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A quantitative second order estimate for (weighted) p-harmonic functions in manifolds under curvature-dimension condition ☆

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ABSTRACT

We build up a quantitative second-order Sobolev estimate of $\ln w$ for positive p-harmonic functions w in Riemannian manifolds under Ricci curvature bounded from below and also for positive weighted p-harmonic functions w in weighted manifolds under the Bakry-Émery curvature-dimension condition.

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1. Introduction

Let (M^n, g) be a complete non-compact Riemannian manifold with dimension $n \geq 2$. Suppose that the Ricci curvature is bounded from below, that is, $Ric_g \geq -\kappa$ for some $\kappa \geq 0$. For any positive harmonic function w in a domain $\Omega \subset M^n$, Cheng-Yau [2] established the following famous gradient estimate:

$$|\nabla \ln w| = \frac{|\nabla w|}{w} \leq C(n)\frac{1+\sqrt{\kappa}r}{r} \quad \text{in } B(z,r) \subset B(z,2r) \subset \Omega. \tag{1.1}$$

Recall that a harmonic function w in Ω is a weak solution to the Laplace equation

$$\Delta w := \operatorname{div}(\nabla w) = 0 \text{ in } \Omega.$$

We also refer to [17, Theorem 1.3] for a quantitative $W^{2,2}_{\text{loc}}$-regularity of harmonic functions.

Motivated by the application in the inverse mean curvature flow (see [11,15]), Cheng-Yau type gradient estimate was extended by [16,11,21,15] to p-harmonic functions in Ω for $1 < p < \infty$, that is, weak solutions to the p-Laplace equation

$$\Delta_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w) = 0 \text{ in } \Omega.$$

Precisely, if (M^n, g) is flat (that is, the Euclidean space $\mathbb{R}^n$) or its sectional curvature is bounded from below by $-\kappa$, via Cheng-Yau's approach Moser [16] and Kotschwar-Ni [11] showed that any positive p-harmonic function w in Ω satisfies

$$|\nabla \ln w| \leq C(n)\frac{1+\sqrt{\kappa}r}{r} \quad \text{in } B(z,r) \subset B(z,2r) \subset \Omega, \tag{1.2}$$

where the constant $C(n) > 0$ is independent of $p \in (1, \infty)$. Under the Ricci curvature lower bound $Ric_g \geq -\kappa$, it was asked in [11] whether (1.2) holds or not. Some progress was made as below. Based on Cheng-Yau's argument, Wang-Zhang [21] proved that

$$|\nabla \ln w|^{\frac{p-\gamma}{2}} \in W^{1,2}_{\text{loc}} \text{ with } \gamma < 0 \tag{1.3}$$

and the following weaker revision of (1.2):

$$|\nabla \ln w| \leq C(n,p)\frac{1+\sqrt{\kappa}r}{r} \quad \text{in } B(z,r) \subset B(z,2r) \subset \Omega, \tag{1.4}$$

where the constant $C(n,p) > 0$ blows up as $p \to 1$. Recently, with the aid of the fake distance coming from capacity, $C(n,p)$ was proved by Mari-Rigoli-Setti [15] to be bounded by $\frac{n-1}{p-1}$ as $p \to 1$. Moreover, (1.3) and (1.4) were generalized to weighted manifolds $(M^n, g, e^{-h}d\text{vol}_g)$. A weighted p-harmonic function w in a domain $\Omega \subset M^n$ is a weak solution to the weighted p-harmonic equation

$$\Delta_{p,h} w := e^h \mathrm{div}(e^{-h} |\nabla w|^{p-2} \nabla w) = 0 \text{ in } \Omega.$$

Under the Bakry-Émery curvature-dimension condition $Ric_h^N \geq -\kappa$ for some $N \in [n, \infty)$ and $\kappa \geq 0$ (see Section 2 for details), Dung-Dat [5] showed that if $w > 0$, then $|\nabla \ln w|^{\frac{p-\gamma}{2}} \in W^{1,2}_{\mathrm{loc}}$ with $\gamma < 0$ and also

$$|\nabla \ln w| \leq C(n, N, p) \frac{1 + \sqrt{\kappa} r}{r} \quad \text{in } B(z, r) \subset B(z, 2r) \subset \Omega. \tag{1.5}$$

The main aim of this paper is to build up a quantitative second-order Sobolev estimate of $\ln w$ for positive p-harmonic functions w in Riemannian manifolds under Ricci curvature bounded from below and also for positive weighted p-harmonic functions w in weighted manifolds under the Bakry-Émery curvature-dimension condition. See Theorem 1.1 and Theorem 1.2 separately. These improve the corresponding second-order Sobolev regularity in [21,5] mentioned above.

To be precise, under the Ricci curvature lower bound, we have the following result. For convenience, below we write $\fint_E f\, dm$ as the average of f in the set E with respect to the measure m, that is, $\fint_E f\, dm = \frac{1}{m(E)} \int_E f\, dm$. We use $C(a_1, \cdots, a_m)$ to denote a positive constant depending on absolute constants $a_1, \cdots, a_m$.

Theorem 1.1. *Suppose that (M^n, g) satisfies $Ric_g \geq -\kappa$ for some $\kappa \geq 0$. Let $1 < p < \infty$ and $\gamma < 3 + \frac{p-1}{n-1}$. For any positive p-harmonic function w in a domain $\Omega \subset M$, we have $|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and*

$$\fint_{B(z,r)} \left| \nabla [|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w] \right|^2 \, \mathrm{dvol}_g \leq C(n, p, \gamma) \left[\frac{1 + \sqrt{\kappa} r}{r} \right]^{p-\gamma+4} e^{\sqrt{\kappa} r} \tag{1.6}$$

whenever $B(z, 4r) \Subset \Omega$.

In particular, if $1 < p < 3 + \frac{2}{n-2}$, then $\nabla^2 \ln w \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\fint_{B(z,r)} |\nabla^2 \ln w|^2 \, \mathrm{dvol}_g \leq C(n, p) \left[\frac{1 + \sqrt{\kappa} r}{r} \right]^4 e^{\sqrt{\kappa} r} \tag{1.7}$$

whenever $B(z, 4r) \Subset \Omega$.

Here and throughout the paper for domains A and B, the notation $A \Subset B$ stands for that A is a bounded subdomain of B and its closure $A \subset B$.

Recall that if (M^n, g) is flat, that is, the Euclidean space $\mathbb{R}^n$, p-harmonic functions w in a domain $\Omega \subset \mathbb{R}^n$ are proved to satisfy $|\nabla w|^{\frac{p-\gamma}{2}} \nabla w \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with some quantitative bound whenever $\gamma < 3 + \frac{p-1}{n-1}$ see [13,9,4,14] and also the references therein for some earlier partial results. In particular, if $1 < p < 3 + \frac{2}{n-2}$, noting $p < 3 + \frac{p-1}{n-1}$ and taking $\gamma = p$, one has $w \in W^{2,2}_{\mathrm{loc}}(\Omega)$. When $n \geq 3$ and $p \geq 3 + \frac{2}{n-2}$, it is not clear whether

$w \in W^{2,2}_{\mathrm{loc}}(\Omega)$ or not. When $n = 2$, the range $\gamma < 3 + \frac{p-1}{n-1} = p + 2$ is optimal as witnessed by some construction in [9].

Moreover, we extend Theorem 1.1 to weighted manifolds satisfying Bakry-Émery curvature-dimension condition,

Theorem 1.2. *Let $(M^n, g, e^{-h}\mathrm{vol}_g)$ be a weighted manifold with $Ric_h^N \geq -\kappa$ for some $n \leq N < \infty$ and $\kappa \geq 0$. Let $1 < p < \infty$ and $\gamma < 3 + \frac{p-1}{N-1}$. For any positive weighted p-harmonic function w in a domain $\Omega \subset M$, we have $|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and*

$$\fint_{B(z,r)} \left|\nabla[|\nabla \ln w|^{\frac{p-\gamma}{2}} \nabla \ln w]\right|^2 \, d\mathrm{vol}_h \leq C(n, N, p, \gamma) \left[\frac{1 + \sqrt{\kappa} r}{r}\right]^{p-\gamma+4} e^{\sqrt{\kappa} r} \tag{1.8}$$

whenever $B(z, 4r) \Subset \Omega$.

In particular, if $p \in (1, 3 + \frac{2}{N-2})$, then $\nabla^2 \ln w \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\fint_{B(z,r)} |\nabla^2 \ln w|^2 \, d\mathrm{vol}_h \leq C(n, N, p) \left[\frac{1 + \sqrt{\kappa} r}{r}\right]^4 e^{\sqrt{\kappa} r} \tag{1.9}$$

whenever $B(z, 4r) \Subset \Omega$.

As a consequence of Theorem 1.1 and Theorem 1.2, one gets that $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W^{1,2}_{\mathrm{loc}}$ for $\gamma < 3 + \frac{p-1}{n-1}$ or $\gamma < 3 + \frac{p-1}{N-1}$, while in [21,5], one has $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W^{1,2}_{\mathrm{loc}}$ for all $\gamma < 2$ (see (1.3) and the line above (1.5)). Thus our range for γ obviously improves the one obtained in [21,5] respectively.

Now we sketch the ideas to prove Theorem 1.1 and Theorem 1.2. Note that when $N = n$ and $h \equiv 1$, we have $Ric_h^N = Ric_g$, and hence Theorem 1.1 corresponds to the special case $N = n$ and $h \equiv 1$ in Theorem 1.2. We only need to prove Theorem 1.2. As usual, we approximate $u = -(p-1)\ln w$ by smooth solution u^ϵ to the standard approximation/regularized equation (3.3), that is,

$$e^h \mathrm{div}(e^{-h}[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-2}{2}} \nabla u^\epsilon) = [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-2}{2}} |\nabla u^\epsilon|^2.$$

(i) Using Bochner formula and the approximation equation (3.3), for $0 < \eta < 1/2$ we bound the integral of

$$(1-\eta)|\nabla^2 u^\epsilon|^2 + (p-\gamma)\frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p-2)(2-\gamma)\frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \tag{1.10}$$

from above by the integral of

$$Ric_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon \rangle$$

and other first order terms, where all integrals are taken against $[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}}\phi^2 e^{-h} d\mathrm{vol}_g$ where $\phi \in C_c^\infty(U)$ is a test function and $U \Subset \Omega$; see Lemma 3.2. Here in (1.10) and in what follows, for any C^2 function f, $\Delta_\infty f := \langle \nabla^2 f \nabla f, \nabla f\rangle$.

(ii) If $\gamma < 3 + \frac{p-1}{N-1}$, via a fundamental inequality given in Lemma 2.1 and the approximation equation (3.3), for sufficiently small $\eta > 0$ we bound (1.10) as below

$$(1.10) \geq \eta |\nabla^2 u^\epsilon|^2 - \frac{\langle \nabla h, \nabla u^\epsilon\rangle^2}{N-n} - C\frac{1}{\eta}|\nabla u^\epsilon|^4 \quad \text{everywhere;}$$

see Lemma 3.4. This is crucial to get Theorem 1.2. Note that the approach in [21,5] could not give Lemma 3.4; see Remark 3.8 for details.

(iii) Combining (i)&(ii) together, the integral of $\eta|\nabla^2 u^\epsilon|^2$ is bounded from above by the integral of $-Ric_h^N(\nabla u^\epsilon, \nabla u^\epsilon)$ and other first order terms, where all integrals are taken against $[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}}\phi^2 e^{-h} d\mathrm{vol}_g$; see Corollary 3.6.
Under the assumption $Ric_h^N \geq -\kappa$, in Lemma 3.7 we obtain an upper L^2_{loc} bound for $\nabla[|\nabla u^\epsilon|^{\frac{p-\gamma}{2}}\nabla u^\epsilon]\phi$ by the integral of some first order terms, where all integrals are against $e^{-h} d\mathrm{vol}_g$. A standard argument then leads to the proof of Theorem 1.2.

Finally, we also notice that the Cheng-Yau gradient estimate (1.1) was generalized to positive harmonic functions w in Alexandrov spaces with curvature bounded from below by Zhang-Zhu in [22], where the authors showed $|\nabla \ln w|^2 \in W^{1,2}_{\mathrm{loc}}(\Omega)$ as a key step. Furthermore, one could study the regularity of p-harmonic functions in more general metric measure spaces. In these spaces, a natural generalization of the (weighted) Ricci curvature bound is the curvature-dimension condition $RCD(\kappa, N)$ in the sense of Bakry-Émery or Ambrosio-Gigli-Savaré. The two senses turned out to be equivalent by the work of Erbar-Kuwada-Sturm [6] (in the finite dimensional case) and Ambrosio-Gigli-Savaré [1] and the spaces satisfying one of the two equivalent conditions are known as $RCD(\kappa, N)$ spaces. Some progress was made in $RCD(\kappa, N)$ spaces. The Cheng-Yau gradient estimate was established by Jiang in [10] for positive harmonic functions w in $RCD(\kappa, N)$ spaces; recently, Gigli-Violo in [7] established $|\nabla \ln w|^{\beta/2} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ under $RCD(0, N)$ spaces if $\beta > \frac{N-2}{N-1}$. However, when $p \neq 2$, it remains open to prove the Cheng-Yau type gradient estimates for positive p-harmonic functions in Alexandrov spaces and also $RCD(\kappa, N)$ spaces.

2. Preliminaries

Let $n \geq 2$ and M^n be a Riemannian manifold, and g be the Riemannian metric. By abuse of notation we also write $|\xi|^2 = g(\xi, \xi)$ and $\langle \xi, \eta\rangle = g(\xi, \eta)$ for all $\xi, \eta \in T_x M^n$. The corresponding Riemannian volume measure is written as $d\mathrm{vol}_g$, and the volume of a set E is written as $\mathrm{vol}_g(E)$. Denote by Ric_g the Ricci curvature 2-tensor and write $Ric_g \geq -\kappa$ if $Ric_g(\xi, \xi) \geq -\kappa|\xi|^2$ for all $\xi \in T_x M^n$.

For $1 < p < \infty$, the p-Laplace operator Δ_p in M^n is given by

$$\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f) \quad \forall f \in C^2(M^n).$$

Obviously, Δ_2 is exactly the Laplace-Beltrami operator Δ in (M^n, g). A function w defined in a domain $\Omega \subset M^n$ is called p-harmonic if $w \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to the p-Laplace equation $\Delta_p w = 0$ in Ω, that is,

$$\int_\Omega |\nabla w|^{p-2}\langle \nabla w, \nabla \phi\rangle d\text{vol}_g = 0 \quad \forall \phi \in C_c^\infty(\Omega).$$

Note that 2-harmonic functions are the well-known harmonic functions.

Next we recall some basic facts of weighted Riemannian manifolds $(M^n, g, e^{-h}d\text{vol}_g)$, where the weight h is a positive smooth function in M^n. The weighted measure $d\text{vol}_h = e^{-h}d\text{vol}_g$ can be viewed as the volume form of a suitable conformal change of the metric g. Denote by $\text{vol}_h(E)$ the weighted volume of a set E. For $n \le N < \infty$, the corresponding N-Bakry-Émery curvature tensor is

$$Ric_h^N = Ric_g + \nabla^2 h - \frac{\nabla h \otimes \nabla h}{N - n},$$

where when $N = n$, by convention, h is a constant function and hence $Ric_h^N = Ric_g$. We say that $(M^n, g, e^{-h}d\text{vol}_g)$ satisfies the Bakry-Émery curvature-dimension condition $Ric_h^N \ge -\kappa$ if

$$Ric_h^N(\xi, \xi) = Ric_g(\xi, \xi) + \langle \nabla^2 h\xi, \xi\rangle - \frac{\langle \nabla h, \xi\rangle^2}{N - n} \ge -\kappa\langle \xi, \xi\rangle \; \forall \xi \in T_x M^n$$

By [18], under $Ric_h^N \ge -\kappa$, one has the following volume comparison result

$$\text{vol}_h(B_{2r}(x)) \le C(N)e^{\sqrt{\kappa}r}\text{vol}_h(B_r(x)) \quad \forall x \in M, \; r > 0. \tag{2.1}$$

For $1 < p < \infty$, the weighted p-Laplacian $\Delta_{h,p}$ is defined as

$$\Delta_{p,h} f = e^h \operatorname{div}(e^{-h}|\nabla f|^{p-2}\nabla f) = \Delta_p f - |\nabla f|^{p-2}\langle \nabla f, \nabla h\rangle \quad \forall f \in C^2(M^n).$$

In the case $p = 2$, one writes $\Delta_{2,h}$ as Δ_h, and hence

$$\Delta_h f = \Delta f - \langle \nabla h, \nabla f\rangle.$$

A function w in a domain $\Omega \subset M^n$ is called as a weighted p-harmonic function if $w \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution to the weighted p-harmonic equation $\Delta_{p,h} w = 0$ in Ω, that is,

$$\int_{\Omega} |\nabla w|^{p-2} \langle \nabla w, \nabla \phi \rangle \, e^{-h} d\mathrm{vol}_g = 0 \quad \forall \phi \in C_c^{\infty}(\Omega). \tag{2.2}$$

By a density argument, we can relax $\phi \in C_c^{\infty}(\Omega)$ to $\phi \in W_0^{1,p}(\Omega)$ in (2.2).

We also recall the following Bochner formula in $(M^n, g, e^{-h} d\mathrm{vol}_g)$:

$$\frac{1}{2}\Delta_h |\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta_h f \rangle + Ric_g(\nabla f, \nabla f) + \langle \nabla^2 h \nabla f, \nabla f \rangle \quad \forall f \in C^3(M), \tag{2.3}$$

which will be used in Section 3.

Finally, we recall the following fundamental inequality; see for example [21,5,14]. For the reader's convenience we include it here. Recall that $\Delta_\infty f = \langle \nabla^2 f \nabla f, \nabla f \rangle$.

Lemma 2.1. *Let $n \geq 2$ and Ω be a domain of M^n. For any $f \in C^2(\Omega)$, we have*

$$|\nabla f|^4 |\nabla^2 f|^2 \geq 2|\nabla f|^2 |\nabla^2 f \nabla f|^2 + \frac{[|\nabla f|^2 \Delta f - \Delta_\infty f]^2}{n-1} - (\Delta_\infty f)^2 \ \text{in } \Omega, \tag{2.4}$$

where when $n = 2$, "$\geq$" becomes "$=$".

Proof. It suffices to prove that for any symmetric $n \times n$ matrix A one has

$$|A|^2 |\xi|^4 \geq \frac{1}{n-1}(\mathrm{tr} A |\xi|^2 - \langle A\xi, \xi \rangle)^2 + 2|A\xi|^2 |\xi|^2 - \langle A\xi, \xi \rangle^2 \quad \forall \xi \in \mathbb{R}^n. \tag{2.5}$$

Note that if $\xi = 0$, (2.5) holds obviously. Below assume that $\xi \neq 0$. Up to a scaling we may assume $|\xi| = 1$. By a change of coordinates, we may further assume $\xi = e_n = (0, \cdots, 0, 1)$; in this case, (2.5) reads as

$$|A|^2 \geq \frac{1}{n-1}(\mathrm{tr} A - \langle Ae_n, e_n \rangle)^2 + 2|Ae_n|^2 - \langle Ae_n, e_n \rangle^2.$$

Denoting by A_{n-1} the $(n-1)$ order principal submatrix of A, one has

$$|A|^2 = |A_{n-1}|^2 + 2|Ae_n|^2 - \langle Ae_n, e_n \rangle^2.$$

Noting that

$$|A_{n-1}|^2 \geq \frac{1}{n-1}(\mathrm{tr} A_{n-1})^2 = \frac{1}{n-1}(\mathrm{tr} A - \langle Ae_n, e_n \rangle)^2,$$

where when $n = 2$, one has $|A_{n-1}|^2 = (\mathrm{tr} A_{n-1})^2$, one concludes (2.4). $\square$

3. Proof of Theorem 1.2

Let w be a positive weighted p-harmonic function in a domain Ω. Set $u = -(p-1)\ln w$. Then u is a weak solution to the equation

$$\Delta_p u - |\nabla u|^{p-2}\langle \nabla u, \nabla h\rangle = |\nabla u|^p \quad \text{in } \Omega, \tag{3.1}$$

that is,

$$-\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle e^{-h} d\mathrm{vol}_g = \int_\Omega |\nabla u|^p \phi e^{-h} d\mathrm{vol}_g \quad \forall \phi \in C_c^\infty(\Omega).$$

Given any smooth domain $U \Subset \Omega$ and $\epsilon \in (0,1]$, consider the approximation/regularized equation defined by

$$e^h \mathrm{div}(e^{-h}[|\nabla v|^2 + \epsilon]^{\frac{p-2}{2}} \nabla v) = [|\nabla v|^2 + \epsilon]^{\frac{p-2}{2}} |\nabla v|^2 \quad \text{in } U;\ v = u \text{ on } \partial U. \tag{3.2}$$

It is well known that if u is the solution to (3.1), then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$; see [3,12,19,20]. Moreover, in the following lemma, we summarize some properties of the solution u to (3.1) and u^ϵ to (3.3), which result from [3] as a special case. See also [19].

Lemma 3.1. *For any $\epsilon \in (0,1]$, there exists a unique solution $u^\epsilon \in C^\infty(U) \cap C^0(\overline{U})$ to (3.3), and moreover, $u^\epsilon \to u$ in $C^0(\overline{U})$ and $u^\epsilon \to u$ in $C^{1,\alpha}(V)$ uniformly in $\epsilon > 0$ as $\epsilon \to 0$ for all $V \Subset U$ where u is the solution to (3.1).*

To show Lemma 3.1, we just need to check that equations (3.1) and (3.3) are special cases of those considered in [3]. We put this verification in the appendix.

By Lemma 3.1, the solution u^ϵ to (3.2) is C^∞, which implies that u^ϵ satisfies (3.2) pointwise. Hence by a direct computation, (3.2) is equivalent to

$$\Delta_h u^\epsilon + (p-2)\frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} = |\nabla u^\epsilon|^2 \quad \text{in } U;\ u^\epsilon = u \text{ on } \partial U. \tag{3.3}$$

To prove Theorem 1.2 we first build up the following upper bound.

Lemma 3.2. *Let u^ϵ be the solution to (3.3). For any $\gamma \in \mathbb{R}$, $\eta > 0$ and $\phi \in C_c^\infty(U)$, we have*

$$\begin{aligned}
&\int_U \left\{(1-\eta)|\nabla^2 u^\epsilon|^2 + (p-\gamma)\frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + (p-2)(2-\gamma)\frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2 + \epsilon]^2}\right\} \\
&\qquad \times [|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\mathrm{vol}_g \\
&\quad \le -\int_U [Ric_g(\nabla u^\epsilon, \nabla u^\epsilon) + \langle \nabla^2 h \nabla u^\epsilon, \nabla u^\epsilon\rangle][|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\mathrm{vol}_g
\end{aligned}$$

$$+ C(p,\gamma)\frac{1}{\eta}\int_U ([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+1}|\nabla\phi|^2 + [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+2}\phi^2)\, e^{-h} d\mathrm{vol}_g. \quad (3.4)$$

To prove this, we need the following identity.

Lemma 3.3. *For any $v \in C^3(U)$ and $\psi \in C_c^\infty(U)$, one has*

$$\int_U |\nabla^2 v|^2 \psi\, e^{-h} d\mathrm{vol}_g = -\int_U \langle \nabla^2 v \nabla v - \Delta_h v \nabla v, \nabla\psi\rangle\, e^{-h} d\mathrm{vol}_g + \int_U (\Delta_h v)^2 \psi\, e^{-h} d\mathrm{vol}_g$$
$$- \int_U [Ric_g(\nabla v, \nabla v) + \langle \nabla^2 h \nabla v, \nabla v\rangle]\psi\, e^{-h} d\mathrm{vol}_g. \quad (3.5)$$

Proof. Applying the Bochner formula to v, one has

$$|\nabla^2 v|^2 + Ric_g(\nabla v, \nabla v) = \frac{1}{2}\Delta_h |\nabla v|^2 - \langle \nabla v, \nabla \Delta_h v\rangle - \langle \nabla^2 h \nabla v, \nabla v\rangle$$

and hence

$$|\nabla^2 v|^2 = [\frac{1}{2}\Delta_h|\nabla v|^2 - (\Delta_h v)^2 - \langle \nabla v, \nabla\Delta_h v\rangle] + (\Delta_h v)^2$$
$$- [Ric_g(\nabla v, \nabla v) + \langle \nabla^2 h \nabla v, \nabla v\rangle].$$

By this, to get (3.5), it suffices to show the following identity

$$\int_U [\frac{1}{2}\Delta_h |\nabla v|^2 - (\Delta_h v)^2 - \langle \nabla v, \nabla \Delta_h v\rangle]\psi e^{-h} d\mathrm{vol}_g$$
$$= -\int_U \langle \nabla^2 v \nabla v - \Delta_h v \nabla v, \nabla \psi\rangle\, e^{-h} d\mathrm{vol}_g. \quad (3.6)$$

Note that

$$-[(\Delta_h v)^2 + \langle \nabla v, \nabla(\Delta_h v)\rangle] = -e^h \mathrm{div}(e^{-h}\nabla v)(\Delta_h v) - e^h \langle e^{-h}\nabla v, \nabla(\Delta_h v)\rangle$$
$$= -e^h \mathrm{div}(e^{-h}\nabla v \Delta_h v).$$

Via integration by parts, one has

$$-\int_U [(\Delta_h v)^2 + \langle \nabla v, \nabla(\Delta_h v)\rangle]\psi\, e^{-h} d\mathrm{vol}_g = -\int_U \mathrm{div}(e^{-h}\nabla v \Delta_h v)\psi\, d\mathrm{vol}_g$$
$$= \int_U \langle \Delta_h v \nabla v, \nabla\psi\rangle\, e^{-h} d\mathrm{vol}_g.$$

Similarly, via integration by parts one also has

$$\begin{aligned}\frac{1}{2}\int_U \Delta_h|\nabla v|^2\psi e^{-h}\,d\mathrm{vol}_g &= \int_U \frac{1}{2}\mathrm{div}(e^{-h}\nabla|\nabla v|^2)\psi\,d\mathrm{vol}_g \\ &= -\int_U \frac{1}{2}\langle e^{-h}\nabla|\nabla v|^2,\nabla\psi\rangle\,d\mathrm{vol}_g \\ &= -\int_U \langle\nabla^2 v\nabla v,\nabla\psi\rangle\, e^{-h}d\mathrm{vol}_g.\end{aligned}$$

Combining together we obtain (3.6) and hence, (3.5) as desired. □

We are ready prove Lemma 3.2 as below.

Proof of Lemma 3.2. Taking $v = u^\epsilon$ and $\psi = [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2$ in (3.5) we get

$$\begin{aligned}&\int_U |\nabla^2 u^\epsilon|^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\,e^{-h}d\mathrm{vol}_g \\ &\quad= -\int_U \langle\nabla^2 u^\epsilon\nabla u^\epsilon - \Delta_h u^\epsilon\nabla u^\epsilon, \nabla[[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2]\rangle\,e^{-h}d\mathrm{vol}_g \\ &\qquad+\int_U (\Delta_h u^\epsilon)^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\,e^{-h}d\mathrm{vol}_g \\ &\qquad-\int_U [Ric(\nabla u^\epsilon,\nabla u^\epsilon)+\langle\nabla^2 h\nabla u^\epsilon,\nabla u^\epsilon\rangle][|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\,e^{-h}d\mathrm{vol}_g. \qquad (3.7)\end{aligned}$$

To bound the second term in the right-hand side in (3.7), recalling (3.3), that is,

$$\Delta_h u^\epsilon = |\nabla u^\epsilon|^2 - (p-2)\frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2+\epsilon}, \qquad (3.8)$$

by Cauchy-Schwarz's inequality one has

$$(\Delta_h u^\epsilon)^2 \le (p-2)^2\frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2+\epsilon]^2} + \frac{\eta}{4}|\nabla^2 u^\epsilon|^2 + C(p)\frac{1}{\eta}|\nabla u^\epsilon|^4,$$

where $0<\eta<1$ is any constant. Thus

$$\begin{aligned}\int_U (\Delta_h u^\epsilon)^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\,e^{-h}d\mathrm{vol}_g &\le (p-2)^2\int_U (\Delta_\infty u^\epsilon)^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}-2}\phi^2\,e^{-h}d\mathrm{vol}_g \\ &\quad+\frac{\eta}{4}\int_U |\nabla^2 u^\epsilon|^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\,e^{-h}d\mathrm{vol}_g\end{aligned}$$

$$+\frac{C(p)}{\eta}\int_U [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+2}\phi^2\, e^{-h} d\mathrm{vol}_g. \tag{3.9}$$

The first term in the right-hand side in (3.7) is further written as

$$\begin{aligned}
&-\int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon - \Delta_h u^\epsilon \nabla u^\epsilon, \nabla[[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2]\rangle\, e^{-h} d\mathrm{vol}_g \\
&= -(p-\gamma)\int_U \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\quad + (p-\gamma)\int_U \Delta_h u^\epsilon \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2+\epsilon}[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\quad - \int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon, \nabla \phi^2\rangle [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\, e^{-h} d\mathrm{vol}_g \\
&\quad + \int_U \langle \Delta_h u^\epsilon \nabla u^\epsilon, \nabla \phi^2\rangle [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\, e^{-h} d\mathrm{vol}_g.
\end{aligned} \tag{3.10}$$

Using (3.8) and Cauchy-Schwarz's inequality, we obtain the following upper bound for the second term in (3.10):

$$\begin{aligned}
&(p-\gamma)\int_U \Delta_h u^\epsilon \frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2+\epsilon}[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&= -(p-\gamma)(p-2)\int_U (\Delta_\infty u^\epsilon)^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}-2}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\quad + (p-\gamma)\int_U \Delta_\infty u^\epsilon |\nabla u^\epsilon|^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}-1}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\le -(p-\gamma)(p-2)\int_U (\Delta_\infty u^\epsilon)^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}-2}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\quad + \frac{\eta}{4}\int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\quad + \frac{C(p)}{\eta}|p-\gamma|^2\int_U [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+2}\phi^2\, e^{-h} d\mathrm{vol}_g.
\end{aligned} \tag{3.11}$$

For the third term in the right-hand side of (3.10), by Cauchy-Schwarz's inequality, one has

$$\left|\int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon, \nabla\phi^2\rangle [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\mathrm{vol}_g\right| \leq \frac{\eta}{4}\int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\mathrm{vol}_g + C\frac{1}{\eta}\int_U [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+1} |\nabla\phi|^2 e^{-h} d\mathrm{vol}_g. \tag{3.12}$$

For the fourth term in the right-hand side of (3.10), in a similar way, using (3.8), one has

$$\begin{aligned}
&\left|\int_U \langle \Delta_h u^\epsilon \nabla u^\epsilon, \nabla\phi^2\rangle [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\mathrm{vol}_g\right| \\
&= \left|\int_U \left\langle |\nabla u^\epsilon|^2 \nabla u^\epsilon - (p-2)\frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2+\epsilon}\nabla u^\epsilon, \nabla\phi^2\right\rangle [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\mathrm{vol}_g\right| \\
&\leq \frac{\eta}{4}\int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\mathrm{vol}_g \\
&\quad + C(p)\frac{1}{\eta}\int_U \left([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+1}|\nabla\phi|^2 + [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+2}\phi^2\right) e^{-h} d\mathrm{vol}_g.
\end{aligned} \tag{3.13}$$

From (3.13), (3.12), (3.11) and (3.10) we attain

$$\begin{aligned}
&-\int_U \langle \nabla^2 u^\epsilon \nabla u^\epsilon - \Delta_h u^\epsilon \nabla u^\epsilon, \nabla[[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2]\rangle e^{-h} d\mathrm{vol}_g \\
&= \frac{3}{4}\eta\int_U |\nabla^2 u^\epsilon|^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\mathrm{vol}_g \\
&\quad -(p-\gamma)\int_U \frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}} \phi^2 e^{-h} d\mathrm{vol}_g \\
&\quad -(p-\gamma)(p-2)\int_U (\Delta_\infty u^\epsilon)^2 [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}-2} \phi^2 e^{-h} d\mathrm{vol}_g \\
&\quad + \frac{C(p)}{\eta}\int_U \left([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+1}|\nabla\phi|^2 + [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+2}\phi^2\right) e^{-h} d\mathrm{vol}_g.
\end{aligned} \tag{3.14}$$

Obviously from (3.14), (3.9) and (3.7) we conclude (3.4). □

If $\gamma < 3 + \frac{p-1}{N-1}$, we get the following pointwise lower bound. Recall that when $N = n$, we always assume that h is a constant function and $\frac{\langle \nabla u^\epsilon, \nabla h\rangle^2}{N-n} = 0$.

Lemma 3.4. *Let u^ϵ be the solution to (3.3). If $\gamma < 3 + \frac{p-1}{N-1}$ for some $N \geq n$, then for sufficiently small $\eta > 0$ we have*

$$
\begin{aligned}
&(1-\eta)|\nabla^2 u^\epsilon|^2 + (p-\gamma)\frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon} + (p-2)(2-\gamma)\frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2+\epsilon]^2} \\
&\geq \eta|\nabla^2 u^\epsilon|^2 - \frac{\langle \nabla u^\epsilon, \nabla h\rangle^2}{N-n} - C(n,N,p,\gamma)\frac{1}{\eta}|\nabla u^\epsilon|^4. \qquad (3.15)
\end{aligned}
$$

To prove this, we need the following pointwise lower bound for $|\nabla^2 u^\epsilon|^2 |\nabla u^\epsilon|^4$.

Lemma 3.5. *Let u^ϵ be the solution to (3.3). If $N \geq n$, then for $0 < \eta < 1$ we have*

$$
\begin{aligned}
(1+\eta)|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4 &\geq 2|\nabla^2 u^\epsilon \nabla u^\epsilon|^2 |\nabla u^\epsilon|^2 \\
&\quad + \left(\frac{1}{N-1}\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2 - 1\right)(\Delta_\infty u^\epsilon)^2 \\
&\quad - (1+\eta)\frac{\langle \nabla u^\epsilon, \nabla h\rangle^2}{N-n}|\nabla u^\epsilon|^4 - C(n,N,p)\frac{1}{\eta}|\nabla u^\epsilon|^8. \qquad (3.16)
\end{aligned}
$$

Proof. Applying (2.4) to u^ϵ one has

$$
|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4 \geq 2|\nabla u^\epsilon|^2|\nabla^2 u^\epsilon \nabla u^\epsilon|^2 + \frac{[|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n-1} - (\Delta_\infty u^\epsilon)^2 \qquad (3.17)
$$

By (3.8) and $\Delta u^\epsilon = \Delta_h u^\epsilon + \langle \nabla h, \nabla u^\epsilon\rangle$, we have

$$
\Delta u^\epsilon = |\nabla u^\epsilon|^2 + \langle \nabla u^\epsilon, \nabla h\rangle - (p-2)\frac{\Delta_\infty u^\epsilon}{|\nabla u^\epsilon|^2+\epsilon}.
$$

Thus

$$
|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon = |\nabla u^\epsilon|^2\left(|\nabla u^\epsilon|^2 + \langle \nabla u^\epsilon, \nabla h\rangle\right) - \left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]\Delta_\infty u^\epsilon,
$$

and hence,

$$
\begin{aligned}
[|\nabla u^\epsilon|^2 \Delta u^\epsilon - \Delta_\infty u^\epsilon]^2 &= \left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2 (\Delta_\infty u^\epsilon)^2 \\
&\quad + |\nabla u^\epsilon|^4\left(|\nabla u^\epsilon|^2 + \langle \nabla u^\epsilon, \nabla h\rangle\right)^2 \\
&\quad - 2\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]|\nabla u^\epsilon|^4 \Delta_\infty u^\epsilon \\
&\quad - 2\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]|\nabla u^\epsilon|^2 \Delta_\infty u^\epsilon \langle \nabla u^\epsilon, \nabla h\rangle \\
&=: I_1 + I_2 + I_3 + I_4. \qquad (3.18)
\end{aligned}
$$

Note that

$$\left|(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right|^2 \le 4p^2, \tag{3.19}$$

which can be obtained by considering $p > 2$ and $1 < p < 2$ separately. Using this, Cauchy-Schwarz inequality, for $0 < \eta < 1$, we have

$$I_3 \ge -\eta|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4 - C(p)\frac{1}{\eta}|\nabla u^\epsilon|^8. \tag{3.20}$$

If h is a constant function and hence $\nabla h = 0$, $I_2 \ge 0$ and $I_4 = 0$, dividing by $n-1$ in both sides of (3.18), by (3.20) one has

$$\begin{aligned}\frac{[|\nabla u^\epsilon|^2\Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n-1} &\ge -\eta|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4 \\ &+ \left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2\frac{(\Delta_\infty u^\epsilon)^2}{n-1} - \frac{C(p)}{\eta}|\nabla u^\epsilon|^8.\end{aligned}$$

Plugging this in (3.17), noting $N = n$, and adding $\eta|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4$ in both sides, one concludes (3.16).

If h is not a constant function, set $\eta_1 = \frac{N-n}{N-1}$. Then

$$1-\eta_1 = \frac{n-1}{N-1} > 0 \quad \text{and} \quad 1-\frac{1}{\eta_1} = -\frac{n-1}{N-n} < 0. \tag{3.21}$$

For any $0 < \eta < 1$ one has

$$\begin{aligned} I_2 &\ge |\nabla u^\epsilon|^4\langle\nabla u^\epsilon, \nabla h\rangle^2 + 2|\nabla u^\epsilon|^6\langle\nabla u^\epsilon, \nabla h\rangle \\ &\ge [1+\eta(1-\frac{1}{\eta_1})]|\nabla u^\epsilon|^4\langle\nabla u^\epsilon, \nabla h\rangle^2 - \frac{1}{\eta|1-\frac{1}{\eta_1}|}|\nabla u^\epsilon|^8. \end{aligned} \tag{3.22}$$

Using Cauchy-Schwarz inequality, we have

$$I_4 \ge -\eta_1\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2(\Delta_\infty u^\epsilon)^2 - \frac{1}{\eta_1}\langle\nabla u^\epsilon, \nabla h\rangle^2|\nabla u^\epsilon|^4 \tag{3.23}$$

Dividing by $n-1$ in both sides of (3.18), by (3.20), (3.22) and (3.23) one has

$$\begin{aligned}\frac{[|\nabla u^\epsilon|^2\Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n-1} &\ge -\eta|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4 + \frac{1-\eta_1}{n-1}\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2(\Delta_\infty u^\epsilon)^2 \\ &+ (1+\eta)\frac{1-\frac{1}{\eta_1}}{n-1}\langle\nabla u^\epsilon, \nabla h\rangle^2|\nabla u^\epsilon|^4 - C(n,N,p)\frac{1}{\eta}|\nabla u^\epsilon|^8\end{aligned}$$

By (3.21),

$$\frac{[|\nabla u^\epsilon|^2\Delta u^\epsilon - \Delta_\infty u^\epsilon]^2}{n-1} \geq -\eta|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4 + \frac{1}{N-1}\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2(\Delta_\infty u^\epsilon)^2$$
$$-(1+\eta)\frac{1}{N-n}\langle\nabla u^\epsilon,\nabla h\rangle^2|\nabla u^\epsilon|^4 - C(n,N,p)\frac{1}{\eta}|\nabla u^\epsilon|^8.$$

Plugging this in (3.17), and adding $\eta|\nabla^2 u^\epsilon|^2|\nabla u^\epsilon|^4$ in both sides, we conclude (3.16) as desired. □

We now prove Lemma 3.4 by using Lemma 3.5.

Proof of Lemma 3.4. Given any point $x \in U$, if $\nabla u^\epsilon(x) = 0$, then (3.15) holds trivially. Below we assume that $\nabla u^\epsilon(x) \neq 0$. At such point x, we already have (3.16) in Lemma 3.5. Dividing by $|\nabla u^\epsilon|^4$ in both sides of (3.16), for $0 < \eta < 1/2$ we obtain

$$(1+\eta)|\nabla^2 u^\epsilon|^2 \geq 2\frac{|\nabla^2 u^\epsilon\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2} + \left(\frac{1}{N-1}\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2 - 1\right)\frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4}$$
$$-(1+\eta)\frac{\langle\nabla u^\epsilon,\nabla h\rangle^2}{N-n} - \frac{C(n,N,p)}{\eta}|\nabla u^\epsilon|^4.$$

In both sides, multiplying by $\frac{1-2\eta}{1+\eta} > 0$ and adding

$$\eta|\nabla^2 u^\epsilon|^2 + (p-\gamma)\frac{|\nabla^2 u^\epsilon\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon} + (p-2)(2-\gamma)\frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2+\epsilon]^2},$$

we get

$$\begin{aligned}
&(1-\eta)|\nabla^2 u^\epsilon|^2 + (p-\gamma)\frac{|\nabla^2 u^\epsilon\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon} + (p-2)(2-\gamma)\frac{(\Delta_\infty u^\epsilon)^2}{[|\nabla u^\epsilon|^2+\epsilon]^2}\\
&\geq \eta|\nabla^2 u^\epsilon|^2 + \left\{\frac{1-2\eta}{1+\eta}2 + (p-\gamma)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}\right\}\frac{|\nabla^2 u^\epsilon\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2}\\
&\quad + \left\{\frac{1-2\eta}{1+\eta}\left(\frac{1}{N-1}\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon}+1\right]^2 - 1\right)\right.\\
&\quad \left.+(p-2)(2-\gamma)\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2+\epsilon]^2}\right\}\frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4}\\
&\quad - (1-2\eta)\frac{\langle\nabla u^\epsilon,\nabla h\rangle^2}{N-n} - C(n,N,p)\frac{1}{\eta}|\nabla u^\epsilon|^4\\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{3.24}$$

Recall that if $N = n$ that is, h is a constant function, $I_4 = 0$ by our convention. If $N > n$ that is, h is not a constant, then by $1 - 2\eta < 1$, we have

$$I_4 \geq -\frac{\langle \nabla u^\epsilon, \nabla h\rangle^2}{N-n}. \tag{3.25}$$

To bound $I_2 + I_3$ from below, since $\gamma < 3 + \frac{p-1}{N-1}$ and $N \geq 2$ implies

$$p + 2 - \gamma > p + 2 - 3 - \frac{p-1}{N-1} = (p-1)(1 - \frac{1}{N-1}) \geq 0,$$

we can find $0 < \hat{\eta}(p,\gamma) < 1/2$ such that for $0 < \eta < \hat{\eta}$, one has $p + 2\frac{1-2\eta}{1+\eta} - \gamma > 0$. Thus the coefficient of I_2 satisfies

$$(p-\gamma)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 2\frac{1-2\eta}{1+\eta} \geq (p + 2\frac{1-2\eta}{1+\eta} - \gamma)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + \frac{1-2\eta}{1+\eta}\frac{\epsilon}{|\nabla u^\epsilon|^2 + \epsilon} > 0.$$

Using this and observing

$$\frac{|\nabla^2 u^\epsilon \nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2} \geq \frac{|\Delta_\infty u^\epsilon|^2}{|\nabla u^\epsilon|^4},$$

one has

$$\begin{aligned}
& I_2 + I_3 \\
& \geq \Bigg\{ (p-\gamma)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 2\frac{1-2\eta}{1+\eta} \\
& \quad + \frac{1-2\eta}{1+\eta}\left(\frac{1}{N-1}\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 - 1 \right) \\
& \quad + (p-2)(2-\gamma)\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2} \Bigg\} \frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4} \\
& =: H(\eta)\frac{(\Delta_\infty u^\epsilon)^2}{|\nabla u^\epsilon|^4}
\end{aligned}$$

We claim that there exists $0 < \bar{\eta}(n, N, p, \gamma) < \hat{\eta}$ such that $H(\eta) > 0$ for all $0 < \eta < \bar{\eta}$. Assuming this claim holds for the moment, for any $0 < \eta < \bar{\eta}$, one has $I_2 + I_3 > 0$. From this, (3.24) and (3.25) we conclude (3.15) as desired.

Finally we prove the above claim. It suffices to show that

$$\begin{aligned}
H(0) := (p-\gamma)\frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2 + \epsilon]} + 2 + \left(\frac{1}{N-1}\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2 + \epsilon} + 1 \right]^2 - 1 \right) \\
+ (p-2)(2-\gamma)\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2 + \epsilon]^2}
\end{aligned}$$

$$> \delta(N,p,\gamma), \tag{3.26}$$

where $\delta(N,p,\gamma) > 0$ is a constant. Indeed, by (3.19), one has

$$H(\eta) \geq H(0) - 2[1 - \frac{1-2\eta}{1+\eta}] - [1 - \frac{1-2\eta}{1+\eta}][\frac{4p^2}{N-1} - 1] \geq \delta(N,p,\gamma) - 15p^2\eta.$$

If $0 < \eta < \bar{\eta} =: \min\{\hat{\eta}, \delta(N,p,\gamma)/15p^2\}$, one has $H(\eta) > 0$ and hence the claim holds as desired.

We prove (3.26) as below. Since

$$\left[(p-2)\frac{|\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon} + 1\right]^2 = (p-2)^2\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2+\epsilon]^2} + 2(p-2)\frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2+\epsilon]} + 1,$$

we rewrite

$$H(0) = (p-2)[2-\gamma+\frac{p-2}{N-1}]\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2+\epsilon]^2} + [p-\gamma+\frac{2(p-2)}{N-1}]\frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2+\epsilon]} + \frac{N}{N-1}.$$

Observing

$$\frac{|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2+\epsilon]} = \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2+\epsilon]^2} + \frac{\epsilon|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2+\epsilon]^2}$$

and

$$1 = \frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2+\epsilon]^2} + 2\frac{\epsilon|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2+\epsilon]^2} + \frac{\epsilon^2}{[|\nabla u^\epsilon|^2+\epsilon]^2},$$

we further write

$$\begin{aligned} H(0) = &\left\{(p-2)[2-\gamma+\frac{p-2}{N-1}] + [p-\gamma+\frac{2(p-2)}{N-1}] + \frac{N}{N-1}\right\}\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2+\epsilon]^2} \\ &+ \left\{[p-\gamma+\frac{2(p-2)}{N-1}] + 2\frac{N}{N-1}\right\}\frac{\epsilon|\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2+\epsilon]^2} \\ &+ \frac{N}{N-1}\frac{\epsilon^2}{[|\nabla u^\epsilon|^2+\epsilon]^2}. \end{aligned}$$

By a direct calculation, $\gamma < 3 + \frac{p-1}{N-1}$ implies that

$$\begin{aligned} [p-\gamma+\frac{2(p-2)}{N-1}] + 2\frac{N}{N-1} &> p + \frac{2(p-2)}{N-1} + 2\frac{N}{N-1} - 3 - \frac{p-1}{N-1} \\ &= p - 1 + \frac{2(p-2)+2-(p-1)}{N-1} \\ &= (p-1)\frac{N}{N-1} \end{aligned}$$

$$> 0.$$

Moreover, $\gamma < 3 + \frac{p-1}{N-1}$ also implies that

$$\begin{aligned}
&(p-2)[2-\gamma+\frac{p-2}{N-1}]+[p-\gamma+\frac{2(p-2)}{N-1}]+\frac{N}{N-1}\\
&=3(p-1)+\frac{(p-2)^2+2(p-2)+1}{N-1}-(p-1)\gamma\\
&=3(p-1)+\frac{(p-1)^2}{N-1}-(p-1)\gamma\\
&=(p-1)[3+\frac{p-1}{N-1}-\gamma]\\
&>0.
\end{aligned}$$

Thus

$$\begin{aligned}
H(0) &> (p-1)[3+\frac{p-1}{N-1}-\gamma]\frac{|\nabla u^\epsilon|^4}{[|\nabla u^\epsilon|^2+\epsilon]^2}+\frac{N}{N-1}\frac{\epsilon^2}{[|\nabla u^\epsilon|^2+\epsilon]^2}\\
&\geq \frac{1}{2}\min\left\{(p-1)[3+\frac{p-1}{N-1}-\gamma],\frac{N}{N-1}\right\}\\
&=: \delta(N,p,\gamma)\\
&>0
\end{aligned}$$

that is, (3.26) holds. □

Combining (3.15) and (3.4) we have the following. Recall that

$$Ric_h^N(\nabla u^\epsilon,\nabla u^\epsilon)=Ric_g(\nabla u^\epsilon,\nabla u^\epsilon)+\langle\nabla^2 h\nabla u^\epsilon,\nabla u^\epsilon\rangle-\frac{\langle\nabla u^\epsilon,\nabla h\rangle^2}{N-n}.$$

Corollary 3.6. *Let u^ϵ be the solution to (3.3). If $\gamma < 3 + \frac{p-1}{N-1}$ for some $N \geq n$, then for sufficiently small $\eta > 0$ one has*

$$\begin{aligned}
&\eta\int_U |\nabla^2 u^\epsilon|^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h}d\mathrm{vol}_g\\
&\quad\leq -\int_U Ric_h^N(\nabla u^\epsilon,\nabla u^\epsilon)[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h}d\mathrm{vol}_g\\
&\qquad+C(n,N,p,\gamma,\eta)\int_U\left([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+1}|\nabla\phi|^2+[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+2}\phi^2\right)e^{-h}d\mathrm{vol}_g
\end{aligned}\tag{3.27}$$

Under the Bakry-Émery curvature-dimension assumption, we have the following uniform upper bound.

Lemma 3.7. *Let u^ϵ be the solution to (3.3). If $\gamma < 3 + \frac{p-1}{N-1}$ and $Ric_h^N \geq -\kappa$, then one has*

$$\begin{aligned}
&\int_U |\nabla[[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{4}}\nabla u^\epsilon]|^2\phi^2\, e^{-h}d\mathrm{vol}_g \\
&\leq C(n,N,p,\gamma)\int_U \kappa|\nabla u^\epsilon|^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h}d\mathrm{vol}_g \\
&\quad + C(n,N,p,\gamma)\int_U \left([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+1}|\nabla\phi|^2 + [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}+2}\phi^2\right) e^{-h}d\mathrm{vol}_g
\end{aligned} \tag{3.28}$$

Proof. By $Ric_h^N \geq -\kappa$ we know that

$$-Ric_h^N(\nabla u^\epsilon, \nabla u^\epsilon) \leq \kappa|\nabla u^\epsilon|^2$$

Thus the first term in the right-hand side of (3.27) is bounded from above by

$$\kappa\int_U |\nabla u^\epsilon|^2[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\phi^2\, e^{-h}d\mathrm{vol}_g.$$

On the other hand, a direct calculation leads to

$$\begin{aligned}
&|\nabla[[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{4}}\nabla u^\epsilon]|^2 \\
&\quad = [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}\left|\nabla^2 u^\epsilon + \frac{p-\gamma}{2}\frac{\nabla u^\epsilon\otimes\nabla^2 u^\epsilon\nabla u^\epsilon}{|\nabla u^\epsilon|^2+\epsilon}\right|^2 \\
&\quad = [|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}[|\nabla^2 u^\epsilon|^2 + (p-\gamma)\frac{|\nabla^2 u^\epsilon\nabla u^\epsilon|^2}{|\nabla u^\epsilon|^2+\epsilon} + \frac{(p-\gamma)^2}{4}\frac{|\nabla u^\epsilon|^2|\nabla^2 u^\epsilon\nabla u^\epsilon|^2}{[|\nabla u^\epsilon|^2+\epsilon]^2}] \\
&\quad \leq C(n,p,\gamma)[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{2}}|\nabla^2 u^\epsilon|^2.
\end{aligned}$$

Thus, up to a constant multiplier, the left-hand side of (3.27) is bounded by

$$\int_U |\nabla[[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{4}}\nabla u^\epsilon]|^2 e^{-h}d\mathrm{vol}_g.$$

We therefore conclude (3.28) from (3.27). □

Now we are able to prove Theorem 1.2.

Proof of Theorem 1.2. Let $w \in W^{1,p}_{\rm loc}(\Omega)$ be any positive weighted p-harmonic function in the domain Ω and $u = -(p-1)\ln w$. Given any smooth domain $U \Subset \Omega$, for each $\epsilon \in (0,1]$, let $u^\epsilon \in C^\infty(U)$ be the solution to (3.3). By Lemma 3.1, we know that $u^\epsilon \to u \in C^{1,\alpha}(U)$, for some $\alpha \in (0,1)$ uniformly in $\epsilon > 0$ as $\epsilon \to 0$. Using this and choosing suitable test functions $\phi \in C^\infty_c(U)$ in (3.28), one concludes $[|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{4}}\nabla u^\epsilon \in W^{1,2}_{\rm loc}(U)$ uniformly in $\epsilon \in (0,1]$.

Next, we claim that

$$|\nabla u|^{\frac{p-\gamma}{2}}\nabla u \in W^{1,2}_{\rm loc}(U), \tag{3.29}$$

and

$$\nabla([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{4}}\nabla u^\epsilon) \to \nabla(|\nabla u|^{\frac{p-\gamma}{2}}\nabla u) \text{ weakly in } L^2_{\rm loc}(U,\mathbb{R}^{n\times n}) \text{ as } \epsilon \to 0. \tag{3.30}$$

To see this, for any subdomain $V \Subset U$, by Lemma 3.7, we already have

$$\sup_{\epsilon\in(0,1]} \|\nabla([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{4}}\nabla u^\epsilon)\|_{L^2(V,\mathbb{R}^{n\times n})} < C(\kappa,n,N,p,\gamma,V).$$

For any subsequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ which converges to 0, by the weak compactness of $W^{2,2}(V)$, up to some subsequence one has $\nabla([|\nabla u^{\epsilon_j}|^2+\epsilon_j]^{\frac{p-\gamma}{4}}\nabla u^{\epsilon_j}) \to z$ weakly in $L^2(V,\mathbb{R}^{n\times n})$ for some function $z \in L^2(V,\mathbb{R}^{n\times n})$. Let $\{e_1,\cdots,e_n\} \subset T_xU$ be a local orthonormal frame at each $x \in U$. Notice that the $n\times n$ matrix

$$\nabla([|\nabla u^{\epsilon_j}|^2+\epsilon_j]^{\frac{p-\gamma}{4}}\nabla u^{\epsilon_j}) = \Big(\nabla_{e_l}([|\nabla u^{\epsilon_j}|^2+\epsilon_j]^{\frac{p-\gamma}{4}}\nabla_{e_k}u^{\epsilon_j})\Big)_{1\le k,l\le n}.$$

Recalling from Lemma 3.1 that $\nabla u^\epsilon \to \nabla u$ in $C^\alpha(U)$ and $V \Subset U$, for any $\phi \in C^\infty_c(U)$ with $\phi|_V = 1$ and $1 \le k,l \le n$, we have

$$\begin{aligned}
&\lim_{j\to 0}\int_U \nabla_{e_l}([|\nabla u^{\epsilon_j}|^2+\epsilon_j]^{\frac{p-\gamma}{4}}\nabla_{e_k}u^{\epsilon_j})\phi\, e^{-h}d\mathrm{vol}_g \\
&\qquad = -\lim_{j\to 0}\int_U ([|\nabla u^{\epsilon_j}|^2+\epsilon_j]^{\frac{p-\gamma}{4}}\nabla_{e_k}u^{\epsilon_j})\nabla_{e_l}(\phi e^{-h})\, d\mathrm{vol}_g \\
&\qquad = -\int_U (|\nabla u|^{\frac{p-\gamma}{2}}\nabla_{e_k}u)\nabla_{e_l}(\phi e^{-h})\, d\mathrm{vol}_g \\
&\qquad = \int_U \nabla_{e_l}(|\nabla u|^{\frac{p-\gamma}{2}}\nabla_{e_k}u)\phi\, e^{-h}d\mathrm{vol}_g.
\end{aligned}$$

This shows that in the distributional sense

$$\nabla([|\nabla u^{\epsilon_j}|^2+\epsilon_j]^{\frac{p-\gamma}{4}}\nabla u^{\epsilon_j}) \to \nabla(|\nabla u|^{\frac{p-\gamma}{2}}\nabla u).$$

Thus $z = \nabla(|\nabla u|^{\frac{p-\gamma}{2}}\nabla u)|_V \in L^2(V, \mathbb{R}^{n\times n})$ in distributional sense. We therefore have $|\nabla u|^{\frac{p-\gamma}{2}}\nabla u|_V \in W^{1,2}(V)$, which gives (3.29).

Moreover, by the arbitrariness of subsequence $\{\epsilon_j\}$, we have

$$\nabla([|\nabla u^\epsilon|^2+\epsilon]^{\frac{p-\gamma}{4}}\nabla u^\epsilon) \to \nabla(|\nabla u|^{\frac{p-\gamma}{2}}\nabla u)$$

weakly in $L^2(V, \mathbb{R}^{n\times n})$ as $\epsilon \to 0$. Hence by the arbitrariness of $V \Subset U$, (3.30) holds.

Letting $\epsilon \to 0$ in (3.28) and using the convergence in the above verified claim, we obtain

$$\begin{aligned}
&\int_U |\nabla[|\nabla u|^{\frac{p-\gamma}{2}}\nabla u]|^2\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\quad\le C(n,N,p,\gamma)\kappa \int_U |\nabla u|^{p-\gamma+2}\phi^2\, e^{-h} d\mathrm{vol}_g \\
&\qquad + C(n,N,p,\gamma)\int_U \left(|\nabla u|^{p-\gamma+2}|\nabla\phi|^2 + |\nabla u|^{p-\gamma+4}\phi^2\right) e^{-h} d\mathrm{vol}_g. \qquad (3.31)
\end{aligned}$$

Let $\phi \in C_c^\infty(B_{2r})$, where $B_{4r} \subset U$, such that $\phi = 1$ in B_r and $|\nabla\phi| \le \frac{C}{r}$. Then (3.31) becomes

$$\begin{aligned}
&\int_{B_r} |\nabla[|\nabla u|^{\frac{p-\gamma}{2}}\nabla u]|^2\, e^{-h} d\mathrm{vol}_g \\
&\quad\le C(n,N,p,\gamma)\int_{B_{2r}} \left[(\frac{1}{r^2}+\kappa)|\nabla u|^{p-\gamma+2} + |\nabla u|^{p-\gamma+4}\right] e^{-h} d\mathrm{vol}_g.
\end{aligned}$$

Recalling from (1.5) the Cheng-Yau type gradient estimate that $|\nabla u| \le C(n,N,p)\frac{1+\sqrt{\kappa}r}{r}$ and noting that $\gamma < 3 + \frac{p-1}{N-1}$ guarantees $p-\gamma+2>0$, we deduce

$$|\nabla u|^{p-\gamma+2} \le C(n,N,p,\gamma)[\frac{1+\sqrt{\kappa}r}{r}]^{p-\gamma+2}.$$

Together with $\frac{1}{r^2}+\kappa \le (\frac{1+\sqrt{\kappa}r}{r})^2$, we conclude

$$\int_{B_r} |\nabla[|\nabla u|^{\frac{p-\gamma}{2}}\nabla u]|^2\, e^{-h} d\mathrm{vol}_g \le C(n,N,p,\gamma)\mathrm{vol}_h(B_{2r})\left[\frac{1+\sqrt{\kappa}r}{r}\right]^{p-\gamma+4}.$$

Dividing both sides by $\mathrm{vol}_h(B_r)$, noting $\mathrm{vol}_h(B_{2r}) \le e^{\sqrt{\kappa}r}\mathrm{vol}_h(B_r)$ from the volume comparison (2.1), and recalling $u = -(p-1)\ln w$, we conclude (1.8).

Note that (1.9) is just the special case $\gamma = p$ of (1.8), where $p < 3 + \frac{2}{N-2}$ guarantees $p < 3 + \frac{p-1}{N-1}$ and hence one can take $\gamma = p$ in (1.8). $\square$

Finally, we compare our proof with [21,5], in particular, the crucial pointwise lower bound given in Lemma 3.4 and Lemma 3.5.

Remark 3.8. (i) It was well known that a positive (weighted) p-harmonic function w, and hence $\ln w$, is always smooth outside of the null set E_w of $\nabla \ln w$. In $\Omega \setminus E_w$, the proof of Lemma 3.5 works for $\ln w$ so to get (3.16) with u^ϵ replaced by $\ln w$ and $\epsilon = 0$, dividing both sides of which by $|\nabla \ln w|^4$, for $0 < \eta < 1/2$ one gets

$$(1+\eta)|\nabla^2 \ln w|^2 \geq 2\frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} + \left(\frac{(p-1)^2}{N-1} - 1\right)\frac{(\Delta_\infty \ln w)^2}{|\nabla \ln w|^4} - (1+\eta)\frac{\langle \nabla \ln w, \nabla h\rangle^2}{N-n} - C(n,N,p)\frac{1}{\eta}|\nabla \ln w|^4. \tag{3.32}$$

If $\gamma < 3 + \frac{p-1}{N-1}$, using (3.32) and noting that the proof of Lemma 3.4 works for $\ln w$, we get (3.15) with u^ϵ replaced by $\ln w$ and $\epsilon = 0$, that is, for $\eta > 0$ sufficiently small,

$$\begin{aligned}(1-\eta)|\nabla^2 \ln w|^2 + (p-\gamma)\frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} + (p-2)(2-\gamma)\frac{(\Delta_\infty \ln w)^2}{|\nabla \ln w|^4} \\ \geq \eta|\nabla^2 \ln w|^2 - \frac{\langle \nabla \ln w, \nabla h\rangle^2}{N-n} - C(n,N,p,\gamma)\frac{1}{\eta}|\nabla \ln w|^4.\end{aligned} \tag{3.33}$$

From the proof, we see that both of the coefficient 2 of $\frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2}$ and the coefficient $\frac{(p-1)^2}{N-1} - 1$ of $(\Delta_\infty \ln w)^2$ in (3.32) are critical to guarantee the existence of sufficiently small $\eta > 0$ in (3.33) when $\gamma < 3 + \frac{p-1}{N-1}$.

On the other hand, instead of (3.32), recall the following lower bound obtained in [5] by using Lemma 2.1 and the equation (3.1):

$$|\nabla^2 \ln w|^2 \geq \frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} - 2\frac{p-1}{n-1}\Delta_\infty \ln w + \frac{1}{N-1}|\nabla \ln w|^2 - \frac{\langle \nabla \ln w, \nabla h\rangle^2}{N-n}, \tag{3.34}$$

and also, when $N = n$ and $h \equiv 1$, recall the following lower bound derived in [21] via Lemma 2.1 and (3.1):

$$|\nabla^2 \ln w|^2 \geq [1 + \min\{\frac{(p-1)^2}{n-1}, 1\}]\frac{|\nabla^2 \ln w \nabla \ln w|^2}{|\nabla \ln w|^2} - 2\frac{p-1}{n-1}\Delta_\infty \ln w + \frac{1}{n-1}|\nabla \ln w|^2. \tag{3.35}$$

From (3.34) and (3.35), via a direct check one can conclude $|\nabla \ln w|^{\frac{p-\gamma+2}{2}} \in W^{1,2}_{\text{loc}}$ for $\gamma < 2$, but NOT for all $\gamma < 3 + \frac{p-1}{N-1}$.

(ii) Moreover, unlike [21,5] where the authors differentiate the equation (3.1) for $\ln w$, we directly derive an upper bound from Bochner formula for the left-hand side of (3.33) with respect to $[|\nabla u^\epsilon|^2 + \epsilon]^{\frac{p-\gamma}{2}} e^{-h} d\text{vol}_g$.

Data availability

No data was used for the research described in the article.

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Appendix A. Proof of Lemma 3.1

In the appendix, we show Lemma 3.1 by checking equations (3.1) and (3.3) are special cases considered in [3]. To this end, we recall the result in [3].

Let Ω be a domain of M^n. Consider the equation

$$-\operatorname{div} \vec{a}(x, \nabla u) + b(x, \nabla u) = 0 \quad \text{in } \Omega \tag{A.1}$$

where $\vec{a}$ is a map from $\Omega \times \mathbb{R}^n$ to $\mathbb{R}^n$ and b maps $\Omega \times \mathbb{R}^n$ to $\mathbb{R}$. Let $\{e_1, \cdots, e_n\} \subset T_x\Omega$ be a local orthonormal frame at each $x \in \Omega$. By a weak solution of (A.1) we mean a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega [\langle \vec{a}(x, \nabla u), \nabla \phi \rangle + b(x, \nabla u)\phi] \, d\mathrm{vol}_g = 0 \quad \forall \phi \in C_c^\infty(\Omega). \tag{A.2}$$

Assume the following holds for $\vec{a} = (a_1, \cdots, a_n)$ and b.

$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq \gamma_0 |\eta|^{p-2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, p > 1, \tag{A$_1$}$$

$$\left| \frac{\partial a_j}{\partial \eta_i} \right| \leq \gamma_1 |\eta|^{p-2}, \quad 1 \leq i, j \leq n, \tag{A$_2$}$$

$$|\nabla_{e_i} a_j(x, \eta)| \leq \gamma_1 |\eta|^{p-1}, \quad 1 \leq i, j \leq n, \tag{A$_3$}$$

$$|b(x, \eta)| \leq \gamma_1 |\eta|^p, \tag{A$_4$}$$

and

$$|\nabla_{e_i} b(x, \eta)| \leq \gamma_1 |\eta|^p, \ \left| \frac{\partial b}{\partial \eta_i}(x, \eta) \right| \leq \gamma_1 |\eta|^{p-1}, \quad 1 \leq i \leq n, \tag{B}$$

for all $\eta \in \mathbb{R}^n$, where γ_i are positive constants, $i = 0, 1$.

For any smooth domain $U \Subset \Omega$ and $\epsilon \in (0,1]$, consider the regularized equation

$$-\operatorname{div} \vec{a^\epsilon}(x, \nabla u^\epsilon) + b^\epsilon(x, \nabla u^\epsilon) = 0 \quad \text{in } U \text{ and } u^\epsilon = u \text{ on } \partial U \tag{A.3}$$

where $\vec{a^\epsilon}$ is a map from $U \times \mathbb{R}^n$ to $\mathbb{R}^n$ and b^ϵ maps $U \times \mathbb{R}^n$ to $\mathbb{R}$ such that

$$\lim_{\epsilon \to 0} \vec{a^\epsilon}(x,\eta) = \vec{a}(x,\eta) \text{ and } \lim_{\epsilon \to 0} b^\epsilon(x,\eta) = b(x,\eta) \quad \forall (x,\eta) \in \Omega \times \mathbb{R}^n.$$

The weak solution of (A.3) is defined similarly as (A.2). Assume the following holds for $\vec{a^\epsilon} = (a_1^\epsilon, \cdots, a_n^\epsilon)$ and b^ϵ.

$$\sum_{i,j=1}^n \frac{\partial a_j^\epsilon}{\partial \eta_i}(x,\eta)\xi_i\xi_j \geq \gamma_0(\epsilon + |\eta|^2)^{\frac{p-2}{2}}|\xi|^2, \quad \xi \in \mathbb{R}^n, p > 1, \tag{$A_{1,\epsilon}$}$$

$$\left|\frac{\partial a_j^\epsilon}{\partial \eta_i}\right| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p-2}{2}}, \quad 1 \leq i,j \leq n, \tag{$A_{2,\epsilon}$}$$

$$|\nabla_{e_i} a_j^\epsilon(x,\eta)| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p-1}{2}}, \quad 1 \leq i,j \leq n, \tag{$A_{3,\epsilon}$}$$

$$|b^\epsilon(x,\eta)| \leq \gamma_1(\epsilon + |\eta|^2)^{\frac{p}{2}}, \tag{$A_{4,\epsilon}$}$$

for all $\eta \in \mathbb{R}^n \setminus \{0\}$.

We recall the results in [3] as follows.

Theorem A.1. *Let $\epsilon \in (0,1]$ and $U \Subset \Omega$. Assume (A_1)-(A_4), (B) and ($A_{1,\epsilon}$)-($A_{4,\epsilon}$) hold. Then there exists a unique solution $u^\epsilon \in C^\infty(U) \cap C^0(\overline{U})$ to (A.3), and moreover, $u^\epsilon \to u$ in $C^0(\overline{U})$ and $u^\epsilon \to u$ in $C^{1,\alpha}(V)$ uniformly in $\epsilon > 0$ as $\epsilon \to 0$ for all $V \Subset U$ where u is the solution to (A.1). As a consequence, $u \in C^{1,\alpha}(\Omega)$.*

Theorem A.1 is a combination of Theorem 1 and Theorem 2 in [3] and several intermediate results in the proof of these two theorems in [3]. Indeed, the existence, uniqueness and C^∞-regularity of u^ϵ is by elliptic theory in PDE; see for example [8]. Based on these facts, in [3], the author first showed that under (A_1)-(A_4), (B) and ($A_{1,\epsilon}$)-($A_{4,\epsilon}$), $u^\epsilon \to u$ in $W^{1,p}(U)$ uniformly in $\epsilon > 0$ in section 2. Moreover, $\|u^\epsilon\|_{L^\infty(U)} \leq \max_{x \in \partial U}\{|u(x)|\}$. Thus recalling that $u^\epsilon|_{\partial U} = u|_{\partial U}$, we know $u^\epsilon \to u$ in $C^0(\overline{U})$. See the discussion around (2.7) in [3]. Then the author showed that $\|u^\epsilon\|_{C^{1,\alpha}(V)}$ is uniformly bounded independently of $\epsilon \in (0,1]$ and finally showed that $u^\epsilon \to u$ in $C^{1,\alpha}(V)$ and $u \in C^{1,\alpha}(U)$ for all $V \Subset U$. By the arbitrariness of $U \Subset \Omega$, one has $u \in C^{1,\alpha}(\Omega)$.

Proof of Lemma 3.1. It suffices to check equations (3.1) and (3.2) are special ones of (A.1) and (A.3) respectively. To this end, let $\vec{a}(x,\eta) = e^{-h(x)}|\eta|^{p-2}\eta$, $b(x,\eta) = -e^{-h(x)}|\eta|^p$, $\vec{a^\epsilon}(x,\eta) = e^{-h(x)}(|\eta|^2 + \epsilon)^{\frac{p-2}{2}}\eta$, and $b^\epsilon(x,\eta) = -e^{-h(x)}(|\eta|^2 + \epsilon)^{\frac{p-2}{2}}|\eta|^2$ for all $x \in U$ and $\eta \in \mathbb{R}^n$. Then in the weak sense, the equations

$$\int_\Omega [\langle \vec{a}(x, \nabla u), \nabla \phi \rangle + b(x, \nabla u)\phi]\, d\mathrm{vol}_g = 0, \quad \forall \phi \in C_c^\infty(\Omega)$$

and

$$\int_\Omega [\langle \vec{a^\epsilon}(x, \nabla u), \nabla \phi \rangle + b^\epsilon(x, \nabla u)\phi]\, d\mathrm{vol}_g = 0, \quad \forall \phi \in C_c^\infty(\Omega)$$

are exactly (3.1) and (3.2) respectively.

We show $\vec{a}$ satisfies (A_1). Noting that $a_j(x, \eta) = e^{-h(x)}|\eta|^{p-2}\eta_j$, we compute

$$\frac{\partial a_j}{\partial \eta_i}(x, \eta) = e^{-h(x)}[(p-2)|\eta|^{p-4}\eta_i\eta_j + \delta_{ij}|\eta|^{p-2}], \quad \forall 1 \leq i, j \leq n$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Thus

$$\begin{aligned}\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j &= e^{-h(x)} \sum_{i,j=1}^n [((p-2)|\eta|^{p-4}\eta_i\eta_j + \delta_{ij}|\eta|^{p-2})\xi_i\xi_j] \\ &= e^{-h(x)}|\eta|^{p-4}[(p-2)(\sum_{i=1}^n \eta_i\xi_i)^2 + |\eta|^2|\xi|^2], \quad \forall \xi \in \mathbb{R}^n.\end{aligned}$$

If $1 < p < 2$, we have

$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq e^{-h(x)}(p-1)|\eta|^{p-2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

And if $p \geq 2$, we have

$$\sum_{i,j=1}^n \frac{\partial a_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq e^{-h(x)}|\eta|^{p-2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

By taking $\gamma_0 := \min_{x \in \overline{U}}\{e^{-h(x)}\}$, we conclude that a satisfies (A_1). By direct computations, one can also check $\vec{a}, \vec{a^\epsilon} \in C^\infty(U \times \mathbb{R}^n, \mathbb{R}^n)$, $b, b^\epsilon \in C^\infty(U \times \mathbb{R}^n)$ satisfy (A_2)-(A_4), (B) and ($A_{1,\epsilon}$)-($A_{4,\epsilon}$) respectively. We omit the details. Thus by Theorem A.1, we get the desired result. □

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