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RESEARCH



# Pair correlations of logarithms of complex lattice points

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## Abstract

We study the correlations of pairs of complex logarithms of  $\mathbb{Z}$ -lattice points in  $\mathbb{C}$  at various scalings, proving the existence of pair correlation functions. We prove that at the linear scaling, the pair correlations exhibit level repulsion, as it sometimes occurs in statistical physics. We prove total loss of mass phenomena at superlinear scalings, and Poissonian behaviour at sublinear scalings. The case of Euler weights has applications to the pair correlation of the lengths of common perpendicular geodesic arcs from the maximal Margulis cusp neighbourhood to itself in the Bianchi orbifold  $\mathrm{PSL}_2(\mathbb{Z}[i]) \backslash \mathbb{H}_{\mathbb{R}}^3$ .

**Keywords:** Pair correlation, Lattice point counting, Complex logarithm, Level repulsion, Euler function, Imaginary quadratic number field

**Mathematics Subject Classification:** 11K38, 11J83, 11N37, 53C22.

## 1 Introduction

When studying the asymptotic distribution of a sequence of finite subsets of  $\mathbb{R}$ , finer information is sometimes given by the statistics of the spacing (or gaps) between pairs or  $k$ -tuples of elements, seen at an appropriate scaling. These problems often arise in quantum chaos, including energy level spacings or clusterings, and in statistical physics, including molecular repulsion or interstitial distribution. See for instance [2, 3, 13–15, 18, 23, 25]. This paper may be seen as a complex version of our paper [22] where we study the pair correlation of logarithms of pairs of natural integers, though new phenomena occur, including the necessity to take limits of the underlying spaces, as we now explain.

The general setting for our study may be described as follows. Let  $E$  be an abelian locally compact group. Let  $\mathcal{A} = (A_N, \omega_N)_{N \in \mathbb{N}}$  be a sequence of finite subsets  $A_N$  of  $E$ , endowed with a *weight function*  $\omega_N : A_N \rightarrow ]0, +\infty[$  (or multiplicity function when its values are positive integers). When studying the asymptotic distribution of differences of elements of  $A_N$ , looking at them at various scalings is often desirable. As explained by Gromov (see for instance [10]), scaling a metric space sometimes requires to change the space, especially at the limit (unless this space has a nice family of homotheties, as the Euclidean space  $\mathbb{R}^n$  does). We thus introduce a sequence  $(E_N)_{N \in \mathbb{N}}$  of abelian locally compact groups converging for the pointed Hausdorff-Gromov convergence to an abelian locally compact group  $E_\infty$  (see for instance [11]). Let  $\mathrm{Haar}_{E_\infty}$  be a Haar measure on  $E_\infty$ .

Let  $\psi : N \mapsto \psi(N)$  be a *scaling function*, that is, for every  $N \in \mathbb{N} \setminus \{0\}$ , let  $\psi(N) : E \rightarrow E_N$  be any map, typically a dilating homeomorphism for appropriate distances, that we think of as “scaling” the space  $E$ . Let  $\psi' : \mathbb{N} \setminus \{0\} \rightarrow [1, +\infty[$  be an appropriately chosen function, called a *renormalising function*. The *pair correlation measure of  $\mathcal{A}$  at time  $N$  with scaling  $\psi(N)$*  is the measure on  $E_N$  with finite support

$$\mathcal{R}_N^{\mathcal{A}, \psi} = \sum_{x, y \in A_N, x \neq y} \omega_N(x) \omega_N(y) \Delta_{\psi(N)(y-x)}, \tag{1}$$

where  $\Delta_z$  denotes the unit Dirac mass at  $z$  in any measurable space. When the sequence of measures  $(\mathcal{R}_N^{\mathcal{A}, \psi})_{N \in \mathbb{N}}$ , renormalised by  $\psi'(N)$ , converges for the pointed Hausdorff-Gromov weak-star convergence (see Sect. 3 for background definitions) to a measure  $g_{\mathcal{A}, \psi} \text{ Haar}_{E_\infty}$  absolutely continuous with respect to  $\text{Haar}_{E_\infty}$ , the Radon-Nikodym derivative  $g_{\mathcal{A}, \psi}$  is called the *asymptotic pair correlation function of  $\mathcal{A}$  for the scaling  $\psi$  and renormalisation  $\psi'$* . When  $g_{\mathcal{A}, \psi}$  is a positive constant, we say that  $\mathcal{A}$  has a *Poissonian behaviour for the scaling  $\psi$  and renormalisation  $\psi'$* . When  $g_{\mathcal{A}, \psi}$  vanishes on a neighbourhood of 0 in  $E_\infty$  but is not the constant 0-function, we say that the pair  $(\mathcal{A}, \psi)$  *exhibits a strong level repulsion*. The standard level repulsion only requires  $g_{\mathcal{A}, \psi}$  to vanish at 0.

Recall that a  $\mathbb{Z}$ -lattice in  $\mathbb{C}$  is a discrete (free abelian) subgroup of  $(\mathbb{C}, +)$  generating  $\mathbb{C}$  as an  $\mathbb{R}$ -vector space. Let  $\Lambda$  be a  $\mathbb{Z}$ -grid in  $\mathbb{C}$  (or an *affine (Euclidean) lattice* in the terminology of [8, 15]), that is, a translate  $\Lambda = a + \bar{\Lambda}$  of a  $\mathbb{Z}$ -lattice  $\bar{\Lambda}$  in the Euclidean space  $\mathbb{C}$  for some  $a \in \mathbb{C}$  (well defined modulo  $\bar{\Lambda}$ ), see for instance [1]. We denote by  $\text{covol}_{\bar{\Lambda}} = \text{Vol}(\mathbb{C}/\bar{\Lambda})$  the area of a fundamental parallelogram for  $\bar{\Lambda}$ . We denote by

$$\text{Sys}_{\bar{\Lambda}} = \min \{ |z| : z \in \bar{\Lambda} \setminus \{0\} \} > 0$$

the *systole* of the  $\mathbb{Z}$ -lattice  $\bar{\Lambda}$ . Recall that the complex logarithm is an isomorphism of abelian topological groups  $\log : \mathbb{C}^\times \rightarrow E = \mathbb{C}/(2\pi i\mathbb{Z})$ . Given  $N \in \mathbb{N} \setminus \{0\}$  and a function  $\psi : \mathbb{N} \setminus \{0\} \rightarrow ]0, +\infty[$ , we again denote by  $\psi(N)$  the scaling map from  $E$  to  $E_N = \mathbb{C}/(2\pi i\psi(N)\mathbb{Z})$  defined by  $z \bmod 2\pi i\mathbb{Z} \mapsto \psi(N)z \bmod 2\pi i\psi(N)\mathbb{Z}$ . In Sects. 2 and 3, we study the pair correlations of the family of the complex logarithms of grid points

$$\mathcal{L}_\Lambda = (L_N^\Lambda = \{ \log z : z \in \Lambda, 0 < |z| \leq N \}, \omega_N = 1)_{N \in \mathbb{N}}$$

without multiplicities.

In order to simplify the statements in this introduction, we only consider power scalings  $\psi : N \mapsto N^\alpha$  for  $\alpha \geq 0$ , and we denote them by  $\text{id}^\alpha$ . We use the notation  $\text{Leb}_A$  for the Lebesgue measures on  $A = \mathbb{C}$  and  $A = \mathbb{C}/(2\pi i\mathbb{Z})$ .

**Theorem 1.1** *Let  $\alpha \geq 0$  and let  $\Lambda$  be a  $\mathbb{Z}$ -grid. As  $N \rightarrow +\infty$ , the normalised pair correlation measures  $\frac{1}{N^{4-2\alpha}} \mathcal{R}_N^{\mathcal{L}_\Lambda, \text{id}^\alpha}$  on the cylinder  $E_N = \mathbb{C}/(2\pi iN^\alpha\mathbb{Z})$  converge for the pointed Hausdorff-Gromov weak-star convergence to the measure  $g_{\mathcal{L}_\Lambda, \text{id}^\alpha} \text{ Leb}_{E_\infty}$  on  $E_\infty = \mathbb{C}/(2\pi i\mathbb{Z})$  if  $\alpha = 0$  and  $E_\infty = \mathbb{C}$  otherwise, with pair correlation function given by*

$$g_{\mathcal{L}_\Lambda, \text{id}^\alpha} : z \mapsto \begin{cases} \frac{\pi}{2\text{covol}_{\bar{\Lambda}}^2} e^{-2|\text{Re } z|} & \text{if } \alpha = 0, \\ \frac{\pi}{2\text{covol}_{\bar{\Lambda}}^2} & \text{if } 0 < \alpha < 1, \\ \frac{1}{\text{covol}_{\bar{\Lambda}}^2 |z|^4} \sum_{p \in \bar{\Lambda}: |p| \leq |z|} |p|^2 & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha > 1. \end{cases}$$

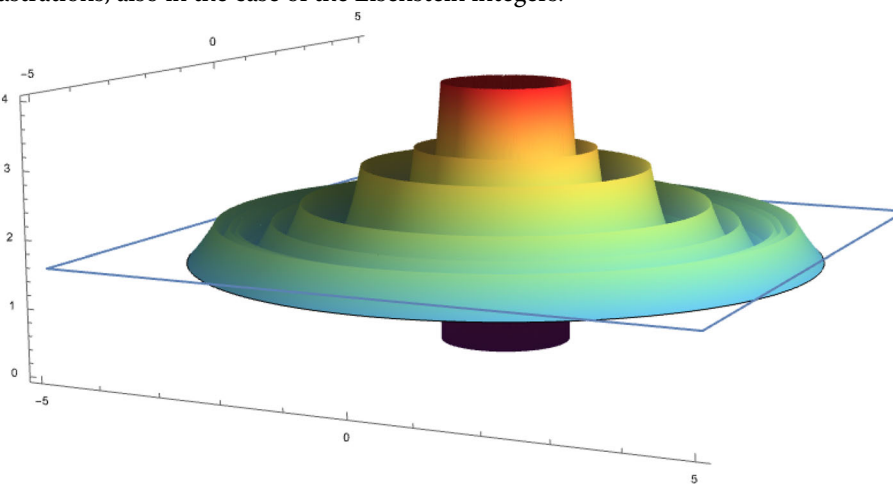
The convergence is uniform on  $\Lambda$  varying in any given compact subset of the set of  $\mathbb{Z}$ -grids of  $\mathbb{C}$  endowed with the Chabauty topology.

The renormalisation by  $\frac{1}{N^{4-2\alpha}}$  in Theorem 1.1 is naturally chosen in order for the pair correlation function to be finite. We refer to Theorems 2.2 and 3.1 for more complete versions of Theorem 1.1, with more general scaling functions, as well as for error terms. These error terms, as well as the ones in Theorems 5.1 and 6.1, constitute the main technical parts of this paper.

A standard scaling function in dimension  $n$  is the inverse of the  $n$ -th root of the *average volume gap*, which is the quotient of the volume of the ball of smallest radius containing  $F_N$  by the number of elements in  $F_N$ . See for instance [3, 13, 14, 18, 25], though these references are in dimension  $n = 1$ . For the family  $\mathcal{L}_\Lambda$ , this average volume gap is equivalent to  $\frac{(\ln N)^2}{N^2}$ , up to a positive multiplicative constant. As we shall see in Theorem 3.1, the corresponding scaling function  $\psi : N \mapsto \frac{N}{\ln N}$  gives, as for  $\psi : N \mapsto N^\alpha$  for  $0 < \alpha < 1$  in the above theorem, a Poissonian behaviour (see also [8, 28] for a similar behaviour).

There is a phase transition from a Poissonian behaviour when  $0 < \alpha < 1$  to a total loss of mass when  $\alpha > 1$ . In fact, the support of the measure itself converges to infinity for  $\alpha > 1$ . The transition occurs at the linear scaling (when  $\alpha = 1$  in Theorem 1.1), where an exotic pair correlation function  $g_{\mathcal{L}_\Lambda, \text{id}^1}$  appears, which has a discontinuity along every circle (centered at 0) through a grid point. Since  $g_{\mathcal{L}_\Lambda, \text{id}^1}(z)$  vanishes when  $z \in \overset{\circ}{B}(0, \text{Sys}_\Lambda^-)$ , the pair  $(\mathcal{L}_\Lambda, \text{id}^1)$  exhibits a strong level repulsion. Hence  $g_{\mathcal{L}_\Lambda, \text{id}^1}$  has near  $z = 0$  a behaviour similar to the case  $\alpha > 1$ . Note that  $g_{\mathcal{L}_\Lambda, \text{id}^1}(z)$  converges to  $\frac{\pi}{2 \text{covol}_\Lambda^2}$  when  $z$  goes to  $\infty$ , corresponding to the Poissonian behaviour of  $0 < \alpha < 1$ , see Lemma 2.1 with  $k = 2$ .

The figure below gives the graph of the pair correlation function  $g_{\mathcal{L}_\Lambda, \psi}$  of  $\mathcal{L}_\Lambda$  for the  $\mathbb{Z}$ -grid (which is a  $\mathbb{Z}$ -lattice)  $\Lambda = \bar{\Lambda} = \mathbb{Z}[i]$  of the Gaussian integers at the linear scaling  $\psi = \text{id}^1 : N \mapsto N$  in the ball of center 0 and radius 5. The blue lines on the bounding box represent the limit  $\frac{\pi}{2 \text{covol}_\Lambda^2} = \frac{\pi}{2}$  at  $+\infty$  of  $g_{\mathcal{L}_\Lambda, \psi}$ . We refer to the end of Sect. 3 for further illustrations, also in the case of the Eisenstein integers.



We now give some existence results of pair correlation functions of logarithms of lattice points with weights, restricting to integral lattices with an arithmetic weight motivated by geometric applications. Let  $K$  be an imaginary quadratic number field  $K$ , with discriminant  $D_K$ , whose ring of integers  $\mathcal{O}_K$  is principal. We fix a nonzero ideal  $\Lambda$  in  $\mathcal{O}_K$ , and we denote

by  $\varphi_K : \mathcal{O}_K \setminus \{0\} \rightarrow \mathbb{N}$  the Euler function  $a \mapsto \text{Card}((\mathcal{O}_K/a\mathcal{O}_K)^\times)$  of  $K$ . In the products below,  $\mathfrak{p}$  runs over the prime ideals of  $\mathcal{O}_K$ . The following result describes the asymptotic behaviour of the pair correlation measures associated with the family

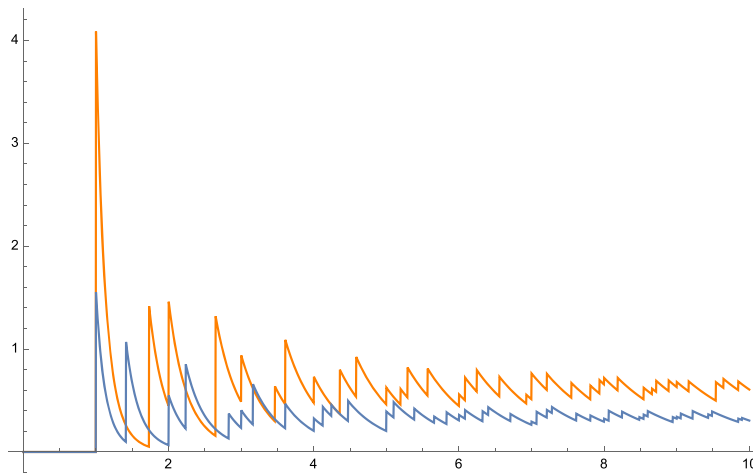
$$\mathcal{L}_\Lambda^{\varphi_K} = (L_N^\Lambda = \{\log z : z \in \Lambda, 0 < |z| \leq N\}, \omega_N = \varphi_K \circ \exp)_{N \in \mathbb{N}}. \tag{2}$$

**Theorem 1.2** (1) As  $N \rightarrow +\infty$ , the pair correlation measures  $\mathcal{R}_N^{\mathcal{L}_\Lambda^{\varphi_K}, 1}$  on the constant cylinder  $E = \mathbb{C}/(2\pi i\mathbb{Z})$ , renormalised to be probability measures, weak-star converge to the probability measure  $g_{\mathcal{L}_\Lambda^{\varphi_K}, 1} \text{Leb}_E$ , with pair correlation function independent of  $\Lambda$  given by  $g_{\mathcal{L}_\Lambda^{\varphi_K}, 1} : z' \mapsto \frac{1}{\pi} e^{-4|\text{Re } z'|}$ .

(2) As  $N \rightarrow +\infty$ , the normalised pair correlation measures  $\frac{1}{N^6} \mathcal{R}_N^{\mathcal{L}_{\mathcal{O}_K}^{\varphi_K}, \text{id}^1}$  on the varying cylinders  $E_N = \mathbb{C}/(2\pi i N \mathbb{Z})$  converge for the pointed Hausdorff-Gromov weak-star convergence to the measure  $g_{\mathcal{L}_{\mathcal{O}_K}^{\varphi_K}, \text{id}^1} \text{Leb}_{\mathbb{C}}$ , with pair correlation function

$$g_{\mathcal{L}_{\mathcal{O}_K}^{\varphi_K}, \text{id}^1} : z \mapsto \frac{2}{|z|^8 \sqrt{|D_K|}} \prod_{\mathfrak{p}} \left(1 - \frac{2}{N(\mathfrak{p})^2}\right) \sum_{\substack{k \in \mathcal{O}_K \\ |k| \leq |z|}} |k|^6 \prod_{\mathfrak{p} | k \mathcal{O}_K} \left(1 + \frac{1}{N(\mathfrak{p})(N(\mathfrak{p})^2 - 2)}\right). \tag{3}$$

We refer to Theorems 5.1 and 6.1 for more complete versions of Theorem 1.2, including possible congruence restrictions, and for error terms. The proof of Theorem 1.2 (2) uses Theorems 1.1 and 4.1 of [24] that describe the asymptotic behaviour in angular sectors in  $\mathbb{C}$  for the Euler function of  $K$ . For the reader’s convenience, we briefly review these results in Sect. 4. In order to simplify the treatment, we only consider the constant and linear scaling in Theorem 1.2.



The pair correlation functions at the linear scaling are radially symmetric by Theorem 1.2 (2). The figure above compares the radial profiles of the pair correlation functions  $g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}$  for  $K = \mathbb{Q}(i)$  and  $\Lambda = \mathcal{O}_K = \mathbb{Z}[i]$  in blue and  $K = \mathbb{Q}(i\sqrt{3})$  and  $\Lambda = \mathcal{O}_K = \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$  in orange. The radial profiles of the pair correlation functions converge to a limit

$$\frac{\pi}{|D_K|} \prod_{\mathfrak{p}} \left(1 - \frac{2}{N(\mathfrak{p})^2}\right) \left(1 + \frac{1}{N(\mathfrak{p})^2(N(\mathfrak{p})^2 - 2)}\right)$$

at infinity, where  $\mathfrak{p}$  ranges over the prime ideals of  $\mathcal{O}_K$ , see Proposition 6.5. This limit is approximately 0.346 for the blue curve and 0.634 for the orange one.

The radial profiles of the pair correlation functions in the weighted and unweighted cases are similar to certain radial distribution functions in statistical physics, see for example [29, Sect. II], [26, Fig. 7], [6, page 199] or [4, page 18]. See also [16]. The unfolding technique (see for instance [4, p. 14] and [15, §3, §5]), though guiding the very first step of the proofs of Theorem 1.1 and 1.2, falls short of giving a complete answer, in particular when varying the scalings and weights and for the error term analysis.

As explained in Sect. 7, our motivation for introducing the weights by the Euler function comes from hyperbolic geometry. We prove in Proposition 7.1 that the pair correlation measures of the lengths (counted with multiplicity) of the common perpendiculars between the maximal Margulis cusp neighbourhood and itself in the (one-cusped) Bianchi orbifold  $\mathrm{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}_{\mathbb{R}}^3$  are closely related to the pair correlation measures of the weighted family  $\mathcal{L}_{\mathcal{O}_K}^{\varphi_K}$ . Theorem 1.2 implies a pair correlation result for the lengths of common perpendiculars of cusps neighbourhoods in the Bianchi orbifold  $\mathrm{PSL}_2(\mathcal{O}_K) / \mathbb{H}_{\mathbb{R}}^3$ , see Corollary 7.2 for a precise statement and a version with congruences.

**Notation.** We introduce here some of the notation used throughout the paper.

All our measures are Borel, positive, regular measures on locally compact spaces. The pushforward of a measure  $\mu$  by a mapping  $f$  is denoted by  $f_*\mu$ , and its total mass by  $\|\mu\|$ . We denote by  $\mathrm{Leb}_B$  the restriction of Lebesgue’s measure of  $\mathbb{C}$  to any Borel subset  $B$  of  $\mathbb{C}$ . For every smooth manifold with boundary  $Y$  and every  $k \in \mathbb{N}$ , we denote by  $C_c^k(Y)$  the set of complex-valued  $C^k$  functions with compact support on  $Y$ .

We equivariantly identify the space  $\mathrm{Grid}_2$  of  $\mathbb{Z}$ -grids in the real Euclidean plane  $\mathbb{C}$ , endowed with the Chabauty topology and the affine action of  $\mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^2$  with the homogeneous space  $(\mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^2) / (\mathrm{GL}_2(\mathbb{Z}) \times \mathbb{Z}^2)$ , which smoothly fibers by the map  $a + \vec{\Lambda} \mapsto \vec{\Lambda}$  over the space of  $\mathbb{Z}$ -lattices  $\mathrm{GL}_2(\mathbb{R}) / \mathrm{GL}_2(\mathbb{Z})$ , with fibers the elliptic curves  $\mathbb{C} / \vec{\Lambda}$ .

We will use the following indexing sets in Sects. 2, 3 and 5. Given a  $\mathbb{Z}$ -grid  $\Lambda$ , for every  $N \in \mathbb{N} \setminus \{0\}$ , let

$$I_N = I_{N,\Lambda} = \{(m, n) \in \Lambda^2 : 0 < |m|, |n| \leq N, m \neq n\},$$

$$I_N^- = \{(m, n) \in I_N : |m| \leq |n|\} \quad \text{and} \quad I_N^+ = \{(m, n) \in I_N : |n| \leq |m|\}.$$

Given a subset  $b$  of the set of ambient parameters, for every positive function  $g$  of a variable in  $\mathbb{N} \setminus \{0\}$ , we will denote by  $O_b(g)$  (and  $O(g)$  when  $b$  is empty) any function  $f$  on  $\mathbb{N} \setminus \{0\}$  such that there exists a constant  $C'$  depending only on the parameters in  $b$  and a constant  $N_0$  possibly depending on the all the parameters (including the ones in  $b$ ) such that for every  $N \geq N_0$ , we have  $|f(N)| \leq C |g(N)|$ .

## 2 Pair correlation of grid points without weight or scaling

In this section, we work on the constant cylinder  $E = \mathbb{C} / (2\pi i\mathbb{Z})$ , endowed with its quotient Riemann surface structure, with its quotient additive abelian locally compact group structure, and with its Haar measure  $d \mathrm{Leb}_E(x' + iy') = dx' dy'$  where  $x' \in \mathbb{R}$  and  $y' \in \mathbb{R} / (2\pi\mathbb{Z})$ . We endow the multiplicative group  $\mathbb{C}^\times$  with its Riemann surface structure as an open subset of  $\mathbb{C}$  and with the restriction of the Lebesgue measure  $\mathrm{Leb}_{\mathbb{C}}$  of  $\mathbb{C}$ . The logarithm map  $\log : \mathbb{C}^\times \rightarrow E$  defined by  $\rho e^{i\theta} \mapsto \ln \rho + i\theta$  is a biholomorphic group isomorphism, whose inverse is the exponential map  $z' = x' + iy' \mapsto \exp(z') = e^{x'} e^{iy'}$ . The real part map

$\text{Re} : E \rightarrow \mathbb{R}$  defined by  $x' + iy' \mapsto x'$  is a smooth (trivial) fibration, and

$$\text{Re}_* \text{Leb}_E = 2\pi \text{Leb}_{\mathbb{R}} . \tag{4}$$

Note that for every  $z \in \mathbb{C} \setminus \{0\}$ , we have

$$\ln(|z|^2) = 2 \text{Re}(\log z) . \tag{5}$$

Since  $d \text{Leb}_{\mathbb{C}}(\rho e^{i\theta}) = \rho d\rho d\theta$ , we have

$$d(\log_* \text{Leb}_{\mathbb{C}})(z') = e^{2\text{Re}(z')} d \text{Leb}_E(z') . \tag{6}$$

Let  $\Lambda = a + \bar{\Lambda}$  be a  $\mathbb{Z}$ -grid. We choose a  $\mathbb{Z}$ -basis  $(v_1, v_2)$  of  $\bar{\Lambda}$  such that the (weak) fundamental parallelogram

$$\mathcal{F}_{\bar{\Lambda}} = \left\{ s v_1 + t v_2 : s, t \in \left[-\frac{1}{2}, \frac{1}{2}\right] \right\}$$

for the action of  $\bar{\Lambda}$  on  $\mathbb{C}$  has smallest diameter. We then denote by

$$\text{diam}_{\bar{\Lambda}} = \text{diam}(\mathcal{F}_{\bar{\Lambda}}) = \max\{|v_1 + v_2|, |v_1 - v_2|\}$$

the diameter of  $\mathcal{F}_{\bar{\Lambda}}$ , which is the length of a longest diagonal of the parallelogram  $\mathcal{F}_{\bar{\Lambda}}$ . We denote by

$$\text{covol}_{\bar{\Lambda}} = \text{Vol}(\mathbb{C}/\bar{\Lambda}) = \text{Area}(\mathcal{F}_{\bar{\Lambda}}) = |\det(v_1, v_2)|$$

the area of the elliptic curve  $\mathbb{C}/\bar{\Lambda}$  for the measure induced by the Lebesgue measure on  $\mathbb{C}$ , or the area of the parallelogram  $\mathcal{F}_{\bar{\Lambda}}$  (which does not depend on the choice of the  $\mathbb{Z}$ -basis  $(v_1, v_2)$  of  $\bar{\Lambda}$ ). We will use several times the following well known result, having a more precise error term that we won't need, and we only give a proof in order to make the dependence on the parameters  $k$  and  $\Lambda$  explicit.

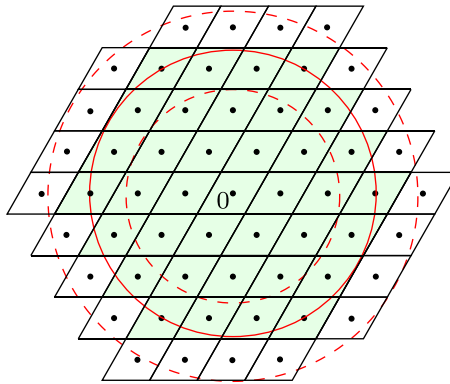
**Lemma 2.1** *For every  $k \in \mathbb{N}$ , there exists a constant  $C_k > 0$  such that for all  $\Lambda \in \text{Grid}_2$  and  $x \geq 1$ , we have*

$$\left| \sum_{p \in \Lambda : |p| \leq x} |p|^k - \frac{2\pi}{(k+2) \text{covol}_{\bar{\Lambda}}} x^{k+2} \right| \leq C_k \left( \frac{(1 + \text{diam}_{\bar{\Lambda}})x + \text{diam}_{\bar{\Lambda}}^2}{\text{covol}_{\bar{\Lambda}}} x^k \right) . \tag{7}$$

*Proof* The case  $k = 0$  of the lemma is the standard Gauss counting result of lattice points in discs. With  $A_x = \{p \in \Lambda : |p| \leq x\}$  and  $B_x = \bigcup_{p \in A_x} (p + \mathcal{F}_{\bar{\Lambda}})$ , so that  $\text{Area}(B_x) = \text{Card}(A_x) \text{Area}(\mathcal{F}_{\bar{\Lambda}})$ , we have

$$B(0, x - \text{diam}_{\bar{\Lambda}}) \subset B_x \subset B(0, x + \text{diam}_{\bar{\Lambda}}),$$

(with the convention that  $B(0, r) = \emptyset$  if  $r < 0$ ) so that the result for  $k = 0$  with a slightly simpler error term  $O\left(\frac{x \text{diam}_{\bar{\Lambda}} + \text{diam}_{\bar{\Lambda}}^2}{\text{covol}_{\bar{\Lambda}}}\right)$  follows by computing the area of the two above discs.



Let now  $k \geq 1$ . We consider the sequence  $(a_n = \text{Card}\{p \in \Lambda : n - 1 < |p| \leq n\})_{n \geq 1}$  and the smooth functions  $f : [1, +\infty[ \rightarrow \mathbb{R}$  defined by  $t \mapsto t^k$  or by  $t \mapsto (t - 1)^k$ . For every  $x \geq 1$ , we have the estimate

$$\sum_{1 \leq n \leq [x]} a_n (n - 1)^k \leq \sum_{p \in \Lambda : |p| \leq x} |p|^k \leq \sum_{1 \leq n \leq [x]} a_n n^k. \tag{8}$$

Using the case  $k = 0$  showing that  $\sum_{1 \leq n \leq t} a_n = \frac{\pi}{\text{covol}_\Lambda} t^2 + O\left(\frac{\text{diam}_\Lambda(t + \text{diam}_\Lambda)}{\text{covol}_\Lambda}\right)$ , the general result follows from Abel's summation formula

$$\sum_{1 \leq n \leq x} a_n f(n) = \left(\sum_{1 \leq n \leq x} a_n\right) f(x) - \int_1^x \left(\sum_{1 \leq n \leq t} a_n\right) f'(t) dt$$

applied to the above sequence  $(a_n)_{n \geq 1}$  and to the two functions  $f$ , the first one for the majoration in Formula (8), the second one for its minoration.  $\square$

For every  $N \in \mathbb{N} \setminus \{0\}$ , the (not normalised) *pair correlation measure* of the logarithms of nonzero grid points in  $\Lambda$ , with trivial multiplicities and with trivial scaling function, is the finite measure on the cylinder  $E$  defined by

$$\nu_N = \nu_{N,\Lambda} = \mathcal{R}_N^{\mathcal{L}_{\Lambda,1}} = \sum_{(m,n) \in I_N} \Delta_{\log m - \log n}.$$

Note that for every  $k \in \mathbb{N} \setminus \{0\}$ , we have  $I_{kN,k\Lambda} = I_{N,\Lambda}$  and  $\nu_{kN,k\Lambda} = \nu_{N,\Lambda}$ . Let us consider the function (actually independent on  $\Lambda$ ) on  $E$  defined by

$$g_{\mathcal{L}_{\Lambda,1}} : z' \mapsto \frac{1}{2\pi} e^{-2|\text{Re}(z')|}.$$

**Theorem 2.2** *As  $N \rightarrow +\infty$ , the measures  $\nu_N$  on  $E$ , renormalised to be probability measures, weak-star converge to  $g_{\mathcal{L}_{\Lambda,1}} \text{Leb}_E$ . The convergence is uniform for  $\Lambda$  varying in any given compact subset of  $\text{Grid}_2$ . Furthermore, for every  $f \in C_c^1(E)$ , we have*

$$\begin{aligned} \frac{\nu_N}{\|\nu_N\|}(f) &= \frac{1}{2\pi} \int_E f(z') e^{-2|\text{Re}(z')|} d \text{Leb}_E(z') \\ &+ O\left(\frac{\text{diam}_\Lambda}{N} (\|f\|_\infty + \|e^{-z} df(z)\|_\infty)\right). \end{aligned}$$

This result implies the case  $\alpha = 0$  of Theorem 1.1 in the introduction, since we will prove in Formula (15) that  $\lim_{N \rightarrow +\infty} \frac{\|\nu_N\|}{N^4} = \frac{\pi^2}{\text{covol}_\Lambda^2}$ .



*Remark 2.3* Theorem 2.2 is still valid if we allow  $n = m$  in the definition of the index set  $I_N$  (this correspond to removing the condition  $p \neq q$  in the definition below of  $J_q$ ), see also Remark (2) in [23, §3] for a general argument. We will use this comment in the proofs of Corollary 2.4 and 2.5.

*Proof of Theorem 2.2.* For all  $N \in \mathbb{N}$  and  $q \in \Lambda$  with  $0 < |q| \leq N$ , let

$$J_q = \{p \in \Lambda : 0 < |p| \leq |q|, p \neq q\} \quad \text{and} \quad \omega_q = \sum_{p \in J_q} \Delta_{\frac{p}{q}}, \quad (9)$$

which is a finitely supported measure on the closed unit disc  $\mathbb{D}$  of  $\mathbb{C}$ . Note that the assumptions  $0 < |p|$  and  $0 < |q|$  are automatic when  $0 \notin \Lambda$ , that is, when  $\Lambda$  is not a  $\mathbb{Z}$ -lattice. As  $q \rightarrow +\infty$ , by Equation (7) with  $k = 0$  (and its slightly better error term), its total mass, which is nonzero since  $-q \in J_q$ , satisfies

$$\text{Card } J_q = \|\omega_q\| = \frac{\pi}{\text{covol}_{\bar{\Lambda}}} |q|^2 + O\left(\frac{\text{diam}_{\bar{\Lambda}}(|q| + \text{diam}_{\bar{\Lambda}})}{\text{covol}_{\bar{\Lambda}}}\right), \quad (10)$$

for some  $O(\cdot)$  uniform in  $\Lambda$ . Note that we need to remove 0 if  $0 \in \Lambda$  and  $q$  from the counting of Equation (7), but this is taken care of by the above  $O(\cdot)$ . In particular, we have  $\|\omega_q\| = O\left(\frac{\text{diam}_{\bar{\Lambda}}^2}{\text{covol}_{\bar{\Lambda}}}\right)$  uniformly in  $\Lambda$  if  $|q| < \text{diam}_{\bar{\Lambda}}$  and otherwise

$$\|\omega_q\| = \frac{\pi}{\text{covol}_{\bar{\Lambda}}} |q|^2 + O\left(\frac{\text{diam}_{\bar{\Lambda}}}{\text{covol}_{\bar{\Lambda}}} |q|\right) = O\left(\frac{|q|^2}{\text{covol}_{\bar{\Lambda}}}\right).$$

We hence have, if  $|q| \geq \text{diam}_{\bar{\Lambda}}$ ,

$$\frac{1}{\|\omega_q\|} = \frac{\text{covol}_{\bar{\Lambda}}}{\pi |q|^2} + O\left(\frac{\text{diam}_{\bar{\Lambda}} \text{covol}_{\bar{\Lambda}}}{|q|^3}\right),$$

for some  $O(\cdot)$  uniform in  $\Lambda$ . We denote by  $\overline{\omega}_q = \frac{\omega_q}{\|\omega_q\|}$  the renormalisation of  $\omega_q$  to a probability measure on  $\mathbb{D}$ .

Let  $f \in C^1(\mathbb{D})$ . Assume that  $|q| \geq \text{diam}_{\bar{\Lambda}}$ . Let

$$C_q = \bigcup_{p \in J_q} (p + \mathcal{F}_{\bar{\Lambda}}).$$

Note that the symmetric difference  $(\mathbb{D} \setminus \frac{C_q}{q}) \cup (\frac{C_q}{q} \setminus \mathbb{D})$  is contained in the union of the annulus  $B(0, 1 + \frac{\text{diam}_{\bar{\Lambda}}}{|q|}) \setminus B(0, 1 - \frac{\text{diam}_{\bar{\Lambda}}}{|q|})$  and (when  $0 \in \Lambda$ ) the parallelogram  $\frac{\mathcal{F}_{\bar{\Lambda}}}{q}$ , hence has area at most

$$\frac{\text{covol}_{\bar{\Lambda}}}{|q|^2} + \pi \left( \left(1 + \frac{\text{diam}_{\bar{\Lambda}}}{|q|}\right)^2 - \left(1 - \frac{\text{diam}_{\bar{\Lambda}}}{|q|}\right)^2 \right) = O\left(\frac{\text{diam}_{\bar{\Lambda}}}{|q|}\right).$$

Also note that  $\frac{\text{diam}_{\bar{\Lambda}} \text{covol}_{\bar{\Lambda}}}{|q|^3} |\omega_q(f)| = O\left(\frac{\text{diam}_{\bar{\Lambda}} \|f\|_{\infty}}{|q|}\right)$  by Equation (10). Therefore

$$\begin{aligned} & \left| \frac{1}{\pi} \int_{\mathbb{D}} f(z) d \text{Leb}_{\mathbb{C}}(z) - \overline{\omega}_q(f) \right| \\ &= \left| \frac{1}{\pi} \int_{\frac{C_q}{q}} f(z) d \text{Leb}_{\mathbb{C}}(z) - \frac{\text{covol}_{\bar{\Lambda}}}{\pi |q|^2} \omega_q(f) \right| + O\left(\frac{\text{diam}_{\bar{\Lambda}} \|f\|_{\infty}}{|q|}\right). \end{aligned}$$

By the mean value inequality, for all  $p \in J_q$  and  $z \in \frac{p+\mathcal{F}_{\bar{\Lambda}}}{q}$ , we have

$$|f(z) - f\left(\frac{p}{q}\right)| \leq \|df\|_{\infty} \left|z - \frac{p}{q}\right|.$$

Hence

$$\begin{aligned} & \left| \frac{1}{\pi} \int_{\frac{C_q}{q}} f(z) d \operatorname{Leb}_{\mathbb{C}}(z) - \frac{\operatorname{covol}_{\bar{\Lambda}}}{\pi |q|^2} \omega_q(f) \right| \\ &= \frac{1}{\pi} \left| \sum_{p \in J_q} \int_{\frac{p+\mathcal{F}_{\bar{\Lambda}}}{q}} \left( f(z) - f\left(\frac{p}{q}\right) \right) d \operatorname{Leb}_{\mathbb{C}}(z) \right| \\ &\leq \frac{1}{\pi} \sum_{p \in J_q} \|df\|_{\infty} \sup_{z \in \frac{p+\mathcal{F}_{\bar{\Lambda}}}{q}} \left|z - \frac{p}{q}\right| \operatorname{Area}\left(\frac{p+\mathcal{F}_{\bar{\Lambda}}}{q}\right) \\ &\leq \frac{1}{\pi} \operatorname{Card}(J_q) \frac{\operatorname{diam}_{\bar{\Lambda}}}{|q|} \frac{\operatorname{covol}_{\bar{\Lambda}}}{|q|^2} \|df\|_{\infty} = O\left(\frac{\operatorname{diam}_{\bar{\Lambda}} \|df\|_{\infty}}{|q|}\right). \end{aligned}$$

Therefore, if  $|q| \geq \operatorname{diam}_{\bar{\Lambda}}$ , then

$$\overline{\omega}_q(f) = \frac{1}{\pi} \int_{\mathbb{D}} f(z) d \operatorname{Leb}_{\mathbb{C}}(z) + O\left(\frac{\operatorname{diam}_{\bar{\Lambda}} (\|f\|_{\infty} + \|df\|_{\infty})}{|q|}\right). \tag{11}$$

In particular, as  $q \rightarrow +\infty$ , we have  $\overline{\omega}_q \xrightarrow{*} \frac{1}{\pi} \operatorname{Leb}_{\mathbb{D}}$ .

Assume that  $N \geq \operatorname{diam}_{\bar{\Lambda}}$ . Let us now define

$$\mu_N^- = \sum_{(m,n) \in I_N^-} \Delta \frac{m}{n} = \sum_{q \in \Lambda, 0 < |q| \leq N} \omega_q$$

which is a finitely supported measure on  $\mathbb{D}$ . By Equations (10) and (7) with  $k = 2, 1, 0$ , a heavy computation since  $N \geq \operatorname{diam}_{\bar{\Lambda}}$  gives that its total mass is equal to

$$\begin{aligned} \|\mu_N^-\| &= \sum_{q \in \Lambda, 0 < |q| \leq N} \|\omega_q\| \\ &= \sum_{q \in \Lambda, 0 < |q| \leq N} \left( \frac{\pi}{\operatorname{covol}_{\bar{\Lambda}}} |q|^2 + O\left(\frac{\operatorname{diam}_{\bar{\Lambda}} |q| + \operatorname{diam}_{\bar{\Lambda}}^2}{\operatorname{covol}_{\bar{\Lambda}}}\right) \right) \\ &= \frac{\pi^2}{2 \operatorname{covol}_{\bar{\Lambda}}^2} N^4 + O\left(\frac{1 + \operatorname{diam}_{\bar{\Lambda}}}{\operatorname{covol}_{\bar{\Lambda}}^2} N^3\right) = O\left(\frac{N^4}{\operatorname{covol}_{\bar{\Lambda}}^2}\right). \end{aligned} \tag{12}$$

It follows that if  $N \geq \operatorname{diam}_{\bar{\Lambda}}$ , then

$$\frac{1}{\|\mu_N^-\|} = \frac{2 \operatorname{covol}_{\bar{\Lambda}}^2}{\pi^2 N^4} + O\left(\frac{(1 + \operatorname{diam}_{\bar{\Lambda}}) \operatorname{covol}_{\bar{\Lambda}}^2}{N^5}\right) = O\left(\frac{\operatorname{covol}_{\bar{\Lambda}}^2}{N^4}\right). \tag{13}$$

Let  $f \in C^1(\mathbb{D})$ . By Equations (11), (13), (12) and (7) with  $k = 1$ , we have, as  $N \geq \operatorname{diam}_{\bar{\Lambda}}$  tends to  $\infty$ ,

$$\frac{\mu_N^-(f)}{\|\mu_N^-\|} = \frac{1}{\|\mu_N^-\|} \left( \sum_{q \in \Lambda, 0 < |q| < \operatorname{diam}_{\bar{\Lambda}}} \|\omega_q\| \overline{\omega}_q(f) + \sum_{q \in \Lambda, \operatorname{diam}_{\bar{\Lambda}} \leq |q| \leq N} \|\omega_q\| \overline{\omega}_q(f) \right)$$

$$\begin{aligned}
 &= O\left(\frac{1}{\|\mu_N^-\|} \|\mu_{\text{diam}_\Lambda^-}^-\| \|f\|_\infty\right) + \left(\frac{\|\mu_N^-\| - \|\mu_{\text{diam}_\Lambda^-}^-\|}{\|\mu_N^-\|}\right) \frac{1}{\pi} \int_{\mathbb{D}} f(z) d \text{Leb}_{\mathbb{C}}(z) \\
 &\quad + \sum_{q \in \Lambda, \text{diam}_\Lambda^- \leq |q| \leq N} \frac{\|\omega_q\|}{\|\mu_N^-\|} O\left(\frac{\text{diam}_\Lambda^- (\|f\|_\infty + \|df\|_\infty)}{|q|}\right) \\
 &= O\left(\frac{\text{covol}_\Lambda^2 \text{diam}_\Lambda^4}{N^4 \text{covol}_\Lambda^2} \|f\|_\infty\right) + \left(1 + O\left(\frac{\text{diam}_\Lambda^4}{N^4}\right)\right) \frac{1}{\pi} \int_{\mathbb{D}} f(z) d \text{Leb}_{\mathbb{C}}(z) \\
 &\quad + O\left(\frac{\text{diam}_\Lambda^-}{N} (\|f\|_\infty + \|df\|_\infty)\right) \\
 &= \frac{1}{\pi} \int_{\mathbb{D}} f(z) d \text{Leb}_{\mathbb{C}}(z) + O\left(\frac{\text{diam}_\Lambda^-}{N} (\|f\|_\infty + \|df\|_\infty)\right). \tag{14}
 \end{aligned}$$

Let  $E^\pm = (\pm[0, \infty[ + i\mathbb{R}) / (2\pi i\mathbb{Z})$  so that  $E = E^- \cup E^+$ . Note that  $\log : \mathbb{D} \setminus \{0\} \rightarrow E^-$  and  $\log : \overline{\mathbb{C}} \setminus \mathbb{D} \rightarrow E^+$  are homeomorphisms. Let us define a measure with finite support on  $E^\pm$  by

$$\nu_N^\pm = \sum_{(m, n) \in I_N^\pm} \Delta_{\log \frac{m}{n}},$$

so that  $\nu_N^- = \log_* \mu_N^- = \nu_N^-|_{E^-}$ , and  $\|\nu_N^-\| = \|\mu_N^-\|$ . For every  $f \in C_c^1(E^-)$ , we have  $f \circ \log \in C_c^1(\mathbb{D} \setminus \{0\})$  (hence  $f \circ \log$  may be extended to a  $C^1$  function on  $\mathbb{D}$  which vanishes on a neighbourhood of 0). By Equations (14) and (6), we have

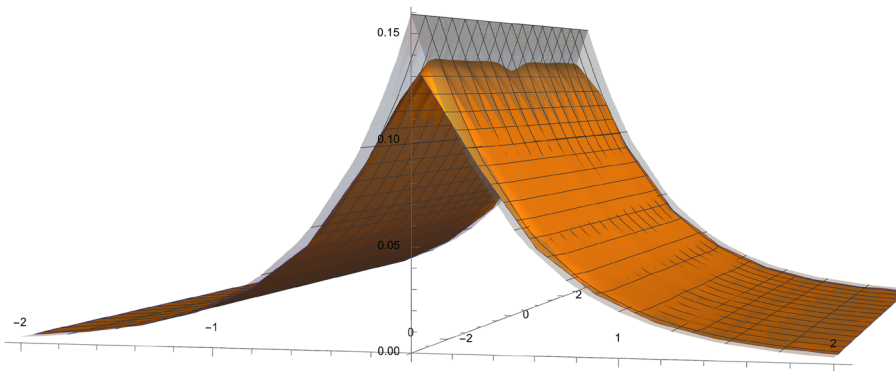
$$\begin{aligned}
 \frac{\nu_N^-(f)}{\|\nu_N^-\|} &= \frac{\mu_N^-(f \circ \log)}{\|\mu_N^-\|} \\
 &= \frac{1}{\pi} \int_{\mathbb{D}} f \circ \log(z) d \text{Leb}_{\mathbb{C}}(z) + O\left(\frac{\text{diam}_\Lambda^-}{N} (\|f \circ \log\|_\infty + \|d(f \circ \log)\|_\infty)\right) \\
 &= \frac{1}{\pi} \int_{E^-} f e^{2\text{Re}(z')} d \text{Leb}_E(z') + O\left(\frac{\text{diam}_\Lambda^-}{N} (\|f\|_\infty + \|e^{-z} df(z)\|_\infty)\right).
 \end{aligned}$$

Let  $\text{sg} : E \rightarrow E$  be the horizontal change of sign map  $x' + iy' \mapsto -x' + iy'$ , which maps  $E^-$  to  $E^+$ . Then  $\nu_N^+ = \text{sg}_* \nu_N^-$  and  $\nu_N = \nu_N^- + \nu_N^+$ . Since  $E^- \cap E^+$  has zero measure for the Haar measure  $\text{Leb}_E$  and since  $\|\nu_N^\pm\| = \frac{1}{2} \|\nu_N\| + O(\text{diam}_\Lambda^- N^3)$ , the last claim of Theorem 2.2 follows. Note that, as needed just after the statement of Theorem 2.2, as  $N \rightarrow +\infty$ , we have

$$\|\nu_N\| \sim 2\|\mu_N^-\| \sim \frac{\pi^2}{\text{covol}_\Lambda^2} N^4. \tag{15}$$

The first claim of Theorem 2.2 follows by approximating continuous functions with compact support by  $C^1$  ones. The uniformity of the convergence on compact subsets of lattices follows from the uniformity of the functions  $O(\cdot)$  and the fact that the constants  $\text{covol}_\Lambda^-$  and  $\text{diam}_\Lambda^-$  vary in a compact subset of  $]0, +\infty[$  when  $\Lambda$  varies in a compact subset of  $\text{Grid}_2$ . □

The following picture illustrates the weak-star convergence statement in Theorem 2.2 when  $\Lambda = \bar{\Lambda} = \mathbb{Z}[i]$  is the ring of Gaussian integers and  $N = 20$ , using as horizontal coordinates  $(x', y') \in E$  with  $x' \in \mathbb{R}$  and  $y' \in [-\pi, \pi[$ . A smooth histogram scaled to a probability density is displayed in orange, and the limiting distribution in grey.



**Arithmetic applications.** (1) Let  $K$  be an imaginary quadratic number field, with discriminant  $D_K$ , ring of integers  $\mathcal{O}_K$  and Dedekind zeta function  $\zeta_K$ . We denote by  $\mathcal{I}_K^+$  the semigroup of nonzero (integral) ideals of the Dedekind ring  $\mathcal{O}_K$  (with unit  $\mathcal{O}_K$ ). We denote by  $N(I) = \text{Card}(\mathcal{O}_K/I)$  the *norm* of an ideal  $I \in \mathcal{I}_K^+$ , which is completely multiplicative. The *norm* of  $a \in \mathcal{O}_K \setminus \{0\}$  is

$$N(a) = N(a\mathcal{O}_K).$$

It coincides with the (relative) norm  $N_{K/\mathbb{Q}}(a)$  of  $a$  (see for instance [20]), and in particular is equal to  $|a|^2$  since  $K$  is imaginary quadratic. The *norm* of a fractional ideal  $\mathfrak{m}$  of  $\mathcal{O}_K$  is  $\frac{1}{|c|^2}N(c\mathfrak{m})$  for any  $c \in \mathcal{O}_K \setminus \{0\}$  such that  $c\mathfrak{m} \subset \mathcal{O}_K$ .

Let  $\mathfrak{m}$  be a nonzero fractional ideal of  $\mathcal{O}_K$ . Note that  $\mathfrak{m}$  is a  $\mathbb{Z}$ -lattice in  $\mathbb{C}$  with

$$\text{covol}_{\mathfrak{m}} = \frac{\sqrt{|D_K|}N(\mathfrak{m})}{2} \quad \text{and} \quad \text{diam}_{\mathfrak{m}} = O(\sqrt{|D_K|}N(\mathfrak{m})), \tag{16}$$

for a  $O(\cdot)$  uniform in  $K$ , since  $\mathcal{O}_K = \mathbb{Z} + \frac{\sqrt{D_K}}{2}\mathbb{Z}$  and  $\text{diam}_{\mathcal{O}_K} = |1 + \frac{\sqrt{D_K}}{2}|$  if  $D_K \equiv 0 \pmod{4}$ , and since  $\mathcal{O}_K = \mathbb{Z} + \frac{1+\sqrt{D_K}}{2}\mathbb{Z}$  and  $\text{diam}_{\mathcal{O}_K} = |\frac{3+\sqrt{D_K}}{2}|$  if  $D_K \equiv 1 \pmod{4}$ . In particular, the Gauss ball counting argument of Equation (7) with  $k = 0$  (with its slightly simpler error term) and  $x = \sqrt{N'}$  gives, as  $N' \geq N(\mathfrak{m})$  tends to  $+\infty$ ,

$$\begin{aligned} (\text{Card}\{m \in \mathfrak{m} : 0 < N(m) \leq N'\})^2 &= \left( \frac{\pi}{\text{covol}_{\mathfrak{m}}} N' + O\left(\frac{\text{diam}_{\mathfrak{m}}\sqrt{N'} + \text{diam}_{\mathfrak{m}}^2}{\text{covol}_{\mathfrak{m}}}\right) \right)^2 \\ &= \frac{4\pi^2 N'^2}{|D_K|N(\mathfrak{m})^2} \left( 1 + O\left(\frac{\sqrt{|D_K|}N(\mathfrak{m})}{\sqrt{N'}}\right) \right). \end{aligned}$$

Hence Theorem 2.2 implies the existence of a pair correlation function (independent of  $\mathfrak{m}$ ) for the family of the complex logarithms of nonzero elements of  $\mathfrak{m}$

$$\mathcal{L}_{\mathfrak{m}} = (A_{N'}^{\mathfrak{m}} = \{\log n : n \in \mathfrak{m}, 0 < N(n) \leq N'\}, \omega_{N'} = 1)_{N' \in \mathbb{N}}$$

without weights or scaling, as stated in the following result, using Remark 2.3.

**Corollary 2.4** *For every  $f \in C_c^1(E)$ , as  $N' \rightarrow +\infty$ , we have*

$$\begin{aligned} &\frac{|D_K|N(\mathfrak{m})^2}{4\pi^2 N'^2} \sum_{m, n \in \mathfrak{m} : 0 < N(m), N(n) \leq N'} f(\log m - \log n) \\ &= \frac{1}{2\pi} \int_E f(z') e^{-2|\text{Re}(z')|} d \text{Leb}_E(z') + O\left(\frac{\sqrt{|D_K|}N(\mathfrak{m})}{\sqrt{N'}} (\|f\|_{\infty} + \|e^{-z} df(z)\|_{\infty})\right). \quad \square \end{aligned}$$

(2) For every positive integer  $d$ , let  $r_{2,d} : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$  be the arithmetic function where

$$r_{2,d}(n) = \text{Card}\{(x, y) \in \mathbb{Z}^2 : x^2 + d y^2 = n\}$$

is the number of integral solutions of the Diophantine equation  $x^2 + d y^2 = n$ , for every  $n \in \mathbb{N}$ . In particular, if  $d = 1$ , then  $r_{2,d} = r_2$  is the well known function counting the sum of two squares representatives of a given positive integer (see for instance [7] or [12, Sect. 16.9]). The following result proves that the map

$$g_{\mathbb{R}} : t \mapsto \frac{1}{2} e^{-|t|}$$

on  $\mathbb{R}$  is the pair correlation function for the family

$$\mathcal{L}_{\mathbb{N}}^{r_{2,d}} = (A_N = \{\ln n : 0 < n \leq N, r_{2,d}(n) \neq 0\}, \omega_N = r_{2,d} \circ \exp)_{N \in \mathbb{N}}$$

of the logarithms of the nonzero natural integers, without scaling but with weights given by  $r_{2,d}$  (removing the zero weights). Other weights have been considered in [22] (including the one given by the Euler function  $\varphi$ ). Note that the following corollary holds also when  $r_{2,d}(n)$  is replaced by the number of representations of  $n$  by the norm form of any imaginary quadratic number field, evaluated on any order of their ring of integers (as for instance the norm form  $(x, y) \mapsto x^2 - xy + y^2$  of the Eisenstein integers).

**Corollary 2.5** *As  $N \rightarrow +\infty$ , we have*

$$\frac{1}{\left(\sum_{0 < m \leq N^2} r_{2,d}(m)\right)^2} \sum_{m,n \in \mathbb{N} : 0 < m,n \leq N^2} r_{2,d}(m) r_{2,d}(n) \Delta_{\ln m - \ln n} \xrightarrow{*} g_{\mathbb{R}} \text{ Leb}_{\mathbb{R}} .$$

*Proof* Let us consider the  $\mathbb{Z}$ -lattice  $\Lambda = \mathbb{Z} + i\sqrt{d} \mathbb{Z}$  in  $\mathbb{C}$ . Using Remark 2.3, we remove the assumptions  $m \neq n$  in the summations defining  $\mathcal{R}_N^{\mathcal{L}_{\Lambda},1}$  as well as  $\mathcal{R}_{N^2}^{\mathcal{L}_{\mathbb{N}}^{r_{2,d}},1}$ .

By the linearity of  $(2 \text{Re})_*$  and  $2 \text{Re}$ , and by Equation (5), for every  $N \in \mathbb{N} \setminus \{0\}$ , we have

$$\begin{aligned} (2 \text{Re})_* \left( \mathcal{R}_N^{\mathcal{L}_{\Lambda},1} \right) &= \sum_{p,q \in \Lambda : 0 < |p|, |q| \leq N} \Delta_{2 \text{Re}(\log p) - 2 \text{Re}(\log q)} \\ &= \sum_{0 < m,n \leq N^2} \sum_{p,q \in \Lambda : |p|^2 = m, |q|^2 = n} \Delta_{\ln(|p|^2) - \ln(|q|^2)} \\ &= \mathcal{R}_{N^2}^{\mathcal{L}_{\mathbb{N}}^{r_{2,d}},1} . \end{aligned}$$

The pushforward map  $(2 \text{Re})_*$  preserves the total mass and is continuous for the weak-star topology, since the map  $2 \text{Re} : E \rightarrow \mathbb{R}$  is proper. Hence by the weak-star convergence statement in Theorem 2.2 and by (4), we have

$$\begin{aligned} \frac{\mathcal{R}_{N^2}^{\mathcal{L}_{\mathbb{N}}^{r_{2,d}},1}}{\|\mathcal{R}_{N^2}^{\mathcal{L}_{\mathbb{N}}^{r_{2,d}},1}\|} &= (2 \text{Re})_* \left( \frac{\mathcal{R}_N^{\mathcal{L}_{\Lambda},1}}{\|\mathcal{R}_N^{\mathcal{L}_{\Lambda},1}\|} \right) \xrightarrow{*} (2 \text{Re})_* \left( \frac{1}{2\pi} e^{-|2 \text{Re}(z')|} d \text{Leb}_E(z') \right) \\ &= \frac{1}{2} e^{-|t|} d \text{Leb}_{\mathbb{R}}(t) . \end{aligned}$$

Corollary 2.5 follows. □

As  $\text{covol}_{\mathbb{Z} + i\sqrt{d} \mathbb{Z}} = \sqrt{d}$ , by Lemma 2.1 with  $k = 0$ , we have

$$\sum_{\ln n \leq t} r_{2,d}(n) = \text{Card} \left( B(0, e^{t/2}) \cap (\mathbb{Z} + i\sqrt{d} \mathbb{Z}) \right) = \frac{\pi}{\sqrt{d}} e^t (1 + O(e^{t/2})) .$$

Thus, the conclusion of Corollary 2.5 also follows from [23, Theo. 1.1], whose proof only uses the exponential growth property of the weighted family  $\mathcal{L}_{\mathbb{N}}^{r_{2,d}}$ .

### 3 Pair correlation of grid points with scaling without weight

In this section, we study the pair correlations of complex logarithms of grid points at various scaling. We fix a positive scaling function  $\psi : \mathbb{N} \setminus \{0\} \rightarrow ]0, +\infty[$  such that  $\lim_{+\infty} \psi = +\infty$ . We consider a normalisation function  $\psi' : \mathbb{N} \setminus \{0\} \rightarrow ]0, +\infty[$  depending on  $\psi$ , which will be made precise later on, but which in most cases will not yield the renormalisation to a probability measure.

We will work on the following family  $(E_N)_{N \in \mathbb{N} \setminus \{0\}}$  of varying cylinders. For every  $N \in \mathbb{N} \setminus \{0\}$ , we consider  $E_N = \mathbb{C}/(2\pi i \psi(N) \mathbb{Z})$ , endowed with its quotient Riemann surface structure and its quotient additive abelian locally compact group structure. Since a real number  $\theta$  is well defined modulo  $2\pi\mathbb{Z}$  if and only if  $\psi(N)\theta$  is well defined modulo  $2\pi\psi(N)\mathbb{Z}$ , the scaled logarithm map  $\psi(N) \log : \mathbb{C}^\times \rightarrow E_N$  defined by  $\rho e^{i\theta} \mapsto \psi(N) \ln \rho + i\psi(N)\theta$  is a biholomorphic group isomorphism, whose inverse is the rescaled exponential map  $z' = x' + iy' \mapsto \exp(\frac{z'}{\psi(N)}) = e^{\frac{x'}{\psi(N)}} e^{i \frac{y'}{\psi(N)}}$ . The real part map  $\text{Re} : \mathbb{C} \rightarrow \mathbb{R}$  induces a map again denoted by  $\text{Re} : E_N \rightarrow \mathbb{R}$ , which is a trivial smooth bundle map with fibers  $i\mathbb{R}/(2\pi i \psi(N)\mathbb{Z})$ , such that for every  $z \in E$ ,

$$\text{Re}(\psi(N)z) = \psi(N) \text{Re}(z) . \tag{17}$$

We consider also  $E_N$  as a pointed metric space, with distance the quotient of the Euclidean distance on  $\mathbb{C}$  and base point its (additive) identity element 0. Note that  $E_N$  is a proper metric space. As  $\lim_{+\infty} \psi = +\infty$ , for every  $R > 0$ , there exists  $N_R \in \mathbb{N} \setminus \{0\}$  such that for every  $N \geq N_R$ , the closed ball  $B(0, R)$  in  $\mathbb{C}$  injects isometrically by the canonical projection  $p_N : \mathbb{C} \rightarrow E_N$ . Hence the sequence  $(E_N)_{N \in \mathbb{N} \setminus \{0\}}$  of proper pointed metric spaces converges to the proper metric space  $\mathbb{C}$  pointed at 0 for the pointed Hausdorff-Gromov convergence (see [11] for background).

Any function  $f \in C_c^0(\mathbb{C})$  defines for all  $N$  large enough a function  $f_N \in C_c^0(E_N)$  as follows. Let  $R_f > 0$  be such that the support of  $f$  is contained in  $B(0, R_f)$ . Then for every  $N \geq N_{R_f}$ , the function  $f_N \in C_c^0(E_N)$  is the function which vanishes outside  $p_N(B(0, R_f))$  and coincides with  $f \circ (p_N|_{B(0, R_f)})^{-1}$  on  $p_N(B(0, R_f))$ . Note that  $f_N$  is  $C^1$  if  $f$  is  $C^1$ .

We say that a sequence  $(\mu_N)_{N \in \mathbb{N} \setminus \{0\}}$  of measures  $\mu_N$  on  $E_N$  converges to a measure  $\mu_\infty$  on  $\mathbb{C}$  for the *pointed Hausdorff-Gromov weak-star convergence* if for every  $f \in C_c^0(\mathbb{C})$ , the sequence  $(\mu_N(f_N))_{N \geq N_{R_f}}$  converges in  $\mathbb{C}$  to  $\mu_\infty(f_\infty)$  (see [11, Chap. 3 $\frac{1}{2}$ ] for background).

We again use the symbol  $\xrightarrow{*}$  in order to denote this convergence.

Let  $\Lambda$  be a  $\mathbb{Z}$ -grid in  $\mathbb{C}$ . For every  $N \in \mathbb{N} \setminus \{0\}$ , the (not normalised, empirical) *pair correlation measure* of the complex logarithms of points in  $\Lambda$  at time  $N$  with *trivial weights and with scaling  $\psi(N)$*  is the measure with finite support in  $E_N$  defined by

$$\mathcal{R}_N^{\mathcal{L}\Lambda, \psi} = \sum_{(m, n) \in I_N} \Delta_{\psi(N) \log m - \psi(N) \log n} ,$$

and the normalised one is  $\frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}\Lambda, \psi}$ .

**Theorem 3.1** *Let  $\Lambda = a + \bar{\Lambda}$  be a  $\mathbb{Z}$ -grid in  $\mathbb{C}$ . Assume that the scaling function  $\psi$  satisfies  $\lim_{N \rightarrow +\infty} \frac{\psi(N)}{N} = \lambda_\psi \in [0, +\infty]$ . As  $N \rightarrow +\infty$ , the measures  $\mathcal{R}_N^{\mathcal{L}\Lambda, \psi}$  on  $E_N$ , normalised by  $\psi'(N)$  as given below, converge for the pointed Hausdorff-Gromov weak-star convergence to a measure  $g_{\mathcal{L}\Lambda, \psi} \text{Leb}_{\mathbb{C}}$  on  $\mathbb{C}$ , absolutely continuous with respect to the Lebesgue measure*

on  $\mathbb{C}$ , with Radon-Nikodym derivative the function

$$g_{\mathcal{L}_{\Lambda}, \psi} : z \mapsto \begin{cases} 0 & \text{if } \lambda_{\psi} = +\infty \text{ and } \psi' = \psi, \\ \frac{\pi}{2 \operatorname{covol}_{\bar{\Lambda}}^2} & \text{if } \lambda_{\psi} = 0 \text{ and } \psi'(N) = \frac{N^4}{\psi(N)^2}, \\ \frac{1}{\operatorname{covol}_{\bar{\Lambda}} |z|^4} \sum_{p \in \bar{\Lambda} : |p| \leq \frac{|z|}{\lambda_{\psi}}} |p|^2 & \text{if } \lambda_{\psi} \neq 0, +\infty \text{ and } \psi'(N) = \psi(N)^2. \end{cases} \tag{18}$$

The convergence

$$\frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_{\Lambda}, \psi} \xrightarrow{*} g_{\mathcal{L}_{\Lambda}, \psi} \operatorname{Leb}_{\mathbb{C}}, \tag{19}$$

is uniform on every compact subset of  $\mathbb{Z}$ -grids  $\Lambda$  in the space  $\operatorname{Grid}_2$ .

Furthermore, if  $\lambda_{\psi} \neq 0, +\infty$ , for all  $A \geq 1$  and  $f \in C_c^1(\mathbb{C})$  with support contained in  $B(0, A)$ , we have

$$\begin{aligned} \frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_{\Lambda}, \psi}(f_N) &= \int_{z \in \mathbb{C}} f(z) g_{\mathcal{L}_{\Lambda}, \psi}(z) d \operatorname{Leb}_{\mathbb{C}}(z) \\ &+ O \left( \frac{A^5 \|f\|_{\infty} \left| \lambda_{\psi} - \frac{\psi(N)}{N} \right|}{\lambda_{\psi}^9 \operatorname{Sys}_{\bar{\Lambda}}^4 \operatorname{covol}_{\bar{\Lambda}}^2} + \frac{A^4 \operatorname{diam}_{\bar{\Lambda}} \|df\|_{\infty}}{\lambda_{\psi}^4 \operatorname{covol}_{\bar{\Lambda}}^2 \operatorname{Sys}_{\bar{\Lambda}} \psi(N)} \right. \\ &\quad \left. + \frac{A^2 (\operatorname{diam}_{\bar{\Lambda}} + \frac{A}{\lambda_{\psi}}) \|f\|_{\infty}}{\lambda_{\psi}^3 \operatorname{covol}_{\bar{\Lambda}}^2 \psi(N)} \right). \end{aligned}$$

Note that the pair correlation function  $g_{\mathcal{L}_{\Lambda}, \psi}$  depends on  $\bar{\Lambda}$  but is independent of  $a$ . The above result shows in particular that renormalizing to probability measures (taking  $\psi'(N) \sim \frac{\pi^2 N^4}{\operatorname{covol}_{\bar{\Lambda}}^2}$  by Equation (7) with  $k = 0$ ) is inappropriate, as the limiting measure would always be 0. We will see during the proof that the above result implies the cases  $\alpha > 0$  of Theorem 1.1 in the introduction.

The fact that  $g_{\mathcal{L}_{\Lambda}, \psi}$  vanishes when  $\lambda_{\psi} = +\infty$  means that the sequence of measures  $\left( \frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_{\Lambda}, \psi} \right)_{N \in \mathbb{N} \setminus \{0\}}$  on  $(E_N)_{N \in \mathbb{N} \setminus \{0\}}$  has a total loss of mass at infinity. For error terms when  $\lambda_{\psi} = +\infty$  and  $\lambda_{\psi} = 0$ , see respectively Equation (37) and Equation (40).

*Proof* Let  $\Lambda = a + \bar{\Lambda}$  be a  $\mathbb{Z}$ -grid in  $\mathbb{C}$ . We may assume that  $a \in \mathcal{F}_{\bar{\Lambda}}$ . Let  $N \in \mathbb{N} \setminus \{0\}$ . Let

$$E_N^{\pm} = (\pm[0, \infty[ + i\mathbb{R}) / (2\pi i \psi(N) \mathbb{Z})$$

(which contains the base point 0) so that  $E_N = E_N^- \cup E_N^+$ . Note that the sequence  $(E_N^{\pm})_{N \in \mathbb{N} \setminus \{0\}}$  converges for the pointed Hausdorff-Gromov convergence to the closed halfplane  $\mathbb{C}^{\pm} = \pm[0, \infty[ + i\mathbb{R}$  and that  $\mathbb{C}^- \cap \mathbb{C}^+$  has measure 0 for any measure absolutely continuous with respect to the Lebesgue measure on  $\mathbb{C}$ . Note that if  $f \in C_c^1(\mathbb{C}^{\pm})$ , then for  $N$  large enough, we have  $f_N \in C_c^1(E_N^{\pm})$ , with the above notation.

Let  $\operatorname{sg}_N : E_N \rightarrow E_N$  be the change of sign map  $z' \mapsto -z'$ , which maps  $E_N^-$  to  $E_N^+$  and converges to the change of sign map  $\operatorname{sg} : z \mapsto -z$  on  $\mathbb{C}$ . The change of variables  $(m, n) \mapsto (n, m)$  in the index set  $I_N$  proves that we have  $\mathcal{R}_N^{\mathcal{L}_{\Lambda}, \psi} |_{E_N^-} = (\operatorname{sg}_N)_* \left( \mathcal{R}_N^{\mathcal{L}_{\Lambda}, \psi} |_{E_N^+} \right)$ .

We will thus only study the convergence of the measures  $\frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_\Lambda, \psi}$  on  $E_N^+$ , and deduce the global result by the symmetry of  $g_{\mathcal{L}_\Lambda, \psi}$  under sg.

For every  $p \in \bar{\Lambda} \setminus \{0\}$ , let

$$J_{p,N} = \{q \in \Lambda : 0 < |q| \leq |p + q| \leq N\}, \tag{20}$$

and let

$$\omega_{p,N} = \sum_{q \in J_{p,N}} \Delta_{\psi(N)\frac{p}{q}} \text{ and } \mu_N^+ = \sum_{p \in \bar{\Lambda} \setminus \{0\}} \omega_{p,N}. \tag{21}$$

Note that  $\omega_{p,N}$  is a measure on  $\mathbb{C}$  with finite support, which vanishes if  $|p| > 2N$  by the triangle inequality, hence  $\mu_N^+$  is also a measure on  $\mathbb{C}$  with finite support.

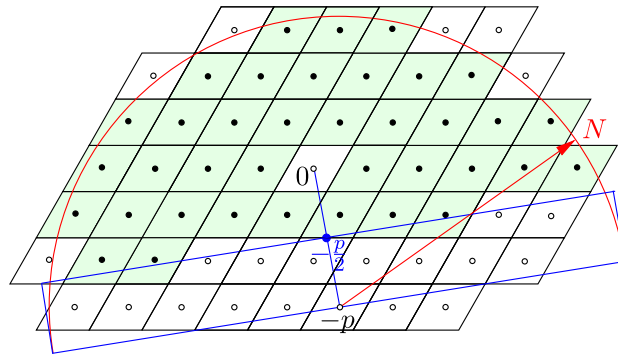
**Lemma 3.2** *As  $N \geq \text{diam}_{\bar{\Lambda}}$  tends to  $+\infty$ , we have*

$$\|\omega_{p,N}\| = \text{Card } J_{p,N} = \frac{\pi N^2}{2 \text{covol}_{\bar{\Lambda}}} + O\left(\frac{(|p| + \text{diam}_{\bar{\Lambda}})N}{\text{covol}_{\bar{\Lambda}}}\right).$$

*Proof* We may assume that  $|p| \leq 2N$ . Note that  $J_{p,N}$  is the finite set of nonzero grid points in the intersection

$$\tilde{C}_{p,N} = \{z \in \mathbb{C} : |z| \leq |p + z| \leq N\} \tag{22}$$

of the disc  $B(-p, N)$  of radius  $N$  centered at  $-p$  with the closed halfplane containing 0 with boundary the perpendicular bisector of 0 and  $-p$  (see the picture below).



Since  $\tilde{C}_{p,N}$  is contained in a halfdisc of radius  $N$  and contains the complement in this halfdisc of its intersection with a rectangle of length  $2N$  and height  $\frac{|p|}{2}$ , we have  $\frac{\pi}{2}N^2 - |p|N \leq \text{Area}(\tilde{C}_{p,N}) \leq \frac{\pi}{2}N^2$ , so that

$$\text{Area}(\tilde{C}_{p,N}) = \frac{\pi}{2}N^2 + O(|p|N).$$

Let

$$C_{p,N} = \bigcup_{q \in J_{p,N}} (q + \mathcal{F}_{\bar{\Lambda}}). \tag{23}$$

By a Gauss counting argument similar to the one in the proof of Equation (7) with  $k = 0$ , we have

$$\|\omega_{p,N}\| = \text{Card } J_{p,N} = \frac{\text{Area}(C_{p,N})}{\text{covol}_{\bar{\Lambda}}} = \frac{\text{Area}(\tilde{C}_{p,N})}{\text{covol}_{\bar{\Lambda}}} + \frac{\text{Area}(C_{p,N}) - \text{Area}(\tilde{C}_{p,N})}{\text{covol}_{\bar{\Lambda}}}$$



$$= \frac{\pi N^2}{2 \operatorname{covol}_{\vec{\Lambda}}} + O\left(\frac{(|p| + \operatorname{diam}_{\vec{\Lambda}})N + \operatorname{diam}_{\vec{\Lambda}}^2}{\operatorname{covol}_{\vec{\Lambda}}}\right).$$

The lemma follows. □

**Lemma 3.3** *For every  $A > 0$  and for every  $f \in C_c^1(\mathbb{C}^+)$  with support contained in  $B(0, A)$ , as  $N \rightarrow +\infty$  and uniformly on  $\Lambda$  varying in a compact subset of  $\operatorname{Grid}_2$ , we have*

$$\left| (\mathcal{R}_N^{\mathcal{L}_{\Lambda, \psi}})_{|E_N^+}(f_N) - \mu_N^+(f) \right| = O\left(\frac{A^4 \|df\|_{\infty} N^4}{\operatorname{covol}_{\vec{\Lambda}}^2 \psi(N)^3}\right).$$

*Proof* Let  $A$  and  $f$  be as in the statement of this lemma. Note that since  $\psi(N) > 0$  and by Equation (5), for every  $(m, n) \in I_N$ , we have  $(m, n) \in I_N^+$ , that is  $|n| \leq |m|$ , if and only if  $\psi(N) \log m - \psi(N) \log n \in E_N^+$ . Hence by the change of variable

$$(p, q) \mapsto (m = p + q, n = q)$$

(which is a bijection from  $\vec{\Lambda} \times \Lambda$  to  $\Lambda \times \Lambda$ ), we have

$$\begin{aligned} (\mathcal{R}_N^{\mathcal{L}_{\Lambda, \psi}})_{|E_N^+}(f_N) &= \sum_{(m, n) \in I_N^+} f_N(\psi(N) \log m - \psi(N) \log n) \\ &= \sum_{\substack{p \in \vec{\Lambda} \setminus \{0\}, q \in \Lambda \\ 0 < |q| \leq |p+q| \leq N}} f_N(\psi(N) \log(p+q) - \psi(N) \log q). \end{aligned}$$

By the assumption on the support of  $f$ , if an index  $(p, q)$  contributes to the above sum, then  $\operatorname{Re}(\psi(N) \log(p+q) - \psi(N) \log q) \leq A$ . Hence by Equations (17) and (5), we have  $\ln \left| 1 + \frac{p}{q} \right| \leq \frac{A}{\psi(N)}$ , which tends to 0 as  $N \rightarrow +\infty$ , since  $\lim_{+\infty} \psi = +\infty$ . In particular, using the assumption on  $q$ , we have

$$\frac{|p|}{|q|} = O\left(\frac{A}{\psi(N)}\right) \quad \text{and} \quad |p| = O\left(\frac{AN}{\psi(N)}\right), \tag{24}$$

so that  $\left| \frac{p}{q} \right| < 1$  if  $N$  is large enough. This allows to use the principal branch, again denoted by  $\log$ , of the complex logarithm in the open ball of center 1 and radius 1. By the analytic expansion of this branch, we have

$$\left| \log\left(1 + \frac{p}{q}\right) - \frac{p}{q} \right| = O\left(\left|\frac{p}{q}\right|^2\right) = O\left(\frac{A^2}{\psi(N)^2}\right).$$

The mean value theorem hence implies that

$$\left| \log\left(1 + \frac{p}{q}\right) - \frac{p}{q} \right| = O\left(\left|\frac{p}{q}\right|^2\right) = O\left(\frac{A^2}{\psi(N)^2}\right). \tag{25}$$

By Lemma 3.2 and Equation (7) with  $k = 0$ , we have

$$\begin{aligned} &\operatorname{Card} \left\{ (p, q) \in \vec{\Lambda} \times \Lambda : 0 < |q| \leq |p+q| \leq N, |p| = O\left(\frac{AN}{\psi(N)}\right) \right\} \\ &= \sum_{p \in \vec{\Lambda} \setminus \{0\} : |p| = O\left(\frac{AN}{\psi(N)}\right)} \operatorname{Card} J_{p, N} \end{aligned}$$

$$\begin{aligned}
 &= O\left(\text{Card}\left\{p \in \bar{\Lambda} \setminus \{0\} : |p| = O\left(\frac{AN}{\psi(N)}\right)\right\} \frac{N^2}{\text{covol}_{\bar{\Lambda}}}\right) \\
 &= O\left(\frac{A^2N^4}{\psi(N)^2 \text{covol}_{\bar{\Lambda}}^2}\right). \tag{26}
 \end{aligned}$$

Similarly, if an index  $(p, q)$  contributes to the sum

$$\mu_N^+(f) = \sum_{\substack{p \in \bar{\Lambda} \setminus \{0\}, q \in \Lambda \\ 0 < |q| \leq |p+q| \leq N}} f\left(\psi(N)\frac{p}{q}\right),$$

then Equation (24) holds. By summing Equation (25) on the set of elements  $(p, q) \in \bar{\Lambda} \times \Lambda$  such that  $0 < |q| \leq |p + q| \leq N$  and  $|p| = O\left(\frac{AN}{\psi(N)}\right)$ , and by using Equation (26), Lemma 3.3 follows.  $\square$

Let us now study the convergence properties (after renormalization) of the measures  $\omega_{p,N}$  and of their sums  $\mu_N^+$  as  $N \rightarrow +\infty$ . We assume in what follows that  $|p| < N$  (which is possible if  $N$  is large enough since we will have  $|p| = O\left(\frac{AN}{\psi(N)}\right)$ ). Let  $\iota : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$  be the involutive diffeomorphism  $z \mapsto \frac{1}{z}$ , which maps  $\mathbb{C}^+ \setminus \{0\}$  to  $\mathbb{C}^+ \setminus \{0\}$ , whose holomorphic derivative at  $z$  is  $-\frac{1}{z^2}$ , hence whose Jacobian at  $z$  is

$$J\iota(z) = \frac{1}{|z|^4}. \tag{27}$$

By the equation on the left in Formula (21), we have

$$\iota_*\omega_{p,N} = \sum_{q \in J_{p,N}} \Delta_{\frac{q}{\psi(N)p}}. \tag{28}$$

When  $q$  varies in  $J_{p,N}$ , as seen in the proof of Lemma 3.2, the above Dirac masses are exactly at the nonzero points of the  $\mathbb{Z}$ -grid  $\Lambda_{p,N} = \frac{1}{\psi(N)p} \Lambda$  that belong to the set

$$\tilde{Y}_{p,N} = \frac{1}{\psi(N)p} \tilde{C}_{p,N}.$$

Note that

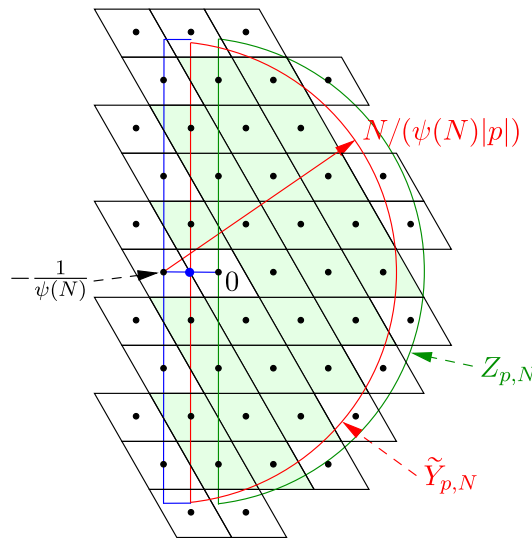
$$\text{covol}_{\bar{\Lambda}_{p,N}} = \text{Area}\left(\frac{\mathcal{F}_{\bar{\Lambda}}}{\psi(N)p}\right) = \frac{\text{covol}_{\bar{\Lambda}}}{\psi(N)^2 |p|^2}. \tag{29}$$

By Equation (22), the set  $\tilde{Y}_{p,N}$  is the intersection of the disc  $B\left(-\frac{1}{\psi(N)}, \frac{N}{\psi(N)|p|}\right)$  with the closed halfplane containing 0 with boundary the perpendicular bisector of 0 and  $-\frac{1}{\psi(N)}$ . Let us define

$$Z_{p,N} = \left\{z \in \mathbb{C} : \text{Re } z \geq 0, |z| \leq \frac{N}{\psi(N)|p|}\right\}.$$

Note that

$$\iota(Z_{p,N}) = \left\{z \in \mathbb{C} : \text{Re } z \geq 0, |z| \geq \frac{\psi(N)|p|}{N}\right\}. \tag{30}$$



The symmetric difference of  $\tilde{Y}_{p,N}$  and  $Z_{p,N}$ , that we denote by  $\tilde{YZ}_{p,N}$ , is contained in the union of the rectangle  $[-\frac{1}{2\psi(N)}, 0] \times [-\frac{N}{\psi(N)|p|}, \frac{N}{\psi(N)|p|}]$  and the half-annulus

$$\left\{ z \in \mathbb{C} : \operatorname{Re} z \geq 0, \frac{N}{\psi(N)|p|} - \frac{1}{\psi(N)} \leq |z| \leq \frac{N}{\psi(N)|p|} \right\}$$

(well defined since  $|p| < N$ ). In particular, its area satisfies  $\operatorname{Leb}_{\mathbb{C}}(\tilde{YZ}_{p,N}) = O\left(\frac{N}{\psi(N)^2|p|}\right)$ . Let

$$Y_{p,N} = \frac{1}{\psi(N)p} C_{p,N}, \tag{31}$$

so that, as in the proof of Lemma 3.2, the symmetric difference of  $Y_{p,N}$  and  $\tilde{Y}_{p,N}$  has area  $O\left(\frac{N \operatorname{diam}_{\bar{\Delta}}}{\psi(N)^2|p|^2}\right)$  as  $N \rightarrow +\infty$ . The symmetric difference of  $Y_{p,N}$  and  $Z_{p,N}$ , that we denote by  $YZ_{p,N}$ , hence has area  $\operatorname{Leb}_{\mathbb{C}}(YZ_{p,N}) = O\left(\frac{N(\operatorname{diam}_{\bar{\Delta}} + |p|)}{\psi(N)^2|p|^2}\right)$  as  $N \rightarrow +\infty$ . In particular, for every  $\phi \in C_c^1(\mathbb{C}^+ \setminus \{0\})$ , since  $Z_{p,N} \subset B(0, \frac{N}{\psi(N)|p|})$  and  $Y_{p,N} \subset B(0, \frac{N + \operatorname{diam}_{\bar{\Delta}}}{\psi(N)|p|})$ , we have

$$\begin{aligned} & \left| \int_{Z_{p,N}} \phi(z) d \operatorname{Leb}_{\mathbb{C}}(z) - \int_{Y_{p,N}} \phi(z) d \operatorname{Leb}_{\mathbb{C}}(z) \right| \\ &= O\left( \frac{N(\operatorname{diam}_{\bar{\Delta}} + |p|) \|\phi\|_{B(0, \frac{N + \operatorname{diam}_{\bar{\Delta}}}{\psi(N)|p|})}}{\psi(N)^2|p|^2} \right). \end{aligned} \tag{32}$$

By Equations (31), (23), (28) and (29), by the mean value theorem and by Lemma 3.2, as  $N \rightarrow +\infty$ , we have

$$\begin{aligned} & \left| \int_{Y_{p,N}} \phi(z) d \operatorname{Leb}_{\mathbb{C}}(z) - \frac{\operatorname{covol}_{\bar{\Delta}}}{\psi(N)^2|p|^2} \iota_* \omega_{p,N}(\phi) \right| \\ &= \left| \sum_{q \in J_{p,N}} \int_{\frac{q + \mathcal{F}_{\bar{\Delta}}}{\psi(N)p}} \left( \phi(z) - \phi\left(\frac{q}{\psi(N)p}\right) \right) d \operatorname{Leb}_{\mathbb{C}}(z) \right| \\ &\leq (\operatorname{Card} J_{p,N}) \frac{\operatorname{covol}_{\bar{\Delta}}}{\psi(N)^2|p|^2} \|\mathbf{d}\phi\|_{B(0, \frac{N + \operatorname{diam}_{\bar{\Delta}}}{\psi(N)|p|})} \frac{\operatorname{diam}_{\bar{\Delta}}}{\psi(N)|p|} \end{aligned}$$

$$= O \left( \frac{\text{diam}_{\bar{\Lambda}} \left\| d\phi_{|B(0, \frac{N+\text{diam}_{\bar{\Lambda}}}{\psi(N)|p|})} \right\|_{\infty} N^2}{\psi(N)^3 |p|^3} \right).$$

Hence by Equation (32), we have

$$\begin{aligned} \iota_* \omega_{p,N}(\phi) &= \frac{\psi(N)^2 |p|^2}{\text{covol}_{\bar{\Lambda}}} \int_{Z_{p,N}} \phi(z) d \text{Leb}_{\mathbb{C}}(z) \\ &+ O \left( \frac{\text{diam}_{\bar{\Lambda}} \left\| d\phi_{|B(0, \frac{N+\text{diam}_{\bar{\Lambda}}}{\psi(N)|p|})} \right\|_{\infty} N^2}{\text{covol}_{\bar{\Lambda}} \psi(N) |p|} + \frac{(\text{diam}_{\bar{\Lambda}} + |p|) \left\| \phi_{|B(0, \frac{N+\text{diam}_{\bar{\Lambda}}}{\psi(N)|p|})} \right\|_{\infty} N}{\text{covol}_{\bar{\Lambda}}} \right). \end{aligned} \tag{33}$$

Let  $f \in C_c^1(\mathbb{C}^+ \setminus \{0\})$  with support contained in  $B(0, A)$ . Note that  $f \circ \iota \in C_c^1(\mathbb{C}^+ \setminus \{0\})$ , that  $\left\| f \circ \iota_{\{|z| \leq \frac{N+\text{diam}_{\bar{\Lambda}}}{\psi(N)|p|}\}} \right\|_{\infty} = \left\| f_{\{|z| \geq \frac{\psi(N)|p|}{N+\text{diam}_{\bar{\Lambda}}}\}} \right\|_{\infty}$  and that

$$\left\| d(f \circ \iota)_{\{|z| \leq \frac{N+\text{diam}_{\bar{\Lambda}}}{\psi(N)|p|}\}} \right\|_{\infty} \leq A^2 \left\| df_{\{|z| \geq \frac{\psi(N)|p|}{N+\text{diam}_{\bar{\Lambda}}}\}} \right\|_{\infty}$$

since the support of  $f$  is contained in  $B(0, A)$ . The change of variable by  $\iota$  in the integral of Equation (33) applied with  $\phi = f \circ \iota$ , together with Equations (30) and (27), hence give

$$\begin{aligned} \omega_{p,N}(f) &= \frac{\psi(N)^2 |p|^2}{\text{covol}_{\bar{\Lambda}}} \int_{|z| \geq \frac{\psi(N)|p|}{N}} f(z) \frac{1}{|z|^4} d \text{Leb}_{\mathbb{C}}(z) \\ &+ O \left( \frac{A^2 \text{diam}_{\bar{\Lambda}} \left\| df_{\{|z| \geq \frac{\psi(N)|p|}{N+\text{diam}_{\bar{\Lambda}}}\}} \right\|_{\infty} N^2}{\text{covol}_{\bar{\Lambda}} \psi(N) |p|} + \frac{(\text{diam}_{\bar{\Lambda}} + |p|) \left\| f_{\{|z| \geq \frac{\psi(N)|p|}{N+\text{diam}_{\bar{\Lambda}}}\}} \right\|_{\infty} N}{\text{covol}_{\bar{\Lambda}}} \right). \end{aligned}$$

For every  $z \in \mathbb{C}^+ \setminus \{0\}$ , let

$$\theta_N(z) = \frac{1}{|z|^4} \sum_{p \in \bar{\Lambda} \setminus \{0\}} |p|^2 \mathbb{1}_{\{|z| \geq \frac{\psi(N)|p|}{N}\}}(z) = \frac{1}{|z|^4} \sum_{p \in \bar{\Lambda} \setminus \{0\} : |p| \leq \frac{N|z|}{\psi(N)}} |p|^2. \tag{34}$$

Note that if  $z$  and  $N$  are fixed, then for  $|p|$  large enough, we have  $|z| < \frac{\psi(N)|p|}{N}$ , thus the first sum above has only finitely many nonzero terms. Let  $\theta_N(0) = 0$ .

Note that  $\theta_N(z)$  vanishes if and only if  $|z| < \frac{\psi(N) \text{Sys}_{\bar{\Lambda}}}{N}$ , by the definition of the systole of  $\bar{\Lambda}$ .

As seen in the proof of Lemma 3.3, the only elements  $p \in \bar{\Lambda}$  that give a nonzero contribution to the sum  $\sum_{p \in \bar{\Lambda} \setminus \{0\}} \omega_{p,N}(f)$  satisfy  $p \neq 0$  and  $|p| = O\left(\frac{AN}{\psi(N)}\right)$ . By Equation (7) with  $k = 0$ , as  $N \rightarrow +\infty$ , we have

$$\text{Card}\{p \in \bar{\Lambda} \setminus \{0\} : |p| = O\left(\frac{AN}{\psi(N)}\right)\} = O\left(\frac{A^2 N^2}{\text{covol}_{\bar{\Lambda}} \psi(N)^2}\right)$$

if  $\lambda_{\psi} < +\infty$ . Otherwise, if  $\lambda_{\psi} = +\infty$ , we have  $O\left(\frac{AN}{\psi(N)}\right) \leq \text{Sys}_{\bar{\Lambda}}$  if  $N$  is large enough, hence if  $N$  is large enough, we have

$$\text{Card}\{p \in \bar{\Lambda} \setminus \{0\} : |p| = O\left(\frac{AN}{\psi(N)}\right)\} = 0. \tag{35}$$

Thus, by the right equality in Formula (21), we have

$$\begin{aligned} \mu_N^+(f) &= \sum_{p \in \bar{\Lambda} \setminus \{0\}} \omega_{p,N}(f) = \frac{\psi(N)^2}{\text{covol}_{\bar{\Lambda}}} \int_{z \in \mathbb{C}^+} f(z) \theta_N(z) d \text{Leb}_{\mathbb{C}}(z) \\ &+ O \left( \frac{A^4 \text{diam}_{\bar{\Lambda}} \left\| df_{\{|z| \geq \frac{\psi(N) \text{Sys}_{\bar{\Lambda}}}{N + \text{diam}_{\bar{\Lambda}}}\}} \right\|_{\infty} N^4}{\text{covol}_{\bar{\Lambda}}^2 \text{Sys}_{\bar{\Lambda}} \psi(N)^3} + \frac{(\text{diam}_{\bar{\Lambda}} + \frac{AN}{\psi(N)}) \left\| f_{\{|z| \geq \frac{\psi(N) \text{Sys}_{\bar{\Lambda}}}{N + \text{diam}_{\bar{\Lambda}}}\}} \right\|_{\infty} A^2 N^3}{\text{covol}_{\bar{\Lambda}}^2 \psi(N)^2} \right). \end{aligned} \tag{36}$$

**Case 1.** Let us first assume that  $\lambda_{\psi} = +\infty$ , that is,  $\lim_{N \rightarrow +\infty} \frac{N}{\psi(N)} = 0$ .

For every  $A \geq 1$ , if  $N$  is large enough (uniformly on  $\Lambda$  varying in a compact subspace of  $\text{Grid}_2$ , since then  $\bar{\Lambda}$  varies in a compact subspace of the space of  $\mathbb{Z}$ -lattices, on which the systole function  $\bar{\Lambda} \mapsto \text{Sys}_{\bar{\Lambda}}$  has a positive lower bound), then for every  $z \in B(0, A)$ , we have  $\theta_N(z) = 0$  by Equation (34), and  $\mu_N^+(f) = 0$  by Formulas (21) and (35), since the sum defining  $\mu_N^+(f)$  is an empty sum. Thus, whatever the (positive) normalizing function  $\psi'$  is, we have a total loss of mass at infinity :

$$\frac{1}{\psi'(N)} \mu_N^+ \xrightarrow{*} 0.$$

Assume that the renormalizing function  $\psi'$  is such that  $\frac{N^4}{\psi(N)^3 \psi'(N)}$  tends to 0 as  $N$  tends to  $\infty$ , for instance  $\psi' = \psi$ , as assumed in the first case of Equation (18). Note that if  $\psi(N) = N^\alpha$  with  $\alpha > 1$ , then we indeed have  $\lambda_{\psi} = +\infty$  and if  $\psi'(N) = N^{4-2\alpha}$  as in the statement of Theorem 1.1, we do have  $\lim_{N \rightarrow +\infty} \frac{N^4}{\psi(N)^3 \psi'(N)} = 0$ .

With Lemma 3.3, the above centered formula proves Formula (19) when  $\lambda_{\psi} = +\infty$ , with a convergence which is uniform on every compact subset of  $\Lambda$  in  $\text{Grid}_2$ , as well as the case  $\alpha > 1$  in Theorem 1.1. Furthermore, it follows from the error term in Lemma 3.3 that for every  $f \in C_c^1(\mathbb{C})$  with support contained in  $B(0, A)$ , as  $N \rightarrow +\infty$  and uniformly on  $\Lambda$  varying in a compact subset of  $\text{Grid}_2$ , we have

$$\frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_{\Lambda}, \psi}(f_N) = O \left( \frac{A^4 \|df\|_{\infty} N^4}{\text{covol}_{\bar{\Lambda}}^2 \psi(N)^3 \psi'(N)} \right). \tag{37}$$

**Case 2.** Let us now assume that  $\lambda_{\psi} = 0$ , that is,  $\lim_{N \rightarrow +\infty} \frac{\psi(N)}{N} = 0$ .

For all  $z \in \mathbb{C}^+ \setminus \{0\}$ , by Equations (34) and (7) for  $k = 2$ , we have

$$\begin{aligned} \frac{\psi(N)^4}{N^4} \theta_N(z) &= \left( \frac{\psi(N)}{N|z|} \right)^4 \sum_{p \in \bar{\Lambda}: |p| \leq \frac{N|z|}{\psi(N)}} |p|^2 \\ &= \frac{\pi}{2 \text{covol}_{\bar{\Lambda}}} + O \left( \frac{(1 + \text{diam}_{\bar{\Lambda}}) \psi(N)}{\text{covol}_{\bar{\Lambda}} N|z|} + \frac{\text{diam}_{\bar{\Lambda}}^2 \psi(N)^2}{\text{covol}_{\bar{\Lambda}} N^2 |z|^2} \right). \end{aligned} \tag{38}$$

In particular, if  $|z| \geq \frac{\psi(N) \text{Sys}_{\bar{\Lambda}}}{N}$ , then  $\frac{\psi(N)^4}{N^4} \theta_N(z)$  is uniformly bounded. Since  $\theta_N(z)$  vanishes if  $|z| < \frac{\psi(N) \text{Sys}_{\bar{\Lambda}}}{N}$ , this proves that the function  $\frac{\psi(N)^4}{N^4} \theta_N$  is uniformly bounded on  $\mathbb{C}^+ \setminus \{0\}$ , and pointwise converges to the constant function  $\frac{\pi}{2 \text{covol}_{\bar{\Lambda}}}$ . Hence by Equation

(36) and by the Lebesgue dominated convergence theorem, we have, with a convergence which is uniform on every compact subset of  $\Lambda$  in  $\text{Grid}_2$ ,

$$\frac{\psi(N)^2}{N^4} \mu_N^+ \xrightarrow{*} \frac{\pi}{2 \text{covol}_\Lambda^2} \text{Leb}_{\mathbb{C}^+}. \tag{39}$$

More precisely, for every  $A \geq 1$ , for every  $f \in C_c^1(\mathbb{C}^+ \setminus \{0\})$  with support in  $B(0, A)$ , and for every  $\Lambda$  in a compact subset of  $\text{Grid}_2$ , we have the following control. At each point  $z \in \mathbb{C}^+$  where  $\theta_N$  does not vanish, the second error term in Equation (38) is at most the first one, as it satisfies

$$\frac{\text{diam}_\Lambda^2 \psi(N)^2}{\text{covol}_\Lambda N^2 |z|^2} \leq \frac{\text{diam}_\Lambda^2 \psi(N)^2}{\text{covol}_\Lambda \text{Sys}_\Lambda N^2 |z|} \leq \frac{(1 + \text{diam}_\Lambda) \psi(N)}{\text{covol}_\Lambda N |z|}$$

for  $N$  large enough since  $\frac{\psi(N)}{N}$  tends to 0. By Equations (36) and (38), and since  $\psi(N) \leq N$  for  $N$  large enough, using the equality  $\int_{-\pi/2}^{\pi/2} \int_0^A \frac{1}{\rho} \rho d\rho d\theta = \pi A$  in order to integrate the first error term in Equation (38), we have

$$\begin{aligned} \frac{\psi(N)^2}{N^4} \mu_N^+(f) &= \frac{\psi(N)^4}{\text{covol}_\Lambda N^4} \int_{z \in \mathbb{C}^+} f(z) \theta_N(z) d \text{Leb}_{\mathbb{C}}(z) \\ &\quad + O\left( \frac{A^4 \text{diam}_\Lambda \|df\|_\infty}{\text{covol}_\Lambda^2 \text{Sys}_\Lambda \psi(N)} + \frac{A^2(\text{diam}_\Lambda + A) \|f\|_\infty}{\text{covol}_\Lambda^2 \psi(N)} \right) \\ &= \frac{\pi}{2 \text{covol}_\Lambda^2} \int_{z \in \mathbb{C}^+} f(z) d \text{Leb}_{\mathbb{C}}(z) + O\left( \frac{A^4 \text{diam}_\Lambda \|df\|_\infty}{\text{covol}_\Lambda^2 \text{Sys}_\Lambda \psi(N)} \right. \\ &\quad \left. + \frac{A^2(\text{diam}_\Lambda + A) \|f\|_\infty}{\text{covol}_\Lambda^2 \psi(N)} + \frac{A(1 + \text{diam}_\Lambda) \psi(N) \|f\|_\infty}{\text{covol}_\Lambda^2 N} \right). \end{aligned}$$

If  $\psi'(N) = \frac{N^4}{\psi(N)^2}$  as assumed in the second case of Equation (18), it follows from Formula (39) and Lemma 3.3 by symmetry that

$$\frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_\Lambda, \psi} \xrightarrow{*} \frac{\pi}{2 \text{covol}_\Lambda^2} \text{Leb}_{\mathbb{C}}.$$

This proves Formula (19) when  $\lambda_\psi = 0$ , with a convergence which is uniform on every compact subset of  $\Lambda$  in  $\text{Grid}_2$ , as well as the case  $0 < \alpha < 1$  in Theorem 1.1. Furthermore, for every  $f \in C_c^1(\mathbb{C})$  with support contained in  $B(0, A)$ , as  $N \rightarrow +\infty$  and uniformly on  $\Lambda$  varying in a compact subset of  $\text{Grid}_2$ , using the error term in Lemma 3.3 with the fact that  $\text{Sys}_\Lambda \leq \text{diam}_\Lambda$ , we have

$$\begin{aligned} \frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_\Lambda, \psi}(f_N) &= \frac{\pi}{2 \text{covol}_\Lambda^2} \int_{\mathbb{C}} f d \text{Leb}_{\mathbb{C}} + O\left( \frac{A^4 \text{diam}_\Lambda \|df\|_\infty}{\text{covol}_\Lambda^2 \text{Sys}_\Lambda \psi(N)} \right. \\ &\quad \left. + \frac{A^2(\text{diam}_\Lambda + A) \|f\|_\infty}{\text{covol}_\Lambda^2 \psi(N)} + \frac{A(1 + \text{diam}_\Lambda) \psi(N) \|f\|_\infty}{\text{covol}_\Lambda^2 N} \right) \tag{40} \end{aligned}$$

**Case 3.** Let us finally assume that  $\lim_{N \rightarrow +\infty} \frac{\psi(N)}{N} = \lambda_\psi$  belongs to  $]0, +\infty[$ .

We consider the function  $\theta_\infty : \mathbb{C} \rightarrow [0, +\infty[$  defined by

$$z \mapsto \frac{1}{|z|^4} \sum_{p \in \bar{\Lambda}: |p| \leq \frac{|z|}{\lambda_\psi}} |p|^2,$$

where by convention  $\theta_\infty(0) = 0$ , and replacing  $p \in \bar{\Lambda}$  by  $p \in \bar{\Lambda} \setminus \{0\}$  makes no difference. Note that  $\theta_\infty$  vanishes on the open disc  $B(0, \lambda_\psi \text{Sys}_{\bar{\Lambda}})$ , is uniformly bounded and tends to  $\frac{\pi}{2 \text{covol}_{\bar{\Lambda}} \lambda_\psi^4}$  as  $|z| \rightarrow +\infty$  by Equation (7) with  $k = 2$ . Furthermore,  $\theta_\infty$  is piecewise continuous, with discontinuities along each circle  $S(0, |p|)$  centered at 0 passing through a nonzero lattice point  $p \in \bar{\Lambda}$ . See the picture in the introduction representing the graph of  $\theta_\infty$  when  $\Lambda = \bar{\Lambda} = \mathbb{Z}[i]$  (so that  $\text{covol}_{\bar{\Lambda}} = 1$ ) and  $\lambda_\psi = 1$ .

By Equation (34), the sequence of uniformly bounded maps  $(\theta_N)_{N \in \mathbb{N}}$  converges almost everywhere to  $\theta_\infty$  (more precisely, it converges at least outside  $\bigcup_{p \in \bar{\Lambda} \setminus \{0\}} S(0, |p|)$ ). Hence by Equation (36) and by the Lebesgue dominated convergence theorem, we have

$$\frac{1}{\psi(N)^2} \mu_N^+ \xrightarrow{*} \frac{1}{\text{covol}_{\bar{\Lambda}}} \theta_\infty \text{Leb}_{\mathbb{C}^+}. \tag{41}$$

Let  $A \geq 1$ . Note that  $|z| \leq A$  implies that  $\frac{|z|}{\lambda_\psi} \leq \frac{A}{\lambda_\psi} \leq \frac{2A}{\lambda_\psi}$ . If  $N$  is large enough so that  $\frac{\psi(N)}{N} \geq \frac{\lambda_\psi}{2}$ , then  $|z| \leq A$  implies that  $\frac{N|z|}{\psi(N)} \leq \frac{2A}{\lambda_\psi}$ . Hence for every  $z \in \mathbb{C}^+ \cap B(0, A)$ , if  $N$  is large enough, we have

$$|\theta_\infty(z) - \theta_N(z)| \leq \frac{1}{|z|^4} \sum_{p \in \bar{\Lambda}: |p| \leq \frac{2A}{\lambda_\psi}} |p|^2 \left| \mathbb{1}_{|z| \geq \lambda_\psi |p|}(z) - \mathbb{1}_{|z| \geq \frac{\psi(N)|p|}{N}}(z) \right|.$$

Note that if  $N$  is large enough, the left term vanishes if  $|z| < \frac{\lambda_\psi}{2} \text{Sys}_{\bar{\Lambda}}$ .

Let  $f \in C_c^1(\mathbb{C}^+)$  with support in  $B(0, A)$ . By integration on annuli and Equation (7) with  $k = 3$ , we have

$$\begin{aligned} \left| \int_{\mathbb{C}^+} f(\theta_\infty - \theta_N) d\text{Leb}_{\mathbb{C}} \right| &= O \left( \frac{\|f\|_\infty}{(\lambda_\psi \text{Sys}_{\bar{\Lambda}})^4} \sum_{p \in \bar{\Lambda}: |p| \leq \frac{2A}{\lambda_\psi}} |p|^2 2\pi \left| \lambda_\psi |p| - \frac{\psi(N)|p|}{N} \right| \right) \\ &= O \left( \frac{A^5 \|f\|_\infty}{\lambda_\psi^9 \text{Sys}_{\bar{\Lambda}}^4 \text{covol}_{\bar{\Lambda}}} \left| \lambda_\psi - \frac{\psi(N)}{N} \right| \right). \end{aligned}$$

Hence by Equation (36), we have

$$\begin{aligned} \frac{1}{\psi(N)^2} \mu_N^+(f) &= \frac{1}{\text{covol}_{\bar{\Lambda}}} \int_{z \in \mathbb{C}^+} f(z) \theta_\infty(z) d\text{Leb}_{\mathbb{C}}(z) + O \left( \frac{A^5 \|f\|_\infty \left| \lambda_\psi - \frac{\psi(N)}{N} \right|}{\lambda_\psi^9 \text{Sys}_{\bar{\Lambda}}^4 \text{covol}_{\bar{\Lambda}}^2} \right) \\ &\quad + O \left( \frac{A^4 \text{diam}_{\bar{\Lambda}} \|df\|_\infty N^4}{\text{covol}_{\bar{\Lambda}}^2 \text{Sys}_{\bar{\Lambda}} \psi(N)^5} + \frac{A^2 (\text{diam}_{\bar{\Lambda}} + \frac{AN}{\psi(N)}) \|f\|_\infty N^3}{\text{covol}_{\bar{\Lambda}}^2 \psi(N)^4} \right) \\ &= \frac{1}{\text{covol}_{\bar{\Lambda}}} \int_{z \in \mathbb{C}^+} f(z) \theta_\infty(z) d\text{Leb}_{\mathbb{C}}(z) + O \left( \frac{A^5 \|f\|_\infty \left| \lambda_\psi - \frac{\psi(N)}{N} \right|}{\lambda_\psi^9 \text{Sys}_{\bar{\Lambda}}^4 \text{covol}_{\bar{\Lambda}}^2} \right) \\ &\quad + \frac{A^4 \text{diam}_{\bar{\Lambda}} \|df\|_\infty}{\lambda_\psi^4 \text{covol}_{\bar{\Lambda}}^2 \text{Sys}_{\bar{\Lambda}} \psi(N)} + \frac{A^2 (\text{diam}_{\bar{\Lambda}} + \frac{A}{\lambda_\psi}) \|f\|_\infty}{\lambda_\psi^3 \text{covol}_{\bar{\Lambda}}^2 \psi(N)}. \end{aligned} \tag{42}$$

If  $\psi'(N) = \psi(N)^2$  as assumed in the third case of Equation (18), it follows from Formula (41) and Lemma 3.3 by symmetry that

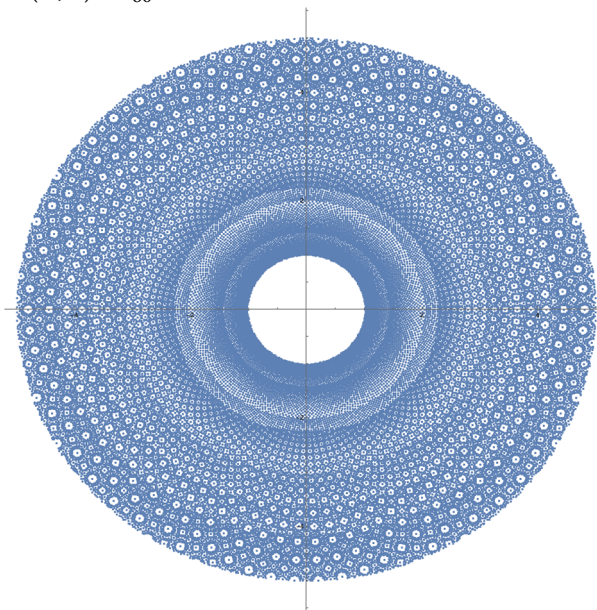
$$\frac{1}{\psi'(N)} \mathcal{R}_N^{\mathcal{L}_{\Lambda, \psi}} \xrightarrow{*} \frac{1}{\text{covol}_{\bar{\Lambda}}} \theta_\infty \text{Leb}_{\mathbb{C}}.$$

This proves Formula (19) when  $\lambda_\psi \neq 0, \infty$ , with a convergence which is uniform on every compact subset of  $\Lambda$  in  $\text{Grid}_2$ , as well as the case  $\alpha = 1$  in Theorem 1.1 (since if  $\psi(N) = N$ , then  $\lambda_\psi = 1$  and  $\psi'(N) = \psi(N)^2 = N^2 = N^{4-2\alpha}$ ). Furthermore, for every  $f \in C_c^1(\mathbb{C}^+)$  with support contained in  $B(0, A)$ , as  $N \rightarrow +\infty$  and uniformly on  $\Lambda$  varying in a compact subset of  $\text{Grid}_2$ , using Equation (42) and the error term in Lemma 3.3 with the fact that  $\text{Sys}_{\bar{\Lambda}} \leq \text{diam}_{\bar{\Lambda}}$ , we have

$$\begin{aligned} \frac{1}{\psi'(N)} (\mathcal{R}_N^{\mathcal{L}_{\Lambda, \psi}})_{|E_N^+}(f_N) &= \frac{1}{\psi(N)^2} \mu_N^+(f) + O\left(\frac{A^4 \|df\|_\infty N^4}{\text{covol}_{\bar{\Lambda}}^2 \psi(N)^5}\right) \\ &= \int_{z \in \mathbb{C}^+} f(z) \frac{\theta_\infty(z)}{\text{covol}_{\bar{\Lambda}}} d \text{Leb}_{\mathbb{C}}(z) \\ &\quad + O\left(\frac{A^5 \|f\|_\infty |\lambda_\psi - \frac{\psi(N)}{N}|}{\lambda_\psi^9 \text{Sys}_{\bar{\Lambda}}^4 \text{covol}_{\bar{\Lambda}}^2} + \frac{A^4 \text{diam}_{\bar{\Lambda}} \|df\|_\infty}{\lambda_\psi^4 \text{covol}_{\bar{\Lambda}}^2 \text{Sys}_{\bar{\Lambda}} \psi(N)} \right. \\ &\quad \left. + \frac{A^2(\text{diam}_{\bar{\Lambda}} + \frac{A}{\lambda_\psi}) \|f\|_\infty}{\lambda_\psi^3 \text{covol}_{\bar{\Lambda}}^2 \psi(N)}\right). \end{aligned}$$

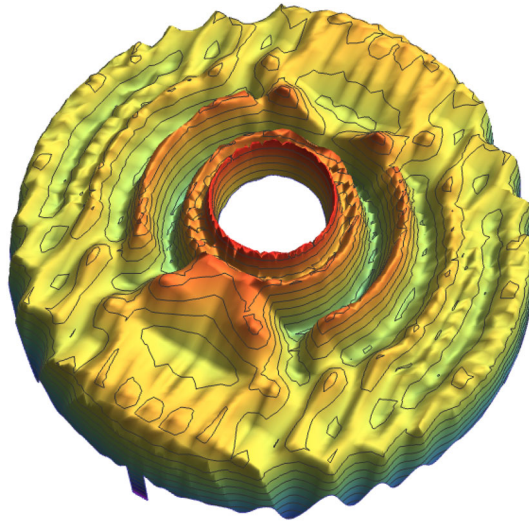
By symmetry, this concludes the proof of Theorem 3.1. □

Let us give a numerical illustration of Theorem 3.1 when  $\Lambda = \bar{\Lambda} = \mathbb{Z}[i]$  and  $\psi(N) = N$ . The following figure shows the points  $60 \log m - 60 \log n$  contained in the ball of radius 5 centered at 0 for  $(m, n) \in I_{60}$ .

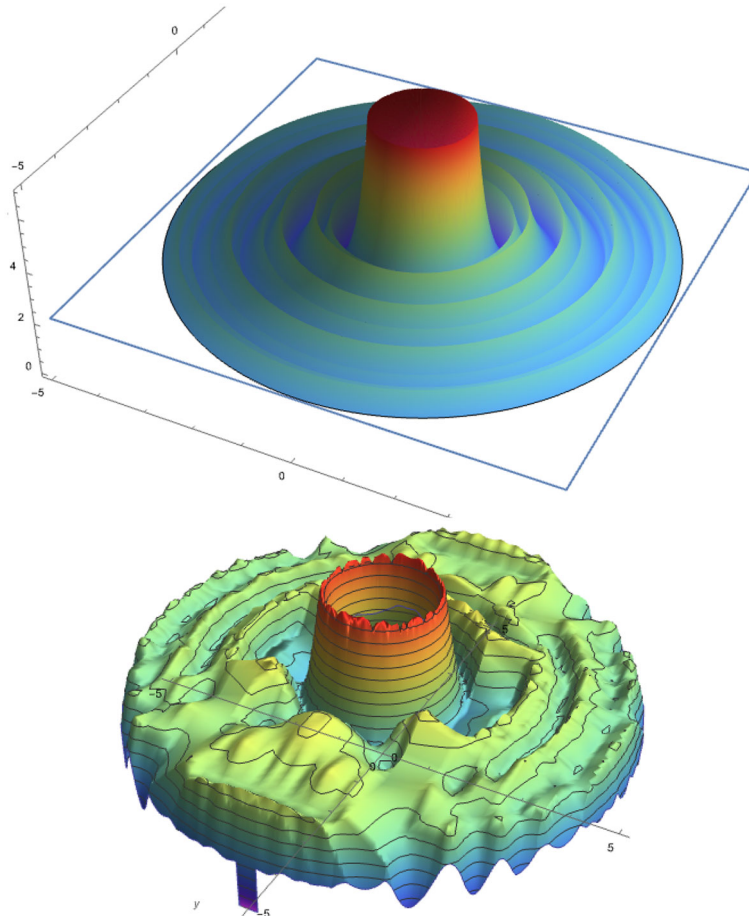


The second figure shows an approximation (given by Mathematica and its smoothing process) of the pair correlation function  $g_{\mathcal{L}_{\Lambda, \psi}}$  computed using the empirical measure  $\frac{1}{60^2} \mathcal{R}_{60}^{\mathcal{L}_{\Lambda, \psi}}$  in the ball of center 0 and radius 5. We refer to the first picture in the introduction for the actual graph of the pair correlation function  $g_{\mathcal{L}_{\Lambda, \psi}}$ .





The first figure below gives the graph of the pair correlation function  $g_{\mathcal{L}_\Lambda, \psi}$  of the  $\mathbb{Z}$ -lattice  $\Lambda = \bar{\Lambda} = \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$  of the Eisenstein integers at the linear scaling  $\psi : N \mapsto N$  in the ball of center 0 and radius 5. The blue lines on the bounding box represent the limit  $\frac{\pi}{2 \operatorname{covol}_\Lambda^2} = \frac{2\pi}{3}$  at  $+\infty$  of  $g_{\mathcal{L}_\Lambda, \psi}$ , given by Equation (7) with  $k = 2$ . The second figure shows the approximation of the pair correlation function computed with the empirical measure  $\frac{1}{60^2} \mathcal{R}_{60}^{\mathcal{L}_\Lambda, \psi}$



#### 4 Mertens and Mirsky formulae for algebraic number fields

In this short section, we recall the notation and statements of [24] that we will use in Sects. 5 and 6.

Let  $K$  be an imaginary quadratic number field (with  $D_K, \mathcal{O}_K, \zeta_K, \mathcal{I}_K^+, \mathbb{N}$  the notation introduced before Corollary 2.4). We assume in Sects. 4, 5 and 6 that  $\mathcal{O}_K$  is principal (or equivalently factorial (UFD)). This implies, see for instance [20], that  $D_K \in \{-4, -8, -3, -7, -11, -19, -43, -67, -163\}$ . For all  $I, J \in \mathcal{I}_K^+$ , we write  $J \mid I$  if  $I \subset J$ , we denote by  $(I, J) = I + J$  the greatest common ideal divisor of  $I$  and  $J$ , and by  $IJ$  the product ideal of  $I$  and  $J$ .

We denote by  $\varphi_K : \mathcal{I}_K^+ \rightarrow \mathbb{N}$  the *Euler function* of  $K$ , defined (see for instance [20, page 13]) equivalently by

$$\forall \mathfrak{a} \in \mathcal{I}_K^+, \quad \varphi_K(\mathfrak{a}) = \text{Card}((\mathcal{O}_K/\mathfrak{a})^\times) = \mathbb{N}(\mathfrak{a}) \prod_{\mathfrak{p} \mid \mathfrak{a}} \left(1 - \frac{1}{\mathbb{N}(\mathfrak{p})}\right),$$

where, here and thereafter,  $\mathfrak{p}$  ranges over the prime ideals of  $\mathcal{O}_K$ . For every  $a \in \mathcal{O}_K \setminus \{0\}$ , we define  $\varphi_K(a) = \varphi_K(a\mathcal{O}_K)$ .

We first give a version in angular sectors of the Mertens formula on the average of the Euler function that will be needed in the proof of Theorem 5.1. For all  $z \in \mathbb{C}^\times, \theta \in ]0, 2\pi]$  and  $R \geq 0$ , we consider the truncated angular sector

$$C(z, \theta, R) = \left\{ \rho e^{it} z : t \in \left] -\frac{\theta}{2}, \frac{\theta}{2} \right], 0 < \rho \leq \frac{R}{|z|} \right\}. \tag{43}$$

Note that for every  $z' \in \mathbb{C}^\times$ , we have

$$z' C(z, \theta, R) = C(zz', \theta, R|z'|). \tag{44}$$

It is important that the function  $O(\cdot)$  in the following result is uniform in  $\mathfrak{m}, z$  and  $\theta$ . For every  $\mathfrak{m} \in \mathcal{I}_K^+$ , let

$$c_{\mathfrak{m}} = \mathbb{N}(\mathfrak{m}) \prod_{\mathfrak{p} \mid \mathfrak{m}} \left(1 + \frac{1}{\mathbb{N}(\mathfrak{p})}\right).$$

**Theorem 4.1** (*A Sectorial Mertens formula*) For all  $\mathfrak{m} \in \mathcal{I}_K^+, z \in \mathbb{C}^\times$  and  $\theta \in ]0, 2\pi]$ , as  $x \rightarrow +\infty$ , we have

$$\sum_{a \in \mathfrak{m} \cap C(z, \theta, x)} \varphi_K(a) = \frac{\theta}{2\sqrt{|D_K|} \zeta_K(2) c_{\mathfrak{m}}} x^4 + O(x^3).$$

*Proof* See [24, Thm. 1.1]. □

We now give a uniform asymptotic formula for the sum in angular sectors in  $\mathbb{C}$  of the products of two shifted Euler functions with congruences, which is used in the proof of Theorems 5.1 and 6.1. When  $K = \mathbb{Q}$  (the sectorial restriction is then meaningless), this formula is due to Mirsky [17, Thm. 9, Eq. (30)] without congruences, and to Fouvry [22, Appendix] with congruences.

For all  $\mathfrak{m} \in \mathcal{I}_K^+, z \in \mathbb{C}^\times, \theta \in ]0, 2\pi], k \in \mathcal{O}_K$ , and  $x \geq 1$ , let

$$\left| S_{\mathfrak{m}, z, \theta, k}(x) = \sum_{a \in \mathfrak{m} \cap C(z, \theta, x) : a \neq -k} \varphi_K(a) \varphi_K(a + k) \right. \tag{45}$$

Let

$$c_{m,k} = \frac{1}{N(m)} \prod_{\substack{\mathfrak{p} \\ (\mathfrak{p},m) \mid k \mathcal{O}_K}} \left(1 - \frac{N((\mathfrak{p},m))}{N(\mathfrak{p})^2}\right) \prod_{\mathfrak{p}} \left(1 - \frac{\kappa_{m,k}(\mathfrak{p}) \kappa'_k(\mathfrak{p}) N((\mathfrak{p},m))}{N(\mathfrak{p})^2}\right), \tag{46}$$

where

$$\kappa_{m,k}(\mathfrak{p}) = \begin{cases} \left(1 - \frac{N((\mathfrak{p},m))}{N(\mathfrak{p})^2}\right)^{-1} & \text{if } (\mathfrak{p},m) \mid k \mathcal{O}_K \\ 1 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\kappa'_k(\mathfrak{p}) = \begin{cases} 1 - \frac{1}{N(\mathfrak{p})} & \text{if } \mathfrak{p} \mid k \mathcal{O}_K \\ 1 & \text{otherwise.} \end{cases}$$

For instance, if  $m = \mathcal{O}_K$  then by [24, Eq. (15)], we have

$$c_{\mathcal{O}_K,k} = \prod_{\mathfrak{p}} \left(1 - \frac{2}{N(\mathfrak{p})^2}\right) \prod_{\mathfrak{p} \mid k \mathcal{O}_K} \left(1 + \frac{1}{N(\mathfrak{p})(N(\mathfrak{p})^2 - 2)}\right). \tag{47}$$

Since it will be useful in Sect. 6, by [24, Lem. 4.2], we have

$$c'_m = \inf_{k \in \mathcal{O}_K} c_{m,k} > 0. \tag{48}$$

**Theorem 4.2** (A Sectorial Mirsky Formula) *There exists a constant  $C_K > 0$  such that for all  $m \in \mathcal{S}_K^+$ ,  $z \in \mathbb{C}^\times$ ,  $\theta \in ]0, 2\pi]$ ,  $k \in \mathcal{O}_K$  and  $x \geq 1$ , we have*

$$\left| S_{m,z,\theta,k}(x) - \frac{\theta c_{m,k}}{3\sqrt{|D_K|}} x^6 \right| \leq C_K \left( (1 + \sqrt{N(k)}) x^5 + N(k) x^4 + N(k) \ln(N(k)) x^2 \ln(2x) \right).$$

*Proof* See [24, Thm. 4.1 and Lem. 4.2]. □

### 5 Pair correlation of integral lattice points with Euler weight and no scaling

In this section, we fix an imaginary quadratic number field  $K$  whose ring of integers  $\mathcal{O}_K$  is principal. We fix a nonzero ideal  $\Lambda \in \mathcal{S}_K^+$ . Note that  $\Lambda = \bar{\Lambda}$  is a  $\mathbb{Z}$ -lattice (hence a  $\mathbb{Z}$ -grid) in  $\mathbb{C}$ , with  $\text{covol}_\Lambda = N(\Lambda) \frac{\sqrt{|D_K|}}{2}$  as seen in Equation (16). As in Sect. 2, we work on the constant cylinder  $E = \mathbb{C}/(2\pi i\mathbb{Z})$  in this section.

Recall that  $\mathcal{L}_\Lambda^{\varphi_K}$  is the family defined in Equation (2). For every  $N \in \mathbb{N} \setminus \{0\}$ , the (not normalised, empirical) pair correlation measure of the logarithms of nonzero elements in  $\Lambda$ , with trivial scaling function  $\Psi = 1$  and multiplicities given by the Euler function, is the measure on  $E$  with finite support defined, with  $I_N = I_{N,\Lambda}$  by

$$\tilde{\nu}_N = \mathcal{R}_N^{\mathcal{L}_\Lambda^{\varphi_K},1} = \sum_{(m,n) \in I_N} \varphi_K(m) \varphi_K(n) \Delta_{\log m - \log n}.$$

**Theorem 5.1** *As  $N \rightarrow +\infty$ , the measures  $\tilde{\nu}_N$  on  $E$ , renormalised to be probability measures, weak-star converge to the measure absolutely continuous with respect to the Lebesgue measure on  $E$ , with Radon-Nikodym derivative the function  $g_{\mathcal{L}_\Lambda^{\varphi_K},1} : z' \mapsto \frac{1}{\pi} e^{-4|\text{Re } z'|}$ , which is independent of  $\Lambda$  and  $K$ :*

$$\frac{\tilde{\nu}_N}{\|\tilde{\nu}_N\|} \xrightarrow{*} g_{\mathcal{L}_\Lambda^{\varphi_K},1} \text{Leb}_E.$$

Furthermore, for all  $f \in C_c^1(E)$  and  $\alpha \in ]0, \frac{1}{2}[$ , with  $c_\Lambda = N(\Lambda) \prod_{p|\Lambda} (1 + \frac{1}{N(p)})$ , we have

$$\frac{\tilde{v}_N}{\|\tilde{v}_N\|}(f) = \int_{z' \in E} \frac{1}{\pi} e^{-4|\operatorname{Re} z'|} f(z') d \operatorname{Leb}_E(z') + O\left(\frac{c_\Lambda \|f\|_\infty}{N^{1-2\alpha}} + \frac{\|e^{-z'} df(z')\|_\infty}{N^\alpha}\right).$$

This result gives the first assertion of Theorem 1.2 in the introduction. As in Remark 2.3, Theorem 5.1 remains valid if we allow  $n = m$  in the definition of the index set  $I_N$ , and we will use this remark in the proof of Corollary 7.2.

*Proof* In this proof, all functions  $O(\cdot)$  are absolute, since there are finitely many fields  $K$  satisfying the assumptions of this section. The first assertion of Theorem 5.1 follows from the second one, by the density of  $C_c^1(E)$  in  $C_c^0(E)$  for the uniform convergence.

For all  $N \in \mathbb{N}$  and  $q \in \Lambda$  with  $0 < |q| \leq N$ , let  $J_q$  be given by the equation on the left in Formula (9). We now define

$$\tilde{\omega}_q = \sum_{p \in J_q} \varphi_K(p) \Delta_{\frac{p}{q}},$$

which is a finitely supported measure on the closed unit disc  $\mathbb{D} = B(0, 1)$  of  $\mathbb{C}$ , and is nonzero since  $-q \in J_q$ .

**Lemma 5.2** *As  $|q| \rightarrow +\infty$ , we have  $\|\tilde{\omega}_q\| = \frac{\pi}{\sqrt{|D_K|} \zeta_K(2) c_\Lambda} |q|^4 + O(|q|^3)$ .*

*Proof* This follows from Theorem 4.1 applied with  $m = \Lambda, z = 1, \theta = 2\pi$  and  $x = |q|$ , since  $\varphi_K(q) = O(N(q))$  and

$$\|\tilde{\omega}_q\| = \sum_{p \in \Lambda : 0 < |p| \leq |q|, p \neq q} \varphi_K(p) = \left( \sum_{p \in \Lambda \cap C(1, 2\pi, |q|)} \varphi_K(p) \right) - \varphi_K(q). \quad \square$$

**Lemma 5.3** *For all  $f \in C^1(\mathbb{D})$  and  $\alpha \in ]0, \frac{1}{2}[$ , as  $|q| \rightarrow +\infty$ , we have*

$$\frac{\tilde{\omega}_q}{\|\tilde{\omega}_q\|}(f) = \int_{\mathbb{D}} \frac{2}{\pi} |z|^2 f(z) d \operatorname{Leb}_{\mathbb{C}}(z) + O\left(\frac{c_\Lambda \|f\|_\infty}{|q|^{1-2\alpha}} + \frac{\|df\|_\infty}{|q|^\alpha}\right).$$

*Proof* Note that  $c_\Lambda \geq 1$  and let us define

$$c''_\Lambda = 2\sqrt{|D_K|} \zeta_K(2) c_\Lambda = O(c_\Lambda).$$

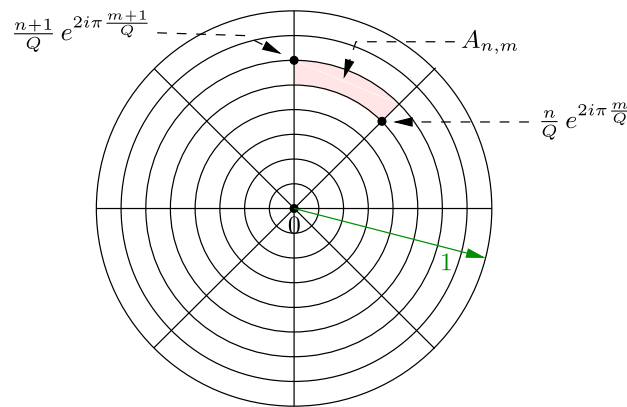
By Lemma 5.2, as  $|q| \rightarrow +\infty$ , we have

$$\frac{1}{\|\tilde{\omega}_q\|} = \frac{c''_\Lambda}{2\pi |q|^4} + O\left(\frac{c_\Lambda^2}{|q|^5}\right). \tag{49}$$

Let  $Q = \lfloor |q|^\alpha \rfloor \geq 1$ , which tends to  $+\infty$  as  $|q| \rightarrow +\infty$ . For all elements  $m$  and  $n$  in  $\{0, \dots, Q - 1\}$ , let

$$A_{n,m} = \left\{ \rho e^{2i\pi t} : \rho \in \left] \frac{n}{Q}, \frac{n+1}{Q} \right], t \in \left] \frac{m}{Q}, \frac{m+1}{Q} \right] \right\},$$

so that  $\mathbb{D} \setminus \{0\}$  is the disjoint union of the sets  $A_{n,m}$  for  $m, n \in \{0, \dots, Q - 1\}$ .



With the notation of Equation (43), we have

$$A_{n,m} = C \left( e^{2i\pi \frac{m+1/2}{Q}}, \frac{2\pi}{Q}, \frac{n+1}{Q} \right) \setminus C \left( e^{2i\pi \frac{m+1/2}{Q}}, \frac{2\pi}{Q}, \frac{n}{Q} \right). \tag{50}$$

Note that since  $n + 1 \leq Q$ , as  $Q$  tends to  $+\infty$ , we have

$$\begin{aligned} \text{diam}(\overline{A_{n,m}}) &\leq \left| \frac{n+1}{Q} e^{2i\pi \frac{m+1}{Q}} - \frac{n+1}{Q} e^{2i\pi \frac{m}{Q}} \right| + \left| \frac{n+1}{Q} e^{2i\pi \frac{m}{Q}} - \frac{n}{Q} e^{2i\pi \frac{m}{Q}} \right| \\ &= O\left(\frac{1}{Q}\right). \end{aligned}$$

Hence for every  $z \in A_{n,m}$ , we have by the mean value theorem

$$f(z) = f\left(\frac{n}{Q} e^{2i\pi \frac{m}{Q}}\right) + O\left(\frac{\|df\|_\infty}{Q}\right). \tag{51}$$

Since

$$\int_{A_{n,m}} |z|^2 d \text{Leb}_{\mathbb{C}}(z) = \int_{2\pi \frac{m}{Q}}^{2\pi \frac{m+1}{Q}} \int_{\frac{n}{Q}}^{\frac{n+1}{Q}} \rho^3 d\rho d\theta = \frac{2\pi}{Q} \frac{(n+1)^4 - n^4}{4Q^4} = O\left(\frac{1}{Q^2}\right),$$

we have therefore

$$\begin{aligned} \int_{A_{n,m}} \frac{1}{\pi} |z|^2 f(z) d \text{Leb}_{\mathbb{C}}(z) &= \left( f\left(\frac{n}{Q} e^{2i\pi \frac{m}{Q}}\right) + O\left(\frac{\|df\|_\infty}{Q}\right) \right) \int_{A_{n,m}} \frac{1}{\pi} |z|^2 d \text{Leb}_{\mathbb{C}}(z) \\ &= \frac{1}{Q^2} \left( \frac{(n+1)^4 - n^4}{2Q^3} f\left(\frac{n}{Q} e^{2i\pi \frac{m}{Q}}\right) + O\left(\frac{\|df\|_\infty}{Q}\right) \right). \end{aligned} \tag{52}$$

By Equations (50) and (44), we have

$$qA_{n,m} = C \left( q e^{2i\pi \frac{m+1/2}{Q}}, \frac{2\pi}{Q}, \frac{(n+1)|q|}{Q} \right) \setminus C \left( q e^{2i\pi \frac{m+1/2}{Q}}, \frac{2\pi}{Q}, \frac{n|q|}{Q} \right).$$

By Equations (51) and (49), applying twice Theorem 4.1 with  $m = \Lambda$ ,  $\theta = \frac{2\pi}{Q}$  and  $x = |q| \frac{n+1}{Q}$ ,  $|q| \frac{n}{Q}$ , and using the fact that  $\frac{|q|}{Q}$  tends to  $+\infty$  as  $|q| \rightarrow +\infty$  since  $\alpha < 1$ , we have, as  $|q| \rightarrow +\infty$ ,

$$\sum_{p \in qA_{n,m} \cap J_q} f\left(\frac{p}{q}\right) \frac{1}{\|\tilde{\omega}_q\|} \varphi_K(p)$$

$$\begin{aligned}
 &= \left( f \left( \frac{n}{Q} e^{2i\pi \frac{m}{Q}} \right) + o \left( \frac{\|df\|_\infty}{Q} \right) \right) \frac{1}{\|\tilde{\omega}_q\|} \sum_{p \in \Lambda \cap qA_{n,m} : p \neq q} \varphi_K(p) \\
 &= \left( f \left( \frac{n}{Q} e^{2i\pi \frac{m}{Q}} \right) + o \left( \frac{\|df\|_\infty}{Q} \right) \right) \left( \frac{c'_\Lambda}{2\pi |q|^4} + o \left( \frac{c_\Lambda^2}{|q|^5} \right) \right) \\
 &\quad \times \left( \frac{2\pi |q|^4 (n+1)^4 - n^4}{Q c'_\Lambda Q^4} + o \left( \frac{|q|^3 (n+1)^3}{Q^3} \right) \right) \\
 &= \frac{1}{Q^2} \left( \frac{(n+1)^4 - n^4}{Q^3} f \left( \frac{n}{Q} e^{2i\pi \frac{m}{Q}} \right) + o \left( \frac{\|df\|_\infty n^3}{Q^4} \right) + o \left( \frac{\|f\|_\infty c_\Lambda n^3}{|q| Q} \right) \right). \tag{53}
 \end{aligned}$$

Note that  $q\mathbb{D} = B(0, |q|)$ . By cutting the sum defining  $\tilde{\omega}_q$  and the integral over  $\mathbb{D}$  into  $Q^2$  subparts, by using Equations (52) and (53), and since  $n \leq Q \leq |q|^\alpha$ , as  $|q| \rightarrow +\infty$ , we have

$$\begin{aligned}
 &\left| \frac{\tilde{\omega}_q}{\|\tilde{\omega}_q\|} (f) - \int_{\mathbb{D}} \frac{2}{\pi} |z|^2 f(z) d \text{Leb}_{\mathbb{C}}(z) \right| \\
 &= \left| \sum_{n,m=0}^{Q-1} \left( \sum_{p \in qA_{n,m} \cap J_q} \frac{\varphi_K(p)}{\|\tilde{\omega}_q\|} f \left( \frac{p}{q} \right) - \int_{A_{n,m}} \frac{2}{\pi} |z|^2 f(z) d \text{Leb}_{\mathbb{C}}(z) \right) \right| \\
 &= o \left( \frac{\|df\|_\infty}{Q} \right) + o \left( \frac{\|f\|_\infty c_\Lambda}{|q|^{1-2\alpha}} \right).
 \end{aligned}$$

This proves Lemma 5.3. □

For every  $N \in \mathbb{N} \setminus \{0\}$ , let us define

$$\tilde{\mu}_N^- = \sum_{(m,n) \in I_N^-} \varphi_K(m) \varphi_K(n) \Delta_{\frac{m}{n}} = \sum_{q \in \Lambda \setminus \{0\} : |q| \leq N} \varphi_K(q) \tilde{\omega}_q,$$

which is a finitely supported measure on  $\mathbb{D}$ . By Theorems 4.1 and 4.2 both with  $m = \Lambda$ ,  $\theta = 2\pi$ ,  $x = N$  and the second one with  $k = 0$ , since  $c_\Lambda \geq 1$  and  $c_{\Lambda,0} \leq 1$  by Equation (46), and since there are finitely many such fields  $K$ , its total mass is

$$\begin{aligned}
 \|\tilde{\mu}_N^-\| &= \sum_{q \in \Lambda \setminus \{0\} : |q| \leq N} \varphi_K(q) \|\tilde{\omega}_q\| = \sum_{(m,n) \in I_N^-} \varphi_K(m) \varphi_K(n) \\
 &= \frac{1}{2} \left( \left( \sum_{\substack{p \in \Lambda \setminus \{0\} \\ |p| \leq N}} \varphi_K(p) \right)^2 - \sum_{\substack{p \in \Lambda \setminus \{0\} \\ |p| \leq N}} \varphi_K(p)^2 \right) = \frac{2\pi^2}{(c'_\Lambda)^2} N^8 + o(N^7).
 \end{aligned}$$

For every  $f \in C^1(\mathbb{D})$ , by Lemmas 5.3 and 5.2, again by Theorem 4.1 with  $m = \Lambda$ ,  $\theta = 2\pi$  and  $x = N$ , we have

$$\begin{aligned}
 \frac{\tilde{\mu}_N^-(f)}{\|\tilde{\mu}_N^-\|} &= \frac{1}{\|\tilde{\mu}_N^-\|} \sum_{q \in \Lambda \setminus \{0\} : |q| \leq N} \varphi_K(q) \|\tilde{\omega}_q\| \frac{\tilde{\omega}_q(f)}{\|\tilde{\omega}_q\|} \\
 &= \int_{\mathbb{D}} \frac{2}{\pi} |z|^2 f(z) d \text{Leb}_{\mathbb{C}}(z) \\
 &\quad + \frac{1}{\|\tilde{\mu}_N^-\|} \sum_{\substack{q \in \Lambda \setminus \{0\} \\ |q| \leq N}} \varphi_K(q) \|\tilde{\omega}_q\| o \left( \frac{c_\Lambda \|f\|_\infty}{|q|^{1-2\alpha}} + \frac{\|df\|_\infty}{|q|^\alpha} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{D}} \frac{2}{\pi} |z|^2 f(z) \, d \text{Leb}_{\mathbb{C}}(z) \\
 &\quad + O \left( \frac{(c''_{\Lambda})^2}{2\pi^2 N^8} \sum_{\substack{q \in \Lambda \setminus \{0\} \\ |q| \leq N}} \varphi_K(q) \frac{2\pi}{c''_{\Lambda}} (N^{3+2\alpha} c_{\Lambda} \|f\|_{\infty} + N^{4-\alpha} \|df\|_{\infty}) \right) \\
 &= \int_{\mathbb{D}} \frac{2}{\pi} |z|^2 f(z) \, d \text{Leb}_{\mathbb{C}}(z) + O \left( \frac{c_{\Lambda} \|f\|_{\infty}}{N^{1-2\alpha}} + \frac{\|df\|_{\infty}}{N^{\alpha}} \right).
 \end{aligned}$$

For every  $N \in \mathbb{N} \setminus \{0\}$ , let us define

$$\tilde{\nu}_N = \mathcal{R}_N^{\mathcal{L}_{\Lambda}^{\varphi_K}, 1} = \sum_{(m, n) \in I_N} \varphi_K(m) \varphi_K(n) \Delta_{\log m - \log n}.$$

which is a measure with finite support on  $E^{\pm} = (\pm[0, \infty[ + i\mathbb{R}) / (2\pi i\mathbb{Z})$ , so that  $\tilde{\nu}_N^{-} = \log_* \tilde{\mu}_N^{-} = \tilde{\nu}_N|_{E^{-}}$ , and  $\|\tilde{\nu}_N^{-}\| = \|\tilde{\mu}_N^{-}\|$ . For every  $f \in C_c^1(E^{-})$ , the function  $f \circ \log$  is a  $C^1$  function on  $\mathbb{D}$  which vanishes on a neighbourhood of 0. By Equation (6), we have

$$\begin{aligned}
 \frac{\tilde{\nu}_N^{-}(f)}{\|\tilde{\nu}_N^{-}\|} &= \frac{\tilde{\mu}_N^{-}(f \circ \log)}{\|\tilde{\mu}_N^{-}\|} \\
 &= \int_{\mathbb{D}} \frac{2}{\pi} |z|^2 f \circ \log(z) \, d \text{Leb}_{\mathbb{C}}(z) + O \left( \frac{c_{\Lambda} \|f \circ \log\|_{\infty}}{N^{1-2\alpha}} + \frac{\|d(f \circ \log)\|_{\infty}}{N^{\alpha}} \right) \\
 &= \int_{E^{-}} \frac{2}{\pi} e^{4 \text{Re}(z')} f(z') \, d \text{Leb}_E(z') + O \left( \frac{c_{\Lambda} \|f\|_{\infty}}{N^{1-2\alpha}} + \frac{\|e^{-z'} df(z')\|_{\infty}}{N^{\alpha}} \right).
 \end{aligned}$$

Since  $\tilde{\nu}_N = \tilde{\nu}_N^{-} + \tilde{\nu}_N^{+}$  on  $E \setminus (i\mathbb{R}) / (2i\pi\mathbb{Z})$  and  $\text{Leb}_E((i\mathbb{R}) / (2i\pi\mathbb{Z})) = 0$ , since  $\tilde{\nu}_N^{+} = \text{sg}_* \tilde{\nu}_N^{-}$  where  $\text{sg} : E \mapsto E$  is the map  $x' + iy' \mapsto -x' + iy'$ , we have  $\|\tilde{\nu}_N^{\pm}\| = \frac{1}{2} \|\tilde{\nu}_N\|$  and the last claim of Theorem 5.1 follows by symmetry.  $\square$

### 6 Pair correlation of integral lattice points with scaling and Euler weight

As in Sect. 5, we fix an imaginary quadratic number field  $K$  whose ring of integers  $\mathcal{O}_K$  is principal, and a nonzero ideal  $\Lambda = \tilde{\Lambda} \in \mathcal{I}_K^{+}$ . We also study the pair correlations of the family  $\mathcal{L}_{\Lambda}^{\varphi_K}$  defined in the introduction, but now with the linear scaling function  $\psi = \text{id}^1 : N \mapsto N$ . We leave to the reader the study of a general scaling  $\psi$ , assumed to converge to  $+\infty$ , proving a Poissonian behaviour for sublinear scalings and total loss of mass behaviour for superlinear scalings. We also leave to the reader a statement similar to Theorem 6.1, replacing the above  $\mathbb{Z}$ -lattice  $\Lambda$  by a  $\mathbb{Z}$ -grid  $a + \Lambda$  for any  $a \in \mathcal{O}_K$ .

As in Sect. 3, we work on the family of varying cylinders  $(E_N = \mathbb{C} / (2\pi i N \mathbb{Z}))_{N \in \mathbb{N} \setminus \{0\}}$ . As in Sect. 3, for every  $f \in C_c^1(\mathbb{C})$ , for every  $N$  large enough such that the support of  $f$  is contained in  $\overset{\circ}{B}(0, \pi N)$ , we denote by  $f_N \in C_c^1(E_N)$  the map that coincides with  $f$  on  $B(0, \pi N)$  modulo  $2\pi i N \mathbb{Z}$  and vanishes elsewhere. For every  $N \in \mathbb{N} \setminus \{0\}$ , we consider the measure on  $E_N$  with finite support defined with  $I_N = I_{N, \Lambda}$  by

$$\tilde{\mathcal{R}}_N = \mathcal{R}_N^{\mathcal{L}_{\Lambda}^{\varphi_K}, \text{id}^1} = \sum_{(m, n) \in I_N} \varphi_K(m) \varphi_K(n) \Delta_N(\log m - \log n),$$

which is the (not normalised) empirical pair correlation measure at time  $N$  of the complex logarithms of the elements of  $\Lambda$  with multiplicities given by the Euler function and with linear scaling  $\psi = \text{id}^1 : N \mapsto N$ .

**Theorem 6.1** *As  $N \rightarrow +\infty$ , the family  $\left(\frac{1}{N^6} \tilde{\mathcal{H}}_N\right)_{N \in \mathbb{N}}$  of measures on  $E_N$  converges (for the pointed Hausdorff-Gromov weak-star convergence) to the measure absolutely continuous with respect to the Lebesgue measure on  $\mathbb{C}$ , with Radon-Nikodym derivative the function*

$$g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1} : z \mapsto \frac{1}{|z|^8} \sum_{k \in \Lambda: |k| \leq |z|} \frac{2 c_{\Lambda, k}}{\sqrt{|D_K|}} |k|^6,$$

that is, as  $N \rightarrow +\infty$ ,

$$\frac{1}{N^6} \tilde{\mathcal{H}}_N \xrightarrow{*} g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1} \text{Leb}_{\mathbb{C}}.$$

Furthermore, for all  $A \geq 1$  and  $f \in C^1(\mathbb{C})$  with compact support contained in  $B(0, A)$ , as  $N \rightarrow +\infty$ , we have

$$\begin{aligned} \frac{1}{N^6} \tilde{\mathcal{H}}_N(f_N) &= \int_{z \in \mathbb{C}} f(z) g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}(z) d \text{Leb}_{\mathbb{C}}(z) \\ &+ O\left(\frac{A^4}{\text{covol}_\Lambda c'_\Lambda N^{1/2}} (\|df\|_\infty + \|f\|_\infty)\right). \end{aligned}$$

The above result with  $\Lambda = \mathcal{O}_K$  gives the second assertion of Theorem 1.2 in the introduction, using the values of  $c_{\mathcal{O}_K, k}$  for  $k \in \mathcal{O}_K$  given in Equation (47).

Note that, as the proof below shows, the total mass of  $\tilde{\mathcal{H}}_N$  is equivalent to  $cN^8$  as  $N \rightarrow +\infty$ , for some constant  $c > 0$ . Hence renormalising  $\tilde{\mathcal{H}}_N$  to be a probability measure would make it converge to the zero measure on  $\mathbb{C}$ .

*Proof* We proceed as in the beginning of the proof of Theorem 3.1 : We only have to prove the second assertion above; We define  $E_N^\pm = (\pm[0, \infty[ + i\mathbb{R}) / (2\pi i N \mathbb{Z})$ ; We only study the convergence of the measures  $\frac{1}{N^6} \tilde{\mathcal{H}}_N$  on the halfcylinder  $E_N^+$  to the measure  $g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1} \text{Leb}_{\mathbb{C}^+}$  on the halfplane  $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$  as  $N \rightarrow +\infty$ ; And we deduce the global result by the symmetry of  $g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}$  under  $z \mapsto -z$ .

For all  $N \in \mathbb{N} \setminus \{0\}$  and  $p \in \Lambda \setminus \{0\}$ , let  $J_{p, N}$  be given by Equation (20). Note that

$$(\Lambda \setminus \{0\}) \cap B(0, N - |p|) \subset J_{p, N} \subset (\Lambda \setminus \{0\}) \cap B(0, N). \tag{54}$$

We now define the key auxiliary measure by

$$\tilde{\omega}_{p, N} = \sum_{q \in J_{p, N}} \varphi_K(q) \varphi_K(q + p) \Delta_{\frac{q}{Np}}.$$

Then  $\tilde{\omega}_{p, N}$  is a measure with finite support on  $B(0, \frac{1}{|p|}) \setminus \{0\}$ , which is nonzero if  $N \geq 2|p|$  (which is the case if  $p$  is bounded and  $N \rightarrow +\infty$ ), and vanishes if  $|p| > 2N$ . If  $N \geq 2|p|$ , by Theorem 4.2 with  $m = \Lambda$ ,  $k = p$  and  $\theta = 2\pi$ , by Formula (54), since  $|p| \geq 1$ , and since  $c_{\Lambda, p} \leq 1$  (see Equation (46)), we have

$$\begin{aligned} \|\tilde{\omega}_{p, N}\| &= \sum_{q \in J_{p, N}} \varphi_K(q) \varphi_K(q + p) = \frac{2\pi c_{\Lambda, p}}{3\sqrt{|D_K|}} (N + O(|p|))^6 \\ &+ O\left(|p| (N + O(|p|))^5 + |p|^2 (N + O(|p|))^4 + |p|^2 \ln |p| (N + O(|p|))^2 \ln(N + O(|p|))\right) \\ &= \frac{2\pi c_{\Lambda, p}}{3\sqrt{|D_K|}} N^6 + O(|p| N^5). \end{aligned} \tag{55}$$



In particular, if  $N \geq 2|p|$ , since  $c'_\Lambda > 0$  by Equation (48), we have

$$\frac{1}{\|\tilde{\omega}_{p,N}\|} = \frac{3\sqrt{|D_K|}}{2\pi c_{\Lambda,p} N^6} \left(1 + O\left(\frac{|p|}{c'_\Lambda N}\right)\right). \quad (56)$$

The next result implies that the measures  $\tilde{\omega}_{p,N}$ , once normalised to be probability measures, weak-star converge to the measure  $d\mu(z) = \frac{3}{\pi} |p|^6 |z|^4 d\text{Leb}_{B(0, \frac{1}{|p|})}(z)$  on  $B(0, \frac{1}{|p|})$  as  $N \rightarrow +\infty$ , uniformly on  $p \in \Lambda \setminus \{0\}$  bounded.

**Lemma 6.2** *For all  $p \in \Lambda \setminus \{0\}$ ,  $\alpha \in ]0, 1[$  and  $f \in C_c^1(\mathbb{C})$ , as  $N \rightarrow +\infty$ , we have*

$$\begin{aligned} \frac{\tilde{\omega}_{p,N}}{\|\tilde{\omega}_{p,N}\|}(f) &= \int_{B(0, \frac{1}{|p|})} \frac{3}{\pi} |p|^6 |z|^4 f(z) d\text{Leb}_{\mathbb{C}}(z) \\ &+ O\left(\frac{\|df\|_\infty}{N^\alpha |p|} + \frac{|p| \|f\|_\infty}{c'_\Lambda N^{1-\alpha}} + \frac{\|f\|_\infty}{N^\alpha}\right). \end{aligned}$$

*Proof* As in the proof of Lemma 5.3, we will estimate the difference of the main terms in the above centered formula by cutting the sum defining the renormalised measure  $\tilde{\omega}_{p,N}$  and by cutting similarly the integral on  $B(0, \frac{1}{|p|})$ . We assume, as we may, that  $N \geq 2|p|$ .

Let  $Q = \lfloor N^\alpha \rfloor \geq 1$ , which tends to  $+\infty$  as  $N \rightarrow +\infty$ . For all  $m, n \in \{0, \dots, Q-1\}$ , let

$$A'_{n,m} = \left\{ \rho e^{2i\pi t} : \rho \in \left] \frac{n}{Q|p|}, \frac{n+1}{Q|p|} \right], t \in \left] \frac{m}{Q}, \frac{m+1}{Q} \right] \right\}, \quad (57)$$

so that  $B(0, \frac{1}{|p|}) \setminus \{0\}$  is the disjoint union of the sets  $A'_{n,m}$  for  $m, n \in \{0, \dots, Q-1\}$ . With the notation of Equation (43), we have

$$A'_{n,m} = C\left(e^{2i\pi \frac{m+1/2}{Q}}, \frac{2\pi}{Q}, \frac{n+1}{Q|p|}\right) \setminus C\left(e^{2i\pi \frac{m+1/2}{Q}}, \frac{2\pi}{Q}, \frac{n}{Q|p|}\right). \quad (58)$$

Note that  $\text{diam}(\overline{A'_{n,m}}) = O\left(\frac{1}{Q|p|}\right)$ . Hence for every  $z \in A'_{n,m}$ , we have by the mean value theorem

$$f(z) = f\left(\frac{n}{Q|p|} e^{2i\pi \frac{m}{Q}}\right) + O\left(\frac{\|df\|_\infty}{Q|p|}\right). \quad (59)$$

If  $|p| \leq N^{1-\alpha}$  (which is the case if  $p$  is bounded and  $N \rightarrow +\infty$ ) and if  $n \leq Q-2$ , then

$$N|p| \frac{n+1}{Q|p|} \leq N \frac{Q-1}{Q} \leq N - N^{1-\alpha} \leq N - |p|.$$

Hence for all  $m, n \in \{0, \dots, Q-1\}$ , by Formula (54), if  $|p| \leq N^{1-\alpha}$  and if  $n \neq Q-1$ , we have

$$(N p A'_{n,m}) \cap J_{p,N} = \Lambda \cap (N p A'_{n,m}). \quad (60)$$

For all  $m, n \in \{0, \dots, Q-1\}$ , let

$$S_{n,m} = \sum_{q \in (N p A'_{n,m}) \cap J_{p,N}} f\left(\frac{q}{N p}\right) \frac{1}{\|\tilde{\omega}_{p,N}\|} \varphi_K(q) \varphi_K(q+p).$$

If  $n \neq Q - 1$ , by Equations (59) and (60) for the first equality, and for the second one, by Equations (56), (58) and (44), by Theorem 4.2 applied twice with  $m = \Lambda$ ,  $k = p$ ,  $\theta = \frac{2\pi}{Q}$  and  $x = \frac{N(n+1)}{Q}, \frac{Nn}{Q}$ , we have, as  $N \rightarrow +\infty$  (so that in particular  $N \geq \max\{2, c'_\Lambda\} |p|$ ),

$$\begin{aligned} S_{n,m} &= \left( f \left( \frac{n}{Q|p|} e^{2i\pi \frac{m}{Q}} \right) + O \left( \frac{\|df\|_\infty}{Q|p|} \right) \right) \frac{1}{\|\tilde{\omega}_{p,N}\|} \sum_{q \in \Lambda \cap (NpA'_{n,m})} \varphi_K(q) \varphi_K(q+p) \\ &= \left( f \left( \frac{n}{Q|p|} e^{2i\pi \frac{m}{Q}} \right) + O \left( \frac{\|df\|_\infty}{Q|p|} \right) \right) \frac{3\sqrt{|D_K|}}{2\pi c_{\Lambda,p} N^6} \left( 1 + O \left( \frac{|p|}{c'_\Lambda N} \right) \right) \\ &\quad \times \frac{2\pi c_{\Lambda,p}}{3\sqrt{|D_K|}} \left( \left( \frac{N(n+1)}{Q} \right)^6 - \left( \frac{Nn}{Q} \right)^6 + O \left( \frac{|p|}{c'_\Lambda} \left( \frac{Nn}{Q} \right)^5 + \frac{|p|^2}{c'_\Lambda} \left( \frac{Nn}{Q} \right)^4 \right. \right. \\ &\quad \left. \left. + \frac{|p|^2}{c'_\Lambda} \ln |p| \left( \frac{Nn}{Q} \right)^2 \ln \left( \frac{Nn}{Q} \right) \right) \right) \\ &= \frac{1}{Q^2} \left( \frac{(n+1)^6 - n^6}{Q^5} f \left( \frac{n}{Q|p|} e^{2i\pi \frac{m}{Q}} \right) + O \left( \frac{\|df\|_\infty}{Q|p|} + \frac{Q|p|\|f\|_\infty}{c'_\Lambda N} \right) \right). \end{aligned} \tag{61}$$

Note that by Equations (58), (54) and (44) for the first inequality, and for the second one, by Equations (56) and twice (55), as  $N \rightarrow +\infty$ , we have

$$\begin{aligned} \left| \sum_{0 \leq m \leq Q-1} S_{Q-1,m} \right| &\leq \|f\|_\infty \frac{1}{\|\tilde{\omega}_{p,N}\|} \sum_{q \in \Lambda \cap (B(0,N) \setminus B(0, N - \frac{N}{Q}))} \varphi_K(q) \varphi_K(q+p) \\ &= \|f\|_\infty \frac{3\sqrt{|D_K|}}{2\pi c_{\Lambda,p} N^6} \left( 1 + O \left( \frac{|p|}{c'_\Lambda N} \right) \right) \\ &\quad \times \frac{2\pi c_{\Lambda,p}}{3\sqrt{|D_K|}} \left( N^6 - \left( N - \frac{N}{Q} \right)^6 + O \left( \frac{|p|}{c'_\Lambda} N^5 \right) \right) = O \left( \frac{\|f\|_\infty}{Q} \right). \end{aligned} \tag{62}$$

For all  $m, n \in \{0, \dots, Q - 1\}$ , let

$$I_{n,m} = \int_{A'_{n,m}} \frac{3}{\pi} |p|^6 |z|^4 f(z) d \text{Leb}_{\mathbb{C}}(z).$$

By Equations (59) and (57), we have

$$\begin{aligned} I_{n,m} &= \left( f \left( \frac{n}{Q|p|} e^{2i\pi \frac{m}{Q}} \right) + O \left( \frac{\|df\|_\infty}{Q|p|} \right) \right) \int_{\frac{2\pi m}{Q}}^{\frac{2\pi(m+1)}{Q}} \int_{\frac{n}{Q|p|}}^{\frac{n+1}{Q|p|}} \frac{3}{\pi} |p|^6 \rho^5 d\rho d\theta \\ &= \frac{1}{Q^2} \left( \frac{(n+1)^6 - n^6}{Q^5} f \left( \frac{n}{Q|p|} e^{2i\pi \frac{m}{Q}} \right) + O \left( \frac{\|df\|_\infty}{Q|p|} \right) \right). \end{aligned} \tag{63}$$

Furthermore,

$$\left| \sum_{0 \leq m \leq Q-1} I_{Q-1,m} \right| \leq \|f\|_\infty \int_0^{2\pi} \int_{\frac{1}{|p|} - \frac{1}{Q|p|}}^{\frac{1}{|p|}} \frac{3}{\pi} |p|^6 \rho^5 d\rho d\theta = O \left( \frac{\|f\|_\infty}{Q} \right). \tag{64}$$

Since  $B(0, \frac{1}{|p|}) \setminus \{0\} = \bigsqcup_{n,m=0}^{Q-1} A'_{n,m}$ , putting together Equations (61), (63), (62) and (64),

and since  $Q = \lfloor N^\alpha \rfloor \in [\frac{N^\alpha}{2}, N^\alpha]$  for  $N$  large enough, we have

$$\left| \frac{\tilde{\omega}_{p,N}}{\|\tilde{\omega}_{p,N}\|}(f) - \int_{B(0, \frac{1}{|p|})} \frac{3}{\pi} |p|^6 |z|^4 f(z) d \text{Leb}_{\mathbb{C}}(z) \right|$$

$$\begin{aligned}
 &= \left| \sum_{n,m=0}^{Q-2} (S_{n,m} - I_{n,m}) + \sum_{m=0}^{Q-1} S_{Q-1,m} - \sum_{m=0}^{Q-1} I_{Q-1,m} \right| \\
 &\leq \sum_{n,m=0}^{Q-2} |S_{n,m} - I_{n,m}| + \left| \sum_{m=0}^{Q-1} S_{Q-1,m} \right| + \left| \sum_{m=0}^{Q-1} I_{Q-1,m} \right| \\
 &= O\left(\frac{\|df\|_\infty}{Q|p|} + \frac{Q|p|\|f\|_\infty}{c'_\Lambda N}\right) + O\left(\frac{\|df\|_\infty}{Q|p|}\right) + O\left(\frac{\|f\|_\infty}{Q}\right) + O\left(\frac{\|f\|_\infty}{Q}\right) \\
 &= O\left(\frac{\|df\|_\infty}{N^\alpha|p|} + \frac{|p|\|f\|_\infty}{c'_\Lambda N^{1-\alpha}} + \frac{\|f\|_\infty}{N^\alpha}\right).
 \end{aligned}$$

This proves Lemma 6.2. □

Now, let us introduce the finitely supported measure on  $\mathbb{C} \setminus \{0\}$  defined by

$$\tilde{\mu}_N^+ = \sum_{p \in \Lambda \setminus \{0\}} \iota_* \tilde{\omega}_{p,N} = \sum_{p,q \in \Lambda \setminus \{0\}: |q| \leq |q+p| \leq N} \varphi_K(q) \varphi_K(q+p) \Delta_{N \frac{p}{q}},$$

where as previously  $\iota : z \mapsto \frac{1}{z}$  (recalling that the measure  $\tilde{\omega}_{p,N}$  vanishes if  $|p| > 2N$  and has finite support contained in  $B(0, \frac{1}{|p|}) \setminus \{0\}$ ).

**Lemma 6.3** *For all  $A \geq 1$  and  $f \in C^1(\mathbb{C}^+)$  with compact support contained in  $B(0, A)$ , as  $N \rightarrow +\infty$ , we have*

$$\left| \tilde{\mathcal{R}}_N|_{E_N^+}(f_N) - \tilde{\mu}_N^+(f) \right| = O\left(\frac{A^4 \|df\|_\infty N^5}{\text{covol}_\Lambda}\right).$$

*Proof* Let us assume that  $N > \frac{A}{\pi}$ , so that the ball  $B(0, A)$  injects by the canonical projection  $\mathbb{C} \rightarrow E_N = \mathbb{C}/(2\pi i N \mathbb{Z})$ . Note that  $f_N$  has support in  $E_N^+$ . Using the change of variables  $(p, q) \mapsto (m = p + q, n = q)$ , we have

$$\begin{aligned}
 \tilde{\mathcal{R}}_N(f_N) &= \sum_{(m,n) \in I_N^+} \varphi_K(m) \varphi_K(n) f_N(N \log m - N \log n) \\
 &= \sum_{p,q \in \Lambda \setminus \{0\}: |q| \leq |q+p| \leq N} \varphi_K(q) \varphi_K(q+p) f_N(N \log(p+q) - N \log q).
 \end{aligned}$$

As in the proof of Lemma 3.3 (see Formulas (24) and (25) with  $\psi(N) = N$ ), if a pair  $(p, q)$  occurs in the index of the sum defining either  $\tilde{\mathcal{R}}_N(f_N)$  or  $\tilde{\mu}_N^+(f)$  with nonzero corresponding summand, then  $\frac{|p|}{|q|} = O\left(\frac{A}{N}\right)$ ,  $|p| = O(A)$ , and

$$\left| f_N(N \log(p+q) - N \log q) - f\left(N \frac{p}{q}\right) \right| = O\left(\frac{A^2 \|df\|_\infty}{N}\right).$$

Hence, by Equation (55), since  $c_{\Lambda,p} \leq 1$  (see Equation (46)) and by Lemma 2.1 with  $k = 0$ , as  $N \geq \text{diam}_{\tilde{\Lambda}}$  tends to  $+\infty$ , we have

$$\begin{aligned}
 \left| \tilde{\mathcal{R}}_N(f_N) - \tilde{\mu}_N^+(f) \right| &\leq \sum_{p \in \Lambda \setminus \{0\}: |p|=O(A), q \in I_{p,N}} \varphi_K(q) \varphi_K(q+p) O\left(\frac{A^2 \|df\|_\infty}{N}\right) \\
 &= \sum_{p \in \Lambda \setminus \{0\}: |p|=O(A)} O(N^6) O\left(\frac{A^2 \|df\|_\infty}{N}\right) = O\left(\frac{A^4 \|df\|_\infty N^5}{\text{covol}_\Lambda}\right)
 \end{aligned}$$

This proves Lemma 6.3. □

**Lemma 6.4** *For all  $A \geq 1$  and  $f \in C^1(\mathbb{C}^+)$  with compact support contained in  $B(0, A)$ , as  $N \rightarrow +\infty$ , we have*

$$\begin{aligned} \frac{1}{N^6} \tilde{\mu}_N^+(f) &= \int_{\mathbb{C}^+} f(z) g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}(z) d\text{Leb}_{\mathbb{C}}(z) \\ &\quad + O\left(\frac{A^4}{\text{covol}_\Lambda c'_\Lambda N^{1/2}} (\|df\|_\infty + \|f\|_\infty)\right). \end{aligned}$$

*Proof* Let  $A$  and  $f$  be as in the statement, let  $N$  be large enough, and let  $\alpha \in ]0, 1[$ . Since the support of  $\tilde{\omega}_{p,N}$  is contained in  $B(0, \frac{1}{|p|})$ , the support of  $\iota_*\tilde{\omega}_{p,N}$  is contained in  $\{z \in \mathbb{C} : |z| \geq |p|\}$ . Since a nonzero element of  $\mathcal{O}_K$  has norm, hence absolute value, at least 1, the measures  $\tilde{\mu}_N^+$  and  $g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}(z) d\text{Leb}_{\mathbb{C}}(z)$  both vanish on  $\overset{\circ}{B}(0, 1)$ . Hence we may assume that the support of  $f$  is contained in  $\{z \in \mathbb{C} : |z| \geq 1\}$ , so that the support of  $f \circ \iota$  is compact. Note that  $\|f \circ \iota\|_\infty = \|f\|_\infty$  and as the support of  $f$  is contained in  $B(0, A)$ , that

$$\|d(f \circ \iota)\|_\infty \leq A^2 \|df\|_\infty.$$

By Equation (55) and by Lemma 6.2, by Equation (27), since  $1 \leq |p| = O(A)$  and  $c_{\Lambda,p} \leq 1$ , as  $N \rightarrow +\infty$ , we hence have

$$\begin{aligned} \tilde{\mu}_N^+(f) &= \sum_{p \in \Lambda \setminus \{0\}} \iota_*\tilde{\omega}_{p,N}(f) = \sum_{p \in \Lambda \setminus \{0\}} \|\tilde{\omega}_{p,N}\| \frac{\tilde{\omega}_{p,N}}{\|\tilde{\omega}_{p,N}\|}(f \circ \iota) \\ &= \sum_{p \in \Lambda \setminus \{0\}} \left( \frac{2\pi c_{\Lambda,p}}{3\sqrt{|D_K|}} N^6 + O(|p|N^5) \right) \\ &\quad \times \left( \int_{B(0, \frac{1}{|p|})} \frac{3}{\pi} |p|^6 |z|^4 f \circ \iota(z) d\text{Leb}_{\mathbb{C}}(z) \right. \\ &\quad \left. + O\left(\frac{A^2 \|df\|_\infty}{N^\alpha |p|} + \frac{|p| \|f\|_\infty}{c'_\Lambda N^{1-\alpha}} + \frac{\|f\|_\infty}{N^\alpha}\right) \right) \\ &= N^6 \left( \sum_{p \in \Lambda} \frac{2c_{\Lambda,p} |p|^6}{\sqrt{|D_K|}} \int_{|z| \geq |p|} \frac{1}{|z|^8} f(z) d\text{Leb}_{\mathbb{C}}(z) \right. \\ &\quad \left. + O\left(\sum_{p \in \Lambda : |p|=O(A)} \frac{|p| \|f\|_\infty}{N} + \frac{A^2 \|df\|_\infty}{N^\alpha} + \frac{A \|f\|_\infty}{c'_\Lambda N^{1-\alpha}} + \frac{\|f\|_\infty}{N^\alpha}\right) \right). \end{aligned}$$

By Lemma 2.1 with  $k = 0$ , as  $N \rightarrow +\infty$ , we hence have

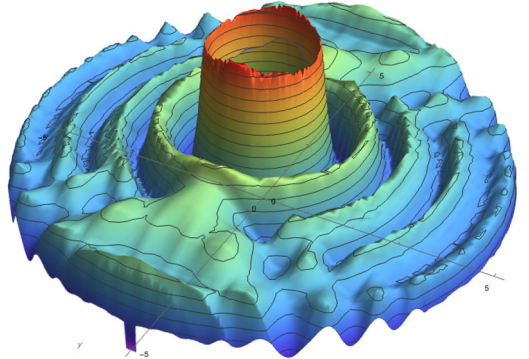
$$\begin{aligned} \frac{\tilde{\mu}_N^+(f)}{N^6} &= \int_{\mathbb{C}} \frac{1}{|z|^8} \sum_{p \in \Lambda : |p| \leq |z|} \frac{2c_{\Lambda,p} |p|^6}{\sqrt{|D_K|}} f(z) d\text{Leb}_{\mathbb{C}}(z) \\ &\quad + O\left(\frac{A^2}{\text{covol}_\Lambda} \left(\frac{A^2 \|df\|_\infty}{N^\alpha} + \frac{A \|f\|_\infty}{c'_\Lambda N^{1-\alpha}} + \frac{\|f\|_\infty}{N^\alpha}\right)\right). \end{aligned}$$

Taking  $\alpha = \frac{1}{2}$ , this proves Lemma 6.4 since  $c'_\Lambda \leq 1$  and  $A \geq 1$ . □

Theorem 6.1 now follows from Lemmas 6.3 and 6.4, as explained in the beginning of the proof. □

The following figure illustrates Theorem 6.1 when we take  $K = \mathbb{Q}(\frac{1+i\sqrt{3}}{2})$  and  $\Lambda = \mathcal{O}_K = \mathbb{Z}[\frac{1+i\sqrt{3}}{2}]$ . It shows an approximation of the pair correlation function  $g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}$

computed using the empirical measure  $\frac{1}{50^6} \tilde{\mathcal{B}}_{50}$  in the ball of radius 5 centered at the origin, to be compared with the orange radial profile of  $g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}$  in the second figure of the introduction.



The graph of  $g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}$  is bounded by Lemma 2.1 with  $k = 6$  since  $c_{\Lambda, p} \leq 1$ . It is asymptotic to a horizontal plane at infinity, by the following result. In its proof, we use the Möbius function  $\mu_K : \mathcal{S}_K^+ \rightarrow \mathbb{Z}$  of  $K$ , defined by

$$\forall \mathfrak{a} \in \mathcal{S}_K^+, \quad \mu_K(\mathfrak{a}) = \begin{cases} 0 & \text{if } \mathfrak{p}^2 \mid \mathfrak{a} \text{ for some prime ideal } \mathfrak{p} \\ (-1)^m & \text{if } \mathfrak{a} = \mathfrak{p}_1 \dots \mathfrak{p}_m \text{ for distinct prime ideals } \mathfrak{p}_1, \dots, \mathfrak{p}_m \end{cases}$$

(in particular  $\mu_K(\mathcal{O}_K) = 1$ ). For every  $a \in \mathcal{O}_K \setminus \{0\}$ , we define  $\mu_K(a) = \mu_K(a\mathcal{O}_K)$ . We have (see for instance [27]) the Möbius inversion formula: for all  $f, g : \mathcal{S}_K^+ \rightarrow \mathbb{C}$ ,

$$f(\mathfrak{a}) = \sum_{\mathfrak{b} \mid \mathfrak{a}} g(\mathfrak{b}) \quad \text{if and only if} \quad g(\mathfrak{a}) = \sum_{\mathfrak{b} \mid \mathfrak{a}} \mu_K(\mathfrak{b}) f(\mathfrak{a}\mathfrak{b}^{-1}). \tag{65}$$

**Proposition 6.5** *We have*

$$\lim_{|z| \rightarrow \infty} g_{\mathcal{L}_\Lambda^{\varphi_K}, \text{id}^1}(z) = \frac{\pi}{|D_K|} \prod_{\mathfrak{p}} \left(1 - \frac{2}{N(\mathfrak{p})^2}\right) \left(1 + \frac{1}{N(\mathfrak{p})^2(N(\mathfrak{p})^2 - 2)}\right).$$

*Proof* Let us consider the multiplicative<sup>1</sup> function on  $\mathcal{S}_K^+$  defined by

$$f : \mathfrak{a} \mapsto \prod_{\mathfrak{p} \mid \mathfrak{a}} \left(1 + \frac{1}{N(\mathfrak{p})(N(\mathfrak{p})^2 - 2)}\right)$$

and the constant

$$C_1 = \frac{2\pi}{|\mathcal{O}_K^\times| \sqrt{|D_K|}} \prod_{\mathfrak{p}} \left(1 + \frac{1}{N(\mathfrak{p})^2(N(\mathfrak{p})^2 - 2)}\right).$$

Let us prove that uniformly in  $x \geq 1$ , we have

$$\sum_{\mathfrak{a} \in \mathcal{S}_K^+ : N(\mathfrak{a}) \leq x} N(\mathfrak{a})^3 f(\mathfrak{a}) = \frac{C_1}{4} x^4 + O(x^{7/2}). \tag{66}$$

Applying this with  $x = |z|^2$ , by Equation (3), since the map  $k \mapsto k\mathcal{O}_K$  from  $\mathcal{O}_K \setminus \{0\}$  onto  $\mathcal{S}_K^+$  is  $|\mathcal{O}_K^\times|$ -to-1, this proves Proposition 6.5.

<sup>1</sup>Recall that a function  $f : \mathcal{S}_K^+ \rightarrow \mathbb{C}^\times$  is *multiplicative* if  $f(\mathcal{O}_K) = 1$  and for all coprime integral ideals  $\mathfrak{a}, \mathfrak{b}$  in  $\mathcal{S}_K^+$ , we have  $f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b})$ .

Let

$$g = f * \mu_K : \mathfrak{a} \mapsto \sum_{\mathfrak{b}|\mathfrak{a}} \mu_K(\mathfrak{b})f(\mathfrak{a}\mathfrak{b}^{-1})$$

be the Dirichlet convolution of  $f$  with the Möbius function  $\mu_K$  of  $K$ . Then  $g$  is multiplicative. For every prime ideal  $\mathfrak{p}$  of  $\mathcal{O}_K$ , we have

$$g(\mathfrak{p}) = f(\mathfrak{p}) \mu_K(\mathcal{O}_K) + f(\mathcal{O}_K) \mu_K(\mathfrak{p}) = \frac{1}{N(\mathfrak{p})(N(\mathfrak{p})^2 - 2)}$$

and  $g(\mathfrak{p}^k) = f(\mathfrak{p}^k) \mu_K(\mathcal{O}_K) + f(\mathfrak{p}^{k-1}) \mu_K(\mathfrak{p}) = 0$  for every  $k \geq 2$ . Therefore, for every  $\mathfrak{b} \in \mathcal{S}_K^+$ , we have

$$g(\mathfrak{b}) = \mu_K(\mathfrak{b})^2 \prod_{\mathfrak{p}|\mathfrak{b}} \frac{1}{N(\mathfrak{p})(N(\mathfrak{p})^2 - 2)}.$$

By for instance Equation (7) with  $\Lambda = \mathcal{O}_K$ ,  $k = 0$  and  $x = \sqrt{y}$ , by Equation (16) with  $\mathfrak{m} = \mathcal{O}_K$ , and again since the map  $k \mapsto k\mathcal{O}_K$  is  $|\mathcal{O}_K^\times|$ -to-1, as  $y \rightarrow +\infty$ , we have (see also [19, Theo. 15])

$$\text{Card}\{\mathfrak{a} \in \mathcal{S}_K^+ : N(\mathfrak{a}) \leq y\} = \frac{2\pi}{|\mathcal{O}_K^\times| \sqrt{|D_K|}} y + O(y^{\frac{1}{2}}). \tag{67}$$

**Lemma 6.6** *For every  $\mathfrak{b} \in \mathcal{S}_K^+$ , we have  $0 \leq g(\mathfrak{b}) \leq N(\mathfrak{b})^{-3} \prod_{\mathfrak{p}} \left(1 - \frac{2}{N(\mathfrak{p})^2}\right)^{-1}$ . In particular,*

$$\sum_{\mathfrak{b} \in \mathcal{S}_K^+ : N(\mathfrak{b}) \geq x} \frac{g(\mathfrak{b})}{N(\mathfrak{b})} = O\left(\frac{1}{x^2}\right).$$

*Proof* This is immediate if  $\mu_K(\mathfrak{b}) = 0$ . Otherwise,  $\mathfrak{b} = \mathfrak{p}_1 \dots \mathfrak{p}_k$  with  $k \in \mathbb{N}$  and  $\mathfrak{p}_1, \dots, \mathfrak{p}_k$  pairwise distinct prime ideals, and

$$\begin{aligned} 0 \leq N(\mathfrak{b})^3 g(\mathfrak{b}) &= \prod_{i=1}^k \frac{N(\mathfrak{p}_i)^3}{N(\mathfrak{p}_i)(N(\mathfrak{p}_i)^2 - 2)} \\ &= \prod_{i=1}^k \left(1 - \frac{2}{N(\mathfrak{p}_i)^2}\right)^{-1} \leq \prod_{\mathfrak{p}} \left(1 - \frac{2}{N(\mathfrak{p})^2}\right)^{-1} < +\infty. \end{aligned}$$

The last claim follows from the well known error term in the Dedekind zeta function summation: as  $a_n = \text{Card}\{\mathfrak{a} \in \mathcal{S}_K^+ : N(\mathfrak{a}) = n\} = O(n)$  (see for instance Equation (67)), we have by the first claim

$$\begin{aligned} \sum_{\mathfrak{b} \in \mathcal{S}_K^+ : N(\mathfrak{b}) \geq x} \frac{g(\mathfrak{b})}{N(\mathfrak{b})} &= O\left(\sum_{\mathfrak{b} \in \mathcal{S}_K^+ : N(\mathfrak{b}) \geq x} \frac{1}{N(\mathfrak{b})^4}\right) = O\left(\sum_{n \geq x} \frac{a_n}{n^4}\right) \\ &= O\left(\int_x^{+\infty} \frac{1}{t^3} dt\right) = O\left(\frac{1}{x^2}\right). \quad \square \end{aligned}$$

Using the Möbius inversion formula of Equation (65) for the first equality, Equation (67) with  $y = \frac{x}{N(\mathfrak{b})}$  for the third equality, Lemma 6.6 for the fifth equality and an Eulerian product (since  $g$  is multiplicative and vanishes on ideals divisible by a nontrivial square) for the sixth equality, with  $S(x) = \sum_{\mathfrak{a} \in \mathcal{S}_K^+ : N(\mathfrak{a}) \leq x} f(\mathfrak{a})$ , uniformly in  $x \geq 1$ , we have

$$S(x) = \sum_{\substack{\mathfrak{b}, \mathfrak{c} \in \mathcal{S}_K^+ \\ N(\mathfrak{b}\mathfrak{c}) \leq x}} g(\mathfrak{b}) \sum_{\substack{\mathfrak{c} \in \mathcal{S}_K^+ \\ N(\mathfrak{c}) \leq x/N(\mathfrak{b})}} 1$$

$$\begin{aligned}
 &= \sum_{\substack{\mathfrak{b} \in \mathcal{I}_K^+ \\ N(\mathfrak{b}) \leq x}} g(\mathfrak{b}) \left( \frac{2\pi x}{|\mathcal{O}_K^\times| \sqrt{|D_K|} N(\mathfrak{b})} + O\left(\frac{x^{1/2}}{N(\mathfrak{b})^{1/2}}\right) \right) \\
 &= \frac{2\pi x}{|\mathcal{O}_K^\times| \sqrt{|D_K|}} \sum_{\substack{\mathfrak{b} \in \mathcal{I}_K^+ \\ N(\mathfrak{b}) \geq x}} \frac{g(\mathfrak{b})}{N(\mathfrak{b})} + O\left(x \sum_{\substack{\mathfrak{b} \in \mathcal{I}_K^+ \\ N(\mathfrak{b}) \geq x}} \frac{g(\mathfrak{b})}{N(\mathfrak{b})}\right) + O\left(x^{1/2} \sum_{\substack{\mathfrak{b} \in \mathcal{I}_K^+ \\ N(\mathfrak{b}) \leq x}} \frac{g(\mathfrak{b})}{N(\mathfrak{b})^{1/2}}\right) \\
 &= \frac{2\pi x}{|\mathcal{O}_K^\times| \sqrt{|D_K|}} \sum_{\mathfrak{b} \in \mathcal{I}_K^+} \frac{g(\mathfrak{b})}{N(\mathfrak{b})} + O(x^{1/2}) \\
 &= \frac{2\pi x}{|\mathcal{O}_K^\times| \sqrt{|D_K|}} \prod_{\mathfrak{p}} \left(1 + \frac{1}{N(\mathfrak{p})^2(N(\mathfrak{p})^2 - 2)}\right) + O(x^{1/2}) = C_1 x + O(x^{1/2}).
 \end{aligned}$$

By summation by parts, we hence have

$$\begin{aligned}
 \sum_{\mathfrak{a} \in \mathcal{I}_K^+ : N(\mathfrak{a}) \leq x} N(\mathfrak{a})^3 f(\mathfrak{a}) &= \int_1^x t^3 d[S(t)] \\
 &= [t^3(C_1 t + O(t^{1/2}))]_1^x - 3 \int_1^x t^2(C_1 t + O(t^{1/2})) dt = \frac{C_1}{4} x^4 + O(x^{7/2}).
 \end{aligned}$$

This proves Equation (66) and concludes the proof of Proposition 6.5. □

### 7 Pair correlations of common perpendiculars in the Bianchi manifolds

#### $\mathrm{PSL}(\mathcal{O}_K) \backslash \mathbb{H}_{\mathbb{R}}^3$

We again fix an imaginary quadratic number field  $K$  whose ring of integers  $\mathcal{O}_K$  is principal, and a nonzero ideal  $\Lambda = \bar{\Lambda} \in \mathcal{I}_K^+$ . In this section, we give a geometric motivation for the introduction of the Euler function as multiplicities in the family  $\mathcal{L}_{\Lambda}^{\varphi_K}$  of complex logarithms of elements of  $\Lambda$  defined in Equation (2), and we give a geometric application of the results in Sect. 5.

We refer to [5, 21] for more information on the following notions. Let  $Y$  be a nonelementary geodesically complete connected proper locally  $\mathrm{CAT}(-1)$  good orbispace, so that the underlying space of  $Y$  is  $\Gamma \backslash \tilde{Y}$  with  $\tilde{Y}$  a geodesically complete proper  $\mathrm{CAT}(-1)$  space and  $\Gamma$  a discrete group of isometries of  $\tilde{Y}$  preserving no point nor pair of points in  $\tilde{Y} \cup \partial_{\infty} \tilde{Y}$ . Let  $D^-$  and  $D^+$  be connected proper nonempty properly immersed locally convex closed subsets of  $Y$ , that is,  $D^-$  and  $D^+$  are locally finite  $\Gamma$ -orbits of proper nonempty closed convex subsets  $\tilde{D}^-$  and  $\tilde{D}^+$  of  $\tilde{Y}$ . A *common perpendicular*  $\alpha$  between  $D^-$  and  $D^+$  is the  $\Gamma$ -orbit of the unique shortest arc  $\tilde{\alpha}$  between  $\tilde{D}^-$  and  $\gamma \tilde{D}^+$  for some  $\gamma \in \Gamma$  such that  $d(\tilde{D}^-, \gamma \tilde{D}^+) > 0$ . The *multiplicity*  $\mathrm{mult}(\alpha)$  of  $\alpha$  is the ratio  $A/B$  where

- $A$  is the number of elements  $(\gamma_-, \gamma_+) \in (\Gamma/\Gamma_{D^-}) \times (\Gamma/\Gamma_{\gamma D^+})$  such that  $\tilde{\alpha}$  is the unique shortest arc between  $\gamma_- \tilde{D}^-$  and  $\gamma_+ \gamma \tilde{D}^+$ , and
- $B$  is the cardinality of the pointwise stabiliser of  $\tilde{\alpha}$  in  $\Gamma$ .

The *length*  $\lambda(\alpha)$  of the common perpendicular  $\alpha$  is the length of the geodesic segment  $\tilde{\alpha}$  in  $\tilde{Y}$ . For every  $\ell$  in the set  $\mathrm{OL}^{\natural}(D^-, D^+)$  of lengths of common perpendiculars, the *length multiplicity* of  $\ell$  is the sum of the multiplicities of the common perpendiculars between

$D^-, D^+$  having the length  $\ell$  :

$$\omega(\ell) = \sum_{\substack{\alpha \text{ common perpendicular} \\ \text{between } D^- \text{ and } D^+ \text{ with } \lambda(\alpha)=\ell}} \text{mult}(\alpha). \tag{68}$$

If  $\text{Perp}(D^-, D^+)$  is the set of all common perpendiculars from  $D^-$  to  $D^+$  with multiplicities, then  $(\lambda(\alpha))_{\alpha \in \text{Perp}(D^-, D^+)}$  is the *marked ortholength spectrum* from  $D^-$  to  $D^+$ , and the set  $\text{OL}(D^-, D^+) = (\text{OL}^\natural(D^-, D^+), \omega)$  of the lengths of the common perpendiculars endowed with the length multiplicity  $\omega$  is the *ortholength spectrum* from  $D^-$  to  $D^+$ .

As defined in [22, §6], the *pair correlation measure of the common perpendiculars from  $D^-$  to  $D^+$*  is the pair correlation measure of the family

$$\mathcal{A}_{D^-, D^+} = \left( (\mathcal{A}_N^{D^-, D^+} = \text{OL}^\natural(D^-, D^+) \cap [0, 2 \ln N])_{N \in \mathbb{N}}, \omega \right).$$

Let us specialise these objects as follows. Let

$$\tilde{Y} = \mathbb{H}_{\mathbb{R}}^3 = \left( \{(z = x + iy, t) \in \mathbb{C} \times \mathbb{R} : t > 0\}, ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2} \right)$$

be the upper halfspace model of the real hyperbolic 3-space with constant curvature  $-1$ . We identify as usual its space at infinity  $\partial_\infty \mathbb{H}_{\mathbb{R}}^3 = (\mathbb{C} \times \{0\}) \cup \{\infty\}$  with  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ .

For every  $\mathfrak{b} \in \mathcal{I}_K^+$ , let  $\Gamma_0[\mathfrak{b}]$  be *Hecke’s congruence subgroup modulo  $\mathfrak{b}$*  of the *Bianchi group*  $\text{PSL}_2(\mathcal{O}_K)$ , which is the preimage of the upper triangular subgroup of  $\text{PSL}_2(\mathcal{O}_K/\mathfrak{b})$  under the reduction morphism  $\text{PSL}_2(\mathcal{O}_K) \rightarrow \text{PSL}_2(\mathcal{O}_K/\mathfrak{b})$ . It acts faithfully on  $\mathbb{H}_{\mathbb{R}}^3$  by Poincaré’s extension, and is a lattice in the isometry group of  $\mathbb{H}_{\mathbb{R}}^3$ . Let  $Y^{\mathfrak{b}} = \Gamma_0[\mathfrak{b}] \backslash \mathbb{H}_{\mathbb{R}}^3$ , which is a finite (possibly ramified) cover of the *Bianchi orbifold*  $\text{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}_{\mathbb{R}}^3$ . Note that since  $\mathcal{O}_K$  is principal, this Bianchi orbifold has only one cusp (the number of cusps being the class number of  $K$ , see for instance [9, Sect. 7.2]).

Let  $\tilde{D}^- = \tilde{D}^+$  be the horoball  $\mathcal{H}_\infty = \{(z, t) \in \mathbb{H}_{\mathbb{R}}^3 : t \geq 1\}$  in  $\mathbb{H}_{\mathbb{R}}^3$ , whose image  $D^- = D^+$  in  $Y^{\mathfrak{b}}$  is a *Margulis neighbourhood* of a cusp of  $Y^{\mathfrak{b}}$ . In order to emphasise the dependence on the ideal  $\mathfrak{b}$ , we will use the notation  $\mathcal{A}_{D^-, D^+}^{\mathfrak{b}} = \mathcal{A}_{D^-, D^+}$  for the family of lengths of common perpendiculars between  $D^-$  and  $D^+$  in  $Y^{\mathfrak{b}}$ .

The following result relates the pair correlation measures of the common perpendiculars from this Margulis cusp neighbourhood to itself to the pair correlation measures of the complex logarithms of the elements of  $\Lambda = \mathfrak{b}$ , with multiplicities given by the Euler function  $\varphi_K$ . As explained in Remark 2.3, in the following result, we remove from the index set  $I_N$  of the summations defining  $\mathcal{R}_N^{\mathcal{A}_{D^-, D^+}^{\mathfrak{b}}, 1}$  and  $\mathcal{R}_N^{\mathcal{L}_{\mathfrak{b}}^{\varphi_K}, 1}$  the assumption that  $m \neq n$ . Recall that the map  $2 \text{Re} : E \rightarrow \mathbb{R}$  is a continuous proper map.

**Proposition 7.1** *For every ideal  $\mathfrak{b} \in \mathcal{I}_K^+$ , we have*

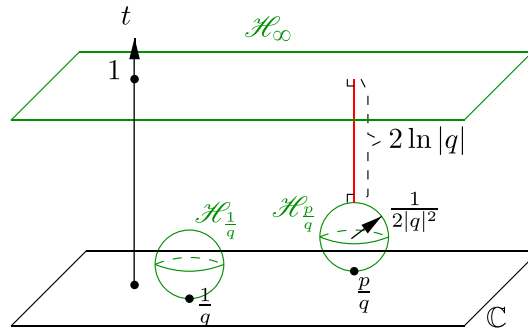
$$\mathcal{R}_N^{\mathcal{A}_{D^-, D^+}^{\mathfrak{b}}, 1} = \frac{4}{|\mathcal{O}_K^\times|^4} (2 \text{Re})_* \left( \mathcal{R}_N^{\mathcal{L}_{\mathfrak{b}}^{\varphi_K}, 1} \right).$$

*Proof* The orbit of  $\mathcal{H}_\infty$  under  $\Gamma_0[\mathfrak{b}]$  consists, besides  $\mathcal{H}_\infty$  itself, of the Euclidean 3-balls  $\mathcal{H}_q^{\mathbb{C}}$  of Euclidean radius  $\frac{1}{2|q|^2}$  tangent to the horizontal plane  $\mathbb{C}$  at the rational elements  $\frac{p}{q}$  (with this point removed), with  $p \in \mathcal{O}_K, q \in \mathfrak{b} \setminus \{0\}$  and  $(p, q) = 1$ . Note that  $\mathcal{H}_\infty$  meets  $\mathcal{H}_q^{\mathbb{C}}$  (and then is tangent to it) if and only if  $q \in \mathcal{O}_K^\times$ , since the hyperbolic distance between  $\mathcal{H}_\infty$  and  $\mathcal{H}_q^{\mathbb{C}}$  is equal to  $2 \ln |q|$ .



Every common perpendicular between  $D^-$  and  $D^+$  has a vertical representative in  $\mathbb{H}_{\mathbb{R}}^3$  which starts from a point in  $\partial \mathcal{H}_{\infty} = \mathbb{C} \times \{1\}$  and ends on the boundary of  $\mathcal{H}_{\frac{p}{q}}$  with  $\frac{p}{q}$  as above and  $q \notin \mathcal{O}_K^{\times}$ . Its hyperbolic length is  $2 \ln |q|$ . In particular, the set  $OL^{\mathbb{Z}}(D^-, D^+)$  is equal to

$$\{2 \ln |q| = 2 \operatorname{Re}(\log q) : q \in \mathfrak{b} \setminus \mathcal{O}_K^{\times}\}.$$



The stabiliser of  $\mathcal{H}_{\infty}$ , or equivalently of  $\infty$ , in  $\Gamma_0[\mathfrak{b}]$  is the upper triangular subgroup  $U$  of  $\Gamma_0[\mathfrak{b}]$ , hence of  $\operatorname{PSL}_2(\mathcal{O}_K)$ . It contains the upper unipotent subgroup consisting of translations by  $\mathcal{O}_K$  with finite index, equal to  $\frac{|\mathcal{O}_K^{\times}|}{2}$ . Hence given a denominator  $q \in \mathfrak{b} \setminus \mathcal{O}_K^{\times}$ , the points at infinity with denominator  $q$  of the vertical geodesic lines containing a lift of a common perpendicular between  $D^-$  and  $D^+$  are, modulo translation by  $\mathcal{O}_K$ , exactly the points  $\frac{p}{q}$  where  $p$  ranges over a set of representatives of  $(\mathcal{O}_K/q\mathcal{O}_K)^{\times}$ . Note that for any unit  $u \in \mathcal{O}_K^{\times}$ , we have  $\frac{up}{uq} = \frac{p}{q}$ . Thus, the number modulo  $U$  of fractions  $\frac{p}{q}$  with  $|q| = n$  is  $\frac{2}{|\mathcal{O}_K^{\times}|^2} \sum_{|q|=n} \varphi_K(q)$ .

By Equation (5), the map  $z \mapsto \ell = 2 \operatorname{Re}(z)$  from  $L_N^{\mathfrak{b}} = \{\log q : q \in (\mathfrak{b} \setminus \{0\}), |q| = n\}$  to the set  $A_n^{D^-, D^+}$  of lengths of the common perpendiculars between  $D^-$  and  $D^+$  with length  $2 \ln n$  hence sends the sum of the Euler weights  $\sum_{|q|=n} \varphi_K(q)$  to  $\frac{|\mathcal{O}_K^{\times}|^2}{2}$  times the multiplicity  $\omega(\ln n)$  defined in Equation (68) of the common perpendiculars of length  $\ell$ . The claim follows.  $\square$

The following result computes the pair correlation function without scaling of the lengths of the common perpendiculars from the Margulis cusp neighbourhood at infinity to itself in the Hecke-Bianchi orbifold  $\Gamma_0[\mathfrak{b}] \backslash \mathbb{H}_{\mathbb{R}}^2$ , giving a new proof of this special case of [23, Cor. 4.2], see also the remark after Corollary 2.5. The maps  $\operatorname{Re} : E_N \rightarrow \mathbb{R}$  for  $N \in \mathbb{N}$  being not uniformly proper, the case with scalings requires a new analysis.

**Corollary 7.2** *For every ideal  $\mathfrak{b} \in \mathcal{I}_K^+$ , as  $N \rightarrow +\infty$ , the pair correlation measures  $\mathcal{R}_N^{\mathcal{A}_{D^-, D^+}^{\mathfrak{b}}}$  on  $\mathbb{R}$ , renormalised to be probability measures, weak-star converge to a measure absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ , with pair correlation function given by  $s \mapsto e^{-2|s|}$ .*

*Proof* This follows from Theorem 5.1 with  $\Lambda = \mathfrak{b}$  as in the proof of Corollary 2.5, using Proposition 7.1.  $\square$

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