ANNALES ACADEMIÆ SCIENTIARUM FENNICÆ

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DISSERTATIONES

73

CAPACITY EXTENSION DOMAINS

PEKKA KOSKELA

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SUOMALAINEN TIEDEAKATEMIA

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Pekka Koskela

Contents

Introduction

F. W. Gehring and O. Martio [GM2] introduced the class of quasiextremal distance domains in connection with the theory of quasiconformal mappings. We generalize their definition from the conformally invariant case $p = n$ to arbitrary $1 < p < \infty$.

A domain $D \subset \mathbb{R}^n$ is called a p-quasiextremal distance (p-QED) domain if there is a constant C such that for any pair K_0, $K_1 \subset D$ of disjoint continua

$$\operatorname{cap}_p(K_0, K_1, \mathbb{R}^n) \leq C \operatorname{cap}_p(K_0, K_1, D), \tag{1}$$

where cap_p is the variational p-capacity.

Together with the class of p-QED domains we study the related class of Sobolev p-capacity (p-SC) domains defined by replacing cap_p in the inequality (1) by the capacity S_p associated with the Sobolev spaces W_p^1; see section 5.

We show that, even though n-QED domains appear to be more regular than p-QED domains for $p \neq n$, these classes still enjoy some of the properties of n-QED domains established in [GM2]. Our primary interest is in the case $p > n-1$, since for $1 < p \leq n-1$ it is possible that $\operatorname{cap}_p(K_0, K_1, \mathbb{R}^n) = 0$ for a pair of disjoint, non-degenerate continua K_0, $K_1 \subset \mathbb{R}^n$.

We mention the following results as examples of properties of p-QED and p-SC domains.

(a) For $p \geq n$: p-QED domains are quasiconvex and p-SC domains are locally quasiconvex (1.3, 3.1, 5.8).

(b) For $p > n-1$: A p-QED domain or a p-SC domain cannot be too thin near its boundary, i.e., it satisfies a uniform measure density condition (4.1, 5.15).

(c) Uniform domains are p-QED and p-SC domains for all $1 < p < \infty$ (1.3, 2.3, 5.7).

(d) For each $p \neq 2$ there is a simply connected, planar, non-uniform p-QED domain while a simply connected, planar 2-QED domain is always uniform (1.3, 2.5, 2.6, 3.7, [GM2, 2.23]).

We show (1.3, 2.2, 2.4, 5.7) that L_p^1-extension and bounded W_p^1-extension domains are p-QED domains, and W_p^1-extension domains are p-SC domains; see 1.3 for definitions. Thus our results for p-QED and p-SC domains contribute to the study of the following problem raised by F. W. Gehring in [G3]: Characterize the domains $D \subset \mathbb{R}^n$ with a Sobolev extension property.

As another result in this direction we have:

(e) Let $D \subset \mathbb{R}^n$ be a domain quasiconformally equivalent to a uniform domain $D' \subset \mathbb{R}^n$. Then D is an L^1_n-extension domain if and only if it is uniform. If also either D or $\mathbb{R}^n \setminus D$ is bounded, then D is a W^1_n-extension domain if and only if it is uniform (6.3).

In the case $p > n$ we introduce a weak version of p-QED (p-SC) domains. We call D a weak p-QED (p-SC) domain if inequality (1) ((1) with cap_p replaced by S_p) holds for all pairs of distinct singletons $K_0 = \{x\}$, $K_1 = \{y\}$ in D. We show (1.3, 7.5):

(f) Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $p > n$. Then the following four conditions are equivalent.
(i) D is a weak p-QED domain.
(ii) D is a weak p-SC domain.
(iii) D is a W^1_p-imbedding domain.
(iv) D is an L^1_p-imbedding domain.

We also study W^1_p-approximation domains, i.e., domains $D \subset \mathbb{R}^n$ for which $C_0^\infty(\mathbb{R}^n)$ is dense in $W^1_p(D)$. We establish a sufficient condition for a domain to be a W^1_p-approximation domain. In particular, for a bounded domain $D \subset \mathbb{R}^n$ we have:

(g) If there is a compact set $K \subset \partial D$ such that $\operatorname{cap}_p(K, B, D) = 0$ for some closed ball $B \subset D$, and if $D \cup V$ is uniform for arbitrarily small neighborhoods V of K, then D is a W^1_q-approximation domain for all $1 < q \le p$ (8.1, 8.8).

Section 1 contains the definitions used and some estimates for the variational p-capacity. We study p-QED, p-SC, L^1_p-extension and W^1_p-extension domains in sections 1–6. In section 7 we study weak p-QED and weak p-SC domains together with L^1_p- and W^1_p-imbedding domains. Section 8 is devoted to the study of W^1_p-approximation domains and finally in section 9 we establish two applications connected with quasiconformal mappings.

1. Preliminaries

1.1. Notation. Our notation is standard and usually as in [Väl]. Throughout this paper D is a domain in $\mathbb{R}^n$, $n \ge 2$, and $p \in (1, \infty)$.

The n-dimensional Lebesgue measure is denoted by m_n or m, and we employ the abbreviations $\Omega_n = m_n(B^n(1))$ and $\omega_{n-1} = m_{n-1}(S^{n-1}(1))$, where $B^n(1) = B^n(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and $S^{n-1}(1) = S^{n-1}(0,1) = \partial B^n(1)$. By $L^p(D)$ we denote the Banach space of all measurable functions $u\colon D \to \mathbb{R} \cup \{-\infty, \infty\}$ for which the norm $\|u\|_{L^p(D)} = (\int_D |u|^p \, dm)^{1/p}$ is finite. Moreover, $L^1_p(D)$ is the space of measurable functions $u\colon D \to \mathbb{R} \cup \{-\infty, \infty\}$ whose first distributional derivatives lie in $L^p(D)$, and we equip $W^1_p(D) = L^1_p(D) \cap L^p(D)$ with the norm $\|u\|_{W^1_p(D)} = \|\nabla u\|_{L^p(D)} + \|u\|_{L^p(D)}$, where ∇u is the distributional gradient of u.

The letters b and C stand for various constants, and if C depends only on α, β, ... we write $C = C(\alpha, \beta, \dots)$.

If γ is a curve, the locus of γ is denoted by $|\gamma|$. A rectifiable curve γ is always parametrized by arc length.

For any pair of disjoint, compact sets $K_0, K_1 \subset \overline{D}$ we define the p-modulus of K_0 and K_1 relative to D by

$$M_p(K_0, K_1, D) = M_p\big(\Delta(K_0, K_1, D)\big),$$

where $\Delta(K_0, K_1, D)$ is the family Γ of curves joining K_0 and K_1 in D, and $M_p(\Gamma)$ is the p-modulus of Γ; see [Väl, 6.1]. Further, the variational p-capacity of K_0 and K_1 relative to D is

$$\operatorname{cap}_p(K_0, K_1, D) = \inf_{u \in L(K_0, K_1, D)} \int_D |\nabla u|^p \, dm,$$

where $L(K_0, K_1, D) = \{u \in L_p^1(D) \cap C(D \cup K_0 \cup K_1) \colon u \equiv 0$ on K_0 and $u \equiv 1$ on $K_1\}$. We write $\operatorname{cap}_p(K, D)$ for $\operatorname{cap}_p(\partial D, K, D)$, $\operatorname{cap}_p(x, D)$ for $\operatorname{cap}_p(\{x\}, D)$, and $\operatorname{cap}_p(x, y, D)$ for $\operatorname{cap}_p(\{x\}, \{y\}, D)$.

1.2. Remark. By [H, 5.5]

$$\operatorname{cap}_p(K_0, K_1, D) = M_p(K_0, K_1, D)$$

for any pair of disjoint, compact sets K_0, $K_1 \subset D$. This result will be tacitly used in what follows. Note that is not known if the above equality holds for K_0, $K_1 \subset \partial D$.

1.3. Definitions.

(i) D is (finitely) locally connected at $x \in \partial D$ if there are arbitrarily small neighborhoods U of x such that $U \cap D$ is (finitely) connected. A set is finitely connected if it has a finite number of components. Further, D is (finitely) locally connected on the boundary if D is (finitely) locally connected at each boundary point.

(ii) D is locally quasiconvex if there are constants $0 < \delta \leq \infty$ and $b \geq 1$ such that any x, $y \in D$ with $|x - y| \leq \delta$ can be joined in D by a curve whose length does not exceed $b|x - y|$. When $\delta = \infty$, we call D b-quasiconvex or quasiconvex.

(iii) D is a (b, δ)-domain [J], $0 < \delta \leq \infty$, $1 \leq b$, if for all x, $y \in D$ with $|x - y| < \delta$ there is a curve $\gamma \colon [0, \ell(\gamma)] \to D$ with $\gamma(0) = x$, $\gamma(\ell(\gamma)) = y$, $\ell(\gamma) \leq b|x - y|$, and $B^n(\gamma(t), \frac{1}{b}\min\{t, \ell(\gamma) - t\}) \subset D$ for $t \in (0, \ell(\gamma))$. A (b, ∞)-domain is called b-uniform; see [GO], [J], [M1], and [MS].

(iv) D is a John domain [MS] if there are constants $a \geq b > 0$ and a point $x_0 \in D$ such that each $x \in D$ can be joined to x_0 by a curve

$\gamma \colon [0, \ell(\gamma)] \to D$ with $\gamma(0) = x$; $\ell(\gamma) \le a$, and $B^n\bigl(\gamma(t), bt/\ell(\gamma)\bigr) \subset D$ for $0 < t \le \ell(\gamma)$; see [M2] and [NV] for various characterizations of John domains.

(v) D is an L_p^1-extension domain if there is a bounded linear operator $E_p \colon L_p^1(D) \to L_p^1(\mathbf{R}^n)$ with $E_p u|_D = u$ for all $u \in L_p^1(D)$. Boundedness of E_p means boundedness with respect to the seminorms $\|\nabla u\|_{L^p(D)}$ and $\|\nabla E_p u\|_{L^p(\mathbf{R}^n)}$.

(vi) D is a W_p^1-extension domain if there is a bounded linear operator $E_p \colon W_p^1(D) \to W_p^1(\mathbf{R}^n)$ with $E_p u|_D = u$ for all $u \in W_p^1(D)$.

(vii) D is an L_p^1-imbedding domain, $p > n$, if

$$|u(x) - u(y)| \le C\,\|\nabla u\|_{L^p(D)}\,|x-y|^{1-(n/p)}$$

for all $u \in L_p^1(D)$ and x, $y \in D$, where u is identified with its continuous refinement and C is independent of u. For the existence of such a refinement we refer the reader to [Mz, 1.1.2] and [A, 5.4].

(viii) D is a W_p^1-imbedding domain, $p > n$, if

$$|u(x) - u(y)| \le C\,\|u\|_{W_p^1(D)}\,|x-y|^{1-(n/p)}$$

for all $u \in W_p^1(D)$ and x, $y \in D$, where u is identified with its continuous refinement and C is independent of u.

(ix) D is a W_p^1-approximation domain if $C_0^\infty(\mathbf{R}^n)$ is dense in $W_p^1(D)$.

1.4. Remarks.

(i) We always have $W_p^1(D) \subset L_p^1(D)$, but it may happen that $L_p^1(D) \not\subset W_p^1(D)$ even if D is bounded; see [Mz, 1.1.4].

(ii) There are W_p^1-extension domains which fail to be L_p^1-extension domains; see Example 6.7. The converse seems to be an open problem.

(iii) A bounded (b,δ)-domain is C-uniform, where $C = C(n, b, \delta, \mathrm{dia}(D))$.

To see this, observe first that a bounded (b,δ)-domain is c-quasiconvex, where $c = c(n, b, \delta, \mathrm{dia}(D))$. Thus, given x, $y \in D$, there is a curve γ joining x and y in D with $\ell(\gamma) \le c\,|x-y|$. Pick points $x = z_1, z_2, \dots, z_k = y \in |\gamma|$ such that $\delta/2b < |z_{i+1} - z_i| < \delta/b$, $i = 1, \dots, k-1$. Connect these points by curves γ_i as in Definition 1.3.(iii). Now $B^n(\omega_i, \delta/4b^2) \subset D$ and $|\omega_{i+1} - \omega_i| < \delta$, $i = 1, \dots, k-1$, where $\omega_i = \gamma_i(\ell(\gamma_i)/2)$. Finally, join ω_i to ω_{i+1} as above, $i = 1, \dots, k-1$.

This process yields a curve $\widetilde{\gamma}$ joining x and y in D with $\ell(\widetilde{\gamma}) \le c\,b^2\,|x-y|$, and it is easy to see that

$$B^n\bigl(\widetilde{\gamma}(t), a\min\{t, \ell(\widetilde{\gamma}) - t\}\bigr) \subset D,$$

$t \in (0, \ell(\tilde{\gamma}))$, where $a = a(b, c, \delta, \mathrm{dia}(D))$; hence D is C-uniform with $C = C(n, b, \delta, \mathrm{dia}(D))$.

(iv) For the readers convenience we chart the known relations between the various classes of domains introduced in Definition 1.3. For simplicity we abbreviate local connectedness on the boundary to LC and finite local connectedness on the boundary to FLC.

$$
\begin{array}{ccccc}
\text{LC} & \Longrightarrow & \text{FLC} & & \\
\Uparrow & & & & \\
\text{locally quasiconvex} & & & & W^1_p\text{-approximation} \\
\Uparrow\ \downarrow & & & & \\
\text{quasiconvex} & & \nwarrow\!\!\nwarrow & & \Uparrow \\
\Uparrow & & & & \\
\text{uniform} & \begin{array}{c}\Longrightarrow\\ \longleftarrow\end{array} & (b, \delta) & \overset{(b)}{\Longrightarrow} & W^1_p\text{-extension} \\
(a)\ \downarrow & & & & \Downarrow\ {\scriptstyle p>n} \\
\text{John} & & & & W^1_p\text{-imbedding} \\
 & & & & \Uparrow \\
 & & L^1_p\text{-extension} & \overset{p\geq n}{\Longrightarrow} & L^1_p\text{-imbedding}
\end{array}
$$

The implications denoted by $\rightarrow$ hold only for bounded domains. For the implications denoted by (a) and (b) the reader is referred to [GM1, 2.18] and [J, Theorem 1], respectively, whereas all the remaining implications are more or less immediate.

We establish the following additions to the implications mentioned above.

$$
\begin{array}{ccccc}
W^1_p\text{-extension} & \underset{p\geq n\ (6.1)}{\Longrightarrow} & \text{locally quasiconvex} & & \\
 & & \Uparrow\ {\scriptstyle (7.10)} & & \\
 & & W^1_p\text{-imbedding} & & \\
 & & \downarrow & & \\
 & & L^1_p\text{-imbedding} & \overset{(7.10)}{\Longrightarrow} & \text{quasiconvex} \\
 & & & & \Uparrow\ {\scriptstyle p\geq n\ (6.1)} \\
 & & & & L^1_p\text{-extension}
\end{array}
$$

The reader is also referred to Examples 2.5, 6.7, 8.10, Theorems 6.3, 6.4, Corollary 8.14, and Remark 6.6 for related results.

1.5. Preliminary lemmas. The purpose of the remainder of this section is to establish estimates for the variational p-capacity $\mathrm{cap}_p(K_0, K_1, D)$ that will be used frequently in what follows.

1.6. Lemma. *Let $p > n-1$. Suppose that K_0, $K_1 \subset S = S^{n-1}(x,r)$ are disjoint, non-empty, compact sets. Then*

$$M_p(K_0, K_1, S) \geq C\, r^{n-p-1},$$

where $C = C(p,n)$ and $M_p(K_0, K_1, S)$ is defined as in [Väl, 10.1].

Proof. The proof of [Väl, 10.2] for the case $p = n$ applies with minor modifications to our case and yields the desired inequality with constant $C(p,n)$, where

$$C(p,2) = (2\pi)^{1-p},$$

and

$$C(p,n) = \frac{\omega_{n-2}}{2}\Big(\int_0^\infty t^{-(n-2)/(p-1)}(1+t^2)^{-(p-n+1)/(p-1)}\,dt\Big)^{1-p}, \quad n > 2.$$

Exactly as in [Väl, 10.12] the preceding lemma implies

1.7. Lemma. *Let $0 < a < b$. If $B^n(x,b)\setminus\overline{B}^n(x,a) \subset D$ and if K_0, $K_1 \subset D$ are disjoint, compact sets such that every sphere $S^{n-1}(x,t)$, $a \leq t \leq b$, meets both K_0 and K_1, then*

$$\operatorname{cap}_p(K_0, K_1, D) \geq \begin{cases} C(n)\log(b/a), & p = n \\ C(p,n)\,|b^{n-p} - a^{n-p}|, & n-1 < p < n \text{ or } p > n. \end{cases}$$

Replacing n by p in the proof of [N2, 3.1], see also [MRV, 3.11], we obtain

1.8. Lemma. *Let K_0, K_1, $K_2 \subset D$ be disjoint, non-empty, compact sets. Then*

$$\begin{aligned}&\operatorname{cap}_p(K_0, K_1, D)\\ &\geq 3^{-p}\min\{\operatorname{cap}_p(K_0, K_2, D), \operatorname{cap}_p(K_1, K_2, D), \inf_{F_0, F_1}\operatorname{cap}_p(F_0, F_1, D)\},\end{aligned}$$

where the infimum is taken over all pairs of continua F_0, $F_1 \subset D$ joining K_0 to K_2 and K_1 to K_2, respectively.

1.9. Corollary. *Let $K \subset B^n(x,r)$ be a continuum with* $\operatorname{dia}(K) \geq br$*, $0 < b$, and let $n-1 < p \leq n$. Then*

$$\operatorname{cap}_p\big(K, B^n(x,r)\big) \geq C\, r^{n-p},$$

where $C = C(p,n,b)$.

Proof. For $p = n$ the claim follows from [GM2, 2.6]. Assume that $n-1 < p < n$.

Let x_1, $x_2 \in K$ satisfy $|x_1 - x_2| = \operatorname{dia}(K)$, and let x_3 be a point of $S^{n-1}(x,r)$ on the line through x_1 and x_2. By symmetry we may assume that $|x_1 - x_3| \leq |x_2 - x_3|$. Then $S^{n-1}(x_3, t)$ intersects both K and $S^{n-1}(x,r)$ for each $|x_1 - x_3| \leq t \leq |x_1 - x_3| + \operatorname{dia}(K)$. Hence Lemma 1.7 and elementary calculus imply

$$\operatorname{cap}_p\left(K, S^{n-1}(x,r), \mathbf{R}^n\right) \geq C_0\left(2^{n-p} - (2-b)^{n-p}\right) r^{n-p},$$

where $C_0 = C_0(p,n)$. Since $\operatorname{cap}_p\left(K, S^{n-1}(x,r), \mathbf{R}^n\right) = \operatorname{cap}_p\left(K, B^n(x,r)\right)$, the desired inequality follows.

1.10. Lemma.

(i) *Let $n-1 < p \leq n$, and let K_0, K_1 be two continua with*

$$\min_{i=0,1} \operatorname{dia}(K_i) \geq A\, d(K_0, K_1),$$

where $A > 0$. Then

$$\operatorname{cap}_p(K_0, K_1, \mathbf{R}^n) \geq C\left(\min_{i=0,1} \operatorname{dia}(K_i)\right)^{n-p},$$

where $C = C(p,n,A)$.

(ii) *Let $p > n$. Then for any pair x, y of distinct points*

$$C^{-1}\, |x-y|^{n-p} \leq \operatorname{cap}_p(x, y, \mathbf{R}^n) \leq C\, |x-y|^{n-p},$$

where $C = C(p,n)$.

Proof. First we prove (i). By symmetry we may assume that $r = \operatorname{dia}(K_0) \leq \operatorname{dia}(K_1)$. Pick a point $x \in K_0$; set $a = (2 + 1/A)r$ and $b = 2a$. Then $K_0 \subset \overline{B}^n(x,a)$, and $K_1 \cap \overline{B}^n(x,a)$ contains a continuum K with $\operatorname{dia}(K) \geq r$. Now $\min\{\operatorname{dia}(K_0), \operatorname{dia}(K)\} = r = b/\left(2(2+1/A)\right)$, and hence by Corollary 1.9

$$\min\{\operatorname{cap}_p\left(K_0, B^n(x,b)\right), \operatorname{cap}_p\left(K, B^n(x,b)\right)\} \geq C\, r^{n-p},$$

where $C = C(p,n,A)$. Thus the claim follows by Lemmas 1.7 and 1.8.

Now we establish (ii). Let x, $y \in \mathbf{R}^n$ be two distinct points. Define $u(z) = \min\{1, |z-x|/|x-y|\}$ for $z \in \mathbf{R}^n$. Then $u \in L(x,y,\mathbf{R}^n)$, and hence

$$\begin{aligned}\operatorname{cap}_p(x,y,\mathbf{R}^n) &\leq \int_{\mathbf{R}^n} |\nabla u|^p \, dm \leq |x-y|^{-p} m_n\left(B^n(x, |x-y|)\right) \\ &= \Omega_n\, |x-y|^{n-p}.\end{aligned}$$

To verify the reverse inequality, let $u \in L(x,y,\mathbf{R}^n)$. We may assume that $0 \leq u \leq 1$. Now $u \in W_p^1\left(B^n(x, 2\,|x-y|)\right)$, and hence the Hölder continuity estimate [BI, 1.7] implies

$$1 = |u(x) - u(y)| \leq C\, \|\nabla u\|_{L^p(B^n(x,2\,|x-y|))}\, |x-y|^{1-n/p},$$

where $C = C(p,n)$. Thus

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dm \geq C^{-p} |x-y|^{n-p},$$

and the claim follows.

In addition to the preceding estimates we frequently use the following well known results; see for example [Mz, 2.2.4].

1.11. Proposition. *Let* $0 < r < R$, *and let* $x \in \mathbb{R}^n$. *Then*

$$\operatorname{cap}_p\left(\overline{B}^n(x,r), B^n(x,R)\right) = \begin{cases} \omega_{n-1}(\log R/r)^{1-n}, & p = n \\ \omega_{n-1}\left|\frac{p-n}{p-1}\right|^{p-1}\left|R^{(p-n)/(p-1)} \right. \\ \qquad \left. -r^{(p-n)/(p-1)}\right|^{1-p}, & p \neq n. \end{cases}$$

Moreover, for $p > n$

$$\operatorname{cap}_p\left(x, B^n(x,r)\right) = \omega_{n-1}\left(\frac{p-n}{p-1}\right)^{p-1} r^{n-p}.$$

2. p-QED domains

In this section we define the class of p-QED domains. Using a result of P. W. Jones, F. W. Gehring and O. Martio showed that uniform domains are n-QED domains, and established that a simply connected planar 2-QED domain is always uniform [GM2, 2.18, 2.23]; see also [GV]. We show that uniform domains are p-QED domains for all $1 < p < \infty$ and we produce examples of non-uniform simply connected planar p-QED domains for each $p \neq 2$.

2.1. Definition. A domain D is called a (C,p)-quasiextremal distance (QED) domain if for each pair K_0, $K_1 \subset D$ of disjoint continua

$$\operatorname{cap}_p(K_0, K_1, \mathbb{R}^n) \leq C \operatorname{cap}_p(K_0, K_1, D),$$

or equivalently,

$$M_p(K_0, K_1, \mathbb{R}^n) \leq C\, M_p(K_0, K_1, D).$$

Finally, D is a p-QED domain if D is a (C,p)-QED domain for some constant C.

2.2. Theorem. *An* L^1_p*-extension domain is a* (C^p,p)*-QED domain, where* C *is the norm of the extension operator.*

Proof. Let K_0, $K_1 \subset D$ be two disjoint continua, and let $\varepsilon > 0$. Take a $u \in L^1_p(D) \cap C(D)$ such that $u \equiv 0$ on U_0, $u \equiv 1$ on U_1, and

$$\int_D |\nabla u|^p \, dm \leq \operatorname{cap}_p(K_0, K_1, D) + \varepsilon,$$

where U_i is a neighborhood of K_i, $i = 0, 1$. Choose a smooth convolution approximation v of $E_p u$ such that $v = u$ on U and

$$\int_{\mathbf{R}^n} |\nabla v|^p \, dm \le \int_{\mathbf{R}^n} |\nabla E_p u|^p \, dm + \varepsilon,$$

where $U \subset U_0 \cup U_1$ is a neighborhood of $K_0 \cup K_1$. Then $v \in L(K_0, K_1, \mathbf{R}^n)$, and hence

$$\begin{aligned} \operatorname{cap}_p(K_0, K_1, \mathbf{R}^n) &\le \int_{\mathbf{R}^n} |\nabla v|^p \, dm \le C^p \int_D |\nabla u|^p \, dm + \varepsilon \\ &\le C^p \big(\operatorname{cap}_p(K_0, K_1, D) + \varepsilon \big) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

2.3. Theorem. *A b-uniform domain is a (C,p)-QED domain, $C = C(p,n,b)$, for all $1 < p < \infty$.*

Proof. By [J, Theorem 2] a b-uniform domain is an L_n^1-extension domain with the norm of the extension operator not exceeding $C_0 = C_0(p,n,b)$. The proof given in [J] applies to all $1 < p < \infty$ provided that D is unbounded. Hence, for unbounded domains, our claim follows from Theorem 2.2. Suppose now that D is bounded, and let K_0, $K_1 \subset D$ be a pair of disjoint continua. Applying an auxiliary stretching if necessary we may assume that $\operatorname{dia}(D) = 1$; see [Väl, 8.2] and [M1, 6.2]. Let $u \in L(K_0, K_1, D)$. We may assume that $u \in C^1(D) \cap W_p^1(D)$. Arguing as in the proof of Theorem 2.2 it suffices to show that there is a $\omega \in L_p^1(\mathbf{R}^n)$ with $\omega|_D = u$ and $\int_{\mathbf{R}^n} |\nabla \omega|^p \, dm \le C \int_D |\nabla u|^p \, dm$, where C is independent of u.

Set $v(x) = u(x) - \int_D u \, dm / m_n(D)$ for $x \in D$. Then $|\nabla v| = |\nabla u|$, $v \in W_p^1(D)$ and by [J, Theorem 1] there is an extension $E_p v \in W_p^1(\mathbf{R}^n)$ satisfying

$$\|\nabla(E_p v)\|_{L^p(\mathbf{R}^n)} \le \|E_p v\|_{W_p^1(\mathbf{R}^n)} \le C_0 \, \|v\|_{W_p^1(D)},$$

where $C_0 = C_0(p,n,b)$. By [GM1, 2.18], a bounded b-uniform domain D is a John domain with constants $a_1 = a_1(b, \operatorname{dia}(D))$ and $a_2 = a_2(b, \operatorname{dia}(D))$. Thus we may apply the Poincaré type inequality [M2, 3.1]

$$\int_D \Big| u - \int_D u \, dm / m_n(D) \Big|^p \, dm \le C_1 \int_D |\nabla u|^p \, dm,$$

where $C_1 = C_1(a_1, a_2, p, n)$, to conclude that

$$\begin{aligned} \int_{\mathbf{R}^n} |\nabla(E_p v)|^p \, dm &\le 2^p \, C_0^p \int_D (|\nabla v|^p + |v|^p) \, dm \\ &= 2^p \, C_0^p \int_D \Big(|\nabla u|^p + \Big| u - \int_D u \, dm / m_n(D) \Big|^p \Big) \, dm \\ &\le 2^p \, C_0^p \, (1 + C_1) \int_D |\nabla u|^p \, dm. \end{aligned}$$

Since $u(x) = (E_p v)(x) + \int_D u\, dm/m_n(D)$ for $x \in D$, the proof is complete.

2.4. Theorem. *A bounded W_p^1-extension domain is a p-QED domain.*

Proof. By the proof of Theorem 2.3 it suffices to show that the Poincaré type inequality

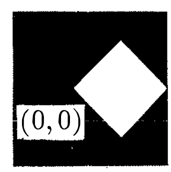
$$\int_D \left| u - \int_D u\, dm/m_n(D) \right|^p dm \le C \int_D |\nabla u|^p\, dm$$

holds for all $u \in C^1(D) \cap W_p^1(D)$ with some constant C independent of u. It is well known, see e.g. [SS, Theorem 12], that this inequality follows if the imbedding $W_p^1(D) \to L^p(D)$ is compact.

Let B be an open ball containing D. Since the imbedding $W_p^1(B) \to L^p(B)$ is compact, see [A, 6.2], and $D \subset B$ is a W_p^1-extension domain, the imbedding $W_p^1(D) \to L^p(D)$ is also compact, and our claim follows.

We close this section by establishing examples of non-uniform planar p-QED domains, $p \neq 2$.

2.5. Example. Let D be the shaded region in our picture. Then D is a p-QED domain for all $1 < p < 2$ but clearly fails to be uniform. By Theorem 2.4 it suffices to show that D is a W_p^1-extension domain for all $1 < p < 2$.



Indeed, the argument in [Mz, 1.5.2] shows that D is a W_p^1-extension domain for all $1 < p < 2$. To be more precise, let $u^+ = u|_{D^+}$ for a given $u \in W_p^1(D)$, $1 < p < 2$, where D^+ is the upper half of D. Denote the upper half of the larger square Q by G, and let $F = \{(x, 0) \colon 0 < x < 1\}$. Since D^+ is a (b, δ)-domain, there is a bounded extension operator $E_p \colon W_p^1(D^+) \to W_p^1(\mathbf{R}^2)$. Define

$$\varphi((x,y)) = \begin{cases} (4/\pi)\arctan(y/x), & 0 < x \le 1/2 \\ (4/\pi)\arctan(y/(1-x)), & 1/2 \le x < 1 \end{cases}$$

for $(x, y) \in G \setminus D^+$, and set φ to be 1 in D^+ and 0 in F. The estimates in [Mz, 1.5.2] imply that

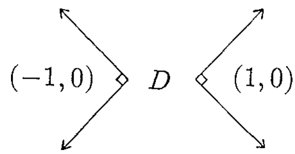
$$\|\varphi E_p u^+\|_{W_p^1(G)} \le C_1 \|E_p u^+\|_{W_p^1(G)}.$$

By symmetry we obtain an extension v of u with $v|_D = u$ and $v \equiv 0$ on F. It follows that $v \in W_p^1(Q)$ and

$$\|v\|_{W_p^1(Q)} \le C_2 \|u\|_{W_p^1(D)}.$$

Since Q is a (b, δ)-domain, we conclude that D is a W_p^1-extension domain for all $1 < p < 2$.

2.6. *Example.* Let $D = \{(x,y)\colon |y| > |x| - 1\}$. Then D is a p-QED domain for all $p > 2$ and clearly not uniform.



Notice first that

$$D_1 = \{(x,y) \in D\colon y > -4\}$$

and

$$D_2 = \{(x,y) \in D\colon y < 4\}$$

are uniform. By the proof of Theorem 2.3

$$\operatorname{cap}_p(K_0, K_1, \mathbb{R}^2) \le C \operatorname{cap}_p(K_0, K_1, D_i) \le C \operatorname{cap}_p(K_0, K_1, D)$$

whenever K_0, $K_1 \subset D_i$, $i = 1,2$, are two disjoint, compact sets. Note that any continuum $K \subset D$ may be written as the union of three compact sets E_1, E_2, E_3 with $E_1 \subset \{(x,y) \in D\colon y \ge 3\}$, $E_2 \subset \{(x,y) \in D\colon |y| \le 3\}$, and $E_3 \subset \{(x,y) \in D\colon y \le -3\}$. Since the variational p-capacity is subadditive [Väl, 6.2], we conclude that D is a p-QED domain provided that

$$\operatorname{cap}_p(K_0, K_1, \mathbb{R}^2) \le C_1 \operatorname{cap}_p(K_0, K_1, D)$$

whenever

$$K_0 \subset \{(x,y) \in D\colon y \ge 3\}$$

and

$$K_1 \subset \{(x,y) \in D\colon y \le -3\}$$

are two compact sets.

Let K_0, $K_1 \subset D$ be as above, and let $u \in L(K_0, K_1, D)$. If $u(0) > 1/2$, then the function $v = \min\{1, 2u\}$ is in $L(K_0, \{0\}, D)$. Otherwise the function $\omega = \max\{0, 2(u - 1/2)\}$ is in $L(\{0\}, K_1, D)$. Thus

$$\begin{aligned}
2^p \operatorname{cap}_p(K_0, K_1, D) &\ge \min\{\operatorname{cap}_p(0, K_1, D), \operatorname{cap}_p(K_0, 0, D)\} \\
&\ge \min\{\operatorname{cap}_p(0, K_1, D_2), \operatorname{cap}_p(K_0, 0, D_1)\} \\
&\ge \frac{1}{C} \min\{\operatorname{cap}_p(0, K_1, \mathbb{R}^2), \operatorname{cap}_p(K_0, 0, \mathbb{R}^2)\}.
\end{aligned}$$

By Lemma 1.10.(ii)

$$\min\{\operatorname{cap}_p(0, K_1, \mathbb{R}^2), \operatorname{cap}_p(K_0, 0, \mathbb{R}^2)\} \ge C_0 \left(\max_{i=0,1} d(K_i, 0)\right)^{2-p},$$

where $C_0 = C_0(p)$. Therefore, it suffices to show that

$$\operatorname{cap}_p(K_0, K_1, \mathbb{R}^2) \le C_2 \left(\max_{i=0,1} d(K_i, 0)\right)^{2-p}$$

for some constant C_2 independent of K_0 and K_1. Note that for $p>2$ there is a $C_3 = C_3(p)$ such that $\operatorname{cap}_p(F_0, F_1, \mathbb{R}^2) \le C_3$ whenever

$$F_0 \subset \{(x,y) \in D\colon y > 0 \text{ and } x^2 + y^2 \ge 4\}$$

and

$$F_1 \subset \{(x,y) \in D\colon y \le 0\}$$

are two compact sets. Indeed, $u = \min\{1, \max\{0, v\}\} \in L(F_0, F_1, \mathbb{R}^2)$, where

$$v((x,y)) = \begin{cases} 1 - y/(|x|-1), & |x| \ge 3/2 \\ 1 - 2y, & |x| < 3/2 \end{cases},$$

and integration in polar coordinates yields

$$\int_{\mathbb{R}^2} |\nabla u|^p \, dm \le C_3(p).$$

Now symmetry and [Väl, 8.2] yield

$$\operatorname{cap}_p(K_0, K_1, \mathbb{R}^2) \le C_3 \, 2^{p-2} \left(\max_{i=0,1} d(K_i, 0) \right)^{2-p},$$

and our reasoning is complete.

2.7. Remarks.

(i) The given bounds for p in Example 2.5 and Example 2.6 are essential; see Remarks 3.7.(i) and (iii).

(ii) There are bounded, non-uniform planar p-QED domains for all $p>2$ as seen by modifying the unbounded W_p^1-extension domain in [Mz, 1.5.2]; these appear more complicated than the domain in Example 2.6.

(iii) An unbounded W_p^1-extension domain may fail to be a p-QED domain; see Example 6.7.

(iv) The proof of Theorem 2.4 does not yield any estimate for the p-QED constant of a bounded W_p^1-extension domain. We refer the reader to Remarks 6.9.(ii) for some results in this direction.

3. Geometric properties of p-QED domains

A uniform domain is a p-QED domain for all $1 < p < \infty$, but a p-QED domain may fail to be uniform. Indeed, $\mathbb{R}^2 \setminus \{(0,i)\colon |i| = 0,1,2,\dots\}$ is clearly p-QED for all $1<p<\infty$ but not uniform. In this section we establish that p-QED domains nevertheless enjoy some of the same geometric properties posessed by uniform domains.

3.1. Theorem. *Let $D \subset \mathbb{R}^n$ be a (C,p)-QED domain with $p \geq n$. Then D is b-quasiconvex, where $b = b(p,n,C)$.*

Proof. For $p = n$ the claim is proved in [GM2, 2.7].

Suppose that $p > n$. Let x_1, x_2 be two distinct points in D and let $r = |x_1 - x_2|$. By Lemma 1.10.(ii)

$$\operatorname{cap}_p(x_1, x_2, \mathbb{R}^n) \geq C_0\, r^{n-p},$$

where $C_0 = C_0(p,n)$. Hence

$$\operatorname{cap}_p(x_1, x_2, D) \geq \frac{C_0}{C}\, r^{n-p}.$$

Let $S = C_1 r$, where

$$C_1 = \Bigl(\frac{C_0}{2\,\omega_{n-1}(\frac{p-n}{p-1})^{p-1}\,C}\Bigr)^{1/(n-p)}.$$

Then

$$\operatorname{cap}_p\bigl(x_1, B^n(x_1,S)\bigr) = \frac{C_0}{2C}\, r^{n-p}.$$

Since $\operatorname{cap}_p(x_1,x_2,D) \geq \dfrac{C_0}{C} r^{n-p}$, it follows by [Väl, 6.2, 6.4] that x_1 and x_2 belong to a component V of $B^n(x_1,S) \cap D$ and that

$$\operatorname{cap}_p(x_1, x_2, V) \geq \frac{C_0}{2C}\, r^{n-p}.$$

Suppose that $\ell(\gamma) \geq \ell > 0$ for every curve γ joining x_1 and x_2 in V. Then by [Väl, 7.1]

$$\operatorname{cap}_p(x_1, x_2, V) \leq \frac{\Omega_n\, S^n}{\ell^p} = \frac{\Omega_n\, C_1^n}{\ell^p}\, r^n,$$

and hence

$$\ell \leq \Bigl(\frac{2\,C\,\Omega_n\,C_1^n}{C_0}\Bigr)^{1/p} r.$$

Consequently, x_1 and x_2 can be joined in D by a curve whose length does not exceed $b\,|x_1 - x_2|$, where

$$b = 2\Bigl(\frac{2\,C\,\Omega_n\,C_1^n}{C_0}\Bigr)^{1/p}.$$

Therefore D is b-quasiconvex.

3.2. Definition. F. W. Gehring has introduced the notion of linear local connectivity; see [G1] and the references therein. A domain $D \subset \mathbb{R}^n$ is b-linearly locally connected if for each $x_0 \in \mathbb{R}^n$ and each $r > 0$

LLC(1) points in $D \cap \overline{B}^n(x_0, r)$ can be joined in $D \cap \overline{B}^n(x_0, br)$,

and

LLC(2) points in $D \setminus B^n(x_0, r)$ can be joined in $D \setminus B^n(x_0, r/b)$.

Further, D is linearly locally connected, or LLC, if D is b-linearly locally connected for some constant b.

Gehring and Martio established [GM2, 2.11] that n-QED domains are LLC. Examples 2.5 and 2.6 show that for $1 < p < n$ a p-QED domain may fail to satisfy LLC(1), and for $p > n$ a p-QED domain may fail to satisfy LLC(2).

We have the following corollary to Theorem 3.1.

3.3. Corollary. *Let D be a (C,p)-QED domain with $p \geq n$. Then D satisfies LLC(1) with a constant $b = b(p,n,C)$.*

3.4. Theorem. *Let D be a (C,p)-QED domain with $n-1 < p \leq n$. Then D satisfies LLC(2) with a constant $b = b(p,n,C)$.*

Proof. The case $p = n$ is proved in [GM2, 2.11].

Assume that $n-1 < p < n$. Let x_1, $x_2 \in S^{n-1}(x_0, r) \cap D$ and choose a curve γ joining x_1 and x_2 in D. Denote by F_i the x_i-component of $|\gamma| \setminus B^n(x_0, r/2)$, $i = 1, 2$.

Suppose that x_1 and x_2 cannot be joined in $D \setminus B^n(x_0, sr)$ for some $s < 1/2$. Then F_1, $F_2 \subset D$ are continua,

$$\min_{i=1,2} \operatorname{dia}(F_i) \geq r/2 \geq d(F_1, F_2)/4,$$

and F_1, F_2 cannot be joined in $D \setminus B^n(x_0, sr)$. Thus, by Lemma 1.10.(i),

$$\operatorname{cap}_p(F_1, F_2, D) \geq \frac{C_0}{C} r^{n-p},$$

where $C_0 = C_0(p,n)$; on the other hand, by [Väl, 6.4]

$$\begin{aligned} \operatorname{cap}_p(F_1, F_2, D) &\leq \operatorname{cap}_p\left(\overline{B}^n(x_0, sr), B^n(x_0, r/2)\right) \\ &= \omega_{n-1}\left(\frac{n-p}{p-1}\right)^{p-1}\left(s^{(p-n)/(p-1)} - (1/2)^{(p-n)/(p-1)}\right)^{1-p} r^{n-p}. \end{aligned}$$

Hence $s \geq b(p, n, C_0, C)$, and therefore x_1 and x_2 can be joined in $D \setminus B^n(x_0, \frac{b}{2}r)$.

Finally, let y_1, $y_2 \in D \setminus B^n(x_0, r)$. Since D is a domain, either $B^n(x_0, r) \cap D = \emptyset$, or we can join y_i to a point $x_i \in S^{n-1}(x_0, r) \cap D$ in $D \setminus B^n(x_0, r)$, $i = 1, 2$. This together with the first part of our proof implies that y_1 and y_2 can be joined in $D \setminus B^n(x_0, \frac{b}{2}r)$, and the claim follows.

3.5. Corollary. *Let $D \subset \mathbb{R}^n$ be a (C_1, p_1)-QED and a (C_2, p_2)-QED domain with $n-1 < p_1 \leq n \leq p_2$. Then D is b-LLC, where $b = b(p_1, p_2, n, C_1, C_2)$.*

Let D, D' be two domains in $\mathbb{R}^n$. Recall that D and D' are quasiconformally equivalent if there is a quasiconformal mapping f of D onto D'. We refer the reader to [Väl] for the definition and basic properties of quasiconformal mappings.

3.6. Corollary. *Let $D \subset \mathbb{R}^n$ be a domain which is quasiconformally equivalent to a uniform domain $D' \subset \mathbb{R}^n$. Then the following conditions are equivalent:*

(i) *D is n-QED.*
(ii) *D is p-QED for all $1 < p < \infty$.*
(iii) *D is p-QED and q-QED with $n-1 < p \leq n \leq q$.*
(iv) *D is LLC.*
(v) *D is uniform.*

Proof. By [Vä2, 5.6] conditions (i), (iv) and (v) are equivalent. By Theorem 2.3.(v) implies (ii) which trivially yields (iii). Finally, (iii) implies (iv) by Corollary 3.5.

The remainder of this section deals with the planar case. Recall that $D \subset \mathbb{R}^2$ is said to be locally connected at infinity if there are arbitrarily large $r > 0$ such that for some open U_r containing the complement of some disk, $U_r \cap B^2(r) = \emptyset$, and $U_r \cap D$ is connected. Denote the one-point compactification of $\mathbb{R}^2$ by $\overline{\mathbb{R}}^2$. We call a domain $D \subset \overline{\mathbb{R}}^2$ a *quasidisk* if it is the image of an open disk under a quasiconformal self-mapping of $\overline{\mathbb{R}}^2$.

3.7. Remarks. Let $D \subsetneq \mathbb{R}^2$ be simply connected.

(i) Suppose that D is a p-QED domain. If $p \geq 2$, then D is locally connected on the boundary by Theorem 3.1. If $1 < p \leq 2$, then Theorem 3.4 and [NV, 2.18, 4.5] imply that D is finitely locally connected on the boundary and locally connected at infinity. Note that by Examples 2.5 and 2.6 our assumptions on p are necessary.

(ii) If D is a bounded p-QED domain with $1 < p \leq 2$, then D is a John domain. Indeed, this follows from Theorem 3.4 and [NV, 4.5].

(iii) Set $D^* = \mathbb{R}^2 \setminus \overline{D}$. Then the following conditions are equivalent:

(a) D is a 2-QED domain.
(b) D is a p-QED domain for all $1 < p < \infty$.
(c) D is a p-QED and a q-QED domain with $1 < p \leq 2 \leq q$.
(d) D is uniform.
(e) D is a quasidisk.
(f) D and D^* are both p-QED domains for some $p \geq 2$.
(g) D is locally connected on the boundary and both D and D^* are p-QED domains for some $1 < p \leq 2$.

The equivalence of the conditions (a)–(d) follows from the Riemann mapping theorem and Corollary 3.6. Further, (d), (e), (f), and (g) are equivalent by (i), Theorems 2.3, 3.1 and 3.4, [NV, 4.5, 9.3], and [MS, 2.33].

We refer the reader to [G2] for a detailed study of quasidisks.

4. A measure property of p-QED domains

We show that for $p > n-1$ each p-QED domain satisfies a uniform measure density condition. Our method of proof is similar to that in [GM2, 2.13].

4.1. Theorem. *Let $D \subset \mathbf{R}^n$ be a (C,p)-QED domain with $p > n-1$. Then for any $x_0 \in \overline{D}$ and any $0 < r < \operatorname{dia}(D)$*

$$m_n\big(D \cap B^n(x_0,r)\big) \geq \frac{C_0}{C} m_n\big(B^n(x_0,r)\big),$$

where $C_0 = C_0(p,n)$.

Proof. Fix $x_0 \in \overline{D}$ and $0 < r < \operatorname{dia}(D)$. Pick a point $x_2 \in D$ such that $|x_2 - x_0| = r/2$. Set $S = r/10$ and choose $x_1 \in D$ such that $|x_1 - x_0| < S$, and let γ be a curve joining x_1 and x_2 in D. Denote the x_1-component of $|\gamma| \cap \overline{B}(x_0, 2S)$ by K_1 and the x_2-component of $|\gamma| \setminus B^n(x_0, 3S)$ by K_2, respectively. Set $u(x) = \min\{1, d(x, \overline{B}^n(x_0, 2S))/S\}$ for $x \in D$. Then $u \in L(K_1, K_2, D)$ and $|\nabla u| \leq 1/S$; hence

$$\begin{aligned}\operatorname{cap}_p(K_1,K_2,D) &\leq \int_D |\nabla u|^p \, dm \leq m_n\big(D \cap B^n(x_0,r)\big)/S^p \\ &= \frac{10^p \, \Omega_n \, r^{n-p} \, m_n\big(D \cap B^n(x_0,r)\big)}{m_n\big(B^n(x_0,r)\big)}.\end{aligned}$$

Since $\min_{i=1,2} \operatorname{dia}(K_i) \geq S \geq d(K_1,K_2)/4$, Lemma 1.10 together with the QED-property of D yield

$$\operatorname{cap}_p(K_1,K_2,D) \geq \frac{C_0}{C}\left(\frac{r}{10}\right)^{n-p},$$

where $C_0 = C_0(p,n)$. Hence

$$m_n\big(D \cap B^n(x_0,r)\big) \geq \frac{C_0}{10^n \, \Omega_n \, C} m_n\big(B^n(x_0,r)\big),$$

and the proof is complete.

4.2. Corollary. *Let $D \subset \mathbf{R}^n$ be a p-QED domain with $p > n-1$. Then $m_n(\partial D) = 0$.*

Proof. It follows from Theorem 4.1 that ∂D cannot contain points of n-density, and hence $m_n(\partial D) = 0$.

4.3. Corollary. *Let $D \subset \mathbf{R}^2$ be a p-QED domain. Then $m_2(\partial D) = 0$.*

5. p-SC domains

In this section we study the class of Sobolev p-capacity (SC) domains. We define this class by introducing an analogue of the p-QED condition replacing the variational p-capacity by the Sobolev p-capacity associated with the Sobolev spaces $W_p^1(D)$. We show that a W_p^1-extension domain is a p-SC domain. We begin by introducing the Sobolev p-capacity $S_p(K_0, K_1, D)$; see also [Me], [Mz] and [R] for various Sobolev capacities.

5.1. Definition. Let K_0, $K_1 \subset \overline{D}$ be disjoint, compact sets. We define the Sobolev p-capacity $S_p(K_0, K_1, D)$ of K_0 and K_1 relative to D by

$$S_p(K_0, K_1, D) = \inf_{u \in W(K_0, K_1, D)} \int_D (|\nabla u|^p + |u|^p)\, dm,$$

where

$$\begin{aligned} W(K_0, K_1, D) = &\{u \in W_p^1(D) \cap C(D \cup K_0 \cup K_1) \colon u \le C_0 \text{ on } K_0, \\ &u \ge C_1 \text{ on } K_1 \text{ for some } C_0, C_1 \text{ with } C_1 - C_0 = 1\}. \end{aligned}$$

If $K \subset D$ is compact, we let $S_p(K, D) = S_p(\partial D, K, D)$.

The following observations are immediate.

5.2. Lemma.

(i) $\operatorname{cap}_p(K_0, K_1, D) \le S_p(K_0, K_1, D)$.
(ii) $S_p(K_0, K_1, D) = S_p(K_1, K_0, D)$.
(iii) *If K_0, $K_1 \subset \overline{D}$ are disjoint, compact sets, $D \subset D'$, and $F_i \subset K_i$, $i = 0, 1$, are compact sets, then*

$$S_p(F_0, F_1, D) \le S_p(K_0, K_1, D').$$

5.3. Lemma. *Let K_0, $K_1 \subset D$ be disjoint, compact sets, and let $\varepsilon > 0$. Then there is an $r_0 > 0$ such that for all $0 < r \le r_0$*

$$S_p\big(K_0(r), K_1(r), D\big) \le S_p(K_0, K_1, D) + \varepsilon,$$

where $K_i(r) = \{x \in \overline{D} \colon d(x, K_i) \le r\}$, $i = 0, 1$.

Proof. Let $u \in W(K_0, K_1, D)$, and let $0 < \delta < 1/2$. Set $d = (1/2) d(K_0 \cup K_1, \partial D)$ if $\partial D \ne \emptyset$ and $d = 1$ otherwise. Then

$$K_{0,\delta} = \{x \in K_0(d) \colon u(x) \le C_0 + \delta\}$$

and

$$K_{1,\delta} = \{x \in K_1(d) \colon u(x) \geq C_1 - \delta\}$$

are disjoint, compact subsets of D and for some $0 < r \leq d$ $K_i(r) \subset K_{i,\delta}$, $i = 0, 1$. Define $u_\delta = u/(1 - 2\delta)$. Now $u_\delta \in W(K_{0,\delta}, K_{1,\delta}, D)$, and hence

$$S_p(K_{0,\delta}, K_{1,\delta}, D) \leq \int_D (|\nabla u_\delta|^p + |u_\delta|^p)\, dm$$
$$\leq (1 - 2\delta)^{-p} \int_D (|\nabla u|^p + |u|^p)\, dm.$$

Thus $S_p(K_{0,\delta}, K_{1,\delta}, D) \leq (1 - 2\delta)^{-p} S_p(K_0, K_1, D)$, and the claim follows by Lemma 5.2.(iii) for $\delta > 0$ sufficiently small.

5.4. *Remark.* Suppose that $m_n(D) < \infty$ and that the Poincaré type inequality $\int_D |u - u_D|^p\, dm \leq C \int_D |\nabla u|^p\, dm$ holds for all $u \in W_p^1(D) \cap C(D)$, where $u_D = \int_D u\, dm / m_n(D)$. Then

$$\operatorname{cap}_p(K_0, K_1, D) \leq S_p(K_0, K_1, D) \leq (C + 1) \operatorname{cap}_p(K_0, K_1, D)$$

for any pair K_0, $K_1 \subset \overline{D}$ of disjoint, compact sets.

By Lemma 5.2.(i) it suffices to verify the right hand side inequality. Let $u \in L(K_0, K_1, D)$. We may assume that $u \in W_p^1(D)$. Now $v = u - u_D$ lies in $W(K_0, K_1, D)$, and $\int_D (|\nabla v|^p + |v|^p)\, dm \leq (C + 1) \int_D |\nabla u|^p\, dm$, hence $S_p(K_0, K_1, D) \leq (C + 1) \operatorname{cap}_p(K_0, K_1, D)$.

Our next result estimates $S_p(K_0, K_1, D)$ in terms of the variational p-capacity without any assumptions on D.

5.5. Theorem. *Let K_0, $K_1 \subset \overline{D}$ be two disjoint, compact sets with $K_1 \subset B^n(x_0, r)$. Then*

$$S_p(K_0, K_1, D) \leq 2^p\big(\operatorname{cap}_p(K_0, K_1, D) + (1 + r^p) \operatorname{cap}_p(K_1, B^n(x_0, r))\big).$$

Proof. Let $u \in L(K_0, K_1, D)$ and let $v \in L\big(K_1, B^n(x_0, r)\big)$. We may assume that $0 \leq u$, $v \leq 1$ and $v \in C_0^1\big(B^n(x_0, r)\big)$.

Set $w(x) = u(x)v(x)$ for $x \in (D \cup K_1) \cap B^n(x_0, r)$ and $w \equiv 0$ on $D \cup K_0 \setminus B^n(x_0, r)$. Then $w \in W(K_0, K_1, D)$ and

$$\int_D (|\nabla w|^p + |w|^p)\, dm \leq \int_{D \cap B^n(x_0, r)} \big((|\nabla u| + |\nabla v|)^p + |v|^p\big)\, dm$$
$$\leq 2^p \Big(\int_D |\nabla u|^p\, dm + \int_{B^n(x_0, r)} (|\nabla v|^p + |v|^p)\, dm\Big).$$

The desired inequality follows since, by the Poincaré inequality [GT, 7.44],

$$\int_{B^n(x_0, r)} |v|^p\, dm \leq r^p \int_{B^n(x_0, r)} |\nabla v|^p\, dm.$$

Next, we define Sobolev p-capacity domains by mimicing the definition for p-QED domains.

5.6. Definition. A domain D is called a Sobolev p-capacity (SC) domain with constant C if for each pair K_0, $K_1 \subset D$ of disjoint continua

$$S_p(K_0, K_1, \mathbb{R}^n) \leq C\, S_p(K_0, K_1, D).$$

Finally, D is a p-SC domain if D is a (C,p)-SC domain for some constant C.

It was shown in Theorem 2.2 that an L^1_p-extension domain is a p-QED domain. Because of Lemma 5.3 we may mimic the argument used to prove Theorem 2.2 thereby establishing

5.7. Theorem. *A W^1_p-extension domain is a (C,p)-SC domain, where C depends only on p and the norm of the extension operator. In particular, a (b,δ)-domain is a (C,p)-SC domain for all $1 < p < \infty$, where $C = C(p,n,b,\delta,d)$ and $d = \min\{1, \operatorname{dia}(D)\}$.*

We proceed to establish some properties of p-SC domains.

5.8. Theorem. *Let $D \subset \mathbb{R}^n$ be a (C,p)-SC domain with $p \geq n$. Then D is locally quasiconvex with constants $\delta = \delta(p,n,C)$ and $b = b(p,n,C)$.*

Proof. Assume first that $p > n$. Let

$$\delta = \left(C\, C_0\, 2^{p+2}\, \omega_{n-1} \left(\frac{p-n}{p-1}\right)^{p-1}\right)^{1/(n-p)},$$

where $C_0 = C_0(p,n)$ is the constant in Lemma 1.10.(ii). Let x_1, x_2 be distinct points in D with $r = (|x_1 - x_2|)/\delta \leq 1$. Then Theorem 5.5 implies

$$\begin{aligned} S_p(x_1, x_2, D) &\leq 2^p\big(\operatorname{cap}_p(x_1, x_2, D) + 2\, \operatorname{cap}_p(x_2, B^n(x_2, r))\big) \\ &= 2^p\left(\operatorname{cap}_p(x_1, x_2, D) + 2\,\omega_{n-1}\left(\frac{p-n}{p-1}\right)^{p-1} r^{n-p}\right) \\ &= 2^p\, \operatorname{cap}_p(x_1, x_2, D) + |x_1 - x_2|^{n-p}/(2\, C\, C_0). \end{aligned}$$

On the other hand, by Lemma 5.2.(i) and Lemma 1.10.(ii)

$$S_p(x_1, x_2, D) \geq \frac{1}{C} S_p(x_1, x_2, \mathbb{R}^n) \geq \frac{|x_1 - x_2|^{n-p}}{C\, C_0}.$$

Thus

$$\operatorname{cap}_p(x_1, x_2, D) \geq \frac{|x_1 - x_2|^{n-p}}{2^{p+1}\, C\, C_0},$$

and hence by the proof of Theorem 3.1 x_1 and x_2 can be joined in D by a curve whose length does not exceed $b\,|x_1 - x_2|$, where $b = b(p,n,C)$.

Next, suppose that $p = n$. Let x_1, x_2 be distinct points in D, and let γ be a curve joining x_1 and x_2 in D. Denote the x_i-component of $|\gamma| \cap \overline{B}^n(x_i, |x_1 - x_2|/4)$

by K_i, $i = 1,2$. Then $\min_{i=1,2} \operatorname{dia}(K_i) \geq d(K_1, K_2)/4$, and hence by Lemma 5.1.(i) and Lemma 1.10.(i)

$$S_n(K_1, K_2, D) \geq \frac{C_0}{C},$$

where $C_0 = C_0(n)$. Further, assuming $|x_1 - x_2| \leq 1$ and using Theorem 5.5

$$\begin{aligned} S_n(K_1, K_2, D) &\leq 2^n \big(\operatorname{cap}_n(K_1, K_2, D) + 2 \operatorname{cap}_n(K_2, B^n(x_2, 1)) \big) \\ &\leq 2^n \Big(\operatorname{cap}_n(K_1, K_2, D) + 2\,\omega_{n-1} \Big(\log \frac{4}{|x_1 - x_2|} \Big)^{1-n} \Big). \end{aligned}$$

Now let $0 < \delta < 1$ be so small that

$$2^{n+1}\,\omega_{n-1} \Big(\log \frac{4}{\delta} \Big)^{1-n} \leq \frac{C_0}{2C}.$$

Then

$$S_n(K_1, K_2, D) \leq 2^n \operatorname{cap}_n(K_1, K_2, D) + \frac{C_0}{2C},$$

provided that $|x_1 - x_2| \leq \delta$, and consequently

$$\operatorname{cap}_n(K_1, K_2, D) \geq \frac{C_0}{2^{n+1} C}.$$

The argument in [GM2, 2.7] now implies that x_1 and x_2 can be joined in D by a curve whose length does not exceed $b\,|x_1 - x_2|$, where $b = b(n, C)$.

As an immediate consequence we have

5.9. Corollary. *Let $D \subset \mathbb{R}^n$ be a (C,p)-SC domain with $p \geq n$. If $\min\{\operatorname{dia}(D), \operatorname{dia}(\mathbb{R}^n \setminus D)\} = d < \infty$, then D is b-quasiconvex and hence satisfies LLC(1) with a constant b, where $b = b(p, n, C, d)$.*

5.10. Theorem. *Let $D \subset \mathbb{R}^n$ be a (C,n)-SC domain. Then for each $\delta > 0$ there is a constant $b = b(n, C, \delta)$ such that whenever $x_0 \in \mathbb{R}^n$ and $0 < r \leq \delta$, points in $D \setminus B^n(x_0, r)$ can be joined in $D \setminus B^n(x_0, r/b)$.*

Proof. Let $0 < r \leq \delta$, and let x_1, $x_2 \in D \cap S^{n-1}(x_0, r)$. Arguing as in the proof of Theorem 3.4, it suffices to show that if x_1 and x_2 cannot be joined in $D \setminus B^n\big(x_0, (s\,r)/2\big)$, $0 < s < 1$, then $1/s < b(n, C, \delta)$.

Suppose that x_1 and x_2 cannot be joined in $D \setminus B^n\big(x_0, (s\,r)/2\big)$, $0 < s < 1$. Let γ be a curve joining x_1 and x_2 in D, and denote the x_i-component of $\big(|\gamma| \cap \overline{B}^n(x_0, r)\big) \setminus B^n(x_0, s\,r)$ by K_i, $i = 1,2$. Then $K_i \cap S^{n-1}(x_0, t) \neq \emptyset$, $i = 1,2$, for all $t \in [s\,r, r]$, and hence by Lemma 1.7 and Lemma 5.2.(i)

$$S_n(K_1, K_2, D) \geq \frac{C_0}{C} \log \frac{1}{s},$$

where $C_0 = C_0(n)$. As in the proof of Theorem 5.9

$$S_n(K_1, K_2, D) \le 2^n \big(\operatorname{cap}_n(K_1, K_2, D) + (1 + (2r)^n) \operatorname{cap}_n \big(K_2, B^n(x_0, 2r)\big)\big).$$

By [Väl, 6.4] we have further

$$\begin{aligned} S_n(K_1, K_2, D) &\le 2^n \big(\operatorname{cap}_n(\overline{B}^n(x_0, (s\,r)/2), B^n(x_0, s\,r)) \\ &\quad + \big(1 + (2r)^n\big) \operatorname{cap}_n \big(\overline{B}^n(x_0, r), B^n(x_0, 2r)\big)\big) \\ &= 2^{n+1}\, \omega_{n-1} (\log 2)^{1-n} \, (1 + 2^{n-1}\, r^n). \end{aligned}$$

Thus

$$\frac{C_0}{C} \log \frac{1}{s} \le 2^{n+1}\, \omega_{n-1} (\log 2)^{1-n} \, (1 + 2^{n-1}\, \delta^n),$$

and consequently $1/s < b(n, C, \delta)$ as desired.

5.11. Corollary. *If $D \subset \mathbb{R}^n$ is a (C, n)-SC domain and if*

$$\min\{\operatorname{dia}(D), \operatorname{dia}(\mathbb{R}^n \setminus D)\} = d < \infty,$$

then D satisfies LLC(2) with a constant b, where $b = b(n, C, d)$.

Proof. It suffices to consider the case $\operatorname{dia}(\mathbb{R}^n \setminus D) = d < \infty$. Let $x_0 \in \mathbb{R}^n$, and let $r > 0$. Note that if $S^{n-1}(x_0, r) \cap (\mathbb{R}^n \setminus D) = \emptyset$ then any two points in $D \setminus B^n(x_0, r)$ can be joined in $D \setminus B^n(x_0, r)$.

Suppose that $S^{n-1}(x_0, r) \cap (\mathbb{R}^n \setminus D) \neq \emptyset$. If $0 < r \le 2d$, then by Theorem 5.10 points in $D \setminus B^n(x_0, r)$ can be joined in $D \setminus B^n(x_0, r/b)$, where $b = b(n, C, d)$. Otherwise $\overline{B}^n(x_0, r/2) \cap (\mathbb{R}^n \setminus D) = \emptyset$, and hence points in $D \setminus B^n(x_0, r)$ can be joined in $D \setminus B^n(x_0, r/2)$. The proof is complete.

5.12. Corollary. *If $D \subset \mathbb{R}^n$ is a (C, n)-SC domain and if*

$$\min\{\operatorname{dia}(D), \operatorname{dia}(\mathbb{R}^n \setminus D)\} = d < \infty,$$

then D is b-LLC, where $b = b(n, C, d)$.

5.13. Remarks.

(i) Corollaries 5.9, 5.11, and 5.12 may fail to hold when both D and $\mathbb{R}^n \setminus D$ are unbounded. Indeed, let $D = \mathbb{R}^n \setminus \{(x_1, \ldots, x_n) \colon 0 \le x_n \le 1 \text{ and } 0 \le x_{n-1}\}$ and $D' = (0,1)^{n-1} \times (0, \infty)$. Then by Theorem 5.7 both D and D' are p-SC domains for all $1 < p < \infty$, but D is not quasiconvex and does not satisfy LLC(1) while D' fails to satisfy LLC(2).

(ii) Let $D \subset \mathbb{R}^n$ be a bounded W^1_p-extension domain. Then by Theorem 2.4 D is a p-QED domain. Hence the properties of p-QED domains imply that D is quasiconvex and satisfies LLC(1) for $p \ge n$ while for

$n-1 < p \leq n$ D satisfies LLC(2); see Theorems 3.1 and 3.4, and Corollary 3.3.

This approach does not yield estimates for the corresponding constants, while Theorem 5.8 and Corollaries 5.9 and 5.11 provide upper bounds for each of these constants in terms of p, n, $\operatorname{dia}(D)$, and the norm of the extension operator. As Example 6.8 shows the LLC(2) constant is not bounded in terms of this data for $n-1 < p < n$.

(iii) Theorem 5.10 and Corollary 5.11 do not hold for $n-1 < p < n$; see Example 6.8.

The following result states that a bounded p-QED domain is a p-SC domain. We show in section 7, see Theorem 7.7, that the converse holds for $p > n$. The reader is referred to Remarks 6.9.(ii) for the case $p < n$.

5.14. Theorem. *A bounded (C,p)-QED domain is a (C_1,p)-SC domain, where $C_1 = C_1(p,n,C,\operatorname{dia}(D))$.*

Proof. Let B be an open ball of radius $\operatorname{dia}(D)$ containing D. Then the Poincaré type inequality of Remark 5.4 holds [GT, 7.45] for B with a constant $C_0 = C_0(p,n,\operatorname{dia}(D))$, and B is a (C_2,p)-SC domain by Theorem 5.7, where $C_2 = C_2(p,n,\operatorname{dia}(D))$. Hence

$$\begin{aligned} S_p(K_0,K_1,\mathbb{R}^n) &\leq C_2\, S_p(K_0,K_1,B) \leq C_2(C_0+1)\operatorname{cap}_p(K_0,K_1,B) \\ &\leq C_2(C_0+1)\operatorname{cap}_p(K_0,K_1,\mathbb{R}^n) \\ &\leq C\,C_2(C_0+1)\operatorname{cap}_p(K_0,K_1,D) \end{aligned}$$

for any pair K_0, $K_1 \subset D$ of disjoint continua. Thus Lemma 5.2.(i) yields

$$S_p(K_0,K_1,\mathbb{R}^n) \leq C_1\, S_p(K_0,K_1,D)$$

for any pair K_0, $K_1 \subset D$ of disjoint continua, where $C_1 = C(C_0+1)\,C_2$, and the proof is complete.

We close this section with the following analogues of Theorem 4.1 and Corollary 4.2.

5.15. Theorem. *Let $D \subset \mathbb{R}^n$ be a (C,p)-SC domain with $p > n-1$. Then for all $x_0 \in \overline{D}$ and $0 < r < b$*

$$m_n(D \cap B^n(x_0,r)) \geq \frac{C_0}{C} m_n(B^n(x_0,r)),$$

where $b = b(p,n,C,\operatorname{dia}(D))$ and $C_0 = C_0(p,n)$.

Proof. Let $x_0 \in \overline{D}$, and let $K_1 \subset \overline{B}^n(x_0,\frac{2}{10}r)$, $K_2 \subset D \setminus B^n(x_0,\frac{3}{10}r)$ be two continua with $\min_{i=1,2}\operatorname{dia}(K_i) \geq r/10 \geq d(K_1,K_2)/4$ as in the proof of Theorem 4.1.

Set

$$u(x) = \min\left\{1, \max\left\{0, \frac{3\,r - 10\,|x - x_0|}{r}\right\}\right\}$$

for $x \in D$. Then $u \in W(K_2, K_1, D)$, $|\nabla u| \le 10/r$ on $B^n(x_0, r) \cap D$, $|\nabla u| \equiv u \equiv 0$ on $D \setminus B^n(x_0, r)$ and $0 \le u \le 1$. Thus

$$\begin{aligned} S_p(K_2, K_1, D) &\le \int_D (|\nabla u|^p + |u|^p)\, dm \\ &\le \frac{10^p\, m_n\big(D \cap B^n(x_0, r)\big)}{r^p} + \Omega_n\, r^n. \end{aligned}$$

On the other hand, by Lemma 1.10 and Lemma 5.2.(i),

$$S_p(K_2, K_1, D) \ge \frac{C_0}{C}\left(\frac{r}{10}\right)^{n-p},$$

where $C_0 = C_0(p, n)$. Choose $b > 0$ small enough so that

$$\Omega_n\, r^n \le \frac{C_0}{2C}\left(\frac{r}{10}\right)^{n-p}$$

whenever $0 < r \le b$. Then for $0 < r \le b$

$$m_n\big(D \cap B^n(x_0, r)\big) \ge \frac{C_0}{2 \cdot 10^n\, \Omega_n\, C} m_n\big((B^n(x_0, r)\big),$$

and the proof is complete.

We have the following corollary to Theorem 5.15

5.16. Corollary. *Let $D \subset \mathbb{R}^n$ be a p-SC domain with $p > n - 1$. Then $m_n(\partial D) = 0$.*

6. L_p^1- and W_p^1-extension domains

An L_p^1-extension domain and a bounded W_p^1-extension domain are both p-QED domains, and a W_p^1-extension domain is a p-SC domain; see Theorems 2.2, 2.4, and 5.8. Therefore the properties of p-QED and p-SC domains established in sections 3, 4, and 5 yield necessary conditions for a domain to be an extension domain.

Theorems 3.1 and 5.8 and Corollaries 3.5, 3.6, and 5.12 imply

6.1. Theorem. *Let $p \ge n$. Then*

(i) *An L_p^1-extension domain is quasiconvex.*

(ii) *A W_p^1-extension domain D is locally quasiconvex. Moreover, if either D or $\mathbb{R}^n \setminus D$ is bounded, then D is quasiconvex.*

6.2. Theorem.

(i) *An L^1_n-extension domain is LLC.*
(ii) *If D is a W^1_n-extension domain and if either D or $\mathbb{R}^n \setminus D$ is bounded, then D is LLC.*

6.3. Theorem. *Let $D \subset \mathbb{R}^n$ be a domain which is quasiconformally equivalent to a uniform domain $D' \subset \mathbb{R}^n$. Then*

(i) *D is an L^1_n-extension domain if and only if it is uniform.*
(ii) *If either D or $\mathbb{R}^n \setminus D$ is bounded, then D is a W^1_n-extension domain if and only if it is uniform.*

We also have

6.4. Theorem. *Let $n-1 < p \le n$. Then*

(i) *An L^1_p-extension domain satisfies LLC(2).*
(ii) *If D is a W^1_p-extension domain and if either D or $\mathbb{R}^n \setminus D$ is bounded, then D satisfies LLC(2).*
(iii) *A bounded, simply connected, planar L^1_p- or W^1_p-extension domain is a John domain.*

Proof. By Theorem 3.4 and Remarks 3.7.(ii), 5.13.(ii) it suffices to show that a W^1_p-extension domain D, $n-1 < p \le n$, satisfies LLC(2) whenever $\mathbb{R}^n \setminus D$ is bounded.

Let D be as above, and let B be an open ball in $\mathbb{R}^n$ containing $\mathbb{R}^n \setminus D$. It follows that $D \cap B$ is a bounded W^1_p-extension domain, and hence satisfies LLC(2). This implies that D satisfies LLC(2) as desired.

We have the following corollary to Theorems 4.1 and 5.15 and Corollaries 4.2 and 5.16.

6.5. Theorem. *Let $D \subset \mathbb{R}^n$ be an L^1_p-extension or a W^1_p-extension domain with $p > n-1$. Then for any $x_0 \in \overline{D}$ and $0 < r < b$*

$$m_n\big(D \cap B^n(x_0,r)\big) \ge C\, m_n\big(B^n(x_0,r)\big).$$

Here C depends only on p, n and the norm of the extension operator, $b = \operatorname{dia}(D)$ for L^1_p-extension domains, and $b = b(p,n,C,\operatorname{dia}(D))$ for W^1_p-extension domains. Moreover, $m_n(\partial D) = 0$.

6.6. Remarks.

(i) Theorems 6.1 and 6.5 have also been established by S. K. Vodop'yanov [Vo] for W^1_p-extension domains, $p > n$. V. M. Gol'dstein [Go2] has announced results similar to Theorem 6.1.

(ii) The following analogue of Remarks 3.7.(iii) seems to be more or less known; see [Go1], [Go2], [GR], [GV], [Vo], [J], and [VGL]. Since some of the conclusions seem to be new and since we have not been able to locate proofs for all known conclusions, we state this analogue for the convenience of the reader.

Let $D \subsetneq \mathbb{R}^2$ be a simply connected domain, and set $D^* = \mathbb{R}^2 \setminus \overline{D}$. Then the following conditions are equivalent:

(a) D is an L_2^1-extension domain.
(b) D is an L_p^1-extension and an L_q^1-extension domain, $1 < p \leq 2 \leq q$.
(c) D is an L_p^1-extension domain for all $1 < p < \infty$.
(d) D is uniform.
(e) D is a quasidisk.
(f) Both D and D^* are L_p^1-extension domains for some $p \geq 2$.
(g) D is locally connected on the boundary and both D and D^* are L_p^1-extension domains for some $1 < p \leq 2$. Moreover, if either D or $\mathbb{R}^2 \setminus \overline{D}$ is bounded then these conditions are equivalent with L_p^1 replaced by W_p^1.

The equivalence of these conditions follows by reasoning as in Remarks 3.7.(iii) using Theorems 3.4, 6.4.(ii), Corollaries 3.3, 5.9, [J, Theorem 1], and the fact that a simply connected, planar, uniform domain is an L_p^1-extension domain for all $1 < p < \infty$ by [Go1, Theorem 1] and the proof of [J, Theorem 2].

Next we show that there are W_p^1-extension domains in $\mathbb{R}^n$ which are neither L_p^1-extension nor p-QED domains for any $1 < p < \infty$. Note that by Theorem 2.4 such a domain has to be unbounded.

6.7. Example. Let $D = (-1,1)^{n-1} \times (-\infty, \infty)$. Then D is a (b,δ)-domain and thus a W_p^1-extension domain for all $1 < p < \infty$. We claim that D is not a p-QED domain for any $1 < p < \infty$, and therefore fails to be an L_p^1-extension domain for any $1 < p < \infty$.

By Theorem 4.1 it suffices to show that D is not a p-QED domain for any $1 < p \leq n-1$.

Let $1 < p \leq n-1$, and define $K_0^i = [-1/2, 1/2]^{n-1} \times [-2^i, -i]$ and $K_1^i = [-1/2, 1/2]^{n-1} \times [i, 2^i]$, $i = 1, 2, \ldots$. Set $u_i(x) = \min\{1, \max\{0, x_n/i\}\}$ for $x \in D$. Then $u_i \in L(K_0^i, K_1^i, D)$, and hence

$$\operatorname{cap}_p(K_0^i, K_1^i, D) \leq \int_D |\nabla u_i|^p \, dm \leq i^{1-p}.$$

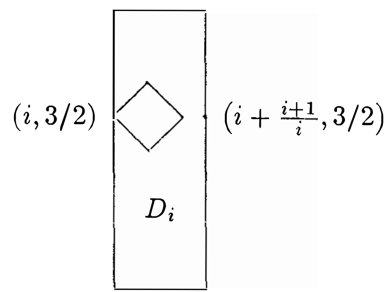
Let $2 \leq n_0 \leq n-1$ be an integer such that $n_0 - 1 < p \leq n_0$, and let T be the n_0-dimensional plane parallel to the $x_{1+(n-n_0)}, \ldots, x_n$-axes passing through a point x with $-1/2 \leq x_1, \ldots, x_{n-n_0} \leq 1/2$. Now $v_i|_T$ is in $L(K_{0_T}^i, K_{1_T}^i, T)$, where $K_{0_T}^i = K_0^i \cap T$, $K_{1_T}^i = K_1^i \cap T$, whenever $v_i \in L(K_0^i, K_1^i, T)$ satisfies $\int_T |\nabla v_i|^p \, dm_{n_0} < \infty$, and hence Lemma 1.7 and Fubini's theorem imply that

$$\operatorname{cap}_p(K_0^i, K_1^i, \mathbb{R}^n) \geq C(p,n,i) \to \infty \qquad \text{as } i \to \infty;$$

a contradiction.

We conclude that D is not a p-QED domain for any $1 < p < \infty$.

6.8. Example. Let $D_i = \left(i, i + \frac{i+1}{i}\right) \times (-1,3) \setminus Q_i$, $i = 2,3,\dots$, where each Q_i is a closed square with $F_i = \{(x, 3/2) \colon i \leq x \leq i+1\}$ as one of its diagonals, and let $D = \cup_2^\infty D_{2i} \cup \{(x,y) \colon y < 0\}$. Note that for any positive constant b there is a D_i whose LLC(2) constant exceeds b. Nevertheless D and all D_i are W_p^1-extension domains for any $1 < p < 2$ with the norms of the extension operators not exceeding $C = C(p)$.



Indeed, a look at Example 2.5 shows that each D_i is a W_p^1-extension domain, $1 < p < 2$, with the norm of the extension operator not exceeding $C = C(p)$. Since $\{(x,y) \colon y < 0\} \cup \left(\cup_2^\infty \left(2i, 2i + \frac{2i+1}{2i}\right) \times (-1,3)\right)$ is a (b,δ)-domain, we conclude that also D is a W_p^1-extension domain for all $1 < p < 2$.

6.9. Remarks.

(i) The various constants in Theorems 6.1, 6.2, and 6.4.(i), (ii) depend for L_p^1-extension domains only on p, n, and the norm of the extension operator, while for W_p^1-extension domains they depend also on $d = \min\{\operatorname{dia}(D), \operatorname{dia}(\mathbb{R}^n \setminus D)\}$, and in Theorem 6.4.(ii) the LLC(2) constant is not bounded in terms of this data for $n-1 < p < n$; see Example 6.8.

(ii) Let $D \subset \mathbb{R}^n$ be a bounded W_p^1-extension domain with the norm of the extension operator not exceeding a constant C. Then D is a (C_1, p)-QED domain for some constant C_1. It follows from Theorem 3.4 and Example 6.8 that C_1 is not bounded in terms of p, n, C, and $\operatorname{dia}(D)$ for $p < n$. We show in section 7, see Corollary 7.9, that for $p > n$ $C_1 = C_1(p,n,C,\operatorname{dia}(D))$. The case $p = n$ is an open problem.

7. L_p^1- and W_p^1-imbedding domains

For $p > n$ the variational p-capacity $\operatorname{cap}_p(K_0, K_1, \mathbb{R}^n)$ and the Sobolev p-capacity $S_p(K_0, K_1, \mathbb{R}^n)$ are positive for singletons $K_0 = \{x\}$ and $K_1 = \{y\}$; see Lemma 1.10.(ii) and Lemma 5.2.(i). This fact inspires the following definition.

7.1. Definition. A domain $D \subset \mathbb{R}^n$ is a *weak* (C,p)-QED domain, $p > n$, if

$$\mathrm{cap}_p(x, y, \mathbb{R}^n) \leq C\ \mathrm{cap}_p(x, y, D)$$

for any pair x, $y \in D$ of distinct points. Analogously, D is a weak (C,p)-SC domain if

$$S_p(x, y, \mathbb{R}^n) \leq C\, S_p(x, y, D)$$

for any pair x, $y \in D$ of distinct points.

Obviously, a p-QED or a p-SC domain is a weak p-QED or a weak p-SC domain, respectively, but we do not know whether or not the converse holds.

In what follows we identify each $u \in W^1_p(D)$ or $u \in L^1_p(D)$, $p > n$, with its continuous refinement.

We shall show that for bounded domains weak p-QED and p-SC conditions and L^1_p- and W^1_p-imbeddings are equivalent. We begin with two results that do not require boundedness of the domain in question.

7.2. Theorem. *A domain is a weak* p*-QED domain if and only if it is an* L^1_p*-imbedding domain.*

Proof. Assume first that D is an L^1_p-imbedding domain with constant C. Let x, $y \in D$ be two distinct points, and let $u \in L(x, y, D)$. Then

$$1 = |u(x) - u(y)| \leq C\, \|\nabla u\|_{L^p(D)}\, |x - y|^{1-(n/p)},$$

and hence

$$C^p \int_D |\nabla u|^p\, dm \geq |x - y|^{n-p}.$$

Thus $C^p\, \mathrm{cap}_p(x, y, D) \geq |x - y|^{n-p}$, and consequently, by Lemma 1.10.(ii), D is a weak p-QED domain.

For the converse, assume that D is a weak (C,p)-QED domain. Let $u \in L^1_p(D)$, and let x, $y \in D$ be two distinct points. We may assume that $u(x) > u(y)$. Now the function v, defined by

$$v(z) = \frac{u(z) - u(y)}{u(x) - u(y)} \qquad \text{for } z \in D,$$

belongs to $L(x, y, D)$; hence by Lemma 1.10.(ii)

$$C \int_D |\nabla v|^p\, dm \geq C\, \mathrm{cap}_p(x, y, D) \geq \mathrm{cap}_p(x, y, \mathbb{R}^n) \geq C_0\, |x - y|^{n-p},$$

where $C_0 = C_0(p,n)$. Since $\int_D |\nabla v|^p \, dm \le |u(x)-u(y)|^{-p} \int_D |\nabla u|^p \, dm$, we obtain

$$|u(x) - u(y)| \le \left(\frac{C}{C_0}\right)^{1/p} \|\nabla u\|_{L^p(D)} \, |x-y|^{1-(n/p)}.$$

The proof is now complete.

7.3. Theorem.

(i) *A weak p-SC domain is a W_p^1-imbedding domain.*

(ii) *If D is a W_p^1-imbedding domain, then for any $\delta > 0$ there is a constant C such that*

$$S_p(x,y,\mathbf{R}^n) \le C \, S_p(x,y,D)$$

whenever x, $y \in D$ are two distinct points with $|x-y| \le \delta$.

Proof. The proof of (i) is similar to that of Theorem 7.2 except that here we define v by $v(z) = u(z)/(u(x)-u(y))$.

For (ii) let D be a W_p^1-imbedding domain. As in the proof of Theorem 7.2 we obtain

$$C^p \, S_p(x,y,D) \ge |x-y|^{n-p}$$

whenever x, $y \in D$ are two distinct points. Then by Lemma 1.10.(ii) and Theorem 5.5

$$S_p(x,y,\mathbf{R}^n) \le C_0 \, |x-y|^{n-p} \, (2 + |x-y|^p),$$

and thus

$$S_p(x,y,\mathbf{R}^n) \le C_0 \, C^p (2+\delta^p) \, S_p(x,y,D),$$

where $C_0 = C_0(p,n)$. This completes the proof.

7.4. Theorem. *A bounded domain is an L_p^1-imbedding domain if and only if it is a W_p^1-imbedding domain.*

Proof. It suffices to show that a bounded W_p^1-imbedding domain is an L_p^1-imbedding domain.

Suppose that D is a bounded W_p^1-imbedding domain. Let $u \in L_p^1(D)$, and let x, $y \in D$ be two points with $u(x) - u(y) > 0$. Define

$$v(z) = \min\left\{1, \max\left\{0, \frac{u(z)-u(y)}{u(x)-u(y)}\right\}\right\}$$

for $z \in D$. Then $v \in W_p^1(D)$ and

$$\begin{aligned} 1 = |v(x) - v(y)| &\le C \, \|v\|_{W_p^1(D)} \, |x-y|^{1-(n/p)} \\ &\le C \, |u(x)-u(y)|^{-1} \, \|\nabla u\|_{L^p(D)} \, |x-y|^{1-(n/p)} + C \, m_n(D)^{1/p} \, |x-y|^{1-(n/p)}. \end{aligned}$$

Thus, if $|x-y| \le \left(2\,C\,m_n(D)^{1/p}\right)^{p/(n-p)}$, then

$$|u(x)-u(y)| \le 2\,C\,\|\nabla u\|_{L^p(D)}\,|x-y|^{1-(n/p)}.$$

Let $\delta = \left(2\,C\,m_n(D)^{1/p}\right)^{p/(n-p)}$. Since D is bounded, there is an integer $k = k(n,\delta,\mathrm{dia}(D)) \ge 2$ such that for any two points $x,\ y \in D$ with $|x-y| > \delta$ there are points $x_1,\dots,x_\ell \in D$, $\ell \le k$, with $x = x_1$, $y = y_\ell$ and $|x_{i+1}-x_i| \le \delta$ for $i = 1,\dots,\ell-1$. Therefore, for $x,\ y \in D$ with $|x-y| > \delta$,

$$\begin{aligned}|u(x)-u(y)| &\le \sum_{i=1}^{\ell-1} 2\,C\,\|\nabla u\|_{L^p(D)}\,|x_{i+1}-x_i|^{1-(n/p)}\\ &\le 2\,C(k-1)\,\|\nabla u\|_{L^p(D)}\,\delta^{1-(n/p)}\\ &\le 2\,C(k-1)\,\|\nabla u\|_{L^p(D)}\,|x-y|^{1-(n/p)}.\end{aligned}$$

The claim follows.

We group the preceding theorems together.

7.5. Corollary. *Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $p > n$. Then the following four conditions are equivalent.*

(i) *D is an L^1_p-imbedding domain.*
(ii) *D is a W^1_p-imbedding domain.*
(iii) *D is a weak p-QED domain.*
(iv) *D is a weak p-SC domain.*

We need the following Poincaré type inequality

7.6. Lemma. *If D is a bounded weak (C,p)-SC domain, $p > n$, then*

$$\int_D |u-u_D|^p\,dm \le C_1 \int_D |\nabla u|^p\,dm$$

for any $u \in W^1_p(D)$, where $C_1 = C_1(p,n,C,\mathrm{dia}(D))$.

Proof. Fix $u \in W^1_p(D)$. Then the proofs of Theorems 7.3 and 7.4 yield for any $x,\ y \in D$

$$|u(x)-u(y)| \le C_2\,|x-y|^{1-(n/p)}\,\|\nabla u\|_{L^p(D)};$$

here $C_2 = C_2(p,n,C,\mathrm{dia}(D))$. Pick $x_0 \in D$ with $u(x_0) = u_D$; this is possible since u is continuous. Now

$$\begin{aligned}\int_D |u-u_D|^p\,dm &= \int_D |u(x)-u(x_0)|^p\,dm\\ &\le C_2^p\,m_n(D)\ \mathrm{dia}(D)^{p-n}\int_D |\nabla u|^p\,dm\\ &\le \Omega_n\,C_2^p\ \mathrm{dia}(D)^p\int_D |\nabla u|^p\,dm\end{aligned}$$

as desired.

By Lemmas 5.2.(i) and 7.6 and Remark 5.4 we obtain

7.7. Theorem. *A bounded (C,p)-SC domain, $p > n$, is a (C_1,p)-QED domain, where $C_1 = C_1(p,n,C,\mathrm{dia}(D))$.*

Theorems 5.14 and 7.7 yield

7.8. Corollary. *Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $p > n$. Then D is a (C_1,p)-QED domain if and only if it is a (C_2,p)-SC domain. Here the constants C_1 and C_2 depend only on p, n, $\mathrm{dia}(D)$, and on each other.*

By Theorem 2.4 a bounded W_p^1-extension domain is a p-QED domain. When $p > n$ we obtain an upper bound for the p-QED constant in terms of p, n, $\mathrm{dia}(D)$, and the norm of the extension operator.

7.9. Corollary. *A bounded W_p^1-extension domain, $p > n$, is a (C,p)-QED domain, where C depends only on p, n, $\mathrm{dia}(D)$, and the norm of the extension operator.*

Proof. The claim follows from Theorems 5.7 and 7.7.

7.10. Remarks.

(i) A look at the proofs of Theorems 3.1, 4.1, 5.8, and 5.15 shows that these results hold for weak p-QED domains and for domains satisfying the local weak p-SC condition of Theorem 7.3.(ii). Hence an L_p^1-imbedding domain is quasiconvex and a W_p^1-imbedding domain is locally quasiconvex. Moreover, both classes satisfy a uniform measure density condition as in Theorem 6.5.

(ii) By [LL], or by combining [GM1] with [BI, 1.7], each Lip_α-extension domain, $0 < \alpha < 1$, is an L_p^1-imbedding domain with $p = n/(1-\alpha)$; see [GM1], [L], and [LL] for the definition and basic properties of Lip_α-extension domains. Thus Lip_α-extension domains are examples of weak p-QED domains for $p = n/(1-\alpha)$.

(iii) Suppose that there exist constants C, k and m such that for any pair x, $y \in D$ of distinct points
(1) there are points $x = x_1, \ldots, x_\ell = y$ in D, $\ell \le k$, with $|x_{i+1} - x_i| \le m\,|x-y|$ for $i = 1, \ldots, \ell - 1$ and
(2) weak (C,p)-QED subdomains $D_1, \ldots, D_{\ell-1}$ of D with x_i, $x_{i+1} \in D_i$ for $i = 1, \ldots, \ell - 1$.
Then it follows easily that D is a weak p-QED domain.

8. W_p^1-approximation domains

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $W_p^1(\mathbb{R}^n)$, it follows that W_p^1-extension domains are W_p^1-approximation domains. Thus, in particular, a (b,δ)-domain is a W_p^1-

approximation domain. We introduce a class of domains with the W_p^1-approximation property which strictly contains the class of (b,δ)-domains. We also show that under a weak additional hypotheses, a W_p^1-approximation domain is locally connected on the boundary.

8.1. Definition. A compact set $K \subset \partial D$ is of p-capacity zero relative to D if for some closed ball $B \subset D$

$$\operatorname{cap}_p(B,K,D) = 0.$$

Further, a closed set $F \subset \partial D$ is of p-capacity zero relative to D if each compact subset of F is of p-capacity zero relative to D.

The following lemma was proved in [HM, 2.6] in the case $p = n$ via a method different from ours.

8.2. Lemma. *If D is bounded and if $K \subset \partial D$ is a compact set of p-capacity zero relative to D, then*

$$\operatorname{cap}_p(F,K,D) = 0$$

for each compact set $F \subset \overline{D} \setminus K$.

Proof. We first show that there is a sequence $\{u_i\}_1^\infty$ of functions in $L(B,K,D)$ with $\|u_i\|_{W_p^1(D)} \to 0$ as $i \to \infty$, where $B \subset D$ is a ball as in Definition 8.1.

Since $\operatorname{cap}_p(B,K,D) = 0$ and $m_n(D) < \infty$, there is a sequence $\{v_j\}_1^\infty$ of functions in $L(B,K,D)$ with $\|v_j\|_{W_p^1(D)} \le m_n(D)^{1/p} + 1$ and $\|\nabla v_j\|_{L^p(D)} \to 0$ as $i \to \infty$. Because $W_p^1(D)$ is weakly compact, there is a subsequence of $\{v_j\}_1^\infty$, which we still denote by $\{v_j\}_1^\infty$, that converges weakly to some $u \in W_p^1(D)$. It follows that $\nabla u = 0$. Since $v_j \to u$ weakly in $W_p^1(D)$, there is a sequence $\{u_i\}_1^\infty$ of convex combinations of v_j's such that $u_i \to u$ in $W_p^1(D)$; see [Ru, 3.13]. As $\nabla u = 0$, u is a constant, and since $u_i \equiv 0$ on B for each i, we conclude that $u = 0$. Now $u_i \in L(B,K,D)$ and $\|u_i\|_{W_p^1(D)} \to 0$ as $i \to \infty$.

Now let $F \subset \overline{D} \setminus K$ be a compact set. Take $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on U_K and $\varphi \equiv 0$ on U_F, where U_K is a neighborhood of K and U_F is a neighborhood of F, respectively. Define

$$v_i(x) = \begin{cases} \varphi(x)\,u_i(x), & x \in D \cup K \\ 0, & x \in F \cap \partial D. \end{cases}$$

Then $v_i \in L(F,K,D)$, and hence

$$\begin{aligned} \operatorname{cap}_p(F,K,D) &\le \int_D |\nabla v_i|^p \, dm \\ &\le \max_{x\in\overline{D}}(|\varphi(x)|^p + |\nabla\varphi(x)|^p)\,2^p \int_D (|\nabla u_i|^p + |u_i|^p)\,dm. \end{aligned}$$

The claim follows by letting $i \to \infty$.

From Hölder's inequality we obtain

8.3. Lemma. *Let $D \subset \mathbb{R}^n$ satisfy $m_n(D) < \infty$. Suppose that K_0, $K_1 \subset \overline{D}$ are disjoint, compact sets. Then for any $1 < q \le p < \infty$*

$$\operatorname{cap}_q(K_0, K_1, D) \le m_n(D)^{(p-q)/p} [\operatorname{cap}_p(K_0, K_1, D)]^{q/p}.$$

8.4. Lemma. *If $F \subset \partial D$ is of p-capacity zero relative to D, then for any $1 < q \le p$ and any two compact sets $K \subset F$ and $E \subset \overline{D} \setminus K$*

$$\operatorname{cap}_q(E, K, D) = 0.$$

In particular, F is of q-capacity zero relative to D.

Proof. Fix two compact sets $K \subset F$ and $E \subset \overline{D} \setminus K$. It suffices to show that $\operatorname{cap}_q(E, K, D) = 0$. Take a closed ball $B \subset D$ with $\operatorname{cap}_p(B, K, D) = 0$, and pick a bounded subdomain $D' \subset D$ containing both B and $U \cap D$ for some neighborhood U of K. Set $d = d(K, E \cup (D \setminus D'))$, and define $E' = \{x \in \overline{D}' : d(x, K) \ge d\}$. Observe that $\operatorname{cap}_p(B, K, D') < \operatorname{cap}_p(B, K, D) = 0$ and $\operatorname{cap}_q(E, K, D) < \operatorname{cap}_q(E', K, D) = \operatorname{cap}_q(E', K, D')$. Now $\operatorname{cap}_p(E', K, D') = 0$, by Lemma 8.2, and hence, by Lemma 8.3,

$$\begin{aligned}\operatorname{cap}_q(E, K, D) &\le \operatorname{cap}_q(E', K, D') \\ &\le m_n(D')^{(p-q)/p} [\operatorname{cap}_p(E', K, D')]^{q/p} = 0\end{aligned}$$

as desired.

8.5. Definition. We say that a domain D is p-weakly (b, δ) if there is a closed set $F \subset \partial D$ of p-capacity zero relative to D with the following property. For any choice of neighborhoods U_i of $F_i = F \cap \overline{B}^n(i) \setminus B^n(i-1)$, $i = 1, 2, \dots$, there are constants b', δ' and neighborhoods V_i of F_i such that $V_i \subset U_i$, $i = 1, 2, \dots$, and $D \cup (\cup_1^\infty V_i)$ is a (b', δ')-domain.

Clearly, a (b, δ)-domain is p-weakly (b, δ) for all $1 < p < \infty$. Note also that by Lemma 8.4 a domain which is p-weakly (b, δ) is q-weakly (b, δ) for all $1 < q < p$.

8.6. Theorem. *If D is p-weakly (b, δ), then D is a W_p^1-approximation domain.*

Proof. Let $u \in W_p^1(D)$ and let $\varepsilon > 0$. It suffices to show that there is a $\psi \in W_p^1(\mathbb{R}^n)$ with $\|u - \psi\|_{W_p^1(D)} < \varepsilon$.

We may assume that $0 \le u \le M$ almost everywhere in D for some $M < \infty$. Let $F \subset \partial D$ be as in Definition 8.5. For each positive integer i, let V_i be a neighborhood of $F_i = F \cap (\overline{B}^n(i) \setminus B^n(i-1))$ such that

$$\|u\|_{W_p^1(V_i \cap D)} < \frac{\varepsilon}{2^{i+2}}$$

and $V_i \cap V_j = \emptyset$ for $j > i+1$. Since F is of p-capacity zero relative to D, Lemma 8.4 implies that there exist functions $\varphi_i \in L^1_p(D)$ such that $0 \le \varphi_i \le 1$, $\varphi_i \equiv 0$ on $V_i' \cap D$ for some neighborhood $V_i' \subset V_i$ of F_i, $\varphi_i \equiv 1$ on $\overline{D} \setminus V_i$, and

$$\int_D |\nabla \varphi_i|^p \, dm < \Big(\frac{\varepsilon}{2^{i+3} M}\Big)^p .$$

Set $\varphi_0 \equiv 0$ on D. Define

$$v(x) = \begin{cases} u(x) & \text{for } x \in D \setminus \cup_{i=1}^{\infty} V_i \text{ and} \\ (u\varphi_{i-1}\,\varphi_i\,\varphi_{i+1})(x) & \text{for } x \in V_i \cap D. \end{cases}$$

Then $v \in W^1_p(D)$ and

$$\begin{aligned} &\|v-u\|_{W^1_p(D)} \le \sum_{i=1}^{\infty} \|v-u\|_{W^1_p(V_i\cap D)} \\ &\le \sum_{i=1}^{\infty} \|v\|_{W^1_p(V_i\cap D)} + \sum_{i=1}^{\infty} \|u\|_{W^1_p(V_i\cap D)} \\ &\le \sum_{i=1}^{\infty} \big(\|u\|_{L^p(V_i\cap D)} + \|\nabla u\|_{L^p(V_i\cap D)} + 3M\,\|\nabla\varphi_i\|_{L^p(V_i\cap D)}\big) + \frac{\varepsilon}{4} \\ &< \varepsilon. \end{aligned}$$

Since D is p-weakly (b,δ), there are neighborhoods U_i of F_i, $i = 1,2,\dots$, such that $\overline{U}_i \subset V_i'$ and $D \cup (\cup_1^{\infty} U_i)$ is a (b',δ')-domain. By extending v as zero to $\cup_1^{\infty} U_i \setminus D$, we have $v \in W^1_p\big(D \cup (\cup_1^{\infty} U_i)\big)$. Thus there is an extension $\psi \in W^1_p(\mathbb{R}^n)$ of v, and the claim follows.

8.7. Corollary. *If D is p-weakly (b,δ), then D is a W^1_q-approximation domain for all $1 < q \le p$.*

8.8. Corollary. *Let D be bounded, and let $K \subset \partial D$ be of p-capacity zero relative to D. If $D \cup V$ is uniform for arbitrarily small neighborhoods V of K, then D is a W^1_q-approximation domain for all $1 < q \le p$.*

8.9. Remark. J. L. Lewis has recently shown [Lw, Theorem 1] that a planar Jordan domain is a W^1_p-approximation domain for all $1 < p < \infty$.

For $\alpha \ge 1$ we denote the standard n-dimensional spire of order α (defined by $\sum_{i=2}^n x_i^2 < x_1^{2\alpha}$, $x_1 > 0$, and $\sum_{i=1}^n x_i^2 < 1$) by Q_α. Define Q_α^- by replacing the requirement $x_1 > 0$ with $x_1 < 0$.

Finally, let $D_\alpha = Q_\alpha \cup Q_\alpha^- \cup \big(B^n(1) \setminus \overline{B}^n(1/2)\big)$ and $D = Q \cup Q^- \cup \big(B^n(1) \setminus \overline{B}^n(1/2)\big)$, where Q is an exponential spire.

8.10. Example. Let D and D_α be as above. Then D is a W_p^1-approximation domain for all $1 < p < \infty$ and D_α is a W_p^1-approximation domain for all $1 < p \leq (n-1)\alpha + 1$.

Indeed, D is p-weakly (b, δ) for all $1 < p < \infty$ and D_α is p-weakly (b, δ) for all $1 < p \leq (n-1)\alpha + 1$ as can easily be seen by taking $K = \{0\}$. Hence Corollary 8.7 implies the desired approximation property.

8.11. Definition. A domain D is called a John domain of order α, $1 \leq \alpha$, if there is a constant L and a point $x_0 \in D$, called the John center of D, such that for any $x \in D$ there exists an L-bilipschitz mapping φ_x of the standard spire Q_α of order α into D with $x \in \varphi_x(Q_\alpha)$ and $x_0 = \varphi_x((0, \dots, 0, 1/2))$. Here $f\colon G \to D$ is called L-bilipschitz if

$$|x - y|/L \leq |f(x) - f(y)| \leq L\,|x - y|$$

for all x, $y \in G$.

8.12. Remark. It follows from [M2, 2.2] that D is a John domain if and only if it is a John domain of order 1.

We show that if D is a John domain of order α and a W_p^1-approximation domain with $p > (n-1)\alpha + 1$, then D is locally connected on the boundary. Note that as Example 8.10 shows D may fail to be locally connected on the boundary when $1 < p \leq (n-1)\alpha + 1$.

8.13. Theorem. *Let $D \subset \mathbb{R}^n$ be a John domain of order α. If $C(\overline{D}) \cap W_p^1(D)$ is dense in $W_p^1(D)$ for some $p > (n-1)\alpha + 1$, then D is locally connected on the boundary.*

Proof. Suppose that D fails to be locally connected at a boundary point z of D. Note that Definition 8.11 implies that D is finitely locally connected at z.

Now a simple limiting argument shows that there is a neighborhood U of z, distinct components V_0 and V_1 of $U \cap D$ and L-bilipschitz mappings φ_0 and φ_1 such that x_0, $z \in \overline{\varphi_i(Q_\alpha)}$ and $\varphi_i(Q_\alpha) \cap U \subset V_i$, $i = 0, 1$. Let $d = (1/2)\,d(z, \partial U)$, and define

$$u(x) = \begin{cases} \max\left\{0, \min\left\{1, 2\,\frac{d - |x - z|}{d}\right\}\right\} & \text{for } x \in V_1, \\ 0 & \text{elsewhere in } D; \end{cases}$$

then $u \in W_p^1(D)$.

Let $\{\psi_j\}_1^\infty$ be a sequence of functions in $W_p^1(D) \cap C(\overline{D})$ converging to u in $W_p^1(D)$. We may assume that for each j there are points x_j, $y_j \in B^n(z, 1/j) \cap \varphi_i(Q_\alpha)$, $i = 0$ or $i = 1$, with $|\psi_j(y_j) - \psi_j(x_j)| \geq 1/3$. It follows from Hölder continuity estimates [A, 5.4, 5.37] that

$$\begin{aligned}\frac{1}{3} &\le C\Big(\frac{2}{j}\Big)^{1-(((n-1)\,\alpha+1)/p)}\,\|\psi_j\|_{W^1_p(\varphi_i(Q_\alpha))}\\ &\le C\Big(\frac{2}{j}\Big)^{1-(((n-1)\,\alpha+1)/p)}\,\|\psi_j\|_{W^1_p(D)},\end{aligned}$$

where C is independent of ψ_j. Hence $\|\psi_j\|_{W^1_p(D)} \to \infty$ as $j \to \infty$, which contradicts our choice of the sequence $\{\psi_j\}_1^\infty$. The claim follows.

8.14. Corollary. *Let $D \subset \mathbb{R}^n$ be a John domain of order α. If D is a W^1_p-approximation domain for some $p > (n-1)\,\alpha + 1$, then D is locally connected on the boundary.*

9. Applications connected with quasiconformal mappings

Our first application considers the boundary behavior of quasiconformal mappings and the second deals with the uniform Hölder continuity of quasiconformal mappings.

Denote the one-point compactification of $\mathbb{R}^n$ by $\overline{\mathbb{R}}^n$. We say that a domain $D \subset \overline{\mathbb{R}}^n$ is locally connected on the boundary if $D \cap \mathbb{R}^n$ is locally connected both on the boundary and at the infinity. Further, we say that D is a p-QED domain if $D \cap \mathbb{R}^n$ is a p-QED domain.

It is known [MV, 6.17] that a quasiconformal mapping of a domain $D \subset \overline{\mathbb{R}}^n$ which is locally connected on the boundary onto an n-QED domain has a continuous extension to $\overline{D}$. The following theorem extends this result.

9.1. Theorem. *Let $D \subset \overline{\mathbb{R}}^n$ be locally connected on the boundary. If $D' \subset \overline{\mathbb{R}}^n$ is a p-QED domain for some $n-1 < p \le n$, $m_n(D') < \infty$, and if f is a quasiconformal mapping of D onto D', then f has a continuous extension to $\overline{D}$.*

Proof. Let z be a finite boundary point of D. Suppose that there are sequences $\{x_i\}_1^\infty$, $\{y_i\}_1^\infty$ of points in D converging to z with

$$v = \lim_{i\to\infty} f(x_i) \ne \lim_{i\to\infty} f(y_i) = w.$$

Since D is locally connected at z, we can find continua K_i, $i = 1, 2, \dots$, joining x_i to y_i in D such that $\operatorname{dia}(K_i) \to 0$ as $i \to \infty$. Then for i large enough

$$\begin{aligned}\operatorname{cap}_n(K_i, K_1, D) &\le \operatorname{cap}_n\big(\overline{B}^n(z, \operatorname{dia}(K_i) + |z - x_i|), B^n(z, d(z, K_1))\big)\\ &= \omega_{n-1}\Big(\log\frac{d(z, K_1)}{\operatorname{dia}(K_i) + |z - x_i|}\Big)^{1-n}.\end{aligned}$$

Since f is a quasiconformal mapping, this implies that

$$\operatorname{cap}_n\big(f(K_i), f(K_1), D'\big) \to 0 \qquad \text{as } i \to \infty.$$

Now, since $v \neq w$, for some $\delta > 0$ and some $i_0 \geq 1$, $\operatorname{dia}\big(f(K_i)\big) \geq \delta$ whenever $i \geq i_0$. Further, either $v \neq \infty$ or $w \neq \infty$. Say $v \neq \infty$. We may assume that $|v - f(x_i)| \leq |v - f(x_1)|$ for each $i \geq i_0$. For $i \geq i_0$ we have

$$\min\{\operatorname{dia}(f(K_i)), \operatorname{dia}(f(K_1))\} \geq \min\{\delta, \operatorname{dia}(f(K_1)\}$$

and

$$d\big(f(K_i), f(K_1)\big) \leq |f(x_i) - v| + |v - f(x_1)| \leq 2\,|v - f(x_1)|;$$

hence by Lemma 1.9.(i)

$$\operatorname{cap}_p\big(f(K_i), f(K_1), \mathbb{R}^n\big) \geq C > 0,$$

where $C = C\big(p, n, \delta, \operatorname{dia}(f(K_1)), |v - f(x_1)|\big)$. Since D is a p-QED domain and $\operatorname{cap}_n\big(f(K_i), f(K_1), D'\big) \to 0$ as $i \to \infty$, Lemma 8.3 yields a contradiction. Thus f has a limit at z.

If $z = \infty$ is a boundary point of D, we may assume that $K_i \subset \mathbb{R}^n \setminus \overline{B}^n(x_1, i)$, where K_i, $i = 1, 2, \dots$, is a continuum as above. Thus

$$\begin{aligned}\operatorname{cap}_n(K_i, K_1, D) &\leq \operatorname{cap}_n\big(\overline{B}^n(x_1, \operatorname{dia}(K_1)), B^n(x, i)\big)\\ &= \omega_{n-1}\log\left(\frac{i}{\operatorname{dia}(K_1)}\right)^{1-n}.\end{aligned}$$

Hence we may apply the reasoning above to show that f has a limit at z.

Therefore f has a limit at each boundary point of D, and the proof is complete.

9.2. Remark. The Riemann mapping theorem, [N1, 4.2], and Theorem 9.1 yield another proof for part of Remarks 3.7.(i) for domains $D \subset \mathbb{R}^2$ with $m_2(D) < \infty$.

9.3. Theorem. *Let D be a bounded domain, and let f be a K-quasiconformal mapping of D onto $B^n(1)$. If D is either n-QED or weakly p-QED for some $p > n$, then f is uniformly Hölder continuous in D with exponent $(2K)^{1/(1-n)}$.*

Proof. By Theorem 3.1 and Remarks 7.9.(i) D is quasiconvex. Hence the Hölder continuity follows from [NP, Theorem 7].

9.4. Remark. Let $f\colon D \to B^2(1)$ be a Riemann mapping function for the domain in Example 2.5. It follows from Theorem 9.1 that f cannot be uniformly Hölder continuous in D. Hence the assumption $p \geq n$ in Theorem 9.3 is necessary.

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