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Regular Article

A quantitative second order estimate for (weighted)  $p$ -harmonic functions in manifolds under curvature-dimension condition <sup>☆</sup>



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ABSTRACT

We build up a quantitative second-order Sobolev estimate of  $\ln w$  for positive  $p$ -harmonic functions  $w$  in Riemannian manifolds under Ricci curvature bounded from below and also for positive weighted  $p$ -harmonic functions  $w$  in weighted manifolds under the Bakry-Émery curvature-dimension condition.

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## 1. Introduction

Let  $(M^n, g)$  be a complete non-compact Riemannian manifold with dimension  $n \geq 2$ . Suppose that the Ricci curvature is bounded from below, that is,  $Ric_g \geq -\kappa$  for some  $\kappa \geq 0$ . For any positive harmonic function  $w$  in a domain  $\Omega \subset M^n$ , Cheng-Yau [2] established the following famous gradient estimate:

$$|\nabla \ln w| = \frac{|\nabla w|}{w} \leq C(n) \frac{1 + \sqrt{\kappa}r}{r} \quad \text{in } B(z, r) \subset B(z, 2r) \subset \Omega. \quad (1.1)$$

Recall that a harmonic function  $w$  in  $\Omega$  is a weak solution to the Laplace equation

$$\Delta w := \operatorname{div}(\nabla w) = 0 \quad \text{in } \Omega.$$

We also refer to [17, Theorem 1.3] for a quantitative  $W_{\text{loc}}^{2,2}$ -regularity of harmonic functions.

Motivated by the application in the inverse mean curvature flow (see [11,15]), Cheng-Yau type gradient estimate was extended by [16,11,21,15] to  $p$ -harmonic functions in  $\Omega$  for  $1 < p < \infty$ , that is, weak solutions to the  $p$ -Laplace equation

$$\Delta_p w = \operatorname{div}(|\nabla w|^{p-2} \nabla w) = 0 \quad \text{in } \Omega.$$

Precisely, if  $(M^n, g)$  is flat (that is, the Euclidean space  $\mathbb{R}^n$ ) or its sectional curvature is bounded from below by  $-\kappa$ , via Cheng-Yau's approach Moser [16] and Kotschwar-Ni [11] showed that any positive  $p$ -harmonic function  $w$  in  $\Omega$  satisfies

$$|\nabla \ln w| \leq C(n) \frac{1 + \sqrt{\kappa}r}{r} \quad \text{in } B(z, r) \subset B(z, 2r) \subset \Omega, \quad (1.2)$$

where the constant  $C(n) > 0$  is independent of  $p \in (1, \infty)$ . Under the Ricci curvature lower bound  $Ric_g \geq -\kappa$ , it was asked in [11] whether (1.2) holds or not. Some progress was made as below. Based on Cheng-Yau's argument, Wang-Zhang [21] proved that

$$|\nabla \ln w|^{\frac{p-\gamma}{2}} \in W_{\text{loc}}^{1,2} \quad \text{with } \gamma < 0 \quad (1.3)$$

and the following weaker revision of (1.2):

$$|\nabla \ln w| \leq C(n, p) \frac{1 + \sqrt{\kappa}r}{r} \quad \text{in } B(z, r) \subset B(z, 2r) \subset \Omega, \quad (1.4)$$

where the constant  $C(n, p) > 0$  blows up as  $p \rightarrow 1$ . Recently, with the aid of the fake distance coming from capacity,  $C(n, p)$  was proved by Mari-Rigoli-Setti [15] to be bounded by  $\frac{n-1}{p-1}$  as  $p \rightarrow 1$ . Moreover, (1.3) and (1.4) were generalized to weighted manifolds  $(M^n, g, e^{-h} d\text{vol}_g)$ . A weighted  $p$ -harmonic function  $w$  in a domain  $\Omega \subset M^n$  is a weak solution to the weighted  $p$ -harmonic equation

$$\Delta_{p,h} w := e^h \operatorname{div}(e^{-h} |\nabla w|^{p-2} \nabla w) = 0 \text{ in } \Omega.$$

Under the Bakry-Émery curvature-dimension condition  $\operatorname{Ric}_h^N \geq -\kappa$  for some  $N \in [n, \infty)$  and  $\kappa \geq 0$  (see Section 2 for details), Dung-Dat [5] showed that if  $w > 0$ , then  $|\nabla \ln w|^{\frac{p-\gamma}{2}} \in W_{\operatorname{loc}}^{1,2}$  with  $\gamma < 0$  and also

$$|\nabla \ln w| \leq C(n, N, p) \frac{1 + \sqrt{\kappa}r}{r} \text{ in } B(z, r) \subset B(z, 2r) \subset \Omega. \tag{1.5}$$

The main aim of this paper is to build up a quantitative second-order Sobolev estimate of  $\ln w$  for positive  $p$ -harmonic functions  $w$  in Riemannian manifolds under Ricci curvature bounded from below and also for positive weighted  $p$ -harmonic functions  $w$  in weighted manifolds under the Bakry-Émery curvature-dimension condition. See Theorem 1.1 and Theorem 1.2 separately. These improve the corresponding second-order Sobolev regularity in [21,5] mentioned above.

To be precise, under the Ricci curvature lower bound, we have the following result. For convenience, below we write  $f_E$  as the average of  $f$  in the set  $E$  with respect to the measure  $m$ , that is,  $f_E = \frac{1}{m(E)} \int_E f \, dm$ . We use  $C(a_1, \dots, a_m)$  to denote a positive constant depending on absolute constants  $a_1, \dots, a_m$ .

**Theorem 1.1.** *Suppose that  $(M^n, g)$  satisfies  $\operatorname{Ric}_g \geq -\kappa$  for some  $\kappa \geq 0$ . Let  $1 < p < \infty$  and  $\gamma < 3 + \frac{p-1}{n-1}$ . For any positive  $p$ -harmonic function  $w$  in a domain  $\Omega \subset M$ , we have  $|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \in W_{\operatorname{loc}}^{1,2}(\Omega)$  and*

$$\int_{B(z,r)} \left| \nabla [|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w] \right|^2 \, d\operatorname{vol}_g \leq C(n, p, \gamma) \left[ \frac{1 + \sqrt{\kappa}r}{r} \right]^{p-\gamma+4} e^{\sqrt{\kappa}r} \tag{1.6}$$

whenever  $B(z, 4r) \Subset \Omega$ .

In particular, if  $1 < p < 3 + \frac{2}{n-2}$ , then  $\nabla^2 \ln w \in L_{\operatorname{loc}}^2(\Omega)$  and

$$\int_{B(z,r)} |\nabla^2 \ln w|^2 \, d\operatorname{vol}_g \leq C(n, p) \left[ \frac{1 + \sqrt{\kappa}r}{r} \right]^4 e^{\sqrt{\kappa}r} \tag{1.7}$$

whenever  $B(z, 4r) \Subset \Omega$ .

Here and throughout the paper for domains  $A$  and  $B$ , the notation  $A \Subset B$  stands for that  $A$  is a bounded subdomain of  $B$  and its closure  $A \subset B$ .

Recall that if  $(M^n, g)$  is flat, that is, the Euclidean space  $\mathbb{R}^n$ ,  $p$ -harmonic functions  $w$  in a domain  $\Omega \subset \mathbb{R}^n$  are proved to satisfy  $|\nabla w|^{\frac{p-\gamma}{2}} \nabla w \in W_{\operatorname{loc}}^{1,2}(\Omega)$  with some quantitative bound whenever  $\gamma < 3 + \frac{p-1}{n-1}$  see [13,9,4,14] and also the references therein for some earlier partial results. In particular, if  $1 < p < 3 + \frac{2}{n-2}$ , noting  $p < 3 + \frac{p-1}{n-1}$  and taking  $\gamma = p$ , one has  $w \in W_{\operatorname{loc}}^{2,2}(\Omega)$ . When  $n \geq 3$  and  $p \geq 3 + \frac{2}{n-2}$ , it is not clear whether

$w \in W_{loc}^{2,2}(\Omega)$  or not. When  $n = 2$ , the range  $\gamma < 3 + \frac{p-1}{n-1} = p + 2$  is optimal as witnessed by some construction in [9].

Moreover, we extend Theorem 1.1 to weighted manifolds satisfying Bakry-Émery curvature-dimension condition,

**Theorem 1.2.** *Let  $(M^n, g, e^{-h} \text{vol}_g)$  be a weighted manifold with  $Ric_h^N \geq -\kappa$  for some  $n \leq N < \infty$  and  $\kappa \geq 0$ . Let  $1 < p < \infty$  and  $\gamma < 3 + \frac{p-1}{N-1}$ . For any positive weighted  $p$ -harmonic function  $w$  in a domain  $\Omega \subset M$ , we have  $|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \in W_{loc}^{1,2}(\Omega)$  and*

$$\int_{B(z,r)} \left| \nabla \left[ |\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \right] \right|^2 d\text{vol}_h \leq C(n, N, p, \gamma) \left[ \frac{1 + \sqrt{\kappa}r}{r} \right]^{p-\gamma+4} e^{\sqrt{\kappa}r} \tag{1.8}$$

whenever  $B(z, 4r) \Subset \Omega$ .

In particular, if  $p \in (1, 3 + \frac{2}{N-2})$ , then  $\nabla^2 \ln w \in L_{loc}^2(\Omega)$  and

$$\int_{B(z,r)} |\nabla^2 \ln w|^2 d\text{vol}_h \leq C(n, N, p) \left[ \frac{1 + \sqrt{\kappa}r}{r} \right]^4 e^{\sqrt{\kappa}r} \tag{1.9}$$

whenever  $B(z, 4r) \Subset \Omega$ .

As a consequence of Theorem 1.1 and Theorem 1.2, one gets that  $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W_{loc}^{1,2}$  for  $\gamma < 3 + \frac{p-1}{n-1}$  or  $\gamma < 3 + \frac{p-1}{N-1}$ , while in [21,5], one has  $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W_{loc}^{1,2}$  for all  $\gamma < 2$  (see (1.3) and the line above (1.5)). Thus our range for  $\gamma$  obviously improves the one obtained in [21,5] respectively.

Now we sketch the ideas to prove Theorem 1.1 and Theorem 1.2. Note that when  $N = n$  and  $h \equiv 1$ , we have  $Ric_h^N = Ric_g$ , and hence Theorem 1.1 corresponds to the special case  $N = n$  and  $h \equiv 1$  in Theorem 1.2. We only need to prove Theorem 1.2. As usual, we approximate  $u = -(p - 1) \ln w$  by smooth solution  $u^\epsilon$  to the standard approximation/regularized equation (3.3), that is,

$$e^h \text{div}(e^{-h} [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-2}{2}} \nabla u^\epsilon) = [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-2}{2}} |\nabla u^\epsilon|^2.$$

- (i) Using Bochner formula and the approximation equation (3.3), for  $0 < \eta < 1/2$  we bound the integral of

$$(1 - \eta) |\nabla^2 u^\epsilon|^2 + (p - \gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p - 2)(2 - \gamma) \frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \tag{1.10}$$

from above by the integral of

$$Ric_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle$$

and other first order terms, where all integrals are taken against  $[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g$  where  $\phi \in C_c^\infty(U)$  is a test function and  $U \Subset \Omega$ ; see Lemma 3.2. Here in (1.10) and in what follows, for any  $C^2$  function  $f$ ,  $\Delta_\infty f := \langle \nabla^2 f \nabla f, \nabla f \rangle$ .

- (ii) If  $\gamma < 3 + \frac{p-1}{N-1}$ , via a fundamental inequality given in Lemma 2.1 and the approximation equation (3.3), for sufficiently small  $\eta > 0$  we bound (1.10) as below

$$(1.10) \geq \eta |\nabla^2 u^\epsilon|^2 - \frac{\langle \nabla h, \nabla u^\epsilon \rangle^2}{N-n} - C \frac{1}{\eta} |\nabla u^\epsilon|^4 \quad \text{everywhere;}$$

see Lemma 3.4. This is crucial to get Theorem 1.2. Note that the approach in [21,5] could not give Lemma 3.4; see Remark 3.8 for details.

- (iii) Combining (i)&(ii) together, the integral of  $\eta |\nabla^2 u^\epsilon|^2$  is bounded from above by the integral of  $-Ric_h^N(\nabla u^\epsilon, \nabla u^\epsilon)$  and other first order terms, where all integrals are taken against  $[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g$ ; see Corollary 3.6.

Under the assumption  $Ric_h^N \geq -\kappa$ , in Lemma 3.7 we obtain an upper  $L_{loc}^2$  bound for  $\nabla[|\nabla u^\epsilon|^{\frac{p-\gamma}{2}} \nabla u^\epsilon] \phi$  by the integral of some first order terms, where all integrals are against  $e^{-h} d\text{vol}_g$ . A standard argument then leads to the proof of Theorem 1.2.

Finally, we also notice that the Cheng-Yau gradient estimate (1.1) was generalized to positive harmonic functions  $w$  in Alexandrov spaces with curvature bounded from below by Zhang-Zhu in [22], where the authors showed  $|\nabla \ln w|^2 \in W_{loc}^{1,2}(\Omega)$  as a key step. Furthermore, one could study the regularity of  $p$ -harmonic functions in more general metric measure spaces. In these spaces, a natural generalization of the (weighted) Ricci curvature bound is the curvature-dimension condition  $RCD(\kappa, N)$  in the sense of Bakry-Émery or Ambrosio-Gigli-Savaré. The two senses turned out to be equivalent by the work of Erbar-Kuwada-Sturm [6] (in the finite dimensional case) and Ambrosio-Gigli-Savaré [1] and the spaces satisfying one of the two equivalent conditions are known as  $RCD(\kappa, N)$  spaces. Some progress was made in  $RCD(\kappa, N)$  spaces. The Cheng-Yau gradient estimate was established by Jiang in [10] for positive harmonic functions  $w$  in  $RCD(\kappa, N)$  spaces; recently, Gigli-Violo in [7] established  $|\nabla \ln w|^{\beta/2} \in W_{loc}^{1,2}(\Omega)$  under  $RCD(0, N)$  spaces if  $\beta > \frac{N-2}{N-1}$ . However, when  $p \neq 2$ , it remains open to prove the Cheng-Yau type gradient estimates for positive  $p$ -harmonic functions in Alexandrov spaces and also  $RCD(\kappa, N)$  spaces.

## 2. Preliminaries

Let  $n \geq 2$  and  $M^n$  be a Riemannian manifold, and  $g$  be the Riemannian metric. By abuse of notation we also write  $|\xi|^2 = g(\xi, \xi)$  and  $\langle \xi, \eta \rangle = g(\xi, \eta)$  for all  $\xi, \eta \in T_x M^n$ . The corresponding Riemannian volume measure is written as  $d\text{vol}_g$ , and the volume of a set  $E$  is written as  $\text{vol}_g(E)$ . Denote by  $Ric_g$  the Ricci curvature 2-tensor and write  $Ric_g \geq -\kappa$  if  $Ric_g(\xi, \xi) \geq -\kappa |\xi|^2$  for all  $\xi \in T_x M^n$ .

For  $1 < p < \infty$ , the  $p$ -Laplace operator  $\Delta_p$  in  $M^n$  is given by

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2} \nabla f) \quad \forall f \in C^2(M^n).$$

Obviously,  $\Delta_2$  is exactly the Laplace-Beltrami operator  $\Delta$  in  $(M^n, g)$ . A function  $w$  defined in a domain  $\Omega \subset M^n$  is called  $p$ -harmonic if  $w \in W_{\text{loc}}^{1,p}(\Omega)$  is a weak solution to the  $p$ -Laplace equation  $\Delta_p w = 0$  in  $\Omega$ , that is,

$$\int_{\Omega} |\nabla w|^{p-2} \langle \nabla w, \nabla \phi \rangle d\operatorname{vol}_g = 0 \quad \forall \phi \in C_c^\infty(\Omega).$$

Note that 2-harmonic functions are the well-known harmonic functions.

Next we recall some basic facts of weighted Riemannian manifolds  $(M^n, g, e^{-h} d\operatorname{vol}_g)$ , where the weight  $h$  is a positive smooth function in  $M^n$ . The weighted measure  $d\operatorname{vol}_h = e^{-h} d\operatorname{vol}_g$  can be viewed as the volume form of a suitable conformal change of the metric  $g$ . Denote by  $\operatorname{vol}_h(E)$  the weighted volume of a set  $E$ . For  $n \leq N < \infty$ , the corresponding  $N$ -Bakry-Émery curvature tensor is

$$\operatorname{Ric}_h^N = \operatorname{Ric}_g + \nabla^2 h - \frac{\nabla h \otimes \nabla h}{N - n},$$

where when  $N = n$ , by convention,  $h$  is a constant function and hence  $\operatorname{Ric}_h^N = \operatorname{Ric}_g$ . We say that  $(M^n, g, e^{-h} d\operatorname{vol}_g)$  satisfies the Bakry-Émery curvature-dimension condition  $\operatorname{Ric}_h^N \geq -\kappa$  if

$$\operatorname{Ric}_h^N(\xi, \xi) = \operatorname{Ric}_g(\xi, \xi) + \langle \nabla^2 h \xi, \xi \rangle - \frac{\langle \nabla h, \xi \rangle^2}{N - n} \geq -\kappa \langle \xi, \xi \rangle \quad \forall \xi \in T_x M^n$$

By [18], under  $\operatorname{Ric}_h^N \geq -\kappa$ , one has the following volume comparison result

$$\operatorname{vol}_h(B_{2r}(x)) \leq C(N) e^{\sqrt{\kappa}r} \operatorname{vol}_h(B_r(x)) \quad \forall x \in M, r > 0. \tag{2.1}$$

For  $1 < p < \infty$ , the weighted  $p$ -Laplacian  $\Delta_{h,p}$  is defined as

$$\Delta_{p,h} f = e^h \operatorname{div}(e^{-h} |\nabla f|^{p-2} \nabla f) = \Delta_p f - |\nabla f|^{p-2} \langle \nabla f, \nabla h \rangle \quad \forall f \in C^2(M^n).$$

In the case  $p = 2$ , one writes  $\Delta_{2,h}$  as  $\Delta_h$ , and hence

$$\Delta_h f = \Delta f - \langle \nabla h, \nabla f \rangle.$$

A function  $w$  in a domain  $\Omega \subset M^n$  is called as a weighted  $p$ -harmonic function if  $w \in W_{\text{loc}}^{1,p}(\Omega)$  is a weak solution to the weighted  $p$ -harmonic equation  $\Delta_{p,h} w = 0$  in  $\Omega$ , that is,

$$\int_{\Omega} |\nabla w|^{p-2} \langle \nabla w, \nabla \phi \rangle e^{-h} d\text{vol}_g = 0 \quad \forall \phi \in C_c^\infty(\Omega). \tag{2.2}$$

By a density argument, we can relax  $\phi \in C_c^\infty(\Omega)$  to  $\phi \in W_0^{1,p}(\Omega)$  in (2.2).

We also recall the following Bochner formula in  $(M^n, g, e^{-h}d\text{vol}_g)$ :

$$\frac{1}{2} \Delta_h |\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta_h f \rangle + Ric_g(\nabla f, \nabla f) + \langle \nabla^2 h \nabla f, \nabla f \rangle \quad \forall f \in C^3(M), \tag{2.3}$$

which will be used in Section 3.

Finally, we recall the following fundamental inequality; see for example [21,5,14]. For the reader’s convenience we include it here. Recall that  $\Delta_\infty f = \langle \nabla^2 f \nabla f, \nabla f \rangle$ .

**Lemma 2.1.** *Let  $n \geq 2$  and  $\Omega$  be a domain of  $M^n$ . For any  $f \in C^2(\Omega)$ , we have*

$$|\nabla f|^4 |\nabla^2 f|^2 \geq 2|\nabla f|^2 |\nabla^2 f \nabla f|^2 + \frac{[|\nabla f|^2 \Delta f - \Delta_\infty f]^2}{n-1} - (\Delta_\infty f)^2 \text{ in } \Omega, \tag{2.4}$$

where when  $n = 2$ , “ $\geq$ ” becomes “ $=$ ”.

**Proof.** It suffices to prove that for any symmetric  $n \times n$  matrix  $A$  one has

$$|A|^2 |\xi|^4 \geq \frac{1}{n-1} (\text{tr} A |\xi|^2 - \langle A\xi, \xi \rangle)^2 + 2|A\xi|^2 |\xi|^2 - \langle A\xi, \xi \rangle^2 \quad \forall \xi \in \mathbb{R}^n. \tag{2.5}$$

Note that if  $\xi = 0$ , (2.5) holds obviously. Below assume that  $\xi \neq 0$ . Up to a scaling we may assume  $|\xi| = 1$ . By a change of coordinates, we may further assume  $\xi = e_n = (0, \dots, 0, 1)$ ; in this case, (2.5) reads as

$$|A|^2 \geq \frac{1}{n-1} (\text{tr} A - \langle Ae_n, e_n \rangle)^2 + 2|Ae_n|^2 - \langle Ae_n, e_n \rangle^2.$$

Denoting by  $A_{n-1}$  the  $(n-1)$  order principal submatrix of  $A$ , one has

$$|A|^2 = |A_{n-1}|^2 + 2|Ae_n|^2 - \langle Ae_n, e_n \rangle^2.$$

Noting that

$$|A_{n-1}|^2 \geq \frac{1}{n-1} (\text{tr} A_{n-1})^2 = \frac{1}{n-1} (\text{tr} A - \langle Ae_n, e_n \rangle)^2,$$

where when  $n = 2$ , one has  $|A_{n-1}|^2 = (\text{tr} A_{n-1})^2$ , one concludes (2.4).  $\square$



### 3. Proof of Theorem 1.2

Let  $w$  be a positive weighted  $p$ -harmonic function in a domain  $\Omega$ . Set  $u = -(p-1) \ln w$ . Then  $u$  is a weak solution to the equation

$$\Delta_p u - |\nabla u|^{p-2} \langle \nabla u, \nabla h \rangle = |\nabla u|^p \quad \text{in } \Omega, \tag{3.1}$$

that is,

$$-\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle e^{-h} d\text{vol}_g = \int_{\Omega} |\nabla u|^p \phi e^{-h} d\text{vol}_g \quad \forall \phi \in C_c^\infty(\Omega).$$

Given any smooth domain  $U \Subset \Omega$  and  $\epsilon \in (0, 1]$ , consider the approximation/regularized equation defined by

$$e^h \text{div}(e^{-h} [|\nabla v|^2 + \epsilon]^{\frac{p-2}{2}} \nabla v) = [|\nabla v|^2 + \epsilon]^{\frac{p-2}{2}} |\nabla v|^2 \quad \text{in } U; v = u \text{ on } \partial U. \tag{3.2}$$

It is well known that if  $u$  is the solution to (3.1), then  $u \in C^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$ ; see [3,12,19,20]. Moreover, in the following lemma, we summarize some properties of the solution  $u$  to (3.1) and  $u^\epsilon$  to (3.3), which result from [3] as a special case. See also [19].

**Lemma 3.1.** *For any  $\epsilon \in (0, 1]$ , there exists a unique solution  $u^\epsilon \in C^\infty(U) \cap C^0(\bar{U})$  to (3.3), and moreover,  $u^\epsilon \rightarrow u$  in  $C^0(\bar{U})$  and  $u^\epsilon \rightarrow u$  in  $C^{1,\alpha}(V)$  uniformly in  $\epsilon > 0$  as  $\epsilon \rightarrow 0$  for all  $V \Subset U$  where  $u$  is the solution to (3.1).*

To show Lemma 3.1, we just need to check that equations (3.1) and (3.3) are special cases of those considered in [3]. We put this verification in the appendix.

By Lemma 3.1, the solution  $u^\epsilon$  to (3.2) is  $C^\infty$ , which implies that  $u^\epsilon$  satisfies (3.2) pointwise. Hence by a direct computation, (3.2) is equivalent to

$$\Delta_h u^\epsilon + (p-2) \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} = |\nabla u^\epsilon|^2 \quad \text{in } U; u^\epsilon = u \text{ on } \partial U. \tag{3.3}$$

To prove Theorem 1.2 we first build up the following upper bound.

**Lemma 3.2.** *Let  $u^\epsilon$  be the solution to (3.3). For any  $\gamma \in \mathbb{R}$ ,  $\eta > 0$  and  $\phi \in C_c^\infty(U)$ , we have*

$$\begin{aligned} & \int_U \left\{ (1-\eta) |\nabla^2 u^\epsilon|^2 + (p-\gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p-2)(2-\gamma) \frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \right\} \\ & \quad \times [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\ & \leq - \int_U [\text{Ric}_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle] [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \end{aligned}$$

$$+ C(p, \gamma) \frac{1}{\eta} \int_U (|\nabla u^\epsilon|^2 + \epsilon)^{\frac{p-\gamma}{2}+1} |\nabla \phi|^2 + (|\nabla u^\epsilon|^2 + \epsilon)^{\frac{p-\gamma}{2}+2} \phi^2 e^{-h} d\text{vol}_g. \tag{3.4}$$

To prove this, we need the following identity.

**Lemma 3.3.** *For any  $v \in C^3(U)$  and  $\psi \in C_c^\infty(U)$ , one has*

$$\begin{aligned} \int_U |\nabla^2 v|^2 \psi e^{-h} d\text{vol}_g &= - \int_U \langle \nabla^2 v \nabla v - \Delta_h v \nabla v, \nabla \psi \rangle e^{-h} d\text{vol}_g + \int_U (\Delta_h v)^2 \psi e^{-h} d\text{vol}_g \\ &\quad - \int_U [\text{Ric}_g(\nabla v, \nabla v) + \langle \nabla^2 h \nabla v, \nabla v \rangle] \psi e^{-h} d\text{vol}_g. \end{aligned} \tag{3.5}$$

**Proof.** Applying the Bochner formula to  $v$ , one has

$$|\nabla^2 v|^2 + \text{Ric}_g(\nabla v, \nabla v) = \frac{1}{2} \Delta_h |\nabla v|^2 - \langle \nabla v, \nabla \Delta_h v \rangle - \langle \nabla^2 h \nabla v, \nabla v \rangle$$

and hence

$$\begin{aligned} |\nabla^2 v|^2 &= \left[ \frac{1}{2} \Delta_h |\nabla v|^2 - (\Delta_h v)^2 - \langle \nabla v, \nabla \Delta_h v \rangle \right] + (\Delta_h v)^2 \\ &\quad - [\text{Ric}_g(\nabla v, \nabla v) + \langle \nabla^2 h \nabla v, \nabla v \rangle]. \end{aligned}$$

By this, to get (3.5), it suffices to show the following identity

$$\begin{aligned} &\int_U \left[ \frac{1}{2} \Delta_h |\nabla v|^2 - (\Delta_h v)^2 - \langle \nabla v, \nabla \Delta_h v \rangle \right] \psi e^{-h} d\text{vol}_g \\ &= - \int_U \langle \nabla^2 v \nabla v - \Delta_h v \nabla v, \nabla \psi \rangle e^{-h} d\text{vol}_g. \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} -[(\Delta_h v)^2 + \langle \nabla v, \nabla(\Delta_h v) \rangle] &= -e^h \text{div}(e^{-h} \nabla v)(\Delta_h v) - e^h \langle e^{-h} \nabla v, \nabla(\Delta_h v) \rangle \\ &= -e^h \text{div}(e^{-h} \nabla v \Delta_h v). \end{aligned}$$

Via integration by parts, one has

$$\begin{aligned} - \int_U [(\Delta_h v)^2 + \langle \nabla v, \nabla(\Delta_h v) \rangle] \psi e^{-h} d\text{vol}_g &= - \int_U \text{div}(e^{-h} \nabla v \Delta_h v) \psi d\text{vol}_g \\ &= \int_U \langle \Delta_h v \nabla v, \nabla \psi \rangle e^{-h} d\text{vol}_g. \end{aligned}$$

Similarly, via integration by parts one also has

$$\begin{aligned} \frac{1}{2} \int_U \Delta_h |\nabla v|^2 \psi e^{-h} \, d\text{vol}_g &= \int_U \frac{1}{2} \text{div}(e^{-h} \nabla |\nabla v|^2) \psi \, d\text{vol}_g \\ &= - \int_U \frac{1}{2} \langle e^{-h} \nabla |\nabla v|^2, \nabla \psi \rangle \, d\text{vol}_g \\ &= - \int_U \langle \nabla^2 v \nabla v, \nabla \psi \rangle e^{-h} \, d\text{vol}_g. \end{aligned}$$

Combining together we obtain (3.6) and hence, (3.5) as desired.  $\square$

We are ready to prove Lemma 3.2 as below.

**Proof of Lemma 3.2.** Taking  $v = u^\epsilon$  and  $\psi = [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2$  in (3.5) we get

$$\begin{aligned} &\int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} \, d\text{vol}_g \\ &= - \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon - \Delta_h u^\epsilon \nabla u^\epsilon, \nabla [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 \rangle e^{-h} \, d\text{vol}_g \\ &\quad + \int_U (\Delta_h u^\epsilon)^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} \, d\text{vol}_g \\ &\quad - \int_U [\text{Ric}(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle] [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} \, d\text{vol}_g. \end{aligned} \tag{3.7}$$

To bound the second term in the right-hand side in (3.7), recalling (3.3), that is,

$$\Delta_h u^\epsilon = |\nabla u^\epsilon|^2 - (p-2) \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon}, \tag{3.8}$$

by Cauchy-Schwarz's inequality one has

$$(\Delta_h u^\epsilon)^2 \leq (p-2)^2 \frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + \frac{\eta}{4} |\nabla^2 u^\epsilon|^2 + C(p) \frac{1}{\eta} |\nabla u^\epsilon|^4,$$

where  $0 < \eta < 1$  is any constant. Thus

$$\begin{aligned} \int_U (\Delta_h u^\epsilon)^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} \, d\text{vol}_g &\leq (p-2)^2 \int_U (\Delta_\infty u^\epsilon)^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}-2} \phi^2 e^{-h} \, d\text{vol}_g \\ &\quad + \frac{\eta}{4} \int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} \, d\text{vol}_g \end{aligned}$$

$$+ \frac{C(p)}{\eta} \int_U [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+2} \phi^2 e^{-h} d\text{vol}_g. \tag{3.9}$$

The first term in the right-hand side in (3.7) is further written as

$$\begin{aligned} & - \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon - \Delta_h u^\epsilon \nabla u^\epsilon, \nabla [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 \rangle e^{-h} d\text{vol}_g \\ & = -(p-\gamma) \int_U \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\ & \quad + (p-\gamma) \int_U \Delta_h u^\epsilon \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\ & \quad - \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon, \nabla \phi^2 \rangle [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\text{vol}_g \\ & \quad + \int_U \langle \Delta_h u^\epsilon \nabla u^\epsilon, \nabla \phi^2 \rangle [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\text{vol}_g. \end{aligned} \tag{3.10}$$

Using (3.8) and Cauchy-Schwarz’s inequality, we obtain the following upper bound for the second term in (3.10):

$$\begin{aligned} & (p-\gamma) \int_U \Delta_h u^\epsilon \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\ & = -(p-\gamma)(p-2) \int_U (\Delta_\infty u^\epsilon)^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}-2} \phi^2 e^{-h} d\text{vol}_g \\ & \quad + (p-\gamma) \int_U \Delta_\infty u^\epsilon |\nabla u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}-1} \phi^2 e^{-h} d\text{vol}_g \\ & \leq -(p-\gamma)(p-2) \int_U (\Delta_\infty u^\epsilon)^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}-2} \phi^2 e^{-h} d\text{vol}_g \\ & \quad + \frac{\eta}{4} \int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\ & \quad + \frac{C(p)}{\eta} |p-\gamma|^2 \int_U [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+2} \phi^2 e^{-h} d\text{vol}_g. \end{aligned} \tag{3.11}$$

For the third term in the right-hand side of (3.10), by Cauchy-Schwarz’s inequality, one has

$$\begin{aligned}
 & \left| \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon, \nabla \phi^2 \rangle [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\text{vol}_g \right| \\
 & \leq \frac{\eta}{4} \int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g + C \frac{1}{\eta} \int_U [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+1} |\nabla \phi|^2 e^{-h} d\text{vol}_g.
 \end{aligned} \tag{3.12}$$

For the fourth term in the right-hand side of (3.10), in a similar way, using (3.8), one has

$$\begin{aligned}
 & \left| \int_U \langle \Delta_h u^\epsilon \nabla u^\epsilon, \nabla \phi^2 \rangle [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\text{vol}_g \right| \\
 & = \left| \int_U \left\langle |\nabla u^\epsilon|^2 \nabla u^\epsilon - (p-2) \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} \nabla u^\epsilon, \nabla \phi^2 \right\rangle [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\text{vol}_g \right| \\
 & \leq \frac{\eta}{4} \int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\
 & \quad + C(p) \frac{1}{\eta} \int_U ([|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+1} |\nabla \phi|^2 + [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+2} \phi^2) e^{-h} d\text{vol}_g.
 \end{aligned} \tag{3.13}$$

From (3.13), (3.12), (3.11) and (3.10) we attain

$$\begin{aligned}
 & - \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon - \Delta_h u^\epsilon \nabla u^\epsilon, \nabla [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 \rangle e^{-h} d\text{vol}_g \\
 & = \frac{3}{4} \eta \int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\
 & \quad - (p-\gamma) \int_U \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\
 & \quad - (p-\gamma)(p-2) \int_U (\Delta_\infty u^\epsilon)^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}-2} \phi^2 e^{-h} d\text{vol}_g \\
 & \quad + \frac{C(p)}{\eta} \int_U ([|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+1} |\nabla \phi|^2 + [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+2} \phi^2) e^{-h} d\text{vol}_g.
 \end{aligned} \tag{3.14}$$

Obviously from (3.14), (3.9) and (3.7) we conclude (3.4).  $\square$

If  $\gamma < 3 + \frac{p-1}{N-1}$ , we get the following pointwise lower bound. Recall that when  $N = n$ , we always assume that  $h$  is a constant function and  $\frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N-n} = 0$ .

**Lemma 3.4.** *Let  $u^\epsilon$  be the solution to (3.3). If  $\gamma < 3 + \frac{p-1}{N-1}$  for some  $N \geq n$ , then for sufficiently small  $\eta > 0$  we have*

$$\begin{aligned} & (1 - \eta)|\nabla^2 u^\epsilon|^2 + (p - \gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p - 2)(2 - \gamma) \frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \\ & \geq \eta |\nabla^2 u^\epsilon|^2 - \frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N - n} - C(n, N, p, \gamma) \frac{1}{\eta} |\nabla u^\epsilon|^4. \end{aligned} \tag{3.15}$$

To prove this, we need the following pointwise lower bound for  $|\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4$ .

**Lemma 3.5.** *Let  $u^\epsilon$  be the solution to (3.3). If  $N \geq n$ , then for  $0 < \eta < 1$  we have*

$$\begin{aligned} (1 + \eta) |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4 & \geq 2 |\nabla^2 u^\epsilon \nabla u^\epsilon|^2 |\nabla u^\epsilon|^2 \\ & + \left( \frac{1}{N - 1} \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 - 1 \right) (\Delta_\infty u^\epsilon)^2 \\ & - (1 + \eta) \frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N - n} |\nabla u^\epsilon|^4 - C(n, N, p) \frac{1}{\eta} |\nabla u^\epsilon|^8. \end{aligned} \tag{3.16}$$

**Proof.** Applying (2.4) to  $u^\epsilon$  one has

$$|\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4 \geq 2 |\nabla u^\epsilon|^2 |\nabla^2 u^\epsilon \nabla u^\epsilon|^2 + \frac{[|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n - 1} - (\Delta_\infty u^\epsilon)^2 \tag{3.17}$$

By (3.8) and  $\Delta u^\epsilon = \Delta_h u^\epsilon + \langle \nabla h, \nabla u^\epsilon \rangle$ , we have

$$\Delta u^\epsilon = |\nabla u^\epsilon|^2 + \langle \nabla u^\epsilon, \nabla h \rangle - (p - 2) \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon}.$$

Thus

$$|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon = |\nabla u^\epsilon|^2 (|\nabla u^\epsilon|^2 + \langle \nabla u^\epsilon, \nabla h \rangle) - \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right] \Delta_\infty u^\epsilon,$$

and hence,

$$\begin{aligned} [|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon]^2 & = \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 (\Delta_\infty u^\epsilon)^2 \\ & + |\nabla u^\epsilon|^4 (|\nabla u^\epsilon|^2 + \langle \nabla u^\epsilon, \nabla h \rangle)^2 \\ & - 2 \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right] |\nabla u^\epsilon|^4 \Delta_\infty u^\epsilon \\ & - 2 \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right] |\nabla u^\epsilon|^2 \Delta_\infty u^\epsilon \langle \nabla u^\epsilon, \nabla h \rangle \\ & =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.18}$$

Note that

$$\left| (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right|^2 \leq 4p^2, \tag{3.19}$$

which can be obtained by considering  $p > 2$  and  $1 < p < 2$  separately. Using this, Cauchy-Schwarz inequality, for  $0 < \eta < 1$ , we have

$$I_3 \geq -\eta |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4 - C(p) \frac{1}{\eta} |\nabla u^\epsilon|^8. \tag{3.20}$$

If  $h$  is a constant function and hence  $\nabla h = 0$ ,  $I_2 \geq 0$  and  $I_4 = 0$ , dividing by  $n - 1$  in both sides of (3.18), by (3.20) one has

$$\begin{aligned} \frac{[|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n - 1} &\geq -\eta |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4 \\ &+ \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 \frac{(\Delta_\infty u^\epsilon)^2}{n - 1} - \frac{C(p)}{\eta} |\nabla u^\epsilon|^8. \end{aligned}$$

Plugging this in (3.17), noting  $N = n$ , and adding  $\eta |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4$  in both sides, one concludes (3.16).

If  $h$  is not a constant function, set  $\eta_1 = \frac{N-n}{N-1}$ . Then

$$1 - \eta_1 = \frac{n - 1}{N - 1} > 0 \quad \text{and} \quad 1 - \frac{1}{\eta_1} = -\frac{n - 1}{N - n} < 0. \tag{3.21}$$

For any  $0 < \eta < 1$  one has

$$\begin{aligned} I_2 &\geq |\nabla u^\epsilon|^4 \langle \nabla u^\epsilon, \nabla h \rangle^2 + 2 |\nabla u^\epsilon|^6 \langle \nabla u^\epsilon, \nabla h \rangle \\ &\geq [1 + \eta(1 - \frac{1}{\eta_1})] |\nabla u^\epsilon|^4 \langle \nabla u^\epsilon, \nabla h \rangle^2 - \frac{1}{\eta |1 - \frac{1}{\eta_1}|} |\nabla u^\epsilon|^8. \end{aligned} \tag{3.22}$$

Using Cauchy-Schwarz inequality, we have

$$I_4 \geq -\eta_1 \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 (\Delta_\infty u^\epsilon)^2 - \frac{1}{\eta_1} \langle \nabla u^\epsilon, \nabla h \rangle^2 |\nabla u^\epsilon|^4 \tag{3.23}$$

Dividing by  $n - 1$  in both sides of (3.18), by (3.20), (3.22) and (3.23) one has

$$\begin{aligned} \frac{[|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n - 1} &\geq -\eta |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4 + \frac{1 - \eta_1}{n - 1} \left[ (p - 2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 (\Delta_\infty u^\epsilon)^2 \\ &+ (1 + \eta) \frac{1 - \frac{1}{\eta_1}}{n - 1} \langle \nabla u^\epsilon, \nabla h \rangle^2 |\nabla u^\epsilon|^4 - C(n, N, p) \frac{1}{\eta} |\nabla u^\epsilon|^8 \end{aligned}$$

By (3.21),

$$\begin{aligned} \frac{[|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n-1} &\geq -\eta |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4 + \frac{1}{N-1} \left[ (p-2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 (\Delta_\infty u^\epsilon)^2 \\ &\quad - (1+\eta) \frac{1}{N-n} \langle \nabla u^\epsilon, \nabla h \rangle^2 |\nabla u^\epsilon|^4 - C(n, N, p) \frac{1}{\eta} |\nabla u^\epsilon|^8. \end{aligned}$$

Plugging this in (3.17), and adding  $\eta |\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4$  in both sides, we conclude (3.16) as desired.  $\square$

We now prove Lemma 3.4 by using Lemma 3.5.

**Proof of Lemma 3.4.** Given any point  $x \in U$ , if  $\nabla u^\epsilon(x) = 0$ , then (3.15) holds trivially. Below we assume that  $\nabla u^\epsilon(x) \neq 0$ . At such point  $x$ , we already have (3.16) in Lemma 3.5. Dividing by  $|\nabla u^\epsilon|^4$  in both sides of (3.16), for  $0 < \eta < 1/2$  we obtain

$$\begin{aligned} (1+\eta) |\nabla^2 u^\epsilon|^2 &\geq 2 \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2} + \left( \frac{1}{N-1} \left[ (p-2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 - 1 \right) \frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4} \\ &\quad - (1+\eta) \frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N-n} - \frac{C(n, N, p)}{\eta} |\nabla u^\epsilon|^4. \end{aligned}$$

In both sides, multiplying by  $\frac{1-2\eta}{1+\eta} > 0$  and adding

$$\eta |\nabla^2 u^\epsilon|^2 + (p-\gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p-2)(2-\gamma) \frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2},$$

we get

$$\begin{aligned} &(1-\eta) |\nabla^2 u^\epsilon|^2 + (p-\gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p-2)(2-\gamma) \frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \\ &\geq \eta |\nabla^2 u^\epsilon|^2 + \left\{ \frac{1-2\eta}{1+\eta} 2 + (p-\gamma) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} \right\} \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2} \\ &\quad + \left\{ \frac{1-2\eta}{1+\eta} \left( \frac{1}{N-1} \left[ (p-2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 - 1 \right) \right. \\ &\quad \left. + (p-2)(2-\gamma) \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \right\} \frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4} \\ &\quad - (1-2\eta) \frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N-n} - C(n, N, p) \frac{1}{\eta} |\nabla u^\epsilon|^4 \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.24}$$



Recall that if  $N = n$  that is,  $h$  is a constant function,  $I_4 = 0$  by our convention. If  $N > n$  that is,  $h$  is not a constant, then by  $1 - 2\eta < 1$ , we have

$$I_4 \geq -\frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N - n}. \tag{3.25}$$

To bound  $I_2 + I_3$  from below, since  $\gamma < 3 + \frac{p-1}{N-1}$  and  $N \geq 2$  implies

$$p + 2 - \gamma > p + 2 - 3 - \frac{p-1}{N-1} = (p-1)\left(1 - \frac{1}{N-1}\right) \geq 0,$$

we can find  $0 < \hat{\eta}(p, \gamma) < 1/2$  such that for  $0 < \eta < \hat{\eta}$ , one has  $p + 2\frac{1-2\eta}{1+\eta} - \gamma > 0$ . Thus the coefficient of  $I_2$  satisfies

$$(p - \gamma) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 2\frac{1-2\eta}{1+\eta} \geq (p + 2\frac{1-2\eta}{1+\eta} - \gamma) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + \frac{1-2\eta}{1+\eta} \frac{\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} > 0.$$

Using this and observing

$$\frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2} \geq \frac{|\Delta_\infty u^\epsilon|^2}{|\nabla u^\epsilon|^4},$$

one has

$$\begin{aligned} & I_2 + I_3 \\ & \geq \left\{ (p - \gamma) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 2\frac{1-2\eta}{1+\eta} \right. \\ & \quad \left. + \frac{1-2\eta}{1+\eta} \left( \frac{1}{N-1} \left[ (p-2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 - 1 \right) \right. \\ & \quad \left. + (p-2)(2-\gamma) \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \right\} \frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4} \\ & =: H(\eta) \frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4} \end{aligned}$$

We claim that there exists  $0 < \bar{\eta}(n, N, p, \gamma) < \hat{\eta}$  such that  $H(\eta) > 0$  for all  $0 < \eta < \bar{\eta}$ . Assuming this claim holds for the moment, for any  $0 < \eta < \bar{\eta}$ , one has  $I_2 + I_3 > 0$ . From this, (3.24) and (3.25) we conclude (3.15) as desired.

Finally we prove the above claim. It suffices to show that

$$\begin{aligned} H(0) := & (p - \gamma) \frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]} + 2 + \left( \frac{1}{N-1} \left[ (p-2) \frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 - 1 \right) \\ & + (p-2)(2-\gamma) \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \end{aligned}$$

$$> \delta(N, p, \gamma), \tag{3.26}$$

where  $\delta(N, p, \gamma) > 0$  is a constant. Indeed, by (3.19), one has

$$H(\eta) \geq H(0) - 2\left[1 - \frac{1 - 2\eta}{1 + \eta}\right] - \left[1 - \frac{1 - 2\eta}{1 + \eta}\right]\left[\frac{4p^2}{N - 1} - 1\right] \geq \delta(N, p, \gamma) - 15p^2\eta.$$

If  $0 < \eta < \bar{\eta} =: \min\{\hat{\eta}, \delta(N, p, \gamma)/15p^2\}$ , one has  $H(\eta) > 0$  and hence the claim holds as desired.

We prove (3.26) as below. Since

$$\left[(p - 2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1\right]^2 = (p - 2)^2\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + 2(p - 2)\frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]} + 1,$$

we rewrite

$$H(0) = (p - 2)\left[2 - \gamma + \frac{p - 2}{N - 1}\right]\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + \left[p - \gamma + \frac{2(p - 2)}{N - 1}\right]\frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]} + \frac{N}{N - 1}.$$

Observing

$$\frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]} = \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + \frac{\epsilon|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2}$$

and

$$1 = \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + 2\frac{\epsilon|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + \frac{\epsilon^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2},$$

we further write

$$\begin{aligned} H(0) &= \left\{ (p - 2)\left[2 - \gamma + \frac{p - 2}{N - 1}\right] + \left[p - \gamma + \frac{2(p - 2)}{N - 1}\right] + \frac{N}{N - 1} \right\} \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \\ &+ \left\{ \left[p - \gamma + \frac{2(p - 2)}{N - 1}\right] + 2\frac{N}{N - 1} \right\} \frac{\epsilon|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \\ &+ \frac{N}{N - 1} \frac{\epsilon^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2}. \end{aligned}$$

By a direct calculation,  $\gamma < 3 + \frac{p-1}{N-1}$  implies that

$$\begin{aligned} \left[p - \gamma + \frac{2(p - 2)}{N - 1}\right] + 2\frac{N}{N - 1} &> p + \frac{2(p - 2)}{N - 1} + 2\frac{N}{N - 1} - 3 - \frac{p - 1}{N - 1} \\ &= p - 1 + \frac{2(p - 2) + 2 - (p - 1)}{N - 1} \\ &= (p - 1)\frac{N}{N - 1} \end{aligned}$$

$$> 0.$$

Moreover,  $\gamma < 3 + \frac{p-1}{N-1}$  also implies that

$$\begin{aligned} & (p-2)[2 - \gamma + \frac{p-2}{N-1}] + [p - \gamma + \frac{2(p-2)}{N-1}] + \frac{N}{N-1} \\ &= 3(p-1) + \frac{(p-2)^2 + 2(p-2) + 1}{N-1} - (p-1)\gamma \\ &= 3(p-1) + \frac{(p-1)^2}{N-1} - (p-1)\gamma \\ &= (p-1)[3 + \frac{p-1}{N-1} - \gamma] \\ &> 0. \end{aligned}$$

Thus

$$\begin{aligned} H(0) &> (p-1)[3 + \frac{p-1}{N-1} - \gamma] \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} + \frac{N}{N-1} \frac{\epsilon^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \\ &\geq \frac{1}{2} \min \left\{ (p-1)[3 + \frac{p-1}{N-1} - \gamma], \frac{N}{N-1} \right\} \\ &=: \delta(N, p, \gamma) \\ &> 0 \end{aligned}$$

that is, (3.26) holds.  $\square$

Combining (3.15) and (3.4) we have the following. Recall that

$$Ric_h^N(\nabla u^\epsilon, \nabla u^\epsilon) = Ric_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle - \frac{\langle \nabla u^\epsilon, \nabla h \rangle^2}{N-n}.$$

**Corollary 3.6.** *Let  $u^\epsilon$  be the solution to (3.3). If  $\gamma < 3 + \frac{p-1}{N-1}$  for some  $N \geq n$ , then for sufficiently small  $\eta > 0$  one has*

$$\begin{aligned} & \eta \int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} dvol_g \\ & \leq - \int_U Ric_h^N(\nabla u^\epsilon, \nabla u^\epsilon) [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} dvol_g \\ & \quad + C(n, N, p, \gamma, \eta) \int_U \left( [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+1} |\nabla \phi|^2 + [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+2} \phi^2 \right) e^{-h} dvol_g \end{aligned} \tag{3.27}$$

Under the Bakry-Émery curvature-dimension assumption, we have the following uniform upper bound.

**Lemma 3.7.** *Let  $u^\epsilon$  be the solution to (3.3). If  $\gamma < 3 + \frac{p-1}{N-1}$  and  $Ric_h^N \geq -\kappa$ , then one has*

$$\begin{aligned} & \int_U |\nabla[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{4}} \nabla u^\epsilon|^2 \phi^2 e^{-h} d\text{vol}_g \\ & \leq C(n, N, p, \gamma) \int_U \kappa |\nabla u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g \\ & \quad + C(n, N, p, \gamma) \int_U \left( [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+1} |\nabla \phi|^2 + [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}+2} \phi^2 \right) e^{-h} d\text{vol}_g \end{aligned} \tag{3.28}$$

**Proof.** By  $Ric_h^N \geq -\kappa$  we know that

$$-Ric_h^N(\nabla u^\epsilon, \nabla u^\epsilon) \leq \kappa |\nabla u^\epsilon|^2$$

Thus the first term in the right-hand side of (3.27) is bounded from above by

$$\kappa \int_U |\nabla u^\epsilon|^2 [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\text{vol}_g.$$

On the other hand, a direct calculation leads to

$$\begin{aligned} & |\nabla[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{4}} \nabla u^\epsilon|^2 \\ & = [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \left| \nabla^2 u^\epsilon + \frac{p-\gamma}{2} \frac{\nabla u^\epsilon \otimes \nabla^2 u^\epsilon \nabla u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} \right|^2 \\ & = [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} [|\nabla^2 u^\epsilon|^2 + (p-\gamma) \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + \frac{(p-\gamma)^2}{4} \frac{|\nabla u^\epsilon|^2 |\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2}] \\ & \leq C(n, p, \gamma) [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} |\nabla^2 u^\epsilon|^2. \end{aligned}$$

Thus, up to a constant multiplier, the left-hand side of (3.27) is bounded by

$$\int_U |\nabla[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{4}} \nabla u^\epsilon|^2 e^{-h} d\text{vol}_g.$$

We therefore conclude (3.28) from (3.27).  $\square$

Now we are able to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let  $w \in W_{\text{loc}}^{1,p}(\Omega)$  be any positive weighted  $p$ -harmonic function in the domain  $\Omega$  and  $u = -(p - 1) \ln w$ . Given any smooth domain  $U \Subset \Omega$ , for each  $\epsilon \in (0, 1]$ , let  $u^\epsilon \in C^\infty(U)$  be the solution to (3.3). By Lemma 3.1, we know that  $u^\epsilon \rightarrow u \in C^{1,\alpha}(U)$ , for some  $\alpha \in (0, 1)$  uniformly in  $\epsilon > 0$  as  $\epsilon \rightarrow 0$ . Using this and choosing suitable test functions  $\phi \in C_c^\infty(U)$  in (3.28), one concludes  $[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{4}} \nabla u^\epsilon \in W_{\text{loc}}^{1,2}(U)$  uniformly in  $\epsilon \in (0, 1]$ .

Next, we claim that

$$|\nabla u|^{\frac{p-\gamma}{2}} \nabla u \in W_{\text{loc}}^{1,2}(U), \tag{3.29}$$

and

$$\nabla([|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{4}} \nabla u^\epsilon) \rightarrow \nabla(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u) \text{ weakly in } L^2_{\text{loc}}(U, \mathbb{R}^{n \times n}) \text{ as } \epsilon \rightarrow 0. \tag{3.30}$$

To see this, for any subdomain  $V \Subset U$ , by Lemma 3.7, we already have

$$\sup_{\epsilon \in (0,1]} \|\nabla([|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{4}} \nabla u^\epsilon)\|_{L^2(V, \mathbb{R}^{n \times n})} < C(\kappa, n, N, p, \gamma, V).$$

For any subsequence  $\{\epsilon_j\}_{j \in \mathbb{N}}$  which converges to 0, by the weak compactness of  $W^{2,2}(V)$ , up to some subsequence one has  $\nabla([|\nabla u^{\epsilon_j}|^2 + \epsilon_j]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon_j}) \rightarrow z$  weakly in  $L^2(V, \mathbb{R}^{n \times n})$  for some function  $z \in L^2(V, \mathbb{R}^{n \times n})$ . Let  $\{e_1, \dots, e_n\} \subset T_x U$  be a local orthonormal frame at each  $x \in U$ . Notice that the  $n \times n$  matrix

$$\nabla([|\nabla u^{\epsilon_j}|^2 + \epsilon_j]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon_j}) = \left( \nabla_{e_l}([|\nabla u^{\epsilon_j}|^2 + \epsilon_j]^{\frac{p-\gamma}{4}} \nabla_{e_k} u^{\epsilon_j}) \right)_{1 \leq k, l \leq n}.$$

Recalling from Lemma 3.1 that  $\nabla u^\epsilon \rightarrow \nabla u$  in  $C^\alpha(U)$  and  $V \Subset U$ , for any  $\phi \in C_c^\infty(U)$  with  $\phi|_V = 1$  and  $1 \leq k, l \leq n$ , we have

$$\begin{aligned} & \lim_{j \rightarrow 0} \int_U \nabla_{e_l}([|\nabla u^{\epsilon_j}|^2 + \epsilon_j]^{\frac{p-\gamma}{4}} \nabla_{e_k} u^{\epsilon_j}) \phi e^{-h} d\text{vol}_g \\ &= - \lim_{j \rightarrow 0} \int_U ([|\nabla u^{\epsilon_j}|^2 + \epsilon_j]^{\frac{p-\gamma}{4}} \nabla_{e_k} u^{\epsilon_j}) \nabla_{e_l}(\phi e^{-h}) d\text{vol}_g \\ &= - \int_U (|\nabla u|^{\frac{p-\gamma}{2}} \nabla_{e_k} u) \nabla_{e_l}(\phi e^{-h}) d\text{vol}_g \\ &= \int_U \nabla_{e_l}(|\nabla u|^{\frac{p-\gamma}{2}} \nabla_{e_k} u) \phi e^{-h} d\text{vol}_g. \end{aligned}$$

This shows that in the distributional sense

$$\nabla([|\nabla u^{\epsilon_j}|^2 + \epsilon_j]^{\frac{p-\gamma}{4}} \nabla u^{\epsilon_j}) \rightarrow \nabla(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u).$$

Thus  $z = \nabla(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u)|_V \in L^2(V, \mathbb{R}^{n \times n})$  in distributional sense. We therefore have  $|\nabla u|^{\frac{p-\gamma}{2}} \nabla u|_V \in W^{1,2}(V)$ , which gives (3.29).

Moreover, by the arbitrariness of subsequence  $\{\epsilon_j\}$ , we have

$$\nabla(|\nabla u^\epsilon|^2 + \epsilon)^{\frac{p-\gamma}{4}} \nabla u^\epsilon \rightarrow \nabla(|\nabla u|^{\frac{p-\gamma}{2}} \nabla u)$$

weakly in  $L^2(V, \mathbb{R}^{n \times n})$  as  $\epsilon \rightarrow 0$ . Hence by the arbitrariness of  $V \Subset U$ , (3.30) holds.

Letting  $\epsilon \rightarrow 0$  in (3.28) and using the convergence in the above verified claim, we obtain

$$\begin{aligned} & \int_U |\nabla[|\nabla u|^{\frac{p-\gamma}{2}} \nabla u]|^2 \phi^2 e^{-h} d\text{vol}_g \\ & \leq C(n, N, p, \gamma) \kappa \int_U |\nabla u|^{p-\gamma+2} \phi^2 e^{-h} d\text{vol}_g \\ & \quad + C(n, N, p, \gamma) \int_U (|\nabla u|^{p-\gamma+2} |\nabla \phi|^2 + |\nabla u|^{p-\gamma+4} \phi^2) e^{-h} d\text{vol}_g. \end{aligned} \tag{3.31}$$

Let  $\phi \in C_c^\infty(B_{2r})$ , where  $B_{4r} \subset U$ , such that  $\phi = 1$  in  $B_r$  and  $|\nabla \phi| \leq \frac{C}{r}$ . Then (3.31) becomes

$$\begin{aligned} & \int_{B_r} |\nabla[|\nabla u|^{\frac{p-\gamma}{2}} \nabla u]|^2 e^{-h} d\text{vol}_g \\ & \leq C(n, N, p, \gamma) \int_{B_{2r}} \left[ \left(\frac{1}{r^2} + \kappa\right) |\nabla u|^{p-\gamma+2} + |\nabla u|^{p-\gamma+4} \right] e^{-h} d\text{vol}_g. \end{aligned}$$

Recalling from (1.5) the Cheng-Yau type gradient estimate that  $|\nabla u| \leq C(n, N, p) \frac{1+\sqrt{\kappa r}}{r}$  and noting that  $\gamma < 3 + \frac{p-1}{N-1}$  guarantees  $p - \gamma + 2 > 0$ , we deduce

$$|\nabla u|^{p-\gamma+2} \leq C(n, N, p, \gamma) \left[ \frac{1+\sqrt{\kappa r}}{r} \right]^{p-\gamma+2}.$$

Together with  $\frac{1}{r^2} + \kappa \leq \left(\frac{1+\sqrt{\kappa r}}{r}\right)^2$ , we conclude

$$\int_{B_r} |\nabla[|\nabla u|^{\frac{p-\gamma}{2}} \nabla u]|^2 e^{-h} d\text{vol}_g \leq C(n, N, p, \gamma) \text{vol}_h(B_{2r}) \left[ \frac{1+\sqrt{\kappa r}}{r} \right]^{p-\gamma+4}.$$

Dividing both sides by  $\text{vol}_h(B_r)$ , noting  $\text{vol}_h(B_{2r}) \leq e^{\sqrt{\kappa r}} \text{vol}_h(B_r)$  from the volume comparison (2.1), and recalling  $u = -(p-1) \ln w$ , we conclude (1.8).

Note that (1.9) is just the special case  $\gamma = p$  of (1.8), where  $p < 3 + \frac{2}{N-2}$  guarantees  $p < 3 + \frac{p-1}{N-1}$  and hence one can take  $\gamma = p$  in (1.8).  $\square$

Finally, we compare our proof with [21,5], in particular, the crucial pointwise lower bound given in Lemma 3.4 and Lemma 3.5.

**Remark 3.8.** (i) It was well known that a positive (weighted)  $p$ -harmonic function  $w$ , and hence  $\ln w$ , is always smooth outside of the null set  $E_w$  of  $\nabla \ln w$ . In  $\Omega \setminus E_w$ , the proof of Lemma 3.5 works for  $\ln w$  so to get (3.16) with  $u^\epsilon$  replaced by  $\ln w$  and  $\epsilon = 0$ , dividing both sides of which by  $|\nabla \ln w|^4$ , for  $0 < \eta < 1/2$  one gets

$$(1 + \eta)|\nabla^2 \ln w|^2 \geq 2 \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} + \left( \frac{(p-1)^2}{N-1} - 1 \right) \frac{(\Delta_\infty \ln w)^2}{|\nabla \ln w|^4} - (1 + \eta) \frac{\langle \nabla \ln w, \nabla h \rangle^2}{N-n} - C(n, N, p) \frac{1}{\eta} |\nabla \ln w|^4. \tag{3.32}$$

If  $\gamma < 3 + \frac{p-1}{N-1}$ , using (3.32) and noting that the proof of Lemma 3.4 works for  $\ln w$ , we get (3.15) with  $u^\epsilon$  replaced by  $\ln w$  and  $\epsilon = 0$ , that is, for  $\eta > 0$  sufficiently small,

$$(1 - \eta)|\nabla^2 \ln w|^2 + (p - \gamma) \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} + (p - 2)(2 - \gamma) \frac{(\Delta_\infty \ln w)^2}{|\nabla \ln w|^4} \geq \eta |\nabla^2 \ln w|^2 - \frac{\langle \nabla \ln w, \nabla h \rangle^2}{N-n} - C(n, N, p, \gamma) \frac{1}{\eta} |\nabla \ln w|^4. \tag{3.33}$$

From the proof, we see that both of the coefficient 2 of  $\frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2}$  and the coefficient  $\frac{(p-1)^2}{N-1} - 1$  of  $(\Delta_\infty \ln w)^2$  in (3.32) are critical to guarantee the existence of sufficiently small  $\eta > 0$  in (3.33) when  $\gamma < 3 + \frac{p-1}{N-1}$ .

On the other hand, instead of (3.32), recall the following lower bound obtained in [5] by using Lemma 2.1 and the equation (3.1):

$$|\nabla^2 \ln w|^2 \geq \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} - 2 \frac{p-1}{n-1} \Delta_\infty \ln w + \frac{1}{N-1} |\nabla \ln w|^2 - \frac{\langle \nabla \ln w, \nabla h \rangle^2}{N-n}, \tag{3.34}$$

and also, when  $N = n$  and  $h \equiv 1$ , recall the following lower bound derived in [21] via Lemma 2.1 and (3.1):

$$|\nabla^2 \ln w|^2 \geq [1 + \min\{\frac{(p-1)^2}{n-1}, 1\}] \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} - 2 \frac{p-1}{n-1} \Delta_\infty \ln w + \frac{1}{n-1} |\nabla \ln w|^2. \tag{3.35}$$

From (3.34) and (3.35), via a direct check one can conclude  $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W_{loc}^{1,2}$  for  $\gamma < 2$ , but NOT for all  $\gamma < 3 + \frac{p-1}{N-1}$ .

(ii) Moreover, unlike [21,5] where the authors differentiate the equation (3.1) for  $\ln w$ , we directly derive an upper bound from Bochner formula for the left-hand side of (3.33) with respect to  $[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\text{vol}_g$ .

**Data availability**

No data was used for the research described in the article.

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**Appendix A. Proof of Lemma 3.1**

In the appendix, we show Lemma 3.1 by checking equations (3.1) and (3.3) are special cases considered in [3]. To this end, we recall the result in [3].

Let  $\Omega$  be a domain of  $M^n$ . Consider the equation

$$-\operatorname{div} \vec{a}(x, \nabla u) + b(x, \nabla u) = 0 \quad \text{in } \Omega \tag{A.1}$$

where  $\vec{a}$  is a map from  $\Omega \times \mathbb{R}^n$  to  $\mathbb{R}^n$  and  $b$  maps  $\Omega \times \mathbb{R}^n$  to  $\mathbb{R}$ . Let  $\{e_1, \dots, e_n\} \subset T_x \Omega$  be a local orthonormal frame at each  $x \in \Omega$ . By a weak solution of (A.1) we mean a function  $u \in W_{\text{loc}}^{1,p}(\Omega)$  such that

$$\int_{\Omega} [ \langle \vec{a}(x, \nabla u), \nabla \phi \rangle + b(x, \nabla u) \phi ] \, d\text{vol}_g = 0 \quad \forall \phi \in C_c^\infty(\Omega). \tag{A.2}$$

Assume the following holds for  $\vec{a} = (a_1, \dots, a_n)$  and  $b$ .

$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta) \xi_i \xi_j \geq \gamma_0 |\eta|^{p-2} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, p > 1, \tag{A_1}$$

$$\left| \frac{\partial a_j}{\partial \eta_i} \right| \leq \gamma_1 |\eta|^{p-2}, \quad 1 \leq i, j \leq n, \tag{A_2}$$

$$|\nabla_{e_i} a_j(x, \eta)| \leq \gamma_1 |\eta|^{p-1}, \quad 1 \leq i, j \leq n, \tag{A_3}$$

$$|b(x, \eta)| \leq \gamma_1 |\eta|^p, \tag{A_4}$$

and

$$|\nabla_{e_i} b(x, \eta)| \leq \gamma_1 |\eta|^p, \quad \left| \frac{\partial b}{\partial \eta_i}(x, \eta) \right| \leq \gamma_1 |\eta|^{p-1}, \quad 1 \leq i \leq n, \tag{B}$$

for all  $\eta \in \mathbb{R}^n$ , where  $\gamma_i$  are positive constants,  $i = 0, 1$ .



For any smooth domain  $U \Subset \Omega$  and  $\epsilon \in (0, 1]$ , consider the regularized equation

$$-\operatorname{div} \vec{a}^\epsilon(x, \nabla u^\epsilon) + b^\epsilon(x, \nabla u^\epsilon) = 0 \quad \text{in } U \text{ and } u^\epsilon = u \text{ on } \partial U \tag{A.3}$$

where  $\vec{a}^\epsilon$  is a map from  $U \times \mathbb{R}^n$  to  $\mathbb{R}^n$  and  $b^\epsilon$  maps  $U \times \mathbb{R}^n$  to  $\mathbb{R}$  such that

$$\lim_{\epsilon \rightarrow 0} \vec{a}^\epsilon(x, \eta) = \vec{a}(x, \eta) \text{ and } \lim_{\epsilon \rightarrow 0} b^\epsilon(x, \eta) = b(x, \eta) \quad \forall (x, \eta) \in \Omega \times \mathbb{R}^n.$$

The weak solution of (A.3) is defined similarly as (A.2). Assume the following holds for  $\vec{a}^\epsilon = (a_1^\epsilon, \dots, a_n^\epsilon)$  and  $b^\epsilon$ .

$$\sum_{i,j=1}^n \frac{\partial a_j^\epsilon}{\partial \eta_i}(x, \eta) \xi_i \xi_j \geq \gamma_0(\epsilon + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2, \quad \xi \in \mathbb{R}^n, p > 1, \tag{A_{1,\epsilon}}$$

$$\left| \frac{\partial a_j^\epsilon}{\partial \eta_i} \right| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p-2}{2}}, \quad 1 \leq i, j \leq n, \tag{A_{2,\epsilon}}$$

$$|\nabla_{e_i} a_j^\epsilon(x, \eta)| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p-1}{2}}, \quad 1 \leq i, j \leq n, \tag{A_{3,\epsilon}}$$

$$|b^\epsilon(x, \eta)| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p}{2}}, \tag{A_{4,\epsilon}}$$

for all  $\eta \in \mathbb{R}^n \setminus \{0\}$ .

We recall the results in [3] as follows.

**Theorem A.1.** *Let  $\epsilon \in (0, 1]$  and  $U \Subset \Omega$ . Assume (A<sub>1</sub>)-(A<sub>4</sub>), (B) and (A<sub>1,\epsilon</sub>)-(A<sub>4,\epsilon</sub>) hold. Then there exists a unique solution  $u^\epsilon \in C^\infty(U) \cap C^0(\bar{U})$  to (A.3), and moreover,  $u^\epsilon \rightarrow u$  in  $C^0(\bar{U})$  and  $u^\epsilon \rightarrow u$  in  $C^{1,\alpha}(V)$  uniformly in  $\epsilon > 0$  as  $\epsilon \rightarrow 0$  for all  $V \Subset U$  where  $u$  is the solution to (A.1). As a consequence,  $u \in C^{1,\alpha}(\Omega)$ .*

Theorem A.1 is a combination of Theorem 1 and Theorem 2 in [3] and several intermediate results in the proof of these two theorems in [3]. Indeed, the existence, uniqueness and  $C^\infty$ -regularity of  $u^\epsilon$  is by elliptic theory in PDE; see for example [8]. Based on these facts, in [3], the author first showed that under (A<sub>1</sub>)-(A<sub>4</sub>), (B) and (A<sub>1,\epsilon</sub>)-(A<sub>4,\epsilon</sub>),  $u^\epsilon \rightarrow u$  in  $W^{1,p}(U)$  uniformly in  $\epsilon > 0$  in section 2. Moreover,  $\|u^\epsilon\|_{L^\infty(U)} \leq \max_{x \in \partial U} \{|u(x)|\}$ . Thus recalling that  $u^\epsilon|_{\partial U} = u|_{\partial U}$ , we know  $u^\epsilon \rightarrow u$  in  $C^0(\bar{U})$ . See the discussion around (2.7) in [3]. Then the author showed that  $\|u^\epsilon\|_{C^{1,\alpha}(V)}$  is uniformly bounded independently of  $\epsilon \in (0, 1]$  and finally showed that  $u^\epsilon \rightarrow u$  in  $C^{1,\alpha}(V)$  and  $u \in C^{1,\alpha}(U)$  for all  $V \Subset U$ . By the arbitrariness of  $U \Subset \Omega$ , one has  $u \in C^{1,\alpha}(\Omega)$ .

**Proof of Lemma 3.1.** It suffices to check equations (3.1) and (3.2) are special ones of (A.1) and (A.3) respectively. To this end, let  $\vec{a}(x, \eta) = e^{-h(x)}|\eta|^{p-2}\eta$ ,  $b(x, \eta) = -e^{-h(x)}|\eta|^p$ ,  $\vec{a}^\epsilon(x, \eta) = e^{-h(x)}(|\eta|^2 + \epsilon)^{\frac{p-2}{2}}\eta$ , and  $b^\epsilon(x, \eta) = -e^{-h(x)}(|\eta|^2 + \epsilon)^{\frac{p-2}{2}}|\eta|^2$  for all  $x \in U$  and  $\eta \in \mathbb{R}^n$ . Then in the weak sense, the equations

$$\int_{\Omega} [ \langle \bar{a}(x, \nabla u), \nabla \phi \rangle + b(x, \nabla u)\phi ] \, d\text{vol}_g = 0, \quad \forall \phi \in C_c^\infty(\Omega)$$

and

$$\int_{\Omega} [ \langle \bar{a}^\epsilon(x, \nabla u), \nabla \phi \rangle + b^\epsilon(x, \nabla u)\phi ] \, d\text{vol}_g = 0, \quad \forall \phi \in C_c^\infty(\Omega)$$

are exactly (3.1) and (3.2) respectively.

We show  $\bar{a}$  satisfies (A<sub>1</sub>). Noting that  $a_j(x, \eta) = e^{-h(x)}|\eta|^{p-2}\eta_j$ , we compute

$$\frac{\partial a_j}{\partial \eta_i}(x, \eta) = e^{-h(x)}[(p-2)|\eta|^{p-4}\eta_i\eta_j + \delta_{ij}|\eta|^{p-2}], \quad \forall 1 \leq i, j \leq n$$

where  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ . Thus

$$\begin{aligned} \sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j &= e^{-h(x)} \sum_{i,j=1}^n [(p-2)|\eta|^{p-4}\eta_i\eta_j + \delta_{ij}|\eta|^{p-2}]\xi_i\xi_j \\ &= e^{-h(x)}|\eta|^{p-4}[(p-2)(\sum_{i=1}^n \eta_i\xi_i)^2 + |\eta|^2|\xi|^2], \quad \forall \xi \in \mathbb{R}^n. \end{aligned}$$

If  $1 < p < 2$ , we have

$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq e^{-h(x)}(p-1)|\eta|^{p-2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

And if  $p \geq 2$ , we have

$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq e^{-h(x)}|\eta|^{p-2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

By taking  $\gamma_0 := \min_{x \in \bar{U}} \{e^{-h(x)}\}$ , we conclude that  $a$  satisfies (A<sub>1</sub>). By direct computations, one can also check  $\bar{a}, \bar{a}^\epsilon \in C^\infty(U \times \mathbb{R}^n, \mathbb{R}^n)$ ,  $b, b^\epsilon \in C^\infty(U \times \mathbb{R}^n)$  satisfy (A<sub>2</sub>)-(A<sub>4</sub>), (B) and (A<sub>1,ε</sub>)-(A<sub>4,ε</sub>) respectively. We omit the details. Thus by Theorem A.1, we get the desired result.  $\square$

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