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To be presented, with the permission of the Faculty of Mathematics and Natural Sciences of the University of Jyväskylä, for public criticism in Auditorium S 212 of the University, on December 16th, 1995, at 12 o'clock noon.

HELSINKI 1995
SUOMALAINEN TIEDEAKATEMIA

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Academia Scientiarum Fennica
ISSN 0355-0087
ISBN 951-41-0789-6

Received 1 November 1995

1991 Mathematics Subject Classification:
Primary 30F40, 30F60, 32G15.

YLIOPISTOPAINO
HELSINKI 1995

**Verkkoversio julkaistu tekijän ja
Suomalaisen Tiedeakatemian luvalla.**

URN:ISBN:978-952-86-0116-6
ISBN 978-952-86-0116-6 (PDF)

Jyväskylän yliopisto, 2024

Acknowledgments

I am grateful to my teacher Professor Tapani Kuusalo and to Docent Mika Seppälä for their support and advice during the preparation of this thesis. Also, I am grateful to Professors Linda Keen and Pekka Tukia for carefully reading the manuscript and for their comments and criticism on the work presented here, and to Dr. Pablo Arés for several discussions on the group constructions that form a major part of this thesis.

Part of the research was done at the State University of New York at Stony Brook during the academic year 1994-1995. I would like to thank the Mathematics Department and IMS for their hospitality, and Yrjö, Vilho ja Kalle Väisälän rahasto and the Academy of Finland for financial support.

Thanks are also due to Mrs. Tuula Blåfield for carefully reading the manuscript and for her comments on linguistics and style, and to Docent Ari Lehtonen for his help with the technicalities of preparing the manuscript for publication.

Jyväskylä, October 7th 1995

Jouni Parkkonen

Contents

1. Introduction

In this paper we apply the techniques used in the setting of terminal regular b-groups by Kra [12,13] and Arés [3] to treat a related class of Koebe groups, namely those that are constructed from hyperbolic triangle groups by AFP and HNN constructions on maximal cyclic elliptic or parabolic subgroups. The main result is a construction algorithm that gives a global complex analytic coordinate on the quasiconformal deformation spaces of these groups and the interpretation of this coordinate as a collection of "natural" gluing parameters associated with a maximal partition of the quotient Riemann surface (with elliptic special points). We also obtain an inside estimate of Teichmüller space in these coordinates by finding an embedded punctured polydisk. Finally, we use the construction algorithm and some elementary geometric observations to find points on the boundary of the punctured polydisks that are also boundary points of the deformation space. The boundary points represent noded Riemann surfaces of the same topological type as the points inside the deformation space.

In the following we briefly define the basic concepts used throughout the paper. For more details about Kleinian groups we refer the reader to Maskit's monograph [17]. For hyperbolic geometry our main reference is Beardon [5].

A subgroup $G \subset \mathrm{PSL}(2,\mathbb{C})$ of Möbius transformations is a *Kleinian group* if there is a point $z \in \widehat{\mathbb{C}}$ that has a neighborhood U with the property that only finitely many translates of U by elements of G intersect U. The maximal set of points in $z \in \widehat{\mathbb{C}}$ with this property is the *set of discontinuity* or the *ordinary set* of G denoted by $\Omega(G)$.

If $\Omega(G)$ has an invariant component Δ (a component of $\Omega(G)$ such that $g(\Delta) = \Delta$ for all elements of G), G is called a *function group.* A function group whose non-invariant components are stabilized by Fuchsian groups is a *Koebe group.* We consider a special class of finitely generated Koebe groups such that the non-invariant components represent spheres with three special points. By a *special point* we mean a *puncture* on the surface (the projection of a horodisk in $\Omega(G)$ invariant under a parabolic cyclic subgroup of G) or an *elliptic special point* (the projection of an elliptic fixed point in the set of discontinuity). G acts without fixed points in the set

$$\Omega^\circ(G) = \Omega(G) \setminus \{x \mid \exists A \in G \text{ elliptic s.t. } A(x) = x\}.$$

For the fixed point-free part of the invariant set we use

$$\Delta^\circ(G) = \Delta(G) \cap \Omega^\circ(G).$$

Let G be a finitely generated Kleinian group. The (*Teichmüller*) *deformation space* of G is

$$\mathbf{T}(G) = \{w\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \text{ quasiconformal} \mid w \circ g \circ w^{-1} \in \mathrm{PSL}(2,\mathbb{C}) \text{ for all } g \in G\}/\sim,$$

where $w_1 \sim w_2$ if there is a transformation $A \in \mathrm{PSL}(2,\mathbb{C})$ so that

$$w_1 \circ g \circ w_1^{-1} = A \circ w_2 \circ g \circ w_2^{-1} \circ A^{-1} \quad \forall g \in G.$$

If G is a Fuchsian group of the first kind representing Riemann surfaces of finite analytic type (p,n), then $\mathbf{T}(G)$ can be naturally identified with $\mathbf{T}(p,n) \times \mathbf{T}(p,n)$, where $\mathbf{T}(p,n)$ is the classical Teichmüller space of Riemann surfaces of type (p,n). Teichmüller space has a natural complex analytic structure induced from the space of Beltrami differentials (see Ahlfors [2]). In the general case of a finitely generated Kleinian group Ahlfors' finiteness theorem (first proved by Ahlfors in [1], see Bers [7] for a different proof) says that the quotient

$$\Omega(G)/G = \bigcup R_i$$

is a finite union of Riemann surfaces of finite analytic type. $\prod \mathbf{T}(R_i)$ is the universal covering space of $\mathbf{T}(G)$, so the deformation space $\mathbf{T}(G)$ inherits a complex analytic manifold structure from the covering

$$\Phi\colon \prod \mathbf{T}(R_i) \to \mathbf{T}(G),$$

see Bers [6], Maskit [15], and Kra [11] for details.

Our aim is to find holomorphic embeddings of deformation spaces of Kleinian groups into $\mathbb{C}^n$, where n is the complex dimension of $\mathbf{T}(G)$, such that given a point in this embedding we can construct a Kleinian group that it represents in $\mathbf{T}(G)$. Coordinates like this are often called *non-variational.* Also, we would like to be able to read off some geometric properties of the corresponding Riemann surfaces and hyperbolic 3-manifolds from the coordinates. Maskit introduced coordinates like this in [16] for the deformation space of terminal b-groups. If G is a terminal b-group, $\mathbf{T}(G)$ is isomorphic with $\mathbf{T}(\Delta(G)/G)$, and $\mathbf{T}(G)$ is called the

Maskit embedding of Teichmüller space. Kra ([12], [13]) has described a geometric coordinate system of the Maskit embedding and an algorithm showing how to reconstruct the b-group from its image under the coordinate map. This work was generalized by Arés [3] for terminal b-groups with torsion. These coordinates are considered *geometric*, because they can be interpreted as parameters for the $zw = t$ plumbing construction with special "geometrically natural" local coordinates. In this paper we extend this method to a larger class of Kleinian groups. Groups of this larger class were already considered in [3] in the case that the groups represent a Riemann surface of genus 0 with 4 punctures. Keen and Series [9] have introduced a different geometric, real analytic coordinate system for the Maskit embedding of $\mathbf{T}(1,1)$.

In special cases (Sections 3.2 and 8.1) we can find a second geometric interpretation of the coordinates considered in this paper. By adding a new generator we can form a Kleinian group G' with no invariant component such that all the components of $\Omega(G)$ are equivalent under the group and the component stabilizers are conjugates of the original group G. The new group G has a number of elliptic axes that have non-cyclic stabilizers in G'. These axes project into the 3-orbifold $\mathbb{H}^3/G'$ as circles, and the coordinates of G give the lengths of these circles and the amount of "twisting" along the circles in $\mathbb{H}^3/G'$.

In the first part of the paper (Sections 2-5) we consider a very special class of Koebe groups: The quotient of the invariant component is a compact Riemann surface with elliptic special points (no punctures). The quotient surface is automatically equipped with a *pants decomposition*, that is, a partition of $\Omega^\circ(G)/G$ by a collection of simple closed curves corresponding to elliptic elements in the group, into parts that are topologically spheres with three holes. These groups can be constructed from a collection of triangle groups by a number of AFP and HNN constructions involving elliptic cyclic subgroups. The constructions are treated in detail in [17]. One of the reasons for this restriction is that we can write down the generators of the Koebe groups in this class using algebraic expressions of

(1) hyperbolic sine and cosine functions of the hyperbolic distances between elliptic special points on the quotient 2-orbifolds of the triangle groups used in the construction. The distances are all finite, because the metric on the spheres with three special points has only "mild singularities", and
(2) the gluing parameters that along with combinatorial data (encoded in *weighted graphs* introduced by Arés in [3]) describe how the group is put together from the original triangle groups.

Also, this is essentially the only case remaining after the work of Kra [13] and Arés [3]. Using the results obtained in the first part and those of [13] and [3], we define global coordinates in deformation spaces of Koebe groups constructed by AFP and HNN constructions from a collection of *hyperbolic* triangle groups. We give an inside estimate of the deformation spaces in these coordinates by finding a non-empty set of the form

$$\prod S_i \subset \tau(\mathbf{T}(G)),$$

where S_i is a suitably chosen punctured disk or a half plane depending on the type of the gluing (Theorem 3). The more general case involving also triangle groups acting discontinuously in $\mathbb{C}$ and $\widehat{\mathbb{C}}$ will not be treated in this paper.

The plan of the paper is as follows: In Section 2 we outline the geometric properties of hyperbolic triangle groups and the corresponding spheres with three special points, some of which can be punctures. Following [13] and [3], we also describe a system of "canonical" local coordinates at special points and punctures.

Section 3 deals with the AFP of two triangle groups and with the HNN extension of a triangle group by a loxodromic Möbius transformation. These basic constructions are interpreted as $zw = t$ plumbing constructions in the local coordinates defined in Section 2. Using methods of [14] we introduce a global complex analytic coordinate in the deformation spaces of Koebe groups of types $(0, 4)$ and $(1, 1)$. We show that the coordinate is *geometric* in the sense that it is closely related with the plumbing parameter of the $zw = t$ plumbing that the group realizes.

In Section 4 we prove an isomorphism theorem (Theorem 1) for one-dimensional deformation spaces. This is the analog of Theorem 1 in [13] in the setting of this paper.

The second part of the paper starting with Section 5 deals with parameters of higher dimensional deformation spaces. In Section 5 we outline a construction algorithm for the class of Koebe groups without parabolic transformations. This algorithm and a version of Maskit's embedding theorem (Theorem 2 in Section 6) is used in Section 6 to prove an inside estimate by a product of punctured disks for the deformation spaces of Koebe groups constructed from hyperbolic triangle groups. The theorem gives an open set embedded in the image of $\mathbf{T}(G)$ of the form

$$\prod S_i \subset \tau\left(\mathbf{T}(G)\right),$$

where S_i is either a punctured disk or a half plane depending on whether it corresponds to an elliptic or a parabolic gluing.

Section 7 illustrates the use of the construction algorithm. We construct all finitely generated Koebe groups of type $(2, 0)$ with a maximal partition and no parabolics and only hyperbolic triangle groups as structure subgroups. We also give an interpretation of the parameters of Example 1 in terms of the geometry of an associated hyperbolic 3-orbifold.

In Section 8 the examples of the previous section are used to show that the estimate of Theorem 3 is sharp in the following sense: If G is a Koebe group of type (p, n) with $p \geq 1$, then there is a parameter

$$\tau_0 \in \partial\Big(\prod S_i\Big) \cap \partial\Big(\tau(\mathbf{T}(G))\Big).$$

We also note that our methods show that Kra's estimate for the deformation spaces of terminal b-groups ([13] Theorem 8.6) is sharp in the same sense.

The methods used in this paper are for the most part very similar to those in Kra's work [12], [13] on horocyclic coordinates of Teichmüller space, parts of the Earle-Marden manuscript [8] and Arés' thesis [3]. The coordinates are defined by the stratification method described in detail by Kra and Maskit in [14]. A treatment of deformation spaces of Koebe groups from a slightly different point of view can be found in [4].

Notation

The *signature* of a Riemann surface X is given by $(p, n; \nu_1, \ldots, \nu_n)$, where p is the genus of X, n is the number of special points and the numbers $\nu_i \in \mathbb{N} \cup \{\infty\}$ give the order of the special point, the value $\nu_j = \infty$ corresponding to a puncture. The pair (p, n) will be called the *type* of X, and for a sphere S with three punctures we call (ν_1, ν_2, ν_3) the *signature* of S.

$\widehat{\mathbb{C}}$	The extended complex plane $\mathbb{C} \cup \{\infty\}$.
$\mathbb{D}$	The unit disk $\{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$.
$\mathbb{D}^*$	The exterior of the unit disk $\{z \in \mathbb{C} \mid \lvert z\rvert > 1\}$.
$\mathbb{H}$	The upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.
$\mathbb{H}^*$	The lower half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.
$\mathbb{H}^3$	The upper half space $\{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$.
$\operatorname{cr}(a, b, c, d)$	$\frac{(a-c)}{(a-b)}\frac{(d-b)}{(d-c)}$ the cross ratio of four points.
$\mathrm{PSL}(2, \mathbb{C})$	Complex 2×2 matrices with determinant 1 and $\pm A$ identified.
$\lvert A\rvert$	The order of a Möbius transformation A.
$\operatorname{fix}_L A$, $\operatorname{fix}_R A$	The left/right fixed point of the elliptic transformation A.
$\operatorname{Ax} A$	The axis of a Möbius transformation A in $\mathbb{H}^3$.
$\operatorname{Isom}(B)$	The isometric circle of B.
$\Omega(G)$	The set of discontinuity of the Kleinian group G.
$\Delta(G)$	An invariant component of the Koebe group G.
$\mathbf{T}(G)$	The Teichmüller space of the Kleinian group G.
$\mathbf{T}(p, n)$	The Teichmüller space of Riemann surfaces of type (p, n).
$\langle G_1, \ldots, G_m \rangle$	The group generated by $G_1, \ldots, G_m$.
$R(\nu_1; \nu_2, \nu_3)$	The maximal radius of a round orbifold disk, see (2.2).
$r(\nu_1; \nu_2, \nu_3)$	The radii of pairwise tangent orbifold disks, see (2.6).
$h(\nu_1, \nu_2)$	The horocyclic radius of a punctured disk, see Lemma 2.5.

2. The geometry of hyperbolic triangle groups

In this section we set the notation for normalized Fuchsian triangle groups and list some elementary properties of hyperbolic triangles. Let Γ be a triangle group of signature (ν_1, ν_2, ν_3) such that

$$\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} < 1.$$

If A and B are primitive elliptic transformations that generate Γ and satisfy

$$|A| = \nu_1, \quad |B| = \nu_2, \text{ and } |AB| = \nu_3,$$

we call them *canonical generators* of Γ.

We need to distinguish between the two fixed points of an elliptic transformation: Let x be a fixed point of an elliptic Möbius transformation M of order at least 3. We call x *the left fixed point* of M,

$$x = \mathrm{fix}_L\, M,$$

if for any point $y \in \mathbb{C}$ not fixed by M the cross ratio

$$\mathrm{cr}(x, y, My, M^2y)$$

has positive imaginary part. The other fixed point is the *right fixed point* of M, denoted by $\mathrm{fix}_R\, M$.

The above definition does not work for order 2 elliptics. To make a consistent choice of left and right fixed points we use the following elementary observation:

Lemma 2.1. *If A and B are elliptics of orders greater than 2 that generate a hyperbolic triangle group Γ with AB an elliptic of order greater than 2, the fixed points $\mathrm{fix}_L\, A$, $\mathrm{fix}_L\, B$ and $\mathrm{fix}_R\, AB$ are in the same component of $\Omega(\Gamma)$.*

In a hyperbolic triangle group, at most one of A, B and AB can have order 2. We define the left and right fixed points of an elliptic of order 2 (as a generator of a fixed hyperbolic triangle group) so that the fixed points of A, B and AB satisfy the above relation.

A hyperbolic triangle group $\Gamma = \langle A, B \rangle$ is said to be *normalized* if

(1) Γ acts in the unit disk $\mathbb{D}$,
(2) A and B form a pair of canonical generators, and
(3) $\mathrm{fix}_L\, A = \infty$, $\mathrm{fix}_R\, A = 0$ and $\mathrm{fix}_L\, B, \mathrm{fix}_R\, B > 0$.

Each hyperbolic triangle group Γ has a fundamental polygon for its action in $\mathbb{D}$ (or in $\mathbb{D}^*$) that consists of two copies of a triangle with angles π/ν_1, π/ν_2 and π/ν_3. In the case of normalized triangle groups with A elliptic, the boundary of the polygon can be taken to consist of parts of two lines from the origin separated by an angle of $2\pi/\nu_1$ and by the *isometric circles* of B and B^{-1}. (If

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$, the isometric circle of T, $\operatorname{Isom} T$ is the Euclidean circle with center $-d/c$ and radius $|c|^{-1}$.) It is easy to see that the quotient 2-orbifold

$$\Omega(\Gamma)/\Gamma = (\mathbb{D}/\Gamma) \cup (\mathbb{D}^*/\Gamma) = S_1 \cup S_2$$

is the disjoint union of two spheres with three special points $P_i^1, P_i^2, P_i^3 \in S_i$, of orders ν_1, ν_2 and ν_3, respectively. The orbifolds can be thought of as two copies of the triangle glued together along the corresponding edges, and the vertices of the triangles correspond to special points on the orbifold with less than a full angle. The hyperbolic cosine rule ([5], Section 7.12) gives

$$\cosh d_i = \frac{\cos(\pi/\nu_j)\cos(\pi/\nu_k) + \cos(\pi/\nu_i)}{\sin(\pi/\nu_j)\sin(\pi/\nu_k)} \tag{2.1}$$

for the distance d_i of two special points P_1^j and P_1^k on S_1 for $\{i,j,k\} = \{1,2,3\}$.

With this notation we can write the canonical generators in a simple form (See Arés [3] for more on canonical generators of triangle groups.)

$$A = \begin{pmatrix} e^{-i\pi/\nu_1} & 0 \\ 0 & e^{i\pi/\nu_1} \end{pmatrix},$$

and

$$B = \begin{pmatrix} i\sin(\pi/\nu_2)\cosh d_3 - \cos(\pi/\nu_2) & -i\sin(\pi/\nu_2)\sinh d_3 \\ i\sin(\pi/\nu_2)\sinh d_3 & -i\sin(\pi/\nu_2)\cosh d_3 - \cos(\pi/\nu_2) \end{pmatrix}.$$

All calculations with matrices in this paper are done in $\mathrm{PSL}(2,\mathbb{C})$: matrices representing the same Möbius transformation will be identified.

The fixed points of the elliptic $B(\nu_1,\nu_2,\nu_3)$ are

$$\operatorname{fix}_L B = \coth\frac{d_3}{2} = \sqrt{\frac{\cos(\pi/\nu_1 - \pi/\nu_2) + \cos(\pi/\nu_3)}{\cos(\pi/\nu_1 + \pi/\nu_2) + \cos(\pi/\nu_3)}},$$

and

$$\operatorname{fix}_R B = \tanh\frac{d_3}{2} = \sqrt{\frac{\cos(\pi/\nu_1 + \pi/\nu_2) + \cos(\pi/\nu_3)}{\cos(\pi/\nu_1 - \pi/\nu_2) + \cos(\pi/\nu_3)}}$$

on the positive real line. We have:

Lemma 2.2. *Let Γ be a hyperbolic triangle group with finite branching indices. Then*

$$\operatorname{cr}(\operatorname{fix}_R A, \operatorname{fix}_R B, \operatorname{fix}_L B, \operatorname{fix}_L A) = \coth^2\frac{d_3}{2}.$$

The following simple lemmas on hyperbolic triangles or, equivalently, on spheres with three special points will prove useful (notation as in Figure 1):

Lemma 2.3. *Let S be a sphere with three special points with hyperbolic signature (ν_1, ν_2, ν_3). The maximal radius of a round orbifold disk centered at the special point P_i is*

$$R = R(\nu_i; \nu_j, \nu_k) = \operatorname{arsinh}\left(\sinh(d_k)\sin\frac{\pi}{\nu_j}\right). \tag{2.2}$$

Proof. Let γ be the shortest geodesic arc joining the vertex with angle P_i to the opposite side. The hyperbolic sine rule gives the hyperbolic length of this segment: (see Figure 1)

$$\sinh R = \sin(\theta_2)\sinh d_3. \quad \square$$

We will also need the following Euclidean estimate to estimate gluing parameters in Section 3:

Lemma 2.4. *Let K be a hyperbolic triangle with angles θ_1, θ_2 and θ_3. If the vertex with angle θ_1 is at 0 in the unit disk $\mathbb{D}$, the Euclidean distance from the origin to the side joining the two other vertices is*

$$\tanh\frac{R}{2} = \frac{\sqrt{\sin^2\theta_2\sinh^2 d_3 + 1} - 1}{\sin\theta_2\sinh d_3}. \tag{2.3}$$

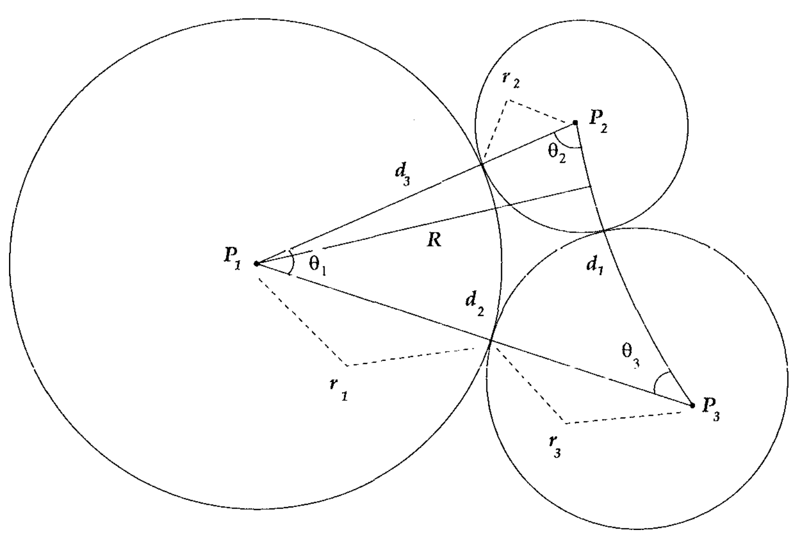


Figure 1. A hyperbolic triangle with finite angles. Notation as in Lemma 2.3 with $\theta_i = \pi/\nu_i$.

There is a unique simple geodesic arc γ on $\mathbb{D}/\Gamma$ joining any two of the three special points (elliptic special points or punctures), P_i and P_j . We can use this property to define a nice coordinate in a neighborhood of a special point as follows: Let $\mathbb{D}_{\nu_i} = \mathbb{D}/\langle z \mapsto e^{2i\pi/\nu_i}\rangle$ be the disk with one special point of order ν_i. The metric on $\mathbb{D}_{\nu_i}$ is

$$ds = \frac{2|d\zeta|}{\nu_i|\zeta|^{1-1/\nu_i}(1-|\zeta|^{2/\nu_i})}. \tag{2.4}$$

If P_i is an elliptic special point of order ν_i we say that an injective holomorphic map $f\colon U \to \mathbb{D}_{\nu_i}$ from an open neighborhood U of P_i is a *natural coordinate at P_i relative to P_j*, if $f(P_i) = 0$ and f maps the geodetic segment $\gamma \cap U$ isometrically into the positive real line in $\mathbb{D}_{\nu_i}$.

If P_i is a puncture, we say that an injective holomorphic map $f\colon U \to \mathbb{D}\setminus\{0\}$ from an open punctured neighborhood U of P_i is a *natural* or *horocyclic coordinate at P_i relative to P_j*, if $f(P_i) = 0$ and f maps the geodetic segment $\gamma \cap U$ isometrically into the positive real line in the metric

$$ds = -\frac{|d\zeta|}{|\zeta|\log|\zeta|} \tag{2.5}$$

of the punctured disk.

The natural and horocyclic coordinates at elliptic special points and punctures are uniquely defined as germs of analytic functions, see Kra [13] and Arés [3]. By choosing the normalization of Γ properly it is easy to find the coordinate maps explicitly. The following lemma will be used in Sections 5–7 to give an estimate of Teichmüller space in the coordinates defined in Section 6. It is a trivial fact (valid in any geometry) that if T is a hyperbolic triangle with vertices v_1, v_2, v_3 and angles π/ν_i at vertex v_i, then the hyperbolic disks centered at the vertices v_i with radius r_i are all pairwise tangent if and only if

$$r_i = r(\nu_i; \nu_j, \nu_k) = \frac{d_j + d_k - d_i}{2}, \tag{2.6}$$

where d_i is the length of the side opposite the vertex v_i given by (2.1).

If some of the vertices of T have angle 0, the triangle has sides of infinite length, and (2.6) does not make sense. In these cases we can use horocyclic coordinates at the vertices at infinity (or equivalently at the punctures of the sphere with three special points obtained by gluing two copies of T together) to measure the "length" of half-infinite geodesic arcs ending at the punctures. The analog of (2.6) for triangles with some of its vertices at infinity is:

Lemma 2.6. *If a triangle T has vertices at infinity, the following triples of hyperbolic or horocyclic radii of disks and horodisks give an arrangement of pairwise tangent disks and horodisks:*

(1) *If T has angles π/ν_1, π/ν_2 and 0 at the vertices v_1, v_2 and v_3, then the radii are*

$$
\begin{aligned}
r_1 &= r(\nu_1;\nu_2,\infty) = \frac{d}{2} + \log\sqrt{\frac{\sin(\pi/\nu_2)}{\sin(\pi/\nu_1)}},\\
r_2 &= r(\nu_2;\nu_1,\infty) = \frac{d}{2} + \log\sqrt{\frac{\sin(\pi/\nu_1)}{\sin(\pi/\nu_2)}},
\end{aligned}
\tag{2.7}
$$

and

$$
h_3 = h(\nu_1,\nu_2) = \exp\left(-\pi e^{\frac{d}{2}}\sqrt{\sin\frac{\pi}{\nu_1}\sin\frac{\pi}{\nu_2}}\left(\cos\frac{\pi}{\nu_1}+\sin\frac{\pi}{\nu_2}\right)^{-1}\right).
\tag{2.8}
$$

(2) *If T has angles π/ν, 0 and 0 at the vertices v_1, v_2 and v_3, then the radii are*

$$
r_1 = r(\nu;\infty,\infty) = -\log\sin\frac{\pi}{2\nu},
\tag{2.9}
$$

and

$$
h_2 = h_3 = h(\infty,\nu) = \exp\left(-\frac{\pi}{\cos(\pi/2\nu)}\right).
\tag{2.10}
$$

(3) *If all the angles are 0, then*

$$
h_1 = h_2 = h_3 = h(\infty,\infty) = e^{-\pi}.
\tag{2.11}
$$

Proof. Case (3) is treated in [13] Proposition 1.6. For the proof of (1) we use the upper half plane model and normalize so that the vertex with angle 0 is at ∞ and the finite vertices are on the unit circle:

$$
v_1 = \cos\frac{\pi}{\nu_1} + i\sin\frac{\pi}{\nu_1},
$$

and

$$
v_2 = -\cos\frac{\pi}{\nu_2} + i\sin\frac{\pi}{\nu_2}.
$$

For this normalization the horocyclic coordinate at the puncture on the corresponding sphere with a puncture and two special points is

$$z = exp\left(i\pi \frac{\zeta}{\cos(\pi/\nu_1) + \cos(\pi/\nu_1)}\right).$$

We need to solve the pair of equations

$$\begin{gathered} r_1 + r_2 = \operatorname{dist}(v_1, v_2), \\ r_1 + \log\sin(\pi/\nu_1) = r_2 + \log\sin(\pi/\nu_2), \end{gathered}$$

where the latter equation is the condition that the points on the circles of hyperbolic radii r_i at v_i have equal imaginary parts, i.e. are on the same horocycle at infinity.

In case (2) we can use the fact that there is an orientation reversing isometric involution of T that exchanges the vertices at infinity. We are looking for an arrangement where the horocycles are tangent at the "midpoint" of their connecting geodesic fixed by the involution. We normalize the triangle so that the vertices with 0-angles are at ∞ and 0 and the finite vertex is $v_1 = \cos\pi/2\nu + i\sin\pi/2\nu$. The solution of case (2) is given by the arrangement where the horocycle at ∞ goes through i and the other two are tangent with it. This immediately gives the expression of r_1 as the hyperbolic distance of v_1 from the line $\{\operatorname{Im} z = i\}$. The horocyclic coordinate at ∞ with respect to 0 is now

$$z = \exp\left(i\pi \frac{\zeta}{\cos(\pi/2\nu)}\right). \quad \square$$

3. Elliptic gluing and 1-dimensional deformation spaces

In this section we use parameters coming from the $zw = t$ plumbing construction to define global complex analytic coordinates on deformation spaces of Koebe groups of types $(0,4)$ and $(1,1)$. This is done by interpreting group theoretical constructions on triangle groups as plumbings of Riemann surfaces with special points.

3.1. The $zw = t$ plumbing construction. In this section we briefly describe the $zw = t$ plumbing construction and show how it can be used to produce "geometric" complex analytic coordinates on a class of 1-dimensional deformation spaces. The construction is discussed by Kra [13] and Earle and Marden [8] in the case of thrice punctured spheres. Arés [3] treats the case of gluing across maximal cyclic parabolic subgroups of terminal regular b-groups and the AFP of two hyperbolic triangle groups of signature (ν, ∞, ∞) across elliptic maximal cyclic subgroups. In this paper we consider the general case of hyperbolic triangle groups. The following construction is, however, quite general.

Let X be a (possibly disconnected) Riemann surface of finite analytic type. Choose two points x_1, x_2 in X and local coordinates

$$z\colon U_1 \to \mathbb{C} \quad \text{and} \quad w\colon U_2 \to \mathbb{C},$$

where U_i is a neighborhood of x_i and

$$z(x_1) = 0 = w(x_2).$$

Assume there are annuli $\mathcal{A}_i \subset U_i$ and a holomorphic homeomorphism $f\colon \mathcal{A}_1 \to \mathcal{A}_2$ so that (see Figure 2)

$$z(x)w(f(x)) = t$$

for some constant $t \in \mathbb{C}$ and f maps the outer boundary of $\mathcal{A}_1$ (the component of $\partial\mathcal{A}_1$ closer to the point x_1) to the inner boundary of $\mathcal{A}_2$ (the component of $\partial\mathcal{A}_2$ farther from x_2). The outer boundaries bound disks on X. Remove these disks to form a new Riemann surface X_{trunc}. Define

$$X_t := X_{\text{trunc}}/\sim,$$

where the equivalence is defined

$$x \sim y \iff z(x)w(y) = t.$$

We say that X_t was obtained from X by the $zw = t$ *plumbing construction with plumbing or gluing parameter* t.

Lemma 3.1. *Let S be a (possibly disconnected) Riemann surface. Let $U \subset S$ and $V \subset S$ be two disjoint open sets and $z\colon U \to \mathbb{C}$ and $w\colon V \to \mathbb{C}$ be local coordinates with $z(P) = 0 = w(Q)$. If $\mathbb{D}(0,r_1) \subset z(U)$ and $\mathbb{D}(0,r_2) \subset w(V)$, and $t \in \mathbb{C}$ satisfies*

$$|t| < r_1\, r_2,$$

then it is possible to do the plumbing construction for the parameter t at P_1 and Q_1 and the construction is limited inside the disks of radii r_1 and r_2 at P_1 and Q_1 respectively.

Proof. The proof is exactly the same as for the case of horocyclic coordinates at punctures in Kra [13].

We cut off a disk of radius $|t|/r_2$ at P and a disk of radius $|t|/r_1$ at Q. The gluing annuli are non-empty, since we assume that

$$|t|/r_2 < r_1$$

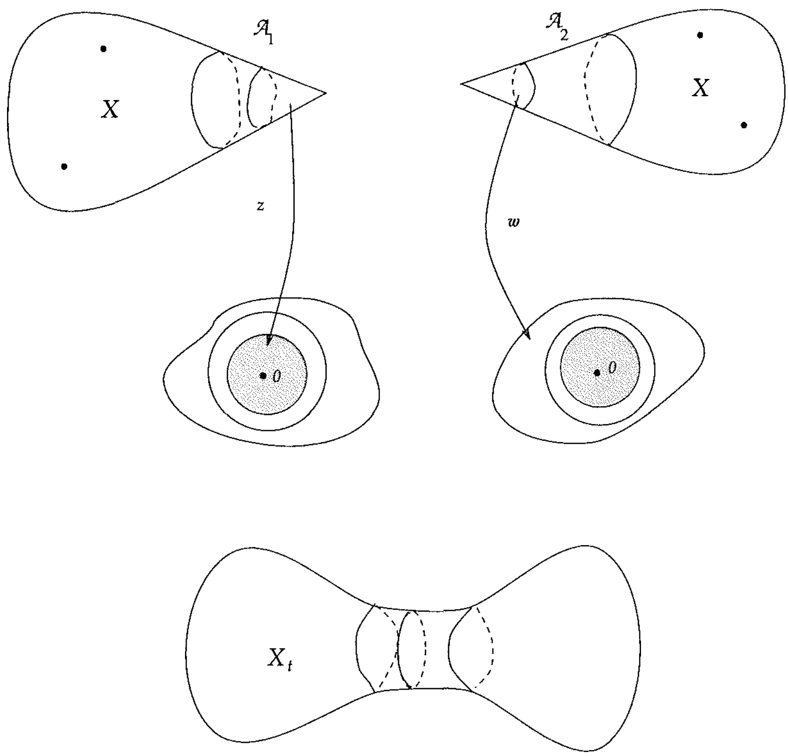


Figure 2. The $zw = t$-construction.

and

$$|t|/r_1 < r_2.$$

Clearly the annuli are inside disks of radii r_1 and r_2 (see Figure 3). $\square$

The following trivial observation will be useful in the examples later: If we use natural or horocyclic coordinates and do the $zw = t$ plumbing construction at the special point or puncture for a $t > 0$, points on the geodesic that is mapped into $\mathbb{D} \cap \mathbb{R}_+$ by z are identified only with points that are on the geodesic that is mapped into $\mathbb{D} \cap \mathbb{R}_+$ by w.

3.2. The AFP construction. We start with two spheres S_1 and S_2, with three special points of signatures (ν_1, ν_2, ν_3) and (ν_1, ν_4, ν_5) respectively. Let $\Gamma_1 = \langle A, B_1 \rangle$ and $\Gamma_2 = \langle A, B_2 \rangle$ be normalized triangle groups uniformizing S_1 and S_2.

Let $\lambda \in \mathbb{C}$ be a complex number with $|\lambda| > 1$ and

$$T_\lambda z = \lambda^2 z.$$

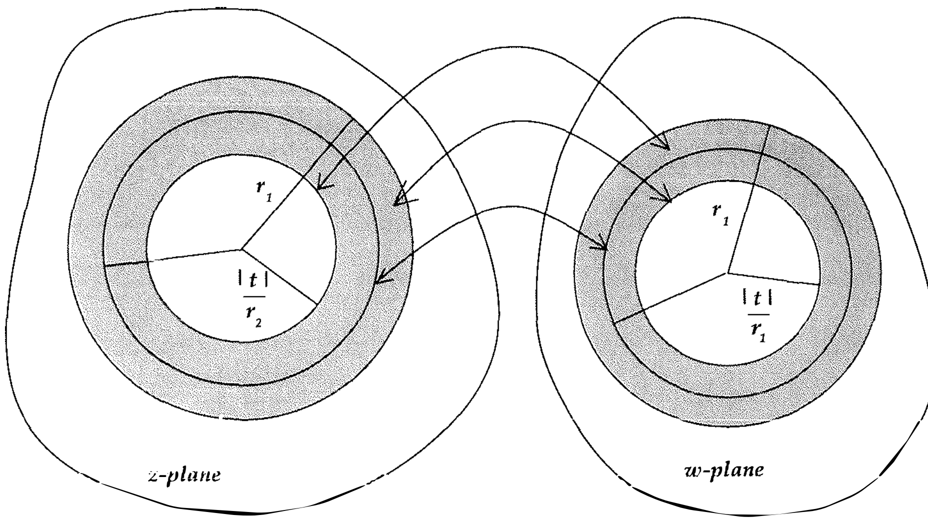


Figure 3. The $zw = t$ plumbing construction restricted to the disks of (2.6) and Lemma 2.5.

Define

$$\Gamma_2(\lambda^2) := T_\lambda \Gamma_2 T_\lambda^{-1} = \langle A, B_{2,\lambda^2}\rangle,$$

where (using the notation $d_3' = d_3(\nu_1, \nu_4, \nu_5)$)

$$B_{2,\lambda^2} = T_\lambda B_2 T_\lambda^{-1} = \begin{pmatrix} i\sin(\pi/\nu_4)\cosh d_3' - \cos(\pi/\nu_4) & -i\lambda^2 \sin(\pi/\nu_4)\sinh d_3' \\ \dfrac{i}{\lambda^2}\sin(\pi/\nu_4)\sinh d_3' & -i\sin(\pi/\nu_4)\cosh d_3' - \cos(\pi/\nu_4) \end{pmatrix}.$$

Obviously

$$\mathbb{D}(0, |\lambda|^2)/\Gamma_2(\lambda^2) = S_2.$$

Consider the group

$$G_1(\lambda^2) = \langle \Gamma_1, \Gamma_2(\lambda^2)\rangle$$

generated by Γ_1 and $\Gamma_2(\lambda^2)$. Maskit's first combination theorem ([17], Theorem VII.C.2) implies that $G_1(\lambda^2)$ is the amalgamated free product of Γ_1 and $\Gamma_2(\lambda^2)$, if we can find a Jordan curve $\mathcal{W} \subset \mathbb{D}(0, |\lambda|^2) \setminus \mathbb{D}$ that is *precisely invariant* in both Γ_1 and Γ_2 under the cyclic group $\langle A\rangle$, that is,

(1) the powers of A map $\mathcal{W}$ into itself, and
(2) $\mathcal{W} \cap g(\mathcal{W}) = \emptyset$ for all $\gamma \in \Gamma_i \setminus \langle A\rangle$.

We will use Lemma 2.4 to find a condition for the existence of a precisely invariant *round* annulus $\mathcal{A} \subset \mathbb{D}(0, |\lambda|^2) \setminus \mathbb{D}$. The existence of this annulus guarantees that the quotient surface is a sphere with four special points. In Section 3.6 we will show that this estimate is sharp if $\Gamma_1 = \Gamma_2$. Let D_i be the a fundamental polygon of Γ_i in $\mathbb{D}$ and $\mathbb{D}^*$ obtained by taking the intersection of the sector

$$\{z \in \widehat{\mathbb{C}} \mid -\frac{\pi}{\nu_1} \leq \arg z \leq \frac{\pi}{\nu_1}\}$$

and the common outside of the isometric circles of B and B^{-1}.

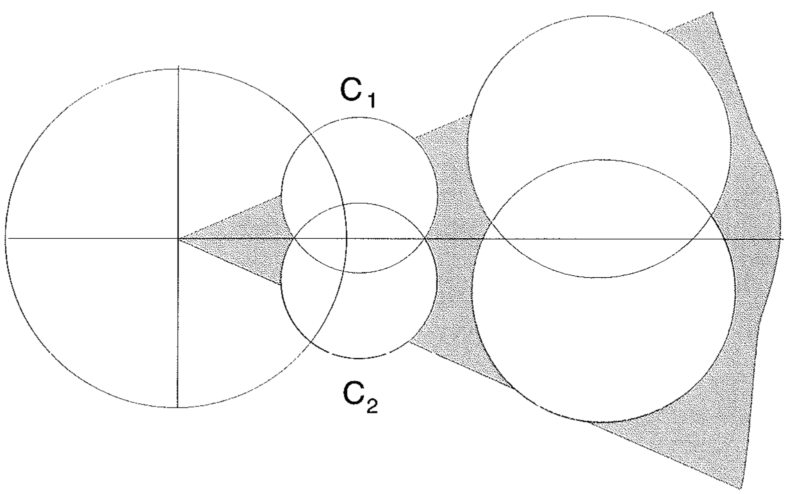


Figure 4. The fundamental domains of an AFP of two hyperbolic triangle groups with a gluing parameter $t \in \mathbb{C} \setminus \mathbb{R}$. The fundamental set extends to infinity in the sector. C_1 and C_2 are the isometric circles of B and B^{-1}.

Lemma 3.2. *Let* $\Gamma_1 = \langle A, B_1 \rangle$ *and* $\Gamma_2 = \langle A, B_2 \rangle$ *be normalized hyperbolic triangle groups of signatures* (ν_1, ν_2, ν_3) *and* (ν_1, ν_4, ν_5). *If*

$$|\lambda|^2 > \coth \frac{R(\nu_1; \nu_2, \nu_3)}{2} \coth \frac{R(\nu_1; \nu_4, \nu_5)}{2}, \tag{3.1}$$

where R *is the maximal radius given by* (2.2), *then*

$$G_1(\lambda^2) = \langle \Gamma_1, \Gamma_2(\lambda^2) \rangle = \Gamma_1 *_A \Gamma_2(\lambda^2).$$

On the other hand, if

$$|\lambda'|^2 < \max\left(\coth \frac{R(\nu_1; \nu_2, \nu_3)}{2}, \coth \frac{R(\nu_1; \nu_4, \nu_5)}{2}\right),$$

then $G_1({\lambda'}^2)$ *is not quasiconformally conjugate with* $G_1(\lambda^2)$.

Proof. For Γ_1 the Euclidean distance from the origin to the isometric circles of B and B^{-1} (see Figure 4) is $\tanh(R(\nu_1;\nu_2,\nu_3)/2)$, so the points on these circles with maximal distance from 0 have modulus $\coth(R(\nu_1;\nu_2,\nu_3)/2)$ The same applies for the conjugated group $\Gamma_2(\lambda^2)$ with the radii multiplied by $|\lambda|^2$. The disk

$$\widehat{\mathbb{C}} \setminus \mathbb{D}\left(0, \coth\frac{R(\nu_1;\nu_2,\nu_3)}{2}\right)$$

is precisely invariant in Γ_1 with respect to the cyclic subgroup generated by A, and the disk

$$\mathbb{D}\left(0, |\lambda|^2\tanh\frac{R(\nu_1;\nu_4,\nu_5)}{2}\right)$$

is precisely invariant in $\Gamma_2(\lambda^2)$ with respect to $\langle A\rangle$. Any circle in the annulus

$$\mathcal{A} = \mathbb{D}\left(0, |\lambda|^2\tanh\frac{R(\nu_1;\nu_4,\nu_5)}{2}\right) \setminus \mathbb{D}\left(0, \coth\frac{R(\nu_1;\nu_2,\nu_3)}{2}\right)$$

is precisely invariant under $\langle A\rangle$ in both groups, so the conditions of Maskit's first combination theorem are satisfied if

$$\coth\frac{R(\nu_1;\nu_2,\nu_3)}{2} < |\lambda|^2\tanh\frac{R(\nu_1;\nu_4,\nu_5)}{2}.$$

This proves the sufficient condition for the group to be the AFP of the original triangle groups.

If $G_1({\lambda'}^2)$ is a quasiconformal conjugate of $G_1(\lambda^2)$ for a parameter λ satisfying the condition (3.1), then $\mathbb{D}$ is precisely invariant in $G_1({\lambda'}^2)$ under Γ_1. If $|\lambda'|^2\tanh(R(\nu_1;\nu_4,\nu_5)/2) < 1$, then the isometric circles of B_2 and B_2^{-1} intersect $\mathbb{D}$. In particular, the points closest to the origin on the isometric circles, $p_1 \in \operatorname{Isom}(B_2)$ and $p_2 \in \operatorname{Isom}(B_2^{-1})$ are in $\mathbb{D}$. Obviously $B_2(p_2) = p_1$. Thus $\mathbb{D}$ is not precisely invariant under Γ_1 in $G_1({\lambda'}^2)$. One obtains the second condition similarly by looking at the outside disk $\mathbb{C}\setminus\lambda^2\mathbb{D}$. $\square$

The group $G_1(\lambda^2) = \langle\Gamma_1, \Gamma_2(\lambda^2)\rangle$ has an infinitely connected invariant component Δ representing a sphere with four elliptic special points with indices ν_2, ν_3, ν_4 and ν_5. The elliptic element A corresponds to a simple closed curve on Δ/Γ: Inside the annulus $\mathcal{A}$ there is a simple closed curve $\mathcal{W}$ that is projected to a simple closed curve on Δ/Γ.

All the other components are divided into two infinite families of disks stabilized by conjugates of the triangle groups Γ_1 and Γ_2, so $\Omega(G_1(\lambda^2))/G_1(\lambda^2)$ is the disjoint union of a surface of type $(0,4)$ and two surfaces of type $(0,3)$.

It is now easy to check that we have constructed the surface $\Delta/G_1(\lambda^2)$ by a $zw = t$ plumbing: Let $\rho_1\colon \mathbb{D}^* \to S_1$ be the canonical projection. The special

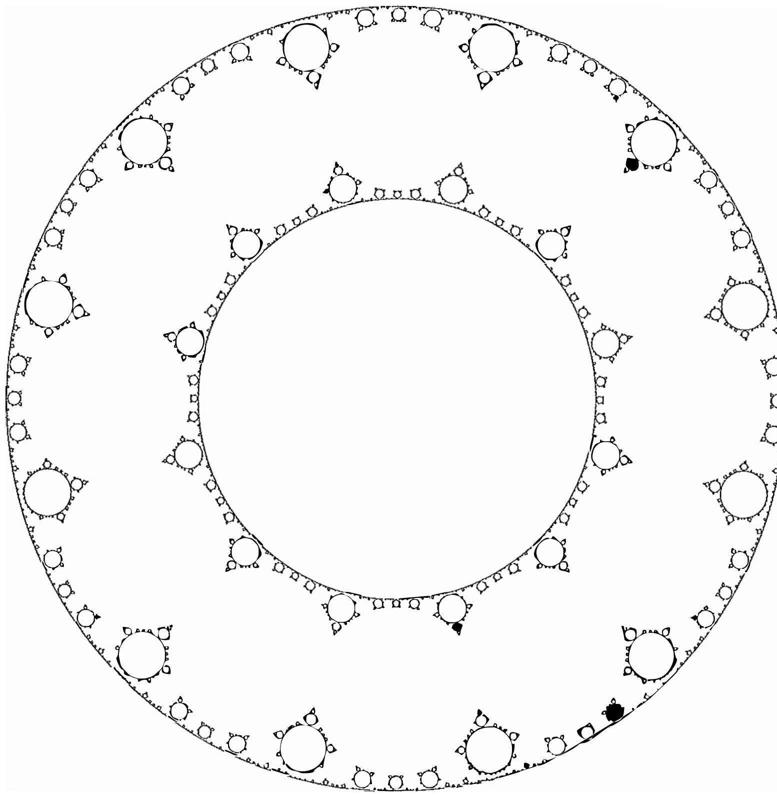


Figure 5. The limit set of a Koebe group $G_1(\lambda^2)$ constructed as the AFP of two triangle groups, both of signature $(6,6,6)$, with $\lambda = 1.4$.

points of S_1 are projections of the fixed points of elliptic transformations of Γ_1 in $\mathbb{D}^*$. On S_1 the special points are

$$P_1^1 = \rho_1(\infty) \ , \quad P_1^2 = \rho_1(\mathrm{fix}_L\, B_1) \quad \text{and} \quad P_1^3 = \rho_1(\mathrm{fix}_R\, AB).$$

The special points on S_2 are

$$P_2^1 = \rho_2(0) \ , \quad P_2^2 = \rho_2(\mathrm{fix}_R\, B_{2,\lambda^2}) \quad \text{and} \quad P_1^3 = \rho_2(\mathrm{fix}_L\, AB_{2,\lambda^2}),$$

where $\rho_2\colon \mathbb{D}(0,|\lambda|^2) \to S_2$ is the canonical projection.

A neighborhood $\widetilde{U}$ of ∞ is projected to a neighborhood U of the special point P_1^1. We get the expression in U of the natural coordinate z at P_1^1 relative to P_1^2 as follows: Choose a branch of ρ_1^{-1} with values in $\widetilde{U}$ and define

$$z(P) = \left(\frac{1}{\rho_1^{-1}(P)}\right)^{\nu_1}.$$

The coordinate can be analytically continued to the complement in S_1 of a simple curve connecting P_1^2 and P_1^3.

Similarly, we choose a branch of ρ_2^{-1} with values in a neighborhood of 0 and find an expression for the natural coordinate w at P_2^1 relative to P_2^2 as

$$w(Q) = \left(\frac{\rho_2^{-1}(Q)}{\lambda^2}\right)^{\nu_1}.$$

If λ^2 satisfies the conditions of Maskit's first combination theorem, we clearly have the relation

$$z(P)\,w\big(\rho_2 \circ \rho_1^{-1}(P)\big) = \left(\frac{1}{\zeta}\right)^{\nu_1}\left(\frac{\zeta}{\lambda^2}\right)^{\nu_1} = \left(\frac{1}{\lambda^2}\right)^{\nu_1} =: t$$

for all points P in a non-empty precisely $\langle A\rangle$-invariant annulus on the invariant component Δ, and thus on its projection to Δ/G_1, which is an annulus as well.

Remarks. (1) The value of the plumbing parameter t depends on the choice of natural coordinates at P_1^1 and P_2^1: The natural coordinate z' at P_1^1 relative to P_1^3 is given by

$$z'(P) = \left(\frac{e^{i\pi/\nu_1}}{\rho_1^{-1}(P)}\right)^{\nu_1} = -z(P). \tag{3.2}$$

(2) If $\nu_1 = \nu_3$ and $\nu_2 = \nu_4$, there is a second geometric interpretation of the parameter λ: The group

$$G_1' = \langle \Gamma_1, T_\lambda\rangle$$

is a Kleinian group with no invariant components. All its components are equivalent under the action of G_1', they are images of the invariant component Δ of the subgroup G_1. The stabilizer of the axis $a = (z,t) \in \mathbb{H}^3 \mid z = 0$ of T_λ is generated by T_λ and A, so its projection in the 3-manifold $\mathbb{H}^3/G_1'$ is an orbifold locus homeomorphic to $\mathbb{S}^1$. The length of this circle is $2\log|\lambda|$, and the imaginary part of the parameter describes the "twisting" in the 3-orbifold along this circle. This construction is similar to the Kleinian group constructions described in [8].

3.3. Deformation spaces of type $(0,4)$.

Lemma 3.3. *Let λ_0 be a complex parameter such that*

$$G_1(\lambda_0^2) = \langle \Gamma_1, \Gamma_2(\lambda_0^2)\rangle = \Gamma_1 *_A \Gamma_2(\lambda_0^2),$$

and $G_1(\lambda_0^2)$ represents a Riemann surface of type $(0,4)$ on the invariant component. Let $w \in [w] \in \mathbf{T}(G_1(\lambda_0^2))$ be a deformation normalized to fix the limit points 1, i and -1. Then the map

$$[w] \mapsto \operatorname{fix}_L(wB_{2,\lambda^2}w^{-1})$$

is a holomorphic injection of $\mathbf{T}(G_1(\lambda_0^2))$ *into* $\mathbb{C}$.

Proof. ([14]) The triangle group Γ_1 has no moduli, so $wAw^{-1} = A$ and $wB_1w^{-1} = B_1$. The group $w\Gamma_2w^{-1}$ is a triangle group generated by A and wB_2w^{-1}. The generator A is known, and we know the orders of B_2 and AB_2, and one of the fixed points of B_2. The fact that A and B generate a triangle group again implies that this is enough to fix B uniquely, and this fixes $G_1(\lambda_0^2)$ uniquely as a point of the deformation space. □

To get a coordinate independent of normalization, we note that λ^2 can be written as a cross ratio of fixed points:

$$\operatorname{cr}(\operatorname{fix}_R B_{2,\lambda^2}, \operatorname{fix}_L A, \operatorname{fix}_R A, \operatorname{fix}_R B_1) = \lambda^2 \frac{\operatorname{fix}_R B_2}{\operatorname{fix}_R B_1}, \tag{3.3}$$

or

$$\operatorname{cr}(\operatorname{fix}_L B_{2,\lambda^2}, \operatorname{fix}_L A, \operatorname{fix}_R A, \operatorname{fix}_L B_1) = \lambda^2 \frac{\operatorname{fix}_R B_1}{\operatorname{fix}_R B_2}. \tag{3.3$'$}$$

The constant

$$\frac{\operatorname{fix}_R B_1}{\operatorname{fix}_R B_2} = \frac{\tanh(d_3'/2)}{\tanh(d_3/2)}$$

is completely determined by the geometry of the two spheres that are glued together, it does not depend on the normalization of the triangle groups.

Definition 3.1. Let A, B_1, and B_2 be a set of generators for G_1 that satisfies

(1) (A, B_1) and (A^{-1}, B_2) are pairs of canonical generators for triangle groups of signatures (ν_1, ν_2, ν_3) and (ν_1, ν_4, ν_5),
(2) $G_1 = \langle A, B_1\rangle *_A \langle A^{-1}, B_2\rangle$,
(3) the part of Δ/G_1 corresponding to $\langle A, B_1\rangle$ lies to the right of the dividing curve $a \subset \Delta/G_1$ corresponding to A.

Let $[w] \in \mathbf{T}(G_1)$. The *gluing coordinate* of $[w]$ in $\mathbf{T}(G_1)$ is

$$\begin{aligned}\lambda^2([w]) &= \operatorname{cr}\left(\operatorname{fix}_L wB_2w^{-1}, \operatorname{fix}_L wAw^{-1}, \operatorname{fix}_R wAw^{-1}, \operatorname{fix}_R wB_1w^{-1}\right) \frac{\tanh(d_3/2)}{\tanh(d_3'/2)},\\ &= \operatorname{cr}\left(w \operatorname{fix}_L B_2, w \operatorname{fix}_L A, w \operatorname{fix}_R A, w \operatorname{fix}_R B_1\right) \frac{\tanh(d_3/2)}{\tanh(d_3'/2)}.\end{aligned}$$

As a result we get

Corollary. *λ^2 is a holomorphic non-variational global coordinate in $\mathbf{T}(G_1)$.*

Remarks. (1) This definition of a coordinate in deformation space is closely related to Kra's work on terminal b-groups in [13]. For torsion-free terminal b-groups the horocyclic coordinate of $\mathbf{T}(0,4)$ is expressed in an invariant form as the cross ratio of four parabolic fixed points, one of which is the accidental parabolic corresponding to the "plumbing curve" and the others correspond to three punctures on the sphere. In the expression above, we have both fixed points of the elliptic element that plays the role of the accidental parabolic and two elliptic fixed points that correspond to special points on the quotient 2-orbifold. The coordinate can be calculated from the cross ratio without ambiguity, as changing the order of the groups Γ_1 and Γ_2 does not change the value of λ^2.

(2) The same formula applies even for parabolic B_i, just replace the left and right fixed points in the expressions by the parabolic fixed point.

3.4. The HNN extension. Let S be a sphere with three special points of signature (ν_1,ν_1,ν_3). Let $\Gamma = \langle A, B\rangle$ be a normalized triangle group uniformizing S. In order to extend Γ to a Kleinian group uniformizing a torus with a special point of order ν_3, we look for a loxodromic element C with the property that the group generated by Γ and C is the HNN extension of Γ:

$$\langle \Gamma, C\rangle = \Gamma *_C .$$

In order to have the required property

$$CB^{-1}C^{-1} = A,$$

the transformation C must map the fixed points of B to the fixed points of A:

$$C(\mathrm{fix}_R\, B) = \mathrm{fix}_L\, A = \infty,$$

and

$$C(\mathrm{fix}_L\, B) - \mathrm{fix}_R\, A = 0.$$

This means that C must be of the form

$$\begin{aligned} C_\tau &= \begin{pmatrix} \tau\sinh(d_3/2) & -\tau\cosh(d_3/2) \\ \dfrac{1}{\tau}\cosh(d_3/2) & -\dfrac{1}{\tau}\sinh(d_3/2) \end{pmatrix} \\ &= \begin{pmatrix} \tau & 0 \\ 0 & \dfrac{1}{\tau} \end{pmatrix} \begin{pmatrix} \sinh(d_3/2) & -\cosh(d_3/2) \\ \cosh(d_3/2) & -\sinh(d_3/2) \end{pmatrix} =: T_\tau H, \end{aligned}$$

where τ is a complex parameter and H a half-turn conjugating B^{-1} and A, and an isometry between $\mathbb{D}$ and $\mathbb{D}^*$. It is clear (see Figure 6) that the requirements of

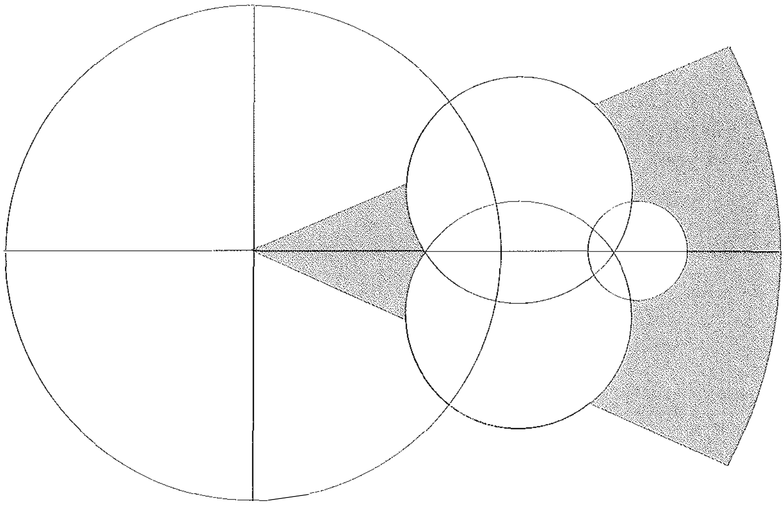


Figure 6. A fundamental set of $G_2(\tau^2)$ for $|\tau| > \coth(d_3/4)$.

the second combination theorem ([17],Theorem VII.E.5) are satisfied for $|\tau|$ big enough.

In fact, we get the following estimate for the parameter τ:

Lemma 3.4. $\langle \Gamma, C_\tau \rangle = \Gamma *_{C_\tau}$ *if*

$$|\tau| > \coth \frac{d_3}{4}. \tag{3.4}$$

Proof. A disk D of hyperbolic radius $r < d_3/2$ at $\mathrm{fix}_L B$ is mapped by the canonical projection to an orbifold disk at the special point corresponding to $\mathrm{fix}_L B$ on $\mathbb{D}^*/\Gamma$. This follows from (2.6), since for triangles with signatures of the form (ν_1, ν_1, ν_2), the radii given by (2.6) are ($d_2 = d_3$)

$$r_1 = r_2 = \frac{d_3}{2}.$$

D is mapped by H to a disk of the same hyperbolic radius at 0. The Euclidean radius of the image is $\tanh(r/2)$. This means that the Euclidean radius of the disk $C_\tau(D)$ is $|\tau|^2 \tanh(r/2)$. The complement of $C_\tau(D)$ is a disk of hyperbolic radius $2\,\mathrm{artanh}(|\tau|^{-2}\coth(r/2))$ at ∞. If

$$2\,\mathrm{artanh}\left(|\tau|^{-2}\coth\frac{r}{2}\right) < \frac{d_3}{2}, \tag{3.5}$$

the disk $\widehat{\mathbb{C}} \setminus C_\tau(D)$ is projected to an orbifold disk disjoint from the projection of D, and the conditions of the second combination theorem ([17],Theorem VII.E.5) are satisfied. Clearly we get (3.5) from $|\tau| > \coth(d_3/4)$. $\square$

The group $\Gamma *_{C_\tau}$ has one invariant component representing a torus with an elliptic special point of order ν_3. The conjugate elliptic elements A and B correspond to a simple closed curve on Δ/G_2 in the free homotopy class determined in the case of the previous lemma by the projections (as point sets) of the boundaries of D and $C_\tau(D)$.

Again we can interpret the group theoretic construction as a $zw = t$ gluing using natural coordinates at the special points of order ν_1. The natural coordinate at $P^1 = \rho(\operatorname{fix}_L B)$ relative to $P^2 = \rho(\infty)$ is

$$z(P) = \big(Z_0(\rho^{-1}(P))\big)^{\nu_1} = \left(\frac{\sinh(d_3/2)\rho^{-1}(P) - \cosh(d_3/2)}{\cosh(d_3/2)\rho^{-1}(P) - \sinh(d_3/2)}\right)^{\nu_1},$$

where we use a branch of ρ^{-1} having values in a neighborhood of $\operatorname{fix}_L B$, and the natural coordinate at P^2 relative to P^1 is (using a branch of ρ^{-1} with values in a neighborhood of ∞)

$$w(Q) = \left(\frac{1}{\rho^{-1}(Q)}\right)^{\nu_1}.$$

If τ satisfies the conditions of Maskit's second combination theorem, there is a non-empty annulus $\widetilde{\mathcal{A}} \subset \Delta$ around 0 that projects to an annulus around P^1 such that $\widetilde{\mathcal{A}} \cap C(\widetilde{\mathcal{A}}) = \emptyset$ and

$$z(P)w\big(\rho\big(C\big(\rho^{-1}(P)\big)\big)\big) = \big(Z_0\big(\rho^{-1}(P)\big)\big)^{\nu_1}\left(\frac{1}{\tau^2 Z_0(\rho^{-1}(P))}\right)^{\nu_1} = \tau^{-2\nu_1} =: t$$

for $P \in \mathcal{A}$.

3.5 Deformation spaces of type $(1,1)$. For parameters τ^2 satisfying the conditions of Maskit's second combination theorem we define

$$G_2(\tau^2) = \Gamma *_{C_\tau}.$$

Using techniques similar to those in Section 3.3, we get

Lemma 3.5. *Let* $\tau_0 \in \mathbb{C}$ *be chosen so that*

$$\langle \Gamma, C_{\tau_0} \rangle = \Gamma *_{C_{\tau_0}},$$

and let $w \in [w] \in \mathbf{T}(G_2(\tau_0^2))$ *be a deformation normalized to fix the limit points* 1, i *and* -1. *Then the map*

$$[w] \mapsto wCw^{-1}(\operatorname{fix}_R wAw^{-1}) = w \circ C(\operatorname{fix}_R A)$$

is a holomorphic injection of $\mathbf{T}(G_2(\tau_0^2))$ *into* $\mathbb{C}$.

We can express τ^2 as a cross ratio of four fixed points of the group $G_2(\tau^2)$ as follows:

$$\operatorname{cr}(C_\tau(\operatorname{fix}_R A), \operatorname{fix}_L A, \operatorname{fix}_R A, \operatorname{fix}_L B) = \operatorname{cr}\left(\tau^2 \coth\frac{d_3}{2}, \infty, 0, \coth\frac{d_3}{2}\right) = \tau^2. \tag{3.6}$$

Definition 3.2. Let A, B and C be a set of generators of G_2 such that

(1) A and B canonically generate a triangle group, and
(2) $CB^{-1}C^{-1} = A$.

Let $[w] \in \mathbf{T}(G_2)$. The *gluing coordinate* of $[w]$ is

$$\begin{aligned}\tau([w]) &= \mathrm{cr}(wCw^{-1}(\mathrm{fix}_R\, wAw^{-1})), \mathrm{fix}_L\, wAw^{-1}, \mathrm{fix}_R\, wAw^{-1}, \mathrm{fix}_L\, wBw^{-1})\\ &= \mathrm{cr}(wC(\mathrm{fix}_R\, A)), w\,\mathrm{fix}_L\, A, w\,\mathrm{fix}_R\, A, w\,\mathrm{fix}_L\, B).\end{aligned}$$

We summarize this in

Corollary. *τ^2 is a holomorphic non-variational global complex analytic coordinate in $T(G_2)$.*

3.6. Boundary points. Let $\Gamma = \langle A, B\rangle$ a normalized hyperbolic triangle group of signature (ν_1, ν_2, ν_3) canonically generated by A and B and let

$$G_1 = G_1(\lambda^2) = \langle \Gamma, \Gamma(\lambda^2)\rangle,$$

and assume for simplicity that A is elliptic. If G_1 is an AFP of the two copies of Γ, then

$$C = (B_{\lambda^2})^{-1}B$$

is a loxodromic element *dual* to A in G_1 with trace

$$\begin{aligned}\mathrm{tr}\, C =& 2\cos^2\frac{\pi}{\nu_2} + \left(2\cosh^2 d_3 - \left(\lambda^2 + \frac{1}{\lambda^2}\right)\sinh^2 d_3\right)\sin^2\frac{\pi}{\nu_2}\\ =& 2 + 2\sinh^2 R - \left(\lambda^2 + \frac{1}{\lambda^2}\right)\sinh^2 R,\end{aligned} \tag{3.7}$$

where R is the hyperbolic distance between 0 and the geodesic connecting the fixed points of B and AB in $\mathbb{D}$ given by (2.2).

Let

$$\lambda_b = \coth\frac{R}{2}.$$

It is a straightforward calculation to show that if $\lambda = \lambda_b$, then $\mathrm{tr}\, C = -2$ and C is parabolic with fixed point

$$\mathrm{fix}\, C = e^{-\pi/2\nu_1}\coth(R/2).$$

If both $\nu_1 > 2$ and $\nu_2 > 2$, the boundary point $\lambda_b^2 \in \partial\mathbf{T}(G_1)$ can be interpreted as giving a degenerated Riemann surface where the dividing geodesic corresponding to the element C (see Figure 7) is pinched to a point: The set of discontinuity of $G_1(\lambda_b^2)$ consists of four non-equivalent families of round disks stabilized by Fuchsian subgroups of $G_1(\lambda_b^2)$. The Fuchsian groups are conjugates of the original triangle groups $\Gamma = \langle A, B_1\rangle$ and $\Gamma(\lambda^2) = \langle A, B_{\lambda_b^2}\rangle$ of signature (ν_1, ν_2, ν_3), and two additional triangle groups

$$\Gamma_1' = \left\langle B, \left(B_{\lambda_b^2}\right)^{-1}\right\rangle \quad \text{and} \quad \Gamma_2' = \left\langle (AB)^{-1}, AB_{\lambda_b^2}\right\rangle$$

of signatures (ν_2, ν_2, ∞) and (ν_3, ν_3, ∞) (with canonical generators). The group generated by C is a maximal cyclic parabolic subgroup of both Γ_1' and Γ_2'. Denote by Δ_i the disk component of $\Omega(G_1)$ stabilized by Γ_i'. We can form a new invariant set

$$\Delta^+ = G_1(\Delta_1 \cup \Delta_2 \cup \{\operatorname{fix} C\}).$$

The quotient Δ^+/G_1 is the union of two thrice punctured spheres and a point P which is the projection of the fixed point of C. The neighborhoods of P are projections of sets of the form $B_1 \cup \{\operatorname{fix} C\} \cup B_2$, where B_i is a topological disk in Δ_i precisely invariant under $\langle C\rangle$ in Γ_i'. Any neighborhood of P is homeomorphic to the set

$$\{(z, w) \in \mathbb{C}^2 \mid zw = 0\}.$$

The point P is called a *node*, and Δ^+/G_1 is a *Riemann surface with nodes.*

If one of $\nu_2 = 2$ or $\nu_3 = 2$, Γ_1' or Γ_2' becomes a finite dihedral group and the quotient space consists of just three spheres with signatures as above. The general case of the AFP of two hyperbolic triangle groups of different signatures requires a more involved treatment, which we will not attempt here.

Let $G_2 = \langle A, B\rangle *_{C_\tau}$ as in Sections 3.4 and 3.5. Again, it is quite easy to see that for the parameter

$$\tau_b = \coth \frac{d_3}{4}$$

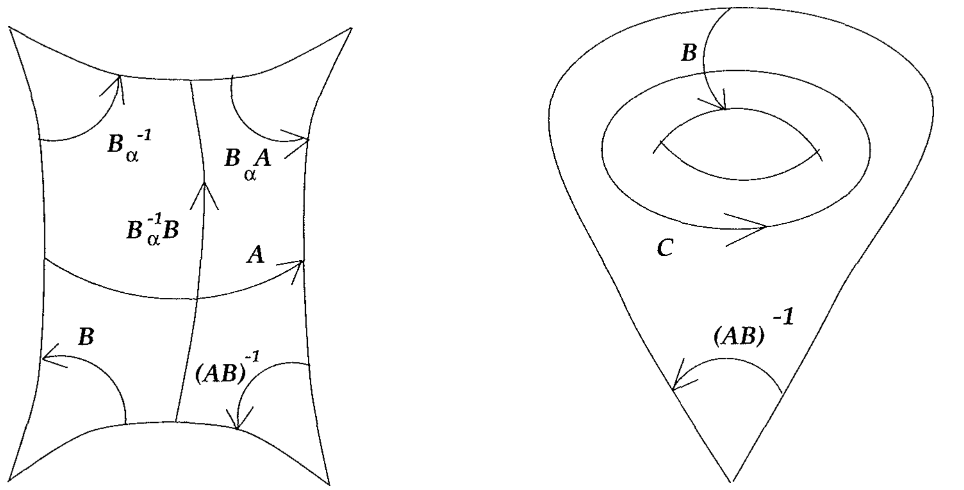
the generator C_τ becomes parabolic, whereas for $|\tau| > \tau_b$ it is loxodromic. The invariant component splits into a collection of disks that are all equivalent under the action of the group. The disks are stabilized by conjugates of the Fuchsian group

$$\Gamma' = \langle A^{-1}CA, C^{-1}\rangle,$$

which is a triangle group of signature (∞, ∞, ν_3). The two parabolic canonical generators of Γ' are conjugate in G_2, so G_2 represents a noded Riemann surface of signature $(1, 1; \nu_3)$.

3.7. AFP with dihedral groups. Let D_{ν_1} be a ν_1-dihedral group generated by

$$A = \begin{pmatrix} e^{-i\pi/\nu_1} & 0 \\ 0 & e^{i\pi/\nu_1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$



(a) (b)

Figure 7. (a) The curves on the sphere $\Delta/G_1(\lambda^2)$ corresponding to the generators and the word ${B_{\lambda^2}}^{-1}B$.
(b) The curves on the torus $\Delta/G_2(\tau^2)$ coming from the generators A, B and C.

Lemma 3.6. *Let Γ_1 be a hyperbolic triangle group of signature (ν_1, ν_2, ν_3) and D_{ν_1} a finite dihedral group normalized as above. Then*

$$G_3(\lambda^2) = \langle \Gamma_1, D_{\nu_1}(\lambda^2)\rangle = \Gamma_1 *_A D_{\nu_1}(\lambda^2),$$

if

$$|\lambda|^2 > \coth \frac{R(\nu_1; \nu_2, \nu_3)}{2}.$$

Proof. The exterior of the disk $\mathbb{D}(0, \coth(R(\nu_1; \nu_2, \nu_3)/2))$ is precisely invariant under $\langle A\rangle$ in Γ. For any $\lambda \neq 0$ the set

$$D = \left\{ z \in \widehat{\mathbb{C}} \mid |z| < |\lambda|^2 \text{ and } -\frac{\pi}{\nu_1} \leq \arg z \leq \frac{\pi}{\nu_1} \right\}$$

is a fundamental domain of $D_{\nu_1}(\lambda^2)$, so it is clear that $\mathbb{D}(0, \coth R(\nu_1; \nu_2, \nu_3)/2)$ is precisely invariant under $\langle A\rangle$ in $D_{\nu_1}(\lambda^2)$, so the conditions of the first combination theorem are satisfied if

$$|\lambda|^2 > \coth \frac{R(\nu_1; \nu_2, \nu_3)}{2}. \quad \square$$

Again, using ideas of [14] we have

Lemma 3.7. *Let λ_0 be a complex parameter such that*

$$G_3(\lambda_0^2) = \langle \Gamma_1, D_{\nu_1}(\lambda_0^2)\rangle = \Gamma_1 *_A D_{\nu_1}(\lambda_0^2),$$

and $G_3(\lambda_0^2)$ represents a Riemann surface of type $(0,4)$ on the invariant component. Let $w \in [w] \in \mathbf{T}(G_3(\lambda_0^2))$ be a deformation normalized to fix the limit points 1, i and -1. Then the map

$$[w] \mapsto \mathrm{fix}_L \left(wB_{\lambda^2}w^{-1}\right)$$

is a holomorphic injection of $\mathbf{T}(G_3(\lambda_0^2))$ into $\mathbb{C}$.

These results are needed in Section 4 in the proof of Theorem 1 on the isomorphisms between 1-dimensional deformation spaces in Section 4.

Remark. The group $G_3(\lambda^2)$ constructed above is an extension of $\Gamma_1 *_A \Gamma_1(\lambda^2)$ by the elliptic

$$E(\lambda) = \begin{pmatrix} 0 & i\lambda \\ i/\lambda & 0 \end{pmatrix}$$

representing the hyperelliptic involution on $\Delta(G_3(\lambda^2))/G_3(\lambda^2)$.

4. Isomorphisms of 1-dimensional deformation spaces

In this section we prove an isomorphism theorem for 1-dimensional deformation spaces. The method of proof is an explicit construction using the fact that for certain Koebe groups of types $(0,4)$ and $(1,1)$, there is a group that contains both of them as subgroups of finite index. This group is constructed by studying the normalizers of these Koebe groups.

Let

$$G_1(\lambda^2) = \Gamma *_A \Gamma(\lambda^2),$$

where $\Gamma = \langle A, B\rangle$ is a normalized triangle group of signature $(0,3;\nu_1, 2\nu_2, 2\nu_2)$. Let $N(G_1(\lambda^2))$ be the normalizer in $\mathrm{PSL}(2,\mathbb{C})$ of $G_1(\lambda^2)$. The square root of A is

$$A^{\frac{1}{2}} = \begin{pmatrix} e^{-i\pi/2\nu_1} & 0 \\ 0 & e^{i\pi/2\nu_1} \end{pmatrix}.$$

A simple calculation shows that

$$A^{\frac{1}{2}} B A^{-\frac{1}{2}} = (AB)^{-1}, \tag{4.1}$$

so

$$A^{\frac{1}{2}} \in N(G_1(\lambda^2)) \setminus G_1(\lambda^2).$$

Also the half-turn

$$E_\lambda = \begin{pmatrix} 0 & i\lambda \\ i/\lambda & 0 \end{pmatrix} \in N(G_1(\lambda^2)),$$

because

$$E_\lambda A E_\lambda = A^{-1}, \text{ and } E_\lambda B E_\lambda = B_{\lambda^2}. \tag{4.2}$$

These elements correspond to conformal automorphisms of the surface $\Delta/G_1(\lambda^2)$ that either preserve or reverse the dividing curve corresponding to the elliptic element A. As a result we know that

$$G_3 = \langle A^{\frac{1}{2}}, B, E_\lambda \rangle \subset N(G_1),$$

and that $G_1 \subset G_3$ is a subgroup of finite index.

Using (4.1) we get

$$(A^{\frac{1}{2}}B)^2 = (A^{\frac{1}{2}}BA^{\frac{1}{2}})B = (A^{\frac{1}{2}}BA^{-\frac{1}{2}})AB = \mathrm{id}, \tag{4.3}$$

so G_3 is the AFP of a triangle group

$$\Gamma' = \langle A^{\frac{1}{2}}, B \mid |A^{\frac{1}{2}}| = 2\nu_1, |B| = 2\nu_2, |A^{\frac{1}{2}}B| = 2 \rangle$$

with a $2\nu_1$-dihedral group

$$D_{2\nu_1} = \langle A^{\frac{1}{2}}, E_\lambda \mid |A^{\frac{1}{2}}| = 2\nu_1, |E_\lambda| = 2 = |A^{\frac{1}{2}}E_\lambda| \rangle$$

across the elliptic cyclic subgroup generated by $A^{\frac{1}{2}}$. We proved in Section 3.7 that λ^2 is a global complex analytic coordinate of the deformation space $\mathbf{T}(G_3)$. Thus the identity map is an isomorphism between $\mathbf{T}(G_1(\lambda^2))$ and $\mathbf{T}(G_3)$.

Again, using (4.1) we have

$$\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)B^{-1} = B^{-2}, \tag{4.4}$$

so the subgroup

$$\Gamma'' = \langle A^{\frac{1}{2}}, BA^{\frac{1}{2}}B^{-1} \rangle$$

is a $(2\nu_1, 2\nu_1, \nu_2)$-triangle group acting on $\widehat{\mathbb{C}} \setminus \mathbb{S}^1$. Let

$$C'_\lambda = E_\lambda B^{-1} =$$

$$\begin{pmatrix} \lambda \sin(\pi/\nu_2) \sinh d_3 & -\lambda\left(\sin(\pi/\nu_2)\cosh d_3 + i\cos(\pi/\nu_2)\right) \\ \frac{1}{\lambda}\left(\sin(\pi/\nu_2)\cosh d_3 - i\cos(\pi/\nu_2)\right) & -\frac{1}{\lambda}\sin(\pi/\nu_2)\sinh d_3 \end{pmatrix}.$$

We claim that

$$G_2 = \langle \Gamma'', C'_\lambda \rangle = \Gamma'' *_{C'_\lambda}.$$

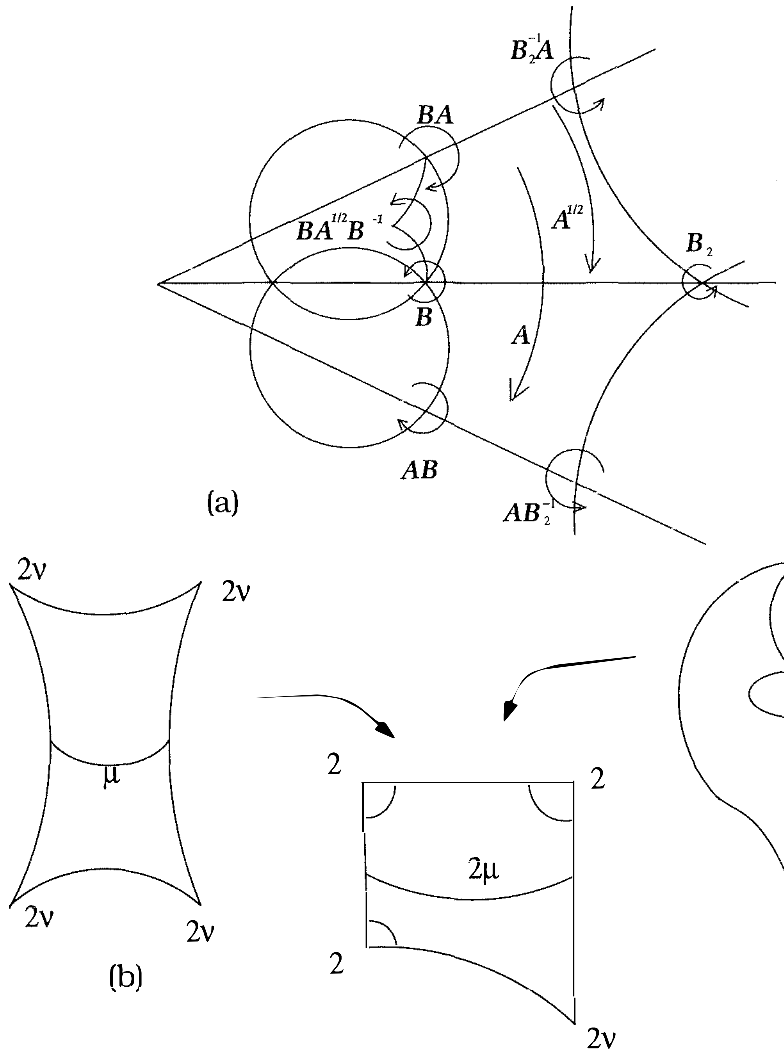


Figure 8. (a) Some of the Möbius transformations involved in the construction of the isomorphism of Theorem 1. (b) The sphere with four special points and the torus with one special point both cover the same sphere with four special points, three of which are of order 2.

The assumption that G_1 is a Koebe group representing a sphere with four special points on the invariant component implies that there is a simple closed curve $W \subset \Delta(G_1)$ that is precisely invariant under $\langle A \rangle$ in both Γ and $\Gamma(\lambda^2)$ such that each $\gamma \in \Gamma \setminus \langle A \rangle$ maps $\operatorname{ext} W$ into $\operatorname{int} W$ and each $\gamma \in \Gamma(\lambda^2) \setminus \langle A \rangle$ maps $\operatorname{int} W$

into $\operatorname{ext} W$. A calculation using (4.2) shows that C'_λ conjugates the generators of Γ'':

$$C'_\lambda(BA^{-\frac{1}{2}}B^{-1})C'^{-1}_\lambda = E_\lambda B^{-1}BA^{-\frac{1}{2}}B^{-1}BE_\lambda = A^{\frac{1}{2}}, \tag{4.5}$$

so we only need to show that Γ'' and C'_λ satisfy the conditions of Maskit's second combination theorem. The left fixed point of $BA^{-\frac{1}{2}}B^{-1}$ is $B(\infty) \in B(\operatorname{ext} W)$. Now $C'_\lambda(B(\operatorname{ext} W)) = E_\lambda(\operatorname{ext} W)$. We can clearly assume that $E_\lambda(W) = W$: For big $|\lambda|^2$ we can choose W to be the circle with center 0 and radius $|\lambda|$, for general λ^2 we can use a quasiconformal image of W. This proves the claim.

The gluing parameter of G_2 is

$$\begin{aligned} &\operatorname{cr}(C(\operatorname{fix}_R A^{\frac{1}{2}}), \operatorname{fix}_L A^{\frac{1}{2}}, \operatorname{fix}_R A^{\frac{1}{2}}, \operatorname{fix}_L BA^{\frac{1}{2}}B^{-1}) = \operatorname{cr}(C(0), \infty, 0, B(\infty)) \\ &= \left(\lambda^2\left(\coth d_3 + i\frac{\cot(\pi/\nu_2)}{\sinh d_3}\right), \infty, 0, \coth d_3 + i\frac{\cot(\pi/\nu_2)}{\sinh d_3}\right) = \lambda^2, \end{aligned} \tag{4.6}$$

so we have a map $\mathbf{T}(G_1) \to \mathbf{T}(G_2)$ defined by

$$\lambda^2 \mapsto \tau^2 = \lambda^2.$$

To see that this is actually an isomorphism of the deformation spaces, we can construct the inverse by noting that

$$G_3 = \langle G_2, E_\lambda \rangle \subset N(G_2),$$

as $E_\lambda = C'_\lambda B = (C'_\lambda B)^{-1}$ conjugates the generators $A^{\frac{1}{2}}$ and C'_λ of G_2 to the following elements of G_2

$$E_\lambda A^{\frac{1}{2}} E_\lambda = A^{-\frac{1}{2}}, \tag{4.7}$$

and

$$E_\lambda C'_\lambda E_\lambda = B^{-1}C'^{-1}_\lambda C'_\lambda B^{-1} C'^{-1}_\lambda = B^{-2}C'^{-1}_\lambda \in G_2. \tag{4.8}$$

We have proved

Theorem 1. *Let G_1 be a Koebe group of signature $(0,4;2\nu_2,2\nu_2,2\nu_2,2\nu_2)$, $\nu_2 \geq 3$, constructed as an AFP of two hyperbolic triangle groups across a maximal elliptic subgroup of order ν_1, and let G_2 be a Koebe group of signature $(1,1;\nu_2)$ constructed as the HNN extension of a hyperbolic triangle group across maximal elliptic cyclic subgroups of order $2\nu_1$. Then in gluing coordinates the map*

$$\lambda^2 \mapsto \tau^2 = \lambda^2$$

is a complex analytic isomorphism between the deformation spaces $\mathbf{T}(G_1)$ and $\mathbf{T}(G_2)$.

It can be shown that G_3 is actually the normalizer of both $G_1(\lambda^2)$ and G_2 in $\mathrm{PSL}(2,\mathbb{C})$.

Remark. The proof is just a modification of the proof in [13] in the case of terminal b-groups.

5. The construction of Koebe groups for weighted graphs with finite weights

The previous sections dealt with the Koebe groups produced by one gluing operation across an elliptic cyclic group. In this section we will look for conditions for the gluing parameters that guarantee that further gluing construction can be performed for the (possibly) remaining special points. This is a step toward proving Theorems 3 and 4 in Section 6. These theorems generalize Theorem 1 in [13]. Kra's result states that given a trivalent graph $\mathcal{G}$ and a d-tuple of gluing parameters $(t_1, \dots, t_d)$ satisfying $|t_i| < e^{-2\pi}$ for all i, it is possible to perform the gluing construction for thrice punctured spheres. The generalization presented here has more complicated conditions for the gluing parameters due to the presence of elliptic elements of various orders.

5.1. Tame gluing constructions. The restriction to groups constructed from triangle groups allows us to replace the signatures (as function groups in the sense of Maskit [17], Chapter X) of these groups using a simpler combinatorial object introduced by Arés in [3]: A *weighted graph* (see Figure 9) is a connected graph such that

(1) every vertex S_i has 3 edges (the graph is *trivalent*), and
(2) every edge a_i is assigned a weight $w_i \in \{2, 3, \dots\} \cup \{\infty\}$

If an edge a_k does not end at a vertex in the set $\{S_1, \dots, S_v\}$, a_k is called a *phantom edge*.

Let $\mathcal{G}$ be a weighted graph. Let S_i be a vertex and $(w_{j_1}, w_{j_2}, w_{j_3})$ the weights of the edges at S_i (if an edge ends at the same vertex S_i, take the weight twice in the list). Replace the vertex S_i by a sphere with three special points of orders $w_{j_1}, w_{j_2}, w_{j_3}$. Suppose that we are given a well-chosen parameter $t_k \in \mathbb{C}$ for each non-phantom edge. Now we do the $zw = t$ plumbing constructions for each non-phantom edge using natural coordinates if the edge has finite weight and horocyclic coordinates if the edge has weight ∞. The main objective of this section is to find a concrete meaning for the word "well-chosen" used above.

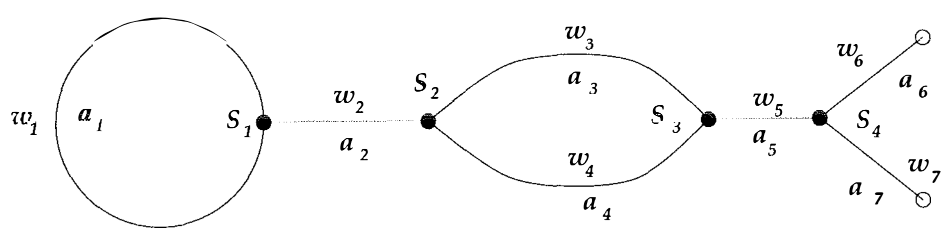


Figure 9. A weighted graph corresponding to a Riemann surface of type $(2, 2)$.

The observations of Section 2 suggest a condition for choosing the disks and the annuli used in the plumbing construction: If the plumbing can be restricted to the pairwise disjoint disks of radii r_i given by (2.6) (that is, if a point P is not in any of the disks, then the equivalence class of P in the gluing construction consists of one point), it is clear that we can use the remaining special points to perform further plumbing operations that are restricted to the corresponding orbifold disks.

Lemma 5.1. *Let S and S' be spheres with three special points with signatures (ν_1, ν_2, ν_3) and (ν_1, ν_4, ν_5). Let $P_1 \in S$ and $Q_1 \in S'$ be the special points of index ν_1 and let $r_1 = r(\nu_1; \nu_2, \nu_3)$ and $r_2 = r(\nu_1; \nu_4, \nu_5)$ be the radii of* (2.6)*. If $t \in \mathbb{C}$ satisfies*

$$|t| < \left(\tanh\frac{r_1}{2}\tanh\frac{r_2}{2}\right)^{\nu_1}, \tag{5.1}$$

then it is possible to do the plumbing construction for the parameter t at the special points P_1 and Q_1 and the construction is limited inside the disks of (*hyperbolic*) *radii r_1 and r_2 at P_1 and Q_1 respectively.*

Proof. The result follows from Lemma 3.1, using natural coordinates at special points. $\square$

If all the gluings are restricted to happen inside disjoint disks on the spheres S_j the plumbing construction is called *tame.* Obviously, the conclusion of Lemma 5.1 holds even for the case $S_1 = S_2$ with the restriction $P_1 \neq Q_1$.

Remarks. (1) When we restrict the gluing to take place inside the disks of Lemma 5.1, we get the same (sharp) lower bound for the absolute value of the gluing parameter in the construction of Koebe groups of type $(1,1)$. For type $(0,4)$ the bound is considerably larger than the sharp bound of Section 3.2. In Section 7 we see that the estimate given by pairwise tangent disks is actually sharp if we have more than one gluing.

(2) The second combination theorem guarantees that for each group G constructed by a number of tame plumbings from triangle groups, there is for each elliptic fixed point $x \in \Delta(G)$ a disk that is precisely invariant under $\mathrm{Stab}_G(x)$ in G. This means that the elliptic fixed points can be used for further gluing constructions. Section 5.2 outlines an algorithm for this and Section 6 gives an estimate for good gluing parameters.

5.2 A construction algorithm. Let $\mathcal{G}$ be a weighted graph. Assume that the non-phantom edges are *semicanonically ordered*, that is, the k:th subgraph $\mathcal{G}_k$ formed by the non-phantom edges $a_1, \ldots, a_k$, and a number of phantom edges, is connected for all k. A non-phantom edge a_i of $\mathcal{G}$ can be of three different types. To construct a Kleinian group corresponding to $\mathcal{G}$ and a gluing parameter $\mathbf{t} \in \mathbb{C}^d$, where d is the number of non-phantom edges of $\mathcal{G}$, each type requires its own procedure. We present the methods for each type of gluing and refer to Kra [13], Sections 3 and 7.5, for a more detailed description of semicanonical ordering and other details needed to devise a nice algorithm for producing Koebe groups (terminal regular b-groups in [13]). Let us denote by G_k the Koebe group constructed by k gluings.

Assume that all the vertices of $\mathcal{G}$ are *hyperbolic*: If w_1, w_2, w_3 are the weights of the edges at a vertex, then

$$\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} < 1.$$

This restriction means that only hyperbolic triangle groups will be allowed in the construction. This is done for technical reasons: The remaining cases (groups acting on the sphere and on the plane) cannot be treated with methods of 2-dimensional hyperbolic geometry.

Type I. The edge a_k disconnects $\mathcal{G}_k$ (edges a_2, a_3 and a_5 in Figure 9). This corresponds to the AFP construction presented in Section 3.2. Choose a primitive elliptic element $A \in G_{k-1}$ such that the special point on $\Omega(G_{k-1})/G_{k-1}$ is the projection of the left fixed point of A in the canonical projection. We can assume that A belongs to one of the triangle groups F used to construct G_{k-1}. There is a unique $B \in G_{k-1}$ such that A and B generate F canonically. Let τ^2 be the gluing parameter. The new triangle group corresponding to the vertex $v \in \mathcal{G}_k \backslash \mathcal{G}_{k-1}$ must have A^{-1} and a parabolic or an elliptic $B_2 \notin G_{k-1}$ as canonical generators. Now solve

$$\tau^2 = \operatorname{cr}(\operatorname{fix}_L B_2, \operatorname{fix}_L A, \operatorname{fix}_R A, \operatorname{fix}_R B)\frac{\tanh(d_3/2)}{\tanh(d_3'/2)}$$

for the left fixed point of B_2. Because we already know the fixed points of the element A, we get B_2 by conjugation from the standard normalization of Section 2.1. Set

$$G_k = G_{k-1} *_A \langle A, B_2 \rangle.$$

An alternative way to determine B_2 would be to use Lemma 2.2: We solve the equation

$$\operatorname{cr}(\operatorname{fix}_R A^{-1}, \operatorname{fix}_R B, \operatorname{fix}_L B, \operatorname{fix}_L A^{-1}) = \coth^2(d_3'/2)$$

for $\operatorname{fix}_R B$. This fixes B_2 uniquely: We know both fixed points and the order of B_2.

Type II. The edge a_k connects two distinct vertices of $\mathcal{G}_k$ but does not disconnect the graph (edge a_4 in Figure 9). We do an HNN construction to produce a subgroup of G_k of type $(0,4)$: The special points on $\Omega(G_{k-1})/G_{k-1}$ that will be used in the gluing construction are the projections of the left fixed points of two elements A_1 and A_2. As in Type I we can assume that A_1 and A_2 belong to triangle groups F_1 with canonical generators A_1 and B_1 and F_2 with canonical generators A_2 and B_2 used in the construction of G_{k-1}. We have to find a loxodromic $C \notin G_k$ that satisfies

$$CA_2^{-1}C^{-1} = A_1,$$

and the group

$$\langle A_1, B_1\rangle *_{A_1} \langle CA_2C^{-1}, CB_2C^{-1}\rangle$$

has gluing coordinate τ.

This reduces to solving for the Möbius transformation C that satisfies

$$\begin{aligned}\operatorname{fix}_L A_2 &\mapsto \operatorname{fix}_R A_1,\\ \operatorname{fix}_R A_2 &\mapsto \operatorname{fix}_L A_1\end{aligned}$$

and

$$\tau^2 = \operatorname{cr}(C(\operatorname{fix}_L B_2), \operatorname{fix}_L A_1, \operatorname{fix}_R A_1, \operatorname{fix}_R B_1)\frac{\tanh(d_3/2)}{\tanh(d_3'/2)}.$$

Set

$$G_k = \langle G_{k-1}, C\rangle.$$

Type III. The edge a_k connects a vertex S_j of the graph with itself. This corresponds to the HNN extension of a triangle group as presented in Section 3.3. Find $A, B \in G_{k-1}$ corresponding to the special points involved in the gluing, such that they generate canonically a triangle group $F \subset G_{k-1}$ and their left fixed points are in the invariant component Δ. Solve for a loxodromic C satisfying

$$\begin{aligned}C(\operatorname{fix}_L B) &= \operatorname{fix}_R A,\\ C(\operatorname{fix}_R B) &= \operatorname{fix}_L A\end{aligned}$$

and

$$\tau^2 = \operatorname{cr}(C(\operatorname{fix}_R(A)), \operatorname{fix}_L(A), \operatorname{fix}_R(A), \operatorname{fix}_L(B)).$$

Set

$$G_k = \langle G_{k-1}, C\rangle.$$

If we drop the requirement that our group must have an invariant component, we can reduce Type I to a HNN construction similar to Types II and III in the special case when the subgraph of type $(0,4)$ has equal weights at both ends. In the case of parabolic cyclic subgroups similar constructions are used by Earle and Marden [8] and Kerckhoff and Thurston [10]. We have

Type I'. Find A and B as in case I. Solve for the Möbius transformation C that satisfies

$$CA_2^{-1}C^{-1} = A_1$$

and

$$\tau^2 = \operatorname{cr}(C(\operatorname{fix}_R B_2), \operatorname{fix}_L A, \operatorname{fix}_R A, \operatorname{fix}_R B).$$

Set

$$G_k' = \langle G_{k-1}, C\rangle.$$

If we use construction I' instead of I we can get the Koebe group by taking the stabilizer of a component that projects to the Riemann surface $S_{\mathbf{t}}$.

In the following section we use the simple observations of Sections 2 and 5.1 in the setting of deformation spaces.

6. Deformation spaces of Koebe groups constructed from hyperbolic triangle groups

In this section we find a condition on the gluing parameters associated to the edges of a weighted graph $\mathcal{G}$ that are sufficient to guarantee that the group G constructed by the algorithm of Section 5, even allowing parabolic gluings, will give a Koebe group of the correct analytical type. We start with the observation that an analog for Maskit's embedding theorem ([16],[12]) for terminal regular b-groups can be proved for the class of Koebe groups constructed from hyperbolic triangle groups, and that the proof in [12] applies in this case as well. We also use the results of the previous section and those of Kra [13] and Arés [3] to find a non-empty open set in the deformation space of the groups constructed from a collection of hyperbolic triangle groups by AFP and HNN constructions. First we review some basic definitions and results on the structure of Koebe groups from [17], Chapter X.

Let G be a finitely generated Koebe group constructed from hyperbolic triangle groups. There is a maximal collection $\Sigma = \{A_1, \dots, A_d\}$ of equivalence classes of elliptic and parabolic elements in G that correspond to simple closed geodesics $\{\alpha_1, \dots, \alpha_d\}$ on the quotient surface Δ/G. By construction Σ is a *maximal partition* of Δ/G: the components of $\Delta^\circ/G \setminus (\cup_j \alpha_j)$ are topologically spheres with three holes. The connected component of $\Delta/G \setminus (\cup_{j\neq i}\alpha_j)$ containing α_i is the *modular part* T_i of α_i.

The components of the preimage of $\Delta^\circ/G \setminus (\cup_j \alpha_j)$ are *structure regions.* Each structure region is stabilized by a *structure subgroup* of G. In the groups considered here the structure subgroups are conjugates of the triangle groups used in the construction.

The common boundary of two adjacent structure regions is a component of the preimage of one of the curves $\alpha_i \in \Sigma$. The preimages of modular parts are *modular regions*, and their stabilizers are modular subgroups of G. If the modular part is of type $(1,1)$, then the modular group is an HNN extension of a triangle group stabilizing a structure region in the modular group. For type $(0,4)$ the modular groups are AFP:s of two structure subgroups across an elliptic or parabolic cyclic subgroup generated by a conjugate of A_i.

We want to define coordinates on the deformation space of a Koebe group G of the type described above. The following theorem is a fundamental observation:

Theorem 2 (The Maskit Embedding Theorem). *Let G be a finitely generated Koebe group constructed from hyperbolic triangle groups. Let $G_1, \ldots, G_d$ be a maximal collection of non-conjugate modular subgroups of G. Then the map*

$$m\colon \mathbf{T}(G) \to \prod_{i=1}^{d} \mathbf{T}(G_i)$$

defined by

$$\mathbf{T}(G) \ni [w] \mapsto ([w], \ldots, [w]) \in \prod_{i=1}^{d} \mathbf{T}(G_i)$$

is injective.

Proof. The proof given by Kra in [12] for the case of b-groups (the Maskit embedding of Teichmüller space) applies also in this setting. Let $A_1, \ldots, A_d \in G$ be a maximal collection of non-conjugate primitive elliptic or parabolic elements corresponding to simple closed curves on the Riemann surface $\Delta(G)/G$. Now the proof in [12] gives the theorem. See Arés [4] for a discussion of the theorem. □

This result means that we can use the coordinates of the 1–dimensional deformation spaces to define coordinates on the deformation space of any finitely generated Koebe group constructed from hyperbolic triangle groups. For elliptic gluing we use the 1-dimensional gluing coordinates (Definitions 3.1 and 3.2). For parabolic gluings we recall the definition of horocyclic coordinates of 1-dimensional Teichmüller spaces from [13] and [3]:

6.1. Terminal b-groups. Let G be a Kleinian group. G is a *b-group* if it has a simply connected invariant component Δ. G is called *terminal* if Δ/G is a Riemann surface of type (p,n) and $(\Omega(G) \setminus \Delta)/G$ is the disjoint union of $2p+n-2$ spheres with three special points (some of which must be punctures) with hyperbolic signatures. In this subsection we briefly review the parametrization of 1-dimensional deformation spaces of terminal b-groups following [13] and [3].

Terminal b-groups of type $(0,4)$ can be constructed as AFP's of two triangle groups across a common maximal cyclic parabolic subgroup. Let $\Gamma_1 = \langle A_0, (B_1)_0 \rangle$ and $\Gamma_2 = \langle A_0^{-1}, (B_2)_0 \rangle$ be triangle groups acting in $\mathbb{H}$ and $\mathbb{H}^*$ with $A_0(z) = z + 2$ parabolic. Let

$$\Gamma_2(\tau) = T\Gamma_2T^{-1},$$

for $T(z) = z + \tau$, $\operatorname{Im}\tau > 0$.

Definition 6.1. Let A, B_1, and B_2 be a set of generators for a terminal b-group G_1 of type $(0,4)$ that satisfies

(1) (A, B_1) and (A^{-1}, B_2) are pairs of canonical generators for triangle groups of signatures (∞, ν_2, ν_3) and (∞, ν_4, ν_5),
(2) $G_1 = \langle A, B_1\rangle *_A \langle A^{-1}, B_2\rangle$,
(3) the part of Δ/G_1 corresponding to $\langle A, B_1\rangle$ lies to the right of the dividing curve $a \subset \Delta/G_1$ corresponding to A with the orientation induced by the action of A on $\widehat{\mathbb{C}}$.

Let $[w] \in \mathbf{T}(G_1)$. Then the *horocyclic coordinate* of the deformation space $\mathbf{T}(G_1(\tau_0))$ is

$$\begin{aligned}\tau([w]) &= \operatorname{cr}(w \operatorname{fix}_R B_2, w \operatorname{fix} A, w \operatorname{fix}_L B_1, w \operatorname{fix}_R AB_1)c_1 + c_2, \\ &= \tau_0([w])c_1 + c_2,\end{aligned} \tag{6.1}$$

where

$$c_1 = \operatorname{fix}_R(AB_1)_0 - \operatorname{fix}_L(B_1)_0,$$

and

$$c_2 = \operatorname{fix}_L(B_1)_0 - \operatorname{fix}_R(B_2)_0.$$

The expression τ_0 can clearly be used as a coordinate on $\mathbf{T}(G_1)$, the normalization by c_1 and c_2 is made to conform with [3]. The constants c_1 and c_2 are determined by the geometry of the spheres. If B_1, B_2 or AB_1 is parabolic, (6.1) applies with the left and right fixed points replaced by the parabolic fixed point. Also, the coordinate τ has a simple relation to the $zw = t$ gluing parameter: The horocyclic coordinate on $\mathbb{H}/\Gamma_1$ at ∞ is

$$z(\zeta) = e^{i\pi\zeta},$$

and the horocyclic coordinate at ∞ on $(\mathbb{H}^* + \tau)/\Gamma_2(\tau)$ is

$$w(\zeta) = e^{-i\pi(\zeta-\tau)}.$$

Thus, repeating the argument used in Section 3.2 for elliptic gluing, we see that the gluing parameter is

$$t = e^{i\pi\tau}.$$

See Kra [13], Section 6, and Arés [3], Section 3.2, for more details on parabolic gluing.

Terminal b-groups of type $(1,1)$ can be constructed as HNN extensions of triangle groups across maximal cyclic parabolic subgroups.

Definition 6.2. Let A, B and C be a set of generators of a terminal b-group G_2 of type $(1,1)$ such that

(1) A and B canonically generate a triangle group,
(2) the puncture used in the HNN construction lies to the left from the curve corresponding to A, and
(3) $CB^{-1}C^{-1} = A$.

Let $[w] \in \mathbf{T}(G_2)$. The *horocyclic coordinate* of $[w]$ is

$$\tau([w]) = \operatorname{cr}(wC(\operatorname{fix} A), w \operatorname{fix} A, w \operatorname{fix} B, w \operatorname{fix}_R AB) \operatorname{fix}_R(AB)_0. \tag{6.2}$$

As in the case of the parabolic AFP construction (Definition 6.1 above), we had to introduce a geometric constant $\operatorname{fix}_R AB$ in order to have the relation

$$t = e^{i\pi\tau},$$

as in [13] and [3].

Remark. The cross ratio expressions used here are different from the ones used in [3] to define horocyclic coordinates on 1-dimensional deformation spaces, but both define the same coordinates. This difference is not essential: Arés uses his parametrization of triangle groups, whereas we prefer to use expressions of cross ratios of fixed points and constants that depend on the geometry of the spheres (with punctures and elliptic special points) that are glued in the construction.

6.2. Gluing coordinates of deformation space. We now treat the general case of gluing coordinates. The algorithm in the general case is the same as in Section 5 appended with parabolic gluing using the parametrization introduced in Definitions 6.1 and 6.2.

Definition 6.3. Let $\mathcal{G}$ be a weighted graph and $G = G(\mathcal{G})$ a Koebe group constructed from triangle groups by the algorithm described above. Let

$$\mathcal{G}_N = \{a_1, \dots a_d\}$$

be the set of non-phantom edges of $\mathcal{G}$ and $d = \#\mathcal{G}_N$. The *gluing coordinate* of $\mathbf{T}(G)$ is the map

$$\tau\colon \mathbf{T}(G) \to \mathbb{C}^d$$

defined by

$$\tau = \tau^d \circ m,$$

where $m\colon \mathbf{T}(G) \to \prod \mathbf{T}(G_1)$ is the Maskit embedding of Theorem 2, and

$$\tau^d\colon \prod_{i=1}^{d} \mathbf{T}(G_i) \to \mathbb{C}^d$$

consists of the appropriate 1-dimensional coordinate maps defined in Definitions 3.1, 3.2, 6.1 and 6.2.

Formula 2.5 gives us a well-defined choice of maximal radii of non-intersecting orbifold disks on a sphere with three special points of finite order. This observation along with the results of Section 5 allows us to estimate the set of parameters that correspond to a tame plumbing construction.

Definition 6.4. Let $\mathcal{G}$ be a weighted graph and let $h(\nu,\mu)$ be the horocyclic radius of Lemma 2.5. If an edge a with weight w connects two vertices of weights (w,ν_1,ν_2) and (w,ν_3,ν_4), the *safe radius* of the edge a is

$$s(\mathcal{G},a) = \coth\frac{r(w;\nu_1,\nu_2)}{2}\coth\frac{r(w;\nu_3,\nu_4)}{2}, \tag{6.3}$$

if $w<\infty$ and

$$h(\mathcal{G},a) = h(\nu_1,\nu_2)\;h(\nu_3,\nu_4), \tag{6.4}$$

if $w=\infty$. If a connects a vertex with weights (w,w,ν_3) to itself,

$$s(\mathcal{G},a) = \coth^2\frac{r(w;w,\nu_3)}{2} = \coth^2\frac{d_3}{4}, \tag{6.5}$$

for $w<\infty$, where d_3 is the distance between the elliptic special points of order w, and

$$h(\mathcal{G},a) = h(\nu_1,\nu_2)^2 \tag{6.6}$$

for $w=\infty$.

Lemma 5.1 generalizes trivially to the case of horocyclic coordinates.

Lemma 6.1. *Let S and S' be spheres with three special points, with signatures (∞,ν_2,ν_3) and (∞,ν_4,ν_5). Let $P\in\overline{S}$ and $Q\in\overline{S'}$. If $t\in\mathbb{C}$ satisfies*

$$|t| < h(\nu_2,\nu_3)\;h(\nu_4,\nu_5), \tag{6.7}$$

then it is possible to do the plumbing construction for the parameter t at the punctures P and Q, and the construction is limited inside the punctured disks of horocyclic radii h_1 and h_2 at P and Q, respectively.

Now we can state the main result of the section:

Theorem 3. *Let G be a Koebe group constructed using a weighted graph $\mathcal{G}$. Denote by $\mathcal{G}_N$ the set of non-phantom edges. Then*

$$\prod_{a\in\mathcal{G}_N} S_a \subset \tau(\mathbf{T}(G)),$$

where

$$S_a = \mathbb{C}\setminus\mathbb{D}(0, s(\mathcal{G}, a)),$$

if the weight of the edge a is finite, and

$$S_a = \mathbb{H} - \frac{i}{\pi}\log h(\mathcal{G}, a),$$

if a has infinite weight.

Proof. The condition that the gluing parameters are in $\prod S_a$ is exactly the criterion of Lemmas 5.1 and 6.1 for the plumbing constructions to be tame. The result follows from the calculations in Sections 3.2, 3.4 and 6.1 relating τ and t for each gluing type.

7. Examples of the use of the construction algorithm

In this section we illustrate the the use of Theorem 4 and the construction algorithm of Koebe groups. We also find maximally pinched groups in the boundaries of the deformation spaces of Koebe groups of type $(2,0)$. There are two different graph types to consider:

7.1. Example 1. Let $\mathcal{G}$ be the weighted graph in Figure 10. Start by forming the AFP of a hyperbolic triangle group $\Gamma = \langle A, B\rangle$ of signature (ν_1,ν_2,ν_3) with a conjugate $\Gamma(\tau_1^2) = \langle A^{-1}, B_{\tau_1^2}^{-1}\rangle$ of the same group across the generator A of order ν_1. We have (using the normalization of Section 2)

$$A = \begin{pmatrix} e^{-i\pi/\nu_1} & 0 \\ 0 & e^{i\pi/\nu_1}\end{pmatrix},$$

$$B = \begin{pmatrix} i\sin(\pi/\nu_2)\cosh d_3 - \cos(\pi/\nu_2) & -i\sin(\pi/\nu_2)\sinh d_3 \\ i\sin(\pi/\nu_2)\sinh d_3 & -i\sin(\pi/\nu_2)\cosh d_3 - \cos(\pi/\nu_2)\end{pmatrix},$$

and

$$B_{\tau_1^2} = \begin{pmatrix} i\sin(\pi/\nu_2)\cosh d_3 - \cos(\pi/\nu_2) & -i\tau_1^2\sin(\pi/\nu_2)\sinh d_3 \\ i\tau_1^{-2}\sin(\pi/\nu_2)\sinh d_3 & -i\sin(\pi/\nu_2)\cosh d_3 - \cos(\pi/\nu_2)\end{pmatrix}.$$

$G_1 = \langle \Gamma, \Gamma(\tau_1^2) \rangle$ is a (non-Fuchsian) Koebe group that represents a sphere with four special points, two of order ν_2 and two of order ν_3. This is an operation of type I.

Then we perform two operations of type II. First we glue together annuli around the special points corresponding to the generators B and $B_2 = B_{\tau_1^2}^{-1}$: The structure subgroups F_1 and F_2 in this case with their canonical generators are

$$F_1 = \langle B_{\tau_1^2}^{-1}, B_{\tau_1^2} A \rangle,$$

and

$$F_2 = \langle B, (AB)^{-1} \rangle.$$

To realize the gluing, we look for a loxodromic element C_1 that satisfies

$$C_1 B C_1^{-1} = B_{\tau_1^2}$$

and the cross ratio condition

$$\tau_2^2 = \operatorname{cr}(C_1(\operatorname{fix}_L (AB)^{-1}), \operatorname{fix}_L B_{\tau_1^2}^{-1}, \operatorname{fix}_R B_{\tau_1^2}^{-1}, \operatorname{fix}_R B_{\tau_1^2} A).$$

For the final gluing, we take

$$F_1 = \langle B_{\tau_1^2} A, A^{-1} \rangle,$$

and

$$F_2 = \langle (AB)^{-1}, A \rangle.$$

We then need to solve the set of equations given by

$$C_2 AB C_2^{-1} = B_{\tau_1^2} A$$

and

$$\tau_3^2 = \operatorname{cr}(C_2(\operatorname{fix}_L A), \operatorname{fix}_L B_{\tau_1^2} A, \operatorname{fix}_R B_{\tau_1^2} A, \operatorname{fix}_R (A)^{-1}).$$

As a result of these calculations we get

$$C_1 = \begin{pmatrix} \dfrac{\tau_1}{2\tau_2}(-1 - \tau_2^2 + (\tau_2^2 - 1)\cosh d_3) & \dfrac{\tau_1(1 - \tau_2^2)\sinh d_3}{2\tau_2} \\ -\dfrac{(1 - \tau_2^2)\sinh d_3}{2\tau_1\tau_2} & \dfrac{-1 - \tau_2^2 - (\tau_2^2 - 1)\cosh d_3}{2\tau_1\tau_2} \end{pmatrix}$$

and

$$C_2 = \begin{pmatrix} \dfrac{e^{i\pi/\nu_1}\tau_1}{2\tau_3}(1 + \tau_3^2 + (1 - \tau_3^2)\cosh d_2) & \dfrac{\tau_1(\tau_3^2 - 1)\sinh d_2}{2\tau_3} \\ -\dfrac{(\tau_3^2 - 1)\sinh d_2}{2\tau_1\tau_3} & e^{-i\pi/\nu_1}\dfrac{1 + \tau_3^2 + (\tau_3^2 - 1)\cosh d_2}{2\tau_1\tau_3} \end{pmatrix}.$$

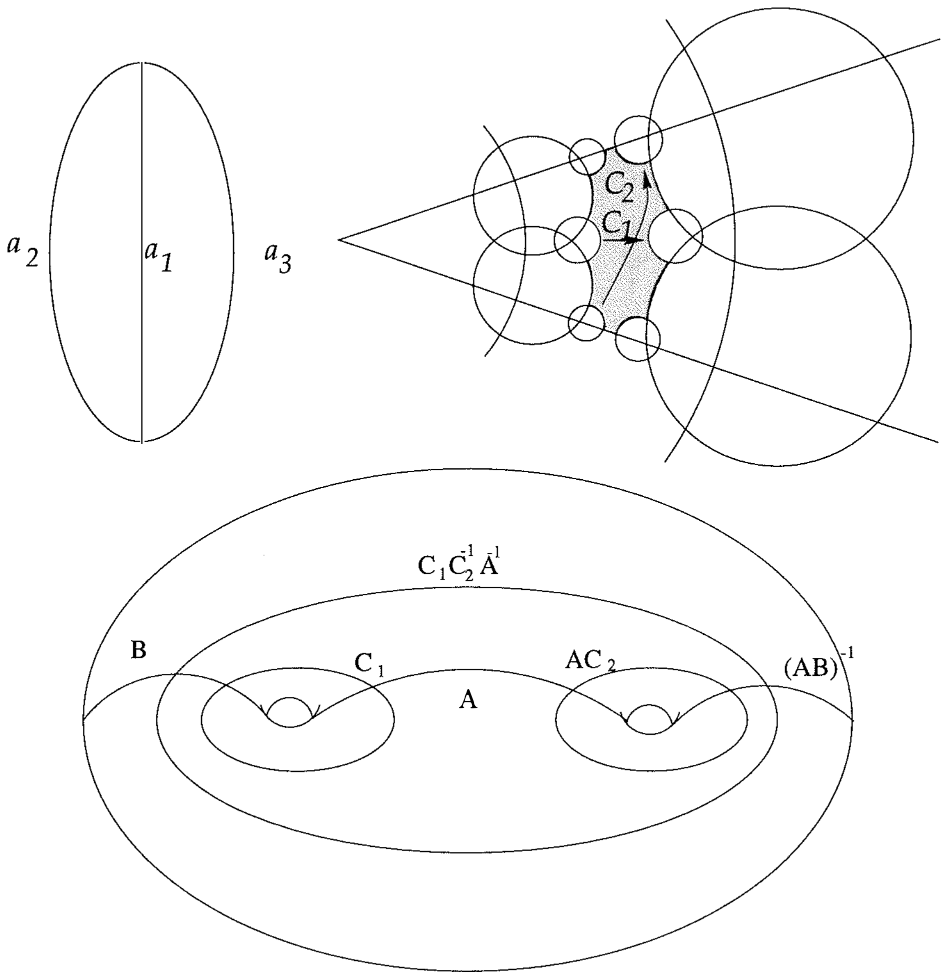


Figure 10. A weighted graph of genus 2 and a fundamental domain of the action of the group of Example 1 in its invariant component. The corresponding Riemann surface has a pants decomposition induced by the elliptic elements A, B and AB. The boundary point of deformation space found in Example 1 corresponds to a Kleinian group representing a noded Riemann surface obtained by pinching the curves C_1, AC_2 and $C_1C_2^{-1}A^{-1}$ to points.

Theorem 3 now gives lower bounds for the absolute values of the parameters

$\tau_1^2, \tau_2^2, \tau_3^2$ for which the group

$$G = \langle A, B, B_2, C_1, C_2 \rangle$$

is a Koebe group of type $(2,0)$.

7.2. Example 2. For the remaining graph type (see Figure 11), we start the construction as in Example 1 with a construction of type I and follow it with two HNN constructions (type III). Since we are going to do type III constructions, the amalgamated groups must have signatures (ν_1, ν_2, ν_2) and (ν_1, ν_3, ν_3). Normalize A and B as in Example 1. We get just as in Example 1

$$B_2 = \begin{pmatrix} -i\sin(\pi/\nu_4)\cosh d_3' - \cos(\pi/\nu_4) & i\tau_1^2 \sin(\pi/\nu_4)\sinh d_3' \\ -\left(\dfrac{i}{\tau_1}\right)^2 \sin(\pi/\nu_4)\sinh d_3' & i\sin(\pi/\nu_4)\cosh d_3' - \cos(\pi/\nu_4) \end{pmatrix}.$$

Next we glue together annular neighborhoods of the special points determined by the left fixed points of $(BA)^{-1}$ and B. This is done by adding a loxodromic generator C_1 that satisfies:

$$C_1 B^{-1} C_1^{-1} = (BA)^{-1}$$

and

$$\tau_2^2 = \operatorname{cr}(C_1(\operatorname{fix}_R(BA)^{-1}), \operatorname{fix}_L(BA)^{-1}, \operatorname{fix}_R(BA)^{-1}, \operatorname{fix}_L B).$$

To do the final gluing we repeat the procedure of the second gluing for the structure subgroup generated by AB_2^{-1} and B_2. We look for C_2 satisfying

$$C_2 B_2^{-1} C_2^{-1} = AB_2^{-1}$$

and

$$\tau_3^2 - \operatorname{cr}(C_2(\operatorname{fix}_R AB_2^{-1}), \operatorname{fix}_L AB_2^{-1}, \operatorname{fix}_R AB_2^{-1}, \operatorname{fix}_L B_2).$$

We get

$$C_1 = \begin{pmatrix} e^{i\pi/2\nu_1} \dfrac{\tau_2^2 - 1}{2i\tau_2} \sinh d_3 & ie^{i\pi/2\nu_1} \dfrac{(\tau_2^2 - 1)\cosh d_3 + \tau_2^2 + 1}{2\tau_2} \\ e^{-i\pi/2\nu_1} \dfrac{(\tau_2^2 - 1)\cosh d_3 - (\tau_2^2 + 1)}{2i\tau_2} & ie^{-i\pi/2\nu_1} \dfrac{\tau_2^2 - 1}{2\tau_2} \sinh d_3 \end{pmatrix}$$

and

$$C_2 = \begin{pmatrix} e^{-i\pi/2\nu_1} \dfrac{\tau_3^2 - 1}{2i\tau_3} \sinh d_3' & i{\tau_1}^2 \dfrac{(\tau_3^2 - 1)\cosh d_3' - (\tau_3^2 + 1)}{2e^{i\pi/2\nu_1}\tau_3} \\ e^{i\pi/2\nu_1} \dfrac{(\tau_3^2 - 1)\cosh d_3' + \tau_3^2 + 1}{2i{\tau_1}^2\tau_3} & ie^{i\pi/2\nu_1} \dfrac{\tau_3^2 - 1}{2\tau_3} \sinh d_3' \end{pmatrix}.$$

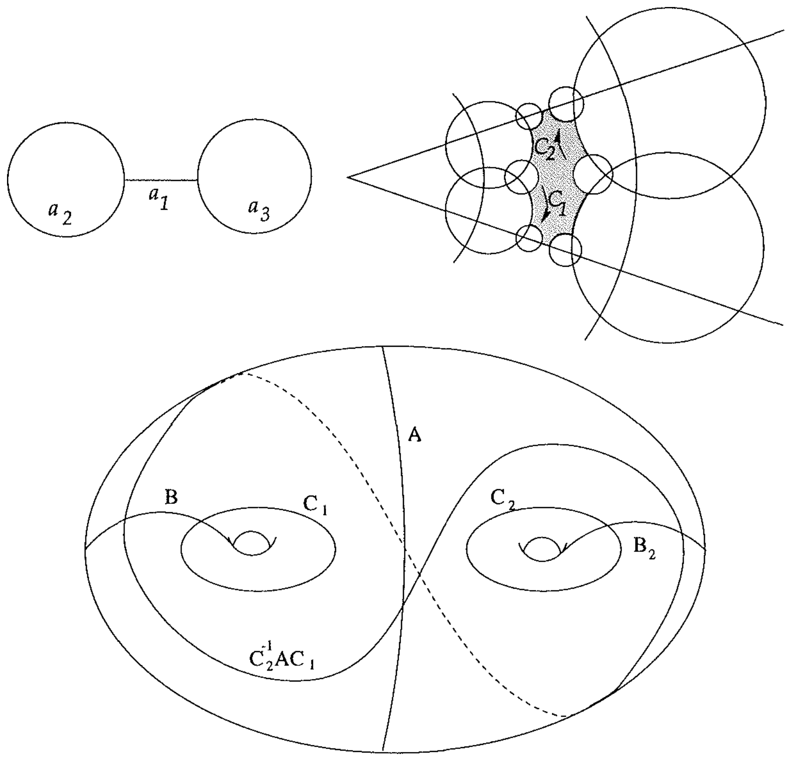


Figure 11. A weighted graph of genus 2 and the partitions of a genus 2 surface as in Example 2.

7.3. A variation of Example 1. In some cases the coordinates (τ_1, τ_2, τ_3) have a 3-dimensional geometric interpretation: Let G_0 be the group of Example 1, and

$$M_0 = \mathbb{H}^3/G.$$

M_0 is a hyperbolic 3-orbifold with three boundary components corresponding to $\Omega(G_0)/G_0$. The orbifold has a simple topological structure:

$$M_0 \setminus (\pi(\mathrm{Ax}\,A \cup \mathrm{Ax}\,B \cup \mathrm{Ax}\,AB)) \cong S(2,0) \times (0,1),$$

where $\pi\colon \mathbb{H}^3 \to M_0$ is the natural projection, $\mathrm{Ax}\,E$ is the axis of the loxodromic or elliptic Möbius transformation E in $\mathbb{H}^3$, and $S(2,0)$ is a surface of genus 2. The two boundary components of type $(0,3)$ have no moduli, so it is easy to form a new orbifold from M_0 by a gluing construction: We can cut out the ends of type $(0,3)$ and glue the resulting boundary components of the convex core of M_0 by an orientation reversing isomorphism of the surfaces. On the level of Kleinian groups this corresponds to adding a new element

$$C_0 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1^{-1} \end{pmatrix},$$

to the group G_0. Clearly, C_0 conjugates Γ with $\Gamma(\tau_1^2)$ and satisfies the conditions of Maskit's second combination theorem ([17], Theorem VII.E.5) for $G = \langle G_0, C_0\rangle$ to be the HNN extension of G_0 by C_0. If F_0 is a fundamental polyhedron of G_0 in $\mathbb{H}^3$, then

$$F_0 \cap \{x \in \mathbb{H}^3 \mid 1 < |x| < \tau_1^2\}$$

is a fundamental polyhedron for G. Similarly, if D is a fundamental set for G_0 in $\Omega(G_0)$, then $D \cap \Delta(G_0)$ is the fundamental set of G in $\Omega(G)$. Thus $M = \mathbb{H}^3/G$ has only one boundary component, $\Omega(G)/G = \Delta(G_0/G_0)$ and G does not have an invariant component.

M has three circles in its *orbifold locus.* These are the projections of the axes of the three G-equivalence classes of elliptic elements:

$$\mathrm{Stab}(\mathrm{Ax}\, A) = \langle A, C_0\rangle,$$
$$\mathrm{Stab}(\mathrm{Ax}\, B) = \langle B, C_0^{-1}C_1\rangle,$$

and

$$\mathrm{Stab}(\mathrm{Ax}\, AB) = \langle AB, C_0^{-1}AC_2\rangle.$$

The complex translation lengths of C_0, $C_0^{-1}C_1$ and $C_0^{-1}AC_2$ are τ_1, τ_2 and τ_3. This means that these axes are projected into $\mathbb{H}^3/G$ as circles length and "twisting" determined by the parameters (τ_1, τ_2, τ_3).

In the torsion free case the complex translation lengths are replaced by the moduli of the three cusp tori.

8. Boundary points

In this section we find maximally pinched boundary points on the boundaries of the deformation spaces of the groups treated in Examples 1 and 2. These boundary points correspond to the $zw = t$ plumbing construction for degenerated pairs of pants constructed by cutting away orbifold disks and punctured disks of maximal radii given by (2.6) – (2.11) from spheres with three special points. The observations made in the case of genus 2 are then used to prove a general theorem that gives a "non-trivial" boundary point of the deformation space of any Koebe group G constructed from hyperbolic triangle groups with the genus of $\Delta(G)/G$ at least 1.

In Example 1 the safe radius for the gluing corresponding to the edge a_i in the graph $\mathcal{G}$ is

$$s(\mathcal{G}, a_i) = \coth^2 \frac{r(\nu_i; \nu_j, \nu_k)}{2} = \coth^2\left(\frac{d_j + d_k - d_i}{4}\right).$$

The choice of safe radii is good in the following sense: For parameters inside $\mathbf{T}(G)$, the transformations C_1, AC_2, $C_1C_2^{-1}A^{-1}$ and $C_2AC_1^{-1}$ are all loxodromics corresponding to simple closed curves on Δ/G (see Figure 10). It is easy to check that all these transformations become parabolic simultaneously for the gluing parameters

$$(\tau_1^2, \tau_2^2, \tau_3^2) = (s(\mathcal{G}, a_1), s(\mathcal{G}, a_2), s(\mathcal{G}, a_3)).$$

The invariant component splits into an invariant collection of disks. Each of these disks is stabilized by a conjugate of one of the torsion-free triangle groups

$$\Gamma_1' = \langle C_1, (AC_2)^{-1} \rangle$$

and

$$\Gamma_2' = \langle C_2A, C_1^{-1} \rangle.$$

The groups Γ_1' and Γ_2' are not conjugate in $G(\tau_1^2, \tau_2^2, \tau_3^2)$. Let $\Delta(\Gamma_i)$ be the component of $\Omega(G)$ stabilized by Γ_i. The parabolic cyclic subgroups of Γ_1' corresponding to the three punctures on $\Delta(\Gamma_1')/\Gamma_1'$ are conjugate to the subgroups corresponding to punctures on $\Delta(\Gamma_2')/\Gamma_2'$:

$$C_1 \in \Gamma_1' \cap \Gamma_2'$$

is a parabolic corresponding to a puncture on both $\Delta(\Gamma_1')/\Gamma_1'$ and $\Delta(\Gamma_2')/\Gamma_2'$, while the other parabolic conjugacy classes of parabolics in Γ_1' are conjugate with those of Γ_2' by different elements of G:

$$AC_2 = A(C_2A)A^{-1},$$

and

$$C_2AC_1^{-1} = B_{\tau_1^2}(C_1C_2^{-1}A^{-1})^{-1}B_{\tau_1^2}^{-1}.$$

As in Section 3.6 we can form a new invariant set Δ^+ from a collection of components of the set of discontinuity and the fixed points of these parabolics:

$$\Delta^+ = G\left(\Delta(\Gamma_1') \cup \Delta(\Gamma_2') \cup \{\operatorname{fix} C_1, \operatorname{fix} C_2, \operatorname{fix} C_1C_2^{-1}A^{-1}\}\right).$$

Thus $(\tau_1^2, \tau_2^2, \tau_3^2)$ can be interpreted as a boundary point of the deformation space $\mathbf{T}(G)$ corresponding to a noded Riemann surface of genus 2 with three nodes.

Similarly, in Example 2 we see that the estimate of Theorem 3 is good: If we take real gluing parameters

$$(\tau_1^2, \tau_2^2, \tau_3^2) = (s(\mathcal{G}, a_1), s(\mathcal{G}, a_2), s(\mathcal{G}, a_3)),$$

the generators C_1 and C_2 become parabolic and a small calculation shows that also the element $C_2^{-1}AC_1$ is a parabolic.

We know from the Remark in Section 5.1 and the results of Section 3.6 that the safe radii are sharp estimates for the gluings of type $(1,1)$. Again, the invariant component splits into two non-equivalent families of disks, each disk stabilized by a conjugate of either

$$\Gamma' = \langle C_1^{-1}A^{-1}, C_2\rangle,$$

or

$$\Gamma'' = \langle C_2^{-1}A, C_1\rangle.$$

Each puncture on the sphere corresponding to Γ' is identified by a puncture on the other sphere:

(1) $C_1 = BC_1AB^{-1}$,
(2) $C_2 = B_2A^{-1}C_2B_2^{-1}$, and
(3) $C_2^{-1}AC_1$ corresponds to a puncture on both surfaces.

The method used here to find boundary points of the deformation spaces of Koebe groups without parabolics can (with the obvious modifications) be used for the deformation spaces Koebe groups constructed from any weighted graph with hyperbolic vertices. For G a terminal b-group of the same graph type as in Example 1 above, the method shows that the estimate

$$(\mathbb{H}+2i)^3 \subset \tau(\mathbf{T}(G))$$

([13] Theorem 8.6) is the best possible: G can be generated by

$$G = \langle A, B, B_2, C_1, C_2\rangle,$$

with A, B, AB parabolic, and where the generators satisfy relations analogous to those of the generators of the group in Example 1. (See Kra [13], Section 7.5 "Two illustrative examples" for a more detailed description of the example.) The generators can be normalized as

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -1+2\tau_1 & -2\tau_1^2 \\ 2 & -1-2\tau_1 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} \tau_1(\tau_2-2)+1 & \tau_1 \\ \tau_2-2 & 1 \end{pmatrix},$$

and

$$C_2 = \begin{pmatrix} (\tau_1-1)\tau_3+1 & (1-\tau_1)\tau_3+\tau_1-2 \\ \tau_3 & 1-\tau_3 \end{pmatrix}.$$

Theorem 4 guarantees that $(\tau_1,\tau_2,\tau_3) \in \mathbf{T}(G)$, if $\operatorname{Im}\tau_i \geq 2$. It is now easy to check that for the parameter

$$(\tau_1,\tau_2,\tau_3) = (2i, 2+2i, 2i)$$

the transformations C_1, AC_2, $C_1C_2^{-1}A^{-1}$ and $C_2AC_1^{-1}$ are parabolic and that the quotient space consists of four thrice punctured spheres.

In all these examples of boundary groups, the first AFP construction produces a non-singular Riemann surface of type $(0,4)$ and the third gluing operation creates two nodes at once. In Example 1 the modular subgroups of the groups corresponding to points in Teichmüller space are all Koebe groups of type $(0,4)$. Let G_b be the boundary group of the graph type of Example 1 constructed above. The subgroups of G_b corresponding to the modular subgroups of G are all Koebe groups of type $(0,4)$ although the set of discontinuity of G_b is a disjoint union of disks stabilized by triangle groups. In the general case we do not necessarily get maximally pinched boundary groups even for compact Riemann surfaces. However, we have the following result:

Theorem 4. *The estimate of Theorem 3 is strict for groups of type* (p,n) *with* $p \geq 1$.

Proof. If the graph of the group contains a subgraph of type $(1,1)$, the statement of the theorem follows from the Remarks in Section 5.1. If this is not the case, there is a subgraph of type $(1,n)$ for some $n \geq 2$. This graph corresponds to a subgroup constructed by $n-1$ operations of type I and an operation of type II.

First we need to fix the local coordinates on the spheres to be glued in the construction: Let the special points on the sphere S_i be P_i^1, P_i^2 and P_i^3, named in such a way that P_i^3 will be a special point on the surface resulting from the gluing, that is, only P_i^1 and P_i^2 are used in the gluing construction corresponding to the subgraph of type $(1,n)$. Let

$$\tau = (\tau_1, \ldots, \tau_n)$$

be a gluing parameter with

$$\tau_i = s(\mathcal{G}, a_i),$$

if the ith gluing is elliptic, and

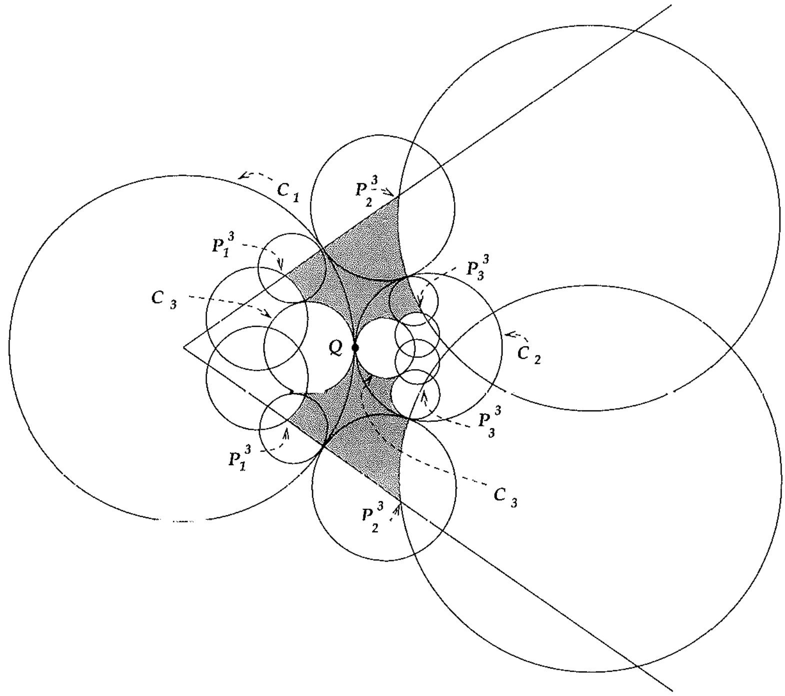
$$\tau_i = \frac{i}{\pi} \log h(\mathcal{G}, a_i),$$

if the ith gluing is parabolic.

This parameter defines a singular circular polygon with an identification pattern as in Figure 12. The transformation that realizes the HNN extension in the construction of type II fixes the point

$$Q = \bigcap C_i, \tag{8.1}$$

where C_i is the properly chosen component of the lift of the boundary curve of the removed orbifold disk (in the case of elliptic gluing) or punctured disk in the



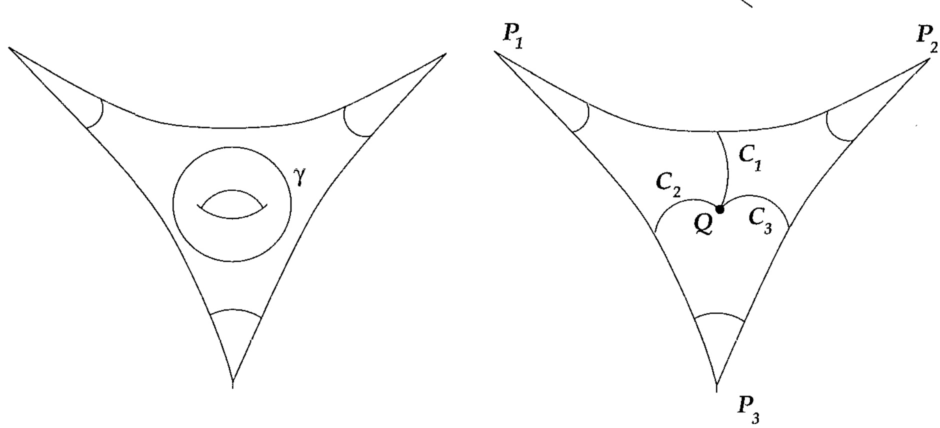


Figure 12. An example of a gluing producing a noded surface representing a boundary point of a deformation space of type $(1,3)$. The points P_1^3, P_2^3 and P_3^3 project to special points on the quotient surface and the point Q projects to a node. The simple closed curve γ on a Riemann surface of type $(1,3)$ has degenerated to the points Q on the surface with nodes.

case of parabolic gluing. The fact that the intersection (8.1) consists of a single point follows from the observation in Section 3.1 about positive gluing parameters.

The group has a finite sided fundamental polyhedron in $\mathbb{H}^3$ that has Q as a boundary point at infinity, so the element fixing Q cannot be loxodromic. Also, the group is an algebraic limit of Kleinian groups corresponding to parameters inside the punctured polydisk given by Theorem 3 and converging to the boundary point. The element corresponding to the one fixing Q is known to be loxodromic in any of the approximating groups, so the element fixing Q must be parabolic. Therefore, it corresponds to a node on the quotient surface and we have found a boundary point of the deformation space. □

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