \frac{1}{C}$ for some fixed absolute constant $C > 1$, we get

$$\mu(I_4) = \frac{\mu(I_{2,l})}{\left[r_{2,l}/R_{4,1}\right]} \leq \frac{\mu(I_{2,l})}{\frac{r_{2,l}}{R_{4,1}} - 1} \leq C\, \frac{\mu(I_{2,l})}{r_{2,l}/R_{4,1}} = C\, \frac{R_{4,1}}{r_{2,l}} r_{2,l}^{\alpha} = C\, R_{4,1}^{\beta}.$$

Let $R_{4,1} \leq r \leq R_{3,1}$ and $x \in \operatorname{spt}\mu$. Then $B(x,r)$ meets at most $2\left[\frac{r}{R_{4,1}}\right]$ intervals I_4, and hence

$$\mu(B(x,r)) \leq 2\left[\frac{r}{R_{4,1}}\right] \mu(I_4) \leq 2C\, \frac{r}{R_{4,1}}\, r_{2,l}^{\alpha-1} R_{4,1} = 2C\, r_{2,l}^{\alpha-1}\, r \leq 2Cr^{\alpha}.$$

On the other hand, $B(x,r)$ contains approximately $\left[\frac{r}{R_{4,1}}\right]$ intervals I_4, and so there exists a constant c' such that

$$\mu(B(x,r)) \geq c'\left[\frac{r}{R_{4,1}}\right] \mu(I_4) \geq \mathrm{c}\frac{r}{R_{4,1}} R_{4,1}^{\beta} \geq \mathrm{c} r^{\beta}.$$

Thus (3.21) also holds for $j=1$.

We continue with the construction of μ by performing the β-S.C.O on each of the intervals I_4. The number of times we must do it will be established later on, since it will depend on the behaviour of ν.

Construction of C_ν.

We start with the unit interval, perform on it n times the β-S.C.O and stop when we get

$$s_{1,n} \leq R_{4,1} \leq s_{1,n-1}.$$

For $1 \leq j \leq n$, let $s_{1,j}$ denote the length of the 2^j intervals obtained after performing j times the β-S.C.O on [0,1]. Hence, for $j=1$ we do have (3.23).

We now proceed as we have done with μ. In other words, we interchange the roles of μ and ν.

Let $R_{5,1} := s_{1,n}^{\frac{\beta}{\alpha}}$. By changing the length of the 2^n subintervals obtained from $s_{1,n}$ to $R_{5,1}$, we change the behaviour of ν from β- to α-dimensional, that is

$$\nu(I_5) = s_{1,n}^{\beta} = R_{5,1}^{\alpha},$$

where I_5 denotes the new intervals. As with μ before and in the case $j = 1$, we see that (3.24) is satisfied.

Next we perform the α-S.C.O on each of the I_5's and stop when we get $s_{5,t} \leq R_{5,1}^{2} = R_{6,1} \leq s_{5,t-1}$. Here $s_{5,t}$ denotes the length of the intervals $I_{5,t}$ obtained after performing t times the α-S.C.O on I_5. One can check that also (3.25) holds for $j = 1$.

In the same way as we have done with μ, we now change the behaviour of ν from α-dimensional to β-dimensional. Define $R_{7,1} := s_{5,t}^{\frac{1-\alpha}{1-\beta}}$ and repeat here the argument made for μ and $R_{4,j}$. Then (3.26) holds for $j = 1$.

As has been done with μ, we continue with the construction of ν by performing the β-S.C.O on the intervals obtained in the last step. The required number of times will be determined by μ.

Construction of C_μ.

Now we are in a good situation to determine the number of steps of the β-S.C.O that has been started in the last stage of the process for the construction of C_μ.

The last step of our construction was to start to perform the β-S.C.O, and now we go on with it until we have that the intervals of the last generation have lengths less than or equal to $R_{7,1}$. Then (3.22) holds for $j = 1$.

We continue in the same way as we have done with ν and repeat the whole process infinitely many times. It is clear that (3.19)–(3.26) will be satisfied for every $j \in \mathbb{N}^+$.

Remark 3.6. It is clear that in the above example the measures μ and ν satisfy

$$\dim_P(\mu \times \nu)(x, y) \geq \beta + \alpha > \alpha + \alpha \geq \dim_A \mu(x) + \dim_A \nu(y)$$

for μ almost all $x \in [0, 1]$ and ν almost all $y \in [0, 1]$.

In order to obtain

$$\dim_P(\mu \times \nu)(x, y) \geq \beta + \beta > \alpha + \beta \geq \dim_P \mu(x) + \dim_A \nu(y), \tag{3.27}$$

a slight modification of the above example will be enough. First note that $\dim_P \mu(x) \leq \beta$ follows from (3.19)–(3.22) and Definition 2.1. To get (3.27), we just perform once more the β-S.C.O after $R_{j,7}$ in the construction of C_μ. That is, in the last step of our construction we performed the β-S.C.O until we got that the lengths of the intervals of the last generation were less than or equal to $R_{7,1}$, and we defined $R_{1,2} := R_{7,1}$. Now we do this operation once more and define $R_{1,2}$ to be the length of the intervals obtained. Hence, in general, it holds that $R_{j+1,1} < R_{j,7}$ and that

$$cr^\beta \leq \mu(B(x,r)) \leq c' r^\beta \quad \text{for } R_{1,j+1} \leq r \leq R_{7,j}.$$

Consequently, for $R_{1,j+1} \leq r \leq R_{7,j}$, both $\mu(B(x,r))$ and $\nu(B(x,r))$ are less than or equal to cr^β at almost all points $x \in [0,1]$. This is by Definition 2.1 enough to guarantee that $\dim_P(\mu\times\nu)(x,y) \geq \beta+\beta$ at almost all points $(x,y) \in [0,1]\times[0,1]$.

4. Average dimension of intersection measures, similarities and isometries

4.1. Notation and preliminaries. Let μ and ν be Radon measures on $\mathbb{R}^n$. Let $O(n)$ be the orthogonal group of $\mathbb{R}^n$, and let θ_n be the unique invariant measure on $O(n)$ with $\theta_n O(n) = 1$.

We consider intersections $\mu \cap f_\# \nu$, where f is a similarity map on $\mathbb{R}^n$. By a similarity we mean a map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that there is $0 < r < \infty$ with $|f(x) - f(y)| = r|x-y|$ for all $x, y \in \mathbb{R}^n$. Then $r = \operatorname{Lip} f$, where $\operatorname{Lip} f$ is the Lipschitz constant defined in 4.2, and f has a unique decomposition as

$$f = \tau_z \circ g \circ \delta_r, \qquad z \in \mathbb{R}^n,\ g \in O(n),\ r \in \mathbb{R}^+,$$

where $\tau_z : \mathbb{R}^n \to \mathbb{R}^n$ is the translation $\tau_z(x) = x + z$, and $\delta_r : \mathbb{R}^n \to \mathbb{R}^n$ is the homothety $\delta_r(x) = rx$.

In order to construct such measures, we shall first slice the product measure $\mu \times \nu$ by the n-planes

$$W_z = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x - y = z\},$$

parallel to the diagonal $W = \{(x,y) : x = y\}$. We obtain for $\mathcal{L}^n$ almost all $z \in \mathbb{R}^n$ Radon measures σ_z on $\mathbb{R}^n \times \mathbb{R}^n$. Then we project the slices obtained to $\mathbb{R}^n$ by the projection $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\pi(x,y) = x$, as in [21] or [23]. Define

$$\mu \cap \tau_{z\#}\nu = \pi_\# \sigma_z \tag{4.1}$$

provided that the sliced measure σ_z exists. This is the case for $\mathcal{L}^n$ almost all $z \in \mathbb{R}^n$, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure. Clearly,

$$\operatorname{spt}(\mu \cap \tau_{z\#}\nu) \subset \operatorname{spt}\mu \cap \tau_z(\operatorname{spt}\nu). \tag{4.2}$$

If ϕ is a non-negative lower semi-continuous function on $\mathbb{R}^n$, then, as in the sliced case, we have

$$\int \phi \, d(\mu \cap \tau_{z\#}\nu) \leq \lim_{\delta \to 0} \alpha(n)^{-1}\delta^{-n} \int_{W_z(\delta)} \phi(x) \, d(\mu \times \nu)(x,y). \tag{4.3}$$

Here $W_z(\delta) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |S(x,y) - z| \leq \delta\}$, where $\delta > 0$ and $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by $S(x,y) = x - y$. Note that if $S_\#(\mu \times \nu) \ll \mathcal{L}^n$, then $P_{W^\perp}(\mu \times \nu) \ll \mathscr{H}^n|W^\perp$ and the disintegration formula (2.2) implies that

$$\int_{W^\perp} \int \varphi \, d(\mu \cap \tau_{z\#}\nu) \, d\mathcal{L}^n z = \int \varphi \, d(\mu \times \nu) \tag{4.4}$$

provided that φ is a non-negative Borel function with $\int \varphi \, d(\mu \times \nu) < \infty$.

For $z \in \mathbb{R}^n$, $g \in O(n)$, $r \in \mathbb{R}^+$ and $\delta > 0$, we set

$$\begin{aligned} &S_{g,r} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \qquad S_{g,r}(x,y) = x - rgy, \\ &W_{z,g,r}(\delta) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |S_{g,r}(x,y) - z| \leq \delta\}. \end{aligned}$$

For any $g \in O(n)$, $r \in \mathbb{R}^+$, we can apply the above method to the measures μ and $(g \circ \delta_r)_\#\nu$ to conclude that the intersection measure

$$\begin{aligned} &\mu \cap f_\#\nu = \mu \cap \tau_{z\#}((g \circ \delta_r)_\#\nu) = \pi_\#[(\mu \times (g \circ \delta_r)_\#\nu)_{W,z}] \\ &\text{with} \quad f = \tau_z \circ g \circ \delta_r \end{aligned} \tag{4.5}$$

exists for $\mathcal{L}^n$ almost all $z \in \mathbb{R}^n$.

From (4.2), (4.3) and (4.4) we infer that the following three statements hold whenever ϕ is a non-negative lower semi-continuous function on $\mathbb{R}^n$:

$$\operatorname{spt}\mu \cap (\tau_z \circ g \circ \delta_r)_\#\nu \subset \operatorname{spt}\mu \cap \ (\tau_z \circ g \circ \delta_r) \ (\operatorname{spt}\nu), \tag{4.6}$$

$$\int \phi \, d(\mu \cap (\tau_z \circ g \circ \delta_r)_\#\nu) \leq \lim_{\delta \to 0} \alpha(n)^{-1}\delta^{-n} \int_{W_{z,g,r}(\delta)} \phi(x) \, d(\mu \times \nu)(x,y) \tag{4.7}$$

and if $S_{g,r\,\#}(\mu \times \nu) \ll \mathcal{L}^n$, then

$$\int_{W^\perp} \int \varphi \, d(\mu \times (g \circ \delta_r)_\# \nu)_{W,a} \, d\mathcal{H}^n a = \int \varphi \, d(\mu \times (g \circ \delta_r)_\# \nu) \tag{4.8}$$

provided that φ is a non-negative Borel function with $\int \varphi \, d(\mu \times (g \circ \delta_r)_\# \nu) < \infty$.

Remark 4.1. If $I_s(\mu) < \infty$, $I_t(\mu) < \infty$ and $s + t \geq n$, then $S_{g,r\#}(\mu \times \nu) \ll \mathcal{L}^n$ for $\theta_n \times \mathcal{L}^1$ almost all $(g, r) \in O(n) \times (0, \infty)$. See Theorem 6.6 in [21].

Definition 4.2. (1) A map $f : A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, is a *Lipschitz map* if there is a constant $L < \infty$ such that

$$|f(x) - f(y)| \leq L \, |x - y| \qquad \text{for } x, y \in A.$$

The smallest such constant L is called the *Lipschitz constant* of f and is denoted by $\mathrm{Lip}(f)$.

(2) A map $f : A \to B$, $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$, is a *bi-Lipschitz map* if f is Lipschitz and it has Lipschitz inverse $f^{-1} : B \to A$. That is,

$$l|x - y| \leq |f(x) - f(y)| \leq L|x - y| \qquad (x, y \in A)$$

where $0 < l \leq L < \infty$.

Remark 4.3. It is obvious by the definition of the average dimension that, given a Radon measure μ in $\mathbb{R}^n$, we have that

$$\dim_A f_\# \mu(f(x)) \leq \dim_A \mu(x) \quad x \in \mathbb{R}^n$$

for any *Lipschitz* map f. We have equality if f is *bi-Lipschitz.*

4.2. Similarities and intersections. From the way that these new measures are constructed one might expect that similar methods to those used in studying the dimensions of sections of measures can be used here. In [15], the quantity $J_h^{r_1,r_2}(a, b)$ was analyzed by giving an upper bound for it. This bound together with the method used in the proof of Theorem 2.6 will give us a nice lower bound for the average dimension of the measures (4.5).

Lemma 4.4. *Let μ and ν be probability Radon measures on $\mathbb{R}^n$ such that $I_{n-t}(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < t < n$. Then for $\mu \times \nu \times \theta_n \times \mathcal{L}^1$ almost all $(a, b, g, r) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n) \times (0, \infty)$, the measure $\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_\# \nu$ is defined and $0 < (\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_\# \nu)(\mathbb{R}^n) < \infty$.*

Moreover, for any non-negative lower semi-continuous function ϕ on $\mathbb{R}^n$, the function

$$(r, g) \to \int \phi \, d(\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_\# \nu)$$

is a Borel function.

Proof. See Lemma 3.4 in [15].

Definition 4.5. Let $0 < r_1 < r_2 < \infty$, $h > 0$ and $a, b \in \mathbb{R}^n$. We define

$$J_h^{r_1,r_2}(a,b) = \int_{r_1}^{r_2} \int (\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_{\#}\nu)(B(a,h))\, d\theta_n g\, d\mathcal{L}^1 r \tag{4.9}$$

provided that the right-hand side is defined.

Lemma 4.6. *Let μ and ν be probability Radon measures on $\mathbb{R}^n$ such that $I_{n-t}(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < t < n$. Let $0 < r_1 < r_2 < \infty$. Then for all $h > 0$ and $\mu \times \nu$ almost all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$*

$$J_h^{r_1,r_2}(a,b) \leq c \int_0^{4h} r^{\alpha-n-1}\mu(B(a,r))\, dr \int_0^{6h/r_1} \widetilde{r}^{-\alpha-1}\nu(B(b,\widetilde{r}))\, d\widetilde{r}, \tag{4.10}$$

where α is any number $0 < \alpha < n$ and c is a constant depending only on n, r_1 and r_2.

Proof. See 3.12 Corollary in [15].

For the rest of this section, s, t and q will be real numbers such that

$$0 < s < n, \qquad 0 < t < n, \qquad 0 \leq q = s + t - n. \tag{4.11}$$

Theorem 4.7. *Let μ and ν be probability Radon measures on $\mathbb{R}^n$ with compact support such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$. Then*

$$\dim_A(\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_{\#}\nu)(a) \geq \dim_A \mu(a) + \dim_H \nu(b) - n$$

for $\mu \times \nu \times \theta_n \times \mathcal{L}^1$ almost all $(a,b,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n) \times (0,\infty)$.

Proof. Note first that by Theorem 6.6 in [21], under our assumptions we have that $S_{g,r\#}(\mu \times \nu) \ll \mathcal{L}^n$ for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times (0,\infty)$.

We want to prove that

$$d_A^\lambda(\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_{\#}\nu)(a) = 0$$

for $\mu \times \nu \times \theta_n \times \mathcal{L}^1$ almost all $(a,b,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n) \times (0,\infty)$ and $\lambda < \dim_A \mu(a) + \dim_H \nu(b) - n$. Let $0 < r_1 < r_2 < \infty$, $\dim_H \nu(b) > \beta > \kappa > t$ and $\lambda < \dim_A \mu(a) + \kappa - n$. Since $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$, we have $\dim_H \mu \geq s$ and

$\dim_H \nu \geq t$, whence $\dim_H \mu(a) \geq n - t \geq n - \kappa$ for μ almost all $a \in \mathbb{R}^n$. Note that in (4.11) we have assumed $s \geq n - t$.

Using Fatou's lemma, Fubini's theorem, Lemma 4.6 and Remark 2.2, we have that for $\mu \times \nu$ almost all $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$

$$
\begin{aligned}
I &= \int_{r_1}^{r_2} \int d_A^{\lambda}(\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_{\#}\nu)(a)\, d\theta_n g\, d\mathcal{L}^1 r \\
&\leq \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_{\delta}^{1} p^{-\lambda-1} \int_{r_1}^{r_2} \int (\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_{\#}\nu)(B(a,p))\, d\theta_n g\, dr\, dp \\
&\leq \liminf_{\delta \to 0} \frac{c}{|\log \delta|} \int_{\delta}^{1} p^{-\lambda-1} \int_0^{4p} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dr \int_0^{6p/r_1} \widetilde{r}^{-\kappa-1} \nu(B(b,\widetilde{r}))\, d\widetilde{r}\, dp \\
&\leq \liminf_{\delta \to 0} \frac{c}{|\log \delta|} \int_{\delta}^{1} p^{-\lambda-1} \int_0^{4p} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dr \int_0^{6p/r_1} k\, \widetilde{r}^{\beta-\kappa-1}\, d\widetilde{r}\, dp \\
&\leq c \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_{\delta}^{1} p^{-\lambda-1}\, p^{\beta-\kappa} \int_0^{4p} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dr\, dp \\
&\leq c \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_{\delta}^{1} p^{-\lambda-1} \int_0^{4p} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dr\, dp,
\end{aligned}
$$

where the constant k depends on $b \in \mathbb{R}^n$ and $c = c(n, r_1, r_2, \kappa, \beta, k)$. Note that in the last inequality we have used that $\beta - \kappa > 0$ and $p < 1$.

Applying Fubini's theorem, in the same way as in the slicing case, we obtain

$$
\begin{aligned}
I \leq c\ &\liminf_{\delta \to 0} \frac{1}{|\log \delta|} \left[\int_0^{4\delta} \int_{\delta}^{1} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, p^{-\lambda-1}\, dp\, dr \right. \\
&\left. + \int_{4\delta}^{4} \int_{r/4}^{1} p^{-\lambda-1} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dp\, dr \right] \\
\leq c\ &\liminf_{\delta \to 0} \frac{1}{|\log \delta|} \left[\delta^{-\lambda} \int_0^{4\delta} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dr + \int_{4\delta}^{1} \frac{\mu(B(a,r))}{r^{n+1-\kappa+\lambda}}\, dr \right. \\
&\left. + \int_1^4 \frac{\mu(B(a,r))}{r^{n+1-\kappa+\lambda}}\, dr \right] \\
\leq c\ &\liminf_{\delta \to 0} \frac{1}{|\log \delta|} \left[\delta^{-\lambda} \int_0^{4\delta} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dr + \int_{4\delta}^{1} \frac{\mu(B(a,r))}{r^{n+1-\kappa+\lambda}}\, dr \right].
\end{aligned}
$$

Note that in the last inequality we have used that the third integral is bounded and therefore the limit is zero.

Applying Lemma 2.7 with $d = n - \kappa$ and $\alpha + d = \lambda + n - \kappa$, we get for μ almost all $a \in \mathbb{R}^n$

$$\liminf_{\delta \to 0} \frac{1}{|\log \delta|} \left[\delta^{-\lambda} \int_0^{4\delta} \frac{\mu(B(a,r))}{r^{n+1-\kappa}}\, dr + \int_{4\delta}^1 \frac{\mu(B(a,r))}{r^{n+1-\kappa+\lambda}}\, dr \right] = 0, \tag{4.12}$$

which gives the desired result.

Remark 4.8. For $s \geq 0$, consider the function

$$\varphi_s(\mu, x) = \int |x-y|^{-s}\, d\mu(y),$$

which is called the *s-potential of the measure μ at the point x.* In [25] it is observed that the *local Hausdorff dimension of μ at x* is equal to the supremum of values s for which the s-potential of μ at x is finite. In other words,

$$\dim_H \mu(x) := \sup\{s : \varphi_s(\mu, x) < \infty\} = \inf\{s : \varphi_s(\mu, x) = \infty\}. \tag{4.13}$$

A review of the proof of Lemma 4.6 (Corollary 3.12 in [15]) reveals that it is still true if we assume that the $(n-t)$-potential of μ at a and the t-potential of ν at b are finite, provided that the measure $\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_\# \nu$ is defined and $0 < (\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_\# \nu)(\mathbb{R}^n) < \infty$. Hence, the above theorem will remain true if we assume $\dim_H \mu(a) > s$, $\dim_H \nu(b) > t$ and the existence of the measure $\mu \cap (\tau_a \circ g \circ \delta_r \circ \tau_{-b})_\# \nu$ instead of $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$.

The proof of the following theorem is based on the proof of Lemma 5.1 in [17].

Theorem 4.9. *Let μ and ν be probability Radon measures on $\mathbb{R}^n$ with compact support such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$. Then*

$$\dim_A(\mu \cap (\tau_z \circ g \circ \delta_r)_\# \nu)(x) \geq \dim_A \mu(x) + \dim_H \nu((\tau_z \circ g \circ \delta_r)^{-1}(x)) - n$$

for $\mathcal{L}^n \times \theta_n \times \mathcal{L}^1$ almost all $(z, g, r) \in \mathbb{R}^n \times O(n) \times (0, \infty)$ and $\mu \cap (\tau_z \circ g \circ \delta_r)_\# \nu$ almost all $x \in \mathbb{R}^n$.

Proof. Let $\lambda(x,y) = \dim_A \mu(x) + \dim_H \nu(y) - n$ and define for $g \in O(n)$, $r \in (0, \infty)$

$$\begin{aligned} A_{g,r} &= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ \dim_A(\mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\# \nu)(x) < \lambda(x,y)\} \\ B_{g,r} &= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ \dim_A(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\# \nu)(x) < \\ &\qquad \dim_A \mu(x) + \dim_H \nu(g^{-1}(y)/r)\}. \end{aligned}$$

Note that for $z \in \mathbb{R}^n$

$$(4.14)\qquad \begin{aligned}(\tau_{x-g(\delta_r(y))} \circ g \circ \delta_r)(z) &= g(rz) + x - g(ry)\\ &= g(r(z-y)) + x = (\tau_x \circ g \circ \delta_r \circ \tau_{-y})(z).\end{aligned}$$

We know by Theorem 4.7 that for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times (0,\infty)$, we have that $\mu \times \nu(A_{g,r}) = 0$. Thus, using (4.14), we get

$$\begin{aligned}&(\mu \times (g \circ \delta_r)_{\#}\nu)(B_{g,r}) = \mu \times \nu \ \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ (x, g(\delta_r(y))) \in B_{g,r}\}\\ &= \mu \times \nu \ \{(x,y) \in \mathbb{R}^n \ : \ \dim_A(\mu \cap (\tau_{x-g(\delta_r(y))} \circ g \circ \delta_r)_{\#}\nu)(x) < \lambda(x,y)\}\\ &= \mu \times \nu(A_{g,r}) = 0\end{aligned}$$

for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times (0,\infty)$. By Theorem 6.6 in [21], we have that $S_{g,r\#}(\mu \times \nu) \ll \mathcal{L}^n$ for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times (0,\infty)$. Thus, taking in (4.8) $\varphi = \chi_{B_{g,r}}$, we have that for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times (0,\infty)$

$$(4.15)\qquad (\mu \times (g \circ \delta_r)_{\#}\nu)_{W,(z,-z)/2}(B_{g,r}) = 0$$

for $\mathcal{L}^n$ almost all $z \in \mathbb{R}^n$.

Let $g \in O(n)$, $r \in (0,\infty)$ such that (4.15) holds. From (2.1) it follows that for $\mathcal{L}^n$ almost all $z \in \mathbb{R}^n$

$$\begin{aligned}0 =&(\mu \times (g \circ \delta_r)_{\#}\nu)_{W,(z,-z)/2} \Big(\{(x, x-z) \in \mathbb{R}^n \times \mathbb{R}^n\\ &\dim_A(\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu)(x) < \dim_A \mu(x) +\\ &\dim_H \nu\big((g^{-1}(x-z))/r\big) - n\}\Big).\end{aligned}$$

Thus,

$$\begin{aligned}&\pi_{\#}[(\mu \times (g \circ \delta_r)_{\#}\nu)_{W,(z,-z)/2}] \Big(\{x \in \mathbb{R}^n \ : \dim_A(\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu)(x)\\ &\quad < \dim_A \mu(x) + \dim_H \nu\big((\tau_z \circ g \circ \delta_r)^{-1}(x)\big) - n\}\Big) = 0.\end{aligned}$$

Then,

$$\dim_A(\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu)(x) \geq \dim_A \mu(x) + \dim_H \nu\big((\tau_z \circ g \circ \delta_r)^{-1}(x)\big) - n$$

for $\mathcal{L}^n$ almost all $z \in \mathbb{R}^n$ and $\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu$ almost all $x \in \mathbb{R}^n$.

Next we will give an upper bound for the average dimension of the measures (4.5). Since the measures (4.5) are defined as slices of the projection of the product of measures, we will need the results on the behaviour of the average dimension of the product of two measures.

Theorem 4.10. *Let μ and ν be probability Radon measures on $\mathbb{R}^n$ with compact support such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$. Then,*

$$\dim_A(\mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\# \nu)(x) \leq \dim_A \mu(x) + \dim_P \nu(y) - n$$

for $\mu \times \nu \times \theta_n \times \mathcal{L}^1$ almost all $(x, y, g, r) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n) \times (0, \infty)$.

Proof. By Remark 4.1 we have that $S_{g,r\#}(\mu \times \nu) \ll \mathcal{L}^n$. Then, for $\theta_n \times \mathcal{L}^1$ almost all $(g, r) \in O(n) \times \mathbb{R}^+$,

$$P_{W^\perp \#}(\mu \times (g \circ \delta_r)_\# \nu) \ll \mathscr{H}^n|_{W^\perp}. \tag{4.16}$$

A review of Theorem 2.8 reveals that the upper bound for slicing measures holds for any V such that $P_{V^\perp \#}\mu \ll \mathscr{H}^m|_{V^\perp}$, and thus we can apply it in our particular case. Therefore, if $g \in O(n)$ and $r \in (0, \infty)$ satisfy (4.16),

$$\begin{aligned} 0 = \mu \times (g \circ \delta_r)_\# \nu \Big(\{ &(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \\ &\dim_A(\mu \times (g \circ \delta_r)_\# \nu)_{W,(x,y)}(x, y) > \dim_A(\mu \times (g \circ \delta_r)_\# \nu)(x, y) - n \} \Big). \end{aligned} \tag{4.17}$$

Hence, by Remark 4.3, (4.17) and noticing that

$$W_{(x-y,y-x)/2} = W_{(x,y)},$$

we get that for $\theta_n \times \mathcal{L}^1$ almost all $(g, r) \in O(n) \times \mathbb{R}^+$ and $\mu \times (g \circ \delta_r)_\# \nu$ almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} \dim_A(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\# \nu)(x) &= \dim_A \pi_\#(\mu \times (g \circ \delta_r)_\# \nu)_{W,(x-y,y-x)/2)}(x) \\ &\leq \dim_A(\mu \times (g \circ \delta_r)_\# \nu)_{W,(x,y)}(x, y) \\ &\leq \dim_A(\mu \times (g \circ \delta_r)_\# \nu)(x, y) - n. \end{aligned}$$

Consequently, using (4.14), it holds that for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times \mathbb{R}^+$

$$\begin{aligned} 0 &= \mu \times (g \circ \delta_r)_{\#}\nu \Big(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_A(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_{\#}\nu)(x) > \\ &\qquad \dim_A(\mu \times (g \circ \delta_r)_{\#}\nu)(x,y) - n\}\Big) \\ &= \mu \times \nu \Big(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_A(\mu \cap (\tau_{x-g(\delta_r(y))} \circ g \circ \delta_r)_{\#}\nu)(x) > \\ &\qquad \dim_A(\mu \times (g \circ \delta_r)_{\#}\nu)(x, g(\delta_r(y))) - n\}\Big) \\ &= \mu \times \nu \Big(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_A(\mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_{\#}\nu)(x) > \\ &\qquad \dim_A(\mu \times (g \circ \delta_r)_{\#}\nu)(x, g(\delta_r(y))) - n\}\Big). \end{aligned}$$

Applying Theorem 3.2 to the above result, one gets that

$$\dim_A(\mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_{\#}\nu)(x) \leq \dim_A \mu(x) + \dim_P \nu(y) - n$$

for $\mu \times \nu \times \theta_n \times \mathcal{L}^1$ almost all $(x,y,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n) \times (0,\infty)$. Note that in the the last equality we have used that for all $y \in \mathbb{R}^n$, $\dim_P \nu(y) = \dim_P ((g \circ \delta_r)_{\#}\nu)\ (x, g(\delta_r(y)))$. This follows directly from the definition of the local packing dimension. □

The same method used in the proof of Theorem 4.9 can be used to prove the following Corollary:

Corollary 4.11. *Under the assumptions of Theorem 4.10, we have that*

$$\dim_A(\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu)(x) \leq \dim_A \mu(x) + \dim_P \nu((\tau_z \circ g \circ \delta_r)^{-1}(x)) - n$$

for $\mathcal{L}^n \times \theta_n \times \mathcal{L}^1$ almost all $(z,g,r) \in \mathbb{R}^n \times O(n) \times (0,\infty)$ and $\mu \cap (\tau_z \circ g \circ \delta_r)_{\#}\nu$ almost all $x \in \mathbb{R}^n$.

4.3. Isometries and intersections. In this section we give some results equivalent to those in Section 4.2 for the average dimension of intersection measures but considering isometries instead of similarities. Since all the theorems and proofs in this case are analogous to the previous ones, we are going to skip most of the proofs giving just an idea of the changes needed to obtain the new results. Here we have to add the condition that the t-energy of ν is finite for all $\frac{1}{2}(n+1) < t < n$. In this section we will keep the notation of the previous one.

Let μ and ν be probability Radon measures in $\mathbb{R}^n$ with compact support and $f : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Then f has a unique representation in the form

$$f = \tau_z \circ g \qquad \text{with } z \in \mathbb{R}^n,\ g \in O(n).$$

Applying Section 4.1 to the measures μ and $g_{\#}\nu$, we get the measure $\mu\cap(\tau_z\circ g)_{\#}\nu$, that is

$$\mu \cap (\tau_z \circ g)_{\#}\nu := \pi_{\#}[(\mu \times g_{\#}\nu)_{W,(z,-z)/2}]. \tag{4.18}$$

Adding the assumption $(n+1)/2 < t$, we have an analogy to Lemma 4.6 (see Lemma 3.5 in [16]) that allows us to prove the following Lemma.

Lemma 4.12. *Let μ and ν be probability Radon measures in $\mathbb{R}^n$ with compact support such that*

$$I_s(\mu) < \infty \quad \textit{and} \quad I_t(\nu) < \infty,$$

where $(n+1)/2 < t$, $0 < s < t$ and $t + s > n$. Then

$$\dim_A(\mu \cap (\tau_a \circ g \circ \tau_{-b})_{\#}\nu)(a) \geq \dim_A \mu(a) + \dim_H \nu(b) - n$$

$\mu \times \nu \times \theta_n$ almost all $(a, b, g) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n)$.

We skip the proof since it is just a slight modification of the proof of Theorem 4.7.

Now we may use the same kind of arguments as in the proof of Theorem 4.9 and get the following theorem:

Theorem 4.13. *Let μ and ν be probability Radon measures in $\mathbb{R}^n$ with compact support such that*

$$I_s(\mu) < \infty \quad \textit{and} \quad I_t(\nu) < \infty.$$

Provided that $(n+1)/2 < t$, $0 < s < t$ and $t + s > n$, then

$$\dim_A(\mu \cap (\tau_z \circ g)_{\#}\nu)(x) \geq \dim_A \mu(x) + \dim_H \nu\ ((\tau_z \circ g)^{-1}(x)) - n$$

for $\mathcal{L}^n \times \theta_n$ almost all $(z, g) \in \mathbb{R}^n \times O(n)$ and $\mu \cap (\tau_z \circ g)_{\#}\nu$ almost all $x \in \mathbb{R}^n$.

Proof. Let $\lambda(x, y) = \dim_A \mu(x) + \dim_H \nu(y) - n$ and define

$$\begin{aligned} A_g &= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_A(\mu \cap (\tau_x \circ g \circ \tau_{-y})_{\#}\nu(x) < \lambda(x, y)\} \\ &= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_A(\mu \cap (\tau_{x-g(y)} \circ g)_{\#}\nu(x) < \lambda(x, y)\}, \\ B_g &= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_A(\mu \cap (\tau_{x-y} \circ g)_{\#}\nu(x) < \lambda(x, g^{-1}(y))\}. \end{aligned}$$

By repeating the arguments of the proof of Theorem 4.9 we get the desired result. □

An inspection of the proof of Theorem 4.10, reveals that, adding the assumption $t > \frac{n+1}{2}$, it remains true when $r = 1$. Note that in both the above theorem and the following corollary, the condition $t > \frac{n+1}{2}$ will be needed in order to have that $S_{g\,\#}(\mu \times \nu) \ll \mathcal{L}^n$ for θ_n almost all $g \in O(n)$. Here $S_g(x,y) = x - gy$, for $(x,y) \subset \mathbb{R}^n \times \mathbb{R}^n$. Also, in the same way as in the previous section, we will have an analogy of Corollary 4.11, which can be proved with the same arguments we used in the proof of Theorem 4.13. Therefore, we have the following Corollary:

Corollary 4.14. *Let μ and ν be probability Radon measures in $\mathbb{R}^n$ with compact support such that*

$$I_s(\mu) < \infty \quad \text{and} \quad I_t(\nu) < \infty,$$

where $t > \frac{n+1}{2}$, $0 < s < t$ and $t + s > n$. Then,

$$\dim_A(\mu \cap (\tau_x \circ g \circ \tau_{-y})_{\#}\nu)(x) \le \dim_A \mu(x) + \dim_P \nu(y) - n$$

for $\mu \times \nu \times \theta_n$ almost all $(x,y,g) \in \mathbb{R}^n \times \mathbb{R}^n \times O(n)$.

Moreover,

$$\dim_A(\mu \cap (\tau_z \circ g)_{\#}\nu)(x) \le \dim_A \mu(x) + \dim_P \nu\big((\tau_z \circ g)^{-1}(x)\big) - n$$

for $\mathcal{L}^n \times \theta_n$ almost all $(z,g) \in \mathbb{R}^n \times O(n)$ and $\mu \cap (\tau_z \circ g)_{\#}\nu$ almost all $x \in \mathbb{R}^n$.

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