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Series A

I. MATHEMATICA

DISSERTATIONES

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Lip_h -EXTENSION DOMAINS

VESA LAPPALAINEN

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VESA LAPPALAINEN

Contents

Introduction

Let h be a modulus of continuity and let $\mathrm{Lip}_h(D)$ denote the space of functions $f: D \to \mathbb{R}$ which have the modulus of continuity h. As regards the Hölder classes $h(x,y) = |x-y|^\alpha$, F.W. Gehring and O. Martio [GM2] showed that for certain domains D each function $f: D \to \mathbb{R}$ in a corresponding local space belongs to the space $\mathrm{Lip}_h(D)$. These domains are called Lip_h-extension domains. In this paper we study general moduli of continuity and extend the result of F.W. Gehring and O. Martio to this situation.

F.W. Gehring and O. Martio applied their theory to quasiconformal mappings of $\mathbb{R}^n$. These extension properties can also be applied to imbedding theorems in Sobolev spaces $W^{1,p}(D)$ or even in Orlicz-Sobolev spaces (see [A, Theorem VIII.8.36]), because it can be proved by classical methods that functions f in $W^{1,p}(D)$ are Hölder-continuous with exponent $1-n/p$ ($p > n$) in smooth parts (like cubes or balls) of D (see [A, Section V]). It then follows from [GM2] and from the results of this paper that the functions are actually Hölder-continuous in D for a very large class of domains D. However, it may happen that f does not belong to the same Hölder class in D as in the smooth parts of D.

After some preliminaries we study $\mathrm{Lip}_{h,g}$-extension domains in Section 3. These are domains where locally $\mathrm{Lip}_h(D)$-continuous functions are also $\mathrm{Lip}_g(D)$-continuous functions. A Lip_h-extension domain is then simply a $\mathrm{Lip}_{h,h}$-extension domain. We give an integral condition for $\mathrm{Lip}_{h,g}$-extension domains (analogous to the one in [GM2]). Using the integral condition we show, in Sections 4 and 5, some geometrical properties of $\mathrm{Lip}_{h,g}$-extension domains. We give a sufficient condition for the moduli of continuity h and g such that a Lip_h-extension domain is also a Lip_g-extension domain. We also show that if for a given h there exist Lip_h-extension domains, their class will be larger than the class of uniform domains.

In Section 6 we examine the special case $h(t) = t^{\alpha}$ studied in [GM2]. We show that the class of Lip_{β}-extension domains is larger than the class of Lip_{α}-extension domains if $0 < \alpha < \beta \leq 1$. We also define total extension domains.

In the last section we consider certain theorems discussed in the papers [GM1], [J] and [St]. We give another equivalent condition for $\mathrm{Lip}_{h,g}$-extension domains based on the maximum derivative (see [GM1]).

Most of the notation used in this paper is presented in Appendix A. In Appendix B and C there are some graphical illustrations relating to the examples in Section 6.

1. Preliminaries

For the details of notation refer to Appendix A.

1.1. Notation. We shall write $\gamma(x,y) \subset D$ for a rectifiable curve joining x to y in a domain $D \subset \mathbb{R}^n$.

$\ell(\gamma)$ denotes the arc length of γ and

$\gamma(s)$ its arc length representation with

$\gamma(0) = x$ and

$\gamma(\ell(\gamma)) = y$.

By analogy, $J(x,y)$ denotes the line segment joining x to y. If g is a real valued function in D, we let

$$\int_{\gamma(x,y)} g(z)\, ds \ := \int_0^{\ell(\gamma)} g(\gamma(s))\, ds$$

be the line integral of g along γ (provided the integral exists). Note that the measure ds depends on the curve γ.

1.2. Definition. A domain $D \subset \mathbb{R}^n$ is said to be *c-quasiconvex* if every $x,y \in D$ can be joined by a rectifiable curve γ in D with

$$\ell(\gamma) \le c|x-y| \ .$$

1.3. Definition. Let $r > 0$ and $\gamma(x,y) \subset D$. By $\mathrm{cig}(\gamma,r)$ (*r-cigar neighbourhood* of γ) we mean the set

$$\mathrm{cig}(\gamma,r) := \bigcup_{0<t<\ell(\gamma)} B(\gamma(t), r\cdot\min(t,\ell(\gamma)-t)) \cup \{x,y\}.$$

The euclidean distance from $\gamma(t)$ to the boundary of $\mathrm{cig}(\gamma,r)$ satisfies the inequality

$$d(\gamma(t),\partial\mathrm{cig}(\gamma,r)) \ge r\cdot\min(t,\ell(\gamma)-t) \ . \tag{1.4}$$

1.5. Definition. Let $c \ge 1$. A domain $U \subset \mathbb{R}^n$ is called *c-uniform* if each $x,y \in U$ can be joined by a rectifiable curve γ in U such that

$$\ell(\gamma) \le c|x-y| \ , \tag{1.6}$$

$$\mathrm{cig}(\gamma,1/c) \subset U \ . \tag{1.7}$$

Using hyperbolic geodesics it can be proved that an open ball in $\mathbb{R}^n$ is $\pi/2$ -uniform (see the proof of Theorem 2.2 in [GM2]).

Remark. Uniform domains were defined by O. Martio and J. Sarvas in [MS]. Definition 1.5 for uniform domains and the definition for $\mathrm{cig}(\gamma, r)$ are from an unpublished paper of J. Väisälä's. For other characterizations of uniform domains see [GO] and [M].

2. Modulus of continuity and Lip_h-classes

We can extend all theorems in [GM2] to general moduli of continuity. This is done in Sections 3 and 4. Here we present the definitions and basic properties of the moduli of continuity.

2.1. Definition. A continuous function $h\colon [0,\infty[\to \mathbb{R}$ is said to be a *modulus of continuity* if it satisfies the following conditions:

(2.2) $\quad h(0) = 0, \quad h(t) > 0, \quad t > 0$,

(2.3) $\quad h$ is increasing and

(2.4) $\quad h'$ exists and is decreasing in $]0,\infty[$.

We begin with some results concerning the modulus of continuity. The next theorem is obtained by elementary calculus.

2.5. Theorem. *Let* h *be a modulus of continuity. Then the following conditions are true:*

(2.6) $\quad h'(t)\, t \le h(t) \; , \quad t > 0$,

(2.7) $\quad h(ct) \le c \cdot h(t)$ *for every* $c > 1$, $t > 0$,

(2.8) $\quad \dfrac{h(t)}{t}$ *is decreasing,* $t > 0$,

(2.9) $\quad h(x,y) := h(|x-y|)$ *defines a metric in* $\mathbb{R}^n$. □

Remark. For our purposes, the conditions (2.7) - (2.9) are enough for the modulus of continuity. However, for simplicity we use Definition 2.1 (see [J]).

2.10. Definition. Let $D \subset \mathbb{R}^n$. A function $f\colon D \to \mathbb{R}^p$ belongs to the *Lipschitz class* $\mathrm{Lip}_h(D)$ if there exists a constant $M < \infty$ such that the inequality

$$(2.11) \qquad |f(x)-f(y)| \le Mh(x,y)$$

holds in D.

If $h(t) = t^\alpha$, we shall use the notation $\mathrm{Lip}_\alpha(D)$ instead of $\mathrm{Lip}_h(D)$. The condition (2.11) is called the Lipschitz condition with the modulus of continuity h and a constant M.

2.12. Definition. Let h and g be moduli of continuity and let $D \subset \mathbb{R}^n$ be a domain. We say that g *dominates* h in D and write $h \prec g$ if there is a constant $A < \infty$ such that for each $x,y \in D$

$$h(x,y) \le A \cdot g(x,y) \ .$$

2.13. Lemma. *If* $h \prec g$ *in* $D \subset \mathbb{R}^n$, *then*

$$\mathrm{Lip}_h(D) \subset \mathrm{Lip}_g(D) \ . \qquad \square$$

2.14. Lemma. *If the domain* $D \subset \mathbb{R}^n$ *is bounded and if* $0 < \alpha \le \beta \le 1$, *then*

$$\mathrm{Lip}_\beta(D) \subset \mathrm{Lip}_\alpha(D) \ . \qquad \square$$

We shall use the abbreviation

$$B_b(x) := B(x, b\cdot d(x,\partial D)), \quad b \le 1,$$

for an open ball in D with the radius $b\cdot d(x,\partial D)$ and the centre at $x \in D$.

2.15. Definition. A function $f\colon D \to \mathbb{R}^p$ belongs to the *local Lipschitz class* $\text{loc Lip}_h(D)$ if there exist constants $b > 0$ and $m_b < \infty$ such that for each $x \in D$ and $y \in B_b(x)$

$$(2.16) \qquad |f(x)-f(y)| \le m_b h(x,y) \quad .$$

Usually a Lipschitz (semi)norm of the function $f \in \text{Lip}_h(D)$ is defined to be the smallest constant M for which (2.11) holds. In the class $\text{loc Lip}_h(D)$ the constant m_b depends on the constant b and no smallest m_b exists. However, the constant b is superfluous:

2.17. Theorem. *A function* $f\colon D \to \mathbb{R}^p$ *belongs to the class* $\text{loc Lip}_h(D)$ *if and only if there exists a constant* $m < \infty$ *such that* (2.16) *holds for each* $x \in D$ *and* $y \in B_{1/2}(x)$.

Proof. The sufficiency is immediate. For the necessity let $f \in \text{loc Lip}_h(D)$ with constants $b < 1/2$ and m_b. Fix an open ball $B_{1/2}(x) \subset D$ and a point $y \in B_{1/2}(x)$. Set $r = d(x,\partial D)/2 > |x-y|$ and choose the open balls $B(z_i, br)$, with $i = 0,\dots,k \le 1/b < k+1$ and $z_i = y + ib\cdot(x-y)$. Since $d(z_i,\partial D) > r$, $z_{i-1} \in B_b(z_i)$, and $f \in \text{loc Lip}_h(D)$,

$$(2.18) \qquad |f(y)-f(x)| \le \sum_{i=1}^{k} |f(z_i)-f(z_{i-1})| + |f(z_k)-f(x)|$$

$$\le \sum_{i=1}^{k} m_b h(|z_i-z_{i-1}|) + m_b h(|z_k-x|)$$

$$\le m_b(k+1)\cdot h(|x-y|) \le m_b \frac{b+1}{b}\, h(x,y) = mh(x,y). \qquad \square$$

2.19. Theorem. *A function* $f\colon D \to \mathbb{R}^p$ *belongs to the class* $\operatorname{loc}\operatorname{Lip}_h(D)$ *if and only if there exists a constant* $m < \infty$ *such that for each* $x, y \in B_{1/2}(z) \subset D$

$$(2.20)\qquad |f(x)-f(y)| \le mh(x,y)\ .$$

Proof. The sufficiency is immediate. For the necessity let $f \in \operatorname{loc}\operatorname{Lip}_h(D)$ with constants m and $b = 1/2$. Fix points $x, y \in B_{1/2}(z)$. Set $r = d(z, \partial D)/2$. As in the proof of Theorem 2.17, we can choose the open balls $B(z_i, r/2)$, where $z_i = x + i\cdot(y-x)/4$, $i = 0,\dots,4$. So by repeating (2.18) we obtain

$$|f(x)-f(y)| \le 4mh(x,y). \qquad \square$$

We shall employ both Theorem 2.17 and Theorem 2.19 to characterize the class $\operatorname{loc}\operatorname{Lip}_h(D)$.

Now we can define the $\operatorname{loc}\operatorname{Lip}_h(D)$ seminorm to be the smallest constant m for which (2.20) holds.

Remark. Our definition for the class $\operatorname{loc}\operatorname{Lip}_h(D)$ is not the same as the following definition: For every $z \in D$ there is a neighbourhood V_z and a constant m_z such that

$$(2.21)\qquad |f(x)-f(y)| \le m_z h(x,y) \quad \text{whenever} \quad x, y \in V_z\ .$$

In general the definition is not the same even if we replace the constant m_z by a uniform constant m_D . For instance, functions $h(t) = t^\beta$, $0 < \alpha < \beta < 1$ and $f\colon]0,1[\to \mathbb{R}$, $f(x) := x^\alpha$ give a counterexample of this: If we take $y = ax$, $a < 1$, then the quotient

$$\frac{|f(x)-f(y)|}{h(x,y)} = \frac{x^\alpha(1-a^\alpha)}{x^\beta(1-a)^\beta}$$

tends to infinity if x tends to 0. But (2.21) holds if y is close to a fixed $x > 0$.

We gave the definition for locally Lipschitz-continuous functions by means of the distance from the boundary of D. The following theorem is trivially true.

2.22. Theorem. *If there exists a constant* $m < \infty$ *such that*

$$(2.23) \qquad |f(x)-f(y)| \le m \cdot h(x,y)$$

whenever x *and* y *belong to a ball contained in* D, *then* f *belongs to the class* $\operatorname{loc}\operatorname{Lip}_h(D)$. □

In Section 4 we shall show that $f \in \operatorname{loc}\operatorname{Lip}_h(D)$ implies (2.23) in some cases and hence Theorem 2.22 has a converse. Next, we shall construct a counterexample of a function belonging to the class $\operatorname{loc}\operatorname{Lip}_h(D)$ for which (2.23) does not hold. We start with the following definition.

2.24. Theorem. *The modulus of continuity* h *defines in* D *the metric*

$$(2.25) \qquad h_D(x,y) := \inf_{\gamma(x,y)} \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds$$

and the semimetric

$$(2.26) \qquad h'_D(x,y) := \inf_{\gamma(x,y)} \int_\gamma h'(d(z,\partial D))\, ds \,.$$

Proof. It is obvious that h_D is both positive and symmetric, and that $h_D(x,y) = 0$ if and only if $x = y$. The infimum over all $\gamma(x,y)$ takes care of the triangle inequality. Because $h'(t)$ can be zero even if $t \neq 0$, h'_D is not necessarily a metric. □

2.27. Theorem. $h'_D(x,y) \le h_D(x,y)$.

Proof. The inequality follows from (2.6). □

2.28. Theorem. *Let* $D \subset \mathbb{R}^n$ *be a domain and* $x_0 \in D$. *Then the following functions belong to the class* $\mathrm{loc\,Lip}_h(D)$ *with a constant* $m \le 2$ *:*

$$u_h(x) := h_D(x_0, x) \quad ,$$
$$u_{h'}(x) := h'_D(x_0, x) \quad .$$

Proof. Let $x, y \in B_{1/2}(z_0)$. By the triangle inequality

$$(2.29) \qquad |u_h(x) - u_h(y)| = |h_D(x_0, x) - h_D(x_0, y)| \le h_D(x, y) \ .$$

Let $r := |x-y|/2$. Now

$$d(z, \partial D) \ge r \quad \text{whenever} \quad z \in J(x,y) \ .$$

Then, using (2.8), we obtain

$$(2.30) \qquad h_D(x,y) \le \int_{J(x,y)} \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \le \int_J \frac{h(r)}{r}\, ds$$
$$= 2r \cdot \frac{h(r)}{r} \le 2 \cdot h(x,y) \ ,$$

and hence by combining (2.29) and (2.30) we obtain $u_h \in \mathrm{loc\,Lip}_h(D)$. The same is true for $u_{h'}$ due to Theorem 2.27. □

2.31. Lemma. *There exists a modulus of continuity* h *such that for every* $a > 0$ *there is a point* t_a *with*

$$(2.32) \qquad h'(t) < a \cdot \frac{h(t)}{t} \qquad \text{if} \quad 0 < t \le t_a \ .$$

Proof. Set

$$h(t) := \begin{cases} -\dfrac{1}{\ln t} \ , & t < \dfrac{1}{e} \\ 0 \ , & t = 0 \end{cases} \ .$$

Now

$$h'(t) = \frac{1/t}{\ln^2 t} \geq 0 , \quad \lim_{t\to 0} h(t) = 0 ,$$

and $h(t) > 0$ if $t > 0$. Hence h is a modulus of continuity if we define it in such a way that the conditions of Definition 2.1 hold also when $t \geq 1/e$.

On the other hand,

$$\frac{h'(t)}{h(t)/t} = - \frac{1}{\ln t} = h(t) \leq a \quad \text{if} \quad t \leq e^{-1/a},$$

and hence (2.32) holds. □

2.33. Counterexample. Let h be a modulus of continuity satisfying the conditions of Lemma 2.31. We show that there is a function $u_h \in \text{loc}\,\text{Lip}_h(B(x_0,r))$ which does not belong to the class $\text{Lip}_h(B(x_0,r))$:

Set $B := B(x_0,r)$ and $u_h(x) := h_B(x_0,x)$. Suppose that u_h belongs to the class $\text{Lip}_h(B)$ with a constant M.

Choose $t_M < r$ such that $h'(t) < (1/2M)\cdot h(t)/t$ when $t \leq t_M$, and let $\varepsilon > 0$ be such that

$$2(h(t_M)-h(\varepsilon)) > h(t_M-\varepsilon) .$$

Let x be a point in B with $d(x,\partial B) = \varepsilon$ and y a point in $J(x,x_0)$ with $d(y,\partial B) = t_M$. Obviously the line segments $J(x,x_0)$, $J(x_0,y)$ and $J(x,y)$ are the best possible curves to join the corresponding points, and so

$$|u_h(x) - u_h(y)| = |h_B(x_0,x) - h_B(x_0,y)|$$

$$= h_B(x,y) = \int_{J(x,y)} \frac{h(d(z,\partial B))}{d(z,\partial B)}\, ds = \int_{\varepsilon}^{t_M} \frac{h(t)}{t}\, dt$$

$$> 2M \int_{\varepsilon}^{t_M} h'(t)\, dt = 2M\cdot(h(t_M) - h(\varepsilon))$$

$$> M\cdot h(t_M - \varepsilon) = M\cdot h(x,y) .$$

Hence u_h is not Lip_h-continuous in B with any constant M.

2.34. Remark. The above counterexample leads to the following observation: Let h be as in Lemma 2.31 and let D be an arbitrary domain in $\mathbb{R}^n$, $D \neq \mathbb{R}^n$. Then $f \in \mathrm{loc\,Lip}_h(D)$ does not imply the condition (2.23). We can see that if we replace the ball B in the proof of Counterexample 2.33 with the complement of a point G and use the trivial inequality

(2.35) $$h_D(x,y) \geq h_G(x,y) \quad \text{whenever} \quad x,y \in D \subset G .$$

2.36. Theorem. *Let* $U \subset \mathbb{R}^n$ *be a* c*-uniform domain and* $x_0 \in U$. *The function*

$$u_{h'}(x) := h'_U(x_0,x)$$

is in the class $\mathrm{Lip}_h(U)$ *with a constant* $2c$.

Proof. Let $x,y \in U$ and let $\gamma(x,y)$ be a curve satisfying (1.6) and (1.7). Since h' is decreasing, we obtain, by using (1.4)-(1.7),

$$|u_{h'}(x) - u_{h'}(y)| \leq h'_U(x,y) \leq \int_\gamma h'(d(z,\partial U))\,ds$$

$$\leq \int_\gamma h'(d(z,\partial\mathrm{cig}(\gamma,1/c)))\,ds$$

$$\leq \int_0^{\ell(\gamma)} h'(\tfrac{1}{c}\min(s,\ell(\gamma)-s))\,ds = 2\int_0^{\ell(\gamma)/2} h'(s/c)\,ds$$

$$= 2c\cdot(h(\ell(\gamma)/2c) - h(0)) \leq 2c\cdot h(|x-y|/2)$$

$$\leq 2c\cdot h(x,y) .$$ □

3. $\mathrm{Lip}_{h,g}$-extension domains

3.1. Definition. Let h and g be moduli of continuity and $D \subset \mathbb{R}^n$ a domain. Let $h \prec g$ in D. D is said to be a $\mathrm{Lip}_{h,g}$-*extension domain* if there is a constant $E = E(D,h,g) < \infty$ satisfying the following condition:

If $f\colon D \to \mathbb{R}$ belongs to the class $\mathrm{loc\,Lip}_h(D)$ with a constant m (see 2.20), then f belongs to the class $\mathrm{Lip}_g(D)$ with a constant $M = Em$.

The name 'extension domain' is motivated by the next theorem.

3.2. Theorem. *If* $D \subset \mathbb{R}^n$ *is a* $\mathrm{Lip}_{h,g}$-*extension domain, then every* $f\colon D \to \mathbb{R}^p$ *in* $\mathrm{loc\,Lip}_h(D)$ *with a constant* m *has an extension* $f^*\colon \mathbb{R}^n \to \mathbb{R}^p$ *such that*

$$f^* \in \mathrm{Lip}_g(\mathbb{R}^n) \text{ with a constant } M \le Em\sqrt{p}\ . \tag{3.3}$$

Proof. (See [GM2, Section 2] and [McS, Theorem 1].) Let $f\colon D \to \mathbb{R}$, $f \subset \mathrm{loc\,Lip}_h(D)$ with a constant m. By definition $f \in \mathrm{Lip}_g(D)$ with a constant $M = Em$. Set

$$f^*(x) := \inf \{\, f(z) + Mg(x,z) \mid z \in D \,\}\ .$$

Let $\varepsilon > 0$. We shall prove that

$$|f^*(x) - f^*(y)| \le Mg(x,y) + \varepsilon \quad \text{in } \mathbb{R}^n, \tag{3.4}$$

and then by letting $\varepsilon \to 0$ we obtain (3.3).

Let $x,y \in \mathbb{R}^n$. Take $z_y \in D$ such that

$$f(z_y) + Mg(y,z_y) \le f^*(y) + \varepsilon\ .$$

Using the triangle inequality for the metric g, we obtain

$$\begin{aligned} f^*(x) &\le f(z_y) + Mg(x,z_y) \\ &\le f(z_y) + Mg(y,z_y) + Mg(x,y) \\ &\le f^*(y) + \varepsilon + Mg(x,y) \quad . \end{aligned}$$

Then by exchanging the roles of x and y, we obtain (3.4).

If f is vector valued, we can repeat the proof for components, which increases the constant only by a factor $\sqrt{p}$. □

Remark. In [S, Theorem VI.3] it is proved that there is a linear extension operator from $\mathrm{Lip}_g(D)$ to $\mathrm{Lip}_g(\mathbb{R}^n)$ if D is bounded.

In this section we shall derive other characterizations of the $\mathrm{Lip}_{h,g}$-extension domains, and study some of their properties.

There are domains which are not extension domains:

3.5. Example. Let $D := B(0,1) \setminus \mathbb{R}_+ \subset \mathbb{R}^2$. D is not a $\mathrm{Lip}_{h,g}$-extension domain. To show this define a function $f\colon D \to \mathbb{R}$,

$$f(r,\varphi) := \varphi r/2\pi \qquad \text{(in polar coordinates)},$$

which is locally $\mathrm{Lip}_h(D)$-continuous. This follows from h being concave and so (2.23) holds. Clearly f is not in the class $\mathrm{Lip}_g(D)$, since there is no continuous extension of f to $\mathbb{R}^2$.

Remark. The quasiconvexity of the domain D also breaks down.

Now by using the metric h_D given in (2.25) we show an integral inequality condition for $\mathrm{Lip}_{h,g}$-extension domains. The idea is taken from [GM2, Theorem 2.2].

3.6. Theorem. *A domain* $D \subset \mathbb{R}^n$ *is a* $\mathrm{Lip}_{h,g}$*-extension domain if and only if there is a constant* $1 \le K(D,h,g) < \infty$ *such that*

$$(3.7) \qquad h_D(x,y) \le K \cdot g(x,y)$$

holds in D.

First, we prove the following lemma.

3.8. Lemma. *If* $h_D(x,y) \le K \cdot g(x,y)$ *in* D, *then* $h \prec g$ *in* D *(with a constant* $A \le 4K$ *).*

Proof. Let $x,y \in D$. We may assume without loss of generality that $d(x,\partial D) \le |x-y|$. Choose a curve $\gamma(x,y)$ such that

$$(3.9) \qquad \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \le 2K \cdot g(x,y) .$$

Now

$$\ell(\gamma) \ge |x-y| \quad \text{and}$$

$$d(\gamma(s),\partial D) \le d(x,\partial D) + \ell(\gamma) \le 2\ell(\gamma) ,$$

and so by using (2.8) and (3.9) we obtain

$$4K \cdot g(x,y) \ge 2 \int_{\gamma(x,y)} \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \ge 2\ell(\gamma) \frac{h(2\ell(\gamma))}{2\ell(\gamma)}$$

$$\ge h(2|x-y|) \ge h(x,y) . \qquad \square$$

Proof of Theorem 3.6. To prove that a $\mathrm{Lip}_{h,g}$-extension domain satisfies (3.7) take a point $y \in D$. The function

$$u_h(x) := h_D(y,x)$$

is locally Lip_h-continuous in D with a constant $m \le 2$ (see 2.28). So by the definition of $\mathrm{Lip}_{h,g}$-extension domains we obtain (3.7)

$$h_D(x,y) = |h_D(x,y) - h_D(y,y)| = |u_h(x)-u_h(y)|$$
$$\le 2E\cdot g(x,y) .$$

To prove the sufficiency of (3.7), suppose that a domain D satisfies (3.7) and let $f \in \mathrm{loc\,Lip}_h(D)$ with a constant m as in (2.20). Let $x,y \in D$ and let $\gamma(x,y)$ satisfy (3.9). Set $\ell_0 := 0$ and choose balls $B(z_i,r_i)$ as follows:

$$z_i := \gamma(\ell_{i-1}) , \quad r_i := d(z_i,\partial D)/4 \quad \text{and}$$

$$\ell_i := \max \{ s \in [0,\ell(\gamma)] \mid \gamma(s) \in \overline{B}(z_i,r_i) \} .$$

Because D is a domain and $\gamma \subset D$ is compact, there is a finite number k such that $\ell_k = \ell(\gamma)$ and the process stops. Set $z_{k+1} := y$. Set

$$A_i := \{ s \in [\ell_{i-1},\ell_i] \mid \gamma(s) \in \overline{B}(z_i,r_i) \} ,$$
$$i = 1,\ldots,k-1 .$$

Now the linear measure of A_i satisfies

$$\text{(3.10)} \qquad m(A_i) \ge r_i = |z_{i+1} - z_i| .$$

For $s \in A_i$ we have

$$\text{(3.11)} \qquad d(\gamma(s),\partial D) \le d(z_i,\partial D) + d(z_i,\gamma(s)) \le d(z_i,\partial D) + r_i$$
$$= (4+1)\cdot r_i .$$

Using (2.8), (3.11) and (3.10) we observe that

$$(3.12)\qquad \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\,ds \ \ge\ \sum_{i=1}^{k-1} \int_{A_i} \frac{h(d(\gamma(s),\partial D))}{d(\gamma(s),\partial D)}\,ds$$

$$\ge \sum_{i=1}^{k-1} \int_{A_i} \frac{h(5r_i)}{5r_i}\,ds \ =\ \sum_{i=1}^{k-1} m(A_i)\cdot\frac{h(5r_i)}{5r_i}$$

$$\ge \frac{1}{5}\cdot\sum_{i=1}^{k-1} h(5r_i) \ \ge\ \frac{1}{5}\cdot\sum_{i=1}^{k-1} h(r_i)\ .$$

Since f is locally Lip_h-continuous and $z_{i+1} \in B_{1/2}(z_i)$,

$$(3.13)\qquad |f(x)-f(y)| \le \sum_{i=1}^{k-1} |f(z_i)-f(z_{i+1})| + |f(z_k)-f(y)|$$

$$\le m \sum_{i=1}^{k-1} h(z_i,z_{i+1}) + m\cdot h(z_k,y)$$

$$= m \sum_{i=1}^{k-1} h(r_i) + m\cdot h(|z_k-y|)\ .$$

Now if $|z_k-y| \le |x-y|$, we can combine (3.9), (3.12), (3.13) and Lemma 3.10 as

$$(3.14)\qquad |f(x)-f(y)| \ \le\ 5m \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\,ds \ +\ m\cdot h(|z_k-y|)$$

$$\le\ 5m\cdot 2K\cdot g(x,y) + mA\cdot g(x,y) \ =\ Em\cdot g(x,y)\ ,$$

where $E := 10K+A < 14K$ (A is from $h \prec g$).

If $|z_k-y| > |x-y|$, then $x,y \in B_{1/2}(z_k)$, and (3.14) holds since $f \in \mathrm{loc}\ Lip_h(D)$ and $h \prec g$ in D. This completes the proof. □

From the proof of Theorem 3.6 we obtain the following theorem:

3.15. Theorem. *A* $\mathrm{Lip}_{h,g}$*-extension domain* D *has the following property:*

If X *is any metric space and* $f\colon D \to X$ *belongs to* $\mathrm{loc}\,\mathrm{Lip}_h(D)$ *with a constant* m, *then* f *belongs to* $\mathrm{Lip}_h(D)$ *with a constant* $E'm$ *(where* $E' < 28E$*).* □

The next theorem shows why domains such as those in Example 3.5 are not extension domains.

3.16. Theorem. *Let the domain* $D \subset \mathbb{R}^n$ *be a* $\mathrm{Lip}_{h,g}$*-extension domain and a constant* K *as in* (3.7). *If* $h\colon [0,\infty[\to [0,\infty[$ *is a homeomorphism, then the points* $x,y \in \overline{B}(x_0,r) \cap D$ *can be joined by a curve* $\gamma(x,y) \subset \overline{B}(x_0,b) \cap D$ *with*

$$\int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\,ds \le K'\cdot g(x,y)\ , \tag{3.17}$$

where $b = \max(2r, h^{-1}(16K'g(r)))$ *and* $K' < 4K$.

Proof. (See [GM2, Theorem 2.15].) Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Choose points $x,y \in \overline{B}(x_0,r) \cap D$. First, assume that $\overline{B}(x_0,2r) \subset D$. Then we can choose $\gamma(x,y) = J(x,y)$ and, as in the proof of Theorem 2.28, we can show that (3.17) holds with a constant $A < 4K$.

If there is a point z_0 in $\overline{B}(x_0,2r) \cap \partial D$, we can choose $\gamma(x,y) \subset D$ for which (3.17) holds. Suppose that γ is not contained in $\overline{B}(x_0,b)$. Then

$$\ell(\gamma) \ge 2(b-r)\ . \tag{3.18}$$

For every $s \in [0,\ell(\gamma)]$, $z = \gamma(s)$, the following estimates hold:

$$\begin{aligned} d(z,\partial D) &\le d(z,z_0) \le d(z,x_0) + d(x_0,z_0) \\ &\le d(z,x_0) + 2r\ , \end{aligned} \tag{3.19}$$

and

$$(3.20) \qquad d(z,x_0) \;\le\; d(z,x) + d(x,x_0) \;\le\; s + r\ .$$

By combining the inequalities (3.19) and (3.20) we obtain

$$(3.21) \qquad d(\gamma(s),\partial D) \;\le\; s + 3r \;\le\; \ell(\gamma) + 3r \;<\; 4\ell(\gamma).$$

Now by using (2.8), (3.18) and (3.21) we obtain

$$\int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\,ds \;=\; \int_0^{\ell(\gamma)} \frac{h(d(\gamma(s),\partial D))}{d(\gamma(s),\partial D)}\,ds$$

$$\ge \int_0^{\ell(\gamma)} \frac{h(\ell(\gamma)+3r)}{\ell(\gamma)+3r}\,ds \;>\; \frac{h(\ell(\gamma)+3r)}{4\ell(\gamma)} \cdot \cancel{\ell(\gamma)}$$

$$\ge \frac{h(2b+r)}{4} \;\ge\; \frac{h(h^{-1}(16K'g(r)))}{4} \;=\; 4K'g(r)$$

$$\ge 2K'g(2r) \;\ge\; 2K'g(x,y)\ ,$$

which contradicts (3.17). So γ is contained in $\overline{B}(x_0,b) \cap D$. □

In Section 5 we shall show that, in a sense, Theorem 3.16 is the best possible.

4. Lip_h-extension domains

In this section we study the special case $g = h$.

4.1. Definition. A domain $D \subset \mathbb{R}^n$ is a Lip_h-*extension domain* if it is a $\mathrm{Lip}_{h,h}$-extension domain.

4.2. Theorem. *A domain* $D \subset \mathbb{R}^n$ *is a* Lip_h*-extension domain if and only if there is a constant* $1 \le K(D,h) < \infty$ *such that*

$$h_D(x,y) \le K \cdot h(x,y) \tag{4.3}$$

holds in D. □

We start with another version of 3.16, where the constant b depends linearly on the radius r.

4.4. Theorem. *Let the domain* $D \subset \mathbb{R}^n$ *be a* Lip_h*-extension domain and a constant* K *as in* (4.3). *Then there is a constant* $b \le (3/2)e^{2K}$ *such that the points* $x, y \in \overline{B}(x_0, r) \cap D$ *can be joined by a curve* $\gamma(x,y) \subset \overline{B}(x_0, br) \cap D$ *with*

$$\int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \le 2K \cdot h(x,y) . \tag{4.5}$$

Proof. The proof of Theorem 3.16 is valid up to the inequality (3.21) (replace b by br). Let us recall that

$$d(\gamma(s), \partial D) \le s + 3r . \tag{3.21'}$$

Now by using (2.8), (3.18) and (3.21') we obtain

$$\int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds = \int_0^{\ell(\gamma)} \frac{h(d(\gamma(s),\partial D))}{d(\gamma(s),\partial D)}\, ds$$

$$\ge \int_0^{\ell(\gamma)} \frac{h(s+3r)}{s+3r}\, ds \ge h(3r) \int_0^{\ell(\gamma)} \frac{1}{s+3r}\, ds$$

$$\ge h(3r) \int_0^{2(b-1)r} \frac{1}{s+3r}\, ds = h(3r) \cdot \ln \frac{2br+r}{3r}$$

$$> h(2r)\cdot \ln \frac{2b}{3} \ \geq \ h(x,y)\cdot 2K,$$

which contradicts (4.5). So γ is contained in $\overline{B}(x_0,br) \cap D$. □

Now we can show a sufficient condition for the inclusion of the classes of Lip_h-extension domains.

4.6. Theorem. *Let* $D \subset \mathbb{R}^n$ *be a* Lip_h*-extension domain and* g *a modulus of continuity such that the function*

$$h/g \quad \textit{is decreasing.}$$

Then D *is also a* Lip_g*-extension domain.*

Proof. Fix $x,y \in D$, and choose $\gamma(x,y) \subset D$ as in Theorem 4.4 with $x_0 := (x+y)/2$ and $2r := |x-y|$.

If $\overline{B}(x_0,br) \cap \partial D = \emptyset$, (4.5) holds for every modulus of continuity with a constant $2K \leq 2$ ($x,y \in B(x_0,r) \subset B_{1/2}(x_0)$; see the proof of Theorem 2.28).

If $\overline{B}(x_0,br) \cap \partial D \neq \emptyset$, we have the estimate

$$d(z,\partial D) \leq 2br \ , \quad \text{for every} \quad z \in \gamma(x,y) \subset \overline{B}(x_0,br),$$

and hence

$$\frac{h(d(z,\partial D))}{g(d(z,\partial D))} \geq \frac{h(2br)}{g(2br)} \quad .$$

Now by using (4.5) we obtain

$$2K\cdot h(2r) \ = \ 2K\cdot h(x,y) \ \geq \ \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds$$

$$= \int_\gamma \frac{g(d(z,\partial D))}{d(z,\partial D)} \frac{h(d(z,\partial D))}{g(d(z,\partial D))}\, ds$$

$$\geq \frac{h(2br)}{g(2br)} \int_\gamma \frac{g(d(z,\partial D))}{d(z,\partial D)}\, ds \geq \frac{h(2br)}{g(2br)}\, g_D(x,y) .$$

Therefore,

$$g_D(x,y) \leq 2K \cdot \frac{h(2r)}{h(2br)} \cdot g(2br) \leq 2Kb \cdot g(2r) ,$$

which completes the proof. □

4.7. Corollary. *A* Lip_h*-extension domain is quasiconvex.*

Proof. A domain $D \subset \mathbb{R}^n$ is a Lip_1-extension domain if and only if D is quasiconvex (use Theorem 4.2). Let D be a Lip_h-extension domain. The function $h(t)/t$ is decreasing by (2.8), and so D is a Lip_1-extension domain by Theorem 4.6. Therefore D is quasiconvex (with a constant $c \leq 3e^{2K}K$). □

4.8. Corollary. *Let* $D \subset \mathbb{R}^n$ *be a* Lip_α*-extension domain. If* h *is a modulus of continuity satisfying the inequality*

(4.9) $\quad h'(t) \cdot t \geq \alpha h(t) , \quad t > 0 ,$

then D *is a* Lip_h*-extension domain.*

Proof. The function

$$h_\alpha(t) := \frac{t^\alpha}{h(t)}$$

is decreasing:

$$h_\alpha'(t) = \frac{\alpha t^{\alpha-1} h(t) - t^\alpha h'(t)}{h(t)^2}$$

$$= \frac{t^{\alpha-1}}{h(t)^2}\,(\alpha h(t) - t h'(t)) \leq 0 ,$$

and hence by 4.6 D is a Lip_h-extension domain. □

4.10. Corollary. *Let* $D \subset \mathbb{R}^n$ *be a* Lip_α*-extension domain and* $0 < \alpha \le \beta \le 1$. *Then* D *is also a* Lip_β*-extension domain.*

Proof. The function

$$\frac{t^\alpha}{t^\beta} = t^{\alpha-\beta}$$

is decreasing. □

Lip_h-extension domains do not exist for every modulus of continuity h :

4.11. Lemma. *Let* h *be a modulus of continuity satisfying the conditions of Lemma 2.31. Then there are no* Lip_h*-extension domains.*

Proof. Let $D \subset \mathbb{R}^n$ and $x_0 \in D$ and choose

$$u_h(x) := h_D(x_0, x) .$$

In Theorem 2.28 we proved that u_h is locally $\mathrm{Lip}_h(D)$-continuous. In 2.33 and 2.34 we observed that the Lipschitz-condition does not hold in the balls $B_1(z) \subset D$. So $u_h(x)$ is not $\mathrm{Lip}_h(D)$-continuous and D cannot be a Lip_h-extension domain. □

On the other hand, if the order of growth as in (4.9) holds in a weak sense, there is a large set of Lip_h-extension domains.

4.12. Theorem. *Let* h *be a modulus of continuity. Then the following conditions are equivalent:*

(4.13) *There are constants* $K < \infty$ *and* $t_K > 0$ *such that*

$$\int_0^t \frac{h(s)}{s}\, ds \quad \le \quad K\cdot h(t) \quad \textit{holds if} \quad 0 < t \le t_K \ .$$

(4.14) *All bounded uniform domains are* Lip_h*-extension domains.*

(4.15) *The unit ball in* $\mathbb{R}^n$ *is a* Lip_h*-extension domain.*

(4.16) *There exists at least one* Lip_h*-extension domain.*

Proof. First, we shall show that (4.14) follows from (4.13). Let $D \subset \mathbb{R}^n$ be a c-uniform domain with the diameter d_D . Choose $x,y \in D$ and $\gamma(x,y)$ as in Definition 1.5. From (1.4) and (2.8) we obtain

$$h_D(x,y) \quad \le \int_0^{\ell(\gamma)} \frac{h(d(\gamma(s),\partial cig(\gamma,1/c)))}{d(\gamma(s),\partial cig(\gamma,1/c))}\, ds$$

$$\le \quad 2 \int_0^{\ell(\gamma)/2} \frac{h(s/c)}{s/c}\, ds \quad \le \quad 2c \int_0^{\ell(\gamma)/2} \frac{h(s)}{s}\, ds \quad ,$$

because $c \ge 1$. If $\ell(\gamma)/2 \le t_K$, then (4.3) holds with a constant $2c^2K$. If $\ell(\gamma)/2 > t_K$, then

$$2c \int_0^{\ell(\gamma)/2} \frac{h(s)}{s}\, ds \quad = \quad 2c \int_0^{t_K} \frac{h(s)}{s}\, ds + 2c \int_{t_K}^{\ell(\gamma)/2} \frac{h(s)}{s}\, ds$$

$$\le \quad 2cK\ h(t_K) + 2c\ h(\ell(\gamma)) \int_{t_K}^{\ell(\gamma)/2} \frac{ds}{s}$$

$$\le \quad 2c\ (\ K\ h(\ell(\gamma)) + h(\ell(\gamma))\ \ln \frac{\ell(\gamma)}{2t_K}\ )$$

$$\le \quad 2c\ (\ K + \ln \frac{c|x-y|}{2t_K}\ )\ h(c|x-y|)$$

$$\leq \; 2c^2 \left(K + \ln \frac{cd_D}{2t_K} \right) h(x,y) \qquad ,$$

and therefore (4.3) holds for the domain D.

Next, we show that (4.16) implies (4.13). Let D be a Lip_h-extension domain. Take a point $y_0 \in D$ and choose a point $x_0 \in \partial D$ such that $J(x_0,y_0) \subset D \cup \{x_0\}$. Let G be the complement of x_0 and $t_K := |x_0-y_0|$. Let $0 < t \leq t_K$ and $0 < \varepsilon < t$. Choose points $x,y \in J(x_0,y_0)$ such that $d(x,x_0) = \varepsilon$ and $d(y,x_0) = t$. Now by (2.35) and (4.3) we obtain

$$\int_{\varepsilon}^{t} \frac{h(s)}{s}\, ds = h_G(x,y) \leq h_D(x,y) \leq Kh(x,y) = Kh(t-\varepsilon).$$

So (4.13) holds by letting $\varepsilon \to 0$.

This completes the proof, since (4.15) follows trivially from (4.14) and (4.16) follows from (4.15). □

Now, repeating the proof of Theorem 4.12, we have the following theorem:

4.17. Theorem. *Let* h *be a modulus of continuity. Then the following conditions are equivalent:*

(4.18) *There is a constant* K *such that*

$$\int_{0}^{t} \frac{h(s)}{s}\, ds \;\leq\; K \cdot h(t) \quad \textit{holds if} \quad t > 0.$$

(4.19) *All uniform domains are* Lip_h*-extension domains.*

(4.20) *The complement of a point is a* Lip_h*-extension domain.* □

So if there exists one Lip_h-extension domain, then at least all bounded uniform domains are extension domains. In fact, the class of Lip_h-extension domains is larger than the class of uniform domains.

4.21. Theorem. *Let* $D \subset \mathbb{R}^n$ *be a union of* Lip_h*-extension domains* D_j *for which* (4.3) *holds with the same constant* K. *Suppose that* $k < \infty$ *and* $c \ge 1$ *are fixed constants. If for each* $x, y \in D$ *there exist domains* D_{i_j}, $i = 1,\dots,k'$ *and points* z_i *such that*

$$z_i\,,\ z_{i+1} \in D_{j_i}\,,\ x = z_1\,,\ y = z_{k'+1}\,,\ k' \le k\,,$$

and

$$|z_i - z_{i+1}| \le c|x-y|\,,$$

then D *is a* Lip_h*-extension domain.*

Proof. (See [GM2, Theorem 2.25].) For the given x and y choose the domains D_{j_i} and the points z_i. Choose the curves $\gamma_i(z_i, z_{i+1}) \subset D_{j_i}$ such that

$$\int_{\gamma_i} \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \le \int_{\gamma_i} \frac{h(d(z,\partial D_{j_i}))}{d(z,\partial D_{j_i})}\, ds$$

$$\le 2Kh(z_i, z_{i+1})\,.$$

Now $\gamma(x,y) = \gamma_1 + \dots + \gamma_j \subset D$ and

$$\int_{\gamma} \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \le \sum_{i=1}^{k'} 2Kh(|z_i - z_{i+1}|)$$

$$\le 2Kk' \cdot h(c|x-y|) \le 2Kkc \cdot h(x,y)\,;$$

hence $h_D(x,y) \le 2Kkc \cdot h(x,y)$ in D. □

4.22. Definition. Let $D \subset \mathbb{R}^n$ and $x,y \in D$. We say that if

$$\gamma(x,y) = \sum_{i \in I} \gamma_i \quad ,$$

then

$$\operatorname{cigc}(\gamma, r) := \bigcup_{i \in I} \operatorname{cig}(\gamma_i, r)$$

is an *r-cigar chain neighbourhood* of $\gamma(x,y)$. If

$$\gamma(x,y) = \sum_{i=1}^{k} \gamma_i \quad ,$$

we write

$$\operatorname{cigc}(\gamma, r, k) := \bigcup_{i=1}^{k} \operatorname{cig}(\gamma_i, r) \quad .$$

4.23. Theorem. *Let* h *be a modulus of continuity for which* (4.18) *holds. Let* $D \subset \mathbb{R}^n$ *be a domain. If there exists a constant* $c \geq 1$ *such that every* $x,y \in D$ *can be joined by a curve* $\gamma(x,y)$ *with*

(4.24) $\operatorname{cigc}(\gamma, 1/c) \subset D$, *and*

(4.25) $\sum_{i \in I} h(\ell(\gamma_i)) \leq c \cdot h(x,y)$,

then D *is a* Lip_h*-extension domain.*

Proof. Let $x,y \in D$ and choose $\gamma(x,y)$ for which (4.24) and (4.25) hold. As in the proof of Theorem 4.12 we have

$$h_D(x,y) \leq \sum_{i \in I} 2c \int_0^{\ell(\gamma_i)/2} \frac{h(s)}{s}\, ds \leq \sum_{i \in I} 2cK \cdot h(\ell(\gamma_i)/2)$$

$$\leq 2cK \sum_{i \in I} h(\ell(\gamma_i)) \leq 2c^2 K \cdot h(x,y).$$

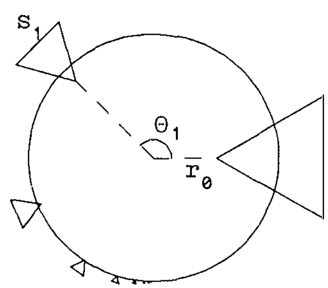
□

4.26. Remark. If (4.24) is in the form $\mathrm{cigc}(\gamma, 1/c, k) \subset D$, (4.25) can be replaced by $\ell(\gamma) \le c|x-y|$, since then

$$\sum_{i=1}^{k} h(\ell(\gamma_i)) \le \sum_{i=1}^{k} h(\ell(\gamma)) \le ck \cdot h(x,y) .$$

4.27. Remark. If D is bounded, it is enough that (4.13) holds instead of (4.18) in Theorem 4.23.

4.28. Lemma. *There are* Lip_h*-extension domains which are not uniform.*



Proof. See [GM2, Example 2.26(c)]. Take the unit disk in $\mathbb{R}^2$ and the interiors of equilateral triangles Δ_i such that the length of the sides of Δ_i is $s_i := 2^{-i}$, the polar angle for the centre and the closest vertex of Δ_i is

$$\Theta_i := \frac{3\pi}{2} \cdot (1-2^{-i}) ,$$

and the distance from the closest vertex of Δ_i to the origin is $r_i := 1 - 4^{-i}/2$. Now let $D := B(0,1) \cup \bigcup_{i=0}^{\infty} \Delta_i$. By Remark 4.27, D is a Lip_h-extension domain for every h (if there exist Lip_h-extension domains). However, the gaps in D are too narrow for D to be uniform. □

We can now return to the question raised in Theorem 2.22.

4.29. Theorem. *Let* $D \subset \mathbb{R}^n$ *be a domain and* h *a modulus of continuity such that* (4.18) *holds (or* (4.13) *if* D *is bounded). Then* $f \in \mathrm{loc}\,\mathrm{Lip}_h(D)$ *if and only if there exists a constant* $m < \infty$ *such that*

(4.30) $\quad |f(x)-f(y)| \le m\cdot h(x,y)$,

whenever x *and* y *belong to a ball contained in* D.

Proof. Let $f \in \mathrm{loc\,Lip}_h(D)$. If $B:=B(z,r) \subset D$, then, as in the proof of 4.12, $h_B(x,y) \le (2\pi/2)\cdot(\pi/2)\cdot K\cdot h(x,y)$ in B. Because $f \in \mathrm{loc\,Lip}_h(B)$, $f \in \mathrm{Lip}_h(B)$. Therefore (4.30) holds with some constant m', which depends only on the constant K in (4.18) and the constant m in (2.20). The converse is trivial. □

4.31. Remark. The condition (4.30) can be replaced by the condition

(4.30′) $\quad |f(x)-f(y)| \le m_c\cdot h(x,y)$

in c-uniform subdomains $U \subset D$.

To finish this section, we prove that the inclusion given in Theorem 4.6 is generally proper. Obviously if $h(t)$ is constant on $[t_0,\infty[$, then there is no unbounded Lip_h-extension domain. And, as mentioned in [GM2, Example 2.26], the domain between two parallel planes is quasiconvex; so it is a Lip_1-extension domain but not a Lip_α-extension domain for any $0 < \alpha < 1$.

We now show that the inclusion is proper also for general bounded domains.

4.32. Counterexample. Let $D \subset \mathbb{R}^n$ be as follows:

$$D := \{ (x,y) \subset \mathbb{R}^2 \mid 0 < x < e^{-2} , \quad |y| < x^2 \} \quad .$$

Choose the moduli of continuity

$$h(t) := t^\alpha \quad , \; 0 < \alpha < 1 ,$$

and

$$g(t) := \begin{cases} -t \ln t \, , & 0 < t \le e^{-2} \\ t + e^{-2} \, , & t > e^{-2} \, . \end{cases}$$

Now h/g is a decreasing function and $g(t) \ge t$. First, we prove that D is not a Lip_h-extension domain. If $x \in \,]0,e^{-2}[$, then

$$d((x,0),\partial D) \le x^2 \, .$$

Thus ($J = J((t/2,0),(t,0))$)

$$(4.33) \qquad h_D((t/2,0),(t,0)) = \int_J \frac{h(d(z,\partial D))}{d(z,\partial D)} \, ds$$

$$= \int_{t/2}^{t} d((x,0),\partial D)^{\alpha-1} \, dx \ \ge \int_{t/2}^{t} (x^2)^{\alpha-1} \, dx$$

$$\ge \int_{t/2}^{t} (t^2)^{\alpha-1} \, dx = t^{2(\alpha-1)} \cdot \frac{t}{2} = (2t)^{\alpha-1} \cdot (t/2)^{\alpha} \, .$$

Here the factor $(2t)^{\alpha-1}$ tends to infinity as t approaches zero. Hence (4.3) does not hold for any constant $K < \infty$.

Next, we shall prove that D is a Lip_g-extension domain. By using elementary calculus for $x \in \,]0,e^{-4}[$, we see that

$$d((x,0),\partial D) \ge x^2/2 \quad .$$

Thus, for every $z_1 = (x_1,0)$ and $z_2 = (x_2,0)$, $0 < x_1 < x_2 < e^{-4}$, the following estimate holds:

$$(4.34) \qquad g_D(z_1,z_2) \le \int_{J(z_1,z_2)} \frac{g(d(z,\partial D))}{d(z,\partial D)} \, ds \le \int_{x_1}^{x_2} \frac{g(x^2/2)}{x^2/2} \, dx$$

$$= \int_{x_1}^{x_2} -\ln(x^2/2)\,dx = 2\int_{x_1}^{x_2} -\ln x\,dx + (x_2-x_1)\cdot\ln 2$$

$$= 2\,\Big|_{x_1}^{x_2}(-x\cdot\ln x + x) + (x_2-x_1)\cdot\ln 2$$

$$= 2\cdot(-x_2\ln x_2 + x_2 + x_1\ln x_1 - x_1) + (x_2-x_1)\cdot\ln 2$$

$$= 2\cdot(g(x_2)-g(x_1)) + \ln(2e^2)\cdot(x_2-x_1)$$

$$\le 2g(x_2-x_1) + \ln(2e^2)\cdot(x_2-x_1)$$

$$\le g(x_2-x_1)\cdot(2+\ln(2e^2)) = K\cdot g(z_1,z_2) \quad .$$

For other points z_1, $z_2 \in D$, the condition (4.3) is proved as follows: The modulus of continuity g clearly satisfies (4.13). So (4.3) holds in every c-uniform subdomain of D (for a 'suitable' constant c). If now the points $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ are 'far enough' from each other, we can use the curve

$$\gamma(z_1,z_2) = J(z_1,(x_1,0)) + J((x_1,0),(x_2,0)) + J((x_2,0),z_2)$$

to prove that (4.3) holds. If the distance between the points z_1 and z_2 is 'small enough', they belong to the same c-uniform subdomain of D. So $g_D(x,y) \le K\cdot g(x,y)$ holds in D. □

4.35. Remark. The trick used in 4.32 is that any t^α, $0 < \alpha < 1$, increases faster than the modulus of continuity g (if t is small). It can be proved by the same techniques as in (4.33) that a Lip_α-extension domain ($0 < \alpha < 1$) cannot contain outward-directed cusps with the angle zero. So it is not very easy to find a Lip_β-extension domain which is not a Lip_α-extension domain for some $0 < \alpha < \beta < 1$. We shall study this question in Section 6.

5. Some geometrical properties of $\mathrm{Lip}_{h,g}$-extension domains

In Section 4 we found out that a Lip_h-extension domain must be quasiconvex. This property does not necessarily hold in $\mathrm{Lip}_{h,g}$-extension domains.

5.1. Theorem. *Let* h *and* g *be moduli of continuity such that* $h\colon [0,\infty[\to [0,\infty[$ *is a homeomorphism and* h *satisfies the condition* (4.18). *Suppose that* D *is a domain in* $\mathbb{R}^n$ *and* $c < \infty$ *a constant such that every* $x, y \in D$ *can be joined by a curve* $\gamma(x,y) \subset D$ *satisfying the properties*

(5.2) $$d(\gamma(t), \partial D) \geq \min(t, \ell(\gamma)-t)/c ,$$

and

(5.3) $$\ell(\gamma) \leq h^{-1}(g(c|x-y|)) .$$

Then D *is a* $\mathrm{Lip}_{h,g}$*-extension domain.*

Proof. (See the proof of Theorem 4.23.) By (5.2) and (5.3) we obtain

$$\int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \leq 2cK\cdot h(\ell(\gamma)) \leq 2Kc\cdot g(c|x-y|)$$

$$\leq 2Kc^2 g(x,y) .$$ □

We can prove a similar result for outward-directed cusps.

5.4. Theorem. *Let* h *and* g *be moduli of continuity such that* $h'\colon]0,\infty[\to]0,\infty[$ *is a homeomorphism and* h *satisfies the condition* (4.9). *Suppose that* D *is a domain in* $\mathbb{R}^n$ *and* $c < \infty$ *a constant such that every* $x, y \in D$ *can be joined by a curve* $\gamma(x,y) \subset D$ *satisfying the properties*

(5.5) $$d(\gamma(t),\partial D) \geq (h')^{-1}(g'(\min(t,\ell(\gamma)-t)/c)) \ ,$$

and

(5.6) $$\ell(\gamma) \leq c\cdot|x-y| \ .$$

Then D *is a* $\mathrm{Lip}_{h,g}$*-extension domain.*

Proof. By (4.9), (5.5) and (5.6) we obtain

$$\int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\,ds \leq \frac{1}{\alpha}\cdot\int_\gamma h'(d(z,\partial D))\,ds$$

$$\leq \frac{2}{\alpha}\cdot\int_0^{\ell(\gamma)/2} g'(t/c)\,dt = (2c/\alpha)\cdot g(\ell(\gamma)/2c)$$

$$\leq (2c/\alpha)\cdot g(x,y) \ . \qquad \square$$

5.7. Example. By an 'order of cusp' $f(t)$ we mean that in $\mathbb{R}^2$ the boundary of the domain $D \subset \mathbb{R}^2$ is (locally) the set

$$\{\ (t,f(t))\ |\ t \geq 0\ \} \cup \{\ (t,-f(t))\ |\ t \geq 0\ \} \ .$$

It is easy to see, in view of Theorems 5.1 and 5.4, that a $\mathrm{Lip}_{\beta,\alpha}$-extension domain can contain 'inward-directed cusps of order'

$$t^{\beta/\alpha}$$

and 'outward-directed cusps of order'

$$t^{\frac{\alpha-1}{\beta-1}}$$

if $\alpha < \beta$.

Remark. The domain D in Example 4.32 is a $\mathrm{Lip}_{\alpha,2\alpha-1}$-extension domain ($\frac{1}{2} < \alpha \leq 1$).

6. Lip_α-extension and total extension domains

In this section we shall study the special case of a modulus of continuity $h(t) = t^\alpha$ with $0 < \alpha \le 1$ (studied in [GM2]). First, we employ the previous results for this special case.

6.1. Lemma. *The condition* (4.18) *holds for* t^α, $0 < \alpha \le 1$.

Proof. The result follows from

$$\int_0^t s^{\alpha-1}\,ds = \frac{1}{\alpha}\cdot \Big|_0^t s^\alpha\,ds = \frac{1}{\alpha}\cdot t^\alpha . \qquad \square$$

Lemma 6.1, Corollary 4.20, Theorem 4.2 and Theorem 4.29 imply the following results:

6.2. Theorem. *Let* $D \subset \mathbb{R}^n$ *be a domain and* $0 < \alpha \le 1$. *A function* $f\colon D \to \mathbb{R}^p$ *belongs to the class* $\mathrm{loc\,Lip}_\alpha(D)$ *if and only if there exists a constant* $m < \infty$ *such that*

$$|f(x)-f(y)| \le m\cdot|x-y|^\alpha$$

whenever x *and* y *belong to a ball contained in* D. $\square$

Theorem 6.2 is the original definition for the local Lipschitz class $\mathrm{loc\,Lip}_h(D)$ in [GM2]. The next theorem is [GM2, Theorem 2.2].

6.3. Theorem. *A domain* $D \subset \mathbb{R}^n$ *is a* Lip_α*-extension domain if and only if there is a constant* $K = K(D,\alpha) < \infty$ *such that for every* $x, y \in D$ *there exists a curve* $\gamma(x,y) \subset D$ *with*

$$\int_\gamma d(z,\partial D)^{\alpha-1}\,ds \le K|x-y|^\alpha . \tag{6.4}$$

$\square$

6.5. Definition. A domain $D \subset \mathbb{R}^n$ is said to be a *total extension domain* if it is a Lip_α-extension domain for every $0 < \alpha \le 1$.

6.6. Theorem. *Uniform domains and the domains mentioned in Remark* 4.26 *are total extension domains.* □

By Corollary 4.10, every Lip_α-extension domain is also a Lip_β-extension domain if $0 < \alpha \le \beta \le 1$. Are all Lip_α-extension domains total extension domains? The answer is 'no', which will be proved in the following example. To construct such a domain we must destroy the inequality (4.25).

6.7. Counterexample. Let $\beta \le 1$ and $0 < \alpha < \beta$. There is a Lip_β-extension domain which is not a Lip_α-extension domain.

The construction. See the figures in Appendix B. Let $0 < \alpha < \beta < 1$. First, we choose a set $D' \subset \mathbb{R}^2$ as follows: Let

$$r_0 := 1/2 \ , \quad r_i := 2^{-i/\alpha}/2 \quad , \ i = 1,2,\dots \quad ,$$

$$\ell_0 := \frac{1}{1-2^{1-1/\alpha}} \quad ,$$

$$J_0 := \{ \ (x,y) \in \mathbb{R}^2 \mid y = 0 \ , \ |x| < \ell_0/2 \ \} \quad .$$

Let (for $z_0 = (x_0,y_0)$ and $r > 0$)

$$B_1(z_0,r) := \{ \ (x,y) \in \mathbb{R}^2 \mid \ |x_0-x| + |y_0-y| < r \ \}.$$

Now we carry out the following Cantor-type construction:

Let $D' := J_0$, $L := J_0$ and $J_{0,1} := J_0$. We proceed by induction. For $i \ge 0$ the set L is a union of line segments $J_{i,j}$, $j = 1,2,\dots,2^i$. We choose the midpoint $z_{i,j}$ of every line segment $J_{i,j}$ and join the boxes $B_1(z_{i,j},r_i)$ to the set D'. Then we take the sets $]\, x_{i,j} - r_i \ , \ x_{i,j} + r_i \,[$ away from the set L.

The set D' contains the boxes $B_1(z_{i,j}, r_i)$ and the line segment J_0 (see Figure 1). The set which is constructed on the line segment $J_{i,j}$ is similar to the set D' (on the scale $1:2^{-i/\alpha}$).

Set $\ell_i := \ell(J_{i,j})$. It follows from the construction that $\ell_i = 2\ell_{i+1} + 2r_i$, and we have

$$\ell_i = \frac{2^{-i/\alpha}}{1-2^{1-1/\alpha}} .$$

The number of boxes joined to D' in step n is 2^n and the number of boxes which meet the line segment $J_{n,j}$ in step i is 2^{i-n} , $i \geq n$. The (Cantor-) set $L_{\infty,n,j}$ of the points of the line segment $J_{n,j}$ that are not covered by any of the boxes has the linear measure

$$\ell_{\infty,n} = \ell_n - \sum_{i=n}^{\infty} 2^{i-n}\cdot 2r_i = \ell_n - \sum_{i=n}^{\infty} 2^{i-n}\cdot 2^{-i/\alpha}$$

$$= \ell_n - \frac{2^{-n/\alpha}}{1-2^{1-1/\alpha}} = 0 .$$

Now there exists a constant $K = K(\alpha,\beta)$ such that each pair of points z_1 , $z_2 \in D'$ can be joined by a curve γ in D' with the property

$$\text{(6.8)} \qquad \int_\gamma d(z,\partial D')^{\beta-1}\, ds \leq K\cdot |z_1-z_2|^\beta .$$

First, we prove that (6.8) holds if z_1 and z_2 are endpoints of the line segment $J_{n,j}$. Obviously the best curve to join z_1 to z_2 is the line segment $J(z_1,z_2) = J_{n,j}$. By combining the results from the construction we obtain

$$\text{(6.9)} \qquad \int_J d(z,\partial D')^{\beta-1}\, ds = \sum_{i=n}^{\infty} (2^{i-n}\cdot 2 \int_0^{r_i} d(J(s),\partial D')^{\beta-1}\, ds)$$

$$= 2\cdot 2^{-n}\cdot \sum_{i=n}^{\infty} \left(2^{i} \int_{0}^{r_i} \left(\frac{s}{\sqrt{2}}\right)^{\beta-1} ds \right)$$

$$= 2\cdot 2^{-n} 2^{(1-\beta)/2}\cdot \sum_{i=n}^{\infty} \left(2^{i}\cdot\frac{1}{\beta} \Big|_{0}^{r_i} s^{\beta} \right)$$

$$= 2\cdot 2^{-n}\cdot 2^{(1-\beta)/2}\cdot\frac{1}{\beta}\cdot \sum_{i=n}^{\infty} \left(2^{i}\cdot\left(\frac{2^{-i/\alpha}}{2}\right)^{\beta} \right) .$$

Since $2^{1-\beta/\alpha} < 1$, the last sum converges, and we conclude

$$\int_J d(z,\partial D')^{\beta-1} ds = \frac{2^{3(1-\beta)/2}}{\beta}\cdot 2^{-n}\cdot\frac{2^{n(1-\beta/\alpha)}}{1-2^{1-\beta/\alpha}}$$

$$= \frac{2^{3(1-\beta)/2}}{\beta(1-2^{1-\beta/\alpha})}\cdot(2^{-n/\alpha})^{\beta} = K'|z_1-z_2|^{\beta} ,$$

where the last equation follows from

$$|z_1-z_2| = \ell_n = \frac{1}{(1-2^{1-1/\alpha})^{\beta}}\cdot(2^{-n/\alpha}) .$$

Next, we show that (6.8) holds if $|z_1-z_2| = \ell_n$ for some n and $z_1 , z_2 \in J_0$. The minimizing curve is again the line segment $J(z_1,z_2)$ and so

$$(6.10) \qquad \int_J d(z,\partial D')^{\beta-1} ds \le \int_{J_{n,j}} d(z,\partial D')^{\beta-1} ds = K'|z_1-z_2|^{\beta}.$$

If $z_1 , z_2 \in J_0$ are arbitrary, there is a number n such that $\ell_{n+1} < |z_1-z_2| \le \ell_n$, $\ell_n = 2^{1/\alpha}\ell_{n+1}$. Hence (6.10) yields

$$\int_{J(z_1,z_2)} d(z,\partial D')^{\beta-1} ds \le \int_{J_{n,j}} d(z,\partial D')^{\beta-1} ds$$

$$= K'\ell_n^{\beta} = K'2^{\beta/\alpha}\ell_{n+1}^{\beta} \le 2^{\beta/\alpha}K'|z_1-z_2|^{\beta} .$$

Finally, if the points z_1, $z_2 \in D'$ are arbitrary, we use the method described at the end of Example 4.32.

In conclusion, it follows that (6.8) holds. However, (6.8) does not hold in the case $\beta \le \alpha$, since then the last sum in (6.9) diverges.

To complete our example, we must 'open' the set D' so that it becomes a domain. We define a domain (see Figure 2).

$$D^* := \{ (x,y) \in \mathbb{R}^2 \mid |y| < 2^{\frac{1}{|x|-\ell_0/2}} , |x| < \ell_0/2 \}.$$

If we now set $D := D' \cup D^*$, then D is a domain (see Figure 3) and (6.8) holds for the exponent $\beta > \alpha$ in D. But if we come close enough to the point $z_0 := (\ell_0/2,0)$, then D is almost similar to D', and (6.8) does not hold if $\beta \le \alpha$.

6.11. Remark. For example, if $D \subset \mathbb{R}^2$ is the upper half plane, it can be proved that curves minimizing the integral (2.25) are restrictions of the curve

$$(6.12)\qquad \left|\begin{array}{l} x(t) = c\,\frac{1}{1-\alpha}\int_0^t (\sin s)^{\frac{1}{1-\alpha}}\,ds + x_0 \\ y(t) = c\,(\sin t)^{\frac{1}{1-\alpha}} , \qquad 0 \le t \le \pi , \end{array}\right.$$

for some constants c and x_0 $(0 < \alpha < 1)$.

If we let $\alpha \to 0$, we obtain a quasi-hyperbolic geodesic, which is a circular arc meeting the x-axis at the angle $\pi/2$. Also for $\alpha > 0$ the curve (6.12) is perpendicular to the x-axis (see the graphs in Appendix C).

7. $\mathrm{Lip}_{h,g}$-extension domains and the Hardy-Littlewood property

In this section we shall study some extensions of the results in [GM1], [J] and [St]. Let $|\partial f|$ be the maximum derivative of a function $f: D \to \mathbb{R}^p$

(7.1) $$|\partial f(z)| := \limsup_{|y| \to 0} \frac{|f(z+y)-f(z)|}{|y|} .$$

In [GM1] F.W. Gehring and O. Martio studied domains in which the condition

(7.2) $$|\partial f(z)| \le m \cdot d(z, \partial D)^{\alpha-1}$$

implies that the function f belongs to the class $\mathrm{Lip}_\alpha(D)$. We now show that these domains are exactly Lip_α-extension domains. E. Johnston has also proved the same kinds of results in [J] using different methods.

7.3. Theorem. *Let* $D \subset \mathbb{R}^n$ *be a domain and* h *and* g *moduli of continuity. Then the following two conditions are equivalent:*

(7.4) D *is a* $\mathrm{Lip}_{h,g}$*-extension domain.*

(7.5) *If a function* $f: D \to \mathbb{R}^p$ *satisfies*

(7.6) $$|\partial f(z)| \le m \cdot \frac{h(d(z,\partial D))}{d(z,\partial D)} \quad \textit{whenever} \quad z \in D,$$

then $f \in \mathrm{Lip}_g(D)$ *with a constant* $K \cdot m$, *where* K *depends only on the domain* D *(and the moduli of continuity* h *and* g*).*

Remark. We have two methods to prove that a $\mathrm{Lip}_{h,g}$-extension domain satisfies (7.5). The first one is to prove that a function satisfying (7.6) is in the class loc $\mathrm{Lip}_h(D)$ in the sense of Theorem 2.17. The second one is to prove the theorem by using Theorem 3.6. We shall use the second method (which is used in [GM1]), because we shall have use for the proof later.

Proof. Let $D \subset \mathbb{R}^n$ be a $\mathrm{Lip}_{h,g}$-extension domain, and let $x, y \in D$. Suppose that a function $f\colon D \to \mathbb{R}^p$ satisfies (7.6). By Theorem 3.6 we can choose a curve $\gamma(x,y)$ such that

$$(7.7) \qquad \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \le 2K \cdot g(x,y) \quad .$$

We have an estimate for the maximum derivative of the function $f \circ \gamma$:

$$|\partial(f\circ\gamma)(s)| = \limsup_{|r|\to 0} \frac{|(f\circ\gamma)(s+r)-(f\circ\gamma)(s)|}{|r|}$$

$$\le \limsup_{|r|\to 0} \frac{|f(\gamma(s+r))-f(\gamma(s))|}{|\gamma(s+r)-\gamma(s)|} \limsup_{|r|\to 0} \frac{|\gamma(s+r)-\gamma(s)|}{|r|}$$

$$\le |\partial f(\gamma(s))| \cdot 1 \ .$$

Choose $z_0 \in \gamma$ closest possible to ∂D. By (7.6)

$$|\partial(f\circ\gamma)(s)| \le m \cdot \frac{h(d(\gamma(s),\partial D))}{d(\gamma(s),\partial D)} \le m \cdot \frac{h(d(z_0,\partial D))}{d(z_0,\partial D)} \ ,$$

and hence $|\partial(f\circ\gamma)(s)|$ is uniformly bounded in $[0, \ell(\gamma)]$ wherefore $f\circ\gamma$ is absolutely continuous in $[0,\ell(\gamma)]$ and

$$(7.8) \qquad |f(x) - f(y)| = |(f\circ\gamma)(\ell(\gamma)) - (f\circ\gamma)(0)|$$

$$\le \int_0^{\ell(\gamma)} |\partial(f\circ\gamma)(s)|\, ds \le m \cdot \int_0^{\ell(\gamma)} \frac{h(d(\gamma(s),\partial D))}{d(\gamma(s),\partial D)}\, ds$$

$$= m \cdot \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \;\le\; m2K \cdot g(x,y) \;.$$

Thus f is $\mathrm{Lip}_g(D)$-continuous with a constant $2Km$ which depends only on the domain D (and the moduli of continuity h and g).

Next, suppose that (7.5) holds. Let $x_0 \in D$ and set

$$u_h(x) := h_D(x_0, x) \;.$$

Now

$$\frac{|u_h(x+y) - u_h(x)|}{|y|} < \frac{h_D(x+y,x)}{|y|}$$

$$\le \frac{1}{|y|} \cdot \int_{J(x,x+y)} \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds \;,$$

where the last term tends to $\dfrac{h(d(x,\partial D))}{d(x,\partial D)}$ if $|y| \to 0$. So

$$|\partial u_h(x)| \le \frac{h(d(x,\partial D))}{d(x,\partial D)} \;,$$

and, by (7.5), $u_h \in \mathrm{Lip}_g(D)$ with a constant K independent of the point x_0. Hence

$$h_D(x,x_0) = |u_h(x) - u_h(x_0)| \le K \cdot g(x,x_0) \;,$$

and, by Theorem 3.6, D is a $\mathrm{Lip}_{h,g}$-extension domain. □

7.9. Definition. A domain $D \subset \mathbb{R}^2$ is said to have the *Hardy-Littlewood property* if for some constant L and for all $\alpha \in\,]0,1]$ every analytic function f with

(7.10) $\quad |f'(z)| \le m \cdot d(z,\partial D)^{\alpha - 1}$

in D is in the class $\mathrm{Lip}_\alpha(D)$ with a constant $M \le Lm/\alpha$ (see [GM1, Section 3]).

7.11. Corollary. *The domains mentioned in Remark* 4.26 *have the Hardy-Littlewood property.*

Proof. Let D be as in 4.26 and $h(t) = t^\alpha$, $0 < \alpha \le 1$. Then

$$h_D(x,y) \le \sum_{i=1}^{k} 2c \int_0^{\ell(\gamma_i)/2} s^{\alpha-1} ds \le \frac{2c}{\alpha} \sum_{i=1}^{k} \ell(\gamma_i)^\alpha$$

$$\le \frac{2c^2k}{\alpha} |x-y|^\alpha$$

holds in D. By the proof of Theorem 7.3

$$f \in \mathrm{Lip}_\alpha(D) \quad \text{with a constant} \quad \frac{2c^2k}{\alpha} m$$

whenever (7.10) holds for f. So we can choose $L = 2c^2k$. □

By Corollary 7.11 the non-uniform domain in the proof of Lemma 4.28 has the Hardy-Littlewood property.

7.12. Remark. Let K_α be the constant for which (4.3) holds for the modulus of continuity $h(t) = t^\alpha$. We can construct a total extension domain D for which the quantity

$$\frac{K_\alpha}{K_\beta} \cdot \frac{\alpha}{\beta}$$

tends to infinity if $\alpha \to 0$. So it is not obvious whether every total extension domain has the Hardy-Littlewood property.

7.13. Remark. If D is a domain in $\mathbb{R}^2$ and f is harmonic in D (or analytic in D), then f belongs to $\mathrm{loc\,Lip}_h(D)$ if and only if (7.6) (or (7.10)) holds. (Modify the proof of [GM1, Theorem 1.1].)

In [St] H. Stegbuchner has studied domains where $h(t)/t$ is replaced by $h'(t)$ in (7.6). Now we can give an equivalent condition for these domains.

7.14. Corollary. *Let* $D \subset \mathbb{R}^n$ *be a domain. Then the following two conditions are equivalent:*

(7.15) *There is a constant* $K(D,h,g) < \infty$ *such that*

$$h'_D(x,y) \le Kg(x,y) \text{ , whenever } x,y \in D.$$

(7.16) *If a function* $f\colon D \to \mathbb{R}^p$ *satisfies*

(7.17) $$|\partial f(z)| \le m \cdot h'(d(z,\partial D)) \text{ whenever } z \in D,$$

then $f \in Lip_g(D)$ *with a constant* Km, *where* K *depends only on the domain* D *(and the moduli of continuity* h *and* g*).*

Proof. Repeat the proof of Theorem 7.3 using the integral

$$\int_\gamma h'(d(z,\partial D))\, ds \quad .$$ □

7.18. Remark. Uniform domains always satisfy the property $h'_D \le Kh$ (see the proof of Theorem 2.36). The same is true also for domains in Remark 4.26. So the class of domains satisfying (7.16) is larger than the class of uniform domains (if $h \prec g$), and for some h and g also larger than the class of $Lip_{h,g}$-extension domains (which may be empty). But if the modulus of continuity h satisfies the order of growth as in (4.9), then the metrics h'_D and h_D are equivalent, and in this case the domains satisfying (7.16) are $Lip_{h,g}$-extension domains.

Notation

i,j,k	= indices in $\mathbb{N}$
n	= dimension of $\mathbb{R}^n$
p	= dimension of the range space $\mathbb{R}^p$, $f: D \to \mathbb{R}^p$
x,y,z,x_i,y_i,z_i	= points in $\mathbb{R}^n$ (in examples $z = (x,y) \in \mathbb{R}^2$)
$\|x-y\|$	= euclidean distance between points x and y
D	= domain in $\mathbb{R}^n$, $D \neq \emptyset$, $D \neq \mathbb{R}^n$
a,r,s,t	= positive real numbers
for γ, $\gamma(x,y)$, $\gamma(s)$, $\ell(\gamma)$, J, $J(x,y)$ see Definition 1.1	
$cig(\gamma,1/c)$	= $1/c$-cigar neighbourhood of γ (1.3)
∂D	= boundary of the domain D
$d(x,\partial D)$	= $\inf \{ \|x-z\| \mid z \in \partial D \}$
α,β	= exponents in $]0,1]$
c	= quasiconvexity (1.2) or uniformity (1.5) constant ($c \geq 1$)
$B(x,r)$	= open ball with centre at x and radius r
$B_b(x)$	= $\{ y \in \mathbb{R}^n \mid \|x-y\| < b \cdot d(x,\partial D) \}$
b	= constant, see above (see also Theorem 3.16)
h,g	= moduli of continuity (2.1)
$h(x,y)$	= $h(\|x-y\|)$, metric defined by h (2.9)
$h \prec g$ in D	= $h(x,y) \leq A \cdot g(x,y)$ whenever $x,y \in D$ (2.12)
A	= constant $< \infty$, see above
$h_D(x,y)$	= $\inf\limits_{\gamma(x,y) \subset D} \int_\gamma \frac{h(d(z,\partial D))}{d(z,\partial D)}\, ds$, (2.24)
$h'_D(x,y)$	= $\inf\limits_{\gamma(x,y) \subset D} \int_\gamma h'(d(z,\partial D))\, ds$, (2.24)
$u_h(x)$	= $h_D(x_0,x)$, $x_0 \in D$ (2.28)
$Lip_h(D)$	= $\{f: D \to \mathbb{R}^p \mid \|f(x)-f(y)\| \leq M h(x,y) \text{ in } D\}$ (2.10)
M	= constant for $f \in Lip_h(D)$ (2.10)
$loc\ Lip_h(D)$	= as above but in $B_{1/2}(z) \subset D$ (2.15, 2.17, 2.20)
m,m_b	= constants for $loc\ Lip_h(D)$ (2.20, 2.15)
f	= $f: D \to \mathbb{R}^p$, Lip_h- or $loc\ Lip_h$- continuous
f^*	= extension of f to $\mathbb{R}^n$
E	= constant to extend $f \in loc\ Lip_h(D)$ (3.1)
K	= $h_D(x,y) \leq K \cdot h(x,y)$ (3.7)

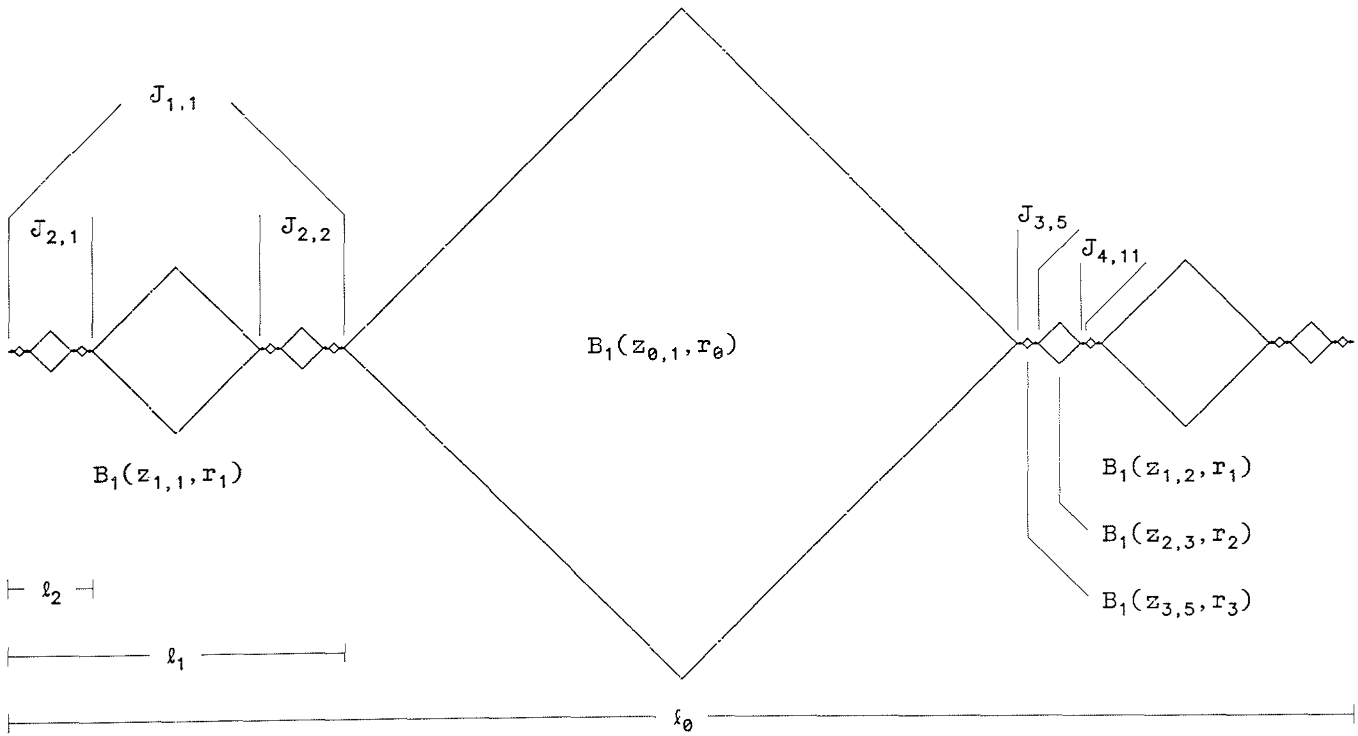


Figure 1. The set D' in case $\alpha = 0.5$.

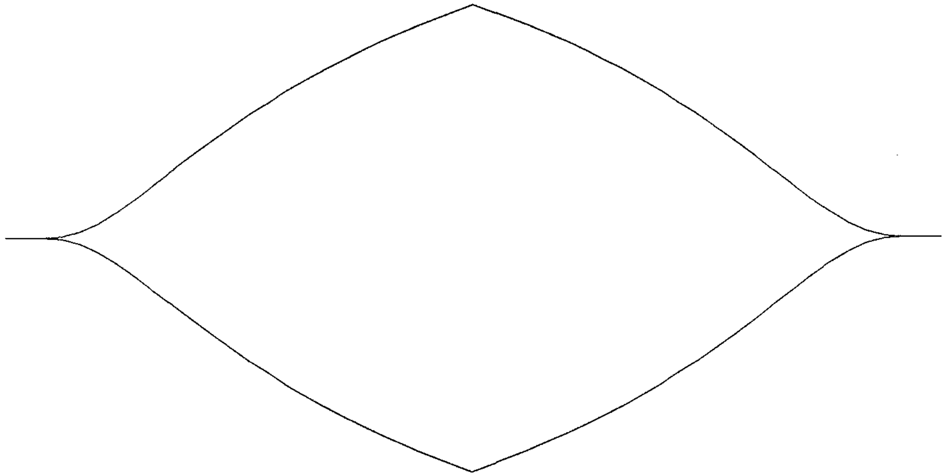


Figure 2. The domain D^* in case $\alpha = 0.5$.

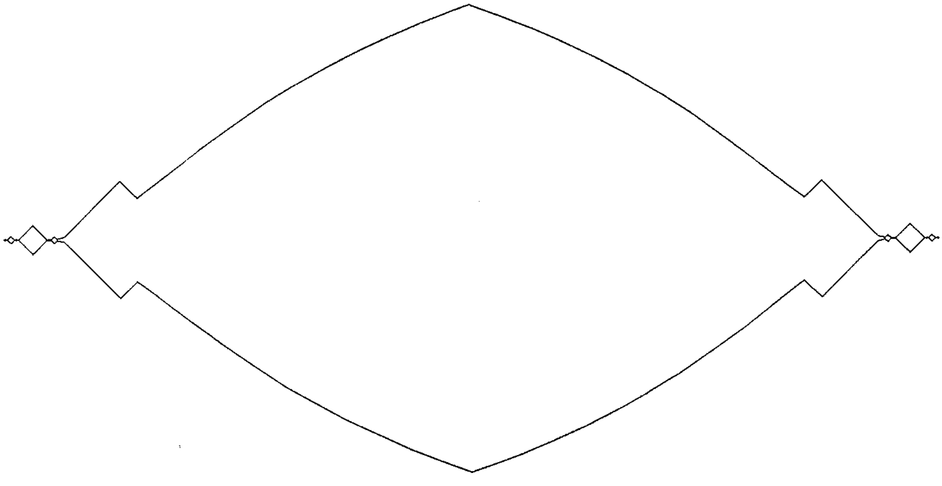


Figure 3. The domain D in case $\alpha = 0.5$.

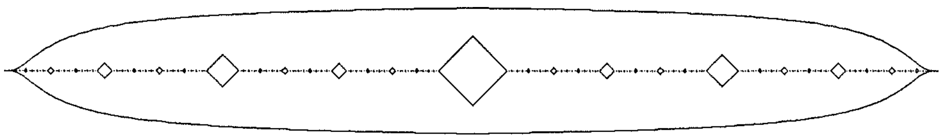


Figure 4. The set D′ and the domain D^* in case $\alpha = 0.9$.

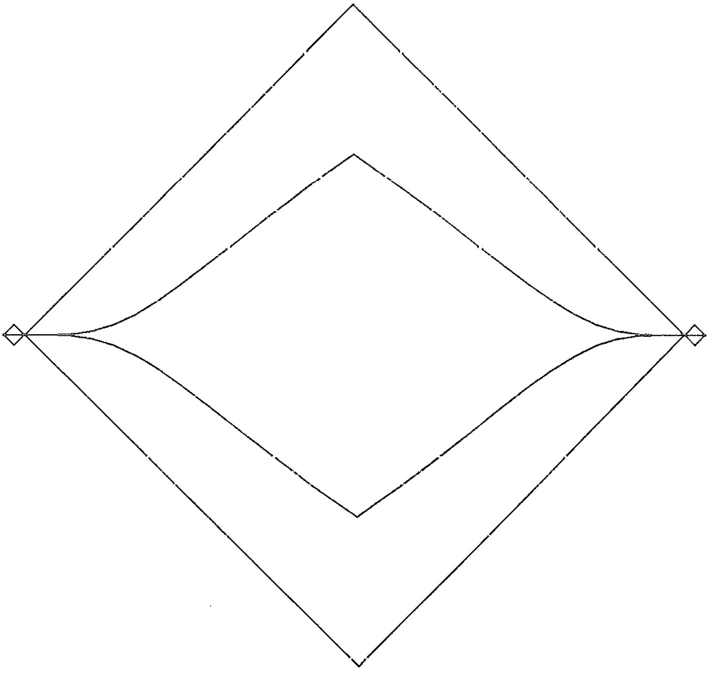


Figure 5. The set D′ and the domain D^* in case $\alpha = 0.2$.

Remark. The Figures 1–5 are not on the same scale.

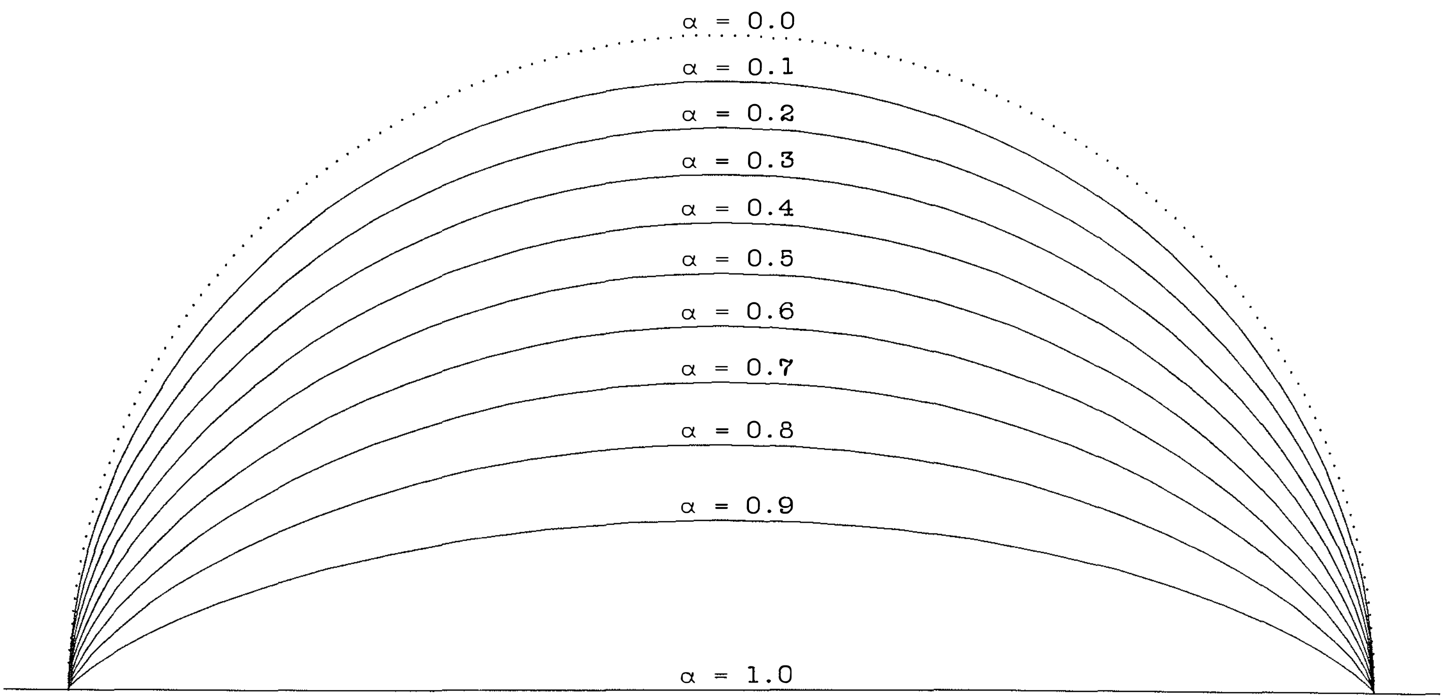


Figure 6. Graphs of curves minimizing the integral $\int_{\gamma} d(z,\partial D)^{\alpha-1}\,ds$ in the upper half plane.

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