JYU DISSERTATIONS 763

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UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Jiayin Liu

List of included articles

This dissertation consists of an introductory part and the following three publications:

- **[A]** J. Liu, *On the dimension of Kakeya sets in the first Heisenberg group,* Proc. Amer. Math. Soc., 150(8):3445-3455, 2022.
- **[B]** J. Liu, *Dimension estimates on circular* (s,t)*-Furstenberg sets*, Annales Fennici Mathematici, 48(1):299-324, Mar. 2023.
- **[C]** K. Fässler, J. Liu, and T. Orponen, *On the Hausdorff dimension of circular Furstenberg sets*, arXiv e-prints, arXiv.2305.11587, 2023.

The author of this dissertation has actively taken part in the work of the joint paper [C].

Abstract

This thesis studies the Hausdorff dimension of variants of Kakeya sets in $\mathbb{R}^n$. It consists of an introduction and three papers.

In paper [A], we define Kakeya sets in the n-th Heisenberg group and show the sharp lower bound is 3 for the Heisenberg Hausdorff dimension of Kakeya sets in the first Heisenberg group.

In papers [B] and [C], we define circular (s,t)-Furstenberg sets F in $\mathbb{R}^2$. We prove that the Hausdorff dimension $\dim_{\mathrm{H}}(F) \geqslant \frac{t}{3} + s$ if $0 < s \leqslant 1$ and $0 < t \leqslant 3$ and $\dim_{\mathrm{H}}(F) \geqslant (2s-1)t+s$ if $1/2 < s \leqslant 1$ and $0 < t \leqslant 1$ in [B]. Moreover, we show the sharp lower bound $\dim_{\mathrm{H}}(F) \geqslant s+t$ if $0 < t \leqslant s \leqslant 1$ in [C].

Tiivistelmä

Tämä väitöskirja tarkastelee Kakeya-joukkojen muunnelmien Hausdorff-dimensiota. Väitöskirja koostuu johdannosta ja kolmesta paperista.

Paperissa [A] määrittelemme Kakeya-joukot n:nnessä Heisenbergin ryhmässä ja näytämme ensimmäisen Heisenberg-ryhmän Kakeya-joukkojen Heisenberg-Hausdorff-dimension tarkan alarajan olevan 3.

Papereissa [B] ja [C] määrittelemme ympyräiset (s, t)-Furstenberg-joukot $F \subset \mathbb{R}^2$. Paperissa [B] osoitamme, että Hausdorffin dimensio $\dim_{\mathrm{H}}(F) \geqslant \frac{t}{3} + s$, jos $0 < s \leqslant 1$ ja $0 < t \leqslant 3$ ja $\dim_{\mathrm{H}}(F) \geqslant (2s-1)t + s$, jos $1/2 < s \leqslant 1$ ja $0 < t \leqslant 1$. Paperissa [C] näytämme tarkan alarajan $\dim_{\mathrm{H}}(F) \geqslant s + t$, jos $0 < t \leqslant s \leqslant 1$.

INTRODUCTION

Estimating the dimension of various fractal sets is one of the major topics in geometric measure theory. This thesis focuses on studying the dimension of fractal sets of certain types, which originate from Kakeya sets. To be precise, the first part of the thesis concentrates on Kakeya sets in the first Heisenberg group. The second part is devoted to the study of circular Furstenberg sets.

1. Kakeya Set and Its Generalizations

We begin with the definition of Kakeya sets in $\mathbb{R}^n$.

Definition 1.1 (Kakeya Set)**.** A set $E \subset \mathbb{R}^n$ is a Kakeya set if for every $e \in S^{n-1}$ there exists a unit line segment I_e parallel to e such that $I_e \subset E$.

Kakeya sets are also known as Besicovitch sets. Directly from the definition, we know that the unit ball $B(0,1) \subset \mathbb{R}^n$ is a Kakeya set which reaches the full dimension n. So it is natural to investigate the dimension lower bound for Kakeya sets. Here the dimension we consider is Hausdorff dimension.

Definition 1.2 (Hausdorff dimension)**.** Let X be a metric space. For $\alpha \in [0, \infty)$ and $E \subset X$, define the α-Hausdorff measure of E by

$$\mathcal{H}^\alpha(E) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E)$$

where

$$\mathcal{H}^\alpha_\delta(E) := \inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} U_j)^\alpha : \bigcup_j U_j \supset E, \ \operatorname{diam} U_j < \delta \right\}$$

where the infimum is taken over all countable covers $U = \{U_j\}$ of E.

The Hausdorff dimension of E is

$$\dim_{\mathrm{H}}(E) := \sup\{\alpha : \mathcal{H}^\alpha(E) = \infty\} = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\}.$$

The following conjecture is known as the Kakeya conjecture.

Conjecture 1.3. *Every Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n.*

Among those who were devoted to resolving this conjecture, the first successful attempt was by Davies [4] in 1971. He confirmed this conjecture for the case $n = 2$. For $n \geqslant 3$, the conjecture is still open and many mathematicians have made partial progress: Bourgain used two different methods to provide lower bounds [1, 2], which were further improved by Wolff [31] and Katz-Tao [14] respectively. Recently, Katz-Zahl [18] and Guth-Zahl [9] enhanced the results of [31] in $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively, which show that the best known lower bound is $5/2 + \epsilon_0$ in $\mathbb{R}^3$ with ϵ_0 an absolute constant and $3 + 1/40$ in $\mathbb{R}^4$. In $\mathbb{R}^n$ with $n \geqslant 5$, the best known lower bound is given by a combination of the results obtained in [16] by Katz-Tao and in [13] by Hickman-Rogers-Zhang. In addition, Wang-Zahl in [30] verified Conjecture 1.3 for the special class of sticky Kakeya sets in $\mathbb{R}^3$. For another special class of SL_2 Kakeya sets in $\mathbb{R}^3$, Fässler-Orponen [6] and Katz-Wu-Zahl [17] verified Conjecture 1.3 using two different approaches.

Since this thesis does not aim to study Kakeya sets in $\mathbb{R}^n$, we will move on to its generalizations. And for a comprehensive study of Kakeya sets and its role in geometric measure theory, we refer to two books [21, 22] by Mattila and references therein.

1.1. **Linear Furstenberg Sets.** From the definition of Kakeya sets, one can also extend this notion to similar types of sets. First, one can generalize this notion to a broader class of sets that contain Kakeya Sets as a special class. We name these sets *linear Furstenberg sets* in this thesis. We first need the following concept.

Definition 1.4 ($\mathcal{A}(n,k)$ and $\mathcal{G}(n,k)$)**.** The space of all affine k-dimensional hyperplanes in $\mathbb{R}^n$ is denoted $\mathcal{A}(n,k)$. The space of k-dimensional subspaces of $\mathbb{R}^n$ is denoted $\mathcal{G}(n,k)$. Every plane $W \in \mathcal{A}(n,k)$ can be expressed uniquely as $W = a + V$, where $V \in \mathcal{G}(n,k)$, and $a = a(W) \in V^\perp$. This observation allows us to metrize $\mathcal{A}(n,k)$ by setting

$$d(W,W') := \|\pi_V - \pi_{V'}\|_{\mathrm{op}} + |a - a'|,$$

where $W = V + a$, $W' = V' + a'$, and $\|\cdot\|_{\mathrm{op}}$ refers to the operator norm.

Definition 1.5 (Linear Furstenberg sets)**.** Let $1 \leqslant k \leqslant n$, $0 < s \leqslant k$ and $0 < t \leqslant \dim \mathcal{A}(n,k)$. A set $F \subset \mathbb{R}^n$ is called a linear (s,t,k)-Furstenberg set if there exists a *parameter set* $K \subset \mathcal{A}(n,k)$ with

$$\dim_{\mathrm{H}} K \geqslant t$$

such that for every $W \in K$,

$$\dim_{\mathrm{H}}(F \cap W) \geqslant s.$$

From Definition 1.5, we see that Kakeya sets in $\mathbb{R}^n$ are special linear $(1, n-1, 1)$-Furstenberg sets with the parameter set K satisfying $\dim K = \dim \mathcal{G}(n,1) = n-1$.

Since most studies of linear Furstenberg sets are concentrated on linear $(s,t,1)$-Furstenberg sets in $\mathbb{R}^2$ in the literature, we write linear (s,t)-Furstenberg sets for simplicity if $k = 1$ and $n = 2$.

In 1999, Wolff [33] initiated the study and showed that linear $(s,1)$-Furstenberg sets with parameter set K containing lines in every direction have Hausdorff dimension at least

$$\max\{\frac{1}{2} + s, 2s\} \text{ for all } 0 < s \leqslant 1. \tag{1.1}$$

After Wolff, general linear (s,t)-Furstenberg sets have been constantly studied. Noting that $n = 2, k = 1$ implies that $\dim \mathcal{A}(2,1) = 2$, this shows all possible values of s and t are $s \in (0,1]$ and $t \in (0,2]$. It is conjectured that the sharp lower bounds for the Hausdorff dimension of linear (s,t)-Furstenberg sets is

$$\min\{s+t, \frac{3s+t}{2}, s+1\}. \tag{1.2}$$

If $0 < s \leqslant t \leqslant 1$, the sharp lower bound is $s + t$, which was shown by Héra-Shmerkin-Yavicoli [12] and Lutz-Stull [20] using two very different proofs. Besides, if $s + t \geqslant 2$, the sharp lower bound is $s + 1$, which was obtained by Fu-Ren [8]. For other values of s and t, the sharp lower bound is $\frac{3s+t}{2}$ is verified by Ren-Wang [25]. We also refer to [3, 8, 10, 24, 27] and references therein for partial progress.

In terms of linear (s, t, k)-Furstenberg sets in $\mathbb{R}^n$, we refer to [3, 10, 11] and references therein.

1.2. **Circular Furstenberg Sets.** In this subsection, we restrict ourselves to $\mathbb{R}^2$. For the second variant of Kakeya sets, we substitute "lines" by "circles" in the definition.

Definition 1.6 (Circular Kakeya Sets)**.** A set $F \subset \mathbb{R}^2$ is called a circular Kakeya set if it contains circles of every radius.

Wolff in [32] showed that circular Kakeya sets in $\mathbb{R}^2$ have full dimension 2 employing techniques from harmonic analysis.

After paper [32], more general families of circles have been studied. Noting that a circle $S(x, r)$ in $\mathbb{R}^2$ is uniquely determined by its center $x \in \mathbb{R}^2$ and its radius $r \in \mathbb{R}_+ = (0, +\infty)$, we can identify a circle in $\mathbb{R}^2$ by a point $(x, r) \in \mathbb{R}^3_+ := \mathbb{R}^2 \times \mathbb{R}_+$. This is a one-to-one correspondence. In the following, we say $\mathcal{S} = \{S(x, r)\}$ is a t-dimensional family of circles in $\mathbb{R}^2$ if $\{(x, r)\}_{S(x,r)\in\mathcal{S}}$ forms a t-dimensional set in $\mathbb{R}^3_+$.

In [34], Wolff proved that a subset in $\mathbb{R}^2$ consisting of circles with 1-dimensional family of radii has Hausdorff dimension at least 2. Also, in [19], motivated by a paper [26] of Schlag, Käenmäki-Orponen-Venieri were able to show that the sharp lower bound $1 + t$ in [32] holds true for any analytic t-dimensional family of circles.

Continuing the study in [19], in papers [B] and [C], we extend the above study to more general circular Furstenberg sets and study their dimension.

Definition 1.7 (Circular Furstenberg sets)**.** Let $0 < s \leqslant 1$ and $0 < t \leqslant 3$. A set $F \subset \mathbb{R}^2$ is called a circular (s, t)-Furstenberg set if there exists a *parameter set* $K \subset \mathbb{R}^3_+$ with

$$\dim_{\mathrm{H}} K \geqslant t$$

such that for every $(x, r) \in K$,

$$\dim_{\mathrm{H}}(F \cap S(x, r)) \geqslant s.$$

In papers [B] and [C], we proved the following dimension lower bounds for circular Furstenberg sets.

Theorem 1.8 (Main result of [B])**.** *For any $0 < s \leqslant 1$ and $0 < t \leqslant 3$, the Hausdorff dimension of a circular (s, t)-Furstenberg set F in $\mathbb{R}^2$ is at least $\frac{t}{3} + s$.*

Theorem 1.9 (Main result of [B])**.** *For any $1/2 < s \leqslant 1$ and $0 < t \leqslant 1$, the Hausdorff dimension of a circular (s, t)-Furstenberg set F in $\mathbb{R}^2$ is at least $(2s - 1)t + s$.*

Theorem 1.10 (Main result of [C])**.** *For $0 < t \leq s \leq 1$, the Hausdorff dimension of a circular (s, t)-Furstenberg set F in $\mathbb{R}^2$ is at least $s + t$.*

We remark that in paper [B], the range of t in Theorem 1.8 is stated for $t \in (0,1]$. However, the proof works for all $t \in (0,3]$.

Recently, after paper [C], Zahl [35] extends Theorem 1.10 to more general Furstenberg set of curves.

1.3. **Kakeya Sets in Heisenberg Groups.** For the third generalization of Kakeya sets, we change the ambient space in Definition 1.1 from Euclidean spaces to Heisenberg groups. We denote by $\mathbb{H}^n$ the n-th Heisenberg group. When $n = 1$, we write $\mathbb{H}$ instead of $\mathbb{H}^1$ for simplicity.

Definition 1.11 (Heisenberg Kakeya Sets)**.** A set $E \subset \mathbb{H}^n$ is a Heisenberg Kakeya set if for every unit line segment $I \subset \mathbb{R}^{2n} \times \{0\}$ centred at the origin, there exists $q \in \mathbb{H}^n$ such that $qI \subset E$.

For an introduction to Heisenberg groups, we refer the readers to Section 2. Compared with the study in the Euclidean case, the study of Heisenberg Kakeya sets has relatively limited results in the literature. Indeed, in [28], Venieri studied the Heisenberg Hausdorff dimension $\dim_{\mathrm{H}}^{\mathbb{H}} E$ of Euclidean Kakeya sets E in $\mathbb{H}^n = \mathbb{R}^{2n+1}$ and showed that $\dim_{\mathrm{H}}^{\mathbb{H}} E \geqslant \frac{2n+5}{2}$ if $n \leqslant 3$ and $\dim_{\mathrm{H}}^{\mathbb{H}} E \geqslant \frac{8n+14}{7}$ if $n \geqslant 4$. In [29], Venieri further studied Kakeya sets for general metric spaces in an axiomatic sense.

In paper [A], we proved the sharp lower bound for Heisenberg Kakeya sets in the first Heisenberg group.

Theorem 1.12 (Main result of [A])**.** *In the first Heisenberg group $\mathbb{H}$ equipped with the Korányi metric, every Heisenberg Kakeya set has Heisenberg Hausdorff dimension at least 3 and this lower bound is sharp.*

The sharpness of the bound in Theorem 1.12 can be easily seen, since the $\{xoy\}$-plane in $\mathbb{H}$, which has Heisenberg Hausdorff dimension 3, is a Heisenberg Kakeya set.

After paper [A], Fässler-Pinamonti-Wald reproved Theorem 1.12 as a corollary of their Heisenberg Kakeya maximal function inequality established in [7]. For $0 < t \leqslant 3$ and a general set E consisting of a t-dimensional family of horizontal lines in $\mathbb{H}$, Fässler-Orponen showed that the Euclidean Hausdorff dimension of E is $\min\{t+1, 3\}$ in [6].

Below in Section 2, we review the contents of paper [A] and in Section 3, we review the contents of papers [B] and [C].

Notation. The notation $|A|$ refers to the cardinality of set $A \subset \mathbb{R}^d$ if A is a finite set and refers to the d-dim Lebesgue measure of $A \subset \mathbb{R}^d$ if A is an infinite set. For $r \in 2^{-\mathbb{N}} = \{2^{-k} : k \in \mathbb{N}\}$, the notation $|E|_r$ refers to the r-covering number of E.

The notation $A \lesssim B$ means that there exists an absolute constant $C \geq 1$ such that $A \leq CB$. Since this introduction aims to provide heuristical ideas of the proof, for $\delta \in (0,1]$, we will abuse the notation $A \lesssim B$ to also denote

$$A \leq C \cdot \left(1 + \log\left(\tfrac{1}{\delta}\right)^C\right) B.$$

And below (3.12), when the auxiliary constant ϵ is chosen, we keep using $A \lesssim B$ to denote $A \leq \delta^{-C\epsilon}B$ for some absolute constant $C \geq 1$. The two-sided inequality $A \lesssim B \lesssim A$ is abbreviated to $A \sim B$. If the constant C is allowed to depend on a parameter "θ", we indicate this by writing $A \lesssim_\theta B$.

2. Dimension of Kakeya Sets in Heisenberg Groups

In this section, we review the contents of paper [A]. We aim to explain the idea of showing Theorem 1.12. We begin with the introduction of the first Heisenberg group $\mathbb{H}$.

The first Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ equipped with the group multiplication, for any $w = (x, y, t)$ and $w' = (x', y', t')$, as follows

$$w \cdot w' = \left(x + x', y + y', t + t' + \frac{1}{2}[xy' - x'y]\right). \tag{2.1}$$

We introduce the Korányi metric on the first Heisenberg group. This is the left invariant metric given by

$$d_{\mathbb{H}}(w, w') := \|(w')^{-1} \cdot w\|_{\mathbb{H}} \tag{2.2}$$

where $\|\cdot\|_{\mathbb{H}}$ is defined as

$$\|(x, y, t)\|_{\mathbb{H}} = ((x^2 + y^2)^2 + 16t^2)^{1/4}.$$

Since we will need the Hausdorff measure and dimension induced by both the Euclidean metric and $d_{\mathbb{H}}$, we denote by $\mathcal{H}^s_{\mathbb{H}}$ (resp. $\mathcal{H}^s_{\mathbb{R}}$) the s-dimensional Hausdorff measure induced by the Korányi metric (resp. Euclidean metric) and by $\dim^{\mathbb{H}}_{\mathrm{H}}$ (resp. $\dim^{\mathbb{R}}_{\mathrm{H}}$) the Hausdorff dimension of sets induced by Korányi metric (resp. Euclidean metric).

We define horizontal lines in the first Heisenberg group $\mathbb{H}$ as lines which can be obtained by a left translation of some line passing through the origin and lying in the $\{xoy\}$-plane, the 2-dimensional subspace spanned by the first two coordinates.

By the definition of horizontal lines, we know that for any $b \in \mathbb{R}$ and $q \in \mathbb{H}$, qI_b and qJ_b are a horizontal line and a horizontal line segment respectively where

$$I_b(\tau) = (\tau, b\tau, 0),\ \ \tau \in \mathbb{R} \tag{2.3}$$

and

$$J_b(\tau) = (\tau, b\tau, 0),\ \ \tau \in \Big(-\frac{1}{2\sqrt{b^2+1}}, \frac{1}{2\sqrt{b^2+1}}\Big). \tag{2.4}$$

The domain $(-\frac{1}{2\sqrt{b^2+1}}, \frac{1}{2\sqrt{b^2+1}})$ of τ in (2.4) guarantees that J_b has unit length with respect to $d_{\mathbb{H}}$.

In fact, (2.3) characterizes all the horizontal lines passing through the origin except the y-axis. Thus, $L(\mathbb{H}) := \{qI_b\}_{b\in\mathbb{R}, q\in\mathbb{H}}$ is the family of all the horizontal lines that are not parallel to the $\{yot\}$-plane since the Heisenberg multiplication restricted to the first two coordinates coincides with the addition in $\mathbb{R}^2$. Furthermore, the horizontal lines in $L(\mathbb{H})$ have the following parametrization,

$$L(\mathbb{H}) = \{l_{(a,b,d)} := (s, bs + a, -\frac{as}{2} + d),\ \ s \in \mathbb{R} : a, b, d \in \mathbb{R}\}. \tag{2.5}$$

Similarly, all the horizontal line segments $\{qJ_b\}_{b\in\mathbb{R},q\in\mathbb{H}}$ have following parametrization,

$$l^{\epsilon}_{(a,b,d)} := \left\{(s, bs+a, -\frac{as}{2}+d) \in \mathbb{H} : s \in (\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}})\right\}.$$

A merit of the above parametrization is that, for a Kakeya set $E \subset \mathbb{H}$, we know for each J_b, $b \in \mathbb{R}$, there exists a copy of J_b under left translation that is contained in E. This implies that for each $b \in \mathbb{R}$, there exists $a = a(b), d = d(b), \epsilon = \epsilon(b)$ such that $l^{\epsilon}_{(a,b,d)} \subset E$.

The above observation motivates us to define the following set

$$L(E) := \{(a, b, d, \epsilon) \in \mathbb{R} \times (-\sqrt{3}, \sqrt{3}) \times \mathbb{R} \times \mathbb{R} : l^{\epsilon}_{(a,b,d)} \subset E\}. \tag{2.6}$$

The reason why we restrict $b \in (-\sqrt{3}, \sqrt{3})$ is that the range $(-\sqrt{3}, \sqrt{3})$ guarantees the Euclidean orthogonal projection of $l^{\epsilon}_{(a,b,d)}$ to x-axis has length larger than $1/2$. This property, combined with some fundamental measure theory, enables us to further find an interval $[c_0, c_0+1/4]$ and a Borel set $B \subset \mathbb{R}^3$ with $\dim^{\mathbb{R}}_{\mathrm{H}}(B) \geqslant 1$ such that

$$B \times \{c\} \subset L(E), \qquad c \in [c_0, c_0+1/4]$$

and

$$l^{\epsilon}_{(a,b,d)} \cap \{x = c\} \neq \emptyset, \qquad (a, b, d) \in B,\ c \in [c_0, c_0+1/4].$$

We have found a 1-dimensional family of parallel planes $\{x = c\}_{c\in[c_0,c_0+1/4]}$ such that $E_c := l^{\epsilon}_{(a,b,d)} \cap \{x = c\} \neq \emptyset$. In particular, $E_c \subset E \cap \{x = c\} \subset E$. Thus, if we can show, for almost every $c \in [c_0, c_0+1/4]$,

$$\dim^{\mathbb{H}}_{\mathrm{H}}(E_c) = 2, \tag{2.7}$$

then, noting that the map $f : (\mathbb{H}, d_{\mathbb{H}}) \to \mathbb{R}, (x, y, t) \to (x, 0, 0)$ is 1-Lipschitz and letting $F = E \cap \{(x, y, t) \in \mathbb{H} \mid x \in [c_0, c_0+1/4]\}$, for any $0 < \alpha < 2$, using a co-area inequality from [5], that is,

$$\mathcal{H}^{\alpha+1}_{\mathbb{H}}(F) \geqslant \int^{*}_{[c_0,c_0+1/4]} \mathcal{H}^{\alpha}_{\mathbb{H}}(F \cap f^{-1}(y))\, dy = \infty, \tag{2.8}$$

we derive

$$\dim^{\mathbb{H}}_{\mathrm{H}}(E) = 3. \tag{2.9}$$

This concludes the proof.

Here we remark that in (2.8), $\int^{*}_{\mathbb{R}} g\, dy$ is the upper integral of $g : \mathbb{R} \to [0, +\infty)$. That is

$$\int^{*}_{\mathbb{R}} g(y)\, dy = \inf \int_{\mathbb{R}} h(y)\, dy$$

where the infimum is taken over all measurable functions $h : \mathbb{R} \to [0, +\infty)$ satisfying $0 \leqslant g(y) \leqslant h(y)$ for a.e. $y \in \mathbb{R}$.

Indeed, (2.7) is established with the help of the following Marstrand-type projection theorem in $\mathbb{R}^3$ by Käenmäki-Orponen-Venieri in [19, Theorem 1.2].

Theorem 2.1. *Suppose that $\gamma : [0, 2\pi) \to S^2, \theta \mapsto \gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$. If $K \subset \mathbb{R}^3$ is a Borel set, then $\dim_{\mathrm{H}}^{\mathbb{R}} \rho_{\gamma(\theta)}(K) = \min\{\dim_{\mathrm{H}}^{\mathbb{R}} K, 1\}$ for almost every $\theta \in [0, 2\pi)$ where for any $x \in S^2$, $\rho_x : \mathbb{R}^3 \to Span(x)$ denotes the Euclidean orthogonal projection to the line passing through the origin and x.*

We explain how to obtain (2.7) from Theorem 2.1. The idea is that using Heisenberg left translation to translate E_c to $\{yot\}$-plane for each $c \in [c_0, c_0 + 1/4]$, we get a new set E_c^1 satisfying

$$\dim_{\mathrm{H}}^{\mathbb{H}}(E_c^1) = \dim_{\mathrm{H}}^{\mathbb{H}}(E_c) \qquad c \in [c_0, c_0 + 1/4]. \tag{2.10}$$

Then we find that the third coordinate of E_c^1 (i.e. the Euclidean orthogonal projection of E_c^1 to t-axis) is precisely the Euclidean projection of B to $\mathrm{Span}(-c, -\frac{c^2}{2}, 1)$, i.e. $\rho_{(-c,-\frac{c^2}{2},1)}(B)$, which implies

$$\dim_{\mathrm{H}}^{\mathbb{H}}(E_c^1) \geqslant \dim_{\mathrm{H}}^{\mathbb{H}}(\rho_{(-c,-\frac{c^2}{2},1)}(B)) \qquad c \in [c_0, c_0 + 1/4]. \tag{2.11}$$

Now $\{\rho_{(-c,-\frac{c^2}{2},1)}(B)\}_{c\in[c_0,c_0+1/4]}$ is a family of orthogonal projections of B to the one parameter family of lines $\{\mathrm{Span}(-c, -\frac{c^2}{2}, 1)\}_{c\in[c_0,c_0+1/4]}$ in $\mathbb{R}^3$. Using some basic geometry, one can find that this one parameter family of lines happens to be the one in Theorem 2.1. Applying this theorem, we arrive at

$$1 = \dim_{\mathrm{H}}^{\mathbb{R}}(\rho_{(-1,\frac{c}{2},c^2)}(B)) = \frac{1}{2}\dim_{\mathrm{H}}^{\mathbb{H}}(\rho_{(-c,-\frac{c^2}{2},1)}(B)) \quad a.e.\ c \in [c_0, c_0 + 1/4] \tag{2.12}$$

where in the second equality we use the property that the Heisenberg Hausdorff dimension for any subset in t-axis is twice as its Euclidean Hausdorff dimension.

Combining (2.10), (2.11) and (2.12), we arrive at (2.7). This completes the sketch of the main ideas in the proof of Theorem 1.12.

3. Dimension of Circular Furstenberg Sets

In this section, we review the contents in papers [B] and [C]. We aim to sketch the proofs of Theorem 1.8, Theorem 1.9 and Theorem 1.10.

One common point of the above three theorems is that these theorems are all proved using a δ-discretized version of circular Furstenberg sets for arbitrary small $\delta > 0$. However, in Theorem 1.8 and Theorem 1.9, the δ-discretized version of circular Furstenberg sets we used is defined using the so called Katz-Tao (δ, s)-sets introduced by Katz-Tao in [15] while in Theorem 1.10, we use another type of (δ, s)-sets introduced by Orponen-Shmerkin in [23]. We give the definitions.

Definition 3.1 (Katz-Tao (δ, s, C)-set)**.** Let $s \geq 0$, $C > 0$, and $\delta > 0$. A bounded δ-separated set $A \subset \mathbb{R}^n$ is called a Katz-Tao (δ, s, C)-set if for all $r \in [\delta, 1]$ and $x \in \mathbb{R}^n$,

$$|A \cap B(x, r)| \leqslant C\left(\frac{r}{\delta}\right)^s.$$

Definition 3.2 (((δ, s, C)-set)**.** Let $s \geq 0$, $C > 0$, and $\delta > 0$. A bounded δ-separated set $A \subset \mathbb{R}^n$ is called a (δ, s, C)-set if for all $r \in [\delta, 1]$ and $x \in \mathbb{R}^n$,

$$|A \cap B(x, r)| \leq C r^s |A|.$$

Here $|\cdot|$ denotes the cardinality of a set and we note that the constant C in the above two definitions can depend on δ and when C is absolute we also write Katz-Tao (δ, s)-set or (δ, s)-set instead.

Remark 3.3. Note that if a Katz-Tao (δ, s)-set A has cardinality $|A| = C\delta^{-s}$ for some absolute constant $C > 0$, then A is also a (δ, s)-set and vice versa. Indeed, in the proof of the three theorems, we finally need to assume a (δ, s)-set A has cardinality $|A| = C\delta^{-s}$ though in many middle steps we only need properties of A being Katz-Tao (δ, s)-set or (δ, s)-set. Thus, in this introduction, we always assume both types of (δ, s)-set A satisfy $|A| = C\delta^{-s}$ and there is no need to distinguish two concepts of (δ, s)-set.

For $p = (x, r) \in \mathbb{R}^2 \times (0, \infty)$ (typically $p \in \mathbf{D}$ where $\mathbf{D}$ is defined in (3.1)), we write $S(p) = S(x, r)$ for the circle centred at x and radius $r > 0$. The notation $S^\delta(p)$ refers to the δ-annulus around $S(p)$, thus $S^\delta(p) = \{w \in \mathbb{R}^2 : \operatorname{dist}(w, S(p)) \leq \delta\}$.

Definition 3.4. Let $s, t \in (0, 1]$, $C > 0$, and $\delta \in 2^{-\mathbb{N}} := \{2^{-k} : k \in \mathbb{N}\}$. A (δ, s, t, C)-*configuration* is a set $\Omega \subset \mathbb{R}^5$ such that

$$\Omega := \bigcup_{p \in P} (p, E(p)).$$

Here,

(i) $P := \pi_{\mathbb{R}^3}(\Omega)$ is a non-empty (δ, t, C)-subset of

$$\mathbf{D} := \{(x, r) \in \mathbb{R}^2 \times [0, \infty) : |x| \leq \tfrac{1}{4} \text{ and } r \in [\tfrac{1}{2}, 1]\}, \tag{3.1}$$

where $\pi_{\mathbb{R}^3}(x_1, \ldots, x_5) = (x_1, x_2, x_3)$;

(ii) $E(p) := \{v \in \mathbb{R}^2 : (p, v) \in \Omega\}$ is a non-empty (δ, s, C)-subset of $S(p)$ for all $p \in P$.

Additionally, we require that the sets $E(p)$ have constant cardinality: there exists $M \geq 1$ such that $|E(p)| = M$ for all $p \in P$.

Remark 3.5. The reduction to only considering the parameter set P in the domain $\mathbf{D}$ in (3.1) is standard (see, for example, Remark 2.1 in [B] for an explanation). This reduction already appeared in [33] by Wolff.

The following theorem reveals why we can estimate the Hausdorff dimension of a circular Furstenberg set by discretizing the set at a fixed scale δ.

Theorem 3.6. *Let $s \in (0, 1]$, $t \in (0, 3]$ and $\alpha \in (0, 2]$. If for every $\kappa > 0$, there exist $\epsilon(\kappa), \delta_0(\kappa) \in (0, \frac{1}{2}]$ such that for all $\delta \in (0, \delta_0]$ and all $(\delta, s, t, \delta^{-\epsilon})$-configurations Ω*

$$|\mathcal{F}|_\delta \geq \delta^{\kappa - \alpha},$$

where

$$\mathcal{F} := \bigcup_{p \in P} E(p),$$

and we recall $|\mathcal{F}|_\delta$ refers to the δ-covering number of $\mathcal{F}$, then every circular (s,t)-Furstenberg set F has Hausdorff dimension at least α.

Note that in paper [C], the symbol $|\mathcal{F}|_\delta$ refers to the number of dyadic δ-cubes intersecting $\mathcal{F}$, which is comparable to the δ-covering number of $\mathcal{F}$. The choice of using dyadic δ-cubes instead of δ-balls will make it easier to formulate several proofs in paper [C].

Thanks to Theorem 3.6, we will focus on providing a lower bound of $|\mathcal{F}|_\delta$ associated with an arbitrary configuration Ω. Indeed, the proof of Theorem 3.6 is standard (see the proof Theorem 1.2 in [C]) and this argument has already been employed to study the Hausdorff dimension of linear Furstenberg sets, for example, in [12].

Specifically, in Theorem 1.8, we need to show $|\mathcal{F}|_\delta \geq \delta^{\kappa-(s+t/3)}$. To this end, we adapt the approach for showing the lower bound for the Hausdorff dimension of linear $(s,1)$-Furstenberg sets used by Wolff in [33] together with some geometric observations from planar geometry. However, the proofs of $|\mathcal{F}|_\delta \geq \delta^{\kappa-[(2t+1)s-t]}$ in Theorem 1.9 and $|\mathcal{F}|_\delta \geq \delta^{\kappa-(s+t)}$ in Theorem 1.10 differ from the above method significantly but share some similar ideas with each other. In fact, we will further transfer the proofs of these two theorems to control the upper bound of a multiplicity function. This idea was first used by Schlag [26]. In Theorem 1.9, we will directly apply a result involving upper bounds of multiplicity functions for whole circles by Käenmäki-Orponen-Venieri in [19], which is further based on the ideas of Schlag [26]. In Theorem 1.10, we deeply investigate the intersections and other properties of circles and obtain a better upper bound for the multiplicity function, which leads to the sharp lower bound for circular (s,t)-Furstenberg sets when $0 < t \leqslant s \leqslant 1$.

3.1. **Outline of Theorem 1.8.** Our goal is to show $|\mathcal{F}|_\delta \gtrsim \delta^{-(s+t/3)}$ where $\mathcal{F} = \cup_{p\in P} E(p)$ is induced by an arbitrary (δ,s,t)-configuration Ω. This is slightly inaccurate compared with Theorem 3.6 since we omit the parameters κ and ϵ. However, it is enough to illustrate the ideas behind the proof. Let $\mathcal{I}$ be the family of δ-balls covering $\mathcal{F}$ and $|\mathcal{I}| = |\mathcal{F}|_\delta$. We recall that Ω being a (δ,s,t)-configuration implies that $|P| = |\pi_{\mathbb{R}^3}(\Omega)| \sim \delta^{-t}$ and $E(p)$ is contained in the circle $S(p)$ with $|E(p)| \sim \delta^{-s}$ for all $p \in P$.

We begin by recalling the fact that three non-collinear points determine a unique circle in $\mathbb{R}^2$. Inspired by this fact, we could deduce that three well-separated δ-balls $(B_i,\ B_j,\ B_k) \in \mathcal{I}$ "determine a unique circle" $S(p)$ with $p \in P$. Here the meaning of "determine a unique circle" is understood in the way that there are $\lesssim 1$ many $p \in P$ such that

$$S(p) \cap B_l \neq \emptyset, \qquad l = i,j,k, \tag{3.2}$$

or equivalently, those points $p \in P$ such that the circles $S(p)$ enjoying (3.2) belong to a $C\delta$-ball $B_{C\delta} \cap P$ for an absolute constant $C \geqslant 1$. Furthermore, this fact can be utilised to identify the circle $S(p)$ with the triple (B_i, B_j, B_k). Indeed, the above operations are stimulated by Wolff's argument [33] to show that the lower bound for the Hausdorff dimension of linear $(s,1)$-Furstenberg sets is $1/2 + s$ in (1.1) where $1/2$ arises from the fact that two points determine a unique line in the plane. For circular $(s,1)$-Furstenberg sets, we merely obtain the lower bound $1/3 + s$ due to the fact that three points determine a circle.

In the meantime, since $|E(p)| \sim \delta^{-s}$ for all $p \in P$, we need, roughly speaking, $\sim \delta^{-s}$ many δ-balls in $\mathcal{I}$ to cover $E(p)$. Hence we could use the triples $(B_i, B_j, B_k) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I}$

to represent each $E(p)$ where $E(p) \cap B_l \neq \emptyset$ for $l = i, j, k$. Then each $E(p)$ leads to $\delta^{-s}(\delta^{-s}-1)(\delta^{-s}-2) \sim \delta^{-3s}$ many distinct triples $(B_i, B_j, B_k) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I}$ on behalf of three distinct δ-balls in $\mathcal{I}$ and as a result we get a total number of $|P| \times \delta^{-3s} = \delta^{-3s-t}$ many distinct triples. Consequently, since all these triples belong to $\mathcal{I} \times \mathcal{I} \times \mathcal{I}$, we can infer that $|\mathcal{I}|^3 \gtrsim \delta^{-3s-t}$, which gives $|\mathcal{F}|_\delta = |\mathcal{I}| \gtrsim \delta^{-s-t/3}$ as desired.

3.2. **Outline of Theorem 1.9 and Theorem 1.10.** As mentioned above, the proofs of these two theorems are based on some good control of the upper bound of the multiplicity function associated with a configuration Ω.

Definition 3.7 (Total multiplicity function)**.** Fix an arbitrary (δ, s, t)-configuration Ω. For $w \in \mathbb{R}^2$, define

$$m_\delta(w \mid \Omega) := |\{(p, v) \in \Omega : w \in B(v, \delta)\}|. \tag{3.3}$$

We briefly explain the meaning of (3.3). By the definition of Ω, we know for each $p \in P$, $E(p)$ is a (δ, s)-set. In particular, $E(p) \subset S(p)$ is δ-separated. Thus for each p, there are at most $\lesssim 1$ many $v \in E(p)$ such that $w \in B(v, \delta)$. This observation gives

$$|\{v \in E(p) : w \in B(v, \delta)\}| \lesssim 1 \qquad p \in P.$$

As a result,

$$m_\delta(w \mid \Omega) \leqslant \sum_{p \in P} |\{v \in E(p) : w \in B(v, \delta)\}| \lesssim |P| \lesssim \delta^{-t}.$$

Hence $m_\delta(w \mid \Omega)$ can be interpreted as, up to an absolute constant, the number of circles $S(p)$ with $p \in P$ such that the associated set $E(p) \subset S(p)$ is δ close to w.

Again, let $\mathcal{I}$ be the family of δ-balls covering $\mathcal{F}$ and $|\mathcal{I}| = |\mathcal{F}|_\delta$. In fact, since one needs at least $\sim \delta^{-s}$ δ-balls in $\mathcal{I}$ to cover $E(p)$ for each $p \in P$, if each δ-ball in $\mathcal{I}$ only intersects one $E(p)$ for some $p \in P$, then $\mathcal{I}$ consists of at least $\delta^{-s}|P| \sim \delta^{-s-t}$ many δ-balls. However, this may not be the case. In general, if each δ-ball in $\mathcal{I}$ intersects no more than $\delta^{-\kappa}$ $(0 < \kappa \leqslant t)$ many sets from the family $\{E(p)\}_{p \in P}$, then we can deduce that $\mathcal{I}$ consists of at least $\frac{\delta^{-s}|P|}{\delta^{-\kappa}} \sim \delta^{-(s+t-\kappa)}$ many δ-balls.

The above discussion reveals that a smaller κ will lead to a larger cardinality of $|\mathcal{I}|$, which further gives a better lower bound for the Hausdorff dimension of the circular Furstenberg sets. Indeed, if we can show

$$m_\delta(w \mid \Omega) \lesssim \delta^{-\kappa} \qquad w \in \mathbb{R}^2, \tag{3.4}$$

then letting w be the center in each δ-ball of $\mathcal{I}$, we deduce that each δ-ball in $\mathcal{I}$ intersects no more than $\delta^{-\kappa}$ many sets from the family $\{E(p)\}_{p \in P}$ and the bound $|\mathcal{I}| \gtrsim \delta^{-(s+t-\kappa)}$. This is the idea of transferring the estimate to the multiplicity function. However, it is not necessary for (3.4) to be true for all $w \in \mathbb{R}^2$. In reality, instead of (3.4), in the proof of Theorem 1.9 and Theorem 1.10, we will show two weaker versions of (3.4), which could guarantee the desired lower bound for $|\mathcal{I}|$.

3.2.1. *Outline of Theorem 1.9.* Recall from Theorem 3.6 that the desired bound for $|\mathfrak{I}|$ is $\delta^{-[(2t+1)s-t]} = \frac{\delta^{-s-t}}{\delta^{t(2-2s)}}$. This indicates that the correct choice of κ is $\kappa = t(2-2s) > 0$ where we recall the range of s in this theorem is $1/2 \leqslant s \leqslant 1$. And the weaker version of (3.4) in this proof is the following: for each (δ, s, t)-configuration Ω, there is a (δ, s, t)-configuration $G \subset \Omega$ such that $|G| \geqslant |\Omega|/4$ and for each $v \in \mathbb{R}^2$ with $(p, v) \in G$, we have that

$$m_\delta(v \mid \Omega) \lesssim \delta^{-\kappa-\eta t} = \delta^{t(2-2s)-\eta t} \tag{3.5}$$

holds for any $0 < \eta \ll 1$. Moreover, writing $G = \cup_{p\in\bar{P}}\{p\} \times \bar{E}(p)$ where $\bar{E}(p) \subset E(p)$ for each $p \in \bar{P} := \pi_{\mathbb{R}^3}(G)$, we have

$$|\bar{P}| \geqslant \frac{1}{2}|P| \text{ and } |\bar{E}(p)| \geqslant \frac{1}{2}|E(p)| \qquad p \in \bar{P}. \tag{3.6}$$

Let $\bar{\mathfrak{I}}$ be the subfamily of $\mathfrak{I}$ which covers $\cup_{p\in\bar{P}}\bar{E}(p)$. We have $|\bar{\mathfrak{I}}| \leqslant |\mathfrak{I}|$. On the other hand, (3.5) will imply $|\bar{\mathfrak{I}}| \gtrsim \frac{\delta^{-s-t}}{\delta^{t(2-2s)}} = \delta^{-[(2t+1)s-t]}$. Hence we obtain the desired lower bound for $|\mathfrak{I}|$.

We are left to show the existence of (δ, s, t)-configuration $G \subset \Omega$ satisfying the above conditions. Indeed, this is an application of [19, Lemma 5.1], which is a variant of Schlag's weak type inequality [26, Lemma 8] and the main lemma in [32] by Wolff. Since in this introductory part we aim to avoid some technical parts of the proof, we will formulate [19, Lemma 5.1] in a simplified and discretized version as below.

Lemma 3.8. *Fix $t \in (0,1]$, $\delta > 0, \eta > 0$, and $P \subset \mathbf{D}$ be a (δ, t)-set. The for any $(\delta, 1, t)$-configuration Ω with $\pi_{\mathbb{R}^3}(\Omega) = P$ and $\lambda \in (0,1]$, there is a set $\bar{P} = \bar{P}(\eta, \delta, \lambda) \subset P$ with*

$$|P \setminus \bar{P}(\eta, \delta, \lambda)| < \delta^{\eta t/3}|P|$$

such that the following holds for all $p \in \bar{P}(\eta, \delta, \lambda)$:

$$|S^\delta(p) \cap \{w : m_\delta(w \mid \Omega) \geqslant \delta^{-\eta t}\lambda^{-2t}\}| \leqslant \lambda |S^\delta(p)|. \tag{3.7}$$

Here we remark that Lemma 3.8 is also used to prove Theorem 2.1 in the previous section. Hence it is ultimately involved in the proof of Theorem 1.12.

Lemma 3.8 states that if we choose $\eta \ll 1$, then for a (δ, t)-set $P \subset \mathbf{D}$ representing a family of circles in $\mathbb{R}^2$, we could always find a subfamily $\bar{P}$ consisting almost all the circles in P such that for all the circles $S(p)$ in this subfamily, the points w in the δ-neighbourhood of $S(p)$ (i.e. $w \in S^\delta(p)$) with the property that $m_\delta(w \mid \Omega) \geqslant \delta^{-\eta t}\lambda^{-2t}$ has λ proportion in $S^\delta(p)$ in the sense of 2-dimensional Lebesgue measure.

However, there is no direct information involving $E(p)$. Since for a (δ, s, t)-configuration Ω, $E(p)$ is a (δ, s)-set for all $p \in P$ with $E(p) \sim \delta^{-s}$, we know the δ-neighbourhood $E^\delta(p)$ of $E(p)$ has measure $|E^\delta(p)| \geqslant c_1\delta^{2-s}$. Moreover, since $|S^\delta(p)| \leqslant c_0\delta$ for all $p \in P$, by choosing $\lambda = c_1\delta^{1-s}/(2c_0)$, (3.7) becomes

$$|S^\delta(p) \cap \{w : m_\delta(w \mid \Omega) \gtrsim \delta^{-\eta t}\delta^{-2t(1-s)}\}| \leqslant \lambda|S^\delta(p)| \leqslant \lambda c_0\delta \leqslant c_1\delta^{2-s}/2 \leqslant |E^\delta(p)|/2.$$

Here c_0 and c_1 are two absolute constants. This further gives

$$|E^\delta(p) \cap \{w : m_\delta(w \mid \Omega) \gtrsim \delta^{-\eta t}\delta^{-2t(1-s)}\}| \leqslant |E^\delta(p)|/2$$

and hence
$$|E^\delta(p) \cap \{w : m_\delta(w \mid \Omega) \lesssim \delta^{-\eta t}\delta^{-2t(1-s)}\}| \geqslant |E^\delta(p)|/2.$$
Now, form a (δ, s, t)-configuration G from Ω by letting $\pi_{\mathbb{R}^3}(G) = \bar{P} = \bar{P}(\eta, \delta, \lambda)$ with $\lambda = c_1\delta^{1-s}/(2c_0)$ and $\bar{E}(p)$ is a (δ, s)-set in $E^\delta(p) \cap \{w : m_\delta(w \mid \Omega) \lesssim \delta^{-\eta t}\delta^{-2t(1-s)}\}$ with $E(p) \sim \delta^{-s}$ for each $p \in P$. We arrive at the desired configuration G and conclude the sketch of the proof of Theorem 1.9.

3.2.2. *Outline of Theorem 1.10.* In this theorem we need to show $|\mathcal{J}| \sim \delta^{-(s+t)}$. Indeed, similar to the reasoning in the proof of Theorem 1.9, the following weaker estimate will be enough to conclude that $|\mathcal{J}| \gtrsim \delta^{-(s+t)}$: for any $\kappa > 0$, $0 < \delta < \delta_0(\kappa)$ and each (δ, s, t)-configuration Ω, there is a (δ, s, t)-configuration $G \subset \Omega$ such that $|G| \gtrsim_\kappa |\Omega|$ and for each $v \in \mathbb{R}^2$ with $(p, v) \in G$, we have
$$m_\delta(v \mid G) \lesssim \delta^{-\kappa}. \tag{3.8}$$
Since we only study the range in $0 < t \leqslant s \leqslant 1$, it turns out that if we can show the above statement for all (δ, s, s)-configurations Ω, then the above statement holds for all (δ, s, t)-configuration Ω. Thus in the following, we only consider a (δ, s, s)-configurations Ω and write (δ, s)-configurations instead.

Note that we need to show (3.8) for any $\kappa > 0$ arbitrarily small. This is much stronger than the previous theorem where we choose $\kappa = t(2-2s) > 0$ as a fixed number. Therefore, we have to investigate more deeply the factors that influence the value of the multiplicity function. Recall that the multiplicity function $m_\delta(w \mid G)$ counts the number of circles $S(p)$ such that the associated set $E(p) \subset S(p)$ is δ close to w. This motivates us to study the intersection of δ-neighbourhoods of circles in the plane, which has already been done by Wolff when studying circular Kakeya sets.

In fact, the shape of the intersection $S^\delta(p) \cap S^\delta(q)$, $p = (x, r), q = (x', r') \in \mathbf{D} \subset \mathbb{R}^2 \times (0, \infty)$ is determined by the following two quantities.
$$\Delta(p, q) := ||x - x'| - |r - r'|| \quad \text{and} \quad |p - q| := |x - x'| + |r - r'|. \tag{3.9}$$
Here $\Delta(p, q)$ is called the *tangency* parameter and $|p - q|$ is called the *distance* parameter. Intuitively, if $\Delta(p, q) = 0$, then the circles $S(p), S(q)$ are internally tangent, and if $\Delta(p, q) \sim 1$, the circles $S(p), S(q)$ intersect transversally. We recall the following definition and lemma by Wolff.

Definition 3.9 ((δ, σ)-rectangle)**.** Given $p \in \mathbf{D}$ and $v \in S(p)$, we call $R^\delta_\sigma(p, v)$ a (δ, σ)-rectangle that is the intersection of the δ-annulus $S^\delta(p)$ with the disc $B(v, \sigma)$ of radius σ, that is,
$$R^\delta_\sigma(p, v) = S^\delta(p) \cap B(v, \sigma).$$
For any $C > 0$, we define
$$CR^\delta_\sigma(p, v) := R^{C\delta}_{C\sigma}(p, v) = S^{C\delta}(p) \cap B(v, C\sigma).$$
We also write $R(p, v)$ instead of $R^\delta_\sigma(p, v)$ if we do not aim to emphasis the parameter δ and σ.

In [33, Lemma 3.1], Wolff showed that

Lemma 3.10. *For $p, q \in \mathbf{D}$, $S^\delta(p) \cap S^\delta(q)$ consists of at most two connected components and $|S^\delta(p) \cap S^\delta(q)| \lesssim \delta^2/\sqrt{(\delta + \Delta(p,q))(\delta + |p-q|)}$. Moreover, $S^\delta(p) \cap S^\delta(q)$ can be covered by boundedly many $(\delta, \delta/\sqrt{(\delta + \Delta(p,q))(\delta + |p-q|)})$-rectangles.*

In the following, we will define the partial multiplicity functions $m_{\delta,\lambda,t}$. We first give some motivations for defining them. Heuristically, given $p \in P$, let $P_1 := \{q \in P : \Delta(p,q) = 0 \text{ and } |p-q| \sim 1\}$. Then by Lemma 3.10, for any $q \in P_1$, we know $|S^\delta(p) \cap S^\delta(q)| \sim \delta^{3/2}$. Thus if $E(p) \subset S^\delta(p) \cap S^\delta(q)$ is a $(\delta, 1)$-set, then there exists $\lesssim \delta^{-1/2}$ many $v \in E(p)$ such that $q \in P_1$ may make one contribution to the total multiplicity function $m_\delta(v \mid G)$, which means that there exists some $w \in E(q)$ such that $v \in B(w, \delta)$ (recalling (3.3)).

However, if we consider the set $P_2 := \{q \in P : \Delta(p,q) \sim 1 \text{ and } |p-q| \sim 1\}$, then $|S^\delta(p) \cap S^\delta(q)| \sim \delta^2$. In this case, there exists $\lesssim 1$ many $v \in E(p)$ such that $q \in P_2$ may make a contribution to the total multiplicity function $m_\delta(v \mid G)$.

This observation motivates us to count the total multiplicity function $m_\delta(v \mid G)$ separately by using the following partial multiplicity functions (this is a vague version and a more detailed version will be given later). For $(p, v) \in G$, we write

$$m_{\delta,\lambda,t}((p,v) \mid G) := |\{(p',v') \in G : \Delta(p,p') \sim \lambda,\ |p-p'| \sim t \text{ and } |v-v'| \leqslant 2\delta\}|.$$

Here and in the following $t \in (\delta, 1]$ always denotes the value of the distance parameter instead of the t in the (δ, s, t)-configuration and there will be no ambiguity since we only discuss (δ, s)-configurations now.

Usually, for example, in the proof of (3.7) in Lemma 3.8, one may choose the series of partial multiplicity functions $m_{\delta,\lambda,t}$ dyadically, where $\lambda, t \in [\delta, 2\delta, \cdots, 1]$. Here, by showing

$$m_{\delta,\lambda,t}((p,v) \mid G) \lesssim \delta^{-\kappa}$$

for each pair (λ, t), it also seems possible to conclude

$$m_\delta(v \mid G) \leqslant \sum_{\lambda,t} m_{\delta,\lambda,t}((p,v) \mid G) \lesssim (\log(1/\delta))^2 \delta^{-\kappa} \le \delta^{-\kappa}\delta^{-\kappa} = \delta^{-2\kappa}$$

provided that $\delta \ll 1$ such that $(\log(1/\delta))^2 < \delta^{-\kappa}$.

However, this is not the series of partial multiplicity functions we adopt in reality. We explain the reason below (in a heuristic way). Note that we need to prove that $m_{\delta,\lambda,t}((p,v) \mid G) \lesssim \delta^{-\kappa}$ simultaneously holds for all pairs (λ, t) and (p, v) in some configuration G with $|G| \sim_\kappa |\Omega|$. In reality, we will first show $m_{\delta,\lambda_1,t_1}((p,v) \mid G_1) \lesssim \delta^{-\kappa}$ holds for a fixed pair (λ_1, t_1) and for all (p, v) in some configuration G_1 with $|G_1| \sim |\Omega|/2$. And then show $m_{\delta,\lambda_2,t_2}((p,v) \mid G_2) \lesssim \delta^{-\kappa}$ holds for the second pair (λ_2, t_2) and for all (p, v) in some configuration G_2 with $|G_2| \sim |G_1|/2$. Since there are $\sim N := (1/\log\delta)^2$ many pairs, after repeating this process for all pairs (λ, t), we will obtain a configuration G_N with $|G_N| \sim (1/2)^N |\Omega| \sim (1/2)^{(1/\log\delta)^2} |\Omega| \leqslant 1$ for $\delta \ll 1$. Thus by this choice of partial multiplicity functions the final configuration G_N would have cardinality too small compared with Ω.

Instead, we choose the partial multiplicity functions as follows. Let $0 < \epsilon \ll \kappa$ be sufficiently small. Let $\Lambda \subset [\delta, 1]$ be a finite set of cardinality $|\Lambda| \sim 1/\epsilon$ which is *multiplicatively*

$\delta^{-\epsilon}$-*dense* in the following sense: if $\lambda \in [\delta, 1]$ is arbitrary, then there exists $\underline{\lambda} \in \Lambda$ with $\underline{\lambda} \leq \lambda \leq \delta^{-\epsilon}\underline{\lambda}$. Next, for every $\lambda \in \Lambda$ fixed, we associate a finite set $\mathcal{T}(\lambda) \subset [\lambda, 1]$ of cardinality $|\mathcal{T}(\lambda)| \sim 1/\epsilon$ which is multiplicatively $\delta^{-\epsilon}$-dense on the interval $[\lambda, 1]$ in the same sense as above: if $t \in [\lambda, 1]$ is arbitrary, then there exists $\underline{t} \in \mathcal{T}(\lambda)$ such that $\underline{t} \leq t \leq \delta^{-\epsilon}\underline{t}$. For each $\lambda \in \Lambda$ and $t \in \mathcal{T} := \cup_{\lambda\in\Lambda}\mathcal{T}(\lambda)$ and $(p, v) \in G$, define the partial multiplicity function

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon}}((p, v) \mid G) := |\{(p', v') \in G : \Delta(p, p') \in [\delta^{\epsilon}\lambda, \lambda],\ |p - p'| \in [\delta^{\epsilon}t, t],\ |v - v'| \leqslant \delta^{1-\epsilon}\}|. \tag{3.10}$$

Thus there are $\sim (1/\epsilon)^2$ many partial multiplicity functions. And in the following we will find a configuration G with $|G| \sim (1/2)^{(1/\epsilon)^2}|\Omega| \gtrsim \delta^{\epsilon}|\Omega| \gtrsim \delta^{\kappa}|\Omega|$ by choosing $\delta < \delta_0(\epsilon)$ so small such that

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon}}(\omega \mid G) \leqslant \delta^{-\kappa}, \qquad \omega \in G, \lambda \in \Lambda, t \in \mathcal{T}. \tag{3.11}$$

And ultimately, we have

$$m_{\delta}(\omega \mid G) \leqslant \sum_{\lambda\in\Lambda, t\in\mathcal{T}} m_{\delta,\lambda,t}^{\delta^{-\epsilon}}((p, v) \mid G) \lesssim (1/\epsilon)^2\delta^{-\kappa} \leq \delta^{-\epsilon}\delta^{-\kappa} = \delta^{-2\kappa}$$

with $\delta^{-2\kappa}$ instead of $\delta^{-\kappa}$ in (3.8), which does no harm to conclude the proof.

We remark that the partial multiplicity function in (3.10) is still not the one used in the actual proof in [C]. The precise one is in Definition 5.29 in [C] and would need extra notions. However, (3.10) is enough to give the rough idea for the following proof.

Our final goal is to show (3.11). Since the final configuration G has cardinality $|G| \sim_{\kappa} |\Omega|$, we will not distinguish G and Ω in the following and keep using Ω. Also, since $\kappa > 0$ can be arbitrarily small in (3.11), we will show $m_{\delta,\lambda,t}^{\delta^{-\epsilon}}(\omega \mid \Omega) \lesssim 1$ to simplify the presentation. From now on, we fix $\epsilon > 0$ and choose $\delta \leqslant \delta_0$ such that

$$1/\epsilon < \delta_0^{-\epsilon}. \tag{3.12}$$

We remark that the actual choice of parameters is more complicated than this and we refer to Section 7.1 in [C]. Since $\epsilon > 0$ is fixed, in the following, for $\delta \in (0, 1]$, we remind the reader that the notation $A \lesssim B$ means

$$A \leq \delta^{-C\epsilon}B.$$

for some absolute constant $C \geq 1$ and change $A \gtrsim B$ and $A \sim B$ correspondingly.

The two main ingredients of the proof will be an application of Wolff's famous tangency bound of circles in [34] and an induction process in an increasing order on $\lambda \in \Lambda$ for each $t \in \mathcal{T}$ fixed. Since for different $t \in \mathcal{T}$, the proof will be independent, we will concentrate on the case $t = 1$ in the following and write $m_{\delta,\lambda}^{\delta^{-\epsilon}}$ instead of $m_{\delta,\lambda,1}^{\delta^{-\epsilon}}$ for simplicity.

Step 1. We show the base case $\lambda = \delta$, that is

$$m_{\delta,\delta}^{\delta^{-\epsilon}}(\omega \mid \Omega) \lesssim 1 \tag{3.13}$$

(corresponding to Section 5 in [C]). This will be an application of Wolff's tangency bound, which, roughly speaking, provides an upper bound of total tangencies of a given family of circles. To introduce this bound, we need several notions. Recall that the intersection of two annuli $S^\delta(p)$ and $S^\delta(q)$ with $\Delta(p,q) = \lambda$ and $|p-q| \sim 1$ can be localized to a (δ,σ)-rectangle R^δ_σ with $\sigma = \delta/\sqrt{\lambda}$. Thus it is possible to transfer counting the total $(\lambda,1)$-tangencies of a given family of circles to counting the total number of $(\delta,\delta/\sqrt{\lambda})$-rectangles associated to this family of circles. However, to properly count the $(\delta,\delta/\sqrt{\lambda})$-rectangles, we need the following two notions of rectangles introduced by Wolff.

To introduce the definitions, we make a further reduction that in the remaining part of this subsection, the parameter set $P \subset \mathbf{D}$ is $P = W \cup B$ where

$$W = P \cap B(p_0, \delta^{2\epsilon}) \text{ and } B = P \cap [B(p_0,1) \setminus B(p_0,\delta^\epsilon)] \tag{3.14}$$

for some $p_0 \in \mathbf{D}$. We note that W and B are δ^ϵ separated, that is, $\operatorname{dist}(W,B) \sim \delta^\epsilon$. This property will be needed in the following proof. The set $W \cup B$ is a special kind of almost 1-bipartite sets (see Definition 4.51 in [C]) and for general P, one can always construct a proper bipartite set $W \cup B$ inside P.

To simplify to computation in this introduction, we also assume

$$|B| \lesssim |W| \sim \delta^{-\epsilon}. \tag{3.15}$$

Definition 3.11 (Type)**.** Let $0 < \delta \le \sigma \le 1$, $\epsilon > 0$. Let $P = W \cup B \subset \mathbf{D}$. For $m, n \ge 1$, we say that a (δ,σ)-rectangle $R \subset \mathbb{R}^2$ has *type* $(\ge m, \ge n)_\epsilon$ *relative to* (W,B) if $R \subset S^{\delta^{1-\epsilon}}(p)$ for at least m points $p \in W$, and $R \subset S^{\delta^{1-\epsilon}}(q)$ at least n points $q \in B$.

Definition 3.12 (Comparability)**.** Given a constant $C \ge 1$, we say that two (δ,σ)-rectangles R_1, R_2 are *C-comparable* if there exists a third (δ,σ)-rectangle $R = R^\delta_\sigma(p,v)$ such that $R_1, R_2 \subset CR$. If no such rectangle R exists, we say that R_1 and R_2 are *C-incomparable*.

We give an example to heuristically explain the meaning of the above definitions.

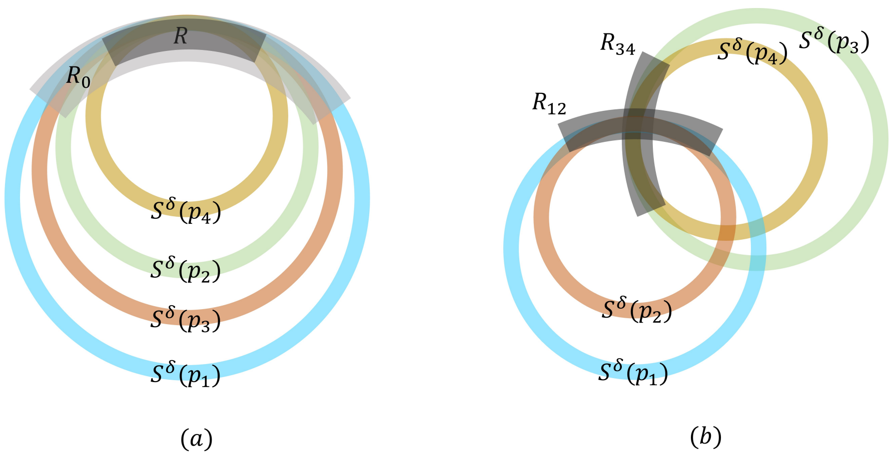

FIGURE 1. Rectangles.

Example 3.13. In Figure 1 (a), the family of four circles $S(p_i)$, $i = 1,2,3,4$ are tangent at one point, so there is only one tangency caused by these four circles. We have $\Delta(p_i, p_j) = 0$ for all pairs $i, j = 1,2,3,4$. By Lemma 3.10 (strictly speaking, Lemma 3.1 in [33]), the intersection of four annuli $S^\delta(p_i)$ can be covered by a $(C\delta, C\sqrt{\delta})$-rectangle R_0. We find that any $(\delta, \sqrt{\delta})$-rectangle R contained in R_0 can represent this tangency formed by these four annuli. By Definition 3.12, any two $(\delta, \sqrt{\delta})$-rectangles contained in R_0 are C-comparable. Thus there is only one C-incomparable $(\delta, \sqrt{\delta})$-rectangle, which coincides with the number of tangency caused by these four circles. Moreover, if we assume $p_1, p_3 \in W$ and $p_2, p_4 \in B$, the situation in Figure 1 (a) can be described as one C-incomparable rectangle of type $(\geqslant 2, \geqslant 2)$ relative to W and B.

In Figure 1 (b), at the tangent point, the circles $S(p_1)$ and $S(p_2)$ are tangent and the circles $S(p_3)$ and $S(p_4)$ are tangent, i.e. $\Delta(p_1, p_2) = 0$ and $\Delta(p_3, p_4) = 0$. But for all other pairs $1 \leqslant i < j \leqslant 4$, $\Delta(p_i, p_j) \sim 1$. We can naturally associate a $(\delta, \sqrt{\delta})$-rectangle R_{12} and a $(\delta, \sqrt{\delta})$-rectangle R_{34} to represent $S^\delta(p_1) \cap S^\delta(p_2)$ and $S^\delta(p_3) \cap S^\delta(p_4)$ respectively. However, for an absolute constant $C > 1$ it is not possible that one can use one $(C\delta, C\sqrt{\delta})$-rectangle R to represent $S^\delta(p_1) \cap S^\delta(p_2)$ and $S^\delta(p_3) \cap S^\delta(p_4)$ simultaneously. This shows that there are two tangencies caused by the pairs (p_1, p_2) and (p_3, p_4) respectively. Also, the $(\delta, \sqrt{\delta})$-rectangles R_{12} and R_{34} are C-incomparable for some absolute constant C. Thus there are two C-incomparable $(\delta, \sqrt{\delta})$-rectangles equal to the number of tangencies caused by the four circles. Moreover, if we assume $p_1, p_3 \in W$ and $p_2, p_4 \in B$, the situation in Figure 1 (b) can be described as two incomparable rectangles of type $(\geqslant 1, \geqslant 1)$ relative to W and B.

We are ready to state the "ϵ"-variant of Wolff's tangency bound for $(\delta, \sqrt{\delta})$-rectangles, which is a simplified version of Lemma 4.53 in [C].

Lemma 3.14. *For every $\epsilon > 0$, there exists $\delta_0 > 0$ such that the following holds for all $\delta \in (0, \delta_0]$. Let $\mathcal{R}^\delta_{\sqrt{\delta}}$ be a family of pairwise 100-incomparable $(\delta, \sqrt{\delta})$-rectangles of type $(\geq m, \geq n)_\epsilon$ relative to (W, B), where $1 \leq m \leq |W|$ and $1 \leq n \leq |B|$. Then,*

$$|\mathcal{R}^\delta_{\sqrt{\delta}}| \lesssim \left(\frac{|W||B|}{mn}\right)^{3/4} + \text{ less important terms.} \tag{3.16}$$

This lemma is the same as [33, Lemma 1.4] by Wolff, except that it allows for constants of form "$\delta^{-\epsilon}$" in Definition 3.11.

Now, in an informal way, we show $m^{\delta^{-\epsilon}}_{\delta,\delta}(\omega \mid \Omega) \lesssim 1$ using Lemma 3.14 by contradiction. We assume there exists $\kappa_0 \gg \epsilon$ such that for all $\omega = (p, v) \in \Omega$,

$$m^{\delta^{-\epsilon}}_{\delta,\delta}(\omega \mid \Omega) = |\{(p', v') \in \Omega : \Delta(p, p') \in [0, \delta],\ |p - p'| \in [\delta^\epsilon, 1],\ |v - v'| \leqslant \delta^{1-\epsilon}\}| = \delta^{-\kappa_0}. \tag{3.17}$$

The idea is to construct a 100-incomparable family of $(\delta, \sqrt{\delta})$-rectangles of type $(\geq m, \geq n)_\epsilon$ relative to (W, B) with $m \leqslant n = \delta^{-\kappa_0}$ that violates (3.16).

Recalling that Ω is a (δ, s)-configuration with $P = \pi_{\mathbb{R}^3}(\Omega)$, from Definition 3.4, we know that for each $p \in P$, $E(p)$ is a (δ, s)-set with $|E(p)| \sim \delta^{-s}$. Since a $(\delta, \sqrt{\delta})$-rectangle can contain at most $\sim (\sqrt{\delta}/\delta)^s = 1/\delta^{s/2}$ many points in $E(p)$, we can associate at least $\sim \delta^{-s}/(1/\delta^{s/2}) = \delta^{-s/2}$ many 100-incomparable $(\delta, \sqrt{\delta})$-rectangles $\{R(p,i)\}_{i=1,\cdots,\delta^{-s/2}}$ on each $S(p)$, $p \in W$. See Figure 2 for an illustration.

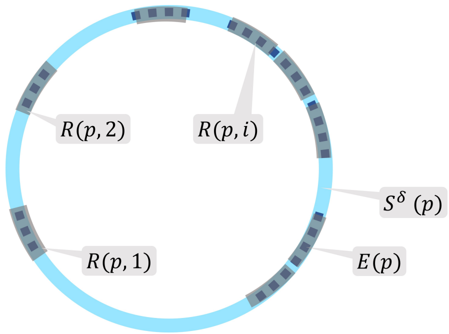

FIGURE 2. The incomparable family $\{R(p,i)\}_{i=1,\cdots,\delta^{-s/2}}$.

For each $R(p,i)$, we investigate its type relative to W and B. First, we show $R(p,i)$ has type $n = \delta^{-\kappa_0}$ relative to B. To this end, fix $R(p,i)$ and choose a point $v \in E(p) \cap R(p,i)$. Recalling the definition of type in Definition 3.11, it suffices to find the number of points $q \in B$ such that $R(p,i) \subset S^{\delta^{1-\epsilon}}(p) \cap S^{\delta^{1-\epsilon}}(q)$. Applying (3.17) to (p,v), we know that for the pair (p,v), there are $\delta^{-\kappa_0}$ many points $(p',v') \in \Omega$ such that $|v - v'| \leqslant \delta^{1-\epsilon}$. Since $v' \in E(p')$ and $E(p')$ is δ-separated, we know for each p', there are $\lesssim 1$ many points in $E(p')$ such that (p',v') satisfies (3.17). This further implies there are $\delta^{-\kappa_0}$ many points $p' \in P$ such that

$$\Delta(p,p') \in [0,\delta],\ |p - p'| \in [\delta^{\epsilon}, 1]. \tag{3.18}$$

Recalling the choice of the sets W and B in (3.14) and noting $p \in W$, we deduce that the points p' satisfying (3.18) are contained in B. Denote the set of points p' satisfying (3.18) by $B_{R(p,i)}$. Thus (3.18) together with Lemma 3.10 implies that $R(p,i) \subset S^{\delta^{1-\epsilon}}(p) \cap S^{\delta^{1-\epsilon}}(q)$ for all $q \in B_{R(p,i)}$. As a result, $R(p,i)$ has type

$$\geqslant n = |B_{R(p,i)}| = \delta^{-\kappa_0} \tag{3.19}$$

relative to B. For an illustration, see Figure 3 where we recall from (3.14) that, W is contained in the ball $B(p_0, \delta^{2\epsilon})$ colored yellow and B is contained in the annulus $B(p_0,1) \setminus B(p_0, \delta^{\epsilon})$ colored grey in Figure 3.

Next, we show $R(p,i)$ has type $m \leqslant \delta^{-\kappa_0}$ relative to W. Similarly as before, it suffices to find the number of points $u \in W$ such that $R(p,i) \subset S^{\delta^{1-\epsilon}}(u)$. From the above paragraph, we can find $q \in B_{R(p,i)}$ such that $R(p,i) \subset S^{\delta^{1-\epsilon}}(q)$ and $w \in E(q)$ such that $w \in \delta^{1-\epsilon}R(p,i)$ (since $v \in R(p,i)$ and by (3.17), $|v - w| \leqslant \delta^{1-\epsilon}$). Hence it suffices to find the number of points in $u \in W$ such that

$$R(p,i) \subset S^{\delta^{1-\epsilon}}(u) \cap S^{\delta^{1-\epsilon}}(q). \tag{3.20}$$

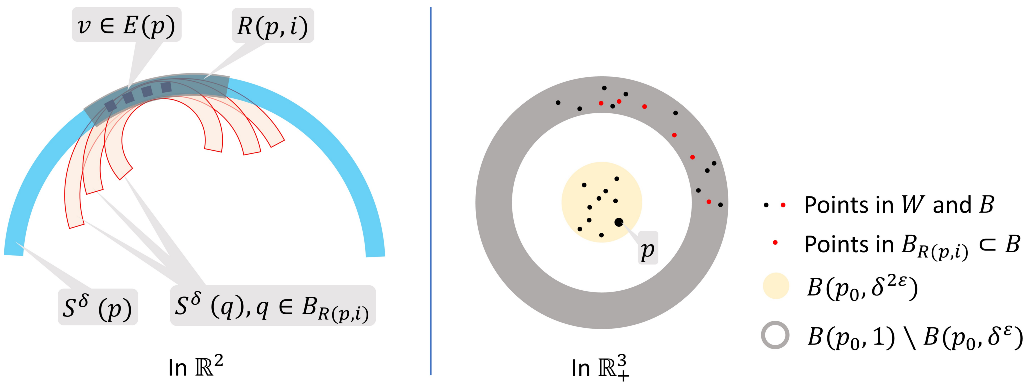

FIGURE 3. The type of $R(p,i)$ relative to B.

Also if u satisfies (3.20), then $|S^{\delta^{1-\epsilon}}(u) \cap S^{\delta^{1-\epsilon}}(q)| \geqslant |R(p,i)| \sim \delta^{3/2}$. By $u \in W$ and $q \in B$ implying $|u-q| \sim 1$ and Lemma 3.10, we know a neccessary condition for u satisfying (3.20) is that $\Delta(u,q) \lesssim \delta$. Applying (3.17) to (q,w) and by a similar reasoning as (3.18), we obtain that there are $\delta^{-\kappa_0}$ many points $u \in P$ such that

$$\Delta(q,u) \in [0,\delta],\ |q-u| \in [\delta^\epsilon, 1]. \tag{3.21}$$

Hence $u \in W$ satisfying (3.20) implies u enjoying (3.21). See Figure 4 for an illustration.

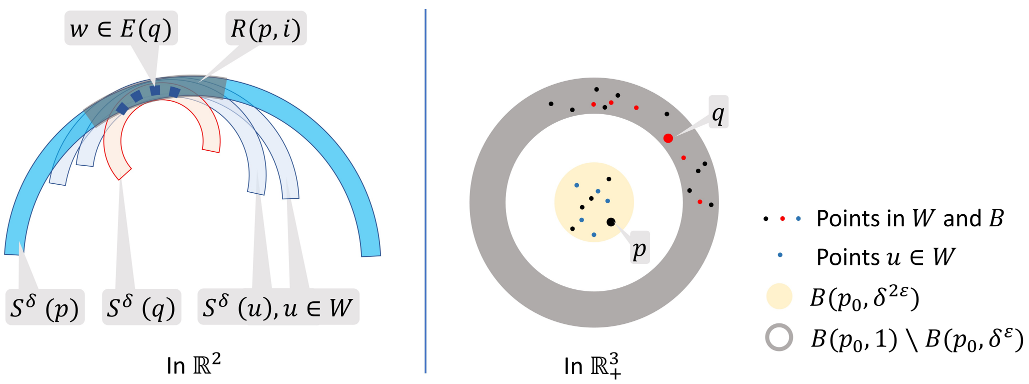

FIGURE 4. The type of $R(p,i)$ relative to W.

By the upper bound in (3.17), we can conclude that there are $\leqslant \delta^{-\kappa_0}$ many $u \in W$ such that $R(p,i) \subset S^{\delta^{1-\epsilon}}(u) \cap S^{\delta^{1-\epsilon}}(q)$. Thus $R(p,i)$ has type $m \leqslant \delta^{-\kappa_0}$ relative to W. Here, for simplicity, we assume that m is independent of the choice among $R(p,i)$.

To summarize, we have constructed a family $\mathcal{R}_0$ of $\sim |W|\delta^{-s/2}$ many $(\delta, \sqrt{\delta})$-rectangles $\{R(p,i)\}_{p\in W, i=1,\cdots,\delta^{-s/2}}$ and for each rectangle, there are at most m circles $S(u)$ with $u \in W$ such that

$$R(p,i) \subset S^{\delta^{1-\epsilon}}(u) \cap S^{\delta^{1-\epsilon}}(p).$$

For each u, if there exists $R(u,i(u)) \in \mathcal{R}_0$ such that

$$R(u,i(u)) \subset S^{\delta^{1-\epsilon}}(u) \cap S^{\delta^{1-\epsilon}}(p), \tag{3.22}$$

then $R(p,i)$ and $R(u,i(u))$ may be 100-comparable. On the other hand, for each u, since by our construction, $\{R(u,i)\}_{i=1,\cdots,\delta^{-s/2}}$ is a 100-incomparable family and $\mathrm{diam} R(u,i) \sim \mathrm{diam}(S^{\delta^{1-\epsilon}}(u) \cap S^{\delta^{1-\epsilon}}(p))$, we know there are at most ~ 1 rectangles $R(u,i(u))$ satisfying (3.22). Therefore, for each $R(p,i)$, there are at most m rectangles in $\mathcal{R}_0$ that are 100-comparable to it. Thus we can find a subfamily $\mathcal{R} \subset \mathcal{R}_0$ of 100-incomparable rectangles with

$$|\mathcal{R}| \sim \frac{|W|\delta^{-s/2}}{m}.$$

Recalling (3.15), we have $|B| \lesssim |W| \sim \delta^{-s}$. Also, we recall that each $R(p,i)$ has type $\geqslant n = \delta^{-\kappa_0}$ relative to B by (3.19). Thanks to $m \leqslant n = \delta^{-\kappa_0}$ and Lemma 3.14, we deduce

$$\frac{|W|\delta^{-s/2}}{m} \lesssim \left(\frac{|W||B|}{mn}\right)^{3/4}, \tag{3.23}$$

which implies

$$1 \lesssim \delta^{\kappa_0/2}.$$

We get a contradiction and this completes the heuristic proof of showing (3.13), that is, $m_{\delta,\delta}^{\delta^{-\epsilon}}(\omega \mid \Omega) \lesssim 1$.

Step 2. Recall $\Lambda = \{\delta = \lambda_1, \lambda_2, \cdots, \lambda_{|\Lambda|}\}$ is the multiplicatively $\delta^{-\epsilon}$-dense set defined above (3.10). We show $m_{\delta,\lambda_k}^{\delta^{-\epsilon}}(\omega \mid \Omega) \lesssim 1$ for all $\lambda_k \in \Lambda$ (corresponding to Section 7.6 in [C]). By induction, assuming

$$m_{\delta,\lambda_l}^{\delta^{-\epsilon}}(\omega \mid \Omega) \lesssim 1 \qquad l = 1, \cdots, k-1, \tag{3.24}$$

we show $m_{\delta,\lambda_k}^{\delta^{-\epsilon}}(\omega \mid \Omega) \lesssim 1$. From *Step 1*, one can expect that a good upper bound for the cardinality of incomparable $(\delta, \delta/\sqrt{\lambda})$-rectangles would be useful to conclude the proof. Actually, we obtain the following bound (a simplified unrigorous version of Theorem 6.5 in [C]). Recall W and B are (δ, s)-sets defined in (3.14) and Ω is the (δ, s)-configuration with parameter set $P = W \cup B$.

Theorem 3.15. *For every $\epsilon > 0$, there exists $\delta_0 > 0$ such that the following holds for all $\delta \in (0, \delta_0]$. For $\delta \leqslant \lambda \leqslant 1$, assume*

$$m_{\lambda,\lambda}^{\lambda^{-\epsilon}}(\omega \mid \mathbf{\Omega}) \lesssim 1, \qquad \omega \in \mathbf{\Omega}. \tag{3.25}$$

where $\mathbf{\Omega} \subset \Omega$ is a properly chosen (λ, s)-configuration. Let $\mathcal{R}^{\delta}_{\delta/\sqrt{\lambda}}$ be a family of pairwise 100-incomparable $(\delta, \delta/\sqrt{\lambda})$-rectangles of type $(\geq m, \geq n)_\epsilon$ relative to (W, B), where $1 \leq m \leq |W|$ and $1 \leq n \leq |B|$. Then,

$$|\mathcal{R}^{\delta}_{\delta/\sqrt{\lambda}}| \lesssim \left(\frac{|W||B|}{mn}\right)^{3/4} \left(\frac{\lambda}{\delta}\right)^{s/2} + \textit{ less important terms.} \tag{3.26}$$

Here the (λ, s)-configuration $\mathbf{\Omega} \subset \Omega$ can be roughly considered as the maximal λ-separated set in Ω. In the following the proof, we will sketch the construction of $\mathbf{\Omega}$ from Ω. We also remark that the assumption (3.25) is not unrealistic, since by *Step 1* (letting $\lambda = \delta$ in (3.13)), one can always find (λ, s)-configurations $\mathbf{\Omega}$ satisfies (3.25). In addition, we substitute the weaker assumption (6.7) made in Theorem 6.5 in [C] by the assumption (3.25) in Theorem 3.15 to simplify many technical steps in this introduction.

Before giving an outline of the proof, we provide some evidence why Theorem 3.15 is true. First, letting $\lambda = \delta$ in Theorem 3.15, then (3.26) becomes

$$|\mathcal{R}^{\delta}_{\sqrt{\delta}}| \lesssim \left(\frac{|W||B|}{mn}\right)^{3/4} + \text{ less important terms,}$$

which coincides with Wolff's tangency bound (3.16) in Lemma 3.14. Another evidence is that, if we assume $|W| = |B| \sim \delta^{-s}$, $m = n = 1$ and let $\lambda \sim 1$ in Theorem 3.15, then (3.26) becomes $|\mathcal{R}^{\delta}_{\delta}| \lesssim \delta^{-2s}$. This bound is also sharp, which can be reached by the case that every circle in W intersects all circles in B transversely. See Figure 5 for an illustration.

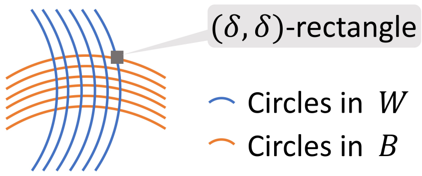

FIGURE 5. Sharp bound for $|\mathcal{R}^{\delta}_{\delta}|$.

The brief idea of the proof of Theorem 3.15 is as follows. Here we will only show the case $m = n = 1$. One can use a random sampling argument to show the other cases $m, n > 1$ (see page 42-44 in paper [C]). We also remark that in reality, the notion of type of rectangles in Theorem 6.5 in [C] is more delicate and takes the sets $\{E(p)\}$ into account.

Let $m = n = 1$. Omitting the "less important terms", we need to show

$$|\mathcal{R}^{\delta}_{\delta/\sqrt{\lambda}}| \lesssim (|W||B|)^{3/4}\left(\frac{\lambda}{\delta}\right)^{s/2} = \left(\frac{|W|}{(\lambda/\delta)^s} \cdot \frac{|B|}{(\lambda/\delta)^s}\right)^{3/4}\left(\frac{\lambda}{\delta}\right)^{2s}. \tag{3.27}$$

Inequality (3.27) will be established with the help of the known upper bound of incomparable $(\lambda, \sqrt{\lambda})$-rectangles with type $(\geqslant 1, \geqslant 1)$ relative to some bipartite set. To this end, consider $p \in W$ and $q \in B$ such that there exists a $(\delta, \delta/\sqrt{\lambda})$-rectangle $R(p,q) \subset S^{C\delta}(p) \cap S^{C\delta}(q)$ of type $(\geqslant 1, \geqslant 1)$ for some absolute constant $C > 1$. By Lemma 3.10, we know

$$\delta^2/\sqrt{\Delta(p,q)+\delta} \gtrsim |S^{C\delta}(p) \cap S^{C\delta}(q)| \geqslant |R(p,q)| \sim \delta^2/\sqrt{\lambda}.$$

This implies that $\Delta(p,q) \lesssim \lambda$ and thus $S^{C\lambda}(p) \cap S^{C\lambda}(q)$ can be covered by boundedly many $(\lambda, \sqrt{\lambda})$-rectangles (where the boundedness is guaranteed by Lemma 4.3 in [C]). As a result,

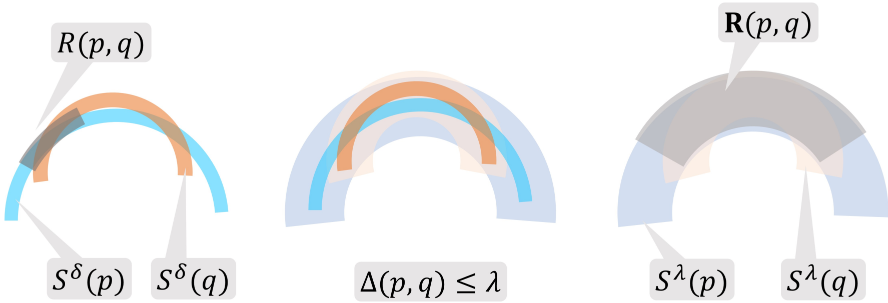

FIGURE 6. The construction of $\mathbf{R}(p,q)$.

we can associate a $(\lambda, \sqrt{\lambda})$-rectangle $\mathbf{R}(p,q)$ such that $R(p,q) \subset \mathbf{R}(p,q) \subset S^{C\lambda}(p) \cap S^{C\lambda}(q)$. See Figure 6 for an illustration.

Next, from the original (δ, s)-configuration Ω, we construct the (λ, s)-configuration $\mathbf{\Omega} \subset \Omega$. Let $\pi_{\mathbb{R}^3}(\mathbf{\Omega}) = \mathbf{W} \cup \mathbf{B}$. Using pigeonholing, we can assume

$$|W \cap B(x,\lambda)| \sim |B \cap B(x,\lambda)| \lesssim \left(\frac{\lambda}{\delta}\right)^s, \quad \forall x \in \mathbb{R}^2$$

and again by pigeonholing, we may choose $\mathbf{W}$ and $\mathbf{B}$ satisfying

$$|W| \sim \left(\frac{\lambda}{\delta}\right)^s |\mathbf{W}| \text{ and } |B| \sim \left(\frac{\lambda}{\delta}\right)^s |\mathbf{B}|. \tag{3.28}$$

Roughly speaking, for each $R(p,q)$, in the parameter set, there exists a "unique" pair ("$\lesssim 1$ many pairs") $\mathbf{p} \in \mathbf{W}$ and $\mathbf{q} \in \mathbf{B}$ such that $p \in B(\mathbf{p},\lambda) \cap W$ and $q \in B(\mathbf{q},\lambda) \cap B$. See Figure 7 for an illustration.

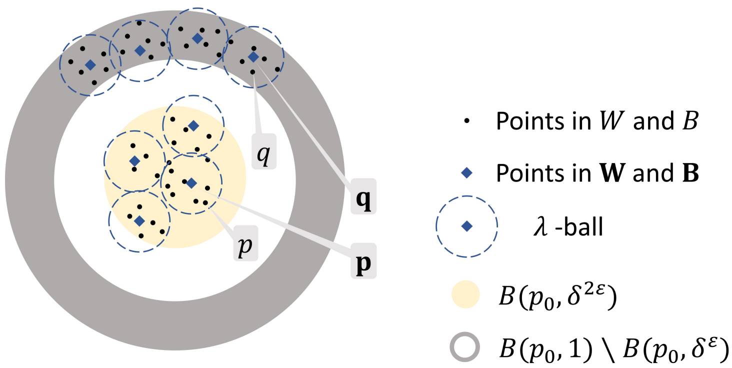

FIGURE 7. $\mathbf{p} \in \mathbf{W}$ and $\mathbf{q} \in \mathbf{B}$.

Moreover, a direct computation shows that $\Delta(\mathbf{p},\mathbf{q}) \lesssim \lambda$ if $\Delta(p,q) \lesssim \lambda$ and we associate a $(\lambda, \sqrt{\lambda})$-rectangle $\mathbf{R}(\mathbf{p},\mathbf{q}) \subset S^{C\lambda}(\mathbf{p}) \cap S^{C\lambda}(\mathbf{q})$. Then an observation is that $\mathbf{R}(\mathbf{p},\mathbf{q})$

and $\mathbf{R}(p,q)$ are comparable and hence $R(p,q)\subset C\mathbf{R}(\mathbf{p},\mathbf{q})$. Thus for each of the $(\lambda,\sqrt{\lambda})$-rectangles $\mathbf{R}(\mathbf{p},\mathbf{q})$, there are

$$\lesssim |B(\mathbf{p},\lambda)\cap W||B(\mathbf{p},\lambda)\cap B|\sim(\lambda/\delta)^{2s} \tag{3.29}$$

many of the $(\delta,\delta/\sqrt{\lambda})$-rectangles $R(p,q)\subset C\mathbf{R}(\mathbf{p},\mathbf{q})$. See Figure 8 for an illustration.

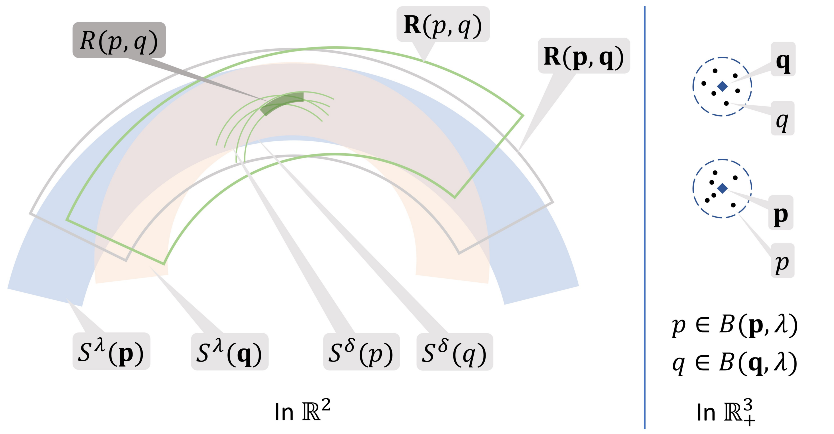

FIGURE 8. The construction of $\mathbf{R}(\mathbf{p},\mathbf{q})$.

Writing

$$\mathcal{R}_\lambda=\{\mathbf{R}(\mathbf{p},\mathbf{q}):\ \text{there exists } R(p,q) \text{ such that } R(p,q)\subset\mathbf{R}(\mathbf{p},\mathbf{q})\},$$

we obtain a family $\mathcal{R}_\lambda$ consisting of $(\lambda,\sqrt{\lambda})$-rectangles. To apply the bound in Lemma 3.14, we need to deduce $\mathcal{R}_\lambda$ consists of incomparable rectangles. This fact is guaranteed by the assumption (3.25). Indeed, (3.25) implies that, generically, the above $\mathbf{R}(\mathbf{p},\mathbf{q})$ have type $(1,1)$ relative to $\mathbf{W}$ and $\mathbf{B}$ and one can then deduce they are incomparable with each other. Thus $\mathcal{R}_\lambda$ is a family of incomparable $(\lambda,\sqrt{\lambda})$-rectangles.

Applying Lemma 3.14 to $\mathbf{W}\cup\mathbf{B}$ we know

$$|\mathcal{R}_\lambda|\lesssim(|\mathbf{W}||\mathbf{B}|)^{3/4}\overset{(3.28)}{\sim}\left(\frac{|W|}{(\lambda/\delta)^s}\cdot\frac{|B|}{(\lambda/\delta)^s}\right)^{3/4}.$$

Combining the above inequality and (3.29), we arrive at the bound (3.27) for $|\mathcal{R}^\delta_{\delta/\sqrt{\lambda}}|$ (in the case $m=n=1$).

With Theorem 3.15 in hand, under the counter assumption $m^{\delta^{-\epsilon}}_{\delta,\lambda_k}(\omega\mid\Omega)=\delta^{-\kappa_0}$, using a similar argument as in *Step 1*, a similar computation as (3.23) will result in a contradiction. We do not repeat the process again here but only highlight the point where we need the induction hypothesis (3.24). Indeed, the induction is employed to show $m\lesssim\delta^{-\kappa_0}$. Using the same notation as in *Step 1*, recall that m is the type relative to W of a $(\delta,\delta/\sqrt{\lambda_k})$-rectangle $R(p,i)$ for some $p\in W$, that is, the number of $p'\in W$ such that

$$R(p,i)\subset S^{\delta^{1-\epsilon}}(p'). \tag{3.30}$$

To upper bound m, since $R(p,i)$ has type $\geqslant n = \delta^{-\kappa_0}$ relative to B, choose $q \in B$ such that $R(p,i) \subset S^{\delta^{1-\epsilon}}(q)$. Together with (3.30), it suffices to find the the number of $p' \in W$ such that

$$R(p,i) \subset S^{\delta^{1-\epsilon}}(p') \cap S^{\delta^{1-\epsilon}}(q). \tag{3.31}$$

As a consequence of (3.31), we have $|S^{\delta^{1-\epsilon}}(p') \cap S^{\delta^{1-\epsilon}}(q)| \geqslant |R(p,i)| \gtrsim \delta^2/\sqrt{\lambda_k}$. Thus for every $p' \in W$ such that (3.31) holds, by Lemma 3.10, we know

$$|S^{\delta}(p') \cap S^{\delta}(q)| \sim \delta^2/\sqrt{\Delta(p',q)} \gtrsim \delta^2/\sqrt{\lambda_k},$$

which implies that $\Delta(p',q) \leqslant \lambda_k$ every $p' \in W$ satisfying (3.31). We can conclude that

$$m \leqslant |\{p' \in W : \Delta(p',q) \leqslant \lambda_k\}|.$$

Choosing $w \in E(q) \cap \delta^{1-\epsilon}R(p,i)$, we know for each $p' \in W$ there exist $\lesssim 1$ many $v' \in E(p') \cap \delta^{1-\epsilon}R(p,i)$ such that $|w - v'| \leqslant \delta^{1-\epsilon}$ since $E(p')$ is δ-separated. We have

$$\begin{aligned} m &\leqslant \sum_{l=1}^{k} |\{(p',v') \in \cup_{p'\in W}(p', E(p')) : \delta^{\epsilon}\lambda_l \leqslant \Delta(p',q) \leqslant \lambda_l,\ |w - v'| \leqslant \delta^{1-\epsilon}\}| \\ &\leqslant \sum_{l=1}^{k} m_{\delta,\lambda_l}^{\delta^{-\epsilon}}(q,w) = \sum_{l=1}^{k-1} m_{\delta,\lambda_l}^{\delta^{-\epsilon}}(q,w) + m_{\delta,\lambda_k}^{\delta^{-\epsilon}}(q,w) \\ &\lesssim \frac{1}{\epsilon} + \delta^{-\kappa_0} \lesssim \delta^{-\kappa_0} \end{aligned}$$

where in the second last inequality we recall $\{\lambda_1, \cdots, \lambda_k\} \subset \Lambda$ with $|\Lambda| \sim 1/\epsilon$ above (3.10) and apply (3.24), and in the last inequality we use (3.12) and $\kappa_0 \gg \epsilon$. This concludes *Step 2* and finishes the rough outline of the proof of Theorem 1.10.

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ON THE DIMENSION OF KAKEYA SETS IN THE FIRST HEISENBERG GROUP

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Abstract. We define Kakeya sets in the Heisenberg group and show that the Heisenberg Hausdorff dimension of Kakeya sets in the first Heisenberg group is at least 3. This lower bound is sharp since, under our definition, the $\{xoy\}$-plane is a Kakeya set with Heisenberg Hausdorff dimension 3.

1. Introduction

The study of Kakeya sets in Euclidean space is one of the central topics in geometric measure theory. A set $E \subset \mathbb{R}^n$ is a Kakeya set if for every $e \in S^{n-1}$ there exists a unit line segment I_e parallel to e such that $I_e \subset E$.

A natural question is to determine the least Hausdorff dimension of Kakeya sets.

The answer is known in 2-dimensional Euclidean space. Indeed, Kakeya sets in $\mathbb{R}^2$ turn out to be of Hausdorff dimension equal to 2 which can be shown by multiple ways. See [7,9,15]. However, for Kakeya sets in higher dimensional Euclidean space, the sharp lower bound is not known. We would like to remark some progress: Bourgain used two different methods to provide lower bounds [3, 4], which were further improved by Wolff [18] and Katz-Tao [11] respectively. Recently, Katz-Zahl [13] and Guth-Zahl [10] enhanced the results of [18] in $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively, which show that the best known lower bound is $5/2 + \epsilon_0$ in $\mathbb{R}^3$ with ϵ_0 an absolute constant and $3 + 1/40$ in $\mathbb{R}^4$. In $\mathbb{R}^n$ with $n \geq 5$, the best known lower bound $3 + (2 - \sqrt{2})(n - 4)$ was established in [12] by Katz-Tao.

As an analogy to Euclidean Kakeya sets, we can define Kakeya sets in the Heisenberg group. In this paper, we denote by $\mathbb{H}^n$ the n-th Heisenberg group. When $n = 1$, we write $\mathbb{H}$ instead of $\mathbb{H}^1$ for simplicity.

Definition 1.1. A set $E \subset \mathbb{H}^n$ is a Kakeya set if for every unit line segment $I \subset \mathbb{R}^{2n} \times \{0\}$ centred at the origin, there exists $q \in \mathbb{H}^n$ such that $qI \subset E$.

Here and in what follows, by a unit line segment we mean an isometric copy of the unit open interval $(0, 1)$. Moreover, by Heisenberg Hausdorff measure we mean the one induced by the Korányi metric on the first Heisenberg group. For the definition of the Korányi metric, we refer the readers to Section 2.

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Remark 1.2. The Kakeya sets we consider here are often also called Besicovitch sets in some other literature. And some authors use Kakeya sets to denote those inside of which a unit line segment can be continuously rotated through all directions. However it is not clear how to well define the latter concept on sub-Riemannian setting. Hence it is quite interesting to consider the latter concept as a further study.

The Heisenberg Hausdorff dimension of Euclidean Kakeya sets has been studied in [16] where the author showed a lower bound on the dimension of Kakeya sets. In [17], the author studied Kakeya sets for general metric spaces in axiomatic sense.

According to Definition 1.1, it is not hard to show that the Euclidean Hausdorff dimension of Kakeya sets in $\mathbb{H}$ is at least 2. This can be done as follows. Under orthogonal projection to the $\{xoy\}$-plane, every Kakeya set E in $\mathbb{H}$ becomes a Kakeya set E' in $\mathbb{R}^2$. Hence the known lower bound of the Euclidean Hausdorff dimension of E' is also the one of E since orthogonal projection in $\mathbb{R}^3$ is Lipschitz. Moreover, 2 is sharp since the $\{xoy\}$-plane is a Kakeya set in $\mathbb{H}$ with Euclidean Hausdorff dimension 2. However, the Heisenberg Hausdorff dimension of the $\{xoy\}$-plane is 3 and the orthogonal projection from $\mathbb{H}$ to the $\{xoy\}$-plane is no longer Lipschitz with respect to the Korányi metric. Hence to calculate a lower bound of Heisenberg Hausdorff dimension of Kakeya sets seems to be a nontrivial problem.

In this note, we will show the following

Theorem 1.3. *In the first Heisenberg group $\mathbb{H}$ equipped with the Korányi metric, every Kakeya set has Heisenberg Hausdorff dimension at least 3 and this lower bound is sharp.*

In the following, E will denote a Kakeya set in the first Heisenberg group.

Our method to show Theorem 1.3 is based on the idea of [2, 9]. We first encode each horizontal line segment in E by a quadruple in $\mathbb{R}^4$ forming a subset $L(E) \subset \mathbb{R}^4$. Then we transfer the computation for dimensions of each intersection of E and a plane belonging to a one parameter family to that of a subset in $\mathbb{R}^3$ obtained by certain projections acting on $L(E)$. This can be seen as a duality principle. Finally we use a recent Marstrand-type projection theorem in $\mathbb{R}^3$ by Käenmäki-Orponen-Venieri [14] and a co-area inequality by Eilenberg-Harrold, Jr. [8] to conclude the proof.

Since every Kakeya set in the first Heisenberg group has Heisenberg Hausdorff dimension at least 3, a further question that may be asked is to find a lower bound of 3-dimensional Heisenberg Hausdorff measure among all Kakeya sets in the first Heisenberg group. Unlike the Euclidean Kakeya set, which may have n-dimensional Lebesque measure zero in every $\mathbb{R}^n$ (for example, see [1]), it is not easy to show a counterpart for Kakeya sets in the Heisenberg group. Hence we would like to ask the following question:

Problem 1.4. Does there exist a Kakeya set in the first Heisenberg group with zero 3-dimensional Heisenberg Hausdorff measure?

The paper is organised as follows. In section 2, we recall some background in Heisenberg groups, the Marstrand-type projection theorem in $\mathbb{R}^3$ and the co-area inequality. In section 3, we prove Theorem 1.3.

2. Preliminaries

The first part of this section is dedicated to a brief introduction to the first Heisenberg group $\mathbb{H}$. For a detailed one, we refer the readers to [5].

The first Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$, equipped with the group multiplication, for any $w = (x, y, t)$ and $w' = (x', y', t')$, as follows

$$ww' = \left(x + x', y + y', t + t' + \frac{1}{2}[xy' - x'y]\right). \tag{2.1}$$

We introduce the Korányi metric on the first Heisenberg group. This is the left invariant metric given by

$$d_{\mathbb{H}}(w, w') := \|(w')^{-1} \cdot w\|_{\mathbb{H}} \tag{2.2}$$

where $\|\cdot\|_{\mathbb{H}}$ is defined as

$$\|(x, y, t)\|_{\mathbb{H}} = ((x^2 + y^2)^2 + 16t^2)^{1/4}.$$

We define horizontal lines in the first Heisenberg group $\mathbb{H}$ as lines which can be obtained by a left translation of some line passing through the origin and lying in the $\{xoy\}$-plane.

By the definition of horizontal lines, we know that for any $b \in \mathbb{R}$ and $q \in \mathbb{H}$, qI_b and qJ_b are horizontal line and horizontal line segment respectively where

$$I_b(\tau) = (\tau, b\tau, 0),\ \tau \in \mathbb{R} \tag{2.3}$$

and

$$J_b(\tau) = (\tau, b\tau, 0),\ \tau \in (-\frac{1}{2\sqrt{b^2+1}}, \frac{1}{2\sqrt{b^2+1}}).$$

The following observation is needed in the proof of Theorem 1.3.

Lemma 2.1. *For any $b \in \mathbb{R}$ and $q = (q_1, q_2, q_3) \in \mathbb{H}$,*

(1) *qI_b and qJ_b can be parameterised as*

$$qI_b(s) = (s, bs + a, -\frac{as}{2} + d),\ s \in \mathbb{R} \tag{2.4}$$

and

$$qJ_b(s) = (s, bs + a, -\frac{as}{2} + d),\ s \in (\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}}) \tag{2.5}$$

where $a = q_2 - bq_1$, $d = q_3 + \frac{1}{2}aq_1$ and $\epsilon = q_1 - \frac{1}{2\sqrt{b^2+1}}$.

(2) *If we denote*

$$l_{(a,b,d)} := \{(s, bs + a, -\frac{as}{2} + d) \in \mathbb{H} \mid s \in \mathbb{R}\} \tag{2.6}$$

and

$$l^{\epsilon}_{(a,b,d)} := \left\{(s, bs + a, -\frac{as}{2} + d) \in \mathbb{H} \mid s \in (\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}})\right\}.$$

Then $l^{\epsilon}_{(a,b,d)}$ has length 1 with respect to $d_{\mathbb{H}}$ for every a, b, d, ϵ.

(3) *If $b \in (-\sqrt{3}, \sqrt{3})$, then the orthogonal projection of $l^{\epsilon}_{(a,b,d)}$ to the x-axis has Euclidean length greater than $\frac{1}{2}$.*

Proof. (1) Let I_b be parameterised as in (2.3). Then by the Heisenberg multiplication law (2.1), we have

$$qI_b = \{(q_1 + \tau, q_2 + b\tau, q_3 + \frac{1}{2}(q_1 b\tau - q_2\tau)) \mid \tau \in \mathbb{R}\ \}.$$

Letting $s = q_1 + \tau$, $a = q_2 - bq_1$ and $d = q_3 + \frac{1}{2}aq_1$, we arrive at (2.4). In addition, letting $\epsilon = q_1 - \frac{1}{2\sqrt{b^2+1}}$, we verify that (2.5) holds.

(2) Using the definition of $d_{\mathbb{H}}$ and the fact that left translation is an isometry with respect to $d_{\mathbb{H}}$, we deduce the result.

(3) From (2.5), the orthogonal projection of $l^{\epsilon}_{(a,b,d)}$ to the x-axis is the interval $(\epsilon, \epsilon + \frac{1}{\sqrt{b^2+1}}) \subset x$-axis. Hence when $b \in (-\sqrt{3}, \sqrt{3})$, the length of the interval is greater than $\frac{1}{2}$.

□

Remark 2.2. In the sense of sub-Riemannian geometry, there exist more general horizontal curves in $\mathbb{H}$ besides horizontal lines. Indeed, associated to the group operation (2.1), we can define the left invariant vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t}.$$

A Lipschitz curve $\gamma = (\gamma_1, \gamma_2, \gamma_3) : [a, b] \to \mathbb{H}$ is said to be horizontal if $\dot{\gamma}(s) \in \mathrm{Span}\{X(\gamma(s)), Y(\gamma(s))\}$, i.e. $\dot{\gamma}(s) = a(s)X(\gamma(s)) + b(s)Y(\gamma(s))$, for almost every $s \in [a, b]$. One can check that horizontal lines are indeed horizontal curves under this definition. For more information from the sub-Riemannian point of view, we refer readers to [5, Chapter 2].

In this paper, we denote by $\mathcal{H}^s_{\mathbb{H}}$ (resp. $\mathcal{H}^s_{\mathbb{R}}$) the s-dimensional Hausdorff measure induced by the Korányi metric (resp. Euclidean metric) and by $\dim^{\mathbb{H}}_{\mathcal{H}}$ (resp. $\dim^{\mathbb{R}}_{\mathcal{H}}$) the Hausdorff dimension of sets induced by Korányi metric (resp. Euclidean metric). In addition, given any set $A \subset \mathbb{H}$, we denote

$$L(A) := \{(a, b, d, \epsilon) \in \mathbb{R} \times (-\sqrt{3}, \sqrt{3}) \times \mathbb{R} \times \mathbb{R} \mid l^{\epsilon}_{(a,b,d)} \subset A\} \tag{2.7}$$

and

$$L(A, c) := \{(a, b, d, \epsilon) \in L(A) \mid l^{\epsilon}_{(a,b,d)} \cap \{x = c\} \neq \emptyset\}. \tag{2.8}$$

Next, we recall a version of a Marstrand-type projection theorem [14, Theorem 1.2]:

Theorem 2.3. *Suppose that* $\gamma : [0, 2\pi) \to \mathbb{R}^3, \theta \mapsto \gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$. *If* $K \subset \mathbb{R}^3$ *is Borel set, then* $\dim^{\mathbb{R}}_{\mathcal{H}} \rho_{\gamma(\theta)}(K) = \min\{\dim^{\mathbb{R}}_{\mathcal{H}} K, 1\}$ *for almost every* $\theta \in [0, 2\pi)$.

Here, and in what follows, for any $x \in \mathbb{R}^3 \setminus \{0\}$, $\rho_x : \mathbb{R}^3 \to \mathrm{Span}(x)$ denotes the Euclidean orthogonal projection to the straight line passing through the origin and x.

We also need the following co-area inequality [8, Theorem 1]:

Theorem 2.4. *Let* X *be an arbitrary metric space,* $0 \leq \alpha < \infty$ *be real numbers and* $F \subset X$ *be any subset. Then, for any* 1*-Lipschitz map* $f : X \to \mathbb{R}$ *we have*

$$\int^{*}_{\mathbb{R}} \mathcal{H}^{\alpha}_{X}(F \cap f^{-1}(y))\, dy \leq \mathcal{H}^{\alpha+1}_{X}(F). \tag{2.9}$$

Here, $\int_{\mathbb{R}}^{*} g\,dy$ *is the upper integral of* $g:\mathbb{R}\to[0,+\infty)$*. That is*

$$\int_{\mathbb{R}}^{*} g(y)\,dy = \inf \int_{\mathbb{R}} h(y)\,dy$$

where the infimum is taken over all measurable functions $h:\mathbb{R}\to[0,+\infty)$ *satisfying* $0\le g(y)\le h(y)$ *for a.e.* $y\in\mathbb{R}$*.*

3. Proof of Theorem 1.2

Proof of Theorem 1.3. Since every set is contained in a G_δ-set of the same dimension, we may assume E to be G_δ.

Step 1 (Properties of $L(E,c)$). First, we need

Claim I. For every $c\in\mathbb{R}$, $L(E,c)$ is a G_δ set in $\mathbb{R}^4$.

Proof of Claim I. Recalling (2.7),

$$L(E,c)=\{(a,b,d,\epsilon)\in L(E)\mid l^{\epsilon}_{(a,b,d)}\cap\{x=c\}\neq\emptyset\}.$$

Since E is a G_δ set, we can find a sequence of open sets $\{E_i\}_{i\in\mathbb{N}}$ such that $E_i\supset E_{i+1}$ for each $i\in\mathbb{N}$ and

$$E=\bigcap_{i\in\mathbb{N}} E_i. \tag{3.1}$$

Consider the sets

$$L(E_i,c)=\{(a,b,d,\epsilon)\in L(E_i)\mid (l^{\epsilon}_{(a,b,d)}\cap\{x=c\})\neq\emptyset\}.$$

By (3.1) it is not hard to see

$$L(E,c)=\bigcap_{i\ge 1} L(E_i,c).$$

We just need to show that $L(E_i,c)$ is open for any $c\in\mathbb{R}$ and $i\in\mathbb{N}$.

Consider an arbitrary quadruple $(a,b,d,\epsilon)\in L(E_i,c)$, i.e.

$$l^{\epsilon}_{(a,b,d)}\cap\{x=c\}\neq\emptyset \quad\text{and}\quad l^{\epsilon}_{(a,b,d)}\subset E_i, \tag{3.2}$$

Thanks to the openness of E_i and the interval $(\epsilon,\epsilon+\frac{1}{\sqrt{b^2+1}})$, we deduce that for (a',b',d',ϵ') close enough to (a,b,d,ϵ), we have

$$l^{\epsilon'}_{(a',b',d')}\cap\{x=c\}\neq\emptyset \quad\text{and}\quad l^{\epsilon'}_{(a',b',d')}\subset E_i$$

and hence $(a',b',d',\epsilon')\in L(E_i,c)$, which implies $L(E_i,c)$ is open. □

Furthermore, we have the following

Claim II. There exists at least one $c_0\in\mathbb{R}$ satisfying

$$\mathcal{H}^1_{\mathbb{R}}(\pi_{123}(L(E,c_0)))>0 \tag{3.3}$$

where π_{123} is the orthogonal projection from $\mathbb{R}^4$ to the subspace spanned by the first three coordinates.

Proof of Claim II. Since E is a Kakeya set in $\mathbb{H}$, by Definition 1.1 and recalling (2.7), we have

$$\pi_2(L(E)) \supset (-\sqrt{3}, \sqrt{3})$$

where π_2 is the orthogonal projection from $\mathbb{R}^4$ to the subspace spanned by the second coordinate, which implies

$$\mathcal{H}^1_{\mathbb{R}}(\pi_2(L(E))) > 0. \tag{3.4}$$

By observing that

$$L(E) = \bigcup_{c\in\mathbb{Q}} L(E,c)$$

and using (3.4), we infer that there exists c_0 such that

$$\mathcal{H}^1_{\mathbb{R}}(\pi_2(L(E,c_0))) > 0. \tag{3.5}$$

Noting that

$$\mathcal{H}^1_{\mathbb{R}}(\pi_{123}(L(E,c))) \geq \mathcal{H}^1_{\mathbb{R}}(\pi_2(L(E,c))),\ \forall c \in \mathbb{R},$$

we conclude the proof. □

We end Step 1 with the following:

Claim III. We can find a Borel set $B \subset \pi_{123}(L(E,c_0))$ with

$$\mathcal{H}^1_{\mathbb{R}}(B) > 0 \tag{3.6}$$

and at least one of the following holds:

$$B \subset \pi_{123}(L(E,c)), \quad \forall c \in [c_0 - \frac{1}{4}, c_0] \tag{3.7}$$

or

$$B \subset \pi_{123}(L(E,c)), \quad \forall c \in [c_0, c_0 + \frac{1}{4}]. \tag{3.8}$$

Proof of Claim III. Using Lemma 2.1(3), we observe that if $l^\epsilon_{(a,b,d)} \cap \{x = c_0\} \neq \emptyset$, then either

$$l^\epsilon_{(a,b,d)} \cap \{x = c_0 - \frac{1}{4}\} \neq \emptyset,$$

or

$$l^\epsilon_{(a,b,d)} \cap \{x = c_0 + \frac{1}{4}\} \neq \emptyset.$$

Hence we have

$$L(E,c_0) \subset [L(E,c_0) \cap L(E,c_0 - \frac{1}{4})] \cup [L(E,c_0) \cap L(E,c_0 + \frac{1}{4})].$$

We conclude

$$\pi_{123}(L(E,c_0)) \subset \pi_{123}(L(E,c_0)\cap L(E,c_0-\frac{1}{4}))\cup\pi_{123}(L(E,c_0)\cap L(E,c_0+\frac{1}{4})). \tag{3.9}$$

From the above inclusion, we can assume, without loss of generality, that

$$\mathcal{H}^1_{\mathbb{R}}\left(\pi_{123}(L(E,c_0) \cap L(E,c_0 + \frac{1}{4}))\right) \geq \frac{1}{2}\mathcal{H}^1_{\mathbb{R}}(L(E,c_0)) > 0$$

where the last inequality results from Claim II.

On the other hand, if $(a,b,d,\epsilon) \in L(E,c_0) \cap L(E,c_0 + \frac{1}{4})$, then for any $c \in [c_0, c_0 + \frac{1}{4}]$, we have $(a,b,d,\epsilon) \in L(E,c)$, which indicates

$$L(E,c_0) \cap L(E,c_0 + \frac{1}{4}) \subset L(E,c) \text{ for any } c \in [c_0, c_0 + \frac{1}{4}].$$

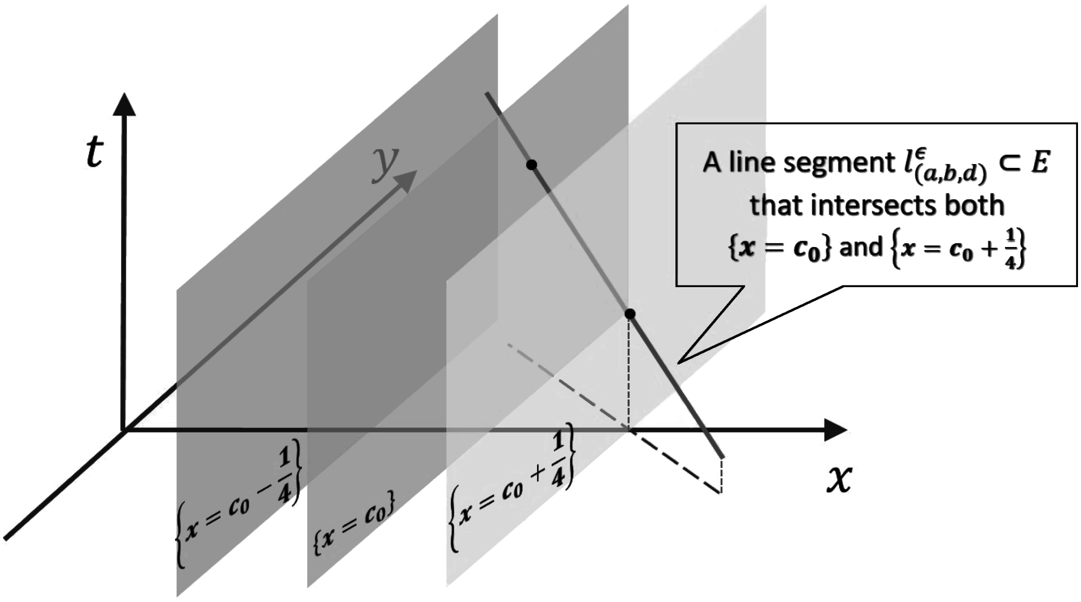

FIGURE 1. A line segment that intersects both $\{x = c_0\}$ and $\{x = c_0 + \frac{1}{4}\}$

By I, for any $c \in \mathbb{R}$, we know that $L(E, c)$ is a G_δ set and hence $\pi_{123}(L(E, c_0) \cap L(E, c_0 + \frac{1}{4}))$ is an analytic set. Hence we can apply Corollary 2 in [6] to choose B to be a closed subset of $\pi_{123}(L(E, c_0) \cap L(E, c_0 + \frac{1}{4}))$ with $\mathcal{H}^1_{\mathbb{R}}(B) > 0$. Therefore B satisfies the assumption of the claim. □

Claim III enables us to choose $c_0 \in \mathbb{R}$ such that, without loss of generality, there exists a Borel set $B \subset \pi_{123}(L(E, c_0))$ satisfying (3.6), i.e.

$$\mathcal{H}^1_{\mathbb{R}}(B) > 0.$$

and (3.8). Therefore, we infer that

$$\dim^{\mathbb{R}}_{\mathcal{H}}(B) \geq 1. \tag{3.10}$$

Step 2 (Establish a duality principle). Recall the definition of $l_{(a,b,d)}$ in (2.6). For every $c \in [c_0, c_0 + 1/4]$, we consider $E_c \subset \{x = c\} \cap E \subset \mathbb{H}$ defined by

$$\begin{aligned} E_c :&= \{l_{a,b,d} \cap \{x = c\} \mid (a, b, d) \in B\} \\ &= \{(c, bc + a, -\frac{ac}{2} + d) \mid (a, b, d) \in B\}. \end{aligned} \tag{3.11}$$

Use left translation $T_{(-c,0,0)}$ to translate E_c to $\{yot\}$-plane, which means $E^1_c := T_{(-c,0,0)}(E_c)$ lies in $\{yot\}$-plane and has the same dimension as E_c. See the above Figure 2.

Recalling (3.11) and the Heisenberg multiplication law (2.1), we deduce that

$$\begin{aligned} E^1_c &= T_{(-c,0,0)}(E_c) \\ &= \{(-c, 0, 0) \cdot (c, bc + a, -\frac{ac}{2} + d) \mid (a, b, d) \in B\} \\ &= \{(0, bc + a, -ac - \frac{bc^2}{2} + d) \mid (a, b, d) \in B\}. \end{aligned} \tag{3.12}$$

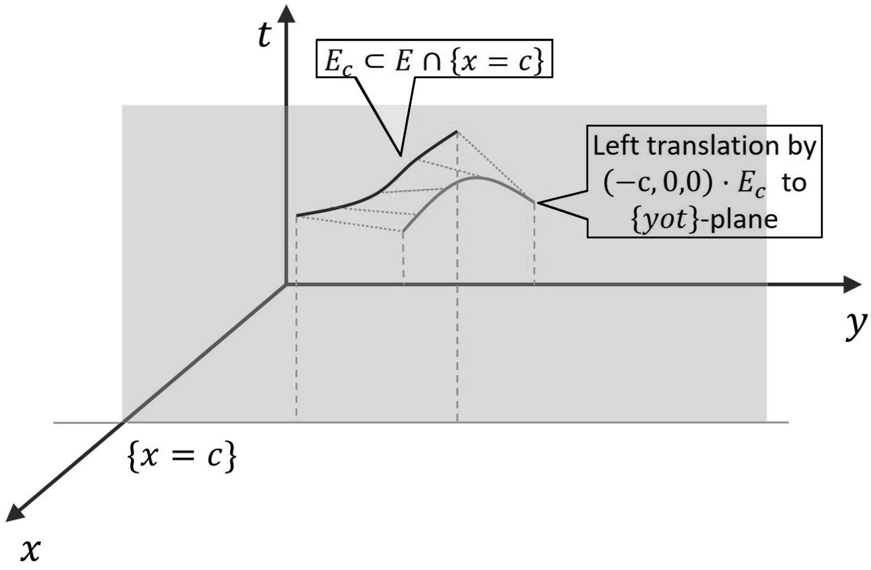

FIGURE 2. Translating E_c to $\{yot\}$-plane

Notice that the third coordinate of points in E_c^1 expressed in (3.12) takes the form

$$-ac - \frac{bc^2}{2} + d = \left\langle (-c, -\frac{c^2}{2}, 1), (a, b, d) \right\rangle$$

where $\langle , \rangle$ is the Euclidean inner product in $\mathbb{R}^3$.

By considering the t-axis as $\mathbb{R}$ and letting

$$\varphi : \{yot\} \to \mathbb{H},\ (0, y, t) \mapsto (0, 0, t), \tag{3.13}$$

we can write

$$\varphi(E_c^1) = \left\{ \left\langle (-c, -\frac{c^2}{2}, 1), (a, b, d) \right\rangle \mid (a, b, d) \in B \right\}$$

$$= \left(1 + \frac{c^2}{2}\right) \rho_{(-c, -\frac{c^2}{2}, 1)}(B). \tag{3.14}$$

Equation (3.14) implies that $\varphi(E_c^1)$ can be viewed as a Euclidean projection of B to the one parameter family of lines $\Gamma = \{\gamma_c : \mathbb{R} \to \mathbb{R}^3 \mid t \mapsto (-ct, -\frac{c^2}{2}t, t), c \in [c_0, c_0 + 1/4]\}$ up to scalings. Letting $t = 1$, we observe that $c \mapsto (-c, -\frac{c^2}{2}, 1)$ forms a part of parabola $\mathcal{H}i$ in $\mathbb{R}^3$. Hence this one parameter family of lines forms part of a cone C_1 in $\mathbb{R}^3$, i.e.

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = -2yz\}.$$

Moreover, the intersection of Γ and the unit sphere in $\mathbb{R}^3$ is contained in a circle and can be parameterised as

$$\widetilde{\gamma}(c) = \left\{ \frac{2}{2 + c^2}(-c, -\frac{c^2}{2}, 1) \mid c \in [c_0, c_0 + 1/4] \right\}.$$

We see the arc $\widetilde{\gamma}$ and the parabola $\mathcal{H}i$ are both conical curves, they can be included in one same cone as Figure 3 depicts.

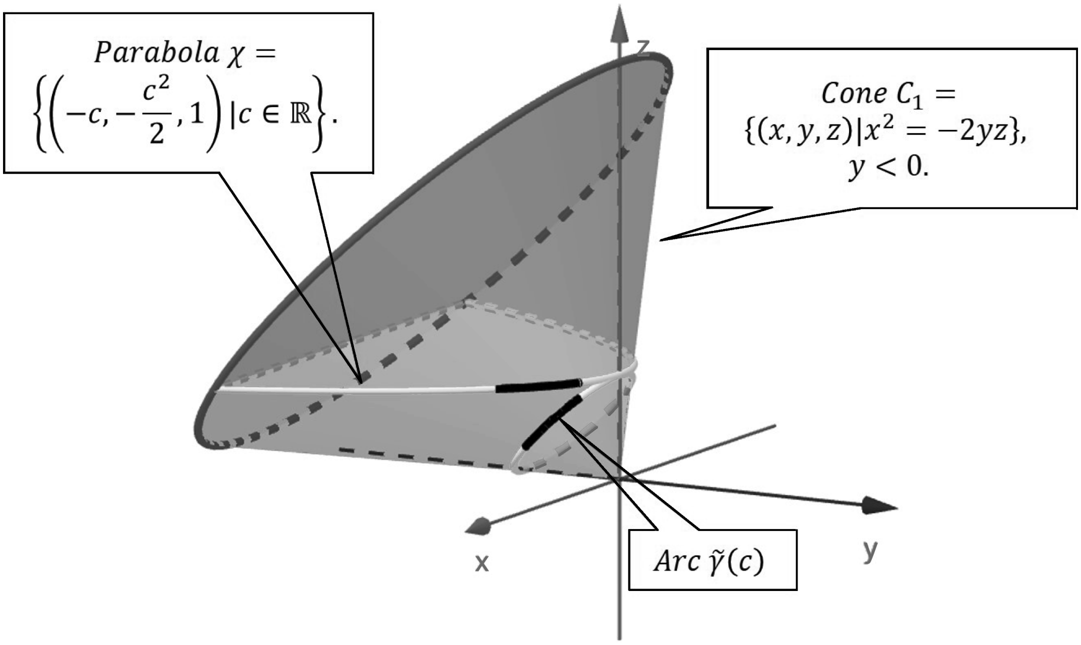

FIGURE 3. Parabola $\mathcal{X}$ and Arc $\widetilde{\gamma}$ can be included in one same cone.

Step 3 (Conclusion). In Theorem 2.3, the family of lines passing through the origin and

$$\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1)$$

also spans a cone $C_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$ in $\mathbb{R}^3$.

We observe that the cone C_2 can be obtained by a rotation $\mathcal{R} : (x, y, z) \mapsto (x, \frac{\sqrt{2}}{2}(y + z), \frac{\sqrt{2}}{2}(z - y))$ acting on C_1 and $\widetilde{\gamma}$ is mapped to an arc of γ, i.e.

$$\begin{aligned}\gamma(\theta(c)) = \mathcal{R} \circ \widetilde{\gamma}(c) &= \frac{2}{2+c^2}\left(-c, \frac{\sqrt{2}}{4}(2-c^2), \frac{\sqrt{2}}{4}(2+c^2)\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{-2\sqrt{2}c}{2+c^2}, \frac{2-c^2}{2+c^2}, 1\right),\end{aligned}$$

for $c \in [c_0, c_0 + \frac{1}{4}]$, where $\theta(c)$ is determined from

$$(\cos(\theta(c)), \sin(\theta(c))) = \left(\frac{-2\sqrt{2}c}{2+c^2}, \frac{2-c^2}{2+c^2}\right).$$

This implies

$$\rho_{(-c,-\frac{c^2}{2},1)}((x, y, z)) = \rho_{\gamma(\theta(c))}(\mathcal{R}(x, y, z)), \quad \forall (x, y, z) \in \mathbb{R}^3. \tag{3.15}$$

By Claim III and (3.10), we know B is Borel and $\dim_{\mathcal{H}}^{\mathbb{R}}(B) \geq 1$. We use (3.15) and apply Theorem 2.3 to the family of lines passing through the origin and $\{\gamma(\theta(c))\}_{c\in[c_0,c_0+1/4]}$ to deduce that

$$\dim_{\mathcal{H}}^{\mathbb{R}}[\rho_{(-c,-\frac{c^2}{2},1)}(B)] = \dim_{\mathcal{H}}^{\mathbb{R}}[\rho_{\gamma(\theta(c))}(\mathcal{R}(B))] = 1 \quad a.e.\ c \in [c_0, c_0 + 1/4]. \tag{3.16}$$

Recalling the definition of φ in (3.13) and according to (2.2), we know φ is 1-Lipschitz with respect to $d_{\mathbb{H}}$ and for any set $A \subset t$-axis,

$$\dim_{\mathcal{H}}^{\mathbb{H}}(A) = 2 \dim_{\mathcal{H}}^{\mathbb{R}}(A).$$

Hence for any $c \in [c_0, c_0 + 1/4]$ such that (3.16) holds, combining (3.11), (3.14), (3.16) and the above equality, we conclude

$$\begin{aligned}\dim_{\mathcal{H}}^{\mathbb{H}}(\{x = c\} \cap E) \geq \dim_{\mathcal{H}}^{\mathbb{H}}(E_c) = \dim_{\mathcal{H}}^{\mathbb{H}}(E_c^1) &\geq \dim_{\mathcal{H}}^{\mathbb{H}}(\phi(E_c^1)) \\ &= 2 \dim_{\mathcal{H}}^{\mathbb{R}}(\rho_{(-1,\frac{c}{2},c^2)}(B)) \\ &= 2\end{aligned}$$

and for any $0 < \alpha < 2$, we deduce

$$\mathcal{H}_{\mathbb{H}}^{\alpha}(\{x = c\} \cap E) = \infty.$$

By definition of $d_{\mathbb{H}}$, the map $f : (\mathbb{H}, d_{\mathbb{H}}) \to \mathbb{R}, (x, y, t) \to (x, 0, 0)$ is 1-Lipschitz. Now letting $X = \mathbb{H}$, $Y = [c_0, c_0 + 1/4]$ and $F = E \cap \{(x, y, t) \in \mathbb{H} \mid x \in [c_0, c_0 + 1/4]\}$ in Theorem 2.4, for any $0 < \alpha < 2$, we derive

$$\mathcal{H}_{\mathbb{H}}^{\alpha+1}(F) \geq \int_{[c_0,c_0+1/4]}^{*} \mathcal{H}_{\mathbb{H}}^{\alpha}(F \cap f^{-1}(y))\, dy = \infty,$$

which implies

$$\dim_{\mathcal{H}}^{\mathbb{H}}(E) \geq \dim_{\mathcal{H}}^{\mathbb{H}}(E \cap \{(x, y, t) \in \mathbb{H} \mid x \in [c_0, c_0 + 1/4]\}) = \dim_{\mathcal{H}}^{\mathbb{H}}(F) \geq 3.$$

We finish the proof. □

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Dimension estimates on circular (s,t)-Furstenberg sets

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Dimension estimates on circular (s,t)-Furstenberg sets

Jiayin Liu

Abstract. In this paper, we show that circular (s,t)-Furstenberg sets in $\mathbb{R}^2$ have Hausdorff dimension at least

$$\max\{\tfrac{t}{3}+s,(2t+1)s-t\} \quad \text{for all } 0<s,t\le 1.$$

This result extends the previous dimension estimates on circular Kakeya sets by Wolff.

Furstenbergin (s,t)-ympyräjoukkojen ulottuvuuden arvioita

Tiivistelmä. Tässä työssä osoitetaan, että tason $\mathbb{R}^2$ Furstenbergin (s,t)-ympyräjoukkojen Hausdorffin ulottuvuus on vähintään

$$\max\{\tfrac{t}{3}+s,(2t+1)s-t\} \quad \text{kaikilla } 0<s,t\le 1.$$

Tämä tulos yleistää Wolffin aiemmin todistamia Kakeyan ympyräjoukkojen ulottuvuusarvioita.

1. Introduction

Let F be a circular (s,t)-Furstenberg set in $\mathbb{R}^2$. That is, there exists a *parameter set* $K\subset\mathbb{R}^3_+$ with Hausdorff dimension

$$\dim_{\mathcal{H}} K\ge t$$

such that for every $(x,r)\in K$,

$$\dim_{\mathcal{H}}(F\cap S(x,r))\ge s \tag{1.1}$$

where $\mathbb{R}^3_+ := \{(x,r)=(x_1,x_2,r)\mid r>0\}$ and $S(x,r)$ is the circle centered at $x\in\mathbb{R}^2$ with radius r. A special class of circular $(1,1)$-Furstenberg sets is the family of circular Kakeya sets, that is, Borel sets in $\mathbb{R}^2$ that contain circles of every radius.

The study on the Hausdorff dimension of Furstenberg sets was initiated from their linear version. In this paper, we call a set $F\subset\mathbb{R}^2$ a linear (s,t)-Furstenberg set if there exists a parameter set K in $A(2,1)$ with

$$\dim_{\mathcal{H}} K\ge t$$

such that for every $L\in K$,

$$\dim_{\mathcal{H}}(F\cap L)\ge s$$

where $A(n,k)$ denotes the family of k-dimensional affine subspaces in $\mathbb{R}^n$.

In 1999, Wolff [16] showed that linear $(s,1)$-Furstenberg sets with parameter set K containing lines in every direction have Hausdorff dimension at least

$$\max\{\tfrac{1}{2}+s,2s\} \quad \text{for all } 0<s\le 1. \tag{1.2}$$

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In the sequel, there is a series of works improving the above lower bound and providing the one for linear (s,t)-Furstenberg sets with some of them only considering special values of s,t. We refer the readers to [9, 1, 11, 10, 12, 7, 3, 2, 13] and references therein. Moreover, in higher dimensions, one can similarly define linear (s,t)-Furstenberg sets with parameter set K in $A(n,k)$. See [5, 6] for some recent progress.

It is not clear whether the above lower bound estimates on the Hausdorff dimension for linear (s,t)-Furstenberg sets in $\mathbb{R}^2$ are sharp for any value of s and t except $s=1$. Hence determining the sharp lower bound remains open for Hausdorff dimension of linear (s,t)-Furstenberg sets.

In terms of circular (s,t)-Furstenberg sets in $\mathbb{R}^2$, Wolff in [17, Corollary 3] showed that circular Kakeya sets in $\mathbb{R}^2$ have full dimension 2 employing techniques from harmonic analysis. Also, in [15, Corollary 3], Wolff proved that Borel sets in $\mathbb{R}^2$ consisting of circles with t-dimensional set of centers have Hausdorff dimension at least $1+t$. Later, in [8], as an application of their techniques to prove a Marstrand-type restricted projection theorem, Käenmäki–Orponen–Venieri were able to show that the above lower bound $1+t$ in [15] holds true for analytic t-dimensional family of circles. Hence they provide an alternative method showing the dimension of sets containing full circles. Since the above results concern special cases of circular $(1,t)$-Furstenberg sets, these bounds are sharp. To the best of the author's knowledge, these works and earlier results on families of full circles are the only ones concerning the Hausdorff dimension for circular Furstenberg sets.

In this paper, we extend the existing result to general $0<s,t\le 1$. We show the following:

Theorem 1.1. *For any $0<s\le 1$ and $0<t\le 1$, the Hausdorff dimension of a circular (s,t)-Furstenberg set F in $\mathbb{R}^2$ is at least*

$$\max\{\tfrac{t}{3}+s,(2s-1)t+s\}. \tag{1.3}$$

We remark that for any $0<t\le 1$, if $0<s\le\frac{2}{3}$, then the maximum in (1.3) is attained by $\frac{t}{3}+s$. Otherwise, it is achieved by $(2s-1)t+s$. Indeed, these two bounds are obtained by different approaches. Hence Theorem 1.1 is a combination of the following two theorems.

Theorem 1.2. *For any $0<s\le 1$ and $0<t\le 1$, the Hausdorff dimension of a circular (s,t)-Furstenberg set F in $\mathbb{R}^2$ is at least*

$$\frac{t}{3}+s.$$

Theorem 1.3. *For any $\frac{1}{2}<s\le 1$ and $0<t\le 1$, the Hausdorff dimension of a circular (s,t)-Furstenberg set F in $\mathbb{R}^2$ is at least*

$$(2s-1)t+s.$$

Below, we briefly outline our ideas of the proof of Theorem 1.2 and Theorem 1.3, which will imply Theorem 1.1. Here, we will focus on explaining some informal ideas on obtaining the Minkowski dimension lower bounds for circular Furstenberg sets. Then we can derive the Hausdorff dimension lower bounds from the Minkowski dimension lower bounds in a standard way. To this end, in the proof, we will work with a discretized version of the circular (s,t)-Furstenberg set F in the following sense. That is, instead of studying the t dimensional parameter set K, we will concentrate on a finite subset $V\subset K$ which is a (δ,t)-set (See Definition 2.2). In brief, V is a

δ-separated set with cardinality δ^{-t} and satisfies a t-dimensional non-concentration condition.

With this discretized circular Furstenberg set $\bigcup_{z\in V} S(z)\cap F$, we consider an arbitrary cover $\mathcal{U}=\{B(x_i,r_i)\}_{i\in\mathcal{I}_{k_1}}$ of this set by balls of radii between $\delta/2$ and δ where $\delta=2^{-k_1}$ ($k_1\in\mathbb{N}$) is sufficiently small. We will give a lower bound of $\#\mathcal{I}_{k_1}$ independent of the choice of the cover $\mathcal{U}$. Recall that the desired lower bound is $\frac{t}{3}+s$ in Theorem 1.2 and $(2t+1)s-t$ in Theorem 1.3, so we need to show that

$$\#\mathcal{I}_{k_1}\gtrsim\left(\frac{1}{\delta}\right)^{\frac{t}{3}+s}\quad\text{in Theorem 1.2}\tag{1.4}$$

and

$$\#\mathcal{I}_{k_1}\gtrsim\left(\frac{1}{\delta}\right)^{(2t+1)s-t}\quad\text{if } \tfrac{1}{2}<s\le 1 \text{ in Theorem 1.3.}\tag{1.5}$$

Indeed, this will imply

$$\sum_{i\in\mathcal{I}_{k_1}} r_i^{\frac{t}{3}+s}\gtrsim\left(\frac{1}{\delta}\right)^{\frac{t}{3}+s}\delta^{\frac{t}{3}+s}\gtrsim 1\quad\text{in Theorem 1.2}$$

and

$$\sum_{i\in\mathcal{I}_{k_1}} r_i^{(2t+1)s-t}\gtrsim\left(\frac{1}{\delta}\right)^{(2t+1)s-t}\delta^{(2t+1)s-t}\gtrsim 1\quad\text{if } \tfrac{1}{2}<s\le 1\text{ in Theorem 1.3,}$$

which further imply that the $\frac{t}{3}+s$ (resp. $(2t+1)s-t$) dimensional Hausdorff measure of F is positive and therefore the Hausdorff dimension of F is at least $\frac{t}{3}+s$ (resp. $(2t+1)s-t$).

To show (1.4), we adapt the approach for showing the lower bound for the Hausdorff dimension of linear $(s,1)$-Furstenberg sets used by Wolff in [16] together with some geometric observations from planar geometry. The heuristic idea is that, since three points determine a unique circle in the plane provided they are not collinear, we can show that three well-separated δ-balls B_i, B_j, B_k determine a “unique” circle $S(z)$ (not necessarily unique in reality, see the statement before (3.22)), $z\in V$, with the help of Lemma 2.5, which intuitively means that there exists a unique circle $S(z)$ with $z\in V$ such that $S(z)\cap B_l\neq\emptyset$ for $l=i,j,k$. This further enables us to identify the circle $S(z)$ with the triple (i,j,k). Indeed, the above manipulations are motivated by Wolff [16] to show the lower bound $1/2+s$ in (1.2) for the Hausdorff dimension of linear $(s,1)$-Furstenberg sets where $1/2$ appears from the fact that two points determine a unique line in the plane. For circular $(s,1)$-Furstenberg sets, we can only get the lower bound $1/3+s$ since we need three points to determine a circle. On the other hand, since $S(z)\cap F$ has Hausdorff dimension no less than s, we need, roughly speaking, at least $\sim\delta^{-s}$ δ-balls in $\mathcal{U}$ to cover $S(z)\cap F$. Hence we can identify each $S(z)\cap F$ by the triples $(i,j,k)\in\mathcal{I}_{k_1}\times\mathcal{I}_{k_1}\times\mathcal{I}_{k_1}$ (or equivalently, $(B_i,B_j,B_k)\in\mathcal{U}\times\mathcal{U}\times\mathcal{U}$) where $S(z)\cap B_l\neq\emptyset$ for $l=i,j,k$. Then each $S(z)\cap F$ gives rise to $\delta^{-s}(\delta^{-s}-1)(\delta^{-s}-2)\sim\delta^{-3s}$ many distinct triples $(i,j,k)\in\mathcal{I}_{k_1}\times\mathcal{I}_{k_1}\times\mathcal{I}_{k_1}$ representing three distinct δ-balls in $\mathcal{U}$ and therefore we obtain a total number $\#V\times\delta^{-3s}=\delta^{-3s-t}$ many distinct triples. Finally, since all these triples are contained in $\mathcal{I}_{k_1}\times\mathcal{I}_{k_1}\times\mathcal{I}_{k_1}$, we deduce that $(\#\mathcal{I}_{k_1})^3\gtrsim\delta^{-3s-t}$, which gives (1.4). This is the rough idea behind the proof of the Minkowski dimesion version of Theorem 1.2.

On the other hand, inequality (1.5) is obtained by applying the result from Käenmäki–Orponen–Venieri in [8] utilised to find the Hausdorff dimension of t-dimensional analytic sets of circles. Heuristically, as discussed above, since one needs at least $\sim \delta^{-s}$ δ-balls in $\mathcal{U}$ to cover $S(z) \cap F$ for each $z \in V$, if each δ-ball in $\mathcal{U}$ only intersects one $S(z) \cap F$ for some $z \in V$, then $\mathcal{U}$ consists of at least $\delta^{-s}\#V \sim \delta^{-s-t}$ many δ-balls. However, this may not be the case. In general, if each δ-ball in $\mathcal{U}$ intersects no more than $\delta^{-\xi}$ $(0 < \xi \leq t)$ many sets from the family $\{S(z) \cap F\}_{z\in V}$, then we can deduce that $\mathcal{U}$ consists of at least $\frac{\delta^{-s}\#V}{\delta^{-\xi}} \sim \frac{\delta^{-s-t}}{\delta^{-\xi}}$ many δ-balls. Actually, by applying [8, Lemma 5.1], we can show that for more than half of points z in V, there exists $S'(z) \subset S(z) \cap F$ with $\dim_{\mathcal{H}} S'(z) = \dim_{\mathcal{H}}[S(z) \cap F] \geq s$ such that each δ-ball in $\mathcal{U}$ intersects no more than $\delta^{t(2s-2)}$ many sets from the family $\{S'(z)\}_{z\in V}$ where $t(2s-2)$ arises from the choice of the parameter λ when applying Lemma 5.1 in [8] to guarantee (4.15) holds. We refer readers to the discussion around (4.17) in Section 4 for details. This fact will imply that there exist at least $\frac{\delta^{-s}\#V}{\delta^{t(2s-2)}} \sim \delta^{-[(2t+1)s-t]}$ many δ-balls in $\mathcal{U}$, which is equivalent to say $\#\mathcal{I}_{k_1} \gtrsim \delta^{-[(2t+1)s-t]}$. Hence (1.5) holds and this concludes a heuristic discussion regarding Theorem 1.3.

Finally, we remark that we do not know if the bound $\max\{\frac{t}{3}+s, (2s-1)t+s\}$ in Theorem 1.1 is sharp and we here make a conjecture that the sharp lower bound for Hausdorff dimension of circular $(s,1)$-Furstenberg sets is $\frac{1}{2}+\frac{3}{2}s$ for $0 < s \leq 1$. Indeed, in the following example, based on the example in [16], we construct a circular $(s,1)$-Furstenberg set whose Hausdorff dimension does not exceed $\frac{1}{2}+\frac{3}{2}s$ for all $0 < s \leq 1$.

Example 1.4. Due to the construction in [16, Section 1] by Wolff, for all $0 < s \leq 1$, there exists a linear $(s,1)$-Furstenberg set $F \subset B(0,4) \setminus B(0,1)$ whose Hausdorff dimension does not exceed $\frac{1}{2}+\frac{3}{2}s$. Now considering $\mathbb{R}^2$ as the complex plane $\mathbb{C}$, using the map $\omega\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \frac{1}{z}$, all lines in $\mathbb{C}$ are mapped to circles through $(0,0)$. Also noticing that $\omega|_{B(0,4)\setminus B(0,1)}$ is a biLipschitz homeomorphism, we deduce that $F' := \omega(F)$ is a circular $(s,1)$-Furstenberg set with same dimension as F. That is, $\dim_{\mathcal{H}}(F') \leq \frac{1}{2}+\frac{3}{2}s$.

The paper is organised as follows. In Section 2, we clarify our notations and symbols, as well as introduce definitions and results employed in the proof. Sections 3 and 4 are devoted to showing the proof of Theorem 1.2 and 1.3 respectively. In the last section, Section 5, we complete the proof of some auxiliary lemmas needed in the proof of Theorem 1.2 using planar geometry.

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2. Preliminaries

In this paper, we denote by $S^\delta(x,r)$ the δ-neighbourhood of $S(x,r)$, i.e.

$$S^\delta(x,r) := B(x,r+\delta) \setminus B(x,r-\delta).$$

We also use the notation $z = (x,r) \in \mathbb{R}^3$. Moreover, we use the notation $f \lesssim g$ (resp. $f \lesssim_h g$) for $f \leq kg$ (resp. $f \leq k(h)g$) where k is a constant that depends only on the ambient space (resp. the parameter h), and may change from line to line. Likewise, $f \gtrsim g$ and $f \sim g$ are understood correspondingly.

The notation $\mathcal{H}^s$ stands for the s-dimensional Hausdorff measure, and $\mathcal{H}^s_\infty$ stands for s-dimensional Hausdorff content. The notation $|\cdot|$ and $\|\cdot\|$ will denote the Lebesgue measure and the Euclidean distance respectively in $\mathbb{R}^2$ or $\mathbb{R}^3$. We also use $\operatorname{dist}(A,B)$ to denote Euclidean distance between A and B where A and B can be either points or sets. $\#A$ will denote the cardinality of a set A.

We have the following observation which makes it possible to restrict ourselves to circular Furstenberg sets with bounded parameter set.

Remark 2.1. (i) Since we are concerned with the Hausdorff dimension of the circular Furstenberg set F, we claim that it is enough to consider the case that F has parameter set $K \subset \mathbf{B}_0$ where

$$\mathbf{B}_0 = \{(x,r) \in \mathbb{R}^3 \mid x \in B(0,\tfrac14) \text{ and } \tfrac12 \le r \le 2\}. \tag{2.1}$$

To see this, consider the following covering of the parameter space $\mathbb{R}^3_+$. For $k,l,m \in \mathbb{Z}$, let

$$D_{k,l,m} := \{(x,r) \in \mathbb{R}^3 \mid x \in B((2^{2m-2}k, 2^{2m-2}l), 2^{2m-2}) \text{ and } 2^{2m-1} \le r \le 2^{2m+1}\}.$$

Then

$$\mathbb{R}^3_+ = \bigcup_{k,l,m} D_{k,l,m}$$

and

$$\mathbf{B}_0 = D_{0,0,0}.$$

Hence for each $\epsilon > 0$ sufficiently small, there exists $k_\epsilon, l_\epsilon, m_\epsilon$ such that

$$\dim_{\mathcal{H}}(K) - \dim_{\mathcal{H}}(K \cap D_{k_\epsilon,l_\epsilon,m_\epsilon}) < \epsilon. \tag{2.2}$$

Let F_ϵ be the circular Furstenberg set with parameter set $K \cap D_{k_\epsilon,l_\epsilon,m_\epsilon}$. Denote by $\mathcal{S}_y\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{S}_y(x) := x - y$ for any $y \in \mathbb{R}^2$ and by $\mathcal{D}_\lambda\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{D}_\lambda(x) := \lambda x$ for any $\lambda > 0$.

Then, letting $y = (2^{2m_\epsilon-2}k_\epsilon, 2^{2m_\epsilon-2}l_\epsilon)$ and $\lambda = 2^{-2m_\epsilon}$, we observe that

$$\widetilde{F}_\epsilon := \mathcal{D}_{2^{-2m_\epsilon}} \circ \mathcal{S}_{(2^{2m_\epsilon-2}k_\epsilon, 2^{2m_\epsilon-2}l_\epsilon)}(F_\epsilon)$$

is a circular Furstenberg set the parameter set $\widetilde{K}_\epsilon$ contained in $\mathbf{B}_0$ and satisfying

$$\dim_{\mathcal{H}}(\widetilde{K}_\epsilon) = \dim_{\mathcal{H}}(K \cap D_{k_\epsilon,l_\epsilon,m_\epsilon}). \tag{2.3}$$

If F is a circular (s,t)-Furstenberg set, then by (2.2) and (2.3), for $0 < \epsilon < t$, we know $\widetilde{F}_\epsilon$ is a circular $(s,t-\epsilon)$-Furstenberg set and

$$\dim_{\mathcal{H}} F \ge \dim_{\mathcal{H}} \widetilde{F}_\epsilon \quad \text{for every } 0 < \epsilon < t. \tag{2.4}$$

Now, assume Theorem 1.1 holds for circular Furstenberg sets with parameter set contained in $\mathbf{B}_0$, then

$$\dim_{\mathcal{H}} \widetilde{F}_\epsilon \ge \max\{\tfrac{t-\epsilon}{3} + s, (2s-1)(t-\epsilon) + s\} \quad \text{for every } 0 < \epsilon < t. \tag{2.5}$$

Combining (2.4) and (2.5), we deduce that

$$\dim_{\mathcal{H}} F \ge \lim_{\epsilon \to 0} \max\{\tfrac{t-\epsilon}{3} + s, (2s-1)(t-\epsilon) + s\} = \max\{\tfrac{t}{3} + s, (2s-1)t + s\}.$$

Hence to show Theorem 1.1, we only need to consider the case that F has parameter set $K \subset \mathbf{B}_0$.

(ii) Note that $|S^\delta(x,r)| \le c_0\delta$ for all $(x,r) \in \mathbf{B}_0$ where c_0 is an absolute constant.

We introduce the following:

Definition 2.2. ((δ,q)-sets) Let $\delta \in (0,1), q>0$, and let $P \subset \mathbb{R}^n$ be a finite δ-separated set. We say that P is a (δ,q)-set, if it satisfies the estimate

$$\#\{P \cap B(x,r)\} \lesssim \left(\frac{r}{\delta}\right)^q, \quad x \in \mathbb{R}^n,\ r>\delta. \tag{2.6}$$

We recall from [4, Lemma 3.13] the following

Lemma 2.3. *Let $\delta, q > 0$, and let $Q \subset \mathbb{R}^n$ be any set with $\mathcal{H}^q_\infty(Q) =: \beta > 0$. Then there exists a (δ,q)-set $P \subset Q$ with cardinality $\#P \gtrsim \beta \cdot \delta^{-q}$.*

Remark 2.4. If $Q \subset \mathbf{B}_0$ and $\mathcal{H}^q_\infty(Q) = \beta$, by Lemma 2.3, we know that for any $\delta > 0$, there exists a (δ,q)-set $P \subset Q$ with cardinality $\#P \gtrsim \beta\delta^{-q}$. Furthermore, letting $r = \operatorname{diam} \mathbf{B}_0$ in (2.6), we know $\#P \lesssim \delta^{-q}$, if $\delta < \operatorname{diam} \mathbf{B}_0$. We conclude that

$$\beta\delta^{-q} \lesssim \#P \lesssim \delta^{-q}.$$

To show Theorem 1.2, we need to establish the following result from planar geometry. Since the proof relies on two more auxiliary lemmas, we postpone it to the last section.

Lemma 2.5. *Let $A,B,C \in \mathbb{R}^2$ such that $\min\{\|A-B\|, \|A-C\|, \|B-C\|, 2\} \geq 2c$. For $a>0$ such that $a < \frac{1}{20}c^2$, define*

$$W := \left\{ (x,b) \in \mathbb{R}^2 \times [\tfrac{1}{2}, 2] : \quad \begin{array}{c} b-a \leq \|x-A\| \leq b+a, \\ b-a \leq \|x-B\| \leq b+a, \\ b-a \leq \|x-C\| \leq b+a \end{array} \right\}. \tag{2.7}$$

Then

$$\operatorname{diam} W \lesssim \frac{a}{c^2}. \tag{2.8}$$

It is worth mentioning that Lemma 2.5 shares a very similar conclusion with the one in [16, Lemma 3.2 (Mastrand's 3-circle lemma)]. Indeed, if we let $\epsilon = \delta = a$, $r = b$, $\lambda = c$, $t = 1/2 - a$ and $r_1 = r_2 = r_3 = a$ therein, then the set W in Lemma 2.5 will be contained in $\Omega_{\epsilon t \lambda}$ defined in [16, Lemma 3.2]. And the conclusion of [16, Lemma 3.2] says that $\Omega_{\epsilon t\lambda}$ is contained in the union of two ellipsoids in $\mathbb{R}^3$ with $\operatorname{diam} \Omega_{\epsilon t \lambda} \lesssim \frac{a}{c^2}$. Since we only consider the case $r_1 = r_2 = r_3 = a$ (that is, $C_\delta(x_i, r_i)$ become balls $B(x_i, 2a)$ for $i = 1,2,3$ in [16, Lemma 3.2]), we can deduce that W lies in one cuboid in $\mathbb{R}^3$ based on an approach which differs completely from the one of [16, Lemma 3.2].

Now, we start the preparation for the proof of Theorem 1.3. Let $P \subset \mathbb{R}^3$ be a (δ,q)-set. For any $p \in P$, let Δ_p be the Dirac measure centered at p. Then

$$\mu_P := \frac{1}{\#P} \sum_{p\in P} \Delta_p \tag{2.9}$$

is a probability measure satisfying the Frostman condition $\mu_P(B(z,r)) \lesssim r^q$ for all $z \in \mathbb{R}^3$ and $r > \delta$. Indeed, for any ball $B(z,r)$ with $r>\delta$ we have

$$\begin{aligned} \mu_P(B(z,r)) &= \frac{1}{\#P}\left(\sum_{p\in P}\Delta_p\right)(B(z,r)) = \frac{1}{\#P}\sum_{p\in P}\Delta_p(B(z,r)) \\ &= \frac{1}{\#P}\#(P\cap B(z,r)) \lesssim r^q. \end{aligned}$$

Below in Section 3 and 4, thanks to Remark 2.1(i), we will assume the circular (s,t)-Furstenberg set F has parameter set $K \subset \mathbf{B}_0$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let F be a circular (s,t)-Furstenberg set with parameter set $K \subset \mathbf{B}_0$. It suffices to show, for any $\epsilon > 0$, $0 < s' < s$ and $0 < t' < t$,

$$\dim_{\mathcal{H}}(F) \geq \frac{t'}{3} + s' - \epsilon.$$

Hence in the following, we fix s', t' and $0 < \epsilon < \frac{t'}{3} + s'$.

We notice that there exists $\alpha > 0$ and $K_1 \subset K$ such that $\mathcal{H}^{t'}_\infty(K_1) > \alpha$, where

$$K_1 := \{z \in K \mid \mathcal{H}^{s'}_\infty(F \cap S(z)) > \alpha\}. \tag{3.1}$$

Indeed, by the subadditivity of Hausdorff content, and the fact

$$K = \bigcup_n \{z \in K \mid \mathcal{H}^{s'}_\infty(F \cap S(z)) > \tfrac{1}{n}\},$$

we deduce the existence of α such that $\mathcal{H}^{t'}_\infty(K_1) > \alpha$ for K_1 defined as in (3.1).

Next, since $\epsilon > 0$, we can find $\delta_0 = \delta_0(\epsilon, s') > 0$ sufficiently small such that for any $0 < \delta < \delta_0$, we have

$$\delta^{-\epsilon} \left(\log \frac{1}{\delta}\right)^{-(\frac{8}{3} + \frac{12}{s'})} > 1. \tag{3.2}$$

and

$$\sqrt{640\delta} < \tau = \tau(\delta) := \pi^{-1} \left(\frac{1}{16}\right)^{1/s'} \left(\frac{1}{\log \frac{1}{\delta}}\right)^{2/s'} < 1. \tag{3.3}$$

Then we choose k_0 to be an integer larger than $\log(\frac{1}{\delta_0})$ also satisfying

$$\alpha > \sum_{k=k_0}^{\infty} \frac{1}{k^2}. \tag{3.4}$$

Now, we outline the main steps of the proof. We start with an arbitrary cover $\mathcal{U} = \{B(x_i, r_i)\}_{i \in \mathcal{I}}$ of F by balls of radius less than 2^{-k_0}. In the sequel, we will derive a lower bound

$$\sum_{i \in \mathcal{I}} r_i^\sigma \gtrsim_{\epsilon, t', s'} 1$$

with $\sigma = t'/3 + s' - \epsilon$ independent of the choice of the particular cover. This will imply

$$\mathcal{H}^\sigma(F) > 0.$$

To this end, we divide the proof into 4 steps. Let

$$\mathcal{I}_k := \{i \in \mathcal{I} \mid 2^{-(k+1)} < r_i \leq 2^{-k}\}, \quad F_k := \left\{\bigcup B(x_i, r_i) \mid i \in \mathcal{I}_k\right\}.$$

First, in Step 1, we will deduce that there exists $k_1 \geq k_0$ and a (δ, t')-set $V \subset K$ with $\delta = 2^{-k_1}$ such that for every circle $z = (x, r) \in V$, we have

$$\mathcal{H}^{s'}_\infty(S(z) \cap F_{k_1}) > k_1^{-2}. \tag{3.5}$$

Then, in Step 2, we modify Wolff's approach for linear $(s,1)$-Furstenberg sets to fit our circular case. For each circle $S(z)$ with $z \in V$, we will extract from $S(z)$ three τ-separated arcs $h_z^+, h_z^-, h_z^\times$ such that

$$\mathcal{H}^{s'}_\infty(h_z^+ \cap F_{k_1}) \gtrsim k_1^{-2}, \quad \mathcal{H}^{s'}_\infty(h_z^- \cap F_{k_1}) \gtrsim k_1^{-2}, \quad \mathcal{H}^{s'}_\infty(h_z^\times \cap F_{k_1}) \gtrsim k_1^{-2}. \tag{3.6}$$

These arcs enable us to define an index set $\mathcal{T} \subset \mathcal{I}_{k_1} \times \mathcal{I}_{k_1} \times \mathcal{I}_{k_1} \times V$ whose cardinality will be estimated in the following steps and will imply the lower bound for $\#\mathcal{I}_{k_1}$.

Next, in Step 3, we will deduce that the cardinality of $\mathcal{T}$ is upper bounded by the cardinality of $\mathcal{I}_{k_1}$ with the help of Lemma 2.5. Indeed, we will show

$$\#\mathcal{T} \lesssim (\#\mathcal{I}_{k_1})^3 \tau^{-6}.$$

Finally, in Step 4, we will estimate the lower bound of $\#\mathcal{T}$ which also serves as the one of $\#\mathcal{I}_{k_1}$, hence $\#\mathcal{I}$ with the aid of (3.6). This will enable us to conclude the proof.

Step 1. Let α be as in (3.4). Hence by pigeonhole principle we deduce that for each $S(z) \in K_1$, there exists $k(z) \geq k_0$ such that $\mathcal{H}^{s'}_\infty(S(z) \cap F \cap F_{k(z)}) > k(z)^{-2}$.

Moreover, by applying pigeonhole principle again we obtain that there exists $k_1 \geq k_0$ such that

$$\mathcal{H}^{t'}_\infty(K_2) > k_1^{-2} \tag{3.7}$$

where $K_2 := \{z \in K_1 \colon k(z) = k_1\}$.

We remark that for every circle $z \in K_2$, we have

$$\infty > \mathcal{H}^{s'}_\infty(S(z) \cap F_{k_1}) \geq \mathcal{H}^{s'}_\infty(S(z) \cap F \cap F_{k_1}) > k_1^{-2}. \tag{3.8}$$

By letting $\delta = 2^{-k_1}$, $q = t'$ and $Q = K_2$ in Lemma 2.3, we know that there exists a (δ, t')-set $V \subset K_2$ with cardinality

$$\#V \gtrsim \mathcal{H}^{t'}_\infty(K_2) \cdot \delta^{-t'}. \tag{3.9}$$

Hence for every $z \in V$, (3.8) implies (3.5), which concludes Step 1.

Step 2. We start the procedure of extracting three disjoint arcs for any $S(z)$, $z = (x, r) \in V$, which is illustrated in Figures 1, 2 and 3. Let

$$\eta := \eta(z) = \mathcal{H}^{s'}_\infty(S(z) \cap F_{k_1}).$$

Also let $\gamma = (\frac{\eta}{16})^{1/s'}$. Divide $S(z)$ into N arcs $I_1, \cdots, I_N$ such that

- the length of $I_1, \cdots, I_{N-1}$ is γ,
- the length of I_N is at most γ,
- and $N\gamma \geq 2\pi r$.

Since $\gamma = (\frac{\eta}{16})^{1/s'} \leq \frac{1}{16}$ and $z = (x, r) \in \mathbf{B}_0$ implies $r > \frac{1}{2}$, we know

$$N \geq \frac{2\pi r}{\gamma} \geq \frac{\pi}{\frac{1}{16}} \geq 16.$$

Note that if I is an arc in $S(z)$, then

$$\mathcal{H}^{s'}_\infty(I) \leq (\operatorname{diam} I)^{s'} \leq (\mathcal{H}^1(I))^{s'}. \tag{3.10}$$

This implies for all $l = 1, \cdots, N$,

$$\mathcal{H}^{s'}_\infty(I_l \cap F_{k_1}) \leq \mathcal{H}^{s'}_\infty(I_l) \leq \gamma^{s'} = \frac{\eta}{16}. \tag{3.11}$$

See Figure 1 for N arcs.

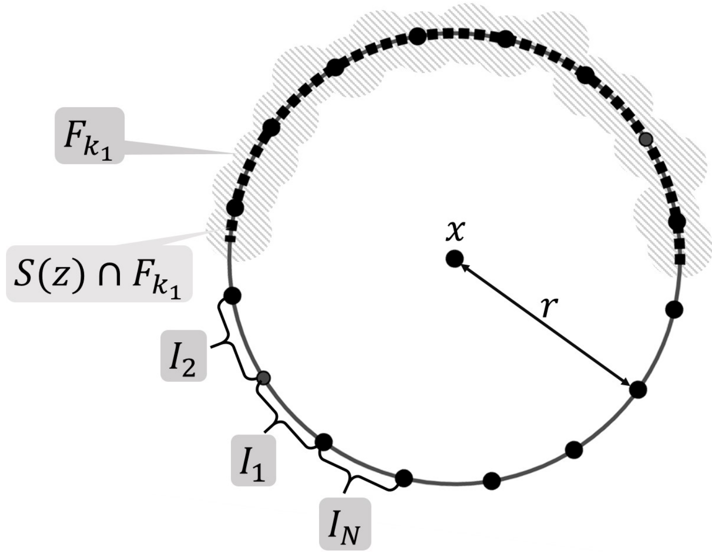

Figure 1. N arcs on $S(z)$.

Since

$$\begin{aligned}\eta &= \mathcal{H}^{s'}_\infty(S(z)\cap F_{k_1}) = \mathcal{H}^{s'}_\infty(\textstyle\bigcup_{l=1}^{N} I_l \cap F_{k_1})\\ &\le \mathcal{H}^{s'}_\infty(\textstyle\bigcup_{l=1}^{N-12} I_l \cap F_{k_1}) + \sum_{l=N-11}^{N} \mathcal{H}^{s'}_\infty(I_l\cap F_{k_1})\\ &\le \mathcal{H}^{s'}_\infty(\textstyle\bigcup_{l=1}^{N-12} I_l \cap F_{k_1}) + 12\frac{\eta}{16}\end{aligned}$$

where in the last inequality we use (3.11), we obtain

$$\mathcal{H}^{s'}_\infty(\textstyle\bigcup_{l=1}^{N-12} I_l \cap F_{k_1}) \ge \frac{1}{4}\eta.$$

This guarantees that there exists $N_1 \in [2, N-12]$ which is the smallest integer satisfying

$$\mathcal{H}^{s'}_\infty(\textstyle\bigcup_{l=1}^{N_1} I_l \cap F_{k_1}) \ge \frac{1}{8}\eta \tag{3.12}$$

and

$$\mathcal{H}^{s'}_\infty(\textstyle\bigcup_{l=1}^{N_1-1} I_l \cap F_{k_1}) < \frac{1}{8}\eta. \tag{3.13}$$

Let $h_z^+ := \bigcup_{l=1}^{N_1} I_l$. By (3.12) and (3.13), we know

$$\begin{aligned}\frac{1}{8}\eta &\le \mathcal{H}^{s'}_\infty(h_z^+) \le \mathcal{H}^{s'}_\infty(\textstyle\bigcup_{l=1}^{N_1-1} I_l \cap F_{k_1}) + \mathcal{H}^{s'}_\infty(I_{N_1}\cap F_{k_1})\\ &\le \frac{1}{8}\eta + \frac{1}{16}\eta = \frac{3}{16}\eta.\end{aligned} \tag{3.14}$$

See Figure 2 for the construction of h_z^+.

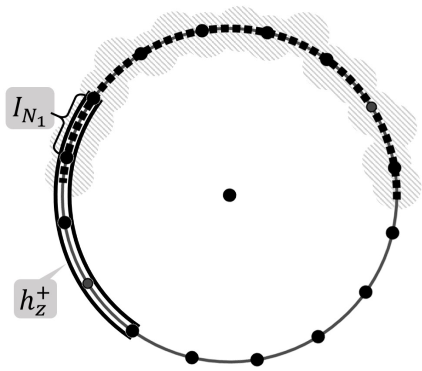

Figure 2. The construction of h_z^+.

Hence the arc h_z^+ satisfies the first inequality in (3.6). We continue to construct the other two arcs. Notice that

$$\mathcal{H}^{s'}_\infty(\bigcup_{l=1}^{N_1+1} I_l \cap F_{k_1}) \le \mathcal{H}^{s'}_\infty(h_z^+) + \mathcal{H}^{s'}_\infty(I_{N_1+1} \cap F_{k_1}) \le \tfrac{3}{16}\eta + \tfrac{1}{16}\eta = \tfrac{1}{4}\eta. \tag{3.15}$$

We remark that since $N_1 \le N - 12$, we know $N_1 + 1 \le N - 11$. Combining this, (3.15) and (3.11), we have

$$\begin{aligned}\eta &= \mathcal{H}^{s'}_\infty(\bigcup_{l=1}^{N} I_l \cap F_{k_1})\\ &\le \mathcal{H}^{s'}_\infty(\bigcup_{l=1}^{N_1+1} I_l \cap F_{k_1}) + \mathcal{H}^{s'}_\infty(\bigcup_{l=N_1+2}^{N-8} I_l \cap F_{k_1}) + \sum_{l=N-7}^{N} \mathcal{H}^{s'}_\infty(I_l \cap F_{k_1})\\ &\le \frac{1}{4}\eta + \mathcal{H}^{s'}_\infty(\bigcup_{l=N_1+2}^{N-8} I_l \cap F_{k_1}) + 8\tfrac{\eta}{16},\end{aligned}$$

which implies

$$\mathcal{H}^{s'}_\infty(\bigcup_{l=N_1+2}^{N-8} I_l \cap F_{k_1}) \ge \tfrac{1}{4}\eta.$$

Hence we can find $N_2 \in [N_1 + 3, N - 8]$ which is the smallest integer satisfying

$$\mathcal{H}^{s'}_\infty(\bigcup_{l=N_1+2}^{N_2} I_l \cap F_{k_1}) \ge \tfrac{1}{8}\eta \tag{3.16}$$

and

$$\mathcal{H}^{s'}_\infty(\bigcup_{l=N_1+2}^{N_2-1} I_l \cap F_{k_1}) < \tfrac{1}{8}\eta. \tag{3.17}$$

Let $h_z^- := \bigcup_{l=N_1+2}^{N_2} I_l$. By (3.16) and (3.17), we know

$$\begin{aligned}\frac{1}{8}\eta &\le \mathcal{H}^{s'}_\infty(h_z^-) \le \mathcal{H}^{s'}_\infty(\bigcup_{l=N_1+2}^{N_2-1} I_l \cap F_{k_1}) + \mathcal{H}^{s'}_\infty(I_{N_2} \cap F_{k_1})\\ &\le \frac{1}{8}\eta + \frac{1}{16}\eta = \frac{3}{16}\eta.\end{aligned} \tag{3.18}$$

The construction of the third arc $h_z^\times \subset S(z)$ is similar. See Figure 3 for an illustration.

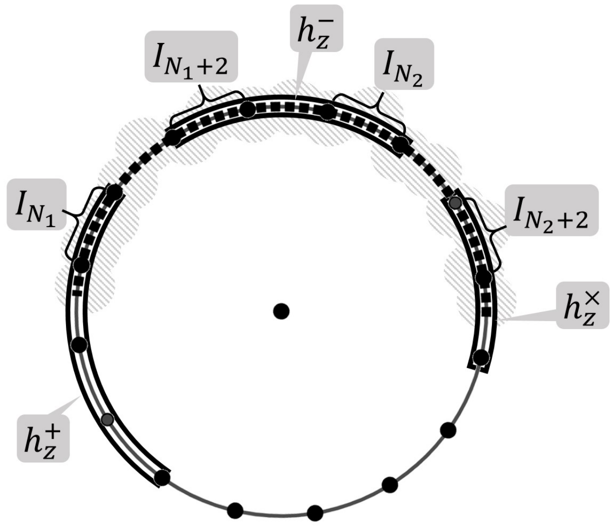

Figure 3. The construction of h_z^- and $h_z^\times$.

That is, we can find $h_z^\times = \bigcup_{l=N_2+2}^{N_3} I_l$ for some integer $N_3 \in [N_2+3, N-2]$ such that

$$\frac{1}{8}\eta \leq \mathcal{H}_\infty^{s'}(h_z^\times) \leq \frac{3}{16}\eta. \tag{3.19}$$

We omit the details here. By the construction, it is clear that

$\mathrm{dist}(h_z^+, h_z^-) = \mathrm{diam} I_{N_1+1}$, $\mathrm{dist}(h_z^-, h_z^\times) = \mathrm{diam} I_{N_2+1}$, and $\mathrm{dist}(h_z^+, h_z^\times) \geq \mathrm{diam} I_{N_3+1}$.

Recall for any $1 \leq l \leq N-1$, $\mathcal{H}^1(I_l) = \gamma$. Hence $\mathrm{diam} I_l \geq \pi^{-1}\gamma$ for any $1 \leq l \leq N-1$. We conclude that

$$\min\{\mathrm{dist}(h_z^+, h_z^-), \mathrm{dist}(h_z^-, h_z^\times), \mathrm{dist}(h_z^+, h_z^\times)\} \geq \pi^{-1}\gamma.$$

Therefore, for each circle $S(z)$, we have found three $\pi^{-1}\gamma$-separated arcs h_z^+, h_z^-, $h_z^\times \subset S(z)$ with the property in (3.14), (3.18) and (3.19) respectively. Furthermore, recalling $\gamma = (\frac{\eta}{16})^{1/s'}$ and $\eta > \frac{1}{k_1^2} = \frac{1}{(\log\frac{1}{\delta})^2}$, we deduce that $h_z^+, h_z^-, h_z^\times$ are $\tau = \pi^{-1}(\frac{1}{16})^{1/s'}(\frac{1}{\log\frac{1}{\delta}})^{2/s'}$-separated. Hence by combining with (3.8), we have showed that (3.6) holds.

We end Step 2 by defining

$$\mathcal{T} := \left\{ (i_+, i_-, i_\times, z) \in \mathcal{I}_{k_1} \times \mathcal{I}_{k_1} \times \mathcal{I}_{k_1} \times V : \begin{array}{c} h_z^+ \cap F_{k_1} \cap B_{i_+} \neq \emptyset, \\ h_z^- \cap F_{k_1} \cap B_{i_-} \neq \emptyset, \\ h_z^\times \cap F_{k_1} \cap B_{i_\times} \neq \emptyset \end{array} \right\}$$

where $B_{i_+} = B(x_{i_+}, r_{i_+})$, $B_{i_-} = B(x_{i_-}, r_{i_-})$, and $B_{i_\times} = B(x_{i_\times}, r_{i_\times})$. In the following, we will write x_+ instead of x_{i_+} for short and other lower indices will be abbreviated correspondingly.

Step 3. We estimate $\#\mathcal{T}$ from above.

First we fix $i_+, i_-, i_\times$ and estimate the upper bound of the number of $z \in V$ such that $(i_+, i_-, i_\times, z) \in \mathcal{T}$, where V is chosen as explained above (3.9).

To this end, we observe that a necessary condition for $(i_+, i_-, i_\times, z) \in \mathcal{T}$ is that

$$S(z) \cap B_{i_+} \neq \emptyset, \quad S(z) \cap B_{i_-} \neq \emptyset, \quad S(z) \cap B_{i_\times} \neq \emptyset \tag{3.20}$$

and

$$\min\{\|x_+ - x_-\|, \|x_+ - x_\times\|, \|x_- - x_\times\|\} \geq \tau - 2\delta = \tau - \sqrt{4\delta} > \frac{\tau}{2} \tag{3.21}$$

since $h_z^+, h_z^-, h_z^\times$ are τ-separated and $B_{i_+} = B(x_+, r_+)$, $B_{i_-} = B(x_-, r_-)$, $B_{i_\times} = B(x_\times, r_\times)$ are balls of radius between $\delta/2$ and δ. Moreover, in the last inequality of (3.21) we recall (3.3).

Hence we will provide an upper bound of z satisfying (3.20) and (3.21) in the following. Assume for some $z = (x, r) \in V$, (3.20) holds. Then we know

$$r - \delta \leq \|x - x_+\| \leq r + \delta,\ r - \delta \leq \|x - x_-\| \leq r + \delta,\ r - \delta \leq \|x - x_\times\| \leq r + \delta,$$

which implies

$$(x, r) \in \Gamma := \left\{ (y, d) \in \mathbb{R}^2 \times [\tfrac{1}{2}, 2] : \quad \begin{array}{c} d - \delta \leq \|y - x_+\| \leq d + \delta, \\ d - \delta \leq \|y - x_-\| \leq d + \delta, \\ d - \delta \leq \|y - x_\times\| \leq d + \delta \end{array} \right\}$$

by the fact that $z \in \mathbf{B}_0$ implies $r \in [\frac{1}{2}, 2]$. Also by (3.21) and by $\delta < \frac{\tau^2}{640}$ from (3.3), we can apply Lemma 2.5 with $\triangle ABC = \triangle x_+ x_- x_\times$, $a = \delta$, $b = r$ and $c = \frac{\tau}{4}$ to deduce that

$$\operatorname{diam} \Gamma \lesssim \frac{\delta}{\tau^2}.$$

Recall V is a δ-separated set in $\mathbf{B}_0 \subset \mathbb{R}^3$. Then for any $z, z' \in V \cap \Gamma$,

$$B(z, \tfrac{\delta}{3}) \cap B(z', \tfrac{\delta}{3}) = \emptyset,$$

which, together with $\operatorname{diam}(V \cap \Gamma) \lesssim \delta\tau^{-2}$, implies

$$\#(V \cap \Gamma)\delta^3 \sim \#(V \cap \Gamma)|B(z, \tfrac{\delta}{3})| = \left| \bigcup_{z \in V \cap \Gamma} B(z, \tfrac{\delta}{3}) \right| \lesssim [\operatorname{diam}(V \cap \Gamma)]^3 \lesssim \delta^3 \tau^{-6}.$$

Hence $\#(V \cap \Gamma) \lesssim \tau^{-6}$. We can deduce that there are at most only $\lesssim \tau^{-6}$ many $z \in V$ satisfying (3.20) for fixed i_+, i_- and $i_\times$. As a consequence, we have

$$\#\mathcal{T} \lesssim \#\mathcal{I}_{k_1} \times \#\mathcal{I}_{k_1} \times \#\mathcal{I}_{k_1} \times \tau^{-6} \lesssim (\#\mathcal{I}_{k_1})^3 \tau^{-6} \lesssim_{s'} (\#\mathcal{I}_{k_1})^3 (\log \tfrac{1}{\delta})^{12/s'}, \tag{3.22}$$

which completes the proof of Step 3.

Step 4. We estimate $\#\mathcal{T}$ from below. To this end, recall $F_{k_1} = \cup_{i \in \mathcal{I}_{k_1}} B(x_i, r_i)$. Hence for any $z \in V$, we have

$$h_z^+ \cap F_{k_1} \subset \bigcup_{i \in \mathcal{I}_{k_1}} B(x_i, r_i), \quad h_z^- \cap F_{k_1} \subset \bigcup_{i \in \mathcal{I}_{k_1}} B(x_i, r_i), \quad h_z^\times \cap F_{k_1} \subset \bigcup_{i \in \mathcal{I}_{k_1}} B(x_i, r_i).$$

For each $z \in V$, define

$$\mathcal{I}_{k_1}^+(z) := \{i \in \mathcal{I}_{k_1} \mid h_z^+ \cap B(x_i, r_i) \neq \emptyset\}, \quad \mathcal{I}_{k_1}^-(z) := \{i \in \mathcal{I}_{k_1} \mid h_z^- \cap B(x_i, r_i) \neq \emptyset\},$$

and

$$\mathcal{I}_{k_1}^\times(z) := \{i \in \mathcal{I}_{k_1} \mid h_z^\times \cap B(x_i, r_i) \neq \emptyset\}.$$

With the help of (3.6), we have

$$(\log \frac{1}{\delta})^{-2} \lesssim_{s'} \mathcal{H}_\infty^{s'}(h_z^+ \cap F_{k_1}) \leq \sum_{i \in \mathcal{I}_{k_1}^+(z)} (\operatorname{diam} B(x_i, r_i))^{s'} \sim \sum_{i \in \mathcal{I}_{k_1}^+(z)} \delta^{s'} \leq \#\mathcal{I}_{k_1}^+(z) \delta^{s'}$$

for all $z \in V$, which implies

$$\#\mathcal{I}_{k_1}^{+}(z) \gtrsim_{s'} \frac{1}{\delta^{s'}}(\log \tfrac{1}{\delta})^{-2} \quad \text{for all } z \in V. \tag{3.23}$$

Similarly, we have

$$\#\mathcal{I}_{k_1}^{-}(z) \gtrsim_{s'} \frac{1}{\delta^{s'}}(\log \tfrac{1}{\delta})^{-2} \quad \text{and} \quad \#\mathcal{I}_{k_1}^{\times}(z) \gtrsim_{s'} \frac{1}{\delta^{s'}}(\log \tfrac{1}{\delta})^{-2} \quad \text{for all } z \in V. \tag{3.24}$$

On the other hand, recalling the definition of $\mathcal{T}$ in thew end of Step 2, we know

$$\mathcal{T} = \bigcup_{z\in V} \mathcal{I}_{k_1}^{+}(z) \times \mathcal{I}_{k_1}^{-}(z) \times \mathcal{I}_{k_1}^{\times}(z) \times \{z\}.$$

Employing the lower bounds in (3.23) and (3.24), we arrive at

$$\#\mathcal{T} \geq \min_{z\in V}\left\{\#\mathcal{I}_{k_1}^{+}(z)\right\} \times \min_{z\in V}\left\{\#\mathcal{I}_{k_1}^{-}(z)\right\} \times \min_{z\in V}\left\{\#\mathcal{I}_{k_1}^{\times}(z)\right\} \times \#V \gtrsim_{s'} \frac{1}{\delta^{3s'}}(\log \tfrac{1}{\delta})^{-6}\#V.$$

Combining (3.7) and (3.9) we conclude

$$\#\mathcal{T} \gtrsim_{s'} \frac{1}{\delta^{3s'+t'}}(\log \tfrac{1}{\delta})^{-8}.$$

Recalling (3.22) we obtain

$$\#\mathcal{I}_{k_1} \gtrsim_{s'} \left(\frac{1}{\delta^{3s'+t'}}(\log \tfrac{1}{\delta})^{-8}\right)^{1/3} (\log \tfrac{1}{\delta})^{-12/s'} = \frac{1}{\delta^{s'+t'/3}}(\log \tfrac{1}{\delta})^{-(\frac{8}{3}+\frac{12}{s'})}.$$

We deduce that

$$\begin{aligned}\sum_{i\in\mathcal{I}} r_i^{s'+t'/3-\epsilon} \geq \sum_{i\in\mathcal{I}_{k_1}} r_i^{s'+t'/3-\epsilon} &\gtrsim_{s'} 2^{-k_1(s'+t'/3-\epsilon)}\frac{1}{\delta^{s'+t'/3}}(\log \tfrac{1}{\delta})^{-(\frac{8}{3}+\frac{12}{s'})}\\ &\gtrsim_{s'} \delta^{-\epsilon}(\log \tfrac{1}{\delta})^{-(\frac{8}{3}+\frac{12}{s'})} > 1.\end{aligned}$$

where in the third inequality we recall $\delta = 2^{-k_1}$ and in the last inequality we recall (3.2). This enables us to deduce

$$\dim_{\mathcal{H}}(F) \geq s' + \frac{t'}{3} - \epsilon$$

for any $0 < s' < s$, $0 < t' < t$ and $\epsilon > 0$. Therefore,

$$\dim_{\mathcal{H}}(F) \geq s + \frac{t}{3}.$$

We conclude the proof. □

4. Proof of Theorem 1.3

To show Theorem 1.3, we define the multiplicity function $m_\delta^\mu(w) : \mathbb{R}^2 \to [0,1]$ with respect to a finite measure μ on $\mathbb{R}^3$:

$$m_\delta^\mu(w) := \mu(\{z \in \mathbb{R}^3 \mid w \in S^\delta(z)\}). \tag{4.1}$$

We recall [8, Lemma 5.1], which is a variant of Schlag's weak type inequaltiy [14, Lemma 8] and the main lemma in [15] by Wolff:

Lemma 4.1. *Fix $t \in (0,1]$, $\delta > 0, \eta > 0, \boldsymbol{C} \geq 1$, and $A \geq C_{\eta,\boldsymbol{C},t} \cdot \delta^{-\eta}$, where $C_{\eta,\boldsymbol{C},t} \geq 1$ is a large constant depending only on $\eta, \boldsymbol{C}$ and t. Let μ be a probability measure on $\mathbb{R}^3$ satisfying the Frostman condition $\mu(B(z,r)) \leq \boldsymbol{C}r^t$ for all $z \in \mathbb{R}^3$ and $r > 0$, and with $D := \operatorname{spt} \mu \subset \boldsymbol{B}_0$ where $\boldsymbol{B}_0$ is defined in (2.1). Then, for $\lambda \in (0,1]$, there is a set $G(A,\delta,\lambda) \subset D$ with*

$$\mu(D \setminus G(A,\delta,\lambda)) < A^{-t/3}$$

such that the following holds for all $z \in G(A,\delta,\lambda)$:

$$|S^\delta(z) \cap \{w \mid m_\delta^\mu(w) \geq A^t\lambda^{-2t}\delta^t\}| \leq \lambda |S^\delta(z)|.$$

Remark 4.2. We remark that the assumptions on μ in Lemma 4.1 can be slightly relaxed, which means we can apply Lemma 4.1 for measures μ satisfying that

(i) μ is a finite measure with total mass smaller or equal to 1 supported on $\mathbf{B}_0$;
(ii) μ enjoys Frostman condition

$$\mu(B(z,r)) \leq \mathbf{C}r^t \quad \text{for all } z \in \mathbb{R}^3 \text{ and } r > \delta.$$

Indeed, in the proof of [8, Lemma 5.1], the fact that the total measure $\mu(D) = 1$ was only used at the beginning to reduce the proof to the case that δ is small. See the first paragraph of the proof therein. Moreover, the Frostman condition was only applied to balls in $\mathbb{R}^3$ with radius $\delta < r \in [C\delta, 1]$ where $C \geq 1$ in their proof. See the inequality above (5.4), the definition of B below (5.22) and inequality (5.24) therein. Hence we can reduce the assumptions in Lemma 4.1 to (i) and (ii) above for the measure μ.

Proof of Theorem 1.3. Let F be a circular (s,t)-Furstenberg set with parameter set $K \subset \mathbf{B}_0$. It suffices to show, for any $\epsilon > 0$, $\frac{1}{2} < s' < s$ and $0 < t' < t$,

$$\dim_{\mathcal{H}}(F) \geq (2s'-1)t' + s' - \epsilon.$$

Hence in the following, we fix ϵ, s', t'.

Let $\alpha > 0$ and K_1 be as in (3.1). Now we clarify the choices of parameters appeared in the ensuing proof and we remind that all parameters are unrelated to those in the proof of Theorem 1.2. First, we choose

$$\eta = \min\{\epsilon/2t', (2s'-1)/2\}. \tag{4.2}$$

Then there exists $\delta_0 = \delta_0(\epsilon, s', t') > 0$ such that for any $0 < \delta < \delta_0$, we have

$$\delta^{\epsilon - t'\eta}(\log \tfrac{1}{\delta})^{6+4t'} \leq \delta^{\frac{\epsilon}{2}}(\log \tfrac{1}{\delta})^{6+4t'} < 1, \tag{4.3}$$

$$\delta^{\frac{\eta t'}{3}}(\log \tfrac{1}{\delta})^2 < \frac{1}{4}, \tag{4.4}$$

and

$$C_{\eta,\mathbf{C},t',s'} = (C_{\eta,\mathbf{C},t'})^{t'}(2c_0 4^{s'})^{2t'}, \tag{4.5}$$

where $\mathbf{C}$ and $C_{\eta,\mathbf{C},t'} \geq 1$ are the constants appeared in Lemma 4.1 and c_0 is as in Remark 2.1(ii), i.e. $|S^\delta(x,r)| \leq c_0\delta$ for all $(x,r) \in \mathbf{B}_0$.

Let k_0 be the smallest integer larger than $(\log \frac{1}{\delta_0})$ also satisfying

$$\alpha > \sum_{k=k_0}^{\infty} \frac{1}{k^2}. \tag{4.6}$$

Now, we outline the main steps of the proof. We start with an arbitrary cover $\mathcal{U} = \{B(x_i, r_i)\}_{i\in\mathcal{I}}$ of F by balls of radius less than 2^{-k_0}. In the sequel, we will derive a lower bound

$$\sum_{i\in\mathcal{I}} r_i^\sigma \gtrsim_{\epsilon,t',s'} 1$$

with $\sigma = (2t'+1)s' - t' - \epsilon$ independent of the choice of the particular cover. This will imply

$$\mathcal{H}^\sigma(F) > 0.$$

To this end, we divide the proof into 3 steps. Let

$$\mathcal{I}_k := \{i \in \mathcal{I} \mid 2^{-(k+1)} < r_i \leq 2^{-k}\}, \quad F_k := \{\bigcup B(x_i, r_i) \mid i \in \mathcal{I}_k\}.$$

First, in Step 1, we will deduce that there exists $k_1 \geq k_0$ and a (δ, t')-set $V \subset K$ with $\delta = 2^{-k_1}$ such that

$$\frac{1}{k_1^2} \cdot \delta^{-t'} \lesssim \#V \lesssim \delta^{-t'}, \tag{4.7}$$

and for every circle $z = (x, r) \in V$, we have

$$\mathcal{H}^{s'}_\infty(S(z) \cap F_{k_1}) > k_1^{-2}. \tag{4.8}$$

Next, in Step 2, we associate a finite measure μ supported on V using (2.9). Then we apply Lemma 4.1 to obtain that there exists $G \subset V$ and $S_2^\delta(z)$ contained in the δ-neighbourhood of $S(z) \cap F_{k_1}$, such that for every $z \in G$ and $w \in S_2^\delta(z)$,

$$\#\{z' \in G \mid w \in S_2^\delta(z')\} \lesssim_{t'} C_{\eta,\mathbf{C},t',s'} \delta^{t'(2s'-2-\eta)} (\log \tfrac{1}{\delta})^{4t'+2}. \tag{4.9}$$

Finally, in Step 3, we will provide a lower bound of the cardinality $\#\mathcal{I}_{k_1}$ by combining the upper bound in Step 2 as well as the lower bounds on the cardinality $\#G$ and the Lebesgue measure $|S_2^\delta(z)|$. Explicitly, we have

$$\#\mathcal{I}_{k_1} \gtrsim_{\epsilon,t',s'} \frac{1}{\delta^{(2t'+1)s'-t'(1+\eta)}} \frac{1}{(\log \frac{1}{\delta})^{6+4t'}}.$$

This will enable us to conclude the proof.

Step 1. Employing the same arguments as in Step 1 in the proof of Theorem 1.2, we can deduce the existence of k_1 and $V \subset K_2 \subset K_1$ satisfying (4.8) and the first inequality in (4.7). The second inequality in (4.7) is derived from Remark 2.4. Here, we omit the details.

Step 2. Define μ_V as in (2.9) applied to $P = V$. Then we know μ_V is a probability measure satisfying the Frostman condition

$$\mu_V(B(z,r)) \leq C\mathcal{H}^{t'}_\infty(K_2)^{-1} r^{t'} < Ck_1^2 r^{t'} = C(\log \tfrac{1}{\delta})^2 r^{t'}$$

for all $z \in \mathbb{R}^3$ and $r > \delta$. Hence by setting

$$\mu := \frac{\mu_V}{(\log \frac{1}{\delta})^2},$$

we know that μ has total measure $(\log \frac{1}{\delta})^{-2} < 1$, $\mathrm{spt}\mu = V \subset \mathbf{B}_0$ and

$$\mu(B(z,r)) \leq Cr^{t'} =: \mathbf{C}r^{t'}$$

for all $z \in \mathbb{R}^3$ and $r > \delta$.

Let m_μ^δ be the corresponding multiplicity function with respect to μ defined as in (4.1).

Applying Lemma 4.1 with $t = t'$, $\delta = 2^{-k_1}$, η as in (4.2), $\mu = (\log \frac{1}{\delta})^{-2}\mu_V$, $D = V$ and

$$\lambda = (2c_0 4^{s'} k_1^2)^{-1}\delta^{1-s'}, \tag{4.10}$$

we obtain that for $A = C_{\eta,\mathbf{C},t'} \cdot \delta^{-\eta}$, there is a set $G = G(k_1, s', t', \epsilon) \subset V$ with

$$\mu(V \setminus G) < A^{-t'/3} \tag{4.11}$$

such that the following holds for all $z \in G$:

$$|S^\delta(z) \cap \{w \mid m_\delta^\mu(w) \geq A^{t'}\lambda^{-2t'}\delta^{t'}\}| \leq \lambda |S^\delta(z)|. \tag{4.12}$$

Because $|S^\delta(z)| \leq c_0\delta$ for all $z \in \mathbf{B}_0$, (4.12) becomes

$$|S^\delta(z) \cap \{w \mid m_\delta^\mu(w) \geq A^{t'}\lambda^{-2t'}\delta^{t'}\}| \leq c_0\lambda\delta. \tag{4.13}$$

Moreover, recalling that $\delta = 2^{-k_1}$ and $k_1 \geq k_0$, we know that $0 < \delta < \delta_0$. Hence by $C_{\eta,\mathbf{C},t'} \geq 1$, (4.4) and the choice of η in (4.2) we deduce

$$A^{-\frac{t'}{3}} \leq \delta^{\frac{\eta t'}{3}} \leq \frac{1}{4}\frac{1}{(\log \frac{1}{\delta})^2} = \frac{1}{4}\mu(V).$$

Hence (4.11) becomes

$$\mu(V \setminus G) < \frac{1}{4}\mu(V). \tag{4.14}$$

For $z \in G$, let $S_1(z) := S(z) \cap F_{k_1}$ and $S_1^\delta(z)$ be the δ-neighbourhood of $S_1(z)$. Our next goal is to substitute the right hand side term $\lambda|S^\delta(z)|$ in (4.12) by the term $\frac{1}{2}|S_1^\delta(z)|$ with the help of the proper choice of λ as in (4.10). This means, in the sense of 2-dimensional Lebesgue measure, more than half of the points in $S_1^\delta(z)$ have low multiplicity. To this end, we claim that

$$|S_1^\delta(z)| \geq \frac{1}{4^{s'}k_1^2}\delta^{2-s'}. \tag{4.15}$$

To see (4.15), let $P(z)$ be a maximal 2δ-separated set in $S_1(z)$. Then $\bigcup_{p\in P(z)} B(p, 2\delta)$ forms a cover of $S_1(z)$. Hence

$$\mathcal{H}_\infty^{s'}(S_1(z)) \leq \#P(z)(4\delta)^{s'}.$$

which, combined with (4.8), implies

$$\#P(z) \geq \mathcal{H}_\infty^{s'}(S_1(z))\frac{1}{(4\delta)^{s'}} \geq \frac{1}{4^{s'}k_1^2}\frac{1}{\delta^{s'}}.$$

On the other hand, we have $\bigcup_{p\in P(z)} B(p, \delta) \subset S_1^\delta(z)$. Hence by $\{B(p, \delta)\}_{p\in P(z)}$ being mutually disjoint, we deduce

$$|S_1^\delta(z)| \geq |\textstyle\bigcup_{p\in P(z)} B(p, \delta)| = \#P\delta^2\pi \geq \frac{\pi}{4^{s'}k_1^2}\delta^{2-s'} > \frac{1}{4^{s'}k_1^2}\delta^{2-s'},$$

which gives (4.15).

Noticing that $|S^\delta(z)| \leq c_0\delta$, $S_1(z) \subset S(z)$ and combining (4.13) as well as (4.15), we arrive at

$$|S_1^\delta(z) \cap \{w \mid m_\delta^\mu(w) \geq A^{t'}\lambda^{-2t'}\delta^{t'}\}| \leq \lambda c_0\delta \leq \lambda c_0 4^{s'}k_1^2\delta^{s'-1}|S_1^\delta(z)|. \tag{4.16}$$

Now recall $A = C_{\eta,\mathbf{C},t'} \cdot \delta^{-\eta}$ and $\lambda = (2c_0 4^{s'}k_1^2)^{-1}\delta^{1-s'} = (2c_0 4^{s'})^{-1}(\log \frac{1}{\delta})^{-2}\delta^{1-s'}$. Then (4.16) becomes

$$|S_1^\delta(z) \cap \{w \mid m_\delta^\mu(w) \geq C_{\eta,\mathbf{C},t',s'}\delta^{t'(2s'-1-\eta)}(\log \tfrac{1}{\delta})^{4t'}\}| \leq \frac{1}{2}|S_1^\delta(z)|$$

where we recall $C_{\eta,\mathbf{C},t',s'}$ defined in (4.5).

For each $z \in G$, define the low-multiplicity set

$$S_2^\delta(z) := \{w \in S_1^\delta(z) \mid m_\delta^\mu(w) < C_{\eta,\mathbf{C},t',s'}\delta^{t'(2s'-1-\eta)}(\log \tfrac{1}{\delta})^{4t'}\}.$$

Then we have

$$|S_2^\delta(z)| \geq \frac{1}{2}|S_1^\delta(z)|. \tag{4.17}$$

See Figure 4 for an illustration of $S_1(z)$, $S_1^\delta(z)$ and $S_2^\delta(z)$.

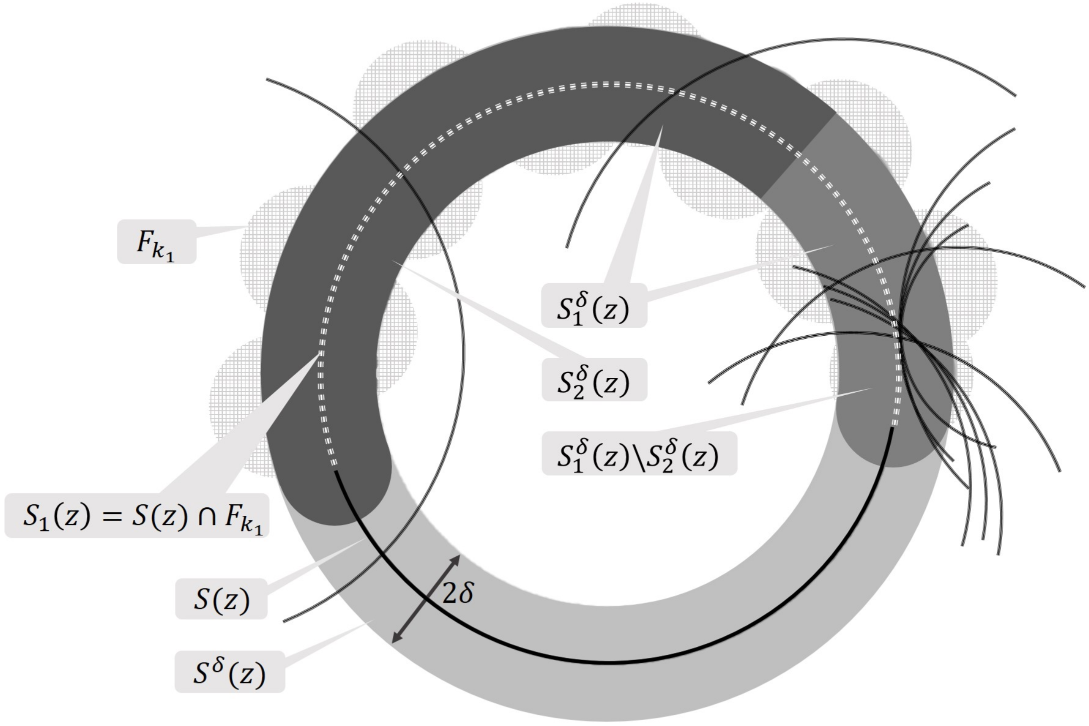

Figure 4. An illustration of $S_1(z)$, $S_1^\delta(z)$ and $S_2^\delta(z)$.

Notice that $m_\delta^\mu(w) < C_{\eta,\mathbf{C},t',s'}\delta^{t'(2s'-1-\eta)}(\log \frac{1}{\delta})^{4t'}$ is equivalent to

$$\mu(\{z' \in \mathbb{R}^3 \mid w \in S^\delta(z')\}) < C_{\eta,\mathbf{C},t',s'}\delta^{t'(2s'-1-\eta)}(\log \tfrac{1}{\delta})^{4t'},$$

which, combined with (4.7), indicates that for $w \in S_2^\delta(z)$, it holds

$$\begin{aligned}\#\{z' \in V \mid w \in S^\delta(z')\} &\leq \#V \cdot C_{\eta,\mathbf{C},t',s'}\delta^{t'(2s'-1-\eta)}(\log \tfrac{1}{\delta})^{4t'+2} \\ &\lesssim_{t'} C_{\eta,\mathbf{C},t',s'}\delta^{t'(2s'-2-\eta)}(\log \tfrac{1}{\delta})^{4t'+2}.\end{aligned}$$

Furthermore, by the inclusions $G \subset V$ and $S_2^\delta(z) \subset S^\delta(z)$, we conclude (4.9), which finishes Step 2.

Step 3. We will lower bound $\#\mathcal{I}_{k_1}$ in the following. First notice that if $\{S^\delta(z)\}_{z\in G}$ were mutually disjoint, we could lower bound $\#\mathcal{I}_{k_1}$ by summing up the number of balls B_i ($i \in \mathcal{I}_{k_1}$) needed to cover each $S_2^\delta(z)$ since no ball could simultaneously intersect two of these sets. However, in general, $\{S^\delta(z)\}_{z\in G}$ may not be mutually disjoint, which needs a bit more efforts to get the lower bound of $\#\mathcal{I}_{k_1}$.

Let

$$\widetilde{F}_{k_1} := \bigcup_{i\in\mathcal{I}_{k_1}} B(x_i, 4r_i).$$

We deduce that

$$\bigcup_{z\in G} S_2^\delta(z) \subset \widetilde{F}_{k_1}. \tag{4.18}$$

Indeed, for any $w \in S_2^\delta(z)$, there exists $w' \in S(z) \cap F_{k_1}$ such that

$$\|w - w'\| < \delta.$$

On the other hand, we know that $w' \in B(x_i, r_i)$ for some $x_i \in \mathcal{I}_{k_1}$ and $r_i > 2^{-(k_1+1)} = \delta/2$, which implies

$$\|w' - x_i\| < r_i$$

and hence

$$\|w - x_i\| < \delta + r_i < 3r_i.$$

In addition, by (4.7) and (4.14), we can infer that

$$\#G \gtrsim \#V \gtrsim \frac{1}{\delta^{t'}} \frac{1}{(\log \frac{1}{\delta})^2}. \tag{4.19}$$

Moreover by recalling (4.9) we obtain that for every $w \in \bigcup_{z\in G} S_2^\delta(z)$,

$$\mathcal{N}(w) := \#\{z' \in G \mid w \in S_2^\delta(z')\} \lesssim_{t'} C_{\eta,\mathbf{C},t',s'} \delta^{t'(2s'-2-\eta)} (\log \tfrac{1}{\delta})^{4t'+2}$$

and hence combining (4.19), we can estimate

$$\begin{aligned}
\left|\bigcup_{z\in G} S_2^\delta(z)\right| &= \sum_{z\in G} \int \chi_{S_2^\delta(z)}(w) \frac{1}{\mathcal{N}(w)}\, dw \\
&\gtrsim_{t'} (C_{\eta,\mathbf{C},t',s'} \delta^{t'(2s'-2-\eta)} (\log \frac{1}{\delta})^{4t'+2})^{-1} \sum_{z\in G} \left|S_2^\delta(z)\right| \\
&\gtrsim_{t'} (C_{\eta,\mathbf{C},t',s'})^{-1} \frac{1}{\delta^{t'(2s'-2-\eta)}} \frac{1}{(\log \frac{1}{\delta})^{4t'+2}} \frac{1}{\delta^{t'}} \frac{1}{(\log \frac{1}{\delta})^2} \min_{z\in G}\{|S_2^\delta(z)|\} \\
&\gtrsim_{\eta,t',s'} \frac{1}{\delta^{t'(2s'-2-\eta)}} \frac{1}{(\log \frac{1}{\delta})^{4t'+2}} \frac{1}{\delta^{t'}} \frac{1}{(\log \frac{1}{\delta})^2} \delta^{2-s'} \frac{1}{(\log \frac{1}{\delta})^2} \qquad (4.20)
\end{aligned}$$

where in the last inequality we employ (4.15) and (4.17). Therefore, combining (4.18) and (4.20) we arrive at

$$\#\mathcal{I}_{k_1}\delta^2 \gtrsim \left|\widetilde{F}_{k_1}\right| \geq \left|\bigcup_{z\in G} S_2^\delta(z)\right| \gtrsim_{\eta,t',s'} \frac{1}{\delta^{t'(2s'-2-\eta)}} \frac{1}{(\log \frac{1}{\delta})^{4t'+2}} \frac{1}{\delta^{t'}} \frac{1}{(\log \frac{1}{\delta})^2} \delta^{2-s'} \frac{1}{(\log \frac{1}{\delta})^2},$$

which implies

$$\#\mathcal{I}_{k_1} \gtrsim_{\eta,t',s'} \frac{1}{\delta^{(2t'+1)s'-t'(1+\eta)}} \frac{1}{(\log \frac{1}{\delta})^{6+4t'}}.$$

Since $\mathcal{I}_{k_1} \subset \mathcal{I}$, we deduce that

$$\begin{aligned}
\sum_{i\in\mathcal{I}} r_i^{(2t'+1)s'-t'-\epsilon} &\geq \sum_{i\in\mathcal{I}_{k_1}} r_i^{(2t'+1)s'-t'-\epsilon} \\
&\gtrsim_{\eta(\epsilon,t',s'),t',s'} 2^{-k_1((2t'+1)s'-t'-\epsilon)} \frac{1}{\delta^{(2t'+1)s'-t'(1+\eta)}} \frac{1}{(\log \frac{1}{\delta})^{6+4t'}} \\
&\gtrsim_{\eta(\epsilon,t',s'),t',s'} \delta^{t'\eta-\epsilon} \frac{1}{(\log \frac{1}{\delta})^{6+4t'}} \\
&\gtrsim_{\epsilon,t',s'} \delta^{-\epsilon/2} \frac{1}{(\log \frac{1}{\delta})^{6+4t'}} > 1,
\end{aligned}$$

where in the third inequality we recall $\delta = 2^{-k_1}$ and in the fourth as well as the last inequality we recall (4.3). This enables us to deduce

$$\dim_{\mathcal{H}}(F) \geq (2t'+1)s' - t' - \epsilon$$

for any $\frac{1}{2} < s' < s$, $0 < t' < t$ and $\epsilon > 0$. Therefore,

$$\dim_{\mathcal{H}}(F) \geq (2t+1)s - t = (2s-1)t + s.$$

We conclude the proof. $\square$

5. Proof of Lemma 2.5

This section is devoted to the proof of Lemma 2.5. For the readers' convenience, we restate Lemma 2.5 in the following.

Lemma 5.1. *Let $A, B, C \in \mathbb{R}^2$ such that* $\min\{\|A-B\|, \|A-C\|, \|B-C\|\} \geq 2c$ *with $c < 1$. For $a > 0$ such that $a < \frac{1}{20}c^2$, define*

$$W := \left\{ (x,b) \in \mathbb{R}^2 \times [\tfrac{1}{2}, 2] : \quad \begin{array}{c} b - a \leq \|x - A\| \leq b + a, \\ b - a \leq \|x - B\| \leq b + a, \\ b - a \leq \|x - C\| \leq b + a \end{array} \right\}.$$

Then

$$\operatorname{diam} W \lesssim \frac{a}{c^2}.$$

We briefly explain the approach. We will decompose W as

$$W = \bigcup_{b \in I \subset [1/2, 2]} W(b) \times \{b\}.$$

Then for each fixed b,

$$W(b) := \left\{ x \in \mathbb{R}^2 : \quad \begin{array}{c} b - a \leq \|x - A\| \leq b + a, \\ b - a \leq \|x - B\| \leq b + a, \\ b - a \leq \|x - C\| \leq b + a \end{array} \right\} = S^a(A,b) \cap S^a(B,b) \cap S^a(C,b)$$

is a subset in $\mathbb{R}^2$ formed by the intersection of three annuli. We will show that $W(b) \neq \emptyset$ only for b ranging in a set I with diameter $\lesssim \frac{a}{c^2}$. Moreover, if $W(b) \neq \emptyset$, then A, B, C form a non-degenerate $\triangle ABC$ with circumcenter M and $W(b)$ is contained in a rhombus centered at M with diameter $\lesssim \frac{a}{c^2}$. This will imply

$$\operatorname{diam} W \lesssim \frac{a}{c^2}.$$

The above justification is contained in next two auxiliary lemmas. In what follows, given $A, B \in \mathbb{R}^2$ and $0 < a < \frac{c^2}{20}$, we denote by $\mathcal{R}^{a,c}_{AB}$ the rectangle centered at the middle point of AB whose short sides have length $\frac{9a}{c}$ and long sides have length 6 parallel to the bisector of AB.

Lemma 5.2. *Let $A, B \in \mathbb{R}^2$ and $b \in [\frac{1}{2}, 2]$. If $c < \min\{1, \frac{\|A-B\|}{2}\}$ and $0 < a < \frac{c^2}{20} < 1$, then*

$$S^a(A,b) \cap S^a(B,b) \subset \mathcal{R}^{a,c}_{AB}.$$

Proof. Let $\|A-B\| = 2u$. Without loss of generality, we assume $A = (-u, 0)$ and $B = (u, 0)$. It is easy to see that

$$\begin{aligned} &S^a(A,b) \cap S^a(B,b) \\ &= \{x \in \mathbb{R}^2 \mid b-a \leq \|x-A\| \leq b+a,\ b-a \leq \|x-B\| \leq b+a\} \\ &\subset U := \{x = (x_1,x_2) \in \mathbb{R}^2 \mid \max\{\|x-A\|, \|x-B\|\} \leq 3, \\ &\qquad\qquad -2a \leq \|x-A\| - \|x-B\| \leq 2a\}. \end{aligned}$$

Since $u = \frac{\|A-B\|}{2} > c > a$, from planar geometry we know that the set

$$\{x \in \mathbb{R}^2 \mid \|x-A\| - \|x-B\| = \pm 2a\}$$

consisting of points, whose absolute difference of distances to the two fixed points A and B is the constant $2a$, is a hyperbola in $\mathbb{R}^2$ determined by the equation

$$y(x) = y(x_1,x_2) = 1$$

where $y : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$y(x) = y(x_1,x_2) \mapsto \frac{x_1^2}{a^2} - \frac{x_2^2}{u^2-a^2}.$$

Then we observe that

$$\{x \in \mathbb{R}^2 \mid -2a \leq \|x-A\| - \|x-B\| \leq 2a\} = \{x \in \mathbb{R}^2 \mid y(x_1,x_2) \leq 1\}$$

and hence

$$U = [B((-u,0),3) \cap B((u,0),3)] \cap \{x \in \mathbb{R}^2 \mid y(x_1,x_2) \leq 1\},$$

which implies

$$U \subset \{x \in \mathbb{R}^2 \mid |x_2| \leq 3,\ y(x_1,x_2) \leq 1\}.$$

Figure 5 shows the case that $u = 2$ and $a = 0.75$.

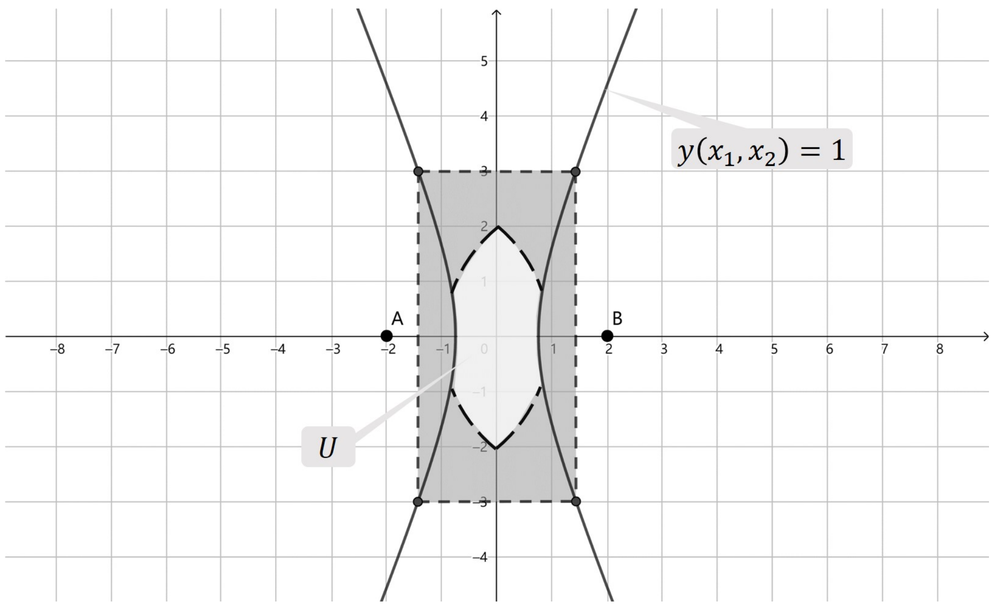

Figure 5. The case $u = 2$ and $a = 0.75$.

Letting $|x_2| = 3$ in the equation $y(x_1, x_2) = 1$, we have $|x_1| = a\sqrt{1 + \frac{9}{u^2 - a^2}}$. Since $20a < c^2 < 1$ and $u > c$, it holds

$$a\sqrt{1 + \frac{9}{u^2 - a^2}} < a\sqrt{1 + \frac{9}{c^2 - a^2}} < a\sqrt{1 + \frac{9}{\frac{8}{9}c^2}} < a\sqrt{\frac{81}{4c^2}} = \frac{9}{2}\frac{a}{c} < 3. \tag{5.1}$$

This implies that the rectangle with four vertices $(\pm\frac{9}{2}\frac{a}{c}, \pm 3)$ has short side length $\frac{9a}{c}$ and long side length 6. By recalling the definition of $\mathcal{R}^{a,c}_{AB}$, we have

$$S^a(A, b) \cap S^a(B, b) \subset U \subset \mathcal{R}^{a,c}_{AB} = \{x \in \mathbb{R}^2 \mid |x_1| \le \tfrac{9}{2}\tfrac{a}{c},\ |x_2| \le 3\},$$

which concludes the proof. □

Lemma 5.3. *Let $A, B, C \in \mathbb{R}^2$ such that* $\min\{\|A - B\|, \|A - C\|, \|B - C\|, 2\} \ge 2c$*. Let $b \in [\frac{1}{2}, 2]$. Then for $a > 0$ such that $a < \frac{1}{20}c^2 < b$, define*

$$W(b) := \left\{ x \in \mathbb{R}^2 : \begin{array}{c} b - a \le \|x - A\| \le b + a, \\ b - a \le \|x - B\| \le b + a, \\ b - a \le \|x - C\| \le b + a \end{array} \right\} \tag{5.2}$$

If the triangle $\triangle ABC$ is degenerate, then

$$W(b) = \emptyset \quad \text{for all } b \in [\frac{1}{2}, 2]. \tag{5.3}$$

If $\triangle ABC$ is non-degenerate, let M be the circumcenter of $\triangle ABC$ and

$$h := \|M - A\| = \|M - B\| = \|M - C\|.$$

Then, we have

$$W(b) \subset B\left(M, K\tfrac{a}{c^2}\right) \quad \text{for all } b \in \left[\tfrac{1}{2}, 2\right]. \tag{5.4}$$

In addition, if $W(b) \ne \emptyset$, then

$$b \in \left[h - K\tfrac{a}{c^2}, h + K\tfrac{a}{c^2}\right] \cap \left[\tfrac{1}{2}, 2\right]. \tag{5.5}$$

Here in (5.4) *and* (5.5)*, K is an absolute constant.*

Proof. Without loss of generality, we assume the side BC of $\triangle ABC$ has maximal length. Then $\angle A := \angle BAC \ge \pi/3$. Since $W(b) = S^a(A, b) \cap S^a(B, b) \cap S^a(C, b)$, from Lemma 5.2 we know

$$W(b) \subset \mathcal{R}^{a,c}_{AB} \cap \mathcal{R}^{a,c}_{AC}. \tag{5.6}$$

Below we estimate $\operatorname{diam}(\mathcal{R}^{a,c}_{AB} \cap \mathcal{R}^{a,c}_{AC})$ from above.

Denote by L_1 and L_2 the bisector of AB and AC respectively. Hence $D := L_1 \cap AB$ is the middle point of AB and $E := L_2 \cap AC$ is the middle point of AC. See Figure 6 for an illustration.

Let $d = \frac{9}{2}\frac{a}{c}$. Since $20a < c^2$, we have

$$d = \frac{9}{2}\frac{a}{c} < \frac{9}{40}c < \frac{1}{4}c. \tag{5.7}$$

Case 1. $\angle A = \pi$. That is, $\triangle ABC$ degenerates. By (5.7), it is easy to see $\mathcal{R}^{a,c}_{AB} \cap \mathcal{R}^{a,c}_{AC} = \emptyset$, which, with help of (5.6), implies

$$W(b) = \emptyset \quad \text{for all } b \in [\tfrac{1}{2}, 2].$$

That is, (5.3) holds.

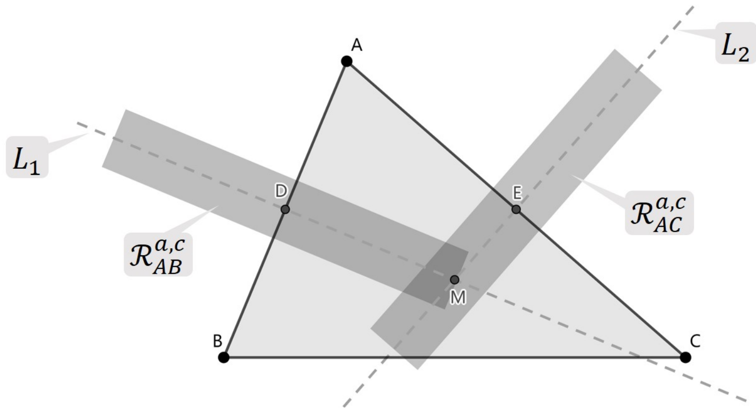

Figure 6. An illustration for L_1, L_2, $\mathcal{R}_{AB}^{a,c}$ and $\mathcal{R}_{AC}^{a,c}$.

Case 2. $\angle A \in (\pi - \arctan(2c/9), \pi)$. We will show that

$$\mathcal{R}_{AB}^{a,c} \cap \mathcal{R}_{AC}^{a,c} = \emptyset. \tag{5.8}$$

Denote $\|A - B\| = 2u$ and $\|A - C\| = 2v$. Since M is the circumcenter of $\triangle ABC$, it is the intersection of lines L_1 and L_2. Then the line L_3 passing through A and M divides $\mathbb{R}^2$ into two connected components. Since the center D of $\mathcal{R}_{AB}^{a,c}$ and the center E of $\mathcal{R}_{AC}^{a,c}$ are contained in different connected components above and $d < \frac{1}{4}c$ by (5.7), a sufficient condition for $\mathcal{R}_{AB}^{a,c} \cap \mathcal{R}_{AC}^{a,c} = \emptyset$ is that

$$\mathcal{R}_{AB}^{a,c} \cap L_3 = \emptyset \quad \text{and} \quad \mathcal{R}_{AC}^{a,c} \cap L_3 = \emptyset. \tag{5.9}$$

See Figure 7 for an illustration.

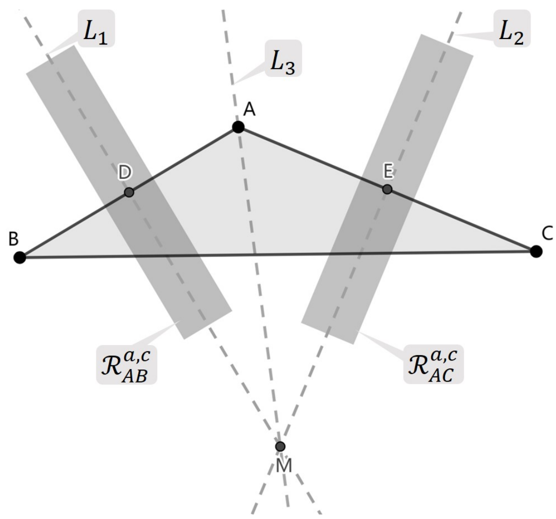

Figure 7. An illustration for Case 2.

Recall that half of the length of the short sides of $\mathcal{R}_{AB}^{a,c}$ and $\mathcal{R}_{AC}^{a,c}$ is $d = \frac{9}{2}\frac{a}{c}$. By assumption $\angle A \in (\pi - \arctan(2c/9), \pi)$, this implies $\angle DMA + \angle EMA \leq \arctan(2c/9)$. Hence

$$\tan \angle DMA < \frac{2c}{9} \leq \frac{c-d}{3} \leq \frac{u-d}{3} \quad \text{and} \quad \tan \angle EMA < \frac{2c}{9} \leq \frac{c-d}{3} \leq \frac{v-d}{3} \tag{5.10}$$

where in the second inequality we apply $d < \frac{c}{3}$ from (5.7). Now we explain how (5.10) implies (5.9). Let D' be the intersection of the line segment AD and the long side of the triangle $\mathcal{R}_{AB}^{a,c}$. Also, let $L_1' := L_1 + (D' - D)$. That is, line L_1' is the translation of line L_1 by the vector $D'-D$ in $\mathbb{R}^2$. Denote the intersection of L_1' and L_3 by M'. See Figure 8 for an illustration.

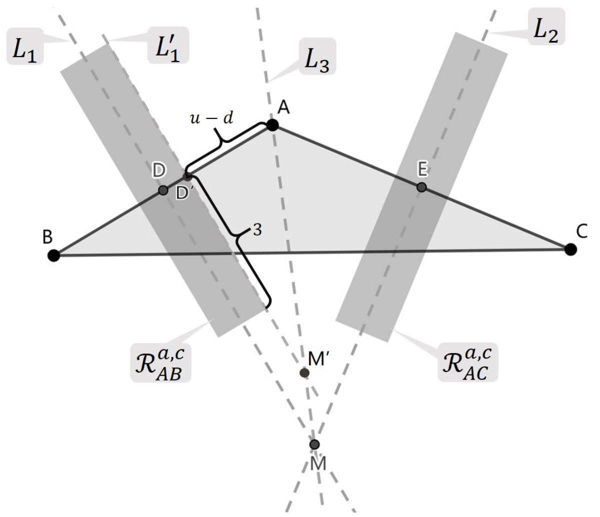

Figure 8. An illustration for D', M' and L_1'.

We observe that

$$\angle D'M'A = \angle DMA \quad \text{and} \quad \tan\angle D'M'A = \frac{\|A-D'\|}{\|D'-M'\|} = \frac{u-d}{\|D'-M'\|} \tag{5.11}$$

where in the last inequality we recall that $\|A-D'\| = \|A-D\|-\|D-D'\|$, $\|A-D\| = u$ and $\|D-D'\| = d$. Combining (5.10) and (5.11), we deduce that

$$\frac{u-d}{\|D'-M'\|} \overset{(5.11)}{=} \tan\angle D'M'A \overset{(5.10)}{<} \frac{u-d}{3},$$

which implies

$$\|D'-M'\| > 3.$$

This, combined with the fact that half of the length of the long sides of $\mathcal{R}_{AB}^{a,c}$ is 3, shows that

$$\mathcal{R}_{AB}^{a,c} \cap L_3 = \emptyset.$$

By a similar argument, we also have $\mathcal{R}_{AC}^{a,c} \cap L_3 = \emptyset$ with the help of (5.10). This shows that (5.9) is true and hence (5.8) holds.

Case 3. $\angle A \in [\pi/3, \pi - \arctan(2c/9)]$. In this case, $W(b)$ may not be empty. Now, we assume that $W(b) \neq \emptyset$, which implies that $\mathcal{R}_{AB}^{a,c} \cap \mathcal{R}_{AC}^{a,c} \neq \emptyset$. Moreover, denote by $\mathcal{V}_{L_i}^d$ the closed d-neighbourhood of lines L_i, $i = 1, 2$. Then $\mathcal{V}_{L_1}^d \cap \mathcal{V}_{L_2}^d$ is a rhombus $\mathcal{T}_M$ centered at M satisfying $\mathcal{R}_{AB}^{a,c} \cap \mathcal{R}_{AC}^{a,c} \subset \mathcal{T}_M$. We will show that

$$\operatorname{diam} \mathcal{T}_M \leq 324\frac{a}{c^2}. \tag{5.12}$$

See Figure 9 for an illustration.

Denote the length of two diagonals of $\mathcal{T}_M$ by d_1 and d_2 and the the length of four sides of $\mathcal{T}_M$ by l. We have

$$\operatorname{diam} \mathcal{T}_M = \max\{d_1, d_2\}, \tag{5.13}$$

$$d_1^2 + d_2^2 = 4l^2 \tag{5.14}$$

and

$$l = \frac{2d}{\sin\angle A}. \tag{5.15}$$

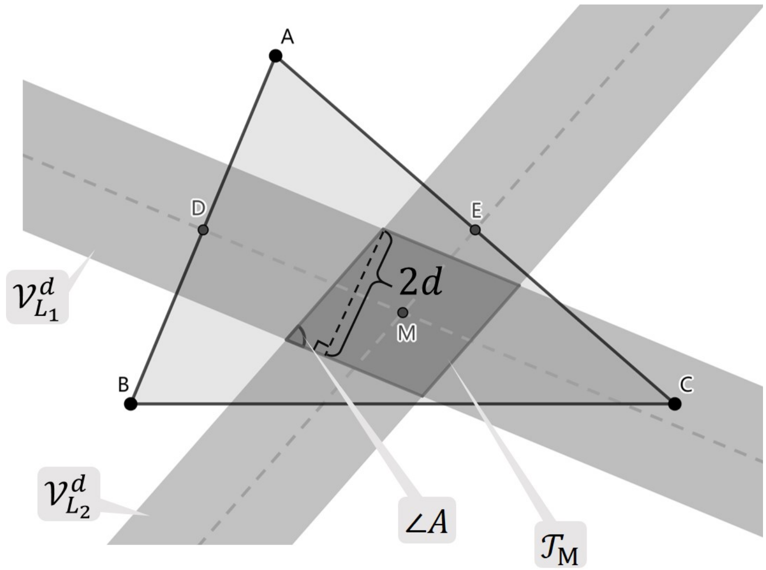

Figure 9. An illustration for the estimate $\|y - M\|$.

Since $\angle A \in [\pi/3, \pi - \arctan(2c/9)]$, we have

$$\sin \angle A \geq \sin\left(\arctan \frac{2c}{9}\right) \geq \sin \frac{c}{9} \geq \frac{c}{18} \tag{5.16}$$

where in the second last inequality we use the fact that $\arctan y > \frac{y}{2}$ if $0 < y < 1$ and in the last inequality we use the fact that $\sin y > \frac{y}{2}$ if $0 < y < 1$.

Combining (5.13), (5.14), (5.15) and (5.16), we obtain

$$\operatorname{diam} \mathcal{T}_M \leq 2l \leq \frac{72d}{c} = 324\frac{a}{c^2} \tag{5.17}$$

where in the last equality we recall $d = \frac{9}{2}\frac{a}{c}$. Therefore, we conclude (5.12).

Combining Case 2 and Case 3, we conclude (5.4).

Finally, we show (5.5). Let $x \in W(b)$. By (5.2) and (5.13), we have

$$|b - h| = |b - \|M - A\|| \leq |b - \|x - A\|| + \|x - M\| \lesssim a + \frac{a}{c^2} \lesssim \frac{a}{c^2}.$$

The proof is complete. □

Now, we are in a position to show:

Proof of Lemma 2.5. For $b \in [\frac{1}{2}, 2]$, define

$$\begin{aligned} \widetilde{W}(b) &:= W(b) \times \{b\} \\ &= \left\{ (x_1, x_2, x_3) = (x, x_3) \in \mathbb{R}^3 : \begin{array}{l} b - a \leq \|x - A\| \leq b + a, \\ b - a \leq \|x - B\| \leq b + a, \ x_3 = b \\ b - a \leq \|x - C\| \leq b + a, \end{array} \right\}. \end{aligned}$$

First we assume $\triangle ABC$ degenerates. Then by (5.3), we know

$$\widetilde{W}(b) = \emptyset \quad \text{for all } b \in [\tfrac{1}{2}, 2].$$

Hence the lemma holds for this case.

Next, we assume $\triangle ABC$ is non-degenerate. Then by (5.4), we have

$$\widetilde{W}(b) \subset B((M, b), K\tfrac{a}{c^2}) \cap \{x_3 = b\} \subset \mathbb{R}^3 \quad \text{for all } b \in [\tfrac{1}{2}, 2], \tag{5.18}$$

where M is the circumcenter of the triangle $\triangle ABC$.

Since $\widetilde{W}(b) \neq \emptyset$ implies $h - K\frac{a}{c^2} \leq b \leq h + K\frac{a}{c^2}$ from Lemma 5.3, we know

$$W \subset \bigcup_{\{b \mid \widetilde{W}(b) \neq \emptyset\}} \widetilde{W}(b) \subset \bigcup_{b \in [h-K\frac{a}{c^2}, h+K\frac{a}{c^2}]} \widetilde{W}(b). \tag{5.19}$$

Then combining (5.18) and (5.19), we deduce (2.8), i.e.

$$\operatorname{diam} W \lesssim \frac{a}{c^2},$$

which finishes the proof. □

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On the Hausdorff dimension of circular Furstenberg sets

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ON THE HAUSDORFF DIMENSION OF CIRCULAR FURSTENBERG SETS

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ABSTRACT. For $0 \leqslant s \leqslant 1$ and $0 \leqslant t \leqslant 3$, a set $F \subset \mathbb{R}^2$ is called a *circular (s,t)-Furstenberg set* if there exists a family of circles $\mathcal{S}$ of Hausdorff dimension $\dim_{\mathrm{H}} \mathcal{S} \geqslant t$ such that

$$\dim_{\mathrm{H}}(F \cap S) \geqslant s, \qquad S \in \mathcal{S}.$$

We prove that if $0 \leqslant t \leqslant s \leqslant 1$, then every circular (s,t)-Furstenberg set $F \subset \mathbb{R}^2$ has Hausdorff dimension $\dim_{\mathrm{H}} F \geqslant s + t$. The case $s = 1$ follows from earlier work of Wolff on circular Kakeya sets.

CONTENTS

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1. Introduction

We start by introducing a few key notions. Throughout the paper, we identify families of circles $\mathcal{S}$ with subsets of $\mathbb{R}^2 \times (0, \infty)$ in the obvious way: the circle $S(x, r)$ with centre $x \in \mathbb{R}^2$ and radius $r > 0$ is identified with the point $(x, r) \in \mathbb{R}^2 \times (0, \infty)$. With this convention, if $E \subset \mathbb{R}^2 \times (0, \infty)$, then the Hausdorff dimension of the circle family $\mathcal{S} = \{S(x, r) : (x, r) \in E\}$ is defined to be

$$\dim_{\mathrm{H}} \mathcal{S} := \dim_{\mathrm{H}} E.$$

Definition 1.1 (Circular Furstenberg sets)**.** Let $0 \leqslant s \leqslant 1$ and $0 \leqslant t \leqslant 3$. A set $F \subset \mathbb{R}^2$ is called a *circular (s, t)-Furstenberg set* if there exists a family of circles $\mathcal{S}$ with $\dim_{\mathrm{H}} \mathcal{S} \geqslant t$ such that $\dim_{\mathrm{H}}(F \cap S) \geqslant s$ for all $S \in \mathcal{S}$.

Equivalently, there exists a set $E \subset \mathbb{R}^2 \times (0, \infty)$ with $\dim_{\mathrm{H}} E \geqslant t$, and with the property that $\dim_{\mathrm{H}}(F \cap S(x, r)) \geqslant s$ for all $(x, r) \in E$.

Our main result is the following:

Theorem 1.2. *Let $0 \leqslant t \leqslant s \leqslant 1$. Then, every circular (s, t)-Furstenberg set $F \subset \mathbb{R}^2$ has Hausdorff dimension $\dim_{\mathrm{H}} F \geqslant s + t$.*

Remark 1.3. After the first version of this paper was posted on the *arXiv*, Zahl [29, Theorem 1.12] proved a significant generalisation of Theorem 1.2, which covers much more general "curvy" Furstenberg sets and yields the same lower bound $\dim_{\mathrm{H}} F \geqslant s + t$.

Theorem 1.2 will be deduced from a more quantitative δ-discretised version, Theorem 1.8 below. To state this version, it is convenient to introduce the following subset of the parameter space $\mathbb{R}^2 \times (0, \infty)$, where the centres are near the origin, and the radii are bounded both from above, and away from zero:

Notation 1.4 (The domain $\mathbf{D}$)**.** We write

$$\mathbf{D} := \{(x, r) \in \mathbb{R}^2 \times [0, \infty) : |x| \leqslant \tfrac{1}{4} \text{ and } r \in [\tfrac{1}{2}, 1]\}. \tag{1.5}$$

A similar normalisation already appears in Wolff's work on circular Kakeya sets, for example [27]. As long as we restrict attention to circles $S(p)$ with $p \in \mathbf{D}$, his geometric estimates will be available to us, including [27, Lemma 3.1].

The following definition will be ubiquitous in the paper:

Definition 1.6. Let $s \geqslant 0$, $C > 0$, and $\delta \in 2^{-\mathbb{N}}$. A bounded set $P \subset \mathbb{R}^d$ is called a *(δ, s, C)-set* if

$$|P \cap B(x, r)|_\delta \leqslant C r^s |P|_\delta, \qquad x \in \mathbb{R}^d,\ r \geqslant \delta.$$

Here, and in the sequel, $|E|_\delta$ refers to the number of dyadic δ-cubes intersecting E. We also extend the definition to the case where P is a finite family of dyadic δ-cubes: such a family is called a (δ, s, C)-set if the union $\cup P$ is a (δ, s, C)-set in the sense above.

The following observations are useful to keep in mind about (δ, s, C)-sets. First, if P is a non-empty (δ, s, C)-set, then $|P|_\delta \geqslant C^{-1}\delta^{-s}$. This follows by applying the defining condition at scale $r = \delta$. Second, a (δ, s, C)-set is a (δ, t, C)-set for all $0 \leqslant t \leqslant s$.

It turns out that the critical case for Theorem 1.2 is the case $s = t$: it will suffice to establish a δ-discretised analogue of the theorem in the case $s = t$ (see Theorem 1.8 below), and the general case $0 \leqslant t \leqslant s$ of Theorem 1.2 will follow from this. With this in mind, we introduce the following δ-discretised variants of a circular (s, s)-Furstenberg sets. In the definition, $\pi_{\mathbb{R}^3} : \mathbb{R}^5 \to \mathbb{R}^3$ stands for the map $\pi_{\mathbb{R}^3}(x_1, \ldots, x_5) = (x_1, x_2, x_3)$.

Definition 1.7. Let $s \in (0,1]$, $C > 0$, and $\delta \in 2^{-\mathbb{N}}$. A (δ, s, C)-*configuration* is a set $\Omega \subset \mathbb{R}^5$ such that $P := \pi_{\mathbb{R}^3}(\Omega)$ is a non-empty (δ, s, C)-subset of $\mathbf{D}$, and $E(p) := \{v \in \mathbb{R}^2 : (p, v) \in \Omega\}$ is a non-empty (δ, s, C)-subset of $S(p)$ for all $p \in P$. Additionally, we require that the sets $E(p)$ have constant cardinality: there exists $M \geqslant 1$ such that $|E(p)| = M$ for all $p \in P$.

If the constant M is worth emphasising, we will call Ω a (δ, s, C, M)-configuration. Conversely, if the constant C is not worth emphasising, we will talk casually about (δ, s)-configurations (but only in heuristic and informal parts of the paper).

We note that automatically $M \geqslant \delta^{-s}/C$, since $E(p)$ is a non-empty (δ, s, C)-set, but it may happen that M is much greater than δ^{-s}.

Theorem 1.8. *For every $\kappa > 0$ and $s \in (0,1]$, there exist $\epsilon, \delta_0 \in (0, \frac{1}{2}]$ such that the following holds for all $\delta \in (0, \delta_0]$. Let Ω be a $(\delta, s, \delta^{-\epsilon}, M)$-configuration. Then, $|\mathcal{F}|_\delta \geqslant \delta^{\kappa - s} M$, where*

$$\mathcal{F} := \bigcup_{p \in P} E(p).$$

The proof of Theorem 1.8 is based on starting with a $(\delta, s, \delta^{-\epsilon})$-configuration Ω, and refining it multiple times (the required number depends on κ and s) until the following *total multiplicity function* of the final refinement is uniformly bounded from above.

Definition 1.9 (Total multiplicity function)**.** Let $\Omega \subset \mathbb{R}^5$ be a bounded set, and let $\delta > 0$. For $w \in \mathbb{R}^2$, we write

$$m_\delta(w \mid \Omega) := |\{(p, v) \in \Omega : w \in B(v, \delta)\}|_\delta. \tag{1.10}$$

The total multiplicity function is called this way, because we will also introduce "partial" multiplicity functions (denoted $m_{\delta,\lambda,t}$) which do not take into account all pairs $(p, v) \in \Omega$, but rather impose certain restrictions on p, depending on the parameters λ and t.

The next theorem contains the technical core of the paper, and it implies Theorem 1.8.

Theorem 1.11. *For every $\kappa > 0$ and $s \in (0,1]$ there exist $\delta_0, \epsilon \in (0, \frac{1}{2}]$ such that the following holds for all $\delta \in (0, \delta_0]$. Let $\Omega \subset \mathbf{D} \times \mathbb{R}^2$ be a $(\delta, s, \delta^{-\epsilon})$-configuration with $|P| \leqslant \delta^{-s-\epsilon}$. Then, there exists a subset $\Omega' \subset \Omega$ such that $|\Omega'|_\delta \geqslant \delta^\kappa |\Omega|_\delta$, and*

$$m_\delta(w \mid \Omega') \leqslant \delta^{-\kappa}, \qquad w \in \Omega'. \tag{1.12}$$

Remark 1.13. In practical applications of Theorem 1.11, it will be important to know that the constant $\epsilon > 0$ stays bounded away from zero as long as $\kappa > 0$ and $s \in (0,1]$ stay bounded away from zero. This is true, and follows from the proof of Theorem 1.11, where the dependence between ϵ and κ, s is always explicit and effective. Since Theorem 1.8 is a consequence of Theorem 1.11, this remark also applies to Theorem 1.8.

Deducing Theorem 1.8 from Theorem 1.11, and finally Theorem 1.2 from 1.8, is accomplished in Section 2.

1.1. **Circular vs. linear Furstenberg sets.** The results in this paper should be contrasted with their (known) counterparts regarding *linear (s,t)-Furstenberg sets*.

A linear (s,t)-Furstenberg set is defined just like a circular (s,t)-Furstenberg set, except that the t-dimensional family of circles is replaced by a t-dimensional family of lines. The main difference between linear and circular Furstenberg sets is that the parameter space of circles is 3-dimensional, whereas the parameter space of lines is only 2-dimensional.

This difference makes linear Furstenberg sets substantially simpler: in particular, the analogue of Theorem 1.2 for linear (s,t)-Furstenberg sets is known, see [8, Theorem A.1] or [11, Theorem 12] for two very different proofs, and [6, 7, 14, 27] for earlier partial results. Furthermore, any results for circular Furstenberg sets imply their own counterparts for linear Furstenberg sets, simply because the map $z \mapsto 1/z$ takes all lines to circles through 0. In particular, Theorem 1.2 gives another – seriously over-complicated – proof for [8, Theorem A.1] and [11, Theorem 12].

Even with Theorem 1.2 in hand, the theory of circular Furstenberg sets remains substantially less developed than its linear counterpart. Theorem 1.2 is obviously sharp in its stated range $0 \leqslant t \leqslant s \leqslant 1$, but gives no new information if $t > s$ (compared to the case $t = s$). In contrast, it is known that linear (s,t)-Furstenberg sets have Hausdorff dimension $\geqslant 2s + \epsilon(s,t)$ for $t > s$ (see [15]). Even stronger results are available for $t > \min\{1, 2s\}$ (see [4, Theorem 1.6] and [20] for the current world records). For circular Furstenberg sets, the only improvement over Theorem 1.2 is known in the range $t \in (3s, 3]$: in an earlier paper [10], the second author proved that every circular (s,t)-Furstenberg set has Hausdorff dimension at least $t/3 + s$, when $s \in (0,1]$ and $t \in (0,3]$ (the result is only stated for $t \in (0,1]$, but the proof actually works for $t \in (0,3]$).

The sharp lower bound for the dimension of linear (s,t)-Furstenberg sets is a major open problem: it seems plausible that every linear (s,t)-Furstenberg has dimension at least $\min\{(3s+t)/2, s+1\}$. The case $t = 1$ of the problem was posed by Wolff in [26, §3] and [27, Remark 1.5]. The $(s+1)$-bound governs the case $s + t \geqslant 2$, and is already known, see [4, Theorem 1.6]. The bound $\min\{(3s+t)/2, s+1\}$ would be sharp if true.

Linear Furstenberg sets can be viewed as special cases of circular Furstenberg sets (as explained above), so at least one cannot hope for something stronger than the lower bound $\min\{(3s+t)/2, s+1\}$ for circular (s,t)-Furstenberg sets. However, it is not clear to us if the optimal lower bounds for linear and circular Furstenberg sets should always coincide. Theorem 1.2 shows that they do in the range $0 \leqslant t \leqslant s \leqslant 1$.

Remark 1.14. After this paper appeared on the *arXiv*, the linear Furstenberg set problem was solved in [16, 18].

1.2. **Relation to previous work.** The main challenge in the proof of Theorem 1.11 is to combine the non-concentration hypotheses inherent in (δ, s)-configurations with the techniques of Wolff [24, 25] developed to treat the case $s = 1$ of Theorem 1.2. Our argument is also inspired by the work of Schlag [19].

To be accurate with the references, Wolff in [24, Corollary 5.4] proved that if $t \in [0,1]$, and $E \subset \mathbb{R}^2$ is a Borel set containing circles centred at all points of a Borel set with Hausdorff dimension $\geqslant t$, then $\dim_{\mathrm{H}} E \geqslant 1 + t$. This is formally weaker than the statement that circular $(1,t)$-Furstenberg sets have dimension $\geqslant 1 + t$, but the distinction is fairly minor: Wolff's technique is robust enough to deal with circular $(1,t)$-Furstenberg sets. The main novelty in the present paper is to consider the cases (s,t) with $0 \leqslant t \leqslant s < 1$.

To illustrate the challenge, consider the case $s = \frac{1}{2}$. Let $\Omega = \{(p,v) : p \in P$ and $v \in E(p)\}$ be a $(\delta, \frac{1}{2})$-configuration. The $(\delta, \frac{1}{2})$-set property of the sets $E(p) \subset S(p)$ implies that $|E(p)|_\delta \gtrapprox \delta^{-1/2}$ for all $p \in P$. Unfortunately, this information alone is far too weak, because all the circles $S(p)$, $p \in P$, may be tangent to a single rectangle $R \subset \mathbb{R}^2$ of dimensions $\delta \times \delta^{1/2}$, and $|R|_\delta \sim \delta^{-1/2}$. So, if we only had access to the information $|E(p)|_\delta \gtrapprox \delta^{-1/2}$, all the sets $E(p)$ might be contained in R. In this case, the resulting

"Furstenberg set" $\mathcal{F}$ in (1.8) would have $|\mathcal{F}|_\delta \leqslant |R|_\delta \sim \delta^{-1/2}$. In other words, we could hope (at best!) to prove the trivial lower bound

$$\dim \mathcal{F} \geqslant \tfrac{1}{2}, \tag{1.15}$$

whereas the "right answer" given by Theorem 1.2 is $\dim \mathcal{F} \geqslant 1$. In a previous work [10], the second author showed that every circular (s,s)-Furstenberg set has Hausdorff dimension at least $\max\{4s/3, 2s^2\}$, and the second bound "$2s^2$" matches (1.15) for $s = \frac{1}{2}$: this bound indeed follows by applying the techniques of Wolff and Schlag without fully exploiting the non-concentration of the sets $E(p)$. The first bound "$4s/3$" used the non-concentration, but only in a non-sharp "two-ends" manner.

Our proof is also inspired by the very recent work of Pramanik, Yang, and Zahl [17]. In fact, [17, Section 1.1] is entitled *A Furstenberg-type problem for circles*, and a special case of Theorem 1.2 follows from [17, Theorem 1.3]. To describe this case, let $s \in [0,1]$, and let $E \subset \mathbb{R}$ be a set with $\dim_{\mathrm{H}} E \geqslant s$. Let $\mathcal{S}$ be a t-dimensional family of circles, with $0 \leqslant t \leqslant s$, and write $E_S := S \cap (E \times \mathbb{R})$ for all $S \in \mathcal{S}$. Assume that $\dim_{\mathrm{H}} E_S = \dim_{\mathrm{H}} E \geqslant s$ for all $S \in \mathcal{S}$. Then

$$F := \bigcup_{S \in \mathcal{S}} E_S$$

is an (s,t)-Furstenberg set, and [17, Theorem 1.3] (with some effort) implies $\dim_{\mathrm{H}} F \geqslant s + t$. In other words, [17, Theorem 1.3] treats the case of (s,t)-Furstenberg sets arising from the specific construction described above. This precursor allowed us to expect Theorem 1.8, but we did not succeed in modifying the argument of [17] to prove it in full generality. Our proof, outlined in the next section, is therefore rather different from [17].

While the existing literature on circular Furstenberg sets is narrow, there are many more works dealing with various aspects of circular – or in general: curvilinear – Kakeya problems. We do not delve into the details or definitions here, but we refer the reader to [1, 2, 5, 9, 12, 13, 21, 22, 23, 28] for more information.

1.3. **Ideas of the proof: key concepts and structure.** When studying circular Kakeya or Furstenberg sets, one needs to understand the geometry of intersecting δ-annuli. If $p = (x,r) \in \mathbb{R}^2 \times (0,\infty)$ and $\delta > 0$, we write $S^\delta(p)$ for the closed δ-annulus around the circle $S(p)$, thus $S^\delta(p) = \{w \in \mathbb{R}^2 : \operatorname{dist}(w, S(p)) \leqslant \delta\}$.

If $p = (x,r), q = (x',r') \in \mathbf{D} \subset \mathbb{R}^2 \times (0,\infty)$, what does this intersection $S^\delta(p) \cap S^\delta(q)$ look like (when non-empty)? Wolff noted that the answer depends on two parameters:

$$\lambda := \lambda(p,q) := ||x - x'| - |r - r'|| \quad \text{and} \quad t := t(p,q) := |p - q|. \tag{1.16}$$

Notice that "t" in (1.16) has a different meaning than the letter "t" in (s,t)-Furstenberg sets. For the majority of the paper (proofs of Theorems 1.8 and 1.11), we only consider (s,s)-Furstenberg sets, so this should not cause confusion. In fact, from now on the letter "t" will always refer to the *distance* parameter defined in (1.16), except for the short proof of Theorem 1.2 in Section 2 (where the distance parameter is not needed).

Here λ is called the *tangency* parameter. If $\lambda(p,q) = 0$, then the circles $S(p), S(q)$ are internally tangent, whereas if $\lambda(p,q) \sim 1$, the circles $S(p), S(q)$ intersect roughly transversally. The intersection $S^\delta(p) \cap S^\delta(q)$ can be covered by boundedly many $(\delta, \delta/\sqrt{\lambda t})$-*rectangles*. In general, a (δ,σ)-*rectangle* is the intersection of a δ-annulus with a disc of radius σ, thus

$$R^\delta_\sigma(p,v) = S^\delta(p) \cap B(v,\sigma)$$

for some $v \in S(p)$. If $\delta \leqslant \sigma \leqslant \sqrt{\delta}$, a (δ,σ)-rectangle looks like a "straight" rectangle of dimensions $\sim \delta \times \sigma$. If $\sigma > \sqrt{\delta}$, then the curvature of the annulus becomes visible, and a (δ,σ)-rectangle is a genuinely "curvy" set of thickness δ and diameter $\sim \sigma$.

When bounding the total multiplicity function m_δ (Definition 1.9), one ends up studying families of (δ,σ)-rectangles, for all possible values $\delta \leqslant \sigma \leqslant 1$. In some form, this problem appears in all previous works related to circular Kakeya sets, but the manner of formalising it varies. For us, the main new twist is to incorporate the information from the "fractal" sets $E(p) \subset S(p)$.

In addition to the total multiplicity function, we introduce a range of *partial multiplicity functions*. The precise definition is Definition 5.29, but we give the idea. For $\delta \leqslant \lambda \leqslant t \leqslant 1$, the partial multiplicity function $m_{\delta,\lambda,t}$ looks like this: for $(p,v) \in \Omega$ (with $p \in P$ and $v \in E(p)$), we write

$$m_{\delta,\lambda,t}(p,v) := |\{(p',v') \in \Omega_\sigma^\delta : \lambda(p,p') \sim \lambda,\, t(p,p') \sim t \text{ and } R_\sigma^\delta(p,v) \cap R_\sigma^\delta(p',v') \neq \emptyset\}|.$$

Here $\sigma := \delta/\sqrt{\lambda t}$, a common notation in the paper. The set Ω_σ^δ is the (δ,σ)-*skeleton* of Ω: slightly vaguely, it is a maximal $(\delta \times \sigma)$-separated set inside the original configuration Ω.

It turns out that the total multiplicity function m_δ is bounded from above by the sum of the partial multiplicity functions $m_{\delta,\lambda,t}$, where the sum ranges over dyadic pairs (λ,t), $\delta \leqslant \lambda \leqslant t \leqslant 1$. There are $\lesssim (\log(1/\delta))^2 \leqslant \delta^{-\kappa}$ such pairs (λ,t). So, to prove the upper bound (1.12) for m_δ, it suffices to prove it separately for all the partial functions $m_{\delta,\lambda,t}$. This is what we do, see Theorem 7.5. Bounding m_δ by the sum of the partial functions $m_{\delta,\lambda,t}$ is straightforward, and is accomplished at the end of the paper, in Section 7.7.

The partial multiplicity functions $m_{\delta,\lambda,t}$ have been normalised so that they might potentially satisfy the same bounds as the total multiplicity function (see Theorem 1.11): after replacing the original (δ,s)-configuration Ω by a suitable refinement Ω' (depending on λ and t), we expect – and will prove in Theorem 7.5 – that

$$\|m_{\delta,\lambda,t}(\cdot \mid \Omega')\|_{L^\infty(\Omega')} \lessapprox 1, \qquad \delta \leqslant \lambda \leqslant t \leqslant 1. \tag{1.17}$$

The proof of (1.17) proceeds in a specific order of the triples (δ,λ,t). In order to cope with a given triple (δ,λ,t), we will need to know *a priori* that the triples (λ,λ,t) and (δ,λ',t) for all $\delta \leqslant \lambda' < \lambda$ have already been dealt with. More precisely: if we have already found a refinement $\Omega' \subset \Omega$ such that (1.17) holds for all the triples (δ,λ',t) with $\delta \leqslant \lambda' < \lambda$, and also for the triple (λ,λ,t), then we are able to refine Ω' further to obtain (1.17) for (δ,λ,t).

We can now explain a technical challenge we need to overcome: the partial multiplicity function $m_{\delta,\lambda,t}$ counts elements in the (δ,σ)-skeleton of Ω, rather than Ω itself. However, our assumptions on the configuration Ω were formulated at scale δ – recall that Ω is a (δ,s)-configuration, which meant that both P, and the sets $E(p)$, are (δ,s)-sets. In order for (1.17) to be plausible, the property of "being a (δ,s)-configuration" needs to be hereditary: the (δ,σ)-skeleton Ω_σ^δ of a (δ,s)-configuration Ω needs to look like a (δ,σ,s)-configuration (whatever that precisely means). This is not literally true, but we develop reasonable substitutes for this idea in Section 3.

We next outline where the "inductive" structure for proving (1.17) stems from. Why do we need information about the triple (λ,λ,t) in order to handle the triple (δ,λ,t)? The reason is one of the main technical results of the paper, Theorem 6.5. This is a generalisation of Wolff's famous "tangency bound" [24, Lemma 1.4]. We sketch the idea of Wolff's result, and our generalisation, in a slightly special case. Namely, we will confine the discussion to the case $t = 1$ to keep the numerology as simple as possible.

In Wolff's terminology, a pair of sets $W, B \subset P \subset \mathbf{D}$ is called *bipartite* if

$$\operatorname{dist}(W, B) \sim 1.$$

If $p \in W$ and $q \in B$, we have $|p - q| \sim 1$, but the tangency parameter $\lambda(p, q)$ may vary freely in $[0, 1]$. If $\lambda(p, q) \sim \lambda \in [0, 1]$, recall that the intersection $S^{\delta}(p) \cap S^{\delta}(q)$ can be covered by boundedly many $(\delta, \delta/\sqrt{\lambda})$-rectangles. When bounding the multiplicity function $m_{\delta,\lambda,1}$, the following turns out to be a key question:

Question 1. *What is the maximal cardinality of incomparable $(\delta, \delta/\sqrt{\lambda})$-rectangles which are incident to at least one pair $(p, q) \in W \times B$ with $\lambda(p, q) \sim \lambda$?*

One of the main results in Wolff's paper [24] contains the answer in the case $\lambda = \delta$. If $\mathcal{R}_{\delta}$ is a collection of incomparable $(\delta, \sqrt{\delta})$-rectangles incident to at least one pair $(p, q) \in W \times B$ with $\lambda(p, q) \lesssim \delta$, then [24, Lemma 1.4] states that

$$|\mathcal{R}_{\delta}| \lessapprox (|W||B|)^{3/4} + \text{lesser terms}. \tag{1.18}$$

This is a highly non-trivial result. In contrast, the case $\lambda \sim 1$ is trivial: the sharp answer is $|\mathcal{R}_1| \lesssim |W||B|$. In this case the $(\delta, \delta/\sqrt{1})$-rectangles are roughly δ-discs, and clearly a generic bipartite pair W, B may generate $\sim |W||B|$ transversal intersections.

Is there a way to "interpolate" between these bounds? One might hope that if $\delta \ll \lambda \ll 1$, then $|\mathcal{R}_{\delta}| \lessapprox (|W||B|)^{\theta(\lambda)}$ for some useful intermediate exponent $\theta(\lambda) \in (\frac{3}{4}, 1)$. Unfortunately, this is not true: if $\lambda \gg \delta$, the best one can say is $|\mathcal{R}_{\lambda}| \lesssim |W||B|$.

FIGURE 1. Scenarios with $|\mathcal{R}_{\lambda}| \sim |W||B|$.

Figure 1 shows two slightly different ways in which $|\mathcal{R}_{\lambda}| \sim |W||B|$ can be realised. In both examples, there are two well-separated collections $\mathcal{W}, \mathcal{B}$ of (thick, λ-separated) λ-annuli, all elements of which are tangent to a common $(\lambda, \sqrt{\lambda})$-rectangle R_{λ}. (A technical comment: to make the figure clearer, we deliberately draw annuli with external tangencies, although formally all our tangency-counting problems and estimates concern numbers of internal tangencies. The distinction between internal and external tangencies is, however, not relevant for the phenomenon we describe here.)

Inside each annulus in $\mathcal{W}$ (respectively $\mathcal{B}$) pick $X_{\mathcal{W}}$ (respectively $X_{\mathcal{B}}$) thinner δ-annuli, shown in darker colours. This way one gets two well-separated collections W, B of δ-annuli with cardinalities

$$|W| = |\mathcal{W}| \cdot X_{\mathcal{W}} \quad \text{and} \quad |B| = |\mathcal{B}| \cdot X_{\mathcal{B}}.$$

The picture on the left of Figure 1 represents the case $X_{\mathcal{W}} = X_{\mathcal{B}} = 1$, the picture on the right represents the case $|\mathcal{W}| = |\mathcal{B}| = 1$. If the δ-annuli in W, B are chosen appropriately, their pairwise intersections (contained in R_{λ}) are located at incomparable

$(\delta, \delta/\sqrt{\lambda})$-rectangles, say $\mathcal{R}$. (To be more accurate, this can be done as long as the total number of intersections $|W||B|$ does not exceed the total number of incomparable $(\delta, \delta/\sqrt{\lambda})$-rectangles contained in R_λ, roughly $(\lambda/\delta)^2$.) Each of the rectangles in $\mathcal{R}$ has type $(\geqslant 1, \geqslant 1)$ relative to (W, B). Therefore, $|\mathcal{R}| \sim |W||B|$, provided $|W||B| \leqslant (\lambda/\delta)^2$.

The trivial upper bound $|\mathcal{R}_\lambda| \lesssim |W||B|$ is useless for $\lambda \ll 1$, but there is a way to improve it. The examples shown in Figure 1 indicate the main obstructions: the high numbers of incomparable $(\delta, \delta/\sqrt{\lambda})$-rectangles are "caused" by either

(a) a high level of tangency of "parent" annuli of thickness λ, or
(b) a high number of "child" δ-annuli contained inside "parent" annuli of thickness λ.

If we stipulate *a priori* bounds on the numbers relevant for problems (a)-(b), we get a non-trivial upper bound for $|\mathcal{R}_\lambda|$, which looks like this (see Theorem 6.5 for a precise statement):

$$|\mathcal{R}_\lambda| \lessapprox (|W||B|)^{3/4} \cdot (X_\lambda Y_\lambda)^{1/2} + \text{lesser terms}, \tag{1.19}$$

Here $X_\lambda = \max |P \cap B_\lambda|$, where the "max" runs over balls of radius λ, and Y_λ is an upper bound for how many λ-annuli can be tangent to any fixed $(\lambda, \sqrt{\lambda})$-rectangle. In fact,

$$Y_\lambda = \|m_{\lambda,\lambda,1}\|_{L^\infty}.$$

In the examples of Figure 1, we have $X_\lambda = 1$ and $Y_\lambda = |W| = |B| \sim \lambda/\delta$ (left picture) or $X_\lambda \sim |W| \sim |B| \sim \lambda/\delta$ and $Y_\lambda = 1$ (right picture). In both cases (1.19) only yields the trivial bound, as it should. On the other hand, if we have already established (1.17) for the triple $(\lambda, \lambda, 1)$, we can rest assured that $Y_\lambda \lessapprox 1$, and (1.19) becomes a useful tool for proving (1.17) for the triple $(\delta, \lambda, 1)$ (bounds for the number X_λ are, more easily, provided by non-concentration conditions on the collections of circles). This explains why our inductive proof of (1.17) needs information about the triples (λ, λ, t) to handle the triples (δ, λ, t). There is a separate reason why all the triples (δ, λ', t), $\lambda' < \lambda$, need to be treated before the triple (δ, λ, t), but we will not discuss this here: the reason will be revealed around Figure 4.

We have now quite thoroughly explained the structure of the paper, but let us summarise. In the short Section 2, we first deduce Theorem 1.8 from Theorem 1.11, and then Theorem 1.2 from Theorem 1.8. Section 3 deals with the question: to what extent is the (λ, σ)-skeleton of a (δ, s)-configuration a (λ, σ, s)-configuration?

Section 4 introduces (δ, σ)-rectangles properly, and studies their elementary geometric properties. For example, what do we exactly mean by two (δ, σ)-rectangles being "incomparable"? The results in Section 4 will look familiar to those readers knowledgeable of Wolff's work, but our (δ, σ)-rectangles are more general than Wolff's $(\delta, \sqrt{\delta/t})$-rectangles, and in some cases we need more quantitative estimates than those recorded in [24].

In Section 5, we establish the cases (λ, λ, t) of the estimate (1.17). The main produce of that section is Theorem 5.31. The geometric input behind Theorem 5.31 is simply Wolff's estimate (1.18), and this is why it can be proven before introducing the general (δ, λ, t)-version in (1.19). The proof of (1.19) occupies Section 6.

Finally, Section 7 applies the estimate (1.19) to prove (1.17) in full generality. The upper bound for the total multiplicity function m_δ is an easy corollary, and the proof Theorem 1.11 is concluded in Section 7.7. In Appendix A we prove some results from Section 4.2.

Notation. Some of the notation in this section has already been introduced above, but we gather it here for ease of reference. If $r \in 2^{-\mathbb{N}}$, the notation $|E|_r$ refers to the number of dyadic r-cubes intersecting E. Here E might be a subset of $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$. We will only ever consider dyadic cubes in $\mathbb{R}^3$ which are subsets of the special region $\mathbf{D}$ introduced in (1.5). Therefore, the notation $\mathcal{D}_r$ will always refer to dyadic r-cubes contained in $\mathbf{D}$.

In general, we will denote points in $\mathbb{R}^3$ (typically in $\mathbf{D}$) by the letters p, p', q, q'. Points in $\mathbb{R}^2$ are denoted by v, v', w, w'.

For $p = (x, r) \in \mathbb{R}^2 \times (0, \infty)$ (typically $p \in \mathbf{D}$), we write $S(p) = S(x, r)$ for the circle centred at x and radius $r > 0$. The notation $S^{\delta}(p)$ refers to the δ-annulus around $S(p)$, thus $S^{\delta}(p) = \{w \in \mathbb{R}^2 : \operatorname{dist}(w, S(p)) \leqslant \delta\}$.

The notation $A \lesssim B$ means that there exists an absolute constant $C \geqslant 1$ such that $A \leqslant CB$. The two-sided inequality $A \lesssim B \lesssim A$ is abbreviated to $A \sim B$. If the constant C is allowed to depend on a parameter "θ", we indicate this by writing $A \lesssim_{\theta} B$.

For $\delta \in (0, 1]$, the notation $A \lessapprox_{\delta} B$ means that there exists an absolute constant $C \geqslant 1$ such that

$$A \leqslant C \cdot \left(1 + \log\left(\tfrac{1}{\delta}\right)^{C}\right) B.$$

We write $A \approx_{\delta} B$ if simultaneously $A \lessapprox_{\delta} B$ and $B \lessapprox_{\delta} A$ hold true. If the constant C is allowed to depend on a parameter "θ", we indicate this by writing $A \lessapprox_{\delta,\theta} B$.

Given $p = (x, r) \in \mathbb{R}^2 \times [0, \infty)$ and $p' = (x', r')$ in $\mathbb{R}^2 \times [0, \infty)$, we write $\Delta(p, p') := ||x - x'| - |r - r'||$. This is slightly inconsistent with our notation from (1.16), but in the sequel we prefer to use the letter "Δ" for this "tangency" parameter.

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2. Proof of Theorem 1.8 and Theorem 1.2

We first use Theorem 1.11 to prove Theorem 1.8.

Proof of Theorem 1.8 assuming Theorem 1.11. Let $\Omega \subset \mathbb{R}^5$ be a $(\delta, s, \delta^{-\epsilon}, M)$-configuration. Write $P := \pi_{\mathbb{R}^3}(\Omega) \subset \mathbf{D}$, and $E(p) = \{v \in \mathbb{R}^2 : (p, v) \in \Omega\} \subset S(p)$. By replacing P and $E(p)$ by maximal δ-separated subsets, we may assume that P, $E(p)$, and Ω are finite and δ-separated to begin with. Furthermore, P contains a $(\delta, s, \delta^{-2\epsilon})$-subset $\bar{P} \subset P$ of cardinality $|\bar{P}| \leqslant \delta^{-s}$ by [15, Lemma 2.7]. Then $\bar{\Omega} := \{(p, v) : p \in \bar{P} \text{ and } v \in E(p)\}$ remains a $(\delta, s, \delta^{-2\epsilon})$-configuration with $|E(p)| \equiv M$. It evidently suffices to prove Theorem 1.8 for this sub-configuration, so we may assume that $|P| \leqslant \delta^{-s}$ to begin with.

With this assumption, we may apply Theorem 1.11 to find a subset $\Omega' \subset \Omega$ with $|\Omega'| \geqslant \delta^{\kappa}|\Omega| = \delta^{\kappa} M|P|$ and the property

$$m_{\delta}(w \mid \Omega') \leqslant \delta^{-\kappa}, \qquad w \in \mathbb{R}^2.$$

For $p \in P$, we write $\Omega'(p) := \{v \in \mathbb{R}^2 : (p, v) \in \Omega'\} \subset E(p)$ (this will become standard notation in the paper).

Let F' be a maximal δ-separated set in

$$\bigcup_{p \in P} \Omega'(p) \subset \mathcal{F},$$

where $\mathcal{F}$ appeared in the statement of Theorem 1.8. We claim that $|F'| \geqslant \delta^{3\kappa - s} M$, if $\delta > 0$ is small enough. This will evidently suffice to prove Theorem 1.8.

First, we notice that $|[\Omega'(p)]_{\delta} \cap F'| \gtrsim |\Omega'(p)|$ for all $p \in P$, where $[A]_{\delta}$ refers to the δ-neighbourhood of A. The reason is that if $w \in \Omega'(p)$, then $\operatorname{dist}(w, F') \leqslant \delta$, and therefore there exists a point $w' \in [\Omega'(p)]_\delta \cap F'$ with $|w - w'| \leqslant \delta$. Moreover, since $\Omega'(p)$ was assumed to be δ-separated, the map $w \mapsto w'$ is at most C-to-1. As a consequence of this observation,

$$\sum_{w \in F'} |\{v \in \Omega'(p) : w \in B(v,\delta)\} \geqslant |[\Omega'(p)]_\delta \cap F'| \gtrsim |\Omega'(p)|.$$

Now,

$$\begin{aligned}\delta^{-\kappa} &\geqslant \frac{1}{|F'|}\sum_{w\in F'} m_\delta(w \mid \Omega') = \frac{1}{|F'|}\sum_{w \in F'} |\{(p,v) \in \Omega' : w \in B(v,\delta)\}| \\ &= \frac{1}{|F'|}\sum_{w\in F'}\sum_{p\in P} |\{v \in \Omega'(p) : w \in B(v,\delta)\}| \gtrsim \frac{1}{|F'|}\sum_{p \in P'} |\Omega'(p)| = \frac{|\Omega'|}{|F'|}.\end{aligned}$$

Now, recalling that $|\Omega'| \geqslant \delta^\kappa M|P| \geqslant \delta^{\kappa+\epsilon-s}M$, and rearranging, we find $|F'| \geqslant \delta^{3\kappa-s}M$, assuming $\delta > 0$ small enough. This is what we claimed. □

Now we use Theorem 1.8 to prove Theorem 1.2. This is virtually the same argument as in the proof of [8, Lemma 3.3], but we give the details for the reader's convenience.

Proof of Theorem 1.2. Fix $0 < t \leqslant s \leqslant 1$, and let $F \subset \mathbb{R}^2$ be a circular (s,t)-Furstenberg set with parameter set $E \subset \mathbb{R}^2 \times (0,\infty)$ satisfying $\dim_{\mathrm{H}} E \geqslant t$. To avoid confusion, we mention already now that the plan is to apply Theorem 1.8 with parameter "t" in place of "s", and with $M \approx \delta^{-s}$ (which is potentially much larger than δ^{-t}).

Translating and scaling F, it is easy to reduce to the case $E \subset \mathbf{D}$. Fix $t' \in [t/2, t]$ and $t' \leqslant s' < s$. Since $\mathcal{H}^{t'}_\infty(E) > 0$, there exists $\alpha = \alpha(E,t') > 0$ and $E_1 \subset E$ such that $\mathcal{H}^{t'}_\infty(E_1) > \alpha$, where

$$E_1 := \{p \in E \mid \mathcal{H}^{s'}_\infty(F \cap S(p)) > \alpha\}. \tag{2.1}$$

This follows from the sub-additivity of Hausdorff content.

We also fix a parameter $\kappa > 0$, and we apply Theorem 1.8 with constants κ and t' (as above). The result is a constant $\epsilon(\kappa, t') > 0$. Recalling Remark 1.13, the constant $\epsilon(\kappa,t') > 0$ stays bounded away from zero for all $t' \in [t/2, t]$. We set

$$\epsilon := \epsilon(\kappa, t) := \inf_{t' \in [t/2,t]} \epsilon(\kappa, t') > 0.$$

Next, we choose $k_0 = k_0(\alpha,\epsilon) = k_0(E,t',\epsilon) \in \mathbb{N}$ satisfying

$$\alpha > \sum_{k=k_0}^{\infty} \frac{1}{k^2} \quad \text{and} \quad k_0^2 \leqslant \min\{2^{\epsilon k_0}/C, 2^{\kappa k_0}/C\}, \tag{2.2}$$

where $C \geqslant 1$ is an absolute constant to be determined later. Let $\mathcal{U} = \{D(x_i, r_i)\}_{i \in \mathcal{I}}$ be an arbitrary cover of F by dyadic r_i-cubes with $r_i \leqslant 2^{-k_0}$ and $F \cap D(x_i, r_i) \neq \varnothing$ for all $i \in \mathcal{I}$. For $k \geqslant k_0$, write

$$\mathcal{I}_k := \{i \in \mathcal{I} : r_i = 2^{-k}\} \quad \text{and} \quad F_k := \{\cup D(x_i, r_i) : i \in \mathcal{I}_k\}.$$

By the pigeonhole principle and (2.2) we deduce that for each $p \in E_1$, there exists $k(p) \geqslant k_0$ such that

$$\mathcal{H}^{s'}_\infty(F \cap S(p) \cap F_{k(p)}) > k(p)^{-2}.$$

Using pigeonhole principle again we obtain that there exists $k_1 \geqslant k_0$ such that

$$\mathcal{H}^{t'}_\infty(E_2) > k_1^{-2} \tag{2.3}$$

where $E_2 := \{p \in E_1 : k(p) = k_1\}$. By the construction of E_2, we have

$$\mathcal{H}^{s'}_\infty(S(p) \cap F_{k_1}) \geqslant \mathcal{H}^{s'}_\infty(F \cap S(p) \cap F_{k_1}) > k_1^{-2}, \qquad p \in E_2.$$

Write $\delta = 2^{-k_1}$. By (2.3) and [3, Lemma 3.13], we know that there exists a δ-separated (δ, t', Ck_1^2)-set $P \subset E_2$ satisfying $(k_1^{-2}/C)\delta^{-t'} \leqslant |P| \leqslant \delta^{-t'}$. Since $P \subset E_2$, we have

$$\mathcal{H}^{s'}_\infty(S(p) \cap F_{k_1}) > k_1^{-2}, \qquad p \in P. \tag{2.4}$$

Applying [3, Lemma 3.13] again to $S(p) \cap F_{k_1}$, $p \in P$, we obtain δ-separated (δ, s', Ck_1^2)-sets $E(p) \subset S(p) \cap F_{k_1}$ such that

$$|E(p)| \equiv M \geqslant (k_1^{-2}/C)\delta^{-s'} \overset{(2.2)}{\geqslant} \delta^{\kappa - s'}, \qquad p \in P.$$

By (2.2), P is a $(\delta, t', \delta^{-\epsilon})$-set, and each $E(p)$ is a $(\delta, s', \delta^{-\epsilon})$-set. Since $s' \geqslant t'$, the sets $E(p)$ are automatically also $(\delta, t', \delta^{-\epsilon})$-sets. Therefore,

$$\Omega := \{(p, v) : p \in P \text{ and } v \in E(p)\} \subset \mathbb{R}^5$$

is a $(\delta, t', \delta^{-\epsilon}, M)$-configuration. Recall that $\epsilon \leqslant \epsilon(\kappa, t')$ by the definition of ϵ. Letting

$$\mathcal{F} := \bigcup_{p \in P} E(p)$$

and applying Theorem 1.8, we deduce that $|\mathcal{F}|_\delta \geqslant \delta^{\kappa - t'} M \geqslant \delta^{2\kappa - s' - t'}$.

Since $E(p) \subset F_{k_1}$ for each $p \in P$, we have $\mathcal{F} \subset F_{k_1}$, which implies

$$|\mathcal{I}_{k_1}| = |F_{k_1}|_\delta \geqslant |\mathcal{F}|_\delta \geqslant \delta^{2\kappa - s' - t'}.$$

Then

$$\sum_{i \in \mathcal{I}} r_i^{s' + t' - 2\kappa} \geqslant \sum_{i \in \mathcal{I}_{k_1}} r_i^{s' + t' - 2\kappa} = \delta^{s' + t' - 2\kappa} |\mathcal{I}_{k_1}| \geqslant 1.$$

As the covering was arbitrary, we infer that $\dim_{\mathrm{H}} F \geqslant s' + t' - 2\kappa$. Sending $s' \nearrow s$, $t' \nearrow t$, and $\kappa \searrow 0$, we arrive at the desired result. $\square$

3. Preliminaries on (δ, s)-configurations

The proof of Theorem 1.11 – the multiplicity upper bound for (δ, s)-configurations – will involve considering such configurations at scales $\Delta \gg \delta$. In a dream world, a (δ, s)-configuration would admit a "dyadic" structure which would enable statements of the following kind: (a) the Δ-parents of a (δ, s)-configuration form a (Δ, s)-configuration, and (b) the Δ-parents of a (δ, s, C, M)-configuration form a (Δ, s, C', M')-configuration. Such claims are not only false as stated, but also seriously ill-defined.

To formulate the problems – and eventually their solutions – precisely, we introduce notation for dyadic cubes.

Definition 3.1 (Dyadic cubes). For $\delta \in 2^{-\mathbb{N}}$, let $\mathcal{D}_\delta$ be the family of dyadic cubes in $\mathbb{R}^3$ of side-length δ which are contained in the set $\mathbf{D}$. We also write $\mathcal{D} := \bigcup_{\delta \in 2^{-\mathbb{N}}} \mathcal{D}_\delta$. If $P \subset \mathbb{R}^3$ is an arbitrary set of points, or a family of cubes, we also write

$$\mathcal{D}_\delta(P) := \{Q \in \mathcal{D}_\delta : Q \cap P \neq \emptyset\}.$$

For $p \in \mathbf{D}$, we write $Q_\delta(p) \in \mathcal{D}_\delta$ for the unique cube in $\mathcal{D}_\delta$ containing p.

We then explain some of the problems we need to overcome. The first one is that if $P \subset \mathbf{D}$ or $P \subset \mathcal{D}_\delta$ is a (δ, s)-set, it is not automatic that $P_\Delta := \mathcal{D}_\Delta(P)$ is a (Δ, s)-set for $\delta < \Delta \leqslant 1$. This is not too serious: it is well-known that there exists a "refinement" $P' \subset P$ such that $|P'| \approx_\delta |P|$, and P'_Δ is a (Δ, s)-set (a proof of this claim will be hidden inside the proof of Proposition 3.14).

There is another problem of the same nature, which seems more complex to begin with, but can eventually be solved with the same idea. Assume that $\Omega = \{(p, v) : p \in P \text{ and } v \in E(p)\}$ is a (δ, s)-configuration, and $\Delta \gg \delta$. In what sense can we guarantee that some "Δ-net" $\Omega_\Delta \subset \Omega$ is a (Δ, s)-configuration? By the fact stated in the previous paragraph, we may start by refining $P \mapsto P'$ such that P'_Δ is a (Δ, s)-set. Then the question becomes: which set $E_\Delta(\mathbf{p}) \subset S(\mathbf{p})$ should we associate to each $\mathbf{p} \in P'_\Delta$ in such a manner that

$$\Omega_\Delta = \{(\mathbf{p}, \mathbf{v}) : \mathbf{p} \in P'_\Delta \text{ and } \mathbf{v} \in E_\Delta(\mathbf{p})\}$$

is a (Δ, s)-configuration – which hopefully still has some useful relationship with Ω? This question will eventually be answered in the main result of this section, Proposition 3.14, but we first need to set up some notation.

For $p = (x, r) \in \mathbb{R}^3_+$ and an arc $I \subset S(p)$, we let $V(p, I)$ be the (one-sided) cone centred at x and spanned by the arc I. That is,

$$V(p, I) := \bigcup_{e \in I} \{x + t(e - x)\}_{t \geqslant 0}.$$

Definition 3.2 (Dyadic arcs)**.** We introduce a dyadic partition on the circles $S(p)$. If $\sigma \in 2^{-\mathbb{N}}$ and $p = (x, r) \in \mathbf{D}$, we let $\mathcal{S}_\sigma(p)$ be a partition of $S(p)$ into disjoint (half-open) arcs of length $2\pi r\sigma$. We also let $\mathcal{S}(p) := \bigcup_{\sigma \in 2^{-\mathbb{N}}} \mathcal{S}_\sigma(p)$. (We note that for $p = (x, r) \in \mathbf{D}$, always $r \in [\frac{1}{2}, 1]$, so the dyadic σ-arcs have length comparable to σ.)

Remark 3.3. The notation of dyadic arcs $\mathcal{S}_\sigma(p)$ will often be applied with parameters such as $\sigma = \sqrt{\delta/t}$ or $\sigma = \delta/\sqrt{\lambda t}$, which are not dyadic rationals to begin with. In such cases, we really mean $\mathcal{S}_{\bar{\sigma}}(p)$, where $\bar{\sigma} \in 2^{-\mathbb{N}}$ is the smallest dyadic rational with $\sigma \leqslant \bar{\sigma}$.

Notation 3.4. In the sequel, it will be very common that the letters $p, q, \mathbf{p}$ refer to dyadic cubes instead of points in $\mathbf{D}$. Regardless, we will use the notation $S(p)$, $\mathcal{S}_\delta(p)$ and $V(p, I)$. This always refers to the corresponding definitions relative to the centre of $p, q, \mathbf{p}$, which is an element of $\mathbf{D}$.

Lemma 3.5. *Let $0 < \delta \leqslant \Delta \leqslant 1$ and $0 < \sigma \leqslant \Sigma \leqslant 1$ be dyadic numbers with $\Delta \leqslant \Sigma$. Assume that $p \in \mathcal{D}_\delta$ and $\mathbf{p} \in \mathcal{D}_\Delta$ with $p \subset \mathbf{p}$, and let $v \in \mathcal{S}_\sigma(p)$. If $\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})$ is such that*

$$v \cap V(\mathbf{p}, \mathbf{v}) \neq \emptyset,$$

then there exists an arc $I_\mathbf{v} \subset S(\mathbf{p})$ of length $\lesssim \Sigma$ such that $v \subset V(\mathbf{p}, I_\mathbf{v})$ and $\mathbf{v} \subset I_\mathbf{v}$.

For all $p, \mathbf{p}$, and v as in the statement of the lemma, there exists at least one $\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})$ such that $v \cap V(\mathbf{p}, \mathbf{v}) \neq \emptyset$, simply because $\mathbb{R}^2 = \cup_{\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})} V(\mathbf{p}, \mathbf{v})$.

Proof. Without loss of generality, we may assume that $\Sigma \leqslant 1/12$, say. We denote

$$\mathcal{S}_\Sigma(\mathbf{p}, v) := \{\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p}) : v \cap V(\mathbf{p}, \mathbf{v}) \neq \emptyset\}.$$

Our goal is to bound the cardinality of $\mathcal{S}_\Sigma(\mathbf{p}, v)$ uniformly from above and prove that $I_\mathbf{v}$ can be obtained as the union of the arcs in $\mathcal{S}_\Sigma(\mathbf{p}, v)$.

Let $(x, r), (\mathbf{x}, \mathbf{r}) \in \mathbf{D}$ be the centers of the cubes $p \in \mathcal{D}_\delta$ and $\mathbf{p} \in \mathcal{D}_\Delta$, respectively. By assumption, $\mathbf{r} \geqslant 1/2$ and $\Delta \leqslant \Sigma \leqslant 1/12$, so that $\mathbf{r} - 3\Delta \gtrsim 1$. Since $S(p) \subset S^{3\Delta}(\mathbf{p})$ by a simple application of the triangle inequality, we find

$$\operatorname{dist}(v, \mathbf{x}) \geqslant \operatorname{dist}(S(p), \mathbf{x}) \gtrsim 1. \tag{3.6}$$

Moreover, using also that $r \geqslant 1/2$, and $\delta \leqslant 1/12$, it follows that $\mathbf{x}$ must be contained in the interior of the disk bounded by $S(p)$.

By the connectedness of v and since $v \cap \{\mathbf{x}\} = \varnothing$, we find that $\cup_{\mathcal{S}_\Sigma(\mathbf{p},v)} \mathbf{v}$ is a connected set in $S(\mathbf{p})$ which implies that $\mathcal{S}_\Sigma(\mathbf{p}, v) = \{\mathbf{v}_i\}_{i=1,\cdots,m}$ is a family of adjacent arcs. If $m \in \{1, 2\}$, then their union is obviously an arc $I_\mathbf{v}$ of length at most $4\pi\mathbf{r}\Sigma$ with $v \subset V(\mathbf{p}, I_\mathbf{v})$ and $\mathbf{v} \subset I_\mathbf{v}$. Thus we assume from now on that $m \geqslant 3$. Letting $\mathbf{v}_i^+$ and $\mathbf{v}_i^-$ be the two endpoints of the arc $\mathbf{v}_i$, we can arrange the arcs $\mathbf{v}_i \in \mathcal{S}_\Sigma(\mathbf{p}, v)$ in such an order that $\mathbf{v}_i^+ = \mathbf{v}_{i+1}^-$ for all $i = 1, \cdots, m-1$.

To conclude the proof of the lemma, it suffices to show that m is bounded from above by a universal constant. As $v \in \mathcal{S}_\sigma(p)$ for $p = (x, r)$, the length $\ell(v)$ of v is $2\pi r\sigma$. As $\mathbf{x}$ lies inside the disk bounded by $S(p)$, the set $v \cap V(\mathbf{p}, \mathbf{v}_i)$ is a curve for every i. Since $\sigma \leqslant \Sigma$ and $r \leqslant 2$, we have that

$$4\pi\Sigma \geqslant \ell(v) = \sum_{i=1}^{m} \ell(v \cap V(\mathbf{p}, \mathbf{v}_i)) \geqslant \sum_{i=2}^{m-1} \ell(v \cap V(\mathbf{p}, \mathbf{v}_i)).$$

Thus, the desired upper bound for m will follow, if we manage to prove that

$$\ell(v \cap V(\mathbf{p}, \mathbf{v}_i)) \gtrsim \Sigma, \qquad 2 \leqslant i \leqslant m-1. \tag{3.7}$$

Note that

$$\partial V(\mathbf{p}, \mathbf{v}_i) = \{\mathbf{x}\} \cup \{\mathbf{x} + t(\mathbf{v}_i^+ - \mathbf{x})\}_{t>0} \cup \{\mathbf{x} + t(\mathbf{v}_i^- - \mathbf{x})\}_{t>0}, \qquad 1 \leqslant i \leqslant m.$$

Write $\bar{\mathbf{v}}_i^+ := \{\mathbf{x} + t(\mathbf{v}_i^+ - \mathbf{x})\}_{t>0}$ and $\bar{\mathbf{v}}_i^- := \{\mathbf{x} + t(\mathbf{v}_i^- - \mathbf{x})\}_{t>0}$. We have

$$\bar{\mathbf{v}}_i^+ \cap \bar{\mathbf{v}}_i^- = \varnothing, \qquad 1 \leqslant i \leqslant m.$$

Recall that $v \cap \{\mathbf{x}\} = \varnothing$. Then, by the arrangement of the arcs $\mathbf{v}_i$, we know for $i = 2, \cdots, m-1$, that v must intersect both $\bar{\mathbf{v}}_i^+$ and $\bar{\mathbf{v}}_i^-$. Let

$$x_i^+ \in \bar{\mathbf{v}}_i^+ \cap v \quad \text{and} \quad x_i^- \in \bar{\mathbf{v}}_i^- \cap v, \qquad 2 \leqslant i \leqslant m-1.$$

We claim that

$$|x_i^+ - x_i^-| \gtrsim \Sigma, \qquad 2 \leqslant i \leqslant m-1, \tag{3.8}$$

which will yield (3.7) and thus conclude the proof of the lemma.

To prove (3.8), recall that $\mathbf{v}_i$ is an arc of length $2\pi\mathbf{r}\Sigma$ in $S(\mathbf{p})$. Thus

$$\angle(\bar{\mathbf{v}}_i^+, \bar{\mathbf{v}}_i^-) = 2\pi\Sigma \leqslant \pi/2.$$

We have

$$\begin{aligned} |x_i^+ - x_i^-| &\geqslant \operatorname{dist}(\{x_i^+\}, \bar{\mathbf{v}}_i^-) = \inf\{|x_i^+ - y| : y \in \bar{\mathbf{v}}_i^-\} = |x_i^+ - \mathbf{x}| \sin \angle(\bar{\mathbf{v}}_i^+, \bar{\mathbf{v}}_i^-) \\ &\overset{(3.6)}{\gtrsim} \frac{\angle(\bar{\mathbf{v}}_i^+, \bar{\mathbf{v}}_i^-)}{2} \gtrsim \Sigma, \end{aligned}$$

where for the second inequality we recall that $x_i^+ \in v \subset S^{3\Delta}(\mathbf{p})$, and we use the fact that $\sin\theta \geqslant \theta/2$ for all $0 \leqslant \theta \leqslant \pi/2$. The proof is complete. $\square$

Dyadic cubes have the well-known useful property that if $Q, Q' \in \mathcal{D}$ with $Q \cap Q' \neq \emptyset$, then either $Q \subset Q'$ or $Q' \subset Q$. For a fixed circle $S(p)$, the dyadic arcs $\mathcal{S}(p)$ have the same property, but things get more complicated when we want to compare dyadic arcs in $\mathcal{S}(p), \mathcal{S}(q)$ for $p \neq q$. The next notation is designed to clarify this issue.

Notation 3.9. Let $0 < \delta \leqslant \Delta \leqslant 1$ and $0 < \sigma \leqslant \Sigma \leqslant 1$ be dyadic numbers. Assume that $p \in \mathcal{D}_\delta$ and $\mathbf{p} \in \mathcal{D}_\Delta$ with $p \subset \mathbf{p}$. For each $v \in \mathcal{S}_\sigma(p)$, we write $v \prec \mathbf{v}$ for the unique arc $\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})$ such that the centre of v is contained in $V(\mathbf{p}, \mathbf{v})$. In particular, $v \cap V(\mathbf{p}, \mathbf{v}) \neq \emptyset$. For two pairs (p, v) and $(\mathbf{p}, \mathbf{v})$, we write

$$(p, v) \prec (\mathbf{p}, \mathbf{v}) \quad \Longleftrightarrow \quad p \subset \mathbf{p} \text{ and } v \prec \mathbf{v}.$$

We remark that by Lemma 3.5, if $\Delta \leqslant \Sigma$ and $(p, v) \prec (\mathbf{p}, \mathbf{v})$, then $v \subset V(\mathbf{p}, I_\mathbf{v})$ for an arc $I_\mathbf{v} \subset S(\mathbf{p})$ of length $\sim \Sigma$ with $\mathbf{v} \subset I_\mathbf{v}$.

The "$\prec$" relation is illustrated in Figure 2. It gives a precise meaning to "dyadic parents" of pairs (p, v) with $p \in \mathcal{D}_\delta$ and $v \in S(p)$. We just have to keep in mind that if $(p, v) \prec (\mathbf{p}, \mathbf{v})$, then it is not quite true that $v \subset \mathbf{v}$. A good substitute is the inclusion $v \subset V(\mathbf{p}, I_\mathbf{v})$.

Definition 3.10 (Skeleton)**.** Let $0 < \delta \leqslant \Delta$ and $0 < \sigma \leqslant \Sigma$ be dyadic rationals. Assume that $p \in \mathcal{D}_\delta$ and $E_\sigma(p) \subset \mathcal{S}_\sigma(p)$. The (Δ, Σ)-*skeleton* of $E_\sigma(p)$ is the set

$$E_\Sigma(p) = \{\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p}) : v \prec \mathbf{v} \text{ for some } v \in E_\sigma(p)\},$$

where $\mathbf{p} \in \mathcal{D}_\Delta$ is the unique dyadic cube with $p \subset \mathbf{p}$. (It is important to note that the (Δ, Σ)-skeleton of $E_\sigma(p)$ is a subset of $\mathcal{S}_\Sigma(\mathbf{p})$ instead of $\mathcal{S}_\Sigma(p)$. These coincide if $\Delta = \delta$.)

We also need the following version of the definition. Let $P \subset \mathcal{D}_\delta$, and assume that we are given a (possibly empty) family $E_\sigma(p) \subset \mathcal{S}_\sigma(p)$ for all $p \in P$. Write $\Omega = \{(p, v) : p \in P \text{ and } v \in E_\sigma(p)\}$. Then, the (Δ, Σ)-*skeleton of* Ω is defined to be

$$\Omega_\Sigma^\Delta := \{(\mathbf{p}, \mathbf{v}) : \mathbf{p} \in \mathcal{D}_\Delta,\ \mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p}), \text{ and } (p, v) \prec (\mathbf{p}, \mathbf{v}) \text{ for some } (p, v) \in \Omega\}.$$

In other words, Ω_Σ^Δ consists of pairs $(\mathbf{p}, \mathbf{v})$ such that $\mathbf{p} \in \mathcal{D}_\Delta(P)$, and $\mathbf{v} \in E_\Sigma(p)$ for some $p \in P$ with $p \subset \mathbf{p}$. We write

$$E_\Sigma(\mathbf{p}) := \{\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p}) : (\mathbf{p}, \mathbf{v}) \in \Omega_\Sigma^\Delta\} \quad \text{and} \quad P_\Delta := \{\mathbf{p} \in \mathcal{D}_\Delta : E_\Sigma(\mathbf{p}) \neq \emptyset\}.$$

Remark 3.11. Note that $E_\Sigma(\mathbf{p})$ is the union of all the (Δ, Σ)-skeletons $E_\Sigma(p)$ for all $p \in P$ with $p \subset \mathbf{p}$. Thus, $E_\Sigma(\mathbf{p})$ may be rather wild, even if the individual sets $E_\sigma(p)$ are nice (say, (σ, s)-sets). Proposition 3.14 will regardless give us useful information about the sets $E_\Sigma(\mathbf{p})$, provided that we are first allowed to prune Ω (and hence the sets $E_\sigma(p)$) slightly.

Let us recap the meaning of (δ, s, C, M)-configurations from Definition 1.7. These were defined to be sets $\Omega \subset \mathbb{R}^5$ such that $P = \pi_{\mathbb{R}^3}(\Omega) \subset \mathbf{D}$ is a non-empty (δ, s, C)-set, and $E(p) = \{v \in \mathbb{R}^2 : (p, v) \in \Omega\}$ is a (δ, s, C)-subset of $S(p)$ for all $p \in P$, satisfying $|E(p)|_\delta \equiv M$. We next pose the following dyadic (and slightly generalised) variant of the definition.

Definition 3.12. Let $0 < s \leqslant 1$, $C > 0$, and let $0 < \delta \leqslant 1$, $0 < \delta, \sigma \leqslant 1$ be dyadic rationals. A $(\delta, \sigma, s, C, M)$-configuration is a set of the form

$$\Omega = \{(p, v) : p \in P \text{ and } v \in E_\sigma(p)\},$$

where $P \subset \mathcal{D}_\delta$ is a (δ, s, C)-set, and $E_\sigma(p) \subset \mathcal{S}_\sigma(p)$, for $p \in P$, is a (σ, s, C)-set of constant cardinality $|E_\sigma(p)| \equiv M$. If Ω is a $(\delta, \sigma, s, C, M)$-configuration for some M, we simply say that Ω is a (δ, σ, s, C)-configuration.

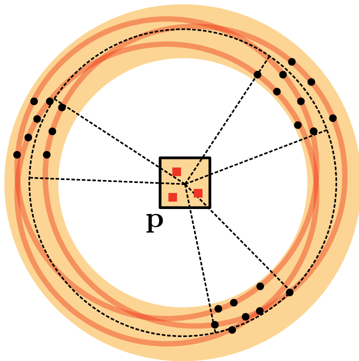

FIGURE 2. The red squares represent the centres of three circles $S(p_1), S(p_2), S(p_3)$, where $p_1, p_2, p_3 \in \mathcal{D}_\delta$. In the figure we have $p_1, p_2, p_3 \subset \mathbf{p}$ for a certain $\mathbf{p} \in \mathcal{D}_\Delta$, where $\Delta > \delta$. Therefore the red δ-annuli $S^\delta(p_1), S^\delta(p_2), S^\delta(p_3)$ are contained in the (yellow) Δ-annulus $S^\Delta(\mathbf{p})$. The black dots on the red circles represent the sets $E_\sigma(p_1), E_\sigma(p_2), E_\sigma(p_3)$, and the three longer arcs spanning the cones form the set $E_\Sigma(\mathbf{p}) \subset S(\mathbf{p})$. As shown in the figure, each pair (p_j, v) with $v \in E_\sigma(p_j)$ satisfies $(p_j, v) \prec (\mathbf{p}, \mathbf{v})$ for some $\mathbf{v} \in E_\Sigma(\mathbf{p})$.

In the new terminology, the (δ, s, C, M)-configurations from Definition 1.7 correspond to $(\delta, \delta, s, C, M)$-configurations. To be precise, we should distinguish between (δ, s, C, M)-configurations and "dyadic" (δ, s, C, M)-configurations, but we will not do this: in the sequel, the terminology will always refer to the dyadic variant in Definition 3.12.

We record the following simple *refinement principle* for $(\delta, \sigma, s, C, M)$-configurations:

Lemma 3.13 (Refinement principle)**.** *Let Ω be a (δ, σ, s, C)-configuration, and let $G \subset \Omega$ be a subset with $|G| \geqslant c|\Omega|$, where $c \in (0, 1]$. Then, there exists a $(\delta, \sigma, s, 2C/c)$-configuration $\Omega' \subset G$ with $|\Omega'| \geqslant (c^2/4)|\Omega|$.*

Proof. Write $\Omega = \{(p, v) : p \in P \text{ and } v \in E_\sigma(p)\}$. For $p \in P$, let $G(p) := \{v \in E_\sigma(p) : (p, v) \in G\}$. Note that (with $M := |E_\sigma(p)|$), we have

$$cM|P| = c|\Omega| \leqslant |G| = \sum_{p \in P} |G(p)| \leqslant M|\{p : |G(p)| \geqslant cM/2\}| + cM|P|/2.$$

It follows that the set $P' := \{p \in P : |G(p)| \geqslant cM/2\}$ has $|P'| \geqslant c|P|/2$. For each $p \in P'$, let $E'_\sigma(p) \subset G(p)$ be a set with $|E'_\sigma(p)| = cM/2 = c|E_\sigma(p)|/2$. Now, P' is a $(\delta, s, 2C/c)$-set, $E'_\sigma(p)$ is a $(\sigma, s, 2C/c)$-set for all $p \in P'$, and

$$\Omega' := \{(p, v) : p \in P' \text{ and } v' \in E'_\sigma(p)\} \subset G$$

is the desired $(\delta, \sigma, 2C/c)$-configuration with $|\Omega'| = c|P'|M/2 \geqslant (c^2/4)|\Omega|$. □

We then arrive at the main result of this section.

Proposition 3.14. *Let $0 < \delta \leqslant \Delta \leqslant 1$ and $0 < \sigma \leqslant \Sigma \leqslant 1$ be dyadic numbers with $\delta \leqslant \sigma$ and $\Delta \leqslant \Sigma$. For every $C \geqslant 1$, there exists a constant $C' \approx_\delta C$ such that the following holds. If Ω is a (δ, σ, s, C)-configuration, then there exists a subset $G \subset \Omega$ with $|G| \approx_\delta |\Omega|$ whose*

(Δ, Σ)-skeleton G_{Σ}^{Δ} is a (Δ, Σ, s, C')-configuration with the property

$$|\{(p, v) \in G : (p, v) \prec (\mathbf{p}, \mathbf{v})\}| \approx_\delta \frac{|\Omega|}{|G_{\Sigma}^{\Delta}|}, \qquad (\mathbf{p}, \mathbf{v}) \in G_{\Sigma}^{\Delta}. \tag{3.15}$$

Remark 3.16. Let $0 < \delta \leqslant \Delta \leqslant 1$ and $0 < \sigma \leqslant \Sigma \leqslant 1$ be dyadic rationals. Let $\Omega = \{(p, v) : p \in P \text{ and } v \in E_\sigma(p)\}$, as in Proposition 3.14. We will use the following notation:

$$\mathbf{p} \otimes \mathbf{v} := \{(p, v) \in \Omega : (p, v) \prec (\mathbf{p}, \mathbf{v}))\}, \qquad \mathbf{p} \in \mathcal{D}_\Delta,\ \mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p}).$$

(So, the sets $\mathbf{p} \otimes \mathbf{v}$ depend on "Ω" even though this is suppressed from the notation). The sets $\mathbf{p} \otimes \mathbf{v}$ are disjoint for distinct $(\mathbf{p}, \mathbf{v})$ with $\mathbf{p} \in \mathcal{D}_\Delta$ and $\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})$. Indeed, if $\mathbf{p} \neq \mathbf{p}'$, evidently no pair (p, v) can lie in $\mathbf{p} \otimes \mathbf{v}$ and $\mathbf{p}' \otimes \mathbf{v}'$ for any $\mathbf{v}, \mathbf{v}' \in \mathcal{S}(\mathbf{p})$. On the other hand, if $\mathbf{p} = \mathbf{p}'$ and $p \in \mathcal{D}_\delta$ with $p \subset \mathbf{p} = \mathbf{p}'$, then for each arc $v \in \mathcal{S}_\sigma(p)$ we have chosen exactly one arc $\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})$ such that $v \prec \mathbf{v}$. That is, $(p, v) \prec (\mathbf{p}, \mathbf{v})$ for only one $(\mathbf{p}, \mathbf{v})$.

To simplify the proof of Proposition 3.14 slightly, we extract the following lemma:

Lemma 3.17. *Let $0 < \delta \leqslant \Delta \leqslant 1$ be dyadic rationals, and let $P \subset \mathcal{D}_\delta$ be a (δ, s, C)-set. Assume that every set*

$$\mathbf{p} \cap P := \{p \in P : p \subset \mathbf{p}\}, \qquad \mathbf{p} \in P_\Delta := \mathcal{D}_\Delta(P),$$

has cardinality $|\mathbf{p} \cap P| \in [m, 2m]$ for some $m \geqslant 1$. Then P_Δ is a (Δ, s, C')-set with $C' \sim C$.

Proof. Let $Q \in \mathcal{D}_r$ with $\Delta \leqslant r \leqslant 1$. Then,

$$m \cdot |Q \cap P_\Delta| \leqslant |Q \cap P| \lesssim C r^s |P| \leqslant 2m \cdot C r^s |P_\Delta|.$$

Dividing by "m" yields a dyadic version of the (Δ, s, C')-set condition for P_Δ. This easily implies the usual (Δ, s, C')-set condition with a slightly worse "C'". $\square$

We then complete the proof of Proposition 3.14.

Proof of Proposition 3.14. In the first part of the proof, we construct certain sets $P_\Delta \subset \mathcal{D}_\Delta$ and $\mathbf{E}(\mathbf{p}) \subset \mathcal{S}_\Sigma(\mathbf{p})$, $\mathbf{p} \in P_\Delta$, by pigeonholing, and we define $\bar{\Omega} = \{(\mathbf{p}, \mathbf{v}) : p \in P_\Delta \text{ and } \mathbf{v} \in \mathbf{E}(\mathbf{p})\}$. The set $G \subset \Omega$ will be defined as

$$G := \bigcup_{(\mathbf{p}, \mathbf{v}) \in \bar{\Omega}} \mathbf{p} \otimes \mathbf{v} \subset \Omega. \tag{3.18}$$

This implies trivially that $G_{\Sigma}^{\Delta} \subset \bar{\Omega}$. In the second part of the proof, we show that $\bar{\Omega}$ is a (Δ, Σ, s, C')-configuration satisfying (3.15), so in particular $\mathbf{p} \otimes \mathbf{v} \neq \emptyset$ for all $(\mathbf{p}, \mathbf{v}) \in \bar{\Omega}$. Therefore also $G_{\Sigma}^{\Delta} \supset \bar{\Omega}$ by definitions, and the proof will be complete.

Write $\Omega = \{(p, v) : p \in P \text{ and } v \in E(p)\}$, where $P \subset \mathcal{D}_\delta$ and $E(p) \subset \mathcal{S}_\delta(p)$. To construct P_Δ, consider initially $P_\Delta^1 := \mathcal{D}_\Delta(P)$. Each $\mathbf{p} \in P_\Delta^1$ may contain different numbers of δ-cubes from P, and to fix this we perform our first pigeonholing. Let $\mathbf{p} \in P_\Delta^1$ and define

$$\mathcal{D}_\delta(\mathbf{p} \cap P) := \{p \in P : p \subset \mathbf{p}\} \text{ and } P_{\Delta,i}^1 := \{\mathbf{p} \in P_\Delta^1 : 2^{i-1} \leqslant |\mathcal{D}_\delta(\mathbf{p} \cap P)| < 2^i\}$$

for $i \geqslant 1$. Observing that

$$|P| = \sum_{i \in \mathbb{N}} \sum_{\mathbf{p} \in P_{\Delta,i}} |\mathcal{D}_\delta(\mathbf{p} \cap P)|$$

and noting that $P_{\Delta,i}$ is empty if $2^{i-1} > |\mathcal{D}_\delta| \sim \delta^{-3}$, we conclude by pigeonholing that there exists $i_0 \lessapprox_\delta 1$ such that

$$|P| \approx_\delta \sum_{\mathbf{p}\in P_{\Delta,i_0}} |\mathcal{D}_\delta(\mathbf{p} \cap P)|.$$

For this index i_0, we then have $|P_{\Delta,i_0}| \approx_\delta |P|/2^{i_0}$.

For each $\mathbf{p} = (\mathbf{x}, \mathbf{r}) \in P^1_\Delta$, we next construct families $\mathcal{S}^j_\Sigma(\mathbf{p})$, $j \in \mathbb{N}$, that will be used for the definition of the sets $\mathbf{E}(\mathbf{p})$. Again, we use pigeonholing to find a subset of $\{\mathbf{p} \otimes \mathbf{v} \colon \mathbf{p} \in P_{\Delta,i_0},\ \mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})\}$ of typical cardinality.

First, since each $\mathbf{p} \in P_{\Delta,i_0}$ contains $\sim 2^{i_0}$ cubes $p \in P$, since we have $|E_\sigma(p)| \equiv M$ for all of them, and since for each such p and $v \in E_\sigma(p)$ there exists a unique $\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p})$ such that $(p, v) \prec (\mathbf{p}, \mathbf{v})$, we obtain

$$\sum_{\mathbf{v}\in\mathcal{S}_\Sigma(\mathbf{p})} |\mathbf{p} \otimes \mathbf{v}| \sim 2^{i_0} M. \tag{3.19}$$

Next, for $j \geqslant 1$, we define

$$\mathcal{S}^j_\Sigma(\mathbf{p}) := \{\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p}) : |\mathbf{p} \otimes \mathbf{v}| \in [2^{j-1}, 2^j)\}. \tag{3.20}$$

Since $\delta \leqslant \sigma$ by assumption, we have $|\mathbf{p} \otimes \mathbf{v}| \lesssim \delta^{-3}\sigma^{-1} \leqslant \delta^{-4}$. It follows that $\mathcal{S}^j_\Sigma(\mathbf{p}) = \varnothing$ if $2^j \gg \delta^{-4}$. Hence, by (3.19) and pigeonholing, there exists $j(\mathbf{p}) \lessapprox_\delta 1$ such that

$$\sum_{\mathbf{v}\in\mathcal{S}^{j(\mathbf{p})}_\Sigma(\mathbf{p})} |\mathbf{p} \otimes \mathbf{v}| \approx_\delta 2^{i_0} M.$$

By a second pigeonholing, since $|P_{\Delta,i_0}| \lesssim \delta^{-3}$, there exists $j_0 \lessapprox_\delta 1$ and $P_\Delta \subset P_{\Delta,i_0} \subset P^1_\Delta$ such that

$$|P_\Delta| \approx_\delta |P_{\Delta,i_0}| \approx_\delta \frac{|P|}{2^{i_0}}, \tag{3.21}$$

and

$$\sum_{\mathbf{v}\in\mathcal{S}^{j_0}_\Sigma(\mathbf{p})} |\mathbf{p} \otimes \mathbf{v}| \approx_\delta 2^{i_0} M, \quad \mathbf{p} \in P_\Delta. \tag{3.22}$$

In (3.25), we will see that all the sets $\mathcal{S}^{j_0}_\Sigma(\mathbf{p})$, $p \in P_\Delta$, have cardinality $\approx_\delta 2^{i_0}M/2^{j_0}$, but the sets $\mathbf{E}(\mathbf{p})$, $\mathbf{p} \in P_\Delta$ are required to have exactly the same cardinality. To this end, we define

$$M_\Sigma := \min\{|\mathcal{S}^{j_0}_\Sigma(\mathbf{p})| : \mathbf{p} \in P_\Delta\}, \tag{3.23}$$

which satisfies $M_\Sigma \geqslant 1$ by (3.22). For each $\mathbf{p} \in P_\Delta$, we choose $\mathbf{E}(\mathbf{p})$ to be an arbitrary subset of $\mathcal{S}^{j_0}_\Sigma(\mathbf{p})$ of cardinality $|\mathbf{E}(\mathbf{p})| = M_\Sigma$. Now, as already announced at the start of the proof, we set $\bar{\Omega} := \{(\mathbf{p}, \mathbf{v}) : \mathbf{p} \in P_\Delta \text{ and } \mathbf{v} \in \mathbf{E}(\mathbf{p})\}$, and we define $G \subset \Omega$ with the formula (3.18). We record that $|\bar{\Omega}| = |P_\Delta| M_\Sigma$.

Keeping in mind Remark 3.16 about the disjointness of the sets $\mathbf{p} \otimes \mathbf{v}$, and using the definition of the sets $\mathbf{E}(\mathbf{p}) \subset \mathcal{S}^{j_0}_\Sigma(\mathbf{p})$, we have

$$|G| = \sum_{\mathbf{p}\in P_\Delta} \sum_{\mathbf{v}\in\mathbf{E}(\mathbf{p})} |\mathbf{p} \otimes \mathbf{v}| \geqslant |P_\Delta| \cdot M_\Sigma \cdot 2^{j_0-1}$$

To conclude that $|G| \gtrapprox_\delta |\Omega| = |P|M$, it suffices to check that

$$M_\Sigma \approx_\delta \frac{|P|M}{|P_\Delta|2^{j_0}}. \tag{3.24}$$

Since $(\mathbf{p} \otimes \mathbf{v}) \cap (\mathbf{p} \otimes \mathbf{v}') = \emptyset$ for $\mathbf{v} \neq \mathbf{v}'$, recalling first (3.20) and (3.22), and then (3.21), we have

$$|\mathcal{S}_\Sigma^{j_0}(\mathbf{p})| \approx_\delta \frac{2^{i_0}M}{2^{j_0}} \approx_\delta \frac{|P|M}{|P_\Delta|2^{j_0}}, \quad \mathbf{p} \in P_\Delta. \tag{3.25}$$

Hence (3.24) holds by the definition of M_Σ in (3.23), and therefore $|G| \approx_\delta |\Omega|$, as desired. In retrospect, (3.25) also implies $M_\Sigma \approx 2^{i_0}M/2^{j_0}$.

We next verify (3.15). The definition of $\mathcal{S}_\Sigma^{j_0}(\mathbf{p})$ in (3.20) results in

$$|\mathbf{p} \otimes \mathbf{v}| \sim 2^{j_0} \overset{(3.24)}{\approx_\delta} \frac{|P|M}{|P_\Delta|M_\Sigma} = \frac{|\Omega|}{|\bar{\Omega}|}, \quad \mathbf{p} \in P_\Delta \text{ and } \mathbf{v} \in \mathbf{E}(\mathbf{p}). \tag{3.26}$$

By the definition of G, we have $(\mathbf{p} \otimes \mathbf{v}) \cap G = \mathbf{p} \otimes \mathbf{v} \neq \emptyset$ for $\mathbf{p} \in P_\Delta$ and $\mathbf{v} \in E_\Sigma(\mathbf{v})$, and evidently $G_\Sigma^\Delta = \bar{\Omega}$. Therefore (3.15) follows from (3.26).

Next we show that P_Δ is a (Δ, s, C')-set and $\mathbf{E}(\mathbf{p})$ is a (Σ, s, C')-set for all $\mathbf{p} \in P_\Delta$. This will show that $\bar{\Omega} = G_\Sigma^\Delta$ is a $(\Delta, \Sigma, s, C', M_\Sigma)$-configuration, and conclude the proof of the proposition. To verify that P_Δ is a (Δ, s, C')-set, note that $P' = \bigcup_{\mathbf{p} \in P_\Delta} \mathcal{D}_\delta(\mathbf{p} \cap P)$ has $|P'| \approx_\delta |P|$ by (3.21). Therefore P' is a (δ, s, C')-set with $C' \approx_\delta C$. But now $P_\Delta = \mathcal{D}_\Delta(P')$, and every cube in P_Δ contains $\sim 2^{i_0}$ elements of P'. Therefore, it follows from Lemma 3.17 that P_Δ is a (Δ, s, C')-set.

It remains to verify that $\mathbf{E}(\mathbf{p})$ is a (Σ, s, C')-set for all $\mathbf{p} \in P_\Delta$. Fix $\mathbf{p} \in P_\Delta$, let $\Sigma \leqslant r \leqslant 1$ be a dyadic number and $\mathbf{v}_r \in \mathcal{S}_r(\mathbf{p})$. Our goal is to show (and it suffices to show) that $|\{\mathbf{v} \in \mathbf{E}(\mathbf{p}) : \mathbf{v} \subset \mathbf{v}_r\}| \lessapprox_\delta Cr^s M_\Sigma$. To this end, we first note that

$$|\{v \in E_\sigma(p) : v \prec \mathbf{v}_r\}| \lesssim Cr^s M, \quad p \in \mathbf{p} \cap P. \tag{3.27}$$

This follows by observing that all $v \in E_\sigma(p)$ with $v \prec \mathbf{v}_r$ are contained in $V(\mathbf{p}, I_\mathbf{v}) \cap S(p)$ by Lemma 3.5, and $\operatorname{diam}(V(\mathbf{p}, I_\mathbf{v}) \cap S(p)) \lesssim r$.

Next, observe that

$$\bigcup_{\substack{\mathbf{v} \in \mathbf{E}(\mathbf{p}) \\ \mathbf{v} \subset \mathbf{v}_r}} \mathbf{p} \otimes \mathbf{v} \subset \bigcup_{p \in \mathbf{p} \cap P} \{(p, v) : v \in \mathbf{E}(p) \text{ and } v \prec \mathbf{v}_r\}. \tag{3.28}$$

Thus recalling that $|\mathbf{p} \otimes \mathbf{v}| \sim 2^{j_0}$ for $\mathbf{v} \in \mathbf{E}(\mathbf{p}) \subset \mathcal{S}_\Sigma^{j_0}(\mathbf{p})$, we have

$$\begin{aligned}
|\{\mathbf{v} \in \mathbf{E}(\mathbf{p}) : \mathbf{v} \subset \mathbf{v}_r\}| \cdot 2^{j_0} &\sim \Big|\bigcup_{\substack{\mathbf{v} \in \mathbf{E}(\mathbf{p}) \\ \mathbf{v} \subset \mathbf{v}_r}} \mathbf{p} \otimes \mathbf{v}\Big| \\
&\leqslant \Big|\bigcup_{p \in \mathbf{p} \cap P} \{(p, v) : v \in E_\sigma(p) \text{ and } v \prec \mathbf{v}_r\}\Big| \\
&\leqslant |\mathbf{p} \cap P| \max_{p \in \mathbf{p} \cap P} |\{(p, v) : v \in E_\sigma(p) \text{ and } v \prec \mathbf{v}_r\}| \\
&\overset{(3.27)}{\lesssim} 2^{i_0} \cdot Cr^s M.
\end{aligned}$$

To conclude the proof, we recall from (3.25) that $M_\Sigma \approx_\delta 2^{i_0}M/2^{j_0}$. Therefore,

$$|\{\mathbf{v} \in \mathbf{E}(\mathbf{p}) : \mathbf{v} \subset \mathbf{v}_r\}| \lessapprox_\delta Cr^s \cdot \frac{2^{i_0}M}{2^{j_0}} \approx_\delta Cr^s M_\Sigma, \qquad \mathbf{v} \in \mathcal{S}_r(\mathbf{p}),$$

as desired. We have now proven that $G_\Sigma^\Delta = \bar{\Omega} = \{(\mathbf{p},\mathbf{v}) : \mathbf{p} \in P_\Delta \text{ and } \mathbf{v} \in \mathbf{E}(\mathbf{p})\}$ is a $(\Delta, \Sigma, s, C', M_\Sigma)$-configuration, as claimed. □

4. Rectangles and geometry

The purpose of this section is to gather facts about curvilinear rectangles (that is: pieces of annuli) and their geometry. Similar considerations are present in every paper regarding curvilinear Kakeya problems and its relatives, for example [17, 24, 25, 27].

4.1. (δ,σ)-rectangles and some basic properties.

Definition 4.1 ((δ,σ)-rectangle)**.** Let $\delta,\sigma \in (0,1]$. By definition, a (δ,σ)-rectangle is a set of the form

$$R_\sigma^\delta(p,v) := S^\delta(p) \cap B(v,\sigma),$$

where $p \in \mathbf{D}$ and $v \in S(p)$. For $C > 0$, we write $CR_\sigma^\delta(p,v) := R_{C\sigma}^{C\delta}(p,v)$.

Remark 4.2. If the reader is familiar with the terminology of Wolff's paper [24], we mention here that Wolff's "(δ,t)-rectangles" are the same as our $(\delta,\sqrt{\delta/t})$-rectangles. While Wolff's notation for these objects is more elegant, the purpose of our terminology is to handle e.g. $(\delta,\delta/\sqrt{\lambda t})$-rectangles without having to introduce further notation.

For the next lemma, we recall that $\Delta(p,p') = ||x-x'|-|r-r'||$ for $p=(x,r) \in \mathbb{R}^2 \times (0,\infty)$ and $p' = (x',r') \in \mathbb{R}^2 \times (0,\infty)$.

Lemma 4.3. *Let $p,q \in \mathbf{D}$ be points with $|p-q| = t$ and $\Delta(p,q) = \lambda$. Then, the intersection $S^\delta(p) \cap S^\delta(q)$ can be covered by boundedly many (δ,σ)-rectangles, where*

$$\sigma := \delta/\sqrt{(\lambda+\delta)(t+\delta)}.$$

Conversely, assume that $v \in S(p) \cap S(q)$. Then, for all $C \geqslant 1$, we have

$$CR_\sigma^\delta(p,v) \subset S^{C'\delta}(q), \tag{4.4}$$

where $C' \lesssim \max\{C, C^2\delta/(\lambda+\delta)\} \leqslant C^2$.

Proof. The first statement is well-known, see for example [27, Lemma 3.1], so we only prove the inclusion (4.4). Recall that $p,q \in \mathbf{D}$, so the radii of the circles $S(p), S(q)$ are bounded between $\frac{1}{2}$ and 1. For this reason, there is no loss of generality in assuming that $S(p)$ is the unit circle $S(p) = S^1$, that the radius of $S(q)$ is $r \in [\frac{1}{2},1)$, and that $S(q)$ is centred at a point $z = (x,0)$ with $x > 0$. These are incidentally the same normalisations as in [27, Lemma 3.1], and our proof is overall very similar to the argument in that lemma. With this notation, we observe that

$$\lambda = \Delta(p,q) = ||z-0| - |1-r|| = |(1-x)-r| \sim |(1-x)^2 - r^2|,$$

and $t = |p-q| \sim x + (1-r)$. Since $S(p) \cap S(q) \neq \emptyset$, we moreover have $x \geqslant 1-r$, and therefore $t \sim x$. We may assume that $x \sim t \geqslant \delta$, since otherwise (4.4) follows from $CR_\sigma^\delta(p,v) \subset S^{C\delta}(p) \subset S^{C\delta+t}(q) \subset S^{2C\delta}(q)$.

We assume that $v \in S(p) \cap S(q)$. Since $S(p) = S^1$, we may therefore write $v = e^{i\theta_0}$ for some $\theta_0 \in (-\pi, \pi]$. Recalling that z is the center of $S(q)$, we have

$$1 - 2x\cos\theta_0 + x^2 = |e^{i\theta_0} - z|^2 = r^2 \quad \Longleftrightarrow \quad \cos\theta_0 = \frac{1 - r^2 + x^2}{2x}. \tag{4.5}$$

We further rewrite this as

$$\cos\theta_0 = \frac{1 - r^2 + x^2}{2x} = 1 - \frac{r^2 - (1-x)^2}{2x} =: 1 - h,$$

where $|h| = |r^2 - (1-x)^2|/(2x) \sim \lambda/t$. We now claim that

$$|\theta - \theta_0| \leqslant C\sigma \sim \frac{C\delta}{\sqrt{(\lambda + \delta)t}} \quad \Longrightarrow \quad ||e^{i\theta} - z|^2 - r^2| \leqslant C'\delta, \tag{4.6}$$

where $C' \sim \max\{C, C^2\delta/\lambda\}$. This means that a circular arc on $S(p)$ of length $C\sigma$ around $v = e^{i\theta_0}$ is contained in $S^{C'\delta}(q)$. Since every point on $CR^\delta_\sigma(p, v)$ is within distance $C\delta \leqslant C'\delta$ from such a circular arc, (4.4) follows immediately.

Revisiting the calculation in (4.5), the condition on the right hand side of (4.6) is equivalent to

$$|1 - 2x\cos\theta + x^2 - r^2| \lesssim C'\delta \quad \Longleftrightarrow \quad |\cos\theta - \cos\theta_0| = \left|\cos\theta - \frac{1 - r^2 + x^2}{2x}\right| \leqslant \frac{C'\delta}{2x}.$$

Moreover, the right hand side here is $\sim C'\delta/t$. To prove that this estimate is valid whenever $|\theta - \theta_0| \leqslant C\sigma$, we note that

$$1 = \sin^2\theta_0 + \cos^2\theta_0 = \sin^2\theta_0 + (1-h)^2 \quad \Longrightarrow \quad \sin^2\theta_0 = 2h - h^2,$$

and therefore

$$|\cos'\theta_0| = |\sin\theta_0| = \sqrt{|2h - h^2|} \lesssim \sqrt{\lambda/t}.$$

recalling that $|h| \sim \lambda/t \leqslant 1$. Finally, for all $|\theta - \theta_0| \leqslant C\sigma$, we have

$$|\cos\theta - \cos\theta_0| = \left|\int_{\theta_0}^{\theta} \cos'\zeta \, d\zeta\right| \leqslant \int_{\theta_0}^{\theta} |\cos'\zeta - \cos'\theta_0| \, d\zeta + |\theta - \theta_0| \cdot |\cos'\theta_0| =: I_1 + I_2.$$

The term I_2 is bounded from above by

$$I_2 \lesssim C\sigma \cdot \sqrt{\lambda/t} \sim \frac{C\delta}{\sqrt{(\lambda+\delta)t}} \cdot \sqrt{\lambda/t} \sim \frac{C\delta}{t}.$$

Since $\cos' = \sin$ is 1-Lipschitz, the term I_1 is bounded from above by

$$I_1 \leqslant \int_{\theta_0}^{\theta} |\zeta - \theta_0| \, d\zeta = \int_0^{|\theta - \theta_0|} |\zeta| \, d\zeta \sim |\theta - \theta_0|^2 \leqslant C^2\sigma^2 = \frac{C^2\delta^2}{(\lambda + \delta)t}.$$

This completes the proof of (4.6) with constant $C' \sim \max\{C\delta, C^2\delta/(\lambda+\delta)\}$. $\square$

Corollary 4.7. *Let $p, q \in \mathbf{D}$ be points with $\lambda = \Delta(p, q)$ and $t = |p - q|$. Write $\sigma := \delta/\sqrt{(\lambda+\delta)(t+\delta)}$. Assume that*

$$CR^\delta_\sigma(p, v) \cap CR^\delta_\sigma(q, w) \neq \emptyset$$

for some $v \in S(p)$, $w \in S(q)$, and $C \geqslant 1$. Then, $CR^\delta_\sigma(p, v) \subset C'R^\delta_\sigma(q, w)$ for some $C' \lesssim C^4$.

Proof. Fix $\mathbf{v} \in CR^\delta_\sigma(p,v) \cap CR^\delta_\sigma(q,w)$. Then

$$\max\{\mathrm{dist}(\mathbf{v}, S(p)), \mathrm{dist}(\mathbf{v}), S(q)\}\} \leqslant C\delta.$$

Consequently, there exist points

$$v' \in S(p) \cap B(v, C\sigma) \quad \text{and} \quad w' \in S(q) \cap B(w, C\sigma)$$

such that $|v' - \mathbf{v}| \leqslant C\delta$ and $|w' - \mathbf{v}| \leqslant C\delta$. Now we shift the circles $S(p)$ and $S(q)$ a little bit so that $\mathbf{v}$ lies in their intersection. The details are as follows. Write $p = (x,r)$, and define $p' = (x', r)$, where $x' = x + (\mathbf{v} - v')$. Thus, $S(p') = S(p) + \mathbf{v} - v'$, and

$$\mathbf{v} = v' + (\mathbf{v} - v') \in S(p) + (\mathbf{v} - v') = S(p').$$

We define similarly $q' := (y', r)$, where $q = (y, r)$, and $y' = y + \mathbf{v} - v'$.

With these definitions, $|p - p'| \leqslant C\delta$ and $|q - q'| \leqslant C\delta$, and

$$\mathbf{v} \in S(p') \cap S(q').$$

Write $\lambda' := \Delta(p', q')$ and $t' := |p' - q'|$, and $\sigma' := \delta/\sqrt{(\lambda' + \delta)(t' + \delta)}$. After a small case chase, it is easy to check that $\sigma \leqslant AC\sigma'$, where $A \geqslant 1$ is absolute (the worst case in the inequality occurs if $\lambda \leqslant t \leqslant \delta$, but $\lambda' \sim t' \sim C\delta$). It now follows from Lemma 4.3 that

$$(AC^2)R^\delta_{\sigma'}(p', \mathbf{v}) \subset S^{C'\delta}(q'),$$

where $C' \lesssim C^4$, since $A \geqslant 1$ is absolute. Finally,

$$\begin{aligned} CR^\delta_\sigma(p,v) &= S^{C\delta}(p) \cap B(v, C\sigma) \\ &\subset S^{2C\delta}(p') \cap B(\mathbf{v}, AC^2\sigma') \\ &\subset (AC^2)R^\delta_{\sigma'}(p', \mathbf{v}) \subset S^{C'\delta}(q') \subset S^{2C'\delta}(q). \end{aligned}$$

Since also $CR^\delta_\sigma(p,v) \subset B(\mathbf{v}, 2C\sigma) \subset B(w, 4C\sigma)$, we have now shown that

$$CR^\delta_\sigma(p,v) \subset S^{2C'\delta}(q) \cap B(w, 4C\sigma) \subset 2C'R^\delta_\sigma(q,w),$$

as claimed. □

4.2. **Comparable rectangles.**

Definition 4.8. Given a constant $C \geqslant 1$, we say that two (δ,σ)-rectangles R_1, R_2 are C-*comparable* if there exists a third (δ,σ)-rectangle $R = R^\delta_\sigma(p,v)$ such that $R_1, R_2 \subset CR$. If no such rectangle R exists, we say that R_1 and R_2 are C-*incomparable*.

The definition of C-comparability raises a few questions. Is it necessary to speak about the third rectangle R, or is it equivalent to require that $R_1 \subset CR_2$ and $R_2 \subset CR_1$ (up to changing constants)? If this definition is equivalent, is it enough to require the one-sided condition $R_1 \subset CR_2$? The answer to both questions is affirmative, and follows from the next lemma.

Lemma 4.9. *Let $0 < \delta \leqslant \sigma \leqslant 1$, and let R_1, R_2 be (δ,σ)-rectangles satisfying $R_1 \subset CR_2$ for some $C \geqslant 1$. Then $CR_2 \subset C'R_1$ for some $C' \lesssim C^5$.*

Proof. Write $R_1 = R^\delta_\sigma(p_1, v_1)$ and $R_2 = R^\delta_\sigma(p_2, v_2)$. Let $t := |p_1 - p_2|$ and $\lambda := \Delta(p_1, p_2)$. Write

$$\bar{\sigma} := C\delta/\sqrt{(\lambda + C\delta)(t + C\delta)}.$$

From Lemma 4.3, we know that the intersection $S^{C\delta}(p_1) \cap S^{C\delta}(p_2)$ can be covered by boundedly many $(C\delta, \bar{\sigma})$-rectangles. Since $R_1 \subset CR_2 \subset S^{C\delta}(p_1) \cap S^{C\delta}(p_2)$, and $\operatorname{diam}(R_1) \gtrsim \sigma$, we may infer that $\bar{\sigma} \gtrsim \sigma$. We set $\bar{C} := \max\{1, C(\sigma/\bar{\sigma})\} \lesssim C$.

It follows from our assumption $R^{\delta}_{\sigma}(p_1, v_1) \subset CR^{\delta}_{\sigma}(p_2, v_2)$ that $R^{\delta}_{\sigma}(p_1, v_1) \subset \bar{C}R^{C\delta}_{\bar{\sigma}}(p_2, v_2)$, and in particular $\bar{C}R^{C\delta}_{\bar{\sigma}}(p_1, v_1) \cap \bar{C}R^{C\delta}_{\bar{\sigma}}(p_2, v_2) \neq \emptyset$. Therefore, applying Corollary 4.7 at scale $C\delta$ and with constant $\bar{C}$, we get

$$CR^{\delta}_{\sigma}(p_2, v_2) \subset \bar{C}R^{C\delta}_{\bar{\sigma}}(p_2, v_2) \overset{\text{C. 4.7}}{\subset} C'R^{C\delta}_{\bar{\sigma}}(p_1, v_1) \subset S^{CC'\delta}(p_1)$$

for some $C' \lesssim \bar{C}^4 \sim C^4$. Finally, since also $CR^{\delta}_{\sigma}(p_2, v_2) \subset B(v_2, C\sigma) \subset B(v_1, 2C\sigma)$, using that $v_1 \in R_1 \subset B(v_2, C\sigma)$, we may infer that

$$CR^{\delta}_{\sigma}(p_2, v_2) \subset S^{CC'\delta}(p_1) \cap B(v_1, 2C\sigma) \subset CC'R^{\delta}_{\sigma}(p_1, v_1).$$

Since $CC' \lesssim C^5$, the proof is complete. □

Remark 4.10. Lemma 4.9 clarifies the (up-to-constants) equivalence of different notions of comparability. If R_1, R_2 are C-comparable (δ, σ)-rectangles according to Definition 4.8, then there exists a third (δ, σ)-rectangle R such that $R_1, R_2 \subset CR$. But now $R \subset C'R_2$ according to Lemma 4.9, so $R_1 \subset C'R_2$. By symmetric reasoning, also $R_2 \subset C'R_1$.

Similarly, if we took as our definition the one-sided inclusion $R_1 \subset CR_2$, then Lemma 4.9 would imply that $R_2 \subset C'R_1$, and consequently we could infer the symmetric condition $R_1 \subset C'R_2$ and $R_2 \subset C'R_1$ (or $R_1, R_2 \subset C'R$ with either $R := R_1$ or $R := R_2$).

We record the following useful corollary of Lemma 4.9:

Corollary 4.11 (Transitivity of comparability). *For every $C \geqslant 1$ there exists $C' \lesssim C^5$ such that the following holds. Let $0 < \delta \leqslant \sigma \leqslant 1$, and let R_1, R_2, R_3 be (δ, σ)-rectangles such that R_1, R_2 and R_2, R_3 are both C-comparable. Then R_1, R_3 are C'-comparable.*

Proof. Since R_1, R_2 and R_2, R_3 are C-comparable, by definition there exist (δ, σ)-rectangles R_{12} and R_{23} such that $R_1, R_2 \subset CR_{12}$ and $R_2, R_3 \subset CR_{23}$. We may infer from Lemma 4.9 that

$$R_1 \subset CR_{12} \subset C'R_2 \quad \text{and} \quad R_3 \subset CR_{23} \subset C'R_2,$$

for some $C' \lesssim C^5$. This means by definition that R_1, R_3 are C'-comparable. □

Next, given a family $\mathcal{R}$ of pairwise 100-incomparable (δ, σ)-rectangles, for $A \geqslant 100$, we will show that there exists a subfamily $\bar{\mathcal{R}} \subset \mathcal{R}$ consisting of A-incomparable rectangles such that $A^{O(1)}|\bar{\mathcal{R}}| \geqslant |\mathcal{R}|$. This result will be proved in Corollary 4.13.

Indeed, Corollary 4.13 is a direct consequence of the following proposition:

Proposition 4.12. *Let $A \geqslant 100$ and $\delta \leqslant \sigma \leqslant 1$, and let $\mathcal{R}$ be a family of pairwise 100-incomparable (δ, σ)-rectangles. Suppose also that there exists a fixed (δ, σ)-rectangle $\mathbf{R}$ such that the union of the rectangles in $\mathcal{R}$ is contained in $A\mathbf{R}$. Then, $|\mathcal{R}| \lesssim A^{10}$.*

After somewhat tedious initial reductions, the proof will be virtually the same as the proof of [17, Lemma 3.15]. We postpone the details to the Appendix A, and only give a short outline here. Since [17, Lemma 3.15] was stated for curvilinear rectangles arising as neighborhoods of arcs of graphs of $C^2(\mathbf{I})$ functions defined on an interval $\mathbf{I} \subset \mathbb{R}$, we need several auxiliary lemmas (see Lemma A.2, A.8, A.12, and A.17 in Appendix A) to reduce our proof to a situation similar to [17, Lemma 3.15]. Then, in the terminology of [17], the (δ, σ)-rectangles we need to consider are called (δ, t)-rectangles with $t = \delta/\sigma^2$

provided that $\sigma \geqslant \sqrt{\delta}$ (hence $t \leqslant 1$). Thus in the range $\sigma \geqslant \sqrt{\delta}$, our proposition would basically follow from [17, Lemma 3.15]. (The comparison between the different types of rectangles is stated more precisely in (A.10)-(A.11).) But we also need to check that the proof works if $\sigma \leqslant \sqrt{\delta}$. In this range our rectangles are shorter than any of the rectangles literally treated by [17, Lemma 3.15]. The argument we give in Appendix A for Proposition 4.12 reveals, however, that the proof sees no essential difference between these cases. Alternatively, one could treat the case $\sigma \leqslant \sqrt{\delta}$ separately, relying on the fact that the (δ, σ)-rectangles in this range look like "ordinary" or "straight" rectangles.

The following consequence of Proposition 4.12 is similar in spirit to [17, Lemma 3.16]. It is not used in this section but will be applied later in the proof of Theorem 6.5.

Corollary 4.13. *Let $A \geqslant 100$ and $\delta \leqslant \sigma \leqslant 1$. Let $\mathcal{R}$ be a pairwise 100-incomparable family of (δ, σ)-rectangles. Then $\mathcal{R}$ contains a subset $\bar{\mathcal{R}}$ of cardinality $|\bar{\mathcal{R}}| \gtrsim A^{-50}|\mathcal{R}|$ consisting of pairwise A-incomparable rectangles.*

Proof. Let $\bar{\mathcal{R}}$ be the maximal A-incomparable subfamily of $\mathcal{R}$. That is, $\bar{\mathcal{R}}$ consists of pairwise A-incomparable rectangles, and any element in $\mathcal{R}$ is A-comparable to at least one rectangle in $\bar{\mathcal{R}}$. For $R \in \bar{\mathcal{R}}$, we define

$$\mathcal{R}_A(R) := \{R' \in \mathcal{R} : R' \subset CA^5 R\},$$

where $C \geqslant 1$ is an absolute constant to be fixed momentarily. By Proposition 4.12,

$$|\mathcal{R}_A(R)| \lesssim A^{50}, \quad R \in \bar{\mathcal{R}}. \tag{4.14}$$

We claim that

$$\mathcal{R} = \bigcup_{R \in \bar{\mathcal{R}}} \mathcal{R}_A(R). \tag{4.15}$$

Once (4.15) has been verified, a combination of (4.14)-(4.15) shows that $|\mathcal{R}| \lesssim |\bar{\mathcal{R}}|A^{50}$, and the proof will be complete.

To prove (4.15), fix $R' \in \mathcal{R}$. Then R' is A-comparable to some $R \in \bar{\mathcal{R}}$ by the maximality of $\bar{\mathcal{R}}$. By Remark 4.10, this gives $R' \subset CA^5 R$, provided that $C > 0$ is a sufficiently large absolute constant. In particular, $R' \in \mathcal{R}_A(R)$, as desired. $\square$

4.3. **A slight generalisation of Wolff's tangency counting bound.** The following definition is due to Wolff [24].

Definition 4.16 (t-bipartite pair). Let $0 < \delta \leqslant t \leqslant 1$. A pair of sets $W, B \subset \mathbf{D}$ is called *t-bipartite* if both W, B are δ-separated, $\max\{\operatorname{diam}(B), \operatorname{diam}(W)\} \leqslant t$, and additionally

$$\operatorname{dist}(B, W) \geqslant t \quad \text{and} \quad \operatorname{diam}(B \cup W) \leqslant 100t.$$

Lemma 4.17. *Let $\delta \leqslant t \leqslant 1$, and let $W, B \subset \mathbf{D}$ be a t-bipartite pair of sets. Let $C \geqslant 1$ be a constant, and assume that $p_1, \ldots, p_k \in W$ and $q_1, \ldots, q_l \in B$ are points satisfying*

$$\Delta(p_i, q_j) \leqslant C\delta, \qquad 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant l.$$

Assume further that there exists a point $v \in \mathbb{R}^2$ which lies on all the circles $S(p_i), S(q_j)$.

Write $\Sigma := \sqrt{\delta/t}$. Then, for suitable $C' \sim C$, every (δ, Σ)-rectangle $R^{\delta}_{\Sigma}(p_i, v)$ is contained in every annulus $S^{C'\delta}(p_m)$ and $S^{C'\delta}(q_n)$ (where i has no relation to m, n).

Proof. We will use the inclusion (4.4). Namely, (4.4) applied with $\lambda := C\delta$ shows immediately that if $(i,j) \in \{1,\ldots,k\} \times \{1,\ldots,l\}$ is a fixed pair, then

$$R^{\delta}_{\Sigma}(p_i,v) \subset S^{C'\delta}(q_j) \tag{4.18}$$

for some $C' \sim C$. (Note that now $\Sigma \lesssim \sqrt{C}\delta/\sqrt{\Delta(p_i,q_j)|p_i - q_j|} \sim \sqrt{C}\sigma$ in the notation of (4.4), so we may apply (4.4) with constant $\sim \sqrt{C}$ to obtain (4.18).) This already proves that every rectangle $R^{\delta}_{\Sigma}(p_i,v)$ is contained in every annulus $S^{C'\delta}(q_j)$. What remains is to prove a similar conclusion about the annuli $S^{C'\delta}(p_m)$ for $m \neq i$.

To proceed, we observe that (4.18) can immediately be upgraded to

$$R^{\delta}_{\Sigma}(p_i,v) \subset R^{C'\delta}_{\Sigma}(q_j,v), \tag{4.19}$$

simply as a consequence of (4.18) and definitions. Further, if $m \in \{1,\ldots,k\}$, we have

$$R^{C'\delta}_{\Sigma}(q_j,v) \subset S^{C''\delta}(p_m), \tag{4.20}$$

by another application of the inclusion (4.4) (here still $C'' \sim C' \sim C$). Now, chaining (4.19)-(4.20), we find $R^{\delta}_{\Sigma}(p_i,v) \subset S^{C''\delta}(p_m)$. Combined with (4.18), this completes the proof. □

In this paper, we will need the following slight relaxation of t-bipartite pairs:

Definition 4.21 (Almost t-bipartite pair)**.** Let $\delta \leqslant t \leqslant 1$. A pair of sets $W,B \subset \mathbf{D}$ is called (δ,ϵ)-*almost t-bipartite* if both W,B are δ-separated, and additionally

$$\operatorname{dist}(W,B) \geqslant \delta^{\epsilon}t \quad \text{and} \quad \operatorname{diam}(B \cup W) \leqslant \delta^{-\epsilon}t.$$

Definition 4.22 (Type)**.** Let $0 < \delta \leqslant \sigma \leqslant 1$, $\epsilon > 0$. Let $W,B \subset \mathbf{D}$ be finite sets. For $m,n \geqslant 1$, we say that a (δ,σ)-rectangle $R \subset \mathbb{R}^2$ has *type* $(\geqslant m, \geqslant n)_{\epsilon}$ *relative to* (W,B) if $R \subset S^{\delta^{1-\epsilon}}(p)$ for at least m points $p \in W$, and $R \subset S^{\delta^{1-\epsilon}}(q)$ at least n points $q \in B$.

Here is a slight variant of [24, Lemma 1.4]:

Lemma 4.23. *For every $\epsilon > 0$, there exists $\delta_0 > 0$ such that the following holds for all $\delta \in (0,\delta_0]$. Let $0 < \delta \leqslant t \leqslant 1$, and let $W,B \subset \mathbf{D}$ be a (δ,ϵ)-almost t-bipartite pair of sets. Let $\Sigma := \sqrt{\delta/t}$, and let $\mathcal{R}^{\delta}_{\Sigma}$ be a family of pairwise 100-incomparable (δ,Σ)-rectangles of type $(\geqslant m, \geqslant n)_{\epsilon}$ relative to (W,B), where $1 \leqslant m \leqslant |W|$ and $1 \leqslant n \leqslant |B|$. Then,*

$$|\mathcal{R}^{\delta}_{\Sigma}| \leqslant \delta^{-C\epsilon}\left(\left(\frac{|W||B|}{mn}\right)^{3/4} + \frac{|W|}{m} + \frac{|B|}{n}\right), \tag{4.24}$$

where $C > 0$ is an absolute constant.

This lemma is the same as [24, Lemma 1.4], except that it allows for constants of form "$\delta^{-\epsilon}$" in both Definition 4.21 and Definition 4.22. In [24, Lemma 1.4], the definition of "t-bipartite pair" is exactly the one we stated in Definition 4.16, and the definition of "type" was defined with a large absolute constant $C_0 \geqslant 1$ in place of $\delta^{-\epsilon}$. As it turns out, Lemma 4.23 can be formally reduced to [24, Lemma 1.4] with a little pigeonholing.

Proof of Lemma 4.23. In this proof, the letter "C" will refer to an absolute constant whose value may change from line to line.

We may assume that $\delta^{1-3\epsilon} \leqslant t$, since if (W,B) is (δ,ϵ)-almost t-bipartite for some $t \leqslant \delta^{1-3\epsilon}$, then both W and B have cardinality $\lesssim \delta^{-12\epsilon}$, and it easily follows that $|\mathcal{R}^{\delta}_{\Sigma}| \leqslant \delta^{-C\epsilon}$.

By assumption, we have $\mathrm{diam}(W) \leqslant \delta^{-\epsilon}t$ and $\mathrm{diam}(B) \leqslant \delta^{-\epsilon}t$. Therefore, we may decompose both W and B into $r \lesssim (\delta^{-\epsilon}t/\delta^{\epsilon}t)^3 = \delta^{-6\epsilon}$ subsets $W_1, \ldots, W_r$ and $B_1, \ldots, B_r$ of diameter $\leqslant \delta^{\epsilon}t$. Now, for each pair W_i, B_j, we have

$$\mathrm{dist}(W_i, B_j) =: \tau_{ij} \in [\delta^{\epsilon}t, \delta^{-\epsilon}t]. \tag{4.25}$$

Each pair (W_i, B_j) is τ_{ij}-bipartite in the terminology of Definition 4.16, since (4.25) holds, and

$$\max\{\mathrm{diam}(W_i), \mathrm{diam}(B_j)\} \leqslant \delta^{\epsilon}t \leqslant \tau_{ij} \quad \text{and} \quad \mathrm{diam}(W_i \cup B_j) \leqslant 3\tau_{ij}.$$

Next, notice that if $R \in \mathcal{R}^{\delta}_{\Sigma}$, then there exists (by the pigeonhole principle) at least one pair (W_i, B_j) such that R^{δ}_{Σ} has type $(\geqslant \bar{m}, \geqslant \bar{n})_{\epsilon}$ relative to (W_i, B_j), where

$$\bar{m} := \max\{\delta^{6\epsilon}m, 1\} \quad \text{and} \quad \bar{n} := \max\{\delta^{6\epsilon}n, 1\}.$$

This means that there exist at least $\bar{m}$ circles $S(p_1), \ldots, S(p_{\bar{m}})$ with $p_k \in W_i$ and and at least $\bar{n}$ circles $S(q_1), \ldots, S(q_{\bar{n}})$ with $q_l \in B_j$ with the property

$$R \subset S^{\delta^{1-\epsilon}}(p_k) \quad \text{and} \quad R \subset S^{\delta^{1-\epsilon}}(q_l). \tag{4.26}$$

Based on what we just said, we have

$$\mathcal{R}^{\delta}_{\Sigma} \subset \bigcup_{i,j} \mathcal{R}^{\delta}_{\Sigma}(i,j) \quad \Longrightarrow \quad |\mathcal{R}^{\delta}_{\Sigma}| \leqslant \sum_{i,j} |\mathcal{R}^{\delta}_{\Sigma}(i,j)|, \tag{4.27}$$

where $\mathcal{R}^{\delta}_{\Sigma}(i,j)$ refers to rectangles of type $(\geqslant \bar{m}, \geqslant \bar{n})_{\epsilon}$ relative to (W_i, B_j). Since the number of pairs (i,j) is $\leqslant \delta^{-C\epsilon}$, it suffices to prove (4.24) for each $\mathcal{R}^{\delta}_{\Sigma}(i,j)$ individually.

Fix $1 \leqslant i,j \leqslant r$, and write $\tau := \tau_{ij} \in [\delta^{\epsilon}t, \delta^{-\epsilon}t]$, and also abbreviate (or redefine) $W := W_i$ and $B := B_j$ and $\mathcal{R}^{\delta}_{\Sigma} := \mathcal{R}^{\delta}_{\Sigma}(i,j)$. Before proceeding further, we deduce information about the "tangency" of $p_k \in W$ and $q_l \in B$ satisfying (4.26). Recall that $|p_k - q_l| \geqslant \tau \geqslant \delta^{\epsilon}t$, and note that $\mathrm{diam}(R) \gtrsim \Sigma = \sqrt{\delta/t}$. Then,

$$\sqrt{\delta/t} \lesssim \mathrm{diam}(R) \overset{\text{L. } 4.3}{\lesssim} \frac{\delta^{1-\epsilon}}{\sqrt{\delta^{\epsilon}t\Delta(p_k, q_l)}},$$

from which we may infer that

$$\Delta(p_k, q_l) \lesssim \delta^{1-3\epsilon}, \qquad 1 \leqslant k \leqslant \bar{m},\ 1 \leqslant l \leqslant \bar{n}. \tag{4.28}$$

For purposes to become apparent in a moment, it would be convenient if W, B were $\delta^{1-3\epsilon}$-separated instead of just δ-separated. This can be arranged, at the cost of reducing $\bar{m}$ and $\bar{n}$ slightly. Indeed, we may partition W and B into $\delta^{1-3\epsilon}$-separated subsets $W_1, \ldots, W_s$ and $B_1, \ldots, B_s$, where $s \leqslant \delta^{-9\epsilon}$. Now, arguing as before, every rectangle $R \in \mathcal{R}^{\delta}_{\Sigma}$ has type $(\geqslant \bar{m}', \geqslant \bar{n}')$ relative to at least one pair (W_i, B_j), where $\bar{m}' := \max\{\delta^{9\epsilon}\bar{m}, 1\}$ and $\bar{n}' := \max\{\delta^{9\epsilon}\bar{n}, 1\}$. After repeating the argument at (4.27), we may focus attention to bounding the number of rectangles associated with a fixed (W_i, B_j). Since the passage from (W, B) to (W_i, B_j) eventually just affects the absolute constant "C" in (4.24), we now assume that W, B are $\delta^{1-3\epsilon}$-separated to begin with, and $\bar{m}' = \bar{m}$ and $\bar{n} = \bar{n}'$.

The improved separation of W, B gives the following benefit: the pair (W, B) is τ-bipartite relative to the scale $\delta^{1-3\epsilon}$ in the strong sense of Definition 4.16. The role of "δ" (or now $\delta^{1-3\epsilon}$) is hardly emphasised, but one of the assumptions in Definition 4.16 was that a τ-bipartite set is δ-separated, and the conclusion of [24, Lemma 1.4] concerns "type" and "tangency" defined for δ-annuli and $(\delta, \sqrt{\delta/t})$-rectangles. Now, since W, B are $\delta^{1-3\epsilon}$-separated, we have access to the conclusion of the same lemma at scale $\delta^{1-3\epsilon}$.

Now, [24, Lemma 1.4] implies that the maximal number of pairwise 100-incomparable $(\delta^{1-3\epsilon}, \sqrt{\delta^{1-3\epsilon}/\tau})$-rectangles of type $(\geqslant \bar{m}, \geqslant \bar{n})$ relative to (W_i, B_j) is bounded from above by the right hand side of (4.24). The definition of "type" here is the one which Wolff is using in the statement of [24, Lemma 1.4]: a $(\delta^{1-3\epsilon}, \sqrt{\delta^{1-3\epsilon}/\tau})$-rectangle $\bar{R}$ has type $(\geqslant \bar{m}, \geqslant \bar{n})$ relative to (W, B) if there are $p_1, \ldots, p_{\bar{m}} \in W$ and $q_1, \ldots, q_{\bar{n}} \in B$ such that

$$\bar{R} \subset S^{C\delta^{1-3\epsilon}}(p_k) \cap S^{C\delta^{1-3\epsilon}}(q_l), \qquad 1 \leqslant k \leqslant \bar{m},\ 1 \leqslant l \leqslant \bar{n}, \tag{4.29}$$

where $C > 0$ is an absolute constant.

What does this conclusion about the rectangles $\bar{R}$ tell us about the cardinality of $\mathcal{R}^{\delta}_{\Sigma}$? We will use the (δ, Σ)-rectangles in $\mathcal{R}^{\delta}_{\Sigma}$ to produce a new family $\bar{\mathcal{R}}$ of pairwise 100-incomparable $(\delta^{1-3\epsilon}, \bar{\Sigma})$-rectangles satisfying (4.29), where $\bar{\Sigma} = \sqrt{\delta^{1-3\epsilon}/\tau}$. Then, we will apply the upper bound for $|\bar{\mathcal{R}}|$ (given by [24, Lemma 1.4]) to conclude the desired estimate for $|\mathcal{R}^{\delta}_{\Sigma}|$.

Recall from (4.26) that each of our (δ, Σ)-rectangles $R \in \mathcal{R}^{\delta}_{\Sigma}$ has type $(\geqslant \bar{m}, \geqslant \bar{n})_{\epsilon}$ relative to (W, B) in the sense $R \subset S^{\delta^{1-\epsilon}}(p_k) \cap S^{\delta^{1-\epsilon}}(q_l)$ for every $1 \leqslant k \leqslant \bar{m}$ and $1 \leqslant l \leqslant \bar{n}$. As we observed in (4.28), this implies $\Delta(p_k, q_l) \lesssim \delta^{1-3\epsilon}$. Recall that further $|p_k - q_l| \sim \tau$ for all $1 \leqslant k \leqslant \bar{m}$ and $1 \leqslant l \leqslant \bar{n}$.

In view of applying Lemma 4.17, we would need that the circles $S(p_k)$ and $S(q_l)$ share a common point. This is not quite true, but it is true for slightly shifted copies of $S(p_k)$ and $S(q_l)$. Namely, take "v" to be an arbitrary point in R, for example its centre (writing $R = R^{\delta}_{\Sigma}(p, v)$ for some $p \in \mathbf{D}$ and $v \in S(p)$). Now, since $v \in R \subset S^{\delta^{1-\epsilon}}(p_k)$, there exists $\bar{p}_k \in B(p_k, \delta^{1-\epsilon})$ such that $v \in S(\bar{p}_k)$ (see the proof of Corollary 4.7). Similarly, there exist points $\bar{q}_l \in B(q_k, \delta^{1-\epsilon})$, $1 \leqslant l \leqslant \bar{n}$, such that $v \in S(\bar{q}_l)$. Note that the crucial hypotheses $\Delta(\bar{p}_k, \bar{q}_l) \lesssim \delta^{1-3\epsilon}$ and $|\bar{p}_k - \bar{q}_l| \sim \tau$ were not violated (since $\tau \geqslant \delta^{\epsilon} t \geqslant \delta^{1-2\epsilon}$).

Now, we are in a position to apply Lemma 4.17 at scale $\delta^{1-3\epsilon}$, and with "τ" in place of "t". The conclusion is that if we set

$$\bar{R} := \bar{R}(R) := R^{\delta^{1-3\epsilon}}_{\bar{\Sigma}}(p_1, v), \qquad \bar{\Sigma} := \sqrt{\delta^{1-3\epsilon}/\tau},$$

then (4.29) holds, provided that the constant $C > 0$ is sufficiently large (initially with the points $\bar{p}_k, \bar{q}_l$, but since $|\bar{p}_k - p_k| \leqslant \delta^{1-\epsilon}$ and $|q_l - \bar{q}_l| \leqslant \delta^{1-\epsilon}$, we also get (4.29) as stated). In other words, $\bar{R}$ is a $(\delta^{1-3\epsilon}, \bar{\Sigma})$-rectangle which has type $(\geqslant \bar{m}, \geqslant \bar{n})$ relative to (W, B) in the terminology of Wolff.

We have now shown that each rectangle $R \in \mathcal{R}^{\delta}_{\Sigma}$ gives rise to a $(\delta^{1-3\epsilon}, \bar{\Sigma})$-rectangle $\bar{R}(R)$ which has type $(\geqslant \bar{m}, \geqslant \bar{n})$ relative to (W, B). We also observe that

$$R \overset{(4.26)}{\subset} S^{\delta^{1-\epsilon}}(p_1) \cap B(v, \Sigma) \subset S^{\delta^{1-3\epsilon}}(p_1) \cap B(v, \bar{\Sigma}) = \bar{R}(R). \tag{4.30}$$

Finally, let $\bar{\mathcal{R}}$ be a maximal pairwise 100-incomparable subset of $\{\bar{R}(R) : R \in \mathcal{R}^{\delta}_{\Sigma}\}$. The rectangles in $\bar{\mathcal{R}}$ have type $(\geqslant \bar{m}, \geqslant \bar{n})$ relative to (W, B), so $|\bar{\mathcal{R}}|$ satisfies the desired upper bound (4.24) by [24, Lemma 1.4]. It remains to show that

$$|\mathcal{R}^{\delta}_{\Sigma}| \leqslant \delta^{-C\epsilon} |\bar{\mathcal{R}}|. \tag{4.31}$$

If $R \in \mathcal{R}^{\delta}_{\Sigma}$, then $\bar{R}(R) \sim_{100} \bar{R}$ for some $\bar{R} \in \bar{\mathcal{R}}$. Combining (4.30) and Lemma 4.9, we may infer that $R \subset \bar{R}(R) \subset C\bar{R}$ for some absolute constant $C > 0$. Therefore, (4.31) will follow if we manage to argue that

$$|\{R \in \mathcal{R}^{\delta}_{\Sigma} : R \subset C\bar{R}\}| \leqslant \delta^{-C\epsilon}, \qquad \bar{R} \in \bar{\mathcal{R}}.$$

But since the rectangles in $\mathcal{R}_\Sigma^\delta$ are pairwise 100-incomparable, this follows immediately from Proposition 4.12. The proof is complete. $\square$

5. Bounding partial multiplicity functions with high tangency

In this section, we will finally introduce the *partial multiplicity functions* $m_{\delta,\lambda,t}$ mentioned in the proof outline, Section 1.3 (see Definition 5.29). The plan of this section is to prove a desirable upper bound for $m_{\lambda,\lambda,t}$ – the partial multiplicity function only taking into account incidences of maximal tangency at scale λ. This will be accomplished in Theorem 5.31, although most of the work is contained in Proposition 5.2.

Notation 5.1 ($G^\rho_{\lambda,t}(\omega)$). Let $0 < \delta \leqslant \sigma \leqslant 1$, and let $P \subset \mathcal{D}_\delta$, $\{E(p)\}_{p \in P}$ be finite sets, where $E(p) \subset \mathcal{S}_\sigma(p)$ for all $p \in P$. Let $\Omega = \{(p,v) : p \in P \text{ and } v \in E(p)\}$. If $G \subset \Omega$ is an arbitrary subset, $\delta \leqslant \lambda \leqslant t \leqslant 1$, and $\rho \geqslant 1$, we define

$$G^\rho_{\lambda,t}(\omega) := \{(p',v') \in G : t/\rho \leqslant |p-p'| \leqslant \rho t \text{ and } \lambda/\rho \leqslant \Delta(p,p') \leqslant \rho\lambda\}, \qquad \omega \in \Omega.$$

The distance $|p-p'|$ and $\Delta(p,p')$ are defined relative to the centres of $p,p' \in \mathcal{D}_\delta$. If $\lambda \in [\delta, \rho\delta]$ (as in Proposition 5.2 below), we remove the lower bound $\Delta(p,p') \geqslant \lambda/\rho$ from the definition.

Proposition 5.2. *For every $\kappa > 0$, and $s \in (0,1]$, there exist $\epsilon = \epsilon(\kappa,s) \in (0,\frac{1}{2}]$ and $\lambda_0 = \lambda_0(\epsilon,\kappa,s) > 0$ such that the following holds for all $\lambda \in (0,\lambda_0]$. Let $\lambda \leqslant t \leqslant 1$ and $\Sigma := \sqrt{\lambda/t}$. Let $\Omega = \{(p,v) : p \in P \text{ and } v \in E(p)\}$ be a $(\lambda,\Sigma,s,\lambda^{-\epsilon})$-configuration (see Definition 3.12). Then, there exists a $(\lambda,\Sigma,s,C_\kappa\lambda^{-\epsilon})$-configuration $G \subset \Omega$ with $|G| \sim_\kappa |\Omega|$ with the property*

$$|\{\omega' \in G^{\lambda^{-\epsilon}}_{\lambda,t}(\omega) : \lambda^{-\epsilon}R^\lambda_\Sigma(\omega) \cap \lambda^{-\epsilon}R^\lambda_\Sigma(\omega') \neq \emptyset\}| \leqslant \lambda^{s-\kappa}|P|, \qquad \omega \in G. \tag{5.3}$$

To be precise, Σ in Proposition 5.2 refers to the smallest dyadic rational $\bar{\Sigma} \in 2^{-\mathbb{N}}$ with $\Sigma \leqslant \bar{\Sigma}$, recall Remark 3.3. Taking this carefully into account has a small impact on some constants in the proof below, but leave this to the reader.

Proof of Proposition 5.2. Write $M_\Sigma := |E(p)|$ for $p \in P$ (this constant is independent of $p \in P$ by Definition 3.12). We start by disposing of the special case where $t \leqslant \lambda^{1-\kappa/3}$. In this case we claim that $G = \Omega$ works. To see this, note that now $\Sigma = \sqrt{\lambda/t} \geqslant \lambda^{\kappa/6}$, so $M_\Sigma = |E(p)| \leqslant |\mathcal{S}_\Sigma(p)| \lesssim \lambda^{-\kappa/6}$. Furthermore,

$$\Omega^{\lambda^{-\epsilon}}_{\lambda,t}(p,v) \subset \{(p',v') \in \Omega : p' \in P \cap B(p,\lambda^{1-\kappa/2})\}, \qquad (p,v) \in \Omega,$$

assuming that $\epsilon \leqslant \kappa/6$. Fix $\omega = (p,v) \in \Omega$. Then, for every $p' \in B(p,\lambda^{1-\kappa/2})$, using the Σ-separation of $E(p')$, there are $\lesssim \lambda^{-\epsilon}$ possible choices $v' \in E(p')$ such that

$$\lambda^{-\epsilon}R^\lambda_\Sigma(\omega) \cap \lambda^{-\epsilon}R^\lambda_\Sigma(p',v') \neq \emptyset.$$

Consequently,

$$\begin{aligned}|\{\omega' \in \Omega^{\lambda^{-\epsilon}}_{\lambda,t}(\omega) : \lambda^{-\epsilon}R^\lambda_\Sigma(\omega) \cap \lambda^{-\epsilon}R^\lambda_\Sigma(\omega') \neq \emptyset\}| &\lesssim \lambda^{-\epsilon} \cdot |P \cap B(p,\lambda^{1-\kappa/2})| \\ &\lesssim \lambda^{-2\epsilon}\lambda^{(1-\kappa/2)s}|P| \leqslant \lambda^{s-5\kappa/6}|P|.\end{aligned}$$

using the $(\lambda,s,\lambda^{-\epsilon})$-set property of P in the final inequality, as well as $\epsilon \leqslant \kappa/6$, and $s \leqslant 1$. We have now proven (5.3) with $G = \Omega$. In the sequel, we may assume that

$$t \geqslant \lambda^{1-\kappa/3}. \tag{5.4}$$

Fix $\epsilon = \epsilon(\kappa, s) > 0$ and $\lambda > 0$ (depending on ϵ, κ, s) be so small that

$$A \cdot 18^{\lceil 20/\kappa \rceil}\epsilon < \kappa s \quad \text{and} \quad A^{18^{\lceil 20/\kappa \rceil}} \lambda^{-18^{\lceil 20/\kappa \rceil}\epsilon} < \lambda^{-s}, \tag{5.5}$$

where $A \geqslant 1$ is a suitable absolute constant. We start by defining a sequence of constants

$$\mathbf{C}_0 \gg \mathbf{C}_1 \gg \ldots \gg \mathbf{C}_h := \lambda^{-\epsilon},$$

where $h = \lceil 20/\kappa \rceil$, and such that $\mathbf{C}_j = A\mathbf{C}_{j+1}^{18}$. Thus,

$$\mathbf{C}_0 \leqslant A^{18^{\lceil 20/\kappa \rceil}} \lambda^{-18^{\lceil 20/\kappa \rceil}\epsilon} < \lambda^{-s}. \tag{5.6}$$

We will abbreviate

$$n_j(\omega \mid G) := |\{\omega' \in G_{\lambda,t}^{\mathbf{C}_j}(\omega) : \mathbf{C}_j R_\Sigma^\lambda(\omega) \cap \mathbf{C}_j R_\Sigma^\lambda(\omega') \neq \emptyset\}|$$

for $G \subset \Omega$ and $\omega \in \Omega$. Note that the constants $\mathbf{C}_j$ are decreasing functions of "j", so $n_h \leqslant n_{h-1} \leqslant \ldots \leqslant n_0$. Also, $n_j(\omega \mid G)$ is an upper bound for the left hand side of (5.3) for each $0 \leqslant j \leqslant h$, since $\mathbf{C}_j \geqslant \lambda^{-\epsilon}$.

We start by recording the "trivial" upper bound

$$n_0(\omega \mid G) \leqslant n_0(\omega \mid \Omega) \lesssim \mathbf{C}_0|P|, \qquad \omega \in \Omega,\ G \subset \Omega. \tag{5.7}$$

The first inequality is clear. To see the second inequality, fix $\omega = (p, v) \in \Omega$ and $p' \in P$. Now, if $v' \in \mathcal{S}_\Sigma(p')$ is such that

$$\mathbf{C}_0 R_\Sigma^\lambda(p', v') \cap \mathbf{C}_0 R_\Sigma^\lambda(\omega) \neq \emptyset,$$

then $|v - v'| \lesssim \mathbf{C}_0\Sigma$. But the points $v' \in S_\Sigma(p')$ are Σ-separated, so there are $\lesssim \mathbf{C}_0$ possible choices for v', for each $p' \in P$. This gives (5.7).

The trivial inequality (5.7) tells us that the estimate (5.3) holds automatically with $G = \Omega$ and $\kappa = 2s$, since $\lambda^{-\epsilon} \leqslant \mathbf{C}_0 < \lambda^{-s}$ by (5.5).

By the previous explanation, if $\kappa > 2s$, there is nothing to prove (we can take $G = \Omega$). Let us then assume that $\kappa \leqslant 2s$. Then, let

$$0 = \kappa_1 < \kappa_2 < \ldots < \kappa_h = 2s$$

be a $(\kappa s/10)$-dense sequence in $[0, 2s]$ (this is why we chose $h = \lceil 20/\kappa \rceil$). We now define a decreasing sequence of sets $\Omega = G_0 \supset G_1 \supset \ldots \supset G_k$, where $k \leqslant h$. We set $G_0 := \Omega$, and in general we will always make sure inductively that $|G_{j+1}| \geqslant \frac{1}{2}|G_j|$ for $j \geqslant 0$. Note that $n_0(\omega \mid G_0) \leqslant \lambda^{-s}|P| = \lambda^{s-\kappa_h}|P|$ by (5.7), for all $\omega \in G_0$.

Let us then assume that the sets $G_0 \supset \ldots \supset G_j$ have already been defined. We also assume inductively that $n_j(\omega \mid G_j) \leqslant \lambda^{s-\kappa_{h-j}}|P|$ for all $\omega \in G_j$. This was true for $j = 0$. Define

$$H_j := \{\omega \in G_j : n_{j+1}(\omega \mid G_j) \geqslant \lambda^{s-\kappa_{h-(j+1)}}|P|\}, \qquad 0 \leqslant j \leqslant k.$$

This is the subset of G_j where the lower bound for the multiplicity nearly matches the (inductive) upper bound – albeit with a slightly different definition of the multiplicity function. There are two options.

(1) If $|H_j| \geqslant \frac{1}{2}|G_j|$, then we set $H := H_j$ and $k := j$ and the construction of the sets G_j terminates.

(2) If $|H_j| < \frac{1}{2}|G_j|$, then the set $G_{j+1} := G_j \setminus H_j$ has $|G_{j+1}| \geqslant \frac{1}{2}|G_j|$, and moreover

$$n_{j+1}(\omega \mid G_{j+1}) \leqslant n_{j+1}(\omega \mid G_j) \leqslant \lambda^{s-\kappa_{h-(j+1)}}|P|, \qquad \omega \in G_{j+1}.$$

In other words, G_{j+1} is a valid "next set" in our sequence $G_0 \supset \ldots \supset G_{j+1}$, and the inductive construction may proceed.

The "hard" case of the proof of Proposition 5.2 occurs when case (1) is reached for some "j" with $\kappa_{h-j} > \kappa$. Namely, if case (1) never takes place for such indices "j", then we can keep constructing the sets G_j until the first index "j" where $\kappa_{h-j} < \kappa$. At this stage, the set $G := G_j$ satisfies $n_j(\omega \mid G) \leqslant \lambda^{s-\kappa}|P|$ for all $\omega \in G$ (so (5.3) is satisfied because $\mathbf{C}_j \geqslant \lambda^{-\epsilon}$), and since $|G| \geqslant 2^{-j}|\Omega| \geqslant 2^{-\lceil 20/\kappa \rceil}|\Omega| \sim_\kappa |\Omega|$, the proof is complete. (To be accurate, G is not quite yet a $(\delta, \Sigma, s, C_\kappa \lambda^{-\epsilon})$-configuration, but this can be fixed by a single application of Lemma 3.13).

In fact, we claim that case (1) cannot occur: more precisely, if $\epsilon = \epsilon(\kappa, s) > 0$ is as small as we declared in (5.5), then case (1) cannot occur for $\kappa_{h-j} \geqslant \kappa$. To prove this, we make a counter assumption: case (1) is reached at some index $j \in \{0, \ldots, h\}$ satisfying $\kappa_{h-j} \geqslant \kappa$. We write $\bar{\kappa} := \kappa_{h-j}$ and

$$\kappa_{h-(j+1)} =: \bar{\kappa} - \zeta, \qquad \text{where } \zeta \leqslant (\kappa s)/10 \leqslant (\bar{\kappa} s)/10. \tag{5.8}$$

We also set

$$\bar{G} := G_j \quad \text{and} \quad H := H_j = \{\omega \in \bar{G} : n_{j+1}(\omega \mid \bar{G}) \geqslant \lambda^{s-\bar{\kappa}+\zeta}|P|\},$$

so that $|H| \geqslant \frac{1}{2}|\bar{G}| \gtrsim_\kappa |\Omega|$ by the assumption that case (1) occurred. Finally, we will abbreviate

$$n := \lambda^{s-\bar{\kappa}+\zeta}|P| \tag{5.9}$$

in the sequel. Thus, to spell out the definitions, we have $H \subset \bar{G}$, and

$$|\{\omega' \in \bar{G}^{\mathbf{C}_{j+1}}_{\lambda,t}(\omega) : \mathbf{C}_{j+1}R^\lambda_\Sigma(\omega) \cap \mathbf{C}_{j+1}R^\lambda_\Sigma(\omega') \neq \emptyset\}| \geqslant n, \qquad \omega \in H. \tag{5.10}$$

On the other hand, by the definition of $\bar{G} = G_j$, and $\bar{\kappa} = \kappa_{h-j}$, we have

$$|\{\omega' \in \bar{G}^{\mathbf{C}_j}_{\lambda,t}(\omega) : \mathbf{C}_j R^\lambda_\Sigma(\omega) \cap \mathbf{C}_j R^\lambda_\Sigma(\omega') \neq \emptyset\}| \leqslant \lambda^{s-\bar{\kappa}}|P| = \lambda^{-\zeta} n, \qquad \omega \in \bar{G}. \tag{5.11}$$

We perform a small refinement to H. Note that

$$\sum_{p \in P} |H(p)| = |H| \gtrsim_\kappa |\Omega| = M_\Sigma |P|,$$

where as usual $H(p) = \{v \in E(p) : (p, v) \in H\}$. Consequently, there exists a subset $\bar{P} \subset P$ of cardinality $|\bar{P}| \gtrsim_\kappa |P|$ and a number $\bar{M}_\Sigma \gtrsim_\kappa M_\Sigma$ such that $|H(p)| \geqslant \bar{M}_\Sigma$ for all $p \in \bar{P}$. For each $p \in \bar{P}$, we further pick (arbitrarily) a subset $\bar{H}(p) \subset H(p)$ of cardinality precisely $|\bar{H}(p)| = \bar{M}_\Sigma$. Then, we define $\bar{H} := \{(p, v) : p \in \bar{P} \text{ and } v \in \bar{H}(p)\} \subset H$. Note that $|\bar{H}| \sim_\kappa |\Omega|$, and now $\bar{H}$ has the additional nice feature compared to H that

$$|\bar{H}(p)| = \bar{M}_\Sigma, \qquad p \in \bar{P}. \tag{5.12}$$

Let $\mathcal{B}$ be a cover of P by balls of radius $\frac{1}{4}t/\mathbf{C}_{j+1}$ such that even the concentric balls of radius $2t\mathbf{C}_{j+1}$ (that is, the balls $\{8\mathbf{C}^2_{j+1}B : B \in \mathcal{B}\}$) have overlap bounded by $\lambda^{-C(\kappa)\epsilon}$

(this is possible, since $\mathbf{C}_j \leqslant \lambda^{-C(\kappa)\epsilon}$ for all $1 \leqslant j \leqslant h$, recall (5.6)). Then, we choose the ball $B(p_0, \frac{1}{4}t/\mathbf{C}_{j+1}) \in \mathcal{B}$ in such a way that the ratio

$$\theta := \frac{|\bar{P} \cap B(p_0, \frac{1}{4}t/\mathbf{C}_{j+1})|}{|P \cap B(p_0, 2\mathbf{C}_{j+1}t)|}$$

is maximised. We claim that $\theta \geqslant \lambda^{C(\kappa)\epsilon}$: this follows from the estimate

$$|\bar{P}| \leqslant \sum_{B \in \mathcal{B}} |\bar{P} \cap B| \leqslant \theta \sum_{B \in \mathcal{B}} |P \cap 8\mathbf{C}_{j+1}^2 B| \leqslant \theta \lambda^{-C(\kappa)\epsilon} |P|,$$

and recalling that $|\bar{P}| \gtrsim_\kappa |P|$. Now, we set

$$W := \bar{P} \cap B(p_0, \tfrac{1}{4}t/\mathbf{C}_{j+1}) \quad \text{and} \quad B := P \cap B(p_0, 2\mathbf{C}_{j+1}t) \setminus B(p_0, \tfrac{1}{2}t/\mathbf{C}_{j+1}), \tag{5.13}$$

so that

$$|B| \leqslant |P \cap B(p_0, 2\mathbf{C}_{j+1}t)| = \theta^{-1}|W| \lesssim_\kappa \lambda^{-C(\kappa)\epsilon}|W|. \tag{5.14}$$

We also set

$$\mathbf{W} := \{(p, v) \in \bar{H} : p \in W\} \quad \text{and} \quad \mathbf{B} := \{(p, v) \in \bar{G} : p \in B\}.$$

Let us note that

$$|\mathbf{W}(p)| = |\{v \in E(p) : (p, v) \in \mathbf{W}\}| \geqslant |\bar{H}(p)| = \bar{M}_\Sigma \sim_\kappa M_\Sigma, \quad p \in W, \tag{5.15}$$

since $W \subset \bar{P}$, recall (5.12). We now claim that

$$\omega \in \mathbf{W} \quad \Longrightarrow \quad \bar{G}_{\lambda,t}^{\mathbf{C}_{j+1}}(\omega) \subset \mathbf{B}_{\lambda,t}^{\mathbf{C}_{j+1}}(\omega). \tag{5.16}$$

Indeed, fix $\omega = (p, v) \in \mathbf{W}$ and $(p', v') \in \bar{G}_{\lambda,t}^{\mathbf{C}_{j+1}}(\omega)$. We simply need to show that $p' \in B$, and this follows from $p \in W \subset B(p_0, \frac{1}{4}t/\mathbf{C}_{j+1})$, and $t/\mathbf{C}_{j+1} \leqslant |p - p'| \leqslant \mathbf{C}_{j+1}t$, and the triangle inequality:

$$\tfrac{3}{4}t/\mathbf{C}_{j+1} \leqslant |p - p'| - |p_0 - p| \leqslant |p_0 - p'| \leqslant |p_0 - p| + |p - p'| \leqslant 2\mathbf{C}_{j+1}t.$$

From (5.16), and since $\mathbf{W} \subset \bar{H} \subset H$, and recalling (5.10), it follows

$$|\{\beta \in \mathbf{B}_{\lambda,t}^{\mathbf{C}_{j+1}}(\omega) : \mathbf{C}_{j+1}R_\Sigma^\lambda(\omega) \cap \mathbf{C}_{j+1}R_\Sigma^\lambda(\beta) \neq \emptyset\}| \geqslant n > 0, \qquad \omega \in \mathbf{W}. \tag{5.17}$$

Next, we consider the rectangles

$$\mathcal{R}_\Sigma^\lambda := \{R_\Sigma^\lambda(\omega) : \omega \in \mathbf{W}\}.$$

To be precise, let $\mathcal{R}_\Sigma^\lambda$ be the maximal family of pairwise 100-incomparable (λ, Σ)-rectangles inside the family indicated above. Below, we will denote the 100-comparability of R, R' by $R \sim_{100} R'$. We now seek to show that every rectangle in $\mathcal{R}_\Sigma^\lambda$ has a high type relative to the pair (W, B), in the terminology of Definition 4.22.

To this end, we first define the quantity

$$m(R) = |\{\omega \in \mathbf{W} : R \sim_{100} R_\Sigma^\lambda(\omega)\}|, \tag{5.18}$$

The value of $m(R)$ may vary between 1 and $\leqslant \lambda^{-4}$, but by pigeonholing, we may find a subset $\bar{\mathcal{R}}_\Sigma^\lambda \subset \mathcal{R}_\Sigma^\lambda$ with the property $m(R) \equiv m \in [1, \lambda^{-4}]$ for all $R \in \bar{\mathcal{R}}_R^\lambda$, and moreover

$$\sum_{\omega \in \mathbf{W}} |\{R \in \bar{\mathcal{R}}_\Sigma^\lambda : R \sim_{100} R_\Sigma^\lambda(\omega)\}| \gtrapprox_\lambda \sum_{\omega \in \mathbf{W}} |\{R \in \mathcal{R}_\Sigma^\lambda : R \sim_{100} R_\Sigma^\lambda(\omega)\}|. \tag{5.19}$$

Now, we have

$$\begin{aligned}|\bar{\mathcal{R}}^{\lambda}_{\Sigma}| &= \frac{1}{m}\sum_{R\in\bar{\mathcal{R}}^{\lambda}_{\Sigma}}\sum_{p\in W}\sum_{\substack{v\in E(p)\\(p,v)\in\mathbf{W}}}\mathbf{1}_{\{R\sim_{100}R^{\lambda}_{\Sigma}(p,v)\}}\\ &\overset{(5.19)}{\gtrapprox_{\lambda}} \frac{1}{m}\sum_{p\in W}\sum_{\substack{v\in E(p)\\(p,v)\in\mathbf{W}}}|\{R\in\mathcal{R}^{\lambda}_{\Sigma} : R\sim_{100}R^{\lambda}_{\Sigma}(p,v)\}|\\ &\overset{(5.15)}{\geqslant} \frac{|W|\bar{M}_{\Sigma}}{m}\sim_{\kappa}\frac{|W|M_{\Sigma}}{m}. \end{aligned} \tag{5.20}$$

The second-to-last inequality is true because every rectangle $R^{\lambda}_{\Sigma}(p,v)$ with $(p,v)\in\mathbf{W}$ is 100-comparable to at least one rectangle in $\mathcal{R}^{\lambda}_{\Sigma}$, by definition of $\mathcal{R}^{\lambda}_{\Sigma}$.

5.0.1. *Proving that $m\lessapprox n$.* We next claim that

$$m(R)\leqslant\lambda^{-\zeta}n,\qquad R\in\mathcal{R}^{\lambda}_{\Sigma}, \tag{5.21}$$

where $n\geqslant 1$ was the constant defined in (5.9). In particular $m\leqslant\lambda^{-\zeta}n$. The estimate (5.21) will eventually follow from the inductive hypothesis (5.11), but the details take some work. Let $R^{\lambda}_{\Sigma}(\omega)\in\mathcal{R}^{\lambda}_{\Sigma}$, with $\omega=(p,v)\in\mathbf{W}$. According to (5.17), there exists at least one element $\beta=(q,w)\in\mathbf{B}^{\mathbf{C}_{j+1}}_{\lambda,t}(\omega)\subset\bar{G}$ such that

$$\mathbf{C}_{j+1}R^{\lambda}_{\Sigma}(\omega)\cap\mathbf{C}_{j+1}R^{\lambda}_{\Sigma}(\beta)\neq\emptyset. \tag{5.22}$$

We claim that if $\omega'=(p',v')\in\mathbf{W}$ is any element such that $R^{\lambda}_{\Sigma}(\omega)\sim_{100}R^{\lambda}_{\Sigma}(\omega')$, then automatically

$$\omega'\in\bar{G}^{\mathbf{C}_j}_{\lambda,t}(\beta)\quad\text{and}\quad\mathbf{C}_jR^{\lambda}_{\Sigma}(\omega')\cap\mathbf{C}_jR^{\lambda}_{\Sigma}(\beta)\neq\emptyset. \tag{5.23}$$

This will show that

$$m(R)\leqslant|\{\omega'\in\bar{G}^{\mathbf{C}_j}_{\lambda,t}(\beta):\mathbf{C}_jR^{\lambda}_{\Sigma}(\beta)\cap\mathbf{C}_jR^{\lambda}_{\Sigma}(\omega')\neq\emptyset\}|\overset{(5.11)}{\leqslant}\lambda^{-\zeta}n,$$

as desired. The points $\omega,\omega'\in\mathbf{W}$ and $\beta\in\mathbf{B}$, as above, will be fixed for the remainder of this subsection.

The second claim in (5.23) is easy: since $R^{\lambda}_{\Sigma}(\omega)\sim_{100}R^{\lambda}_{\Sigma}(\omega')$, it follows from Lemma 4.9 that $R^{\lambda}_{\Sigma}(\omega')\subset AR^{\lambda}_{\Sigma}(\omega)\subset\mathbf{C}_{j+1}R^{\lambda}_{\Sigma}(\omega)$ for a suitable absolute constant $A\geqslant 1$. Lemma 4.9 then yields

$$\mathbf{C}_{j+1}R^{\lambda}_{\Sigma}(\omega)\subset A\mathbf{C}^{5}_{j+1}R^{\lambda}_{\Sigma}(\omega')\subset\mathbf{C}_jR^{\lambda}_{\Sigma}(\omega'). \tag{5.24}$$

The second part of (5.23) follows from this inclusion, and (5.22).

We turn to the first claim in (5.23). Since $\omega'=(p',v')\in\mathbf{W}$ and $\beta=(q,w)\in\mathbf{B}$, we have $p'\in W$ and $q\in B$, so

$$t/\mathbf{C}_j\leqslant\tfrac{1}{4}t/\mathbf{C}_{j+1}\leqslant|p'-q|\leqslant 2\mathbf{C}_{j+1}t\leqslant\mathbf{C}_jt.$$

It therefore only remains to show that $\Delta(p',q)\leqslant\mathbf{C}_j\lambda$. To this end, recall that $\omega=(p,v)$. Then, since $\beta=(q,w)\in\mathbf{B}^{\mathbf{C}_{j+1}}_{\lambda,t}(\omega)$, we have

$$\bar{\lambda}:=\Delta(p,q)\leqslant\mathbf{C}_{j+1}\lambda\quad\text{and}\quad\bar{t}:=|p-q|\leqslant\mathbf{C}_{j+1}t.$$

Consequently,

$$\bar{\Sigma}:=\lambda/\sqrt{(\bar{\lambda}+\lambda)(\bar{t}+\lambda)}\gtrsim\mathbf{C}^{-1}_{j+1}\sqrt{\lambda/t}=\mathbf{C}^{-1}_{j+1}\Sigma,$$

and because of this,

$$A\mathbf{C}_{j+1}^2 R_\Sigma^\lambda(\omega) \cap A\mathbf{C}_{j+1}^2 R_\Sigma^\lambda(\beta) \supset \mathbf{C}_{j+1} R_\Sigma^\lambda(\omega) \cap \mathbf{C}_{j+1} R_\Sigma^\lambda(\beta) \overset{(5.22)}{\neq} \emptyset.$$

It now follows from Corollary 4.7 applied at scale λ and with constant $C = A\mathbf{C}_{j+1}^2$ that

$$R_\Sigma^\lambda(\omega) \subset \mathbf{C}'_{j+1} R_\Sigma^\lambda(\beta) \subset S^{\mathbf{C}'_{j+1}\lambda}(q), \tag{5.25}$$

for some $\mathbf{C}'_{j+1} \lesssim \mathbf{C}_{j+1}^8$. On the other hand, we saw in (5.24) that

$$R_\Sigma^\lambda(\omega) \subset A\mathbf{C}_{j+1}^5 R_\Sigma^\lambda(\omega') \subset S^{A\mathbf{C}_{j+1}^5}(p'),$$

and therefore $R_\Sigma^\lambda(\omega)$ is contained in the intersection $S^{\mathbf{C}_{j+1}^9\lambda}(q) \cap S^{\mathbf{C}_{j+1}^9}(p')$. But this intersection can be covered by boundedly many discs of radius $\mathbf{C}_{j+1}^9\lambda/\sqrt{\Delta(p',q)|p'-q|}$, which shows that

$$\sqrt{\frac{\lambda}{t}} = \Sigma \lesssim \frac{\mathbf{C}_{j+1}^{18}\lambda}{\sqrt{\Delta(p',q)|p'-q|}},$$

and rearranging this we find $\Delta(p',q) \lesssim \mathbf{C}_{j+1}^{12}\lambda$. This proves that $\Delta(p',q) \leqslant \mathbf{C}_j\lambda$, since we chose $\mathbf{C}_j = A\mathbf{C}_{j+1}^{18}$ above (5.6). We have now shown (5.23), and therefore (5.21).

5.0.2. *The type of rectangles in $\bar{\mathcal{R}}_\Sigma^\lambda$.* We claim that that every $R \in \bar{\mathcal{R}}_\Sigma^\lambda$ has type $(\geqslant \bar{m}, \geqslant \bar{n})_\rho$ relative to (W, B), where

$$\bar{m} := \lambda^\rho m, \quad \bar{n} \geqslant \lambda^\rho n, \quad \text{and} \quad \rho = 10 \cdot 18^{\lceil 20/\kappa \rceil}\epsilon. \tag{5.26}$$

Let us recall from Definition 4.22 what this means: a (λ, Σ)-rectangle R has type $(\geqslant \bar{m}, \geqslant \bar{n})_\rho$ relative to W, B if there exists at least $\bar{m}$ points $\{p_1, \ldots, p_{\bar{m}}\} \subset W$ and at least $\bar{n}$ points $\{q_1, \ldots, q_{\bar{n}}\} \subset B$ such that

$$R \subset S^{\lambda^{1-\rho}}(p_k) \cap S^{\lambda^{1-\rho}}(q_l), \qquad 1 \leqslant k \leqslant \bar{m},\ 1 \leqslant l \leqslant \bar{n}. \tag{5.27}$$

To see this, recall that $m(R) \equiv m$ for all $R \in \bar{\mathcal{R}}_\Sigma^\lambda$, where $m(R)$ was defined in (5.18): there exist m pairs $\{\omega_1, \ldots, \omega_m\} \subset \mathbf{W}$ such that $R \sim_{100} R_\Sigma^\lambda(\omega_j)$. Writing $\omega_k = (p_k, v_k)$, and using Lemma 4.9, this implies

$$R \subset AR_\Sigma^\lambda(\omega_k) \subset S^{\lambda^{1-\epsilon}}(p_k),$$

where $A \geqslant 1$ is absolute, and the second inclusion holds for $\lambda > 0$ small enough. This is even better than the first inclusion in (5.27). There is a small problem: some of the points "p_k" may be repeated, even though the pairs $\omega_k = (p_k, v_k) \in \mathbf{W}$ are distinct. However, for $p_k \in \mathbf{D}$ fixed, there are $\lesssim 1$ choices $v_k \in E(p)$ such that $R \subset AR_\Sigma^\lambda(p_k, v_k)$ (since $E(p)$ is Σ-separated), so the number of distinct points "p_k" is $\gtrsim m$, and certainly $\geqslant \bar{m}$.

The proof of the second inclusion in (5.27) is similar, but now based on (5.17): for all $R = R_\Sigma^\lambda(\omega) \in \mathcal{R}_\Sigma^\lambda$, there exist n pairs

$$\{\beta_1, \ldots, \beta_n\} \subset \mathbf{B}_{\lambda,t}^{\mathbf{C}_{j+1}}(\omega) \quad \text{s.t.} \quad \mathbf{C}_{j+1}R \cap \mathbf{C}_{j+1}R_\Sigma^\lambda(\beta_l) \neq \emptyset \text{ for } 1 \leqslant l \leqslant n.$$

If we write $\beta_l = (q_l, w_l)$, then the same argument which we used in (5.25) shows that

$$R_\Sigma^\lambda(\omega) \subset \mathbf{C}_{j+1}^9 R_\Sigma^\lambda(\beta_l) \subset S^{\mathbf{C}_{j+1}^9\lambda}(q_l) \subset S^{\lambda^{1-\rho}}(q_l), \qquad 1 \leqslant l \leqslant n, \tag{5.28}$$

using in the final inclusion that

$$\mathbf{C}_{j+1}^9 \leqslant \mathbf{C}_0^9 \overset{(5.6)}{\leqslant} A^{9\cdot 18^{\lceil 20/\kappa \rceil}} \cdot \lambda^{-9\cdot 18^{\lceil 20/\kappa \rceil}\epsilon} \overset{(5.26)}{\leqslant} \lambda^{-\rho},$$

assuming $\lambda > 0$ small enough (depending on ϵ, κ) in the final inequality. This proves the second inclusion in (5.27). Again, all the "n" points q_l need not be distinct, but for every fixed q_l, the first inclusion in (5.28) can hold for $\lesssim \mathbf{C}_{j+1}^5 \leqslant \lambda^{-\rho}$ choices of "w_l", so $|\{q_1, \ldots, q_l\}| \gtrsim \lambda^\rho n$, as desired. This completes the proof of (5.27).

5.0.3. *Applying Lemma 4.23.* To find a contradiction, and conclude the proof, we aim to apply Lemma 4.23 to bound the cardinality of $\bar{\mathcal{R}}_\Sigma^\lambda$ from above. Notice that, by the definition of W, B, see (5.13), the definition of "ρ" at (5.26), and since $4\mathbf{C}_{j+1} \leqslant \lambda^{-\rho}$, the pair (W, B) is (λ, ρ)-almost t-bipartite. In the previous section, we showed that every rectangle $R \in \bar{\mathcal{R}}_\Sigma^\lambda$ has type $(\geqslant \bar{m}, \geqslant \bar{n})_\rho$ relative to W, B. Therefore, Lemma 4.23 is applicable to $\bar{\mathcal{R}}_\Sigma^\lambda$. This yields the following inequality (see explanations below it):

$$\frac{|W| M_\Sigma}{m} \overset{(5.20)}{\lessapprox_\lambda} |\bar{\mathcal{R}}_\Sigma^\lambda| \lessapprox_\lambda \left(\frac{|B||W|}{mn}\right)^{3/4} + \frac{|B|}{m} + \frac{|W|}{n} \lessapprox_\lambda \left(\frac{|W|^2}{mn}\right)^{3/4} + \lambda^{-\zeta} \frac{|W|}{m}.$$

To make the estimate look neater, we allowed the "$\approx_\lambda$" notation hide constants of the form $\lambda^{-C(\kappa)\epsilon}$, where $C(\kappa) \geqslant 1$ is a constant depending only on κ. In the second inequality, we are hiding the constant $\lambda^{-C\rho} = \lambda^{-C(\kappa)\epsilon}$ produced by Lemma 4.23. In the third inequality, we are hiding the constant $\lambda^{-C(\kappa)\epsilon}$ produced by (5.14). The factor $\lambda^{-\zeta}$ in the second inequality appears from (5.21), and it is a good moment to recall from (5.8) that $\zeta \leqslant (\kappa s)/10$.

We observe immediately that the second term on the right cannot dominate the left hand side for $\epsilon = \epsilon(\kappa, s) > 0$ sufficiently small (the choice in (5.5) should suffice), and $\lambda = \lambda(\epsilon, \kappa, s) > 0$ sufficiently small: this is because $M_\Sigma \geqslant \lambda^\epsilon \Sigma^{-s} = \lambda^\epsilon \sqrt{\lambda/t}^{-s} \geqslant \lambda^{\epsilon - \kappa s/6} \geqslant \lambda^{-\kappa s/7}$ (using our assumption (5.4)), whereas $\lambda^{-\zeta} \leqslant \lambda^{-\kappa s/10}$ by (5.8).

Therefore, the term $|W|^{3/2}/(mn)^{3/4}$ needs to dominate the left hand side. Rearranging this inequality, using again $m \leqslant \lambda^{-\zeta} n$, recalling that $n = \lambda^{s - \bar{\kappa} + \zeta}|P|$, and finally using the $(\lambda, s, \lambda^{-\epsilon})$-set property of P to bound $|W| \leqslant \lambda^{-\epsilon} t^s |P|$ leads to

$$M_\Sigma \lessapprox_\lambda \lambda^{-\zeta/4} n^{-1/2} |W|^{1/2} \leqslant \lambda^{-s/2 - \zeta + \bar{\kappa}/2} \left(\frac{|W|}{|P|}\right)^{1/2} \lessapprox_\lambda \lambda^{-\zeta + \bar{\kappa}/2} \left(\frac{t}{\lambda}\right)^{s/2}.$$

This inequality is impossible for $\epsilon, \lambda > 0$ small enough depending on κ, since $\bar{\kappa} \geqslant \kappa$, and $\zeta \leqslant (\kappa s)/10 \leqslant \bar{\kappa}/10$ – and finally because $M_\Sigma \equiv |E(p)| \geqslant \lambda^\epsilon \Sigma^{-s} = \lambda^\epsilon (t/\lambda)^{s/2}$.

To summarise, we have now shown that the case (1) in the construction of the sequence $\{G_j\}$ cannot occur as long as long as $\kappa_{h-j} \geqslant \kappa$. As we explained below the case distinction, this allows us to set $G := G_j$ for the first index satisfying $\kappa_{h-j} < \kappa$. The proof of Proposition 5.2 is complete. □

We will use Proposition 5.2 via Theorem 5.31 below. First, as promised at the beginning of this section, we introduce *the partial multiplicity functions*. Compare these with the *total multiplicity function* from Definition 1.9.

Definition 5.29 (Partial multiplicity function)**.** Fix $0 < \delta \leqslant \Delta \leqslant \lambda \leqslant t \leqslant 1$ and $\rho \geqslant 1$. Let $P \subset \mathcal{D}_\delta$, and $E(p) \subset \mathcal{S}_\delta(p)$ for all $p \in P$. Write $\Omega = \{(p, v) : p \in P \text{ and } v \in E(p)\}$, and let $\sigma \in 2^{-\mathbb{N}}$ be the smallest dyadic rational larger than $\Delta/\sqrt{\lambda t}$. For $G \subset \Omega$, we define

$$m_{\Delta,\lambda,t}^{\rho,C}(\omega \mid G) := |\{\omega' \in (G_\sigma^\Delta)_{\lambda,t}^\rho(\omega) : CR_\sigma^\Delta(\omega) \cap CR_\sigma^\Delta(\omega') \neq \emptyset\}|, \qquad \omega \in G \cup G_\sigma^\Delta.$$

Here G_σ^Δ is the (Δ, σ)-skeleton of G.

Remark 5.30. The only interesting parameters "Δ" for us will be $\Delta = \delta$ and $\Delta = \lambda$. If $\Delta = \delta$, we will usually write $\sigma = \Delta/\sqrt{\lambda t} = \delta/\sqrt{\lambda t}$, and for $\Delta = \lambda$, we will instead use the capital letter $\Sigma = \Delta/\sqrt{\lambda t} = \sqrt{\lambda/t}$. Also, to be accurate, the notation σ, Σ typically refers to the smallest dyadic rational greater than $\delta/\sqrt{\lambda t}$ and $\sqrt{\lambda/t}$, respectively.

Finding a nice notation for the partial multiplicity functions is a challenge, due to the large number of parameters. In addition to the "range" and "constant" parameters ρ and C, one could add up to 4 further parameters: two "skeleton" parameters and two "rectangle" parameters. In practice, however, if the triple (Δ, λ, t) is given, the only useful rectangles are the $(\Delta, \Delta/\sqrt{\lambda t})$-rectangles. This relationship stems from Lemma 4.3. So, we have decided against introducing the fourth parameter independently.

Theorem 5.31. *For every $\kappa > 0$ and $s \in (0, 1]$, there exist $\epsilon_0 := \epsilon_0(\kappa, s) \in (0, \frac{1}{2}]$ and $\delta_0 = \delta_0(\epsilon, \kappa, s) > 0$ such that the following holds for all $\delta \in (0, \delta_0]$ and $\epsilon \in (0, \epsilon_0]$.*

Let Ω be a $(\delta, \delta, s, \delta^{-\epsilon})$-configuration with $P := \pi_{\mathbb{R}^3}(\Omega)$. Fix $\delta \leqslant \lambda \leqslant t \leqslant 1$. Then, there exists a $(\delta, \delta, s, C\delta^{-\epsilon})$-configuration $G \subset \Omega$ such that $C \approx_{\delta,\kappa} 1$, $|G| \approx_{\delta,\kappa} |\Omega|$, and

$$m_{\lambda,\lambda,t}^{\delta^{-\epsilon_0},\delta^{-\epsilon_0}}(\omega \mid G) \leqslant \delta^{-\kappa}\lambda^s|P|_\lambda, \qquad \omega \in G_\Sigma^\lambda. \tag{5.32}$$

Proof. Let us spell out what (5.32) means: for $\Sigma = \sqrt{\lambda/t}$, we should prove that

$$|\{\omega' \in (G_\Sigma^\lambda)_{\lambda,t}^{\delta^{-\epsilon_0}}(\omega) : \delta^{-\epsilon_0}R_\Sigma^\lambda(\omega) \cap \delta^{-\epsilon_0}R_\Sigma^\lambda(\omega') \neq \emptyset\}| \leqslant \delta^{-\kappa}\lambda^s|P|_\lambda, \qquad \omega \in G_\Sigma^\lambda.$$

We first dispose of the case where $\lambda \geqslant \delta^{\kappa/10}$. In this case we simply take $\epsilon_0 = \kappa/5$ and $G := \Omega$. Now, the left hand side of (5.32) is bounded from above by

$$|G_\Sigma^\lambda| \lesssim \lambda^{-3}\Sigma^{-1} \leqslant \lambda^{-4} \leqslant \delta^{-2\kappa/5} = \delta^{-3\kappa/5}\delta^{\epsilon_0} \leqslant \delta^{-3\kappa/5}\lambda^s|P|_\lambda,$$

using finally the assumption that P is a non-empty $(\delta, s, \delta^{-\epsilon_0})$-set.

Let us then assume that $\lambda \leqslant \delta^{\kappa/10}$. We apply Proposition 3.14 with $\sigma = \delta$ and $\Delta = \lambda$ and $\Sigma = \sqrt{\lambda/t} \geqslant \sigma$. This produces a subset $G_0 \subset \Omega$ of cardinality $|G_0| \approx_\delta |\Omega|$ whose (λ, Σ)-skeleton

$$(G_0)_\Sigma^\lambda = \{(\mathbf{p}, \mathbf{v}) : \mathbf{p} \in P_\lambda \text{ and } \mathbf{v} \in \mathbf{E}(\mathbf{p})\}$$

is a $(\lambda, \Sigma, s, C\delta^{-\epsilon})$-configuration with $C \approx_\delta 1$ (in particular, this skeleton is a $(\lambda, \Sigma, s, C\delta^{-\epsilon_0})$-configuration). Moreover, recall from (3.15) that

$$|\{(p, v) \in G_0 : (p, v) \prec (\mathbf{p}, \mathbf{v})\}| \approx_\delta \frac{|\Omega|}{|(G_0)_\Sigma^\lambda|}, \qquad (\mathbf{p}, \mathbf{v}) \in (G_0)_\Sigma^\lambda. \tag{5.33}$$

It may be worth emphasising a small technical point: we never claimed, and do not claim here either, that G_0 would be a $(\delta, \delta, s, C\delta^{-\epsilon})$-configuration.

Next, we apply Proposition 5.2 with constants "κ, s". This produces a constant $\epsilon_1 := \epsilon_1(\kappa, s) > 0$. Note that since $\lambda \leqslant \delta^{\kappa/10}$ by assumption, we have $C\delta^{-\epsilon_0} \leqslant \lambda^{-20\epsilon_0/\kappa}$ for $\delta > 0$ small enough. Therefore, if we choose "ϵ_0" presently so small that $20\epsilon_0/\kappa < \epsilon_1$, we see that $(G_0)_\Sigma^\lambda$ is a $(\lambda, \Sigma, s, \lambda^{-\epsilon_1})$-configuration. Now, by Proposition 5.2, there exists a $(\lambda, \Sigma, s, C_\kappa\lambda^{-\epsilon_1})$-configuration $\mathbf{G} \subset (G_0)_\Sigma^\lambda$ with $|\mathbf{G}| \sim_\kappa |(G_0)_\Sigma^\lambda|$, and the property

$$|\{\omega' \in \mathbf{G}_{\lambda,t}^{\lambda^{-\epsilon_1}}(\omega) : \lambda^{-\epsilon_1}R_\Sigma^\lambda(\omega) \cap \lambda^{-\epsilon_1}R_\Sigma^\lambda(\omega') \neq \emptyset\}| \leqslant \lambda^{s-\kappa}|P_\lambda|, \qquad \omega \in \mathbf{G}. \tag{5.34}$$

Note that $\delta^{-\epsilon_0} \leqslant \lambda^{-\epsilon_1}$ by our choices of constants, and $\lambda \geqslant \delta$, so (5.34) implies

$$|\{\omega' \in \mathbf{G}_{\lambda,t}^{\delta^{-\epsilon_0}}(\omega) : \delta^{-\epsilon_0}R_\Sigma^\lambda(\omega) \cap \delta^{-\epsilon_0}R_\Sigma^\lambda(\omega') \neq \emptyset\}| \leqslant \delta^{-\kappa}\lambda^s|P|_\lambda, \qquad \omega \in \mathbf{G}. \tag{5.35}$$

We also used that $|P_\lambda| \leqslant |P|_\lambda$. Next, let

$$G_1 := \bigcup_{(\mathbf{p},\mathbf{v})\in\mathbf{G}} \{(p,v) \in G_0 : (p,q) \prec (\mathbf{p},\mathbf{v})\} = \bigcup_{(\mathbf{p},\mathbf{v})\in\mathbf{G}} G_0 \cap (\mathbf{p} \otimes \mathbf{v}).$$

Then $(G_1)^\lambda_\Sigma \subset \mathbf{G}$ by definition, so (5.35) implies (5.32) for G_1. Moreover, as explained in Remark 3.16, the sets $\mathbf{p} \otimes \mathbf{v}$ are disjoint, so

$$|G_1| = \sum_{(\mathbf{p},\mathbf{v})\in\mathbf{G}} |G_0 \cap (\mathbf{p} \otimes \mathbf{v})| \overset{(5.33)}{\approx_\delta} |\mathbf{G}| \cdot \frac{|\Omega|}{|(G_0)^\lambda_\Sigma|} \sim_\kappa |\Omega|.$$

The only problem remaining is that G_1 may not be a $(\delta, \delta, s, C\delta^{-\epsilon})$-configuration. However, $|G_1| \approx_{\delta,\kappa} |\Omega|$, so it follows from the refinement principle (Lemma 3.13) that there exists a $(\delta, \delta, s, C\delta^{-\epsilon})$-configuration $G \subset G_1$ such that $C \approx_{\delta,\kappa} 1$ and $|G| \approx_{\delta,\kappa} |\Omega|$. Now, G continues to satisfy (5.32), so the proof of Theorem 5.31 is complete. □

Remark 5.36. It may be worth remarking that if G is the final (δ, δ, s)-configuration in the previous theorem, the (λ, Σ)-skeleton G^λ_Σ may fail to be a (λ, Σ, s)-configuration. This was not claimed either. It seems generally tricky to ensure that a set $G \subset \Omega$ is simultaneously a (δ, σ, s)-configuration and a (Δ, Σ, s)-configuration for $\delta \ll \Delta$ and $\sigma \ll \Sigma$.

6. AN UPPER BOUND FOR INCOMPARABLE (δ, σ)-RECTANGLES

Notation 6.1. Let $0 < \delta \leqslant \Delta \leqslant 1$ and $0 < \sigma \leqslant \Sigma \leqslant 1$. Let $p \in \mathcal{D}_\delta$, and let $E_\sigma(p) \subset \mathcal{S}_\sigma(p)$ (recall that the notation $S_\delta(p)$ refers to a circle associated to the centre of p). We write

$$\mathcal{E}^\Delta_\Sigma(p) := \bigcup_{\mathbf{v}\in E_\Sigma(p)} R^\Delta_\Sigma(\mathbf{p},\mathbf{v}) \subset S^\Delta(\mathbf{p}),$$

where $\mathbf{p} \in \mathcal{D}_\Delta$ is the unique dyadic Δ-cube with $p \subset \mathbf{p}$, and $E_\Sigma(p)$ is the (Δ, Σ)-skeleton of $E_\sigma(p)$, namely $E_\Sigma(p) = \{\mathbf{v} \in \mathcal{S}_\Sigma(\mathbf{p}) : v \prec \mathbf{v} \text{ for some } v \in E_\sigma(p)\}$.

Lemma 6.2. *Let $C \geqslant 1$, $0 < \delta \leqslant \Delta \leqslant 1$, $0 < \sigma \leqslant \Sigma \leqslant 1$. Assume also that $\Delta \leqslant \Sigma$. Let $p \in \mathcal{D}_\delta$, and let $E_\sigma(p) \subset \mathcal{S}_\sigma(p)$. Then $C\mathcal{E}^\delta_\sigma(p) \subset C'\mathcal{E}^\Delta_\Sigma(p)$ for some $C' \sim C$.*

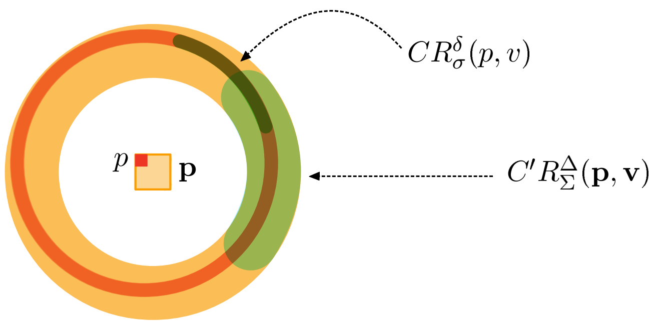

FIGURE 3. The rectangles $CR^\delta_\sigma(p,v)$ and $C'R^\Delta_\Sigma(\mathbf{p},\mathbf{v})$ in the proof of Lemma 6.2.

Proof. The proof is illustrated in Figure 3. The set $\mathcal{E}^{\delta}_{\sigma}(p)$ is a union of the (δ,σ)-rectangles $R^{\delta}_{\sigma}(p,v)$ centred at $v \in E_{\sigma}(p)$. Let $R = R^{\delta}_{\sigma}(p,v)$ be one of these rectangles. By the definition of (Δ,Σ)-skeleton, there exists $\mathbf{v} \in E_{\Sigma}(p)$ such that $v \prec \mathbf{v}$, or in other words $p \subset \mathbf{p} \in \mathcal{D}_{\Delta}$ and $v \cap V(\mathbf{p},\mathbf{v}) \neq \emptyset$. Since $|p - \mathbf{p}| \leqslant 2\Delta$ and $\delta \leqslant \Delta$, we have

$$S^{C\delta}(p) \subset S^{2C\Delta}(\mathbf{p}).$$

Moreover, it follows from $v \cap V(\mathbf{p},\mathbf{v}) \neq \emptyset$ and $\Delta \leqslant \Sigma$ that $|v - \mathbf{v}| \leqslant C'\Sigma$, where $C' > 0$ is absolute. Consequently,

$$B(v, C\sigma) \subset B(v, C\Sigma) \subset B(\mathbf{v}, (C + C')\Sigma).$$

Combining this information, we have

$$CR^{\delta}_{\sigma}(p,v) = S^{C\delta}(p) \cap B(v, C\sigma) \subset S^{2C\Delta}(\mathbf{p}) \cap B(\mathbf{v}, (C + C')\Sigma) = C''R^{\Delta}_{\Sigma}(\mathbf{p},\mathbf{v}),$$

with $C'' := \max\{2C, C + C'\}$. This completes the proof. □

We next define a variant of the "type" introduced in Definition 4.22.

Definition 6.3. Let $0 < \delta \leqslant \sigma \leqslant 1$, and let $P \subset \mathcal{D}_{\delta}$. For every $p \in P$, let $E(p) \subset \mathcal{S}_{\delta}(p)$. Let $W, B \subset P$ be finite sets. For $\delta \leqslant \lambda \leqslant 1$, and $m, n \geqslant 1$, we say that a (δ,σ)-rectangle $R \subset \mathbb{R}^2$ has *λ-restricted type $(\geqslant m, \geqslant n)_{\epsilon}$ relative to* $(W, B, \{E(p)\})$ if there exists a set $W_R \subset W$ of cardinality $|W_R| \geqslant m$, and for every $p \in W_R$ a subset $B_R(p) \subset B$ of cardinality $|B_R(p)| \geqslant n$ such that the following holds:

(1) $\delta^{\epsilon}\lambda \leqslant \Delta(p,q) \leqslant \delta^{-\epsilon}\lambda$ for all $p \in W_R$ and all $q \in B_R(p)$.
(2) $R \subset \delta^{-\epsilon}\mathcal{E}^{\delta}_{\sigma}(p) \cap \delta^{-\epsilon}\mathcal{E}^{\delta}_{\sigma}(q)$ for all $p \in W_R$ and all $q \in B_R(p)$.

If $\lambda = \delta$, then the requirement in (1) is relaxed to $\Delta(p,q) \leqslant \delta^{1-\epsilon}$.

Remark 6.4. The presence of the sets $E(p)$ is a major difference compared to Definition 4.22, and we will distinguish between these two definitions by using the terminology "...relative to (W, B)" in Definition 4.22, and "...relative to $(W, B, \{E(p)\})$ in Definition 6.3. We will make sure that there is never a risk of confusion which definition is meant.

Other differences are (obviously) the condition (1) of Definition 6.3, which is completely absent from Definition 4.22. A more subtle point is the asymmetry of Definition 6.3: even if a rectangle has λ-restricted type $(\geqslant m, \geqslant m)_{\epsilon}$ relative to $(W, B, \{E(p)\})$, it need not have λ-restricted type $(\geqslant m, \geqslant m)_{\epsilon}$ relative to $(B, W, \{E(p)\})$.

Theorem 6.5. *For every $\eta > 0$, there exist $\epsilon = \epsilon(\eta) \in (0,1]$ and $\delta_0 = \delta_0(\eta,\epsilon) \in (0,1]$ such that the following holds for all $\delta \in (0,\delta_0]$. Let $0 < \delta \leqslant \lambda \leqslant t \leqslant 1$ be dyadic rationals with $\lambda \leqslant \delta^{2\epsilon}t$. Let $P \subset \mathcal{D}_{\delta}$ be a set satisfying*

$$|P \cap \mathbf{p}| \leqslant X_{\lambda}, \qquad \mathbf{p} \in \mathcal{D}_{\lambda}, \tag{6.6}$$

where $X_{\lambda} \in \mathbb{N}$. For every $p \in P$, let $E(p) \subset \mathcal{S}_{\delta}(p)$. Write $\Sigma := \sqrt{\lambda/t}$, and let $\Omega^{\lambda}_{\Sigma}$ be the (λ,Σ)-skeleton of $\Omega = \{(p,v) : p \in P \text{ and } v \in E(p)\}$. Assume that, for some $Y_{\lambda} \in \mathbb{N}$,

$$m^{\delta^{-\mathbf{A}\epsilon},\delta^{-\mathbf{A}\epsilon}}_{\lambda,\lambda,t}(\omega \mid \Omega) \leqslant Y_{\lambda}, \qquad \omega \in \Omega^{\lambda}_{\Sigma}, \tag{6.7}$$

where $\mathbf{A} \geqslant 1$ is a sufficiently large absolute constant, in particular $\mathbf{A}$ is independent of the previous parameters δ, η, λ, t. Write $\sigma := \delta/\sqrt{\lambda t}$, and let W, B be a (δ,ϵ)-almost t-bipartite pair *of subsets of P. Let $1 \leqslant m \leqslant |W|$ and $1 \leqslant n \leqslant |B|$. Let $\mathcal{R}^\delta_\sigma$ be a collection of pairwise* 100-*incomparable (δ, σ)-rectangles whose λ-restricted type relative to $(W, B, \{E(p)\})$ is $(\geqslant m, \geqslant n)_\epsilon$. Then,*

$$|\mathcal{R}^\delta_\sigma| \leqslant \delta^{-\eta} \left[\left(\frac{|W||B|}{mn} \right)^{3/4} (X_\lambda Y_\lambda)^{1/2} + \frac{|W|}{m} \cdot X_\lambda Y_\lambda + \frac{|B|}{n} \cdot X_\lambda Y_\lambda \right]. \tag{6.8}$$

Remark 6.9. It is worth noting that the upper bound (6.7) which is **assumed** here looks exactly like the upper bound provided by Theorem 5.31.

Another remark is that (6.8) in the case $\lambda = \delta$ may actually be weaker than Wolff's tangency bound (4.24). In this case evidently $X_\lambda \leqslant 1$, but it may well happen that $Y_\lambda \gg 1$. This is irrelevant for our purposes, since Theorem 6.5 will only be applied in a situation where $Y_\lambda \lesssim 1$. For the interested reader, we mention that the main loss in the proof arises from the estimate (6.31), which is always unsharp if $M_\lambda N_\lambda \cdot (m_\lambda n_\lambda) \gg (\lambda/\delta)^2$.

Proof of Theorem 6.5. We start with the case $m = 1 = n$, and later deal with the general case with a "random sampling" argument. Fix $\eta > 0$. We also choose $\epsilon > 0$ so small that $\sqrt{\epsilon} < c\eta$ for a suitable absolute constant to be determined later (this constant will be determined by the constant in Lemma 4.23).

In this proof, "C" will refer to an absolute constant whose value may change – usually increase – from one line to the next without separate remark. We will also assume, when needed, that "$\delta > 0$ is small enough" without separate remark.

We may assume with no loss of generality that the rectangles in $\mathcal{R}^\delta_\sigma$ are pairwise $\delta^{-C\epsilon}$-incomparable for a suitable absolute constant $C > 0$, instead of just 100-incomparable. This is because by Corollary 4.13, any collection of 100-incomparable rectangles $\mathcal{R}^\delta_\sigma$ contains a $\delta^{-C\epsilon}$-incomparable subset $\bar{\mathcal{R}}^\delta_\sigma$ of cardinality $|\bar{\mathcal{R}}^\delta_\sigma| \geqslant \delta^{O(C\epsilon)}|\mathcal{R}^\delta_\sigma|$, and now it suffices to prove (6.8) for $\bar{\mathcal{R}}^\delta_\sigma$.

By assumption, every rectangle $R \in \mathcal{R}^\delta_\sigma$ has λ-restricted type $(\geqslant 1, \geqslant 1)_\epsilon$ relative to (W, B). Thus, for every $R \in \mathcal{R}^\delta_\sigma$ we may associate a pair $(p, q)_R \in W \times B$ with the properties

$$\delta^\epsilon \lambda \leqslant \Delta(p, q) \leqslant \delta^{-\epsilon}\lambda \quad \text{and} \quad R \subset \delta^{-\epsilon}\mathcal{E}^\delta_\sigma(p) \cap \delta^{-\epsilon}\mathcal{E}^\delta_\sigma(q) \subset S^{\delta^{1-\epsilon}}(p) \cap S^{\delta^{1-\epsilon}}(q). \tag{6.10}$$

(If $\lambda = \delta$, we only have $\Delta(p, q) \leqslant \delta^{1-\epsilon}$.) We record at this point that any fixed pair $(p, q) \in W \times B$ can only be associated to boundedly many rectangles $R \in \mathcal{R}^\delta_\sigma$:

$$|\{R \in \mathcal{R}^\delta_\sigma : (p, q)_R = (p, q)\}| \lesssim 1, \qquad (p, q) \in W \times B. \tag{6.11}$$

Indeed, if there exists at least one rectangle $R_0 \in \mathcal{R}^\delta_\sigma$ such that $(p, q)_{R_0} = (p, q)$, then $|p - q| \geqslant \delta^\epsilon t$ and $\Delta(p, q) \geqslant \delta^\epsilon \lambda$. Under these conditions, Lemma 4.3 implies that the intersection $S^{\delta^{1-\epsilon}}(p) \cap S^{\delta^{1-\epsilon}}(q)$ can be covered by boundedly many $(\delta^{1-C\epsilon}, \delta^{1-C\epsilon}/\sqrt{\lambda t})$-rectangles, and actually they can be selected to be of the form

$$\delta^{-C\epsilon}R_j := \delta^{-C\epsilon}R^\delta_\sigma(q, v_j), \qquad 1 \leqslant j \lesssim 1,$$

where each R_j is a (δ, σ)-rectangle. (We note that this is true also if $\lambda = \delta$, using only $|p - q| \geqslant \delta^\epsilon t$ in that case.) We claim that each rectangle $R \in \mathcal{R}^\delta_\sigma$ with $(p, q)_R = (p, q)$ is $\delta^{-C\epsilon}$-comparable to one of the rectangles R_j. This will imply (6.11), because at most one rectangle in $\mathcal{R}^\delta_\sigma$ can be $\delta^{-C\epsilon}$-comparable to a fixed R_j: indeed any pair of (δ, σ)-rectangles $\delta^{-C\epsilon}$-comparable to R_j would be $\lesssim \delta^{-C\epsilon}$-comparable to each other by Corollary 4.11, contradicting our "without loss of generality" assumption that the rectangles in $\mathcal{R}^\delta_\sigma$ are

$\delta^{-C\epsilon}$-incomparable. Thus, the left hand side of (6.11) is bounded by the number of the rectangles R_j (which is $\lesssim 1$).

We then show that every $R \in \mathcal{R}^\delta_\sigma$ with $(p,q)_R = (p,q)$ is $\delta^{-C\epsilon}$-comparable to some R_j. Namely, if $(p,q)_R = (p,q)$, then $R \subset S^{\delta^{1-\epsilon}}(p) \cap S^{\delta^{1-\epsilon}}(q)$ by definition, and because $\mathrm{diam}(R) \leqslant 2\sigma$, it follows that $R \subset \delta^{-\epsilon} R^\delta_\sigma(q,v)$ for some $v \in S(q)$ (here e.g. v is the closest point on $S(q)$ from the centre of R). On the other hand, since the rectangles $\delta^{-C\epsilon} R_j = \delta^{-C\epsilon} R^\delta_\sigma(q,v_j)$ cover $S^{\delta^{1-\epsilon}}(p) \cap S^{\delta^{1-\epsilon}}(q)$, one of them intersects R, say $R \cap \delta^{-C\epsilon} R_j \neq \emptyset$. Now, it is easy to check that

$$R_j \subset \delta^{-C\epsilon} R^\delta_\sigma(q,v),$$

and therefore $R, R_j \subset \delta^{-C\epsilon} R^\delta_\sigma(q,v)$. In other words, R, R_j are $\delta^{-C\epsilon}$-comparable.

With the proof of (6.11) behind us, we proceed with other preliminaries. Let

$$\mathbf{p} \in \mathcal{D}_\lambda(W) =: \mathcal{W}_\lambda \quad \text{and} \quad \mathbf{q} \in \mathcal{D}_\lambda(B) =: \mathcal{B}_\lambda,$$

and write

$$\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q}) := \{R \in \mathcal{R}^\delta_\sigma : (p,q)_R \in (W \cap \mathbf{p}) \times (B \cap \mathbf{q})\}.$$

With this notation, we have

$$|R^\delta_\sigma| \leqslant \sum_{(\mathbf{p},\mathbf{q}) \in \mathcal{W}_\lambda \times \mathcal{B}_\lambda} |\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q})|. \tag{6.12}$$

We use the pigeonhole principle to find subsets

$$\overline{\mathcal{W}}_\lambda \subset \mathcal{W}_\lambda \quad \text{and} \quad \overline{\mathcal{B}}_\lambda \subset \mathcal{B}_\lambda$$

with the properties

$$\begin{cases} |W \cap \mathbf{p}| \sim M_\lambda, & \mathbf{p} \in \overline{\mathcal{W}}_\lambda, \\ |B \cap \mathbf{q}| \sim N_\lambda, & \mathbf{q} \in \overline{\mathcal{B}}_\lambda, \end{cases} \tag{6.13}$$

(where $M_\lambda, N_\lambda \in \{1, \ldots, X_\lambda\}$ are fixed integers) and such that

$$|\mathcal{R}^\delta_\sigma| \overset{(6.12)}{\leqslant} \sum_{(\mathbf{p},\mathbf{q}) \in \mathcal{W}_\lambda \times \mathcal{B}_\lambda} |\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q})| \approx_\delta \sum_{(\mathbf{p},\mathbf{q}) \in \overline{\mathcal{W}}_\lambda \times \overline{\mathcal{B}}_\lambda} |\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q})|. \tag{6.14}$$

It now suffices to show that

$$\sum_{(\mathbf{p},\mathbf{q}) \in \overline{\mathcal{W}}_\lambda \times \overline{\mathcal{B}}_\lambda} |\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q})| \lessapprox_\delta (|W||B|)^{3/4}(X_\lambda Y_\lambda)^{1/2} + |W|(X_\lambda Y_\lambda) + |B|(X_\lambda Y_\lambda). \tag{6.15}$$

To begin with, we claim that

$$|\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q})| \lesssim M_\lambda N_\lambda, \qquad \mathbf{p} \in \overline{\mathcal{W}}_\lambda,\ \mathbf{q} \in \overline{\mathcal{B}}_\lambda. \tag{6.16}$$

This follows from

$$|\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q})| = \sum_{(p,q) \in (W \cap \mathbf{p}) \times (B \cap \mathbf{q})} |\{R \in \mathcal{R}^\delta_\sigma : (p,q)_R = (p,q)\}|,$$

and the fact recorded in (6.11) that every term in this sum is $\lesssim 1$.

To proceed estimating (6.14), notice that we only need to sum over the pairs $(\mathbf{p},\mathbf{q}) \in \overline{\mathcal{W}}_\lambda \times \overline{\mathcal{B}}_\lambda$ with $\mathcal{R}^\delta_\sigma(\mathbf{p},\mathbf{q}) \neq \emptyset$. In this case there exists at least one pair $p \in W \cap \mathbf{p}$

and $q \in B \cap \mathbf{q}$ satisfying (6.10). It follows from Lemma 6.2 applied with $\Delta = \lambda$ and $\Sigma := \sqrt{\lambda/t} \geqslant \max\{\lambda, \sigma\}$ that

$$\delta^{2\epsilon}t \leqslant |\mathbf{p}-\mathbf{q}| \leqslant \delta^{-2\epsilon}t, \quad \Delta(\mathbf{p},\mathbf{q}) \leqslant \delta^{-2\epsilon}\lambda, \quad \text{and} \quad \delta^{-2\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}) \cap \delta^{-2\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{q}) \neq \emptyset. \tag{6.17}$$

Here the bounds $|\mathbf{p}-\mathbf{q}| \geqslant \delta^{2\epsilon}t$ and $\Delta(\mathbf{p},\mathbf{q}) \leqslant \delta^{-2\epsilon}\lambda$ used our assumption $\lambda \leqslant \delta^{2\epsilon}t$ (and that $|p-q| \geqslant \delta^{\epsilon}t$ for some pair $p \in \mathbf{p}$ and $q \in \mathbf{q}$). To spell out the definitions, here

$$\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}) = \bigcup_{\mathbf{v} \in E_{\Sigma}(\mathbf{p})} R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{v}),$$

where $E_{\Sigma}(\mathbf{p})$ is the (λ,Σ)-skeleton of $E(p)$ (for $p \in P \cap \mathbf{p}$). We record at this point that

$$\mathbf{p} \in \mathcal{W}_{\lambda} \cup \mathcal{B}_{\lambda} \text{ and } \mathbf{v} \in E_{\Sigma}(\mathbf{p}) \quad \Longrightarrow \quad (\mathbf{p},\mathbf{v}) \in \Omega^{\lambda}_{\Sigma},$$

where $\Omega^{\lambda}_{\Sigma}$ is the (Σ,λ)-skeleton of Ω.

We write $\mathbf{p} \sim \mathbf{q}$ if $\mathbf{p} \in \overline{\mathcal{W}}_{\lambda}$, $\mathbf{q} \in \overline{\mathcal{B}}_{\lambda}$, and the conditions (6.17) hold. Then, by (6.14) and the preceding discussion

$$|\mathcal{R}^{\delta}_{\sigma}| \lessapprox \sum_{\mathbf{p}\sim\mathbf{q}} |\mathcal{R}^{\delta}_{\sigma}(\mathbf{p},\mathbf{q})| \overset{(6.16)}{\lessapprox} M_{\lambda}N_{\lambda} \cdot |\{(\mathbf{p},\mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda} : \mathbf{p} \sim \mathbf{q}\}|. \tag{6.18}$$

To estimate the cardinality $|\{(\mathbf{p},\mathbf{q}) : \mathbf{p} \sim \mathbf{q}\}|$, we will infer from (6.17) that whenever $\mathbf{p} \sim \mathbf{q}$, then $S(\mathbf{p})$ and $S(\mathbf{q})$ are "roughly" tangent to a (λ,Σ)-rectangle, denoted $R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{q})$, more precisely satisfying

$$R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{q}) \subset \delta^{-C\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}) \cap \delta^{-C\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{q}) \tag{6.19}$$

for a suitable absolute constant $C \geqslant 1$. Let us justify why $R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{q})$ can be found. Since $\delta^{-2\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}) \cap \delta^{-2\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{q}) \neq \emptyset$, there first of all exist $\mathbf{v} \in E_{\Sigma}(\mathbf{p})$, $\mathbf{w} \in E_{\Sigma}(\mathbf{q})$, and a point

$$v \in \delta^{-2\epsilon}R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{v}) \cap \delta^{-2\epsilon}R^{\lambda}_{\Sigma}(\mathbf{q},\mathbf{w}).$$

Consequently, we may find points $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ with $|\bar{\mathbf{p}}-\mathbf{p}| \leqslant \delta^{-2\epsilon}\lambda$ and $|\bar{\mathbf{q}}-\mathbf{q}| \leqslant \delta^{-2\epsilon}\lambda$ such that $v \in S(\bar{\mathbf{p}}) \cap S(\bar{\mathbf{q}})$. Since $\Sigma \geqslant \lambda$, we also have

$$\max\{\mathrm{dist}(v, E_{\Sigma}(\mathbf{p})), \mathrm{dist}(v, E_{\Sigma}(\mathbf{q}))\} \leqslant \delta^{-2\epsilon}\Sigma. \tag{6.20}$$

Now, it follows from (6.17) and the inclusion (4.4) (and noting that $\Sigma \leqslant \delta^{-\epsilon}\sqrt{\lambda/|\mathbf{p}-\mathbf{q}|}$) that

$$R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{q}) := R^{\lambda}_{\Sigma}(\bar{\mathbf{p}},v) \subset S^{\delta^{-C\epsilon}\lambda}(\bar{\mathbf{p}}) \cap S^{\delta^{-C\epsilon}\lambda}(\bar{\mathbf{q}}).$$

Taking also into account (6.20), we arrive at (6.19).

Now that we have defined the (λ,Σ)-rectangles $R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{q})$, we let $\mathcal{R}^{\lambda}_{\Sigma}$ be a maximal collection of pairwise 100-incomparable rectangles in $\{\mathcal{R}^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{q}) : \mathbf{p} \in \overline{\mathcal{W}}_{\lambda}, \mathbf{q} \in \overline{\mathcal{B}}_{\lambda} \text{ and } \mathbf{p} \sim \mathbf{q}\}$. For $R \in \mathcal{R}^{\lambda}_{\Sigma}$, we then write $R \sim (\mathbf{p},\mathbf{q})$ if $\mathbf{p} \sim \mathbf{q}$ and $R \sim_{100} R^{\lambda}_{\Sigma}(\mathbf{p},\mathbf{q})$. With this notation, we may estimate

$$|\{(\mathbf{p},\mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda} : \mathbf{p} \sim \mathbf{q}\}| \leqslant \sum_{R \in \mathcal{R}^{\lambda}_{\Sigma}} |\{(\mathbf{p},\mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda} : R \sim (\mathbf{p},\mathbf{q})\}|, \tag{6.21}$$

since every pair $(\mathbf{p},\mathbf{q})$ with $\mathbf{p} \sim \mathbf{q}$ satisfies $R \sim (\mathbf{p},\mathbf{q})$ for at least one rectangle $R \in \mathcal{R}^{\lambda}_{\Sigma}$.

To estimate (6.21) further, we consider the following slightly *ad hoc* "type" of the rectangles $R \in \mathcal{R}^{\lambda}_{\Sigma}$ relative to the pair $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$. (This notion will not appear outside this proof.) We say that $R \in \mathcal{R}^{\lambda}_{\Sigma}$ has *type* $(m_{\lambda}, n_{\lambda})$ *relative to* $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$ if the following sets $\mathcal{W}_{\lambda}(R) \subset \overline{\mathcal{W}}_{\lambda}$ and $\mathcal{B}_{\lambda}(R) \subset \overline{\mathcal{B}}_{\lambda}$ have cardinalities $|\mathcal{W}_{\lambda}(R)| = m_{\lambda}$ and $|\mathcal{B}_{\lambda}(R)| = n_{\lambda}$:

- $\mathcal{W}_{\lambda}(R)$ consists of all $\mathbf{p} \in \overline{\mathcal{W}}_{\lambda}$ such that $R \subset \delta^{-\mathbf{C}\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p})$.
- $\mathcal{B}_{\lambda}(R)$ consists of all $\mathbf{q} \in \overline{\mathcal{B}}_{\lambda}$ such that $R \subset \delta^{-\mathbf{C}\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{q})$.

Here

$$\mathbf{C} := 2C \geqslant 2, \tag{6.22}$$

where "C" is the absolute constant from (6.19). We observe at once that the type of every rectangle $R \in \mathcal{R}^{\lambda}_{\Sigma}$ is $(\geqslant 1, \geqslant 1)$ in this terminology, because each $R \in \mathcal{R}^{\lambda}_{\Sigma}$ has the form $R = R^{\lambda}_{\Sigma}(\mathbf{p}, \mathbf{q})$ for some $(\mathbf{p}, \mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda}$, and (6.19) holds for this pair $(\mathbf{p}, \mathbf{q})$.

Remark 6.23. Assume for a moment that $\lambda \leqslant \delta^{\sqrt{\epsilon}}$. Then, if $R \in \mathcal{R}^{\lambda}_{\Sigma}$ has type $(\geqslant m_{\lambda}, n_{\lambda})$ relative to $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$ according to the definition above, then R also has type $(\geqslant m_{\lambda}, n_{\lambda})_{\mathbf{C}\sqrt{\epsilon}}$ relative to $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$ in the sense of Definition 4.22. This is simply because

$$\delta^{-\mathbf{C}\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}) \subset S^{\delta^{-\mathbf{C}\epsilon}\lambda}(p),$$

and $\delta^{-\mathbf{C}\epsilon} \leqslant \lambda^{-\mathbf{C}\sqrt{\epsilon}}$ by the temporary assumption $\lambda \leqslant \delta^{\sqrt{\epsilon}}$. But the "ad hoc" definition here is far more restrictive: it requires R to lie close to the sets $E_{\Sigma}(\mathbf{p})$ and $E_{\Sigma}(\mathbf{q})$.

We now establish two claims related to our *ad hoc* notion of type:

Claim 6.24. *If $R \in \mathcal{R}^{\lambda}_{\Sigma}$ has type $(m_{\lambda}, n_{\lambda})$ relative to $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$, then*

$$|\{(\mathbf{p}, \mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda} : R \sim (\mathbf{p}, \mathbf{q})\}| \leqslant m_{\lambda} n_{\lambda}. \tag{6.25}$$

Proof. Let $\mathcal{W}'_{\lambda}(R) \subset \overline{\mathcal{W}}_{\lambda}$ be the subset of all those $\mathbf{p} \in \overline{\mathcal{W}}_{\lambda}$ such that $R \sim (\mathbf{p}, \mathbf{q})$ for at least one $\mathbf{q} \in \overline{\mathcal{B}}_{\lambda}$. Define $\mathcal{B}'_{\lambda}(R)$ similarly, interchanging the roles of $\overline{\mathcal{W}}_{\lambda}$ and $\overline{\mathcal{B}}_{\lambda}$. Evidently

$$|\{(\mathbf{p}, \mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda} : R \sim (\mathbf{p}, \mathbf{q})\}| \leqslant |\mathcal{W}'_{\lambda}(R)||\mathcal{B}'_{\lambda}(R)|.$$

It remains to show that $\mathcal{W}'_{\lambda}(R) \subset \mathcal{W}_{\lambda}(R)$ and $\mathcal{B}'_{\lambda}(R) \subset \mathcal{B}_{\lambda}(R)$. To see this, fix $\mathbf{p} \in \mathcal{W}'_{\lambda}(R)$. By definition, there exists $\mathbf{q} \in \overline{\mathcal{B}}_{\lambda}$ such that $R \sim (\mathbf{p}, \mathbf{q})$. This means that R is 100-comparable to the rectangle $R^{\lambda}_{\Sigma}(\mathbf{p}, \mathbf{q})$ which satisfies (6.19). According to Lemma 4.9, there exists an absolute constant $C > 0$ such that

$$R \subset CR^{\lambda}_{\Sigma}(\mathbf{p}, \mathbf{q}) \overset{(6.19)}{\subset} C\delta^{-C\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}) \cap C\delta^{-\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{q}) \subset \delta^{-\mathbf{C}\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}).$$

This shows that $\mathbf{p} \in \mathcal{W}_{\lambda}(R)$ by definition. Thus $\mathcal{W}'_{\lambda}(R) \subset \mathcal{W}_{\lambda}(R)$. The proof of the inclusion $\mathcal{B}'_{\lambda}(R) \subset \mathcal{B}_{\lambda}(R)$ is similar. □

Claim 6.26. *Assume that $R \in \mathcal{R}^{\lambda}_{\Sigma}$ has type $(m_{\lambda}, n_{\lambda})$ relative to $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$, and assume that the constant "$\mathbf{A}$" in (6.7) satisfies $\mathbf{A} \geqslant 3(\mathbf{C} + 1)$, where $\mathbf{C}$ is the absolute constant determined at (6.22). Then*

$$\max\{m_{\lambda}, n_{\lambda}\} \leqslant Y_{\lambda}. \tag{6.27}$$

This is where the absolute constant $\mathbf{A}$ in the statement of Theorem 6.5 is determined.

Proof of Claim 6.26. Write $R = \mathcal{R}^{\lambda}_{\Sigma}(\mathbf{p}, \mathbf{q})$ with $\mathbf{p} \in \overline{\mathcal{W}}_{\lambda}$, $\mathbf{q} \in \overline{\mathcal{B}}_{\lambda}$, and $\mathbf{p} \sim \mathbf{q}$. Then, enumerate $\mathcal{W}_{\lambda}(R) = \{\mathbf{p}_1, \ldots, \mathbf{p}_{m_{\lambda}}\}$. Now $\delta^{2\epsilon}t \leqslant |\mathbf{p}_j - \mathbf{q}| \leqslant \delta^{-2\epsilon}t$ for all $1 \leqslant j \leqslant m_{\lambda}$, and moreover

$$R \subset \delta^{-\mathbf{C}\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{p}_j) \cap \delta^{-\mathbf{C}\epsilon}\mathcal{E}^{\lambda}_{\Sigma}(\mathbf{q}), \qquad 1 \leqslant j \leqslant m_{\lambda}.$$

Unraveling this inclusion, there exist $\mathbf{w} \in E_{\Sigma}(\mathbf{q})$, and for each $1 \leqslant j \leqslant m_{\lambda}$ some $\mathbf{v}_j \in E_{\Sigma}(\mathbf{p}_j)$ such that

$$R \subset \delta^{-\mathbf{C}\epsilon} R^{\lambda}_{\Sigma}(\mathbf{p}_j, \mathbf{v}_j) \cap \delta^{-\mathbf{C}\epsilon} R^{\lambda}_{\Sigma}(\mathbf{q}, \mathbf{w}). \tag{6.28}$$

We now claim that

$$(\mathbf{p}_j, \mathbf{v}_j) \in (\Omega^{\lambda}_{\Sigma})^{\delta^{-\mathbf{A}\epsilon}}_{\lambda,t}(\mathbf{q}, \mathbf{w}), \qquad 1 \leqslant j \leqslant m_{\lambda}, \tag{6.29}$$

if $\mathbf{A} \geqslant 3(\mathbf{C} + 1)$. Noting that $\omega := (\mathbf{q}, \mathbf{w}) \in \Omega^{\lambda}_{\Sigma}$, this will prove that

$$m_{\lambda} \leqslant |\{\omega' \in (\Omega^{\lambda}_{\Sigma})^{\delta^{-\mathbf{A}\epsilon}}_{\lambda,t}(\omega) : \delta^{-\mathbf{A}\epsilon} R^{\lambda}_{\Sigma}(\omega') \cap \delta^{-\mathbf{A}\epsilon} R^{\lambda}_{\Sigma}(\omega) \neq \emptyset\}| = m^{\delta^{-\mathbf{A}\epsilon},\delta^{-\mathbf{A}\epsilon}}_{\lambda,\lambda,t}(\omega \mid \Omega) \overset{(6.7)}{\leqslant} Y_{\lambda},$$

and the upper bound $|\mathcal{B}_{\lambda}(R)| = n_{\lambda} \leqslant Y_{\lambda}$ can be established in a similar fashion.

Regarding (6.29), we already know that $\delta^{\mathbf{A}\epsilon} t \leqslant |\mathbf{p}_j - \mathbf{q}| \leqslant \delta^{-\mathbf{A}\epsilon} t$, provided that $\mathbf{A} \geqslant 2$. So, it remains to show that $\Delta(\mathbf{p}_j, \mathbf{q}) \leqslant \delta^{-\mathbf{A}\epsilon}\lambda$. But (6.28) implies that

$$R \subset S^{\delta^{-\mathbf{C}\epsilon}\lambda}(\mathbf{p}_j) \cap S^{\delta^{-\mathbf{C}\epsilon}\lambda}(\mathbf{q}), \qquad 1 \leqslant j \leqslant m_{\lambda}.$$

The set $R \in \mathcal{R}^{\lambda}_{\Sigma}$ has $\operatorname{diam}(R) \sim \Sigma = \sqrt{\lambda/t}$, and on the other hand Lemma 4.3 implies that the intersection of the two annuli above can be covered by boundedly many discs of radius

$$\delta^{-\mathbf{C}\epsilon}\lambda/\sqrt{\Delta(\mathbf{p}_j, \mathbf{q})|\mathbf{p}_j - \mathbf{q}|} \leqslant \delta^{-(\mathbf{C}+1)\epsilon}\lambda/\sqrt{\Delta(\mathbf{p}_j, \mathbf{q}) \cdot t}.$$

This shows that $\sqrt{\lambda/t} \lesssim \delta^{-(\mathbf{C}+1)\epsilon}\lambda/\sqrt{\Delta(\mathbf{p}_j, \mathbf{q}) \cdot t}$, and rearranging gives $\Delta(\mathbf{p}_j, \mathbf{q}) \lesssim \delta^{-2(\mathbf{C}+1)\epsilon}\lambda$. This completes the proof of (6.29), and the lemma. □

Each rectangle $R \in \mathcal{R}^{\lambda}_{\Sigma}$ has some type $(m_{\lambda}, n_{\lambda})$ relative to $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$, with $1 \leqslant m_{\lambda}, n_{\lambda} \lesssim \lambda^{-3}$. By pigeonholing, we may find a subset $\bar{\mathcal{R}}^{\lambda}_{\Sigma} \subset \mathcal{R}^{\lambda}_{\Sigma}$ such that every rectangle $R \in \bar{\mathcal{R}}^{\lambda}_{\Sigma}$ has type between $(m_{\lambda}, n_{\lambda})$ and $(2m_{\lambda}, 2n_{\lambda})$ for some $m_{\lambda}, n_{\lambda} \geqslant 1$, and moreover

$$\sum_{R \in \mathcal{R}^{\lambda}_{\Sigma}} |\{(\mathbf{p}, \mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda} : R \sim (\mathbf{p}, \mathbf{q})\}| \approx_{\delta} \sum_{R \in \bar{\mathcal{R}}^{\lambda}_{\Sigma}} |\{(\mathbf{p}, \mathbf{q}) \in \overline{\mathcal{W}}_{\lambda} \times \overline{\mathcal{B}}_{\lambda} : R \sim (\mathbf{p}, \mathbf{q})\}|. \tag{6.30}$$

When we now combine (6.18) with (6.21), then (6.30), and finally (6.25), we find

$$|\mathcal{R}^{\delta}_{\sigma}| \lessapprox_{\delta} M_{\lambda} N_{\lambda} \cdot (m_{\lambda} n_{\lambda}) \cdot |\bar{\mathcal{R}}^{\lambda}_{\Sigma}|. \tag{6.31}$$

To conclude the proof of (6.8) from here, we consider separately the "main" case $\lambda \leqslant \delta^{\sqrt{\epsilon}}$, and the "trivial" case $\lambda \geqslant \delta^{\sqrt{\epsilon}}$. In the trivial case, we simply apply the following uniform estimates:

$$\max\{m_{\lambda}, n_{\lambda}\} \leqslant \lambda^{-C} \leqslant \delta^{-C\sqrt{\epsilon}} \quad \text{and} \quad |\bar{\mathcal{R}}^{\lambda}_{\Sigma}| \leqslant \lambda^{-C} \leqslant \delta^{-C\sqrt{\epsilon}}$$

Consequently, using also $M_{\lambda} \leqslant \min\{|W|, X_{\lambda}\}$ and $N_{\lambda} \leqslant \min\{|B|, X_{\lambda}\}$, we get

$$|\mathcal{R}^{\delta}_{\sigma}| \lessapprox_{\delta} \delta^{-3C\sqrt{\epsilon}}(M_{\lambda} N_{\lambda}) \leqslant \delta^{-3C\sqrt{\epsilon}}(|W||B|)^{3/4} X_{\lambda}^{1/2}.$$

This is even better than the case $m = 1 = n$ of (6.8), assuming $3C\sqrt{\epsilon} \leqslant \eta$.

Assume then that $\lambda \leqslant \delta^{\sqrt{\epsilon}}$. In this case, as pointed out in Remark 6.23, the family $\bar{\mathcal{R}}^{\lambda}_{\Sigma}$ consists of (λ, Σ)-rectangles of type $(\geqslant m_{\lambda}, \geqslant n_{\lambda})_{\mathbf{C}\sqrt{\epsilon}}$ relative to $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$, in the sense of Definition 4.22. Furthermore, the pair $(\overline{\mathcal{W}}_{\lambda}, \overline{\mathcal{B}}_{\lambda})$ is $(\lambda, \mathbf{C}\sqrt{\epsilon})$-almost t-bipartite by (6.17), and since $\delta^{-2\epsilon} \leqslant \lambda^{-\mathbf{C}\sqrt{\epsilon}}$. Consequently, by Lemma 4.23, we have

$$|\bar{\mathcal{R}}^{\lambda}_{\Sigma}| \leqslant \lambda^{-O(\sqrt{\epsilon})} \left[\left(\frac{|\overline{\mathcal{W}}_{\lambda}||\overline{\mathcal{B}}_{\lambda}|}{m_{\lambda} n_{\lambda}} \right)^{3/4} + \frac{|\overline{\mathcal{W}}_{\lambda}|}{m_{\lambda}} + \frac{|\overline{\mathcal{B}}_{\lambda}|}{n_{\lambda}} \right]. \tag{6.32}$$

In particular, we may choose $\epsilon = \epsilon(\eta) > 0$ so small that $\lambda^{-O(\sqrt{\epsilon})} \leqslant \delta^{-\eta}$.

The estimate (6.32) is not yet the same as the case $m = 1 = n$ of (6.8). To reach (6.8) from here, we consider separately the cases where the first, second, or third terms in (6.32) dominate. In all cases, we will use (recall (6.13)) that

$$|\overline{\mathcal{W}}_\lambda| \lesssim \frac{|W|}{M_\lambda} \quad \text{and} \quad |\overline{\mathcal{B}}_\lambda| \lesssim \frac{|B|}{N_\lambda} \quad \text{and} \quad \max\{M_\lambda, N_\lambda\} \leqslant X_\lambda.$$

Now, if the first ("main") term in (6.32) is the largest, then (omitting the factor $\lambda^{-O(\sqrt{\epsilon})}$ for notational simplicity, and combining (6.31) with (6.32))

$$\begin{aligned}|\mathcal{R}^\delta_\sigma| &\lessapprox_\delta M_\lambda N_\lambda \cdot (m_\lambda n_\lambda) \cdot \left(\frac{|\overline{\mathcal{W}}_\lambda||\overline{\mathcal{B}}_\lambda|}{m_\lambda n_\lambda}\right)^{3/4} \\ &\lesssim (M_\lambda N_\lambda)^{1/4} \cdot (m_\lambda n_\lambda)^{1/4} \cdot (|W||B|)^{3/4} \overset{(6.27)}{\leqslant} (X_\lambda Y_\lambda)^{1/2} \cdot (|W||B|)^{3/4}.\end{aligned}$$

This is what we desired in (6.8) (case $m = n = 1$).

Assume next that the second term in (6.32) dominates. Then,

$$|\mathcal{R}^\delta_\sigma| \lessapprox_\delta M_\lambda N_\lambda \cdot (m_\lambda n_\lambda) \cdot \frac{|\overline{\mathcal{W}}_\lambda|}{m_\lambda} = N_\lambda \cdot n_\lambda \cdot |W| \lesssim X_\lambda Y_\lambda |W|.$$

Similarly, if the third term in (6.32) dominates, we get $|\mathcal{R}^\delta_\sigma| \lessapprox_\delta X_\lambda Y_\lambda |B|$. This concludes the proof of (6.8) in the case $m = 1 = n$.

We then, finally, consider the case of general $1 \leqslant m \leqslant |W|$ and $1 \leqslant n \leqslant |B|$. This is morally the random sampling argument from [24, Lemma 1.4], but the details are more complicated due to our asymmetric definition of "λ-restricted type". Fix a large absolute constant $A \geqslant 1$ (to be determined soon; this constant has no relation to the constant $\mathbf{A}$ introduced in Claim 6.26). Let $\overline{W} \subset W$ be the subset obtained by keeping every element of W with probability A/m. Define the random subset $\bar{B} \subset B$ in the same way, keeping every element of B with probability A/n. However, if $m \leqslant 2A$, we keep all the elements of W, and if $n \leqslant 2A$, we keep all the elements of B. We assume in the sequel that $\min\{m, n\} \geqslant 2A$ and leave the converse special cases to the reader (the case $\max\{m, n\} < 2A$ is completely elementary, but to understand what to do in the case $m < 2A \leqslant n$, we recommend first reading the argument below, and then thinking about the small modification afterwards.)

The underlying probability space is $\{0,1\}^{|W|} \times \{0,1\}^{|B|} =: \Lambda$. The pairs $(\omega, \beta) \in \Lambda$ are in 1-to-1 correspondence with subset-pairs $\overline{W} \times \bar{B} \subset W \times B$, and we will prefer writing "$(\overline{W}, \bar{B}) \in \Lambda$" in place of "$(\omega, \beta) \in \Lambda$". We denote by $\mathbb{P}$ the probability which corresponds to the explanation in the previous paragraph: thus, the probability of a sequence (ω, β) equals

$$\mathbb{P}\{(\omega, \beta)\} = (\tfrac{A}{m})^{|\{\omega_i = 1\}|}(1 - \tfrac{A}{m})^{|\{\omega_i = 0\}|}(\tfrac{A}{n})^{|\{\beta_j = 1\}|}(1 - \tfrac{A}{n})^{|\{\beta_j = 0\}|}.$$

The most central random variables will be $|\overline{W}|$ and $|\bar{B}|$, formally

$$|\overline{W}|(\omega, \beta) := |\{1 \leqslant i \leqslant |W| : \omega_i = 1\}| \quad \text{and} \quad |\bar{B}|(\omega, \beta) := |\{1 \leqslant j \leqslant |B| : \beta_j = 1\}|$$

In expectation $\mathbb{E}|\overline{W}| = A|W|/m$ and $\mathbb{E}|\bar{B}| = A|B|/n$. By Chebychev's inequality, the probability that either $|\overline{W}| \geqslant 4A|W|/m$ or $|\bar{B}| \geqslant 4A|B|/n$ is at most $\frac{1}{2}$. We let $\Lambda' \subset \Lambda$ be sequences in $(\omega, \beta) \in \Lambda$ for which $|\overline{W}(\omega, \beta)| \leqslant 4A|W|/m$ and $|\bar{B}(\omega, \beta)| \leqslant 4A|B|/n$. As we just said, $\mathbb{P}(\Lambda') \geqslant \frac{1}{2}$.

Let $\mathcal{R}^{\delta}_{\sigma}(\overline{W}, \bar{B}) \subset \mathcal{R}^{\delta}_{\sigma}$ be the subset which has λ-restricted type $(\geqslant 1, \geqslant 1)_{\epsilon}$ relative to $(\overline{W}, \bar{B})$. We claim that there exists $(\overline{W}, \bar{B}) \in \Lambda'$ such that

$$|\mathcal{R}^{\delta}_{\sigma}| \leqslant 4|\mathcal{R}^{\delta}_{\sigma}(\overline{W}, \bar{B})|. \tag{6.33}$$

To see this, fix $R \in \mathcal{R}^{\delta}_{\sigma}$, and recall the definition of λ-restricted type $(\geqslant m, \geqslant n)_{\epsilon}$ relative to (W, B). There exists a set $W_R \subset W$ with $|W_R| \geqslant m$, and for each $p \in W_R$ a subset

$$B(p) \subset B \quad \text{with} \quad |B(p)| \geqslant n, \tag{6.34}$$

such that $\delta^{\epsilon}\lambda \leqslant \Delta(p, q) \leqslant \delta^{-\epsilon}\lambda$ for all $p \in W_R$ and $q \in B(p)$, and $R \subset \delta^{-\epsilon}\mathcal{E}^{\delta}_{\sigma}(p) \cap \delta^{-\epsilon}\mathcal{E}^{\delta}_{\sigma}(q)$ for all $p \in W_R$ and $q \in B_R(p)$. We claim that for any $c > 0$, we have

$$\mathbb{P}(\{\exists \text{ at least one pair } (p, q) \in \overline{W} \times \bar{B} \text{ such that } p \in W_R \text{ and } q \in B(p)\}) \geqslant 1 - c, \tag{6.35}$$

assuming that the constant "A" is chosen large enough, depending only on c. Before attempting this, we prove something easier: $\mathbb{P}(\{\overline{W} \cap W_R \neq \emptyset\}) \geqslant 1 - c$. For each $p \in W_R$ fixed, we have

$$\mathbb{P}(\{p \notin \overline{W}\}) = 1 - \frac{A}{m}.$$

Moreover, these events are independent when $p \in W_R$ (or even $p \in W$) varies. Therefore,

$$\mathbb{P}(\{\overline{W} \cap W_R = \emptyset\}) = \prod_{p \in W_R} \mathbb{P}(\{p \notin \overline{W}\}) = (1 - \tfrac{A}{m})^{|W_R|} \leqslant \Big((1 - \tfrac{A}{m})^{m/A}\Big)^{A}. \tag{6.36}$$

Since $m \geqslant 2A$, the right hand side is bounded from above by ρ^A for some (absolute) $\rho < 1$, and in particular the probability is $< c$ as soon as $\rho^A < c$.

To proceed towards (6.35), we partition the event $\{\overline{W} \cap W_R \neq \emptyset\}$ into a union of events of the form $\{\overline{W} \cap W_R = H\}$, where $H \subset W_R$ is a fixed non-empty subset. Clearly the events $\{\overline{W} \cap W_R = H\}$ and $\{\overline{W} \cap W_R = H'\}$ are disjoint for distinct (not necessarily disjoint) $H, H' \subset W_R$. For every $\emptyset \neq H \subset W_R$, we designate a point $p_H \in H$ in an arbitrary manner. For example, we could enumerate the points in W_R, and $p_H \in H$ could be the point with the lowest index in the enumeration. Then, for $H \subset W_R$ fixed, we consider the event $\{\bar{B} \cap B(p_H) \neq \emptyset\}$, where $B(p_H) \subset B$ is the set from (6.34). Since $\mathbb{P}(\{q \notin \bar{B}\}) = 1 - A/n$, and $|B(p_H)| \geqslant n$, a calculation similar to the one on line (6.36) shows that

$$\mathbb{P}(\{\bar{B} \cap B(p_H) \neq \emptyset\}) \geqslant 1 - \rho^A > 1 - c, \qquad \emptyset \neq H \subset W_R, \tag{6.37}$$

assuming that $\rho^A < c$. Furthermore, we notice that for $\emptyset \neq H \subset W_R$ fixed,

$$\mathbb{P}(\{\overline{W} \cap W_R = H\} \cap \{\bar{B} \cap B(p_H) \neq \emptyset\}) = \mathbb{P}(\{\overline{W} \cap W_R = H\})\mathbb{P}(\{\bar{B} \cap B(p_H) \neq \emptyset\}).$$

From a probabilistic point of view, this is because the events $\{\bar{B} \cap B(p_H) \neq \emptyset\}$ and $\{\overline{W} \cap W_R = H\}$ are independent. From a measure theoretic point of view, the set $\{\overline{W} \cap W_R = H\} \cap \{\bar{B} \cap B(p_H) \neq \emptyset\} \subset \{0,1\}^{|W|} \times \{0,1\}^{|B|} = \Lambda$ can be written as a product set.

Now, we may estimate as follows:

$$\begin{aligned}\sum_{\emptyset \neq H \subset W_R} & \mathbb{P}(\{\bar{B} \cap B(p_H) \neq \emptyset\} \cap \{\overline{W} \cap W_R = H\}) \\ & \overset{(6.37)}{\geqslant} (1-c) \sum_{\emptyset \neq H \subset W_R} \mathbb{P}(\{\overline{W} \cap W_R = H\}) \\ & = (1-c) \cdot \mathbb{P}(\{\overline{W} \cap W_R \neq \emptyset\}) \geqslant (1-c)^2.\end{aligned}$$

On the other hand, the events we are summing over on the far left are disjoint, and their union is contained in the event shown in (6.35). This proves (6.35) with $(1-c)^2$ in place of $(1-c)$, which is harmless.

Let $G_R \subset \Lambda$ be the "good" event from (6.35). Note that if $(\overline{W}, \bar{B}) \in G_R$, then R has restricted λ-type $(\geqslant 1, \geqslant 1)_\epsilon$ relative to $(\overline{W}, \bar{B})$ – indeed this is due to the pair $(p,q) \in \overline{W} \times \bar{B}$ with $p \in W_R$ and $q \in B(p)$ whose existence is guaranteed by the definition of $(\overline{W}, \bar{B}) \in G_R$. Thus $R \in \mathcal{R}^\delta_\sigma(\overline{W}, \bar{B})$ (defined above (6.33)) whenever $(\overline{W}, \bar{B}) \in G_R$. This implies that

$$\int_{\Lambda'} |\mathcal{R}^\delta_\sigma(\overline{W}, \bar{B})| \, d\mathbb{P}(\overline{W}, \bar{B}) = \sum_{R \in \mathcal{R}^\delta_\sigma} \mathbb{P}(\Lambda' \cap \{R \in \mathcal{R}^\delta_\sigma(\overline{W}, \bar{B})\}) \geqslant \sum_{R \in \mathcal{R}^\delta_\sigma} \mathbb{P}(\Lambda' \cap G_R).$$

Finally, recall that $\mathbb{P}(\Lambda') \geqslant 1/2$ and $\mathbb{P}(G_R) \geqslant 1 - c$. In particular, if we choose $c < 1/4$ (and thus finally fix "A" sufficiently large), then the integral above is bounded from below by $|\mathcal{R}^\delta_\sigma|/4$. This proves the existence of $(\overline{W}, \bar{B}) \in \Lambda'$ such that (6.33) holds.

Finally, since every $R \in \mathcal{R}^\delta_\sigma(\overline{W}, \bar{B}) =: \bar{\mathcal{R}}^\delta_\sigma$ has λ-restricted type $(\geqslant 1, \geqslant 1)_\epsilon$ relative to $(\overline{W}, \bar{B})$, the first part of the proof implies

$$|\mathcal{R}^\delta_\sigma| \leqslant 4|\bar{\mathcal{R}}^\delta_\sigma| \lesssim \delta^{-\eta} \left[(|\overline{W}||\bar{B}|)^{3/4} (X_\lambda Y_\lambda)^{1/2} + |\overline{W}|(X_\lambda Y_\lambda) + |\bar{B}|(X_\lambda Y_\lambda) \right].$$

Since $(\overline{W}, \bar{B}) \in \Lambda'$, we have $|\overline{W}| \leqslant 4A|W|/m$ and $|\bar{B}| \leqslant 4A|B|/n$. Noting that "$A$" is an absolute constant, the upper bound matches (6.8), and the proof is complete. □

7. Proof of Theorem 1.11

In this section we finally prove Theorem 1.11. In fact, we will prove a stronger statement concerning the partial multiplicity functions $m_{\delta,\lambda,t}$, see Theorem 7.5 below. Theorem 1.11 will finally be deduced from Theorem 7.5 in Section 7.7.

Recall Notation 5.1. We will need the following slight generalisation, where the ranges of the "distance" and "tangency" parameters can be specified independently of each other.

Definition 7.1 ($G^{\rho_\lambda,\rho_t}_{\lambda,t}(\omega)$)**.** Let $\delta \leqslant \lambda \leqslant t \leqslant 1$, and $G \subset \Omega = \{(p,v) : p \in P \text{ and } v \in E(p)\}$. For $\rho_\lambda, \rho_t \geqslant 1$ and $\omega = (p,v) \in \Omega$, we write

$$G^{\rho_\lambda,\rho_t}_{\lambda,t}(\omega) := \{(p',v') \in G : \lambda/\rho_\lambda \leqslant \Delta(p,p') \leqslant \rho_\lambda \lambda \text{ and } t/\rho_t \leqslant |p-p'| \leqslant \rho_t t\}.$$

Similarly, for $Q \subset P \subset \mathbf{D}$, we will also write

$$Q^{\rho_\lambda,\rho_t}_{\lambda,t}(p) := \{q \in Q : \lambda/\rho_\lambda \leqslant \Delta(p,q) \leqslant \rho_\lambda \lambda \text{ and } t/\rho_t \leqslant |p-q| \leqslant \rho_t t\}.$$

Thus, the former notation concerns pairs, and the latter points. The correct interpretation should always be clear from the context (whether $G \subset \Omega$ or $Q \subset P$).

Whenever $\delta \leqslant \lambda \leqslant \delta\rho_\lambda$, we modify both definitions so that the two-sided condition $\lambda/\rho_\lambda \leqslant \Delta(p,q) \leqslant \rho_\lambda\lambda$ is replaced by the one-sided condition $\Delta(p,q) \leqslant \rho_\lambda\lambda$.

Notation 7.2. Thankfully, we can most often (not always) use the definitions in the cases $\rho_\lambda = \rho = \rho_t$. In this case, we abbreviate $G^{\rho_\lambda,\rho_t}_{\lambda,t} =: G^{\rho}_{\lambda,t}$.

Definition 7.3 ($m^{\rho_\lambda,\rho_t,C}_{\delta,\lambda,t}$). Fix $0 < \delta \leqslant \lambda \leqslant t \leqslant 1$ and $\rho_\lambda, \rho_t \geqslant 1$. Let $\Omega = \{(p,v) : p \in P \text{ and } v \in E(p)\}$ as usual, and write $\sigma := \delta/\sqrt{\lambda t}$. For any set $G \subset \Omega$, we define

$$m^{\rho_\lambda,\rho_t,C}_{\delta,\lambda,t}(\omega \mid G) := |\{\omega' \in (G^\delta_\sigma)^{\rho_\lambda,\rho_t}_{\lambda,t} : CR^\delta_\sigma(\omega) \cap CR^\delta_\sigma(\omega') \neq \emptyset\}|, \qquad \omega \in G.$$

Here G^δ_σ is the (δ,σ)-skeleton of G.

Notation 7.4. Consistently with Notation 7.2, in the case $\rho_\lambda = \rho = \rho_t$ we abbreviate

$$m^{\rho_\lambda,\rho_t,C}_{\delta,\lambda,t} =: m^{\rho,C}_{\delta,\lambda,t}.$$

The full generality of the notation will only be needed much later, and we will remind the reader at that point.

Theorem 7.5. *For every $\kappa \in (0,\frac{1}{2}]$ and $s \in (0,1]$, there exist $\epsilon = \epsilon(\kappa,s) > 0$ and $\delta_0 = \delta_0(\epsilon,\kappa,s) > 0$ such that the following holds for all $\delta \in (0,\delta_0]$. Let $\Omega = \{(p,v) : p \in P \text{ and } v \in E(p)\}$ be a $(\delta,\delta,s,\delta^{-\epsilon})$-configuration with*

$$|P| \leqslant \delta^{-s-\epsilon}, \tag{7.6}$$

Then, there exists a subset $G \subset \Omega$ of cardinality $|G| \geqslant \delta^\kappa|\Omega|$ such that the following holds simultaneously for all $\delta \leqslant \lambda \leqslant t \leqslant 1$:

$$m^{\delta^{-\epsilon},\kappa^{-1}}_{\delta,\lambda,t}(\omega \mid G) \leqslant \delta^{-\kappa}, \qquad \omega \in G. \tag{7.7}$$

Theorem 1.11 will be easy to derive from Theorem 7.5. The details are in Section 7.7. Theorem 7.5 will be proven by a sequence of successive refinements to the initial configuration Ω. Every refinement will take care of the inequality (7.7) for one fixed pair (λ,t), but the refinements will need to be performed in an appropriate order, as we will discuss later. After a large but finite number of such refinements, we will be able to check that (7.7) holds for all $\delta \leqslant \lambda \leqslant t \leqslant 1$ simultaneously.

Notation 7.8. Throughout this section, we allow the implicit constants in the "$\approx_\delta$" notation to depend on the constants κ, s and $\epsilon = \epsilon(\kappa,s)$ in Theorem 7.5 (the choice of ϵ is explained in Section 7.1). Thus, the notation $A \lessapprox_\delta B$ means that $A \leqslant C(\log(1/\delta))^C B$, where $C = C(\epsilon,\kappa,s) > 0$. In particular, if $\delta > 0$ is small enough depending on ϵ,κ,s, the inequality $A \lessapprox_\delta B$ implies $A \leqslant \delta^{-\epsilon}B$.

7.1. **Choice of constants.** We explain how ϵ in Theorem 7.5 depends on κ, s. Let $\epsilon_{\max} = \epsilon_{\max}(\kappa,s) > 0$ be an auxiliary constant, which (informally) satisfies $\epsilon \ll \epsilon_{\max} \ll \kappa$. Precisely, the constant $\epsilon_{\max}$ is determined by the following two requirements:

- Let $\mathbf{A}$ be the absolute constant from Theorem 6.5. We require $\epsilon_{\max}$ to be so small that if Theorem 5.31 is applied with parameters $\bar{\kappa} = \kappa s/100$ and s, then $\mathbf{A}\epsilon_{\max} \leqslant \epsilon_0(\bar{\kappa},s)$, where the $\epsilon_0(\bar{\kappa},s)$ is the constant produced by Theorem 5.31.
- We apply Theorem 6.5 with constant $\eta = \kappa s/100$, and we require that $\epsilon_{\max} \leqslant \epsilon(\eta)$ (where $\epsilon(\eta)$ is the constant produced by Theorem 6.5).

- We require that $\epsilon_{\max} < c\kappa s$ for a small absolute constant $c > 0$, whose size will be determined later.

The relationship between the "final" ϵ in Theorem 7.5, and the constant $\epsilon_{\max}$ fixed above, is the following, for a suitable absolute constant $C > 0$:

$$C \cdot 10^{100/\kappa}\epsilon \leqslant \epsilon_{\max}. \tag{7.9}$$

As stated in Theorem 7.5, the threshold $\delta_0 > 0$ may depend on all the parameters ϵ, κ, s. We do not attempt to track the dependence explicitly, and often we will state inequalities of (e.g.) the form "$C \leqslant \delta^{-\epsilon}$" under the implicit assumption that $\delta > 0$ is small enough, depending on ϵ. Here, we only explicitly record that $\delta_0 > 0$ is taken so small that

$$CA(\epsilon, \kappa)^{C/\epsilon} \leqslant \delta_0^{-\epsilon_{\max}}, \tag{7.10}$$

where $A(\epsilon, \kappa) \geqslant 1$ is a constant depending only on κ, and $C \geqslant 1$ is absolute.

7.2. **The case $t \approx \lambda$.** In the "main" argument for Theorem 7.5, we will need to assume that $t \geqslant \delta^{-\kappa/10}\lambda$. The opposite case $t \leqslant \delta^{-\kappa/10}\lambda$ is elementary, and we handle it straight away. So, fix $\delta \leqslant \lambda \leqslant t \leqslant 1$ with $t \leqslant \delta^{-\kappa/10}\lambda$.

Claim 7.11. *There exists a $(\delta, \delta, s, 4\delta^{-\epsilon})$-configuration $G \subset \Omega$ (depending on λ, t) of cardinality $|G| \geqslant |\Omega|/16$ such that (7.7) holds with $\epsilon := \kappa/100$.*

We record that our assumption $t \leqslant \delta^{-\kappa/10}\lambda$ implies

$$\sigma = \delta/\sqrt{\lambda t} \geqslant \delta^{\kappa/20}(\delta/\lambda) \geqslant \delta^{\kappa/5}(\delta/\lambda). \tag{7.12}$$

To save a little space, we abbreviate $\mathbf{R}(p, v) := \kappa^{-1}R^{\delta}_{\sigma}(p, v)$. We also write $M := |E(p)|$ for the common cardinality of the sets $E(p)$, $p \in P$. With this notation, we estimate as follows (the final estimate will be justified carefully below the computation):

$$\begin{aligned}
&\frac{1}{|P|}\sum_{p \in P}\frac{1}{M}\sum_{v \in E(p)} |\{(p', v') \in (\Omega^{\delta}_{\sigma})^{\delta^{-\epsilon}}_{\lambda,t}(p, v) : \mathbf{R}(p, v) \cap \mathbf{R}(p', v') \neq \emptyset\}| \\
&\quad \leqslant \frac{1}{|P|M}\sum_{p \in P}\sum_{p' \in P^{\delta^{-\epsilon}}_{\lambda,t}(p)} |\{(v, v') \in E(p) \times \mathcal{S}_{\sigma}(p') : \mathbf{R}(p, v) \cap \mathbf{R}(p', v') \neq \emptyset\}| \qquad (7.13)\\
&\quad \leqslant \delta^{s-\kappa/2}|P|. \qquad (7.14)
\end{aligned}$$

We justify the final estimate. The easiest part is

$$|P^{\delta^{-\epsilon}}_{\lambda,t}(p)| \leqslant |P \cap B(p, \delta^{-\epsilon}t)| \leqslant |P \cap B(p, \delta^{-\epsilon-\kappa/10}\lambda)| \leqslant \delta^{-2\epsilon-\kappa/10}\lambda^{s}|P|, \tag{7.15}$$

using the $(\delta, s, \delta^{-\epsilon})$-set property of P. A slightly more elaborate argument is needed to estimate the number of pairs (v, v') appearing in (7.13) for (p, p') fixed. Fix $(p, p') \in P \times P$ with $p' \in P^{\delta^{-\epsilon}}_{\lambda,t}(p)$: thus $|p - p'| \geqslant \delta^{\epsilon}t \geqslant \delta^{\epsilon}\lambda$ and $\Delta(p, p') \geqslant \delta^{\epsilon}\lambda$. Lemma 4.3 implies that the intersection

$$S^{\delta/\kappa}(p) \cap S^{\delta/\kappa}(p') \tag{7.16}$$

can be covered by boundedly many discs of radius

$$\frac{\delta/\kappa}{\sqrt{(\Delta(p, p') + \delta/\kappa)(|p - p'| + \delta/\kappa)}} \leqslant \frac{\delta/\kappa}{\sqrt{(\delta^{\epsilon}\lambda)(\delta^{\epsilon}\lambda)}} \leqslant \delta^{-2\epsilon}(\delta/\lambda) =: r.$$

(Here we assumed that $\delta > 0$ is small enough in terms of ϵ, κ.) Let $\{B(z_i, r)\}_{i=1}^{C}$ be an enumeration of these discs. Now, if $\mathbf{R}(p,v) \cap \mathbf{R}(p',v') \neq \emptyset$, then both v, v' must lie at distance $\leqslant 2r$ from one of these discs (the intersection $\mathbf{R}(p,v) \cap \mathbf{R}(p',v')$ is contained in the intersection (7.16), and $\operatorname{diam}(\mathbf{R}) \leqslant \sigma/\kappa \leqslant \delta/(\lambda\kappa) \ll r$). On the other hand,

$$|E(p) \cap B(z_i, r)| \leqslant \delta^{-\epsilon} r^s M \leqslant \delta^{-3\epsilon}(\delta/\lambda)^s M \quad \text{and} \quad |\mathcal{S}_\sigma(p') \cap B(z_i, 4r)| \leqslant \delta^{-\kappa/4},$$

where the first inequality used the $(\delta, s, \delta^{-\epsilon})$-set property of $E(p)$, and the second inequality used (7.12), along with the σ-separation of $\mathcal{S}_\sigma(p')$. This shows that

$$|\{(v,v') \in E(p) \times \mathcal{S}_\sigma(p') : \mathbf{R}(p,v) \cap \mathbf{R}(p',v') \neq \emptyset\}| \leqslant \delta^{-4\epsilon-\kappa/4}(\delta/\lambda)^s M.$$

When this upper bound is plugged into (7.13), then combined with (7.15), we find (7.14).

To conclude the proof, notice that the left hand side of (7.14) is in fact the expectation of the random variable

$$\omega \mapsto m_{\delta,\lambda,t}^{\delta^{-\epsilon},\kappa^{-1}}(\omega \mid \Omega),$$

relative to normalised counting measure on Ω. By Chebychev's inequality, there exists a set $G \subset \Omega$ with $|G| \geqslant \frac{1}{2}|\Omega|$ such that

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon},\kappa^{-1}}(\omega \mid G) \leqslant m_{\delta,\lambda,t}^{\delta^{-\epsilon},\kappa^{-1}}(\omega \mid \Omega) \leqslant \delta^{s-3\kappa/4}|P| \leqslant \delta^{-\kappa}, \qquad \omega \in G,$$

using the assumption (7.6) that $|P| \leqslant \delta^{-s-\epsilon}$ in the final inequality. Finally, we replace "G" by a slightly smaller $(\delta, \delta, s, 4\delta^{-\epsilon})$-configuration by applying Lemma 3.13 with $c = \frac{1}{2}$.

7.3. **Uniform sets.** We start preparing for the proof of Theorem 7.5 (the case of pairs (λ, t) with $t \geqslant \delta^{-\kappa/10}\lambda$) with a few auxiliary definitions and results which allow us to find – somewhat – regular subsets inside arbitrary finite sets $P \subset \mathbf{D}$.

Definition 7.17. Let $n \geqslant 1$, and let

$$\delta = \Delta_n < \Delta_{n-1} < \ldots < \Delta_1 \leqslant \Delta_0 = 1$$

be a sequence of dyadic scales. We say that a set $P \subset \mathbf{D}$ is $\{\Delta_j\}_{j=1}^n$-*uniform* if there is a sequence $\{N_j\}_{j=1}^n$ such that $|\mathcal{D}_{\Delta_j}(P \cap \mathbf{p})| = |P \cap \mathbf{p}|_{\Delta_j} = N_j$ for all $j \in \{1, \ldots, n\}$ and all $\mathbf{p} \in \mathcal{D}_{\Delta_{j-1}}(P)$. As usual, we extend this definition to $P \subset \mathcal{D}_\delta$ (by applying it to $\cup P$).

The following lemma allows us to find $\{\Delta_j\}_{j=1}^n$-uniform subsets inside general finite sets. The result is a special case of [15, Lemma 7.3], which works for more general sequences $\{\Delta_j\}_{j=1}^m$ than the sequence $\{2^{-jT}\}_{j=1}^m$ treated in Lemma 7.18.

Lemma 7.18. *Let $P \subset \mathbf{D}$, $m, T \in \mathbb{N}$, and $\delta := 2^{-mT}$. Let also $\Delta_j := 2^{-jT}$ for $0 \leqslant j \leqslant m$, so in particular $\delta = \Delta_m$. Then, there there is a $\{\Delta_j\}_{j=1}^m$-uniform set $P' \subset P$ such that*

$$|P'|_\delta \geqslant (4T)^{-m}\,|P|_\delta. \tag{7.19}$$

In particular, if $\epsilon > 0$ and $T^{-1}\log(4T) \leqslant \epsilon$, then $|P'|_\delta \geqslant \delta^\epsilon |P|_\delta$.

Proof. The inequality (7.19) follows by inspecting the short proof of [15, Lemma 7.3]. The "in particular" claim follows by noting that

$$(4T)^{-m} = 2^{-m\log(4T)} = 2^{-mT\cdot(T^{-1}\log(4T))} = \delta^{T^{-1}\log(4T)}.$$

This completes the proof. □

7.4. **Initial regularisation for the proof of Theorem 7.5.** We denote the "given" $(\delta, \delta, s, \delta^{-\epsilon})$-configuration in Theorem 7.5 by

$$\Omega_0 = \{(p, v) : p \in P_0 \text{ and } v \in E_0(p)\},$$

where $P_0 \subset \mathcal{D}_\delta$ is a non-empty $(\delta, s, \delta^{-\epsilon})$-set, and $E_0(p) \subset \mathcal{S}_\delta(p)$ is a $(\delta, s, \delta^{-\epsilon})$-set of cardinality $M \geqslant 1$ (for every $p \in P_0$). The purpose of this section is to perform an initial pruning to Ω_0, that is, to find a $(\delta, \delta, s, \delta^{-2\epsilon})$-configuration

$$\Omega = \{(p, v) : p \in P \text{ and } v \in E(p)\} \subset \Omega_0,$$

where P is a $(\delta, s, \delta^{-2\epsilon})$-set, each $E(p)$ is a $(\delta, s, \delta^{-2\epsilon})$-set with constant cardinality, and $|\Omega| \approx_\delta |\Omega_0|$. The subset Ω will have additional useful regularity properties compared to Ω_0. After we are finished constructing Ω, we will focus on finding the "final" set G (as in Theorem 7.5) inside Ω, instead of Ω_0.

There is no loss of generality in assuming that $\delta = 2^{-mT}$ for some $m \geqslant 1$, and some $T \geqslant 1$ whose size depends on ϵ (and therefore eventually κ). We start by applying Lemma 7.18 to the sequence

$$\lambda_j := 2^{-jT}, \qquad 0 \leqslant j \leqslant m.$$

Provided that $T^{-1}\log(4T) \leqslant \epsilon$, the result is a $\{\lambda_j\}_{j=1}^m$-uniform subset $P_0' \subset P_0$ with cardinality $|P_0'| \geqslant \delta^\epsilon |P_0|$. In particular, P_0' is a $(\delta, s, \delta^{-2\epsilon})$-set. We define $\Omega_0' := \{(p, v) : p \in P_0' \text{ and } v \in E_0(p)\}$. Then $|\Omega_0'| \geqslant \delta^\epsilon |\Omega_0|$. From this point on, the proof will see no difference between Ω_0, P_0 and Ω_0', P_0', so we assume that $P_0 = P_0'$ and $\Omega_0 = \Omega_0'$ to begin with – or in other words that P_0 is $\{\lambda_j\}_{j=1}^m$-uniform for $\lambda_j = 2^{-jT}$, $1 \leqslant j \leqslant m$. In particular, the "branching numbers"

$$N_j := |P_0 \cap \mathbf{p}|_{\lambda_j}, \qquad \mathbf{p} \in \mathcal{D}_{2^T\lambda_j}(P_0),\ 1 \leqslant j \leqslant m,$$

are well-defined (that is, independent of "$\mathbf{p}$").

We have slightly overshot our target: the argument above shows that P_0 may be assumed to be $\{2^{-jT}\}_{j=1}^m$-uniform. We only need something weaker. Let $\epsilon > 0$ be so small that the requirement (7.9) is met. Let $\Lambda \subset [\delta, 1]$ be a finite set of cardinality $|\Lambda| \sim 1/\epsilon$ which is *multiplicatively* $\delta^{-\epsilon/2}$*-dense* in the following sense: if $\lambda \in [\delta, 1]$ is arbitrary, then there exists $\underline{\lambda} \in \Lambda$ with $\underline{\lambda} \leqslant \lambda \leqslant \delta^{-\epsilon/2}\underline{\lambda}$. If $\delta > 0$ is so small that $2^T \leqslant \delta^{-\epsilon}$, we may (and will) choose $\Lambda \subset \{2^{-jT}\}_{j=1}^m = \{\lambda_j\}_{j=1}^m$. We agree that $\{\delta, 1\} \in \Lambda$, and for every $\lambda \in \Lambda \setminus \{1\}$, we denote by $\hat{\lambda} \in \Lambda$ the smallest element of Λ with $\hat{\lambda} > \lambda$.

Since $\Lambda \subset \{2^{-jT}\}_{j=1}^m$, the set P_0 is automatically Λ-uniform: the number

$$N_\lambda := |P_0 \cap \mathbf{p}|_\lambda, \qquad \mathbf{p} \in \mathcal{D}_{\hat{\lambda}}(P_0),\ \lambda \in \Lambda \setminus \{1\}, \tag{7.20}$$

is independent of the choice of $\mathbf{p} \in \mathcal{D}_{\hat{\lambda}}(P_0)$. From this point on, the uniformity with respect to the denser sequence $\{2^{-jT}\}_{j=1}^m$ will no longer be required. From (7.20), it follows that also the number

$$X_\lambda := |P_0 \cap \mathbf{p}|_\delta = |P_0|/|P_0|_\lambda, \qquad \mathbf{p} \in \mathcal{D}_\lambda(P_0),\ \lambda \in \Lambda, \tag{7.21}$$

is independent of the choice of $\mathbf{p} \in \mathcal{D}_\lambda(P_0)$ (since X_λ is the product of the numbers $N_{\lambda'}$ for $\lambda' \in \Lambda$ with $\lambda' < \lambda$, recalling that $\delta \in \Lambda$ by definition).

Next, for every $\lambda \in \Lambda$ fixed, we associate a finite set $\mathcal{T}(\lambda) \subset [\lambda, 1]$ of cardinality $|\mathcal{T}(\lambda)| \sim 1/\epsilon$ which is multiplicatively $\delta^{-\epsilon/2}$-dense on the interval $[\lambda, 1]$ in the same sense as above: if $t \in [\lambda, 1]$ is arbitrary, then there exists $\underline{t} \in \mathcal{T}(\lambda)$ such that $\underline{t} \leqslant t \leqslant \delta^{-\epsilon/2}\underline{t}$. For later technical convenience, it will be useful to know that the sets

$$\Lambda(t) := \{\lambda \in \Lambda : t \in \mathcal{T}(\lambda)\}, \qquad t \in \mathcal{T} := \bigcup_{\lambda \in \Lambda} \mathcal{T}(\lambda), \tag{7.22}$$

are multiplicatively $\delta^{-\epsilon/2}$-dense in $[\delta, t]$. This can be accomplished by choosing both the λ's and the t's from some "fixed" multiplicatively $\delta^{-\epsilon/2}$-dense sequence in $[\delta, 1]$, for example $\{\delta, \delta^{1-\epsilon/2}, \delta^{-\epsilon}, \ldots, 1\}$.

We order the pairs (λ, t) with $\lambda \in \Lambda$ and $t \in \mathcal{T}(\lambda)$ arbitrarily. The total number of pairs is $\lesssim \epsilon^{-2}$. Then, we apply Theorem 5.31 with constant $\kappa s/100$ to the first pair (λ_1, t_1). If $\epsilon_{\max} > 0$ is sufficiently small (as small as we stated in Section 7.1), and since $\epsilon \leqslant \epsilon_{\max} \leqslant \mathbf{A}\epsilon_{\max} \leqslant \epsilon_0(\bar{\kappa}, s)$, Theorem 5.31 provides us with a $(\delta, \delta, s, C\delta^{-\epsilon})$-configuration $G \subset \Omega_0$ such that $C \approx_\delta 1$, $|G| \approx_\delta |\Omega_0|$, and

$$m_{\lambda_1,\lambda_1,t_1}^{\delta^{-\mathbf{A}\epsilon_{\max}},\delta^{-\mathbf{A}\epsilon_{\max}}}(\omega \mid G) \leqslant \delta^{-\kappa s/100}\lambda_1^s |P_0|_{\lambda_1}, \qquad \omega \in G_{\Sigma_1}^{\lambda_1}, \tag{7.23}$$

where $\Sigma_1 = \sqrt{\lambda_1/t_1}$, and $\mathbf{A} \geqslant 1$ is the constant from Theorem 6.5.

Assume that we have already found a sequence of $(\delta, \delta, s, C_j\delta^{-\epsilon})$-configurations $G =: G_1 \supset G_2 \supset \ldots G_j$, where $C_j \approx_{\delta,j} 1$ and $|G_j| \approx_{\delta,j} |\Omega_0|$, and (7.23) holds for G_j relative to the pair (λ_j, t_j) (with $\Sigma_j = \sqrt{\lambda_j/t_j}$). We reapply Theorem 5.31 to $\Omega_j := G_j$, and the pair (λ_{j+1}, t_{j+1}). This is legitimate, since $j \lesssim \epsilon^{-2}$, and the constant $C_j\delta^{-\epsilon}$ is smaller than the threshold $\delta^{-\epsilon_{\max}}$ required to apply Theorem 5.31 with constant "$\kappa s/100$" (by our choice of "ϵ"). Thus, Theorem 5.31 outputs a $(\delta, \delta, s, C_{j+1}\delta^{-\epsilon})$-configuration $G_{j+1} \subset G_j$ satisfying (7.23) for the pair (λ_{j+1}, t_{j+1}), and with $|G_{j+1}| \approx_{\delta,j+1} |\Omega_0|$.

After Theorem 5.31 has been applied in this "successive" manner to all the pairs (λ, t) with $\lambda \in \Lambda$ and $t \in \mathcal{T}(\lambda)$, we arrive at a final $(\delta, \delta, s, C_\epsilon\delta^{-\epsilon})$-configuration

$$\Omega = \{(p, v) : p \in P \text{ and } v \in E(p)\}, \tag{7.24}$$

where $C_\epsilon \approx_\delta 1$, $|\Omega| \approx_\delta |\Omega_0|$, and Ω satisfies simultaneously a version of (7.23) for all the pairs (λ_j, t_j). In particular, we note that $|P| \approx_\delta |P_0|$ and $|E(p)| \approx_\delta M$ for all $p \in P$. Therefore, $P, E(p)$ remain $(\delta, s, C_\epsilon\delta^{-\epsilon})$-sets with $C_\epsilon \approx_\delta 1$.

Remark 7.25. It is worth comparing the accomplishment (7.23) with the ultimate goal (7.7) in Theorem 7.5. Roughly speaking, we have now tackled the cases (λ, λ, t) of (7.7) (with the caveat that this has only been done for the pairs (λ, t) with $\lambda \in \Lambda$ and $t \in \mathcal{T}(\lambda)$).

7.5. **Proof of Theorem 7.5.** We just finished constructing the $(\delta, \delta, s, C_\epsilon\delta^{-\epsilon})$-configuration $\Omega = \{(p, v) : p \in P \text{ and } v \in E(p)\} \subset \Omega_0$ with $|\Omega| \approx_\delta |\Omega_0|$ which satisfies property (7.23) (with $G = \Omega$) for all $\lambda \in \Lambda$ and $t \in \mathcal{T}(\lambda)$. We record this once more:

$$m_{\lambda,\lambda,t}^{\delta^{-\mathbf{A}\epsilon_{\max}},\delta^{-\mathbf{A}\epsilon_{\max}}}(\omega \mid \Omega) \leqslant \delta^{-\kappa s/100}\lambda^s |P_0|_\lambda, \quad \omega \in \Omega_\Sigma^\lambda, \tag{7.26}$$

for every $\lambda \in \Lambda$ and $t \in \mathcal{T}(\lambda)$, where $\Sigma = \sqrt{\lambda/t}$.

Remark 7.27. At this point, we remind the reader that the left hand side of (7.26) is shorthand notation for

$$m_{\lambda,\lambda,t}^{\delta^{-\mathbf{A}\epsilon_{\max}},\delta^{-\mathbf{A}\epsilon_{\max}},\delta^{-\mathbf{A}\epsilon_{\max}}}(\omega \mid \Omega),$$

recall Notation 7.4. Soon we will need the full generality of the notation $m_{\delta,\lambda,t}^{\rho_\lambda,\rho_t,C}$.

The main step towards proving Theorem 7.5 for every pair (λ, t) with $\delta \leqslant \lambda \leqslant t \leqslant 1$ is to prove it for the (finitely many) pairs (λ, t) with $\lambda \in \Lambda$ and $t \in \mathcal{T}(\lambda)$. Write

$$\mathcal{T} := \bigcup_{\lambda \in \Lambda} \mathcal{T}(\lambda) \subset [\delta, 1],$$

and for every $t \in \mathcal{T}$, let $\Lambda(t) := \{\lambda \in \Lambda : t \in \mathcal{T}(\lambda)\} \subset [\delta, t]$. Recall from (around) (7.22) that $\Lambda(t)$ is multiplicatively $\delta^{-\epsilon/2}$-dense in $[\delta, t]$. This will be used in the form of the corollary that $\Lambda(t)$ is multiplicatively $\delta^{-\epsilon/2}$-dense in $[\delta, \max \Lambda(t)]$.

Proposition 7.28. *For every fixed $t \in \mathcal{T}$ and $\lambda \in \Lambda(t)$, there exists a $(\delta, \delta, s, C_\epsilon \delta^{-\epsilon})$-configuration $G \subset \Omega$ (depending on λ, t), such that $|G| \approx_\delta |\Omega|$, and*

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon}, C\kappa^{-1}}(\omega \mid G) \leqslant \delta^{-\kappa}, \qquad \omega \in G, \tag{7.29}$$

where $C > 0$ is an absolute constant to be determined in the proof of Proposition 7.30.

We will prove Proposition 7.28 in such a way that the various configurations "G" will form a nested sequence. So, once the proposition has been established for all pairs (λ, t) with $t \in \mathcal{T}$ and $\lambda \in \Lambda(t)$, then the "last" set G will satisfy (7.29) for all pairs $t \in \mathcal{T}$ and $\lambda \in \Lambda(t)$ simultaneously. We start with the easiest cases where $\lambda \approx t$. The value of the constant "C_ϵ" will change many times during the proof, but it will always remain $C_\epsilon \approx_\delta 1$.

Pairs (λ, t) with $t \leqslant \delta^{-\kappa/10}\lambda$. Let $\lambda \in \Lambda$ and $t \in \mathcal{T}$ with $t \leqslant \delta^{-\kappa/10}\lambda$. In this case we apply the claim proved in Section 7.2: the conclusion is that there exists a $(\delta, \delta, s, 4C_\epsilon\delta^{-\epsilon})$-configuration $G_1 \subset \Omega$ satisfying (7.29) for the fixed pair (λ, t). (To be perfectly accurate, one needs to apply the proof of the claim with constant $C\kappa$ in place of κ.) Next, we simply repeat the argument inside G_1, and for all the pairs $(\lambda, t) \in \Lambda \times \mathcal{T}$ with $t \leqslant \delta^{-\kappa/10}\lambda$, in arbitrary order. This involves refining Ω at most $\lesssim \epsilon^{-2}$ times, so the final product of this argument remains a $(\delta, \delta, s, C_\epsilon\delta^{-\epsilon})$-configuration.

Before launching to the main argument – treating the cases $\lambda \leqslant \delta^{\kappa/10}t$ – we use (7.29) to complete the proof of Theorem 7.5.

Proposition 7.30. *Assume that (7.29) holds for simultaneously for all $(\lambda, t) \in \Lambda \times \mathcal{T}$. Then, if the absolute constant $C > 0$ is large enough, we have*

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon/2}, \kappa^{-1}}(\omega \mid G) \leqslant \delta^{-2\kappa}, \qquad \omega \in G \tag{7.31}$$

simultaneously for all $\delta \leqslant \lambda \leqslant t \leqslant 1$ (not necessarily from $\Lambda \times \mathcal{T}$).

Proof. Let $\delta \leqslant \lambda \leqslant t \leqslant 1$. Let $\underline{\lambda} \in \Lambda$ and $\underline{t} \in \mathcal{T}(\lambda)$ be elements with $\underline{\lambda} \leqslant \lambda \leqslant \delta^{-\epsilon/2}\underline{\lambda}$ and $\underline{t} \leqslant t \leqslant \delta^{-\epsilon/2}\underline{t}$. Recall that

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon/2}, \kappa^{-1}}(\omega \mid G) = |\{\omega' \in (G_\sigma^\delta)_{\lambda,t}^{\delta^{-\epsilon/2}}(\omega) : \kappa^{-1}R_\sigma^\delta(\omega) \cap \kappa^{-1}R_\sigma^\delta(\omega') \neq \emptyset\}|, \qquad \omega \in G,$$

where G_σ^δ is the (δ, σ)-skeleton of G (with $\sigma = \delta/\sqrt{\lambda t}$). An unpleasant technicality is that $\bar{\sigma} = \delta/\sqrt{\underline{\lambda}\underline{t}} \in [\sigma, \delta^{-\epsilon/2}\sigma]$ might be a little different from σ, so elements of G_σ^δ are not automatically elements of $G_{\bar{\sigma}}^\delta$. However, for every $\omega' = (q, w) \in G_\sigma^\delta$, we may pick $\bar{\omega}' = (q, \bar{w}) \in G_{\bar{\sigma}}^\delta$ with $(q, w) \prec (q, \bar{w})$, and in particular $|w - \bar{w}| \leqslant C\bar{\sigma}$ for an absolute constant $C \geqslant 1$. Then, it is straightforward to check that

$$\omega' \in (G_\sigma^\delta)_{\lambda,t}^{\delta^{-\epsilon/2}}(\omega) \quad \Longrightarrow \quad \bar{\omega}' \in (G_{\bar{\sigma}}^\delta)_{\underline{\lambda},\underline{t}}^{\delta^{-\epsilon}}(\omega), \qquad \omega \in G, \tag{7.32}$$

and

$$\kappa^{-1}R_{\sigma}^{\delta}(\omega) \cap \kappa^{-1}R_{\sigma}^{\delta}(\omega') \neq \emptyset \quad \Longrightarrow \quad C\kappa^{-1}R_{\bar{\sigma}}^{\delta}(\omega) \cap C\kappa^{-1}R_{\bar{\sigma}}^{\delta}(\bar{\omega}') \neq \emptyset. \tag{7.33}$$

The implication (7.33) follows from the inclusion $\kappa^{-1}R_{\sigma}^{\delta}(\omega') \subset C\kappa^{-1}R_{\bar{\sigma}}^{\delta}(\bar{\omega}')$ (note that $\bar{\sigma} \geqslant \sigma$). Regarding (7.32), it is worth noting that the implication is even true in the special case $\lambda \leqslant \delta^{1-\epsilon/2}$ (recall Definition 7.1) since in that case $\underline{\lambda} \leqslant \delta^{1-\epsilon}$.

Finally, observe that the map $\omega' \mapsto \bar{\omega}'$ is at most $\delta^{-\epsilon}$-to-1: if $(q, w_1), (q, w_2), \ldots, (q, w_N) \in G_{\sigma}^{\delta}$ are distinct, and $\bar{\omega}' = (q, \bar{w})$ is the image of them all, then $|w_i - w_j| \gtrsim N\sigma$ for some $1 \leqslant i \neq j \leqslant N$, and on the other hand $\max\{|\bar{w} - w_i|, |\bar{w} - w_j|\} \lesssim \bar{\sigma} \leqslant \delta^{-\epsilon/2}\sigma$.

Combining this with (7.32)-(7.33), we find

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon/2},\kappa^{-1}}(\omega \mid G) \leqslant \delta^{-\epsilon} m_{\delta,\underline{\lambda},\underline{t}}^{\delta^{-\epsilon},C\kappa^{-1}}(\omega \mid G) \overset{(7.29)}{\leqslant} \delta^{-\kappa-\epsilon}, \qquad \omega \in G.$$

This proves (7.31), since $\epsilon \leqslant \kappa$ (by the choices in Section 7.1). $\square$

7.6. **Proof of Proposition 7.28.** We then arrive at the core of the proof of Theorem 7.5.

7.6.1. *Structure of the proof Proposition 7.28.* Very much like in Section 7.4, we will enumerate the pairs (λ, t) with $t \in \mathcal{T}$, and $\lambda \in \Lambda(t) \cap [\delta, \delta^{\kappa/10}t]$, and we will construct a decreasing sequence of (δ, δ, s)-configurations $G_1 \supset G_2 \supset \ldots$ such that G_j satisfies (7.29) for the pair (λ_j, t_j) – and therefore automatically for all pairs (λ_i, t_i) with $1 \leqslant i \leqslant j$. We will show inductively that $|G_j| \approx_{\delta} |\Omega|$.

In contrast to Section 7.4, this time the ordering of the pairs (λ_j, t_j) matters. We will do this as follows. We enumerate the elements of $\mathcal{T}$ arbitrarily. Then, if $t_j \in \mathcal{T}$ is fixed, we enumerate the pairs (λ, t_j) with $\lambda \in \Lambda(t_j) \cap [\delta, \delta^{\kappa/10}t_j]$ in increasing order. Thus, the first pair is (δ, t_j), the second one $(\delta^{1-\epsilon/2}, t_j)$, and so on. This has the crucial benefit that when we are in the process of proving (7.29) for a fixed pair (λ, t_j), we may already assume that (the current) G satisfies (7.29) for all pairs (λ', t_j) with $\lambda' \in \Lambda(t_j)$ and $\lambda' < \lambda$.

7.6.2. *Setting up the induction.* We will then begin to implement the strategy outlined above. Fix $t := t_j \in \mathcal{T}$ arbitrarily, and for the remainder of the proof. We enumerate $\Lambda(t) \cap [\delta, \delta^{\kappa/10}t]$ in increasing order, with the abbreviation $|\Lambda| := |\Lambda(t) \cap [\delta, \delta^{\kappa/10}t]|$:

$$\delta = \lambda_1 < \lambda_2 < \ldots < \lambda_{|\Lambda|} \leqslant \delta^{\kappa/10}t. \tag{7.34}$$

For each index $1 \leqslant l \leqslant |\Lambda|$ we also define a constant $C_l \geqslant 1$ in such a way that the sequence $C_1 > C_2 > \ldots > C_{|\Lambda|} \geqslant 1$ is very rapidly decreasing, more precisely

$$C_{l+1} = A(\epsilon, \kappa)^{-1}C_l, \qquad 1 \leqslant l < |\Lambda| \tag{7.35}$$

for a suitable constant $A(\epsilon, \kappa) \geqslant 1$, depending only on κ, and to be determined later, precisely right after (7.51). To complete the definition of the sequence $\{C_l\}$, we specify its smallest (last) element:

$$C_{|\Lambda|} := C\kappa^{-1}, \tag{7.36}$$

where $\kappa > 0$ is the parameter given in Theorem 7.5, and $C > 0$ is the absolute constant from (7.29). With these definitions, and noting that $|\Lambda| \leqslant C/\epsilon$ for an absolute constant $C > 0$, we have

$$C_1 = A(\epsilon, \kappa)^{|\Lambda|}C_{|\Lambda|} \leqslant CA(\epsilon, \kappa)^{C/\epsilon}\kappa^{-1} \overset{(7.10)}{\leqslant} \delta^{-\epsilon_{\max}}, \qquad \delta \in (0, \delta_0]. \tag{7.37}$$

We will prove the following by induction on $k \in \{1, \ldots, |\Lambda|\}$: there exists a decreasing sequence of $(\delta, \delta, s, C_\epsilon \delta^{-\epsilon})$-configurations $G_1 \supset \ldots \supset G_k$ such that $|G_l| \approx_\delta |\Omega|$ for all $1 \leqslant l \leqslant k$, and such that the following slightly stronger version of (7.29) holds:

$$m_{\delta,\lambda_l,t}^{C_l\delta^{-\epsilon},C_l}(\omega \mid G_l) \leqslant \delta^{-\kappa}, \qquad \omega \in G_l,\ 1 \leqslant l \leqslant k. \tag{7.38}$$

Once we have accomplished this for $k = |\Lambda|$, we set $G := G_{|\Lambda|}$. Then (7.29) holds for G (by (7.36)), and for all pairs (λ, t) with $\lambda \in \Lambda(t)$. After this, we may repeat the same procedure for all $t \in \mathcal{T}$ in arbitrary order (but always working inside the configurations we have previously constructed). This will complete the proof of Proposition 7.28.

Remark 7.39. Notice that the constants "C_l" in (7.38) decrease (rapidly) as l increases. The idea is that we can prove (7.38) with index "$k+1$" and the smaller constant C_{k+1}, provided that we already have (7.38) for all $1 \leqslant l \leqslant k$, and the much larger constants $C_l \gg C_{k+1}$.

7.6.3. *The case* $k = 1$. This case is a consequence of (7.26) applied with $\lambda = \lambda_1 = \delta$, with $G_1 := \Omega$. Note that in this case $\sigma = \delta/\sqrt{\lambda t} = \sqrt{\lambda/t} = \Sigma$, so (7.26) with $\lambda = \delta$ (and our fixed $t \in \mathcal{T}$) can be rewritten as

$$m_{\delta,\delta,t}^{\delta^{-\mathbf{A}\epsilon_{\max}},\delta^{-\mathbf{A}\epsilon_{\max}}}(\omega \mid \Omega) \leqslant \delta^{-s\kappa/100}\lambda^s|P_0|_\delta \leqslant \delta^{-\kappa}, \qquad \omega \in \Omega_\Sigma^\delta. \tag{7.40}$$

This is actually much stronger than what we need in (7.38), since $\epsilon < \epsilon_{\max}$, and $C_1 \sim_{\epsilon,\kappa} 1$. One small point of concern is that (7.38) is a statement about $\omega \in G_1 = \Omega$, whereas (7.40) deals with $\omega \in \Omega_\Sigma^\delta = \Omega_\sigma^\delta$. This is not a problem thanks to the following elementary lemma, which will also be useful later:

Lemma 7.41. *Let $0 < \delta \leqslant \lambda \leqslant t \leqslant 1$ and $\rho_\lambda, \rho_t, C \geqslant 1$. Let $G \subset \Omega$ and $\omega \in G$. Let $\bar{\omega} \in G_\sigma^\delta$ be the parent of ω in the (δ, σ)-skeleton G_σ^δ, where $\sigma = \delta/\sqrt{\lambda t}$ as usual. Then,*

$$m_{\delta,\lambda,t}^{\rho_\lambda,\rho_t,C/A}(\bar{\omega} \mid G) \leqslant m_{\delta,\lambda,t}^{\rho_\lambda,\rho_t,C}(\omega \mid G) \leqslant m_{\delta,\lambda,t}^{\rho_\lambda,\rho_t,AC}(\bar{\omega} \mid G), \tag{7.42}$$

where $A \geqslant 1$ is absolute.

In particular, (7.40) for $\omega \in \Omega_\Sigma^\delta$ implies (7.38) for all $\omega \in \Omega$, at the cost of replacing the second $\delta^{-\mathbf{A}\epsilon_{\max}}$ by $\delta^{-\mathbf{A}\epsilon_{\max}}/A$ (which is still much bigger than $C_1 \sim_{\epsilon,\kappa} 1$).

Proof of Lemma 7.41. We only prove the upper bound, since the lower bound is established in a similar fashion. Let us spell out the quantities in (7.42):

$$m_{\delta,\lambda,t}^{\rho_\lambda,\rho_t,C}(\omega \mid G) = |\{\omega' \in (G_\sigma^\delta)_{\lambda,t}^{\rho_\lambda,\rho_t}(\omega) : CR_\sigma^\delta(\omega) \cap CR_\sigma^\delta(\omega') \neq \emptyset\}|$$

and

$$m_{\delta,\lambda,t}^{\rho_\lambda,\rho_t,AC}(\bar{\omega} \mid G) = |\{\omega' \in (G_\sigma^\delta)_{\lambda,t}^{\rho_\lambda,\rho_t}(\bar{\omega}) : ACR_\sigma^\delta(\bar{\omega}) \cap ACR_\sigma^\delta(\omega') \neq \emptyset\}|.$$

The crucial observation is that if the point $\omega \in G$ is written as $\omega = (p, v)$, then the parent $\bar{\omega} = (p, \mathbf{v})$, where $|v - \mathbf{v}| \lesssim 1$, and the "$p$-component" remains unchanged. In particular,

$$\omega' \in (G_\sigma^\delta)_{\lambda,t}^{\rho_\lambda,\rho_t}(\omega) \quad \Longleftrightarrow \quad \omega' \in (G_\sigma^\delta)_{\lambda,t}^{\rho_\lambda,\rho_t}(\bar{\omega}),$$

since these inclusions only concern the p-components of $\omega, \omega', \bar{\omega}$. Therefore, (7.42) boils down to the observation

$$CR_\sigma^\delta(\omega) \cap CR_\sigma^\delta(\omega') \neq \emptyset \quad \Longrightarrow \quad ACR_\sigma^\delta(\bar{\omega}) \cap CR_\sigma^\delta(\omega') \neq \emptyset,$$

which follows from $CR_\sigma^\delta(\omega) \subset ACR_\sigma^\delta(\bar{\omega})$ (for $A \geqslant 1$ sufficiently large). $\square$

7.6.4. *Cases* $1 < k+1 \leqslant |\Lambda|$. We then assume that the $(\delta, \delta, s, C_\epsilon \delta^{-\epsilon})$-configurations $G_1 \supset \ldots \supset G_k$ have already been constructed for some $1 \leqslant k < |\Lambda|$. We next explain how to construct the set G_{k+1}. To be precise, our task is to construct a $(\delta, \delta, s, C_\epsilon \delta^{-\epsilon})$-configuration $G_{k+1} \subset G_k$ with the properties $C_\epsilon \approx_\delta 1$, $|G_{k+1}| \approx_\delta |G_k|$, and

$$m_{\delta, \lambda_{k+1}, t}^{C_{k+1}\delta^{-\epsilon}, C_{k+1}}(\omega \mid G_{k+1}) \leqslant \delta^{-\kappa}, \qquad \omega \in G_{k+1}. \tag{7.43}$$

We abbreviate

$$\lambda := \lambda_{k+1} \quad \text{and} \quad \sigma := \delta/\sqrt{\lambda_{k+1} t} \tag{7.44}$$

for the duration of this argument. We write $G_k = \{(p, v) : p \in P_k \text{ and } v \in G_k(p)\}$ with $|G_k(p)| \equiv M_k$ for all $p \in P_k$. Here $|P_k| \approx_\delta |P|$ and $M_k \approx_\delta M$ since $|G_k| \approx_\delta |\Omega| = M|P|$.

Note that the multiplicity function appearing in (7.43) counts elements in the (δ, σ)-skeleton of G_{k+1}. It would be desirable to know that $|E_\sigma(p)| \equiv M_\sigma$ is a constant independent of $p \in P_k$, where

$$E_\sigma(p) = \{\mathbf{v} \in \mathcal{S}_\sigma(p) : v \prec \mathbf{v} \text{ for some } v \in G_k(p)\}$$

is the (δ, σ)-skeleton of $G_k(p)$. This may not be true to begin with, but may be accomplished with a small pruning, as follows. For each $(p, \mathbf{v}) \in (G_k)^\delta_\sigma$, let

$$M(p, \mathbf{v}) = |\{(p', v) \in G_k : (p', v) \prec (p, \mathbf{v})\}| = |\{v \in G_k(p) : v \prec \mathbf{v}\}|.$$

The second equation follows from $p \in \mathcal{D}_\delta$ (that is, $(p', v) \prec (p, \mathbf{v})$ implies $p' = p$). Now, for each $p \in P_k$ fixed, we pigeonhole an integer $M(p) \geqslant 1$ and a subset $E'_\sigma(p) \subset (G_k)^\delta_\sigma(p)$ such that $M(p) \leqslant M(p, \mathbf{v}) \leqslant 2M(p)$ for all $\mathbf{v} \in E'_\sigma(p)$, and further

$$|\{v \in G_k(p) : v \prec \mathbf{v} \text{ for some } \mathbf{v} \in E'_\sigma(p)\}| \approx_\delta |G_k(p)| = M_k. \tag{7.45}$$

It follows that $M(p) \cdot |E'_\sigma(p)| \approx_\delta M_k \approx_\delta M$ for all $p \in P_k$. Next, we pigeonhole an integer $M_\sigma \geqslant 1$, and a subset $\bar{P}_k \subset P_k$ such that $M_\sigma \leqslant |E'_\sigma(p)| \leqslant 2M_\sigma$ for all $p \in \bar{P}_k$, and $|\bar{P}_k| \approx_\delta |P_k|$. With this definition, let

$$\bar{G} := \{(p, v) : p \in \bar{P}_k,\ v \in G_k(p), \text{ and } v \prec \mathbf{v} \text{ for some } \mathbf{v} \in E'_\sigma(p)\}.$$

Thus, the (δ, σ)-skeleton of $\bar{G}$ is $\bar{G}^\delta_\sigma = \{(p, \mathbf{v}) : p \in \bar{P}_k \text{ and } \mathbf{v} \in E'_\sigma(p)\}$, and for each $p \in \bar{P}_k$, the (δ, σ)-skeleton of $\bar{G}(p)$ is $\bar{G}^\delta_\sigma(p) = E'_\sigma(p)$, which has constant cardinality M_σ (up to a factor of 2). To simplify notation, we denote in the sequel $E_\sigma(p) := E'_\sigma(p)$ for $p \in \bar{P}_k$. Note that

$$|\bar{G}| = \sum_{p \in \bar{P}_k} \sum_{\mathbf{v} \in E_\sigma(p)} |\{v \in G_k(p) : v \prec \mathbf{v}\}| \overset{(7.45)}{\approx_\delta} |\bar{P}_k| M_k \approx_\delta |P_k| M_k = |G_k|.$$

To summarise, the procedure above has reduced G_k to a subset $\bar{G} \subset G_k$ of size $|\bar{G}| \approx_\delta |G_k|$, and further we have gained the following properties:

$$|\bar{G}^\delta_\sigma(p)| = |E_\sigma(p)| \in [M_\sigma, 2M_\sigma], \qquad p \in \bar{P}_k, \tag{7.46}$$

and

$$|\{v \in G_k(p) : v \prec \mathbf{v}\}| = M(p) \approx_\delta M/M_\sigma, \qquad p \in \bar{P}_k,\ \mathbf{v} \in E_\sigma(p). \tag{7.47}$$

We also record for future reference that

$$M_\sigma \sim |E_\sigma(p)| \gtrapprox_\delta \delta^\epsilon \sigma^{-s} = \delta^\epsilon \left(\frac{\sqrt{\lambda t}}{\delta} \right)^s, \tag{7.48}$$

since $\bar{G}(p)$ is a non-empty $(\delta, s, C_\epsilon\delta^{-\epsilon})$-set. (It follows from (7.46)-(7.47) that $|\bar{G}(p)| \approx_\delta M$ for all $p \in \bar{P}_k$, but $\bar{G}$ may fail to be a $(\delta, \delta, s, C\delta^{-\epsilon})$-configuration in the strict sense that the sets $|\bar{G}(p)|$ have equal cardinality. This will not be needed, so we make no attempt to prune back this property. The moral here is that the set G_k can be completely forgotten: we will only need $\bar{G} \subset G_k$ in the sequel, and the rough constancy of $|\bar{G}^{\delta}_{\sigma}(p)|$.)

We then begin the construction of the set $G_{k+1} \subset \bar{G}$. This argument requires another induction, in fact very similar to the one we saw during the proof of Proposition 5.2. This is not too surprising, given that the "base case" $\delta = \lambda$, or in other words $k = 1$, of (7.38) followed directly from Proposition 5.2. To reduce confusion with indices, the letters "k, l" will from now on refer to the sets in the sequence $G_1, \ldots, G_k$ already constructed in our "exterior" induction and we will use letters "i, j" are reserved for the "interior" induction required to construct G_{k+1}.

Remark 7.49. It may be worth noting that the "exterior" induction runs $\sim 1/\epsilon$ times, whereas the "interior" induction below runs only $\lceil 20/\kappa \rceil \sim 1/\kappa$ times. This is significant, because it is legitimate to increase (say: double) the constant "ϵ" roughly $1/\kappa$ times and still rest assured that the resulting final constant is $\lesssim 2^{1/\kappa}\epsilon \leqslant \epsilon_{\max}$ (a small number). In contrast, it would not be legitimate to double the constant "ϵ" roughly $1/\epsilon$ times in the "exterior" induction.

We start by setting $h := \lceil 20/\kappa \rceil$, and defining the auxiliary sequence of exponents

$$100\epsilon < \epsilon_h < \epsilon_{h-1} < \ldots < \epsilon_0 < \epsilon_{\max}/100, \tag{7.50}$$

where $\epsilon_j < \epsilon_{j-1}/10$ for all $1 \leqslant j \leqslant h$. This choice of the sequence $\{\epsilon_j\}$ is possible thanks to the relation between the constants "ϵ" and "$\epsilon_{\max}$" explained in Section 7.1. Namely, in (7.9) we required that

$$C \cdot 10^{100/\kappa}\epsilon \leqslant \epsilon_{\max}.$$

In addition to the exponents $\{\epsilon_j\}$, we also define an auxiliary sequence of constants $\{\mathbf{C}_j\}$:

$$C_{k+1} \ll \mathbf{C}_h \ll \mathbf{C}_{h-1} \ll \ldots \ll \mathbf{C}_0 \ll C_k. \tag{7.51}$$

The necessary rate of decay for the sequence $\{\mathbf{C}_j\}$ turns out to be of the form $A\mathbf{C}_{j+1}^5 \leqslant \mathbf{C}_j$ for an absolute constant $A \geqslant 1$. There are $h = \lceil 20/\kappa \rceil$ constants in the sequence, so the sequence $\{\mathbf{C}_j\}$ can be found, satisfying (7.51), since $C_k = A(\epsilon, \kappa)C_{k+1}$ by (7.35). This is the requirement which determines the size of the constant $A(\epsilon, \kappa)$. It may worth remarking that the constant $A(\epsilon, \kappa)$ necessarily depends on both ϵ and κ. This is because the index "k" in C_k, C_{k+1} ranges in $\{1, \ldots, C/\epsilon\}$ for an absolute constant $C \geqslant 1$, so C_{k+1} depends on both ϵ, κ. Given the requirement for the constants $\mathbf{C}_j$ stated below (7.51), we see that the size of the multiplicative gap $A(\epsilon, \kappa) = C_k/C_{k+1}$ also depends on both ϵ, κ.

Recall that our goal is to define the next set "G_{k+1}" satisfying (7.43). To do so (as in the proof of Proposition 5.2), we consider an auxiliary sequence of sets $\bar{G} = \mathbf{G}_0 \supset \mathbf{G}_1 \supset \ldots \supset \mathbf{G}_j$. Finally, we will set $G_{k+1} := \mathbf{G}_j$ for a suitable member of this auxiliary sequence (or in fact a slight refinement of $\mathbf{G}_j$).

Recalling from (7.44) that $\sigma = \delta/\sqrt{\lambda t}$, and $\lambda = \lambda_{k+1}$, and writing

$$\rho_j := \mathbf{C}_j\delta^{-\epsilon}, \tag{7.52}$$

we will abbreviate

$$m_j(\omega \mid \mathbf{G}) := m_{\delta,\lambda,t}^{\delta^{-\epsilon_j},\rho_j,\mathbf{C}_j}(\omega \mid \mathbf{G})$$
$$= |\{\omega' \in (\mathbf{G}_\sigma^\delta)_{\lambda,t}^{\delta^{-\epsilon_j},\rho_j}(\omega) : \mathbf{C}_j R_\sigma^\delta(\omega) \cap \mathbf{C}_j R_\sigma^\delta(\omega') \neq \emptyset\}| \tag{7.53}$$

for $\mathbf{G} \subset \bar{G}$ and $\omega \in \bar{G}$. We recall that the constant $\delta^{-\epsilon_j}$ refers to the range of the tangency parameter "λ", and the constant ρ_j refers to the range of the distance parameter "t". It is worth noting that

$$\mathbf{C}_{j+1} \leqslant \mathbf{C}_j \quad \text{and} \quad \rho_{j+1} \leqslant \rho_j \quad \text{and} \quad \delta^{-\epsilon_{j+1}} \leqslant \delta^{-\epsilon_j},$$

so $m_h \leqslant m_{h-1} \leqslant \ldots \leqslant m_0$. It is also worth noting that since $\epsilon_j > 10\epsilon$, the "tangency" range $\delta^{-\epsilon_j}$ is very much larger than the "distance" range $\rho_j \sim_{\epsilon,\kappa} \delta^{-\epsilon}$, assuming that $\delta > 0$ is sufficiently small in terms of ϵ, κ.

We start by recording the "trivial" upper bound

$$m_0(\omega \mid \mathbf{G}) \leqslant m_0(\omega \mid \Omega_0) \lesssim \mathbf{C}_0 \delta^{-s-\epsilon}, \qquad \omega \in \bar{G},\ \mathbf{G} \subset \bar{G}, \tag{7.54}$$

which has nothing to do with the parameters $\delta^{-\epsilon_0}, \rho_0$, and only has to do with the constant $\mathbf{C}_0 \sim_{\epsilon,\kappa} 1$. The first inequality is clear. To see the second inequality, fix $\omega = (p, v) \in \bar{G}$ and $(p', v') \in (\Omega_0)_\sigma^\delta \subset \{(q, w) : q \in P_0 \text{ and } w \in \mathcal{S}_\sigma(q)\}$ such that

$$\mathbf{C}_0 R_\sigma^\delta(p', v') \cap \mathbf{C}_0 R_\sigma^\delta(p, v) \neq \emptyset.$$

Then $v' \in \mathcal{S}_\sigma(p')$ and $|v' - v| \lesssim \mathbf{C}_0 \sigma$. But $\mathcal{S}_\sigma(p')$ is σ-separated, so this can only happen for $\lesssim \mathbf{C}_0$ choices of v'. This gives (7.54), recalling that $|P_0| \leqslant \delta^{-s-\epsilon}$ by assumption (7.6).

The trivial inequality (7.54) tells us that the estimate (7.43) holds automatically with $G_{k+1} = \bar{G}$ and $\kappa = 2s$ (with room to spare), assuming that $\delta, \epsilon > 0$ is chosen so small that $\mathbf{C}_0 \leqslant \delta^{-\epsilon} \leqslant \delta^{-s/2}$. So, we may assume that $0 < \kappa \leqslant 2s$. Let $0 = \kappa_1 < \kappa_2 < \ldots < \kappa_h = 2s$ be a $(\kappa s/10)$-dense sequence in $[0, 2s]$. Thus $h \leqslant 20/\kappa$. As already hinted above, we now define a decreasing sequence of sets $\bar{G} = \mathbf{G}_0 \supset \mathbf{G}_1 \supset \ldots \supset \mathbf{G}_l$, where $l \leqslant h$. We set $\mathbf{G}_0 := \bar{G}$, and in general we will assume inductively that $|\mathbf{G}_{j+1}| \geqslant \frac{1}{2}|\mathbf{G}_j|$ for $j \geqslant 0$ (whenever $\mathbf{G}_j, \mathbf{G}_{j+1}$ have been defined). Note that $m_0(\omega \mid \mathbf{G}_0) \leqslant \delta^{-2s} = \delta^{-\kappa_h}$ by (7.54), for all $\omega \in \mathbf{G}_0$, provided that $\delta > 0$ is small enough.

Let us then assume that the sets $\mathbf{G}_0 \supset \ldots \supset \mathbf{G}_j$ have already been defined. We also assume inductively that

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon_j},\rho_j,\mathbf{C}_j}(\omega \mid \mathbf{G}_j) = m_j(\omega \mid \mathbf{G}_j) \leqslant \delta^{-\kappa_{h-j}}, \qquad \omega \in \mathbf{G}_j. \tag{7.55}$$

This is true by (7.54) for $j = 0$, as we observed above. Define

$$\mathbf{H}_j := \{\omega \in \mathbf{G}_j : m_{j+1}(\omega \mid \mathbf{G}_j) \geqslant \delta^{-\kappa_{h-(j+1)}}\}.$$

Note that $\kappa_{h-(j+1)} < \kappa_{h-j}$. So, $\mathbf{H}_j$ is the subset of $\mathbf{G}_j$ where the lower bound for the $(j+1)^{st}$ multiplicity nearly matches the (inductive) upper bound on the j^{th} multiplicity.

There are two options.

(1) If $|\mathbf{H}_j| \geqslant \frac{1}{2}|\mathbf{G}_j|$, then we set $\mathbf{H} := \mathbf{H}_j$, and the construction of the sets $\mathbf{G}_j$ terminates. We will see that this case cannot occur as long as $\kappa_{h-j} > \kappa$.

(2) If $|\mathbf{H}_j| < \frac{1}{2}|\mathbf{G}_j|$, then the set $\mathbf{G}_{j+1} := \mathbf{G}_j \setminus \mathbf{H}_j$ has $|\mathbf{G}_{j+1}| \geqslant \frac{1}{2}|\mathbf{G}_j|$, and moreover

$$m_{j+1}(\omega \mid \mathbf{G}_{j+1}) \leqslant m_{j+1}(\omega \mid \mathbf{G}_j) \leqslant \delta^{-\kappa_{h-(j+1)}}, \qquad \omega \in \mathbf{G}_{j+1}.$$

In other words, $\mathbf{G}_{j+1}$ is a valid "next set" in our sequence $\mathbf{G}_0 \supset \ldots \supset \mathbf{G}_{j+1}$, and the inductive construction may proceed.

If (and since) case (1) does not occur for indices $j \geqslant 0$ with $\kappa_{h-j} > \kappa$, we can keep constructing the sets $\mathbf{G}_j$ until the first index "j" where $\kappa_{h-j} \leqslant \kappa$. At this stage, the set

$$G_{k+1} := \mathbf{G}_j \tag{7.56}$$

satisfies $m_j(\omega \mid G_{k+1}) \leqslant \delta^{-\kappa}$ for all $\omega \in G_{k+1}$ by the inductive assumption (7.55). This implies (7.43), since $C_{k+1} \leqslant \mathbf{C}_j$ by (7.51). Moreover, $|G_{k+1}| \geqslant 2^{-j}|\bar{G}| \geqslant 2^{-20/\kappa}|\bar{G}| \approx_\delta |G_k|$, so G_{k+1} is a valid "next set" in the sequence $\{G_k\}$. To be precise, we still need to apply Lemma 3.13, and thereby refine G_{k+1} (as in (7.56)) to a $(\delta, \delta, s, C_\epsilon\delta^{-\epsilon})$-configuration of cardinality $\approx_\delta |G_k|$. This will complete the definition of G_{k+1}.

Thus, to complete the construction of the sequence $\{G_k\}$, and the proof of Theorem 7.5, it suffices to verify that the "hard" case (1) cannot occur for any $j \geqslant 0$ such that $\kappa_{h-j} > \kappa$. To prove this, we make a counter assumption:

Counter assumption: Case (1) occurs at some index $j \in \{0, \ldots, h\}$ with $\kappa_{h-j} > \kappa$.

7.6.5. *Deriving a contradiction.* The overall strategy is similar to the one we have already encountered in the proofs of Proposition 5.2 and Theorem 6.5. We will use the counter assumption to produce a "large" collection of incomparable (δ, σ)-rectangles, each of which has a high (λ-restricted) type relative to a certain $(\delta, \epsilon_{\max})$-almost t-bipartite pair (W, B) of subsets of P. Eventually, the existence of these rectangles will contradict the upper bound established in Theorem 6.5. The hypothesis (6.7) of Theorem 6.5 will be valid thanks to our previous refinements, specifically (7.26).

We write $\bar{\kappa} := \kappa_{h-j}$ and (recalling the $(\kappa s)/10$-density of the sequence $\{\kappa_j\}$),

$$\kappa_{h-(j+1)} =: \bar{\kappa} - \zeta, \qquad \text{where } \zeta \leqslant (\kappa s)/10 \leqslant (\bar{\kappa} s)/10.$$

We also abbreviate

$$\mathbf{G} := \mathbf{G}_j \quad \text{and} \quad \mathbf{H} := \mathbf{H}_j = \{\omega \in \mathbf{G} : m_{j+1}(\omega \mid \mathbf{G}) \geqslant \delta^{-\bar{\kappa}+\zeta}\} \subset \mathbf{G},$$

and we recall that $|\mathbf{H}| \geqslant \frac{1}{2}|\mathbf{G}| \approx_\delta |G_k| \approx_\delta M|P|$ by the assumption that we are in case (1). Finally, we will abbreviate

$$n := \delta^{-\bar{\kappa}+\zeta}. \tag{7.57}$$

To spell out the definition of "m_{j+1}" (recall (7.53)), we have

$$|\{\omega' \in (\mathbf{G}^\delta_\sigma)^{\delta^{-\epsilon_{j+1}}, \rho_{j+1}}_{\lambda,t}(\omega) : \mathbf{C}_{j+1}R^\delta_\sigma(\omega) \cap \mathbf{C}_{j+1}R^\delta_\sigma(\omega') \neq \emptyset\}| \geqslant n, \qquad \omega \in \mathbf{H}. \tag{7.58}$$

On the other hand, by the inductive assumption (7.55) applied to $\mathbf{G} = \mathbf{G}_j$, and recalling that $\bar{\kappa} = \kappa_{h-j}$, we have

$$|\{\omega' \in (\mathbf{G}^\delta_\sigma)^{\delta^{-\epsilon_j}, \rho_j}_{\lambda,t}(\omega) : \mathbf{C}_j R^\delta_\sigma(\omega) \cap \mathbf{C}_j R^\delta_\sigma(\omega') \neq \emptyset\}| \leqslant \delta^{-\bar{\kappa}} = \delta^{-\zeta} n, \quad \omega \in \mathbf{G}. \tag{7.59}$$

The numerology is not particularly important yet, but it is crucial that a certain lower bound for $m_{j+1}(\cdot \mid \mathbf{G})$ holds in a large subset $\mathbf{H} \subset \mathbf{G}$, whereas a nearly matching upper bound for $m_j(\omega \mid \mathbf{G})$ holds for all $\omega \in \mathbf{G}$. Achieving this "nearly extremal" situation was the reason to define the sequence $\{\mathbf{G}_j\}$.

Remark 7.60. In fact, we will need (7.58)-(7.59) for $\omega \in \mathbf{H}_\sigma^\delta$ and $\omega \in \mathbf{G}_\sigma^\delta$ instead of $\omega \in \mathbf{H}$ and $\omega \in \mathbf{G}$, respectively. This is easily achieved, at the cost of changing the constants a little. Indeed, if $A \geqslant 1$ is a sufficiently large absolute constant, then (7.58)-(7.59) imply

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon_{j+1}},\rho_{j+1},A\mathbf{C}_{j+1}}(\omega \mid \mathbf{G}) \geqslant n, \qquad \omega \in \mathbf{H}_\sigma^\delta,$$

and

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon_j},\rho_j,\mathbf{C}_j/A}(\omega \mid \mathbf{G}) \leqslant \delta^{-\zeta} n, \qquad \omega \in \mathbf{G}_\sigma^\delta.$$

These inequalities follow from Lemma 7.41. It will be important that the constant $\mathbf{C}_j$ is substantially larger than $\mathbf{C}_{j+1}$, but we can arrange this so (recall the definition (7.51)) that even $\mathbf{C}_j/A \gg A\mathbf{C}_{j+1}$. To avoid burdening the notation with further constants, we will assume from now on that (7.58)-(7.59) hold as stated for $\omega \in \mathbf{H}_\sigma^\delta$ and $\omega \in \mathbf{G}_\sigma^\delta$, respectively.

The set $\mathbf{H} \subset \bar{G}$ may have lost the uniformity property (7.46) at scale σ. That is, we no longer know that all the (δ,σ)-skeletons $\mathbf{H}_\sigma^\delta(p) \subset \bar{G}_\sigma^\delta(p)$, for $p \in \bar{P}_k$, have roughly constant cardinality (let alone M_σ). (Recall that the set $\bar{P}_k \subset P_k$ was defined below (7.45).) We resuscitate this property by a slight pruning of $\mathbf{H}$. Note that

$$M|P| \approx_\delta |\mathbf{H}| = \sum_{p \in \bar{P}_k} \sum_{\mathbf{v} \in \mathbf{H}_\sigma^\delta(p)} |\{v \in \mathbf{H}(p) : v \prec \mathbf{v}\}|. \tag{7.61}$$

By pigeonholing, choose a number $\bar{M}_\sigma \geqslant 1$, and a subset $\bar{P} \subset \bar{P}_k$ with the properties $\bar{M}_\sigma \leqslant |\mathbf{H}_\sigma^\delta(p)| \leqslant 2\bar{M}_\sigma$ for all $p \in \bar{P}$, and such that the quantity on the right hand side of (7.61) is only reduced by a factor of $\approx_\delta 1$ when replacing $\bar{P}_k$ by $\bar{P}$. Thus,

$$M|P| \approx_\delta \sum_{p \in \bar{P}} \sum_{\mathbf{v} \in \mathbf{H}_\sigma^\delta(p)} |\{v \in \mathbf{H}(p) : v \prec \mathbf{v}\}| \leqslant |\bar{P}| \cdot 2\bar{M}_\sigma \cdot \max_{\mathbf{v}} |\{v \in \mathbf{H}(p) : v \prec \mathbf{v}\}|. \tag{7.62}$$

Here the "max" runs over all $\mathbf{v} \in \mathbf{H}_\sigma^\delta(p)$, with all possible $p \in \bar{P}$. Here $\mathbf{H}(p) \subset G_k(p)$ and $p \in \bar{P}_k$, so we see from (7.47) that the "max" is bounded by $\lessapprox_\delta M/M_\sigma$. Since evidently $\bar{M}_\sigma \leqslant 2M_\sigma$, we may now deduce that $\bar{M}_\sigma \approx_\delta M_\sigma$ and $|\bar{P}| \approx_\delta |\bar{P}_k|$. At this point we define $\bar{\mathbf{H}} := \{(p,v) \in \mathbf{H} : p \in \bar{P}\}$. Then it follows from (7.62) that $|\bar{\mathbf{H}}| \approx_\delta |\mathbf{H}| \approx_\delta M|P|$, and moreover

$$|\bar{\mathbf{H}}_\sigma^\delta(p)| = |\mathbf{H}_\sigma^\delta(p)| \sim \bar{M}_\sigma \approx_\delta M_\sigma, \qquad p \in \bar{P}. \tag{7.63}$$

7.6.6. *Finding a t-bipartite pair.* Next, we proceed to find a $(\delta, \epsilon_{\max})$-almost t-bipartite pair of subsets of P, very much like in the proof of Proposition 5.2. Let $\mathcal{B}$ be a cover of P by balls of radius $t/(4\rho_{j+1})$ such that the concentric balls of radius $2\rho_{j+1}t$ (that is, the balls $\{8\rho_{j+1}^2 B : B \in \mathcal{B}\}$) have overlap bounded by $O(\rho_{j+1}) = O_\epsilon(\delta^{-\epsilon}) \leqslant \delta^{-\epsilon_{\max}}$ (recall that $\rho_{j+1} = \mathbf{C}_{j+1}\delta^{-\epsilon}$). Then, we choose a ball $B(p_0, t/(4\rho_{j+1})) \in \mathcal{B}$ in such a way that the ratio

$$\theta := \frac{|\bar{P} \cap B(p_0, t/(4\rho_{j+1}))|}{|P \cap B(p_0, 2\rho_{j+1}t)|}$$

is maximised. Here $\bar{P} \subset \bar{P}_k \subset P_k$ is the subset of cardinality $|\bar{P}| \approx_\delta |P_k| \approx_\delta |P|$ we just found above, recall (7.63). We claim that $\theta \gtrapprox_\delta \delta^{\epsilon_{\max}}$: this follows immediately from the estimate

$$|\bar{P}| \leqslant \sum_{B \in \mathcal{B}} |\bar{P} \cap B| \leqslant \theta \sum_{B \in \mathcal{B}} |P \cap 8\rho_{k+1}^2 B| \leqslant \theta\delta^{-\epsilon_{\max}}|P|,$$

and since $|\bar{P}| \approx_\delta |P|$. Now, we set

$$W := \bar{P} \cap B(p_0, t/(4\rho_{j+1})) \quad \text{and} \quad B := P \cap B(p_0, 2\rho_{j+1}t) \setminus B(p_0, t/(2\rho_{j+1})), \tag{7.64}$$

so that

$$|B| \leqslant |P \cap B(p_0, 2\rho_{j+1}t)| = \theta^{-1}|W| \lessapprox_\delta \delta^{-\epsilon_{\max}}|W|. \tag{7.65}$$

We record at this point that

$$\operatorname{dist}(W, B) \geqslant \tfrac{1}{4}t/\rho_{j+1} \geqslant \delta^{\epsilon_{\max}}t \quad \text{and} \quad \operatorname{diam}(W \cup B) \leqslant 4\rho_{j+1}t \leqslant \delta^{-\epsilon_{\max}}t, \tag{7.66}$$

so the pair (W, B) is $(\delta, \epsilon_{\max})$-almost t-bipartite, independently of "j" or "k". This will be needed in an upcoming application of Theorem 6.5.

We then set

$$\mathbf{W} := \{(p, v) \in \bar{\mathbf{H}}^\delta_\sigma : p \in W\} \quad \text{and} \quad \mathbf{B} := \{(p, v) \in \mathbf{G} : p \in B\}. \tag{7.67}$$

We note that the "angular" components of $\mathbf{W}$ have separation σ, but the angular components of $\mathbf{B}$ are δ-separated; this is not a typo. Let us note that

$$|\mathbf{W}(p)| = |\bar{\mathbf{H}}^\delta_\sigma(p)|_\sigma \overset{(7.63)}{\sim} \bar{M}_\sigma \approx_\delta M_\sigma, \quad p \in W. \tag{7.68}$$

(For this purpose, it was important to choose $W \subset \bar{P}$.) Also, it follows from definitions of W, B that if $p \in W$, and $q \in P$ is arbitrary with $t/\rho_{j+1} \leqslant |p - q| \leqslant \rho_{j+1}t$, then $q \in B$. Consequently,

$$\omega \in \mathbf{W} \quad \Longrightarrow \quad (\mathbf{G}^\delta_\sigma)^{\delta^{-\epsilon_{j+1}}, \rho_{j+1}}_{\lambda,t}(\omega) \subset (\mathbf{B}^\delta_\sigma)^{\delta^{-\epsilon_{j+1}}, \rho_{j+1}}_{\lambda,t}(\omega).$$

For this inclusion to be true, it is important that in the definition of "B" we take into account all points in $\bar{P}_k$, and not only the refinement $\bar{P}$. Now this is certainly true, because we are even taking along all the points in P. From this, and since $\mathbf{W} \subset \mathbf{H}^\delta_\sigma$, and recalling (7.58), it follows

$$|\{\beta \in (\mathbf{B}^\delta_\sigma)^{\delta^{-\epsilon_{j+1}}, \rho_{j+1}}_{\lambda,t}(\omega) : \mathbf{C}_{j+1}R^\delta_\sigma(\omega) \cap \mathbf{C}_{j+1}R^\delta_\sigma(\beta) \neq \emptyset\}| \geqslant n > 0, \qquad \omega \in \mathbf{W}. \tag{7.69}$$

We also used the reduction explained in Remark 7.60 that we may assume (7.58) to hold for all $\omega \in \mathbf{H}^\delta_\sigma$. Without this reduction, (7.69) would instead hold with constant "$C\mathbf{C}_{j+1}$".

7.6.7. *The rectangles $\mathcal{R}^\delta_\sigma$.* We will produce a family of 100-incomparable (δ, σ)-rectangles with high λ-restricted type relative to (W, B). This will place us in a position to apply Theorem 6.5. Consider the (δ, σ)-rectangles $\{R^\delta_\sigma(\omega) : \omega \in \mathbf{W}\}$, and let

$$\mathcal{R}^\delta_\sigma \subset \{R^\delta_\sigma(\omega) : \omega \in \mathbf{W}\}.$$

be a maximal family of pairwise 100-incomparable elements. Some rectangles in $\mathcal{R}^\delta_\sigma$ may arise as $R^\delta_\sigma(\omega)$ for multiple distinct $\omega \in \mathbf{W}$. We quantify this by considering

$$m(R) = |\{\omega \in \mathbf{W} : R \sim_{100} R^\delta_\sigma(\omega)\}|, \qquad R \in \mathcal{R}^\delta_\sigma, \tag{7.70}$$

where "$\sim_{100}$" refers to 100-comparability. We note that since every $R \in \mathcal{R}^\delta_\sigma$ satisfies $R \sim_{100} R^\delta_\sigma(\omega)$ for some $\omega \in \mathbf{W}$, we have $m(R) \geqslant 1$ (and $m(R) \leqslant |\mathbf{W}| \lesssim \delta^{-4}$). By pigeonholing, we may find a subset $\bar{\mathcal{R}}^\delta_\sigma \subset \mathcal{R}^\delta_\sigma$ with the property $m(R) \equiv m \in [1, C\delta^{-4}]$ for all $R \in \bar{\mathcal{R}}^\delta_\sigma$, and moreover

$$\sum_{\omega \in \mathbf{W}} |\{R \in \bar{\mathcal{R}}^\delta_\sigma : R \sim_{100} R^\delta_\sigma(\omega)\}| \approx_\delta \sum_{\omega \in \mathbf{W}} |\{R \in \mathcal{R}^\delta_\sigma : R \sim_{100} R^\delta_\sigma(\omega)\}|.$$

Now, we have

$$|\bar{\mathcal{R}}^\delta_\sigma| = \frac{1}{m}\sum_{R\in\bar{\mathcal{R}}^\delta_\sigma} m(R) = \frac{1}{m}\sum_{R\in\bar{\mathcal{R}}^\delta_\sigma}\sum_{p\in W}\sum_{\mathbf{v}\in\mathbf{W}(p)} \mathbf{1}_{\{R\sim_{100}R^\delta_\sigma(p,\mathbf{v})\}}$$
$$\approx_\delta \frac{1}{m}\sum_{p\in W}\sum_{\mathbf{v}\in\mathbf{W}(p)} |\{R\in\mathcal{R}^\delta_\sigma : R\sim_{100} R^\delta_\sigma(p,\mathbf{v})\}| \overset{(7.68)}{\gtrsim_\delta} \frac{|W|M_\sigma}{m}. \tag{7.71}$$

(The final lower bound would not necessarily hold for $\bar{\mathcal{R}}^\delta_\sigma$, since every rectangle $R^\delta_\sigma(p,\mathbf{v})$, $(p,\mathbf{v})\in\mathbf{W}$, is not necessarily 100-comparable to at least one rectangle from $\bar{\mathcal{R}}^\delta_\sigma$.)

7.6.8. *Proving that $m \lesssim n$.* Recall the constant $n = \delta^{-\bar{\kappa}+\zeta}$ from (7.57). We next claim that

$$m(R) \lesssim_{\epsilon,\kappa} \delta^{-\zeta} n, \qquad R\in\mathcal{R}^\delta_\sigma, \tag{7.72}$$

and in particular $m \lesssim_{\epsilon,\kappa} \delta^{-\zeta}n$. This inequality is analogous to (5.21) in the proof of Proposition 5.2, but the argument here will be a little harder: now we will finally need the inductive information (7.38) regarding the higher levels of tangency λ_l for $1 \leqslant l \leqslant k$.

Let $R = R^\delta_\sigma(p,v)\in\mathcal{R}^\delta_\sigma$, with $(p,v)\in\mathbf{W}$. According to (7.69), there exists at least one

$$\beta = (q,w)\in(\mathbf{B}^\delta_\sigma)^{\delta^{-\epsilon_{j+1}},\rho_{j+1}}_{\lambda,t}(p,v) \subset (\mathbf{G}^\delta_\sigma)^{\delta^{-\epsilon_{j+1}},\rho_{j+1}}_{\lambda,t}(p,v) \tag{7.73}$$

such that $\mathbf{C}_{j+1}R^\delta_\sigma(p,v)\cap\mathbf{C}_{j+1}R^\delta_\sigma(\beta)\neq\emptyset$. We first claim that if $\omega' = (p',v')\in\mathbf{W}\subset\mathbf{G}^\delta_\sigma$ is any element such that $R^\delta_\sigma(p,v)\sim_{100}R^\delta_\sigma(\omega')$, then automatically

$$t/\rho_j \leqslant \tfrac{1}{4}t/\rho_{j+1} \leqslant |p'-q| \leqslant 4\rho_{j+1}t \leqslant \rho_j t \quad\text{and}\quad \mathbf{C}_jR^\delta_\sigma(\omega')\cap\mathbf{C}_jR^\delta_\sigma(\beta)\neq\emptyset \tag{7.74}$$

The first property follows from the separation (7.64) of the sets W,B, and noting that $\rho_j = \mathbf{C}_j\delta^{-\epsilon}\geqslant 4\mathbf{C}_{j+1}\delta^{-\epsilon} = 4\rho_{j+1}$ (recall (7.52)).

For the second property, note that since $R^\delta_\sigma(p,v)\sim_{100}R^\delta_\sigma(\omega')$, we have $R^\delta_\sigma(\omega')\subset AR^\delta_\sigma(p,v)\subset\mathbf{C}_{j+1}R^\delta_\sigma(p,v)$ for some absolute constant $A>0$ according to Lemma 4.9, and by a second application of the same lemma,

$$\mathbf{C}_{j+1}R^\delta_\sigma(p,v)\subset\mathbf{C}'_{j+1}R^\delta_\sigma(\omega')$$

for some $\mathbf{C}'_{j+1}\lesssim\mathbf{C}^5_{j+1}$. In particular, $\mathbf{C}_{j+1}R^\delta_\sigma(p,v)\subset\mathbf{C}_jR^\delta_\sigma(\omega')$, recalling from (7.51) the rapid decay of the sequence $\{\mathbf{C}_j\}$. The second part of (7.74) follows from this inclusion, recalling that $\mathbf{C}_{j+1}R(p,v)\cap\mathbf{C}_{j+1}R^\delta_\sigma(\beta)\neq\emptyset$.

Let us recap: we have now shown that for $\omega'\in\mathbf{W}$ with $R^\delta_\sigma(p,v)\sim_{100}R^\delta_\sigma(\omega')$, the conditions (7.74) hold relative to the fixed pair $\beta\in\mathbf{B}^\delta_\sigma$ (determined by R). This gives an inequality of the form

$$m(R)\leqslant|\{\omega'\in(\mathbf{G}^\delta_\sigma)^{\rho_j}_t(\beta) : \mathbf{C}_jR^\delta_\sigma(\beta)\cap\mathbf{C}_jR^\delta_\sigma(\omega')\neq\emptyset\}|, \tag{7.75}$$

where the (non-standard) notation $(\mathbf{G}^\delta_\sigma)^{\rho_j}_t(\beta)$ refers to those pairs (p',v') such that $t/\rho_j\leqslant|p'-q|\leqslant\rho_jt$. In particular, we have no information – yet – about the tangency parameter $\Delta(p',q)$. This almost brings us into a position to apply (7.59), except for one problem: (7.59) only gives an upper bound for the cardinality of elements

$$\omega'\in(\mathbf{G}^\delta_\sigma)^{\delta^{-\epsilon_j},\rho_j}_{\lambda,t}(\beta).$$

To benefit directly from this upper bound, we should be able to add the information

$$\delta^{\epsilon_j}\lambda\leqslant\Delta(p',q)\leqslant\delta^{-\epsilon_j}\lambda \tag{7.76}$$

to the properties (7.74). This is a delicate issue: it follows from the choice of $\beta = (q, w)$ in (7.73) that we have excellent two-sided control for $\Delta(p, q)$. Regardless, it is only possible to obtain the upper bound for $\Delta(p', q)$ required by (7.76), given the information that $R^{\delta}_{\sigma}(p, v) \sim_{100} R^{\delta}_{\sigma}(p', v')$. The lower bound may seriously fail: the circles $S(p'), S(q)$ may be much more tangent than the circles $S(p), S(q)$, see Figure 4. This problem will be circumvented by applying our inductive hypothesis. Before that, we however prove the upper bound: we claim that if the properties (7.74) hold, then the upper bound in (7.76) holds. This will be a consequence of Corollary 4.7.

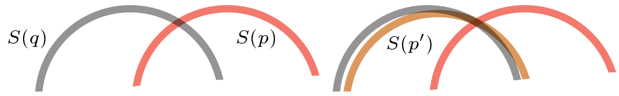

FIGURE 4. The failure of the lower bound in (7.76) in a case where $\lambda \approx 1 \approx t$, thus $\sigma = \delta/\sqrt{\lambda t} \approx \delta$. The black and red annuli $S^{\delta}(p), S^{\delta}(q)$ on the left intersect in a (δ, δ)-rectangle $R = R^{\delta}_{\delta}(p, v)$. On the right, the rectangle R is evidently 100-comparable to a (δ, δ)-rectangle $R' = R^{\delta}_{\delta}(p', v') \subset S^{\delta}(p')$, but nevertheless $\Delta(p', q) \approx \delta \ll \lambda$.

Let p', p, q be as in (7.74). Thus $R^{\delta}_{\sigma}(p', v') \sim_{100} R^{\delta}_{\sigma}(p, v)$, where the point $(p, v) \in \mathbf{W}$ satisfied

$$\mathbf{C}_{j+1} R^{\delta}_{\sigma}(p, v) \cap \mathbf{C}_{j+1} R^{\delta}_{\sigma}(q, w) \neq \emptyset, \tag{7.77}$$

and

$$\delta^{\epsilon_{j+1}} \lambda \leqslant \Delta(p, q) \leqslant \delta^{-\epsilon_{j+1}} \lambda \quad \text{and} \quad t/\rho_j \leqslant |p - q| \leqslant \rho_j t. \tag{7.78}$$

These conditions place us in a position to apply Corollary 4.7. We write $\bar{\lambda} := \Delta(p, q)$ and $\bar{t} := |p - q|$, and $\bar{\sigma} := \delta/\sqrt{(\bar{\lambda} + \delta)(\bar{t} + \delta)}$. If follows from the upper bounds in (7.78), and since $\rho_j = \mathbf{C}_j \delta^{-\epsilon} \leqslant \delta^{-\epsilon_{j+1}}$, that

$$\sigma \lesssim \delta^{-\epsilon_{j+1}} \bar{\sigma}. \tag{7.79}$$

After this observation, it follows from (7.77) that

$$\mathbf{A}_{j+1} R^{\delta}_{\bar{\sigma}}(p, v) \cap \mathbf{A}_{j+1} R^{\delta}_{\bar{\sigma}}(q, w) \neq \emptyset \tag{7.80}$$

for some $\mathbf{A}_{j+1} \lesssim \mathbf{C}_{j+1} \delta^{-\epsilon_{j+1}} \lesssim_{\epsilon,\kappa} \delta^{-\epsilon_{j+1}}$. Using (7.79) again, we may also choose the constant $\mathbf{A}_{j+1}$ (under the same size constraint) so large that

$$A R^{\delta}_{\sigma}(p, v) \subset \mathbf{A}_{j+1} R^{\delta}_{\bar{\sigma}}(p, v),$$

where $A \geqslant 1$ is an absolute constant to be specified momentarily. Now, according to Corollary 4.7, (7.80) implies

$$A R^{\delta}_{\sigma}(p, v) \subset \mathbf{A}_{j+1} R^{\delta}_{\bar{\sigma}}(p, v) \subset \mathbf{A}'_{j+1} R^{\delta}_{\bar{\sigma}}(q, w) \tag{7.81}$$

for some $\mathbf{A}'_{j+1} \lesssim \mathbf{A}^4_{j+1} \lesssim_{\epsilon,\kappa} \delta^{-4\epsilon_{j+1}}$. Finally, since $R^{\delta}_{\sigma}(p', v') \sim_{100} R^{\delta}_{\sigma}(p, v)$, we have

$$R^{\delta}_{\sigma}(p', v') \overset{\text{L. 4.9}}{\subset} A R^{\delta}_{\sigma}(p, v) \subset \mathbf{A}'_{j+1} R^{\delta}_{\bar{\sigma}}(q, w) \subset S^{\mathbf{A}'_{j+1}\delta}(q).$$

Trivially also $R^{\delta}_{\sigma}(p', v') \subset S^{\mathbf{A}'_{j+1}\delta}(p')$, so $R^{\delta}_{\sigma}(p', v') \subset S^{\mathbf{A}'_{j+1}\delta}(q) \cap S^{\mathbf{A}'_{j+1}\delta}(p')$. This implies

$$\frac{\mathbf{A}'_{j+1}\delta}{\sqrt{\Delta(p', q)|p' - q|}} \overset{\text{L. 4.3}}{\gtrsim} \operatorname{diam}(R^{\delta}_{\sigma}(p', v')) \sim \sigma.$$

Recalling that $\mathbf{A}'_{j+1} \lesssim_{\epsilon,\kappa} \delta^{-4\epsilon_{j+1}}$, this can be rearranged to

$$\Delta(p',q) \lesssim_{\epsilon,\kappa} (\delta^{1-4\epsilon_{j+1}}/\sigma)^2|p'-q|^{-1} = \delta^{-8\epsilon_{j+1}}\lambda \cdot (t/|p'-q|) \lesssim_\epsilon \delta^{-9\epsilon_{j+1}}\lambda.$$

In the final inequality we used that $p' \in W$ and $q \in B$, so $|p'-q| \geqslant t/\rho_j \gtrsim_\epsilon \delta^{\epsilon_{j+1}}t$. In (7.50), the sequence $\{\epsilon_j\}$ was chosen to be so rapidly decreasing that $\epsilon_j > 10\epsilon_{j+1}$. Therefore, if $\delta > 0$ is small enough, the inequality above implies the upper bound claimed in (7.76).

Recalling also (7.75), we have now shown that

$$m(R) \leqslant |\{\omega' \in (\mathbf{G}^\delta_\sigma)^{\delta^{-\epsilon_j},\rho_j}_{\leqslant\lambda,t}(\beta) : \mathbf{C}_j R^\delta_\sigma(\beta) \cap \mathbf{C}_j R^\delta_\sigma(\omega') \neq \emptyset\}|, \tag{7.82}$$

where the "$\leqslant \lambda$" symbol refers to the fact that we only have guaranteed the upper bound in (7.76), but not a matching lower bound.

As noted above, the matching lower bound $\Delta(p',q) \gtrapprox \lambda$ may be false. However, recall from (7.44) that $\lambda = \lambda_{k+1}$, and that the sequence $\{\lambda_l\}_{l=1}^k$ is multiplicatively $\delta^{-\epsilon}$-dense (or even $\delta^{-\epsilon/2}$-dense) on the interval $[\delta, \delta^\epsilon\lambda] \subset [\delta, \lambda_k]$. Therefore, we are either in the happy case of the 2-sided bound

$$\delta^{\epsilon_j}\lambda \leqslant \Delta(p',q) \leqslant \delta^{-\epsilon_j}\lambda, \tag{7.83}$$

or otherwise $\Delta(p',q) < \delta^{\epsilon_j}\lambda \leqslant \delta^\epsilon\lambda$, and we can find an index $1 \leqslant l \leqslant k$ such that

$$\delta^\epsilon\lambda_l/C_l \leqslant \lambda_l \leqslant \Delta(p',q) \leqslant \delta^{-\epsilon}\lambda_l \leqslant C_l\delta^{-\epsilon}\lambda_l. \tag{7.84}$$

In fact, there is a small gap in this argument: if $\Delta(p',q) < \delta$, then we cannot guarantee (7.84) for any $1 \leqslant l \leqslant k$. To fix this, we modify (7.84) so that in the case $l = 1$, only the upper bound is claimed. With this convention, the index $l \in \{1,\ldots,k\}$ satisfying (7.84) can always be found whenever $\Delta(p',q) < \delta^{\epsilon_j}\lambda$.

One of the two cases (7.83)-(7.84) is "typical" in the following sense. Since $\Delta(p',q) \leqslant \delta^{-\epsilon_j}\lambda$ for all pairs (p',q) appearing in (7.82), by the pigeonhole principle there exist $\bar{m}(R) \sim_\epsilon m(R)$ pairs $\omega_1,\ldots,\omega_{\bar{m}(R)} \in \mathbf{W}$ with first components $p_1,\ldots,p_{\bar{m}(R)} \in W$, and a fixed index $1 \leqslant l \leqslant k+1$, such that

$$\delta^\epsilon\lambda_l/C_l \leqslant \Delta(p_i,q) \leqslant C_l\delta^{-\epsilon}\lambda_l, \qquad 1 \leqslant i \leqslant \bar{m}(R), \tag{7.85}$$

for some fixed $1 \leqslant l \leqslant k+1$. In the case $l = k+1$, the constant "ϵ" in (7.85) needs to be replaced by ϵ_j, recalling the alternatives (7.83)-(7.84). In the case $l = 1$, the two-sided inequality in (7.85) has to be replaced by the one-sided inequality $\Delta(p_i,q) \leqslant C_1\delta^{1-\epsilon}$.

A subtle point is that even though the pairs $\omega_1,\ldots,\omega_{\bar{m}(R)}$ are distinct, the first components $p_1,\ldots,p_{\bar{m}(R)}$ need not be. However, they "almost" are: for $p \in W$ fixed, there can only exist $\lesssim \mathbf{C}_j$ choices $v \in \mathcal{S}_\sigma(p)$ such that $\mathbf{C}_j R^\delta_\sigma(p,v) \cap \mathbf{C}_j R^\delta_\sigma(\beta) \neq \emptyset$ (as in (7.74)). Thus,

$$|\{p_1,\ldots,p_{\bar{m}(R)}\}| \gtrsim \mathbf{C}_j^{-1}m(R).$$

For this argument, it was important that the "angular" components of the pairs in $\mathbf{W}$ are elements in $\mathcal{S}_\sigma(p)$, recall (7.67). For notational convenience, we will assume in the sequel that the points $p_1,\ldots,p_{\bar{m}(R)}$ are distinct, and we will trade this information for the weaker estimate $\bar{m}(R) \gtrsim_\epsilon \mathbf{C}_j^{-1}m(R)$ (this is harmless, since $\mathbf{C}_j \lesssim_{\epsilon,\kappa} 1$).

Now, we have two separate cases to consider. First, if $l = k+1$, then $\lambda_l = \lambda$, and we have $\delta^{\epsilon_j}\lambda \leqslant \Delta(p_i,q) \leqslant \delta^{-\epsilon_j}\lambda$ for all $1 \leqslant i \leqslant \bar{m}(R)$. In this case

$$m(R) \lesssim_\epsilon \bar{m}(R) \leqslant |\{\omega' \in (\mathbf{G}^\delta_\sigma)^{\delta^{-\epsilon_j},\rho_j}_{\lambda,t}(\beta) : \mathbf{C}_j R^\delta_\sigma(\beta) \cap \mathbf{C}_j R^\delta_\sigma(\omega') \neq \emptyset\}| \overset{(7.59)}{\leqslant} \delta^{-\zeta}n,$$

using that $\beta \in \mathbf{B}_\sigma^\delta \subset \mathbf{G}_\sigma^\delta$ (recall also Remark 7.60 where we explained why (7.59) may be assumed to hold for $\beta \in \mathbf{G}_\sigma^\delta$, not just $\beta \in \mathbf{G}$). This proves (7.72) in the case $l = k+1$.

Assume finally that $1 \leqslant l \leqslant k$. Then, according to (7.85) we have

$$m(R) \lesssim_\epsilon \bar{m}(R) \leqslant |\{\omega' \in (\mathbf{G}_\sigma^\delta)_{\lambda_l,t}^{C_l\delta^{-\epsilon},\rho_j}(\beta) : \mathbf{C}_j R_\sigma^\delta(\beta) \cap \mathbf{C}_j R_\sigma^\delta(\omega') \neq \emptyset\}|. \tag{7.86}$$

We note that $\rho_j = \mathbf{C}_j\delta^{-\epsilon} \leqslant C_l\delta^{-\epsilon}$ by the choice of the intermediate constants $\{\mathbf{C}_j\}$, see (7.51), so the inequality (7.86) implies

$$m(R) \lesssim_\epsilon |\{\omega' \in (\mathbf{G}_\sigma^\delta)_{\lambda_l,t}^{C_l\delta^{-\epsilon}}(\beta) : \mathbf{C}_j R_\sigma^\delta(\beta) \cap \mathbf{C}_j R_\sigma^\delta(\omega') \neq \emptyset\}|. \tag{7.87}$$

(This remains true as stated also in the special case $l = 1$: in this case (7.85) had to be replaced by the one-sided inequality $\Delta(p_i, q) \leqslant C_1\delta^{-\epsilon}\lambda_1 = C_1\delta^{1-\epsilon}$, but this implies $\omega_i = (p_i, v_i) \in (\mathbf{G}_\sigma^\delta)_{\lambda_1,t}^{C_1\delta^{-\epsilon}}(\beta)$ for $\lambda_1 = \delta$, see the last line of Definition 7.1).

The right hand side looks deceptively like $m_{\delta,\lambda_l,t}^{C_l\delta^{-\epsilon},C_l}(\omega \mid \mathbf{G})$ (note also that $\mathbf{C}_j \leqslant C_l$), and since $\mathbf{G} \subset G_l$, the inductive hypothesis (7.38) now appears to show that

$$m(R) \lesssim_\epsilon \delta^{-\kappa} \overset{(7.57)}{\leqslant} \delta^{-\zeta} n,$$

as desired, using here that $\bar{\kappa} = \kappa_{h-j} > \kappa$ by our counter assumption. There is still a small gap in this argument: the definition of $m_{\delta,\lambda_l,t}^{C_l\delta^{-\epsilon},C_l}$ counts elements in the (δ,σ_l)-skeleton of $\mathbf{G}$ with $\sigma_l = \delta/\sqrt{\lambda_l t} > \sigma$, rather than the (δ,σ)-skeleton appearing on the right hand side of (7.87).

This is easy to fix. The solution is to first use the (distinct!) points $p_1, \ldots, p_{\bar{m}(R)}$ found in (7.85) to produce a collection of pairs $\bar{\omega}_1, \ldots, \bar{\omega}_{\bar{m}(R)} \in \mathbf{G}_{\sigma_l}^\delta$. Indeed, for every $1 \leqslant i \leqslant \bar{m}(R)$, we know from (7.74) that there corresponds a pair $\omega_i = (p_i, v_i) \in \mathbf{G}_\sigma^\delta$ such that

$$\mathbf{C}_j R_\sigma^\delta(p_i, v_i) \cap \mathbf{C}_j R_\sigma^\delta(\beta) \neq \emptyset. \tag{7.88}$$

For every $1 \leqslant i \leqslant \bar{m}(R)$, choose $\bar{\omega}_i := (p_i, \mathbf{v}_i) \in \mathbf{G}_{\sigma_l}^\delta$ with $(p_i, v_i) \prec (p_i, \mathbf{v}_i)$. Note that the pairs $\bar{\omega}_1, \ldots, \bar{\omega}_{\bar{m}(R)}$ are all distinct, since the "base" points $p_1, \ldots, p_{\bar{m}(R)}$ are distinct. Further, it follows from (7.88), combined with

$$A\mathbf{C}_j \overset{(7.51)}{\leqslant} C_k \leqslant C_l \quad \Longrightarrow \quad \mathbf{C}_j R_\sigma^\delta(p_i, v_i) \subset C_l R_{\sigma_l}^\delta(p_i, \mathbf{v}_i) = C_l R_{\sigma_l}^\delta(\bar{\omega}_i),$$

(here $A \geqslant 1$ is a sufficiently large absolute constant) that

$$C_l R_{\sigma_l}^\delta(\bar{\omega}_i) \cap C_l R_{\sigma_l}^\delta(\beta) \neq \emptyset, \qquad 1 \leqslant i \leqslant \bar{m}(R). \tag{7.89}$$

(The deduction from (7.88) to (7.89) looks superficially similar to the deduction of the second claim in (7.74), but now the situation is much simpler, because (p_i, v_i) and $(p_i, \mathbf{v}_i)$ have the same "p_i".) We note that the tangency and distance parameters of the pairs $((p_i, v_i), \beta)$ and $(\bar{\omega}_i, \beta)$ are exactly the same, since the "base point" p_i remained unchanged. Consequently, by (7.87) and (7.89), we have

$$\begin{aligned} \bar{m}(R) &\leqslant |\{\bar{\omega}' \in (\mathbf{G}_{\sigma_l}^\delta)_{\lambda_l,t}^{C_l\delta^{-\epsilon}}(\beta) : C_l R_{\sigma_l}^\delta(\beta) \cap C_l R_{\sigma_l}^\delta(\bar{\omega}') \neq \emptyset\}| \\ &= m_{\delta,\lambda_l,t}^{C_l\delta^{-\epsilon},C_l}(\beta \mid \mathbf{G}) \overset{(7.38)}{\leqslant} \delta^{-\kappa} \overset{(7.57)}{\leqslant} \delta^{-\zeta} n. \end{aligned} \tag{7.90}$$

We have finally proven (7.72).

7.6.9. *The type of the rectangles $R \in \bar{\mathcal{R}}^\delta_\sigma$.* We next claim that every rectangle $R \in \bar{\mathcal{R}}^\delta_\sigma$ has λ-restricted type $(\geqslant \bar{m}, \geqslant \bar{n})_{\epsilon_{\max}}$ relative to $(W, B, \{E(p)\})$, where $\bar{m} := \delta^{\epsilon_{\max}} m$ and $\bar{n} := \delta^{\epsilon_{\max}} n$. Recall from Definition 6.3 what this means. Given $R \in \bar{\mathcal{R}}^\delta_\sigma$, we should find a subset $W_R \subset W$ with $|W_R| \geqslant \bar{m}$, and the following property: for every $p \in W_R$, there exists a subset $B_R(p) \subset B$ of cardinality $|B_R(p)| \geqslant \bar{n}$ satisfying

$$\delta^{\epsilon_{\max}} \lambda \leqslant \Delta(p,q) \leqslant \delta^{-\epsilon_{\max}} \lambda \quad \text{and} \quad R \subset \delta^{-\epsilon_{\max}} \mathcal{E}^\delta_\sigma(p) \cap \delta^{-\epsilon_{\max}} \mathcal{E}^\delta_\sigma(q) \tag{7.91}$$

for all $p \in W_R$ and $q \in B_R(p)$. If $\lambda = \delta$, the first requirement in (7.91) is relaxed to $\Delta(p,q) \leqslant \delta^{-\epsilon_{\max}} \lambda$.

Remark 7.92. In (7.91), the definition of the sets $\mathcal{E}^\delta_\sigma(p), \mathcal{E}^\delta_\sigma(q)$ involves the (δ,σ)-skeletons of $E(p)$ and $E(q)$. We emphasise that these sets are not the "original" sets $E_0(p), E_0(q)$ given in Theorem 7.5 (recall the notation from Section 7.4), but rather the subsets found at the end of Section 7.4, see (7.24). This is important, since the upper bound (7.26) will be needed in a moment.

To begin finding W_R and $B_R(p)$ for $p \in W_R$, recall that $m(R) = m$ for all $R \in \bar{\mathcal{R}}^\delta_\sigma$. This mean that there exists a set $\mathbf{W}_R \subset \mathbf{W}$ of m pairs $\{\omega_i\}_{i=1}^m = \{(p_i, v_i)\}_{i=1}^m$ such that $R \sim_{100} R^\delta_\sigma(\omega_i)$ for all $1 \leqslant i \leqslant m$. While the pairs ω_i are all distinct, the first components p_i need not be. This issue is similar to the one we encountered below (7.85), and the solution is also the same: for every p_i fixed, there can only be $\lesssim 1$ possibilities for $v \in \mathcal{S}_\sigma(p_i)$ such that $R \sim_{100} R^\delta_\sigma(p_i, v)$. Therefore the number of distinct elements in $W_R := \{p_1, \ldots, p_m\}$ is $\gtrsim m$, and certainly $|W_R| \geqslant \delta^{\epsilon_{\max}} m = \bar{m}$. To remove ambiguity, for each distinct point $p_i \in W_R$, we pick a single element $v \in \mathcal{S}_\sigma(p_i)$ such that $(p_i, v) \in \mathbf{W}_R$, and we restrict $\mathbf{W}_R$ to this subset without changing notation.

Next, fix $p \in W_R$. Let $v \in \mathcal{S}_\sigma(p)$ be the unique element such that $\omega = (p, v) \in \mathbf{W}_R \subset \mathbf{W}$. Recall from (7.69) that

$$|\{\beta \in (\mathbf{B}^\delta_\sigma)^{\delta^{-\epsilon_{j+1}}, \rho_{j+1}}_{\lambda,t}(\omega) : \mathbf{C}_{j+1} R^\delta_\sigma(\omega) \cap \mathbf{C}_{j+1} R^\delta_\sigma(\beta) \neq \emptyset\}| \geqslant n.$$

Thus, there exists a collection $\{\beta_i\}_{i=1}^n = \{(q_i, w_i)\}_{i=1}^n \subset \mathbf{B}^\delta_\sigma$ of pairs such that

$$\mathbf{C}_{j+1} R^\delta_\sigma(\omega) \cap \mathbf{C}_{j+1} R^\delta_\sigma(q_i, w_i) \neq \emptyset, \tag{7.93}$$

and

$$\delta^{\epsilon_{j+1}} \lambda \leqslant \Delta(p, q_i) \leqslant \delta^{-\epsilon_{j+1}} \lambda \quad \text{and} \quad \delta^{\epsilon_{\max}} t \leqslant |p - q_i| \leqslant \delta^{-\epsilon_{\max}} t \tag{7.94}$$

for all $1 \leqslant i \leqslant n$. In the estimates for $|p - q_i|$, we already plugged in $\rho_{j+1} = \mathbf{C}_{j+1} \delta^{-\epsilon} \leqslant \delta^{-\epsilon_{\max}}$, assuming $\delta > 0$ small enough.

Once more, the q_i-components of the pairs $\{\beta_i\}$ need not all be distinct, but they almost are, by the following familiar argument: for each q_i, there can correspond $\lesssim \mathbf{C}_{j+1}$ distinct choices $w \in \mathcal{S}_\sigma(q_i)$ such that (7.93) holds. Therefore, $B_R(p) := \{q_1, \ldots, q_n\} \subset B$ has $\gtrsim \mathbf{C}_{j+1}^{-1} n$ distinct elements, and certainly $|B_R(p)| \geqslant \bar{n}$.

Let us finally check the conditions (7.91) for $p \in W_R$ and $q \in B_R(p)$. The tangency constraint follows readily from (7.94), and noting that $\epsilon_{j+1} \leqslant \epsilon_{\max}$. So, it remains to check the inclusion in (7.91). Fix $p \in W_R$ and $q \in B_R(p)$. By definition, $p \in W_R$ means that $R \sim_{100} R^\delta_\sigma(\omega)$ for some $\omega = (p, v) \in \mathbf{W}_R \subset \mathbf{G}^\delta_\sigma$, and in particular $v \in E_\sigma(p)$ (the (δ,σ)-skeleton of $E(p)$). Next, $q \in B_R(p)$ means that there exists $\beta = (q, w) \in \mathbf{B}^\delta_\sigma$ (in particular $w \in E_\sigma(q)$) such that (7.93)-(7.94) hold. We now claim that

$$R \subset \delta^{-\epsilon_{\max}} R^\delta_\sigma(\omega) \cap \delta^{-\epsilon_{\max}} R^\delta_\sigma(\beta) \subset \delta^{-\epsilon_{\max}} \mathcal{E}^\delta_\sigma(p) \cap \delta^{-\epsilon_{\max}} \mathcal{E}^\delta_\sigma(q). \tag{7.95}$$

This is a consequence of Corollary 4.7, and the argument is extremely similar to the one we recorded below (7.77)-(7.78). We just sketch the details. Applying Corollary 4.7 with $\bar{\sigma} := \delta/\sqrt{(\Delta(p,q)+\delta)(|p-q|+\delta)}$, it follows from the non-empty intersection (7.93) that

$$AR^{\delta}_{\sigma}(\omega) \subset \mathbf{A}_{j+1}R^{\delta}_{\sigma}(\beta),$$

where $A \geqslant 1$ is a suitable absolute constant, and $\mathbf{A}_{j+1} \lesssim_{\epsilon} \delta^{-O(\epsilon_{j+1})}$ (compare with (7.81)). Next, from $R \sim_{100} R^{\delta}_{\sigma}(\omega)$, we simply deduce that $R \subset AR^{\delta}_{\sigma}(\omega)$. Since $\max\{A, \mathbf{A}_{j+1}\} \leqslant \delta^{-\epsilon_{\max}}$ for $\delta > 0$ small enough, the inclusion (7.95) follows.

We have now proven that every rectangle $R \in \bar{\mathcal{R}}^{\delta}_{\sigma}$ has λ-restricted type $(\geqslant \bar{m}, \geqslant \bar{n})_{\epsilon_{\max}}$ relative to $(W, B, \{E(p)\})$.

7.6.10. *Applying Theorem 6.5.* The constant $\epsilon_{\max} = \epsilon_{\max}(\kappa, s) > 0$ was chosen (recall Section 7.1) in such a way that Theorem 6.5 holds with constant $\eta = \kappa s/100$. Therefore, we may apply the theorem as soon as we have checked that its hypotheses are valid. At the risk of over-repeating, we will apply Theorem 6.5 to the space $\Omega = \{(p,v) : p \in P \text{ and } v \in E(p)\} \subset \Omega_0$ constructed during the "initial regularisation" in Section 7.4. Crucially, we recall that Ω satisfies the upper bounds (7.26) for all $\lambda \in \Lambda$ and $t \in \mathcal{T}(\lambda)$. This means that the hypothesis (6.7) of Theorem 6.5 is valid with constant $Y_{\lambda} = \delta^{-\kappa s/100}\lambda^{s}|P_0|_{\lambda}$.

We also recall from Section 7.4 that our set P_0 is Λ-uniform (without loss of generality), and at (7.21) we denoted $X_{\lambda} := |P_0 \cap \mathbf{p}|_{\delta} = |P_0|/|P_0|_{\lambda}$ for $\mathbf{p} \in \mathcal{D}_{\lambda}(P_0)$, and $\lambda \in \Lambda$.

We have now verified the hypotheses of Theorem 6.5. Recall from the previous section that every rectangle $R \in \bar{\mathcal{R}}^{\delta}_{\sigma}$ has type $(\geqslant \bar{m}, \geqslant \bar{n})_{\epsilon_{\max}}$ relative to $(W, B, \{E(p)\})$. Therefore, we may infer from Theorem 6.5 that

$$\frac{|W|M_{\sigma}}{m} \overset{(7.71)}{\lessapprox_{\delta}} |\bar{\mathcal{R}}^{\delta}_{\sigma}| \leqslant \delta^{-\kappa s/100}\left[\left(\frac{|W||B|}{\bar{m}\bar{n}}\right)^{3/4}(X_{\lambda}Y_{\lambda})^{1/2} + \frac{|W|}{\bar{m}}\cdot X_{\lambda}Y_{\lambda} + \frac{|B|}{\bar{n}}\cdot X_{\lambda}Y_{\lambda}\right].$$

Here

$$X_{\lambda}Y_{\lambda} \leqslant (|P_0|/|P_0|_{\lambda})\cdot(\delta^{-\kappa s/100}\lambda^{s}|P_0|_{\lambda}) = \delta^{-\kappa s/100}\lambda^{s}|P_0|.$$

We also recap from (7.72) that $m \lesssim_{\epsilon,\kappa} \delta^{-\zeta}n \leqslant \delta^{-\zeta-\epsilon_{\max}}\bar{n}$ (where $\zeta < \kappa s/10$), and from (7.65) that $|B| \leqslant \delta^{-2\epsilon_{\max}}|W|$. Recalling from (7.57) that $n \geqslant \delta^{-\kappa+\zeta}$, and from (7.6) that $|P_0| \leqslant \delta^{-s-\epsilon}$, we may rearrange and simplify the estimate above to the form

$$\begin{aligned} M_{\sigma} &\leqslant \delta^{-\kappa s/100-\zeta-O(\epsilon_{\max})}\left[|W|^{1/2}\cdot\delta^{\kappa/2}\cdot(\delta^{-\kappa s/100}\lambda^{s}|P_0|)^{1/2} + \delta^{-\kappa s/100}\lambda^{s}|P_0|\right] \\ &\leqslant \delta^{-\kappa s/100-\kappa s/10-O(\epsilon_{\max})}\left[|W|^{1/2}\cdot(\lambda/\delta)^{s/2}\cdot\delta^{\kappa/2-\kappa s/100} + \delta^{-\kappa s/100}(\lambda/\delta)^{s}\right]. \end{aligned} \tag{7.96}$$

To derive a contradiction from this estimate, recall from (7.48) that

$$M_{\sigma} \geqslant \delta^{2\epsilon}\left(\frac{\sqrt{\lambda t}}{\delta}\right)^{s} \geqslant \delta^{2\epsilon-\kappa s/5}\left(\frac{\lambda}{\delta}\right)^{s}. \tag{7.97}$$

The second inequality follows from our restriction to pairs (λ, t) with $\lambda \leqslant \delta^{\kappa/10}t$ (recall (7.34), and that $\lambda = \lambda_{k+1}$). These inequalities show that the second term in (7.96) cannot dominate the left hand side, provided that $\epsilon_{\max}$ is chosen small enough in terms of κ, s, and finally $\delta > 0$ is sufficiently small in terms of all these parameters.

To produce a contradiction with the counter assumption formulated above Section 7.6.5, it remains to show that the first term in (7.96) cannot dominate M_{σ}. Since P_0 is a $(\delta, s, \delta^{-\epsilon})$-set, and $W \subset P \subset P_0$ is contained in a ball of radius t, we have $|W| \lessdot \delta^{-\epsilon} t^s |P_0| \lessdot \delta^{-\epsilon}(t/\delta)^s$. Therefore, the first term in (7.96) is bounded from above by

$$\delta^{\kappa/2 - \kappa s(1/50 + 1/10) - O(\epsilon_{\max})} \left(\frac{\sqrt{\lambda t}}{\delta} \right)^s \leqslant \delta^{\kappa s/5} \left(\frac{\sqrt{\lambda t}}{\delta} \right)^s,$$

provided that $\epsilon_{\max} > 0$ is small enough in terms of κ, s. Evidently, the number above is smaller than the lower bound for M_σ recorded in (7.97), provided that $\epsilon, \epsilon_{\max}, \delta > 0$ are small enough in terms of κ, s. We have now obtained the desired contradiction.

To summarise, we have now shown that case (1) in the construction of the sequence $\{\mathbf{G}_j\}$ cannot occur as long as $\kappa_{h-j} > \kappa$. As explained at and after (7.56), this shows that we may define $G_{k+1} := \mathbf{G}_j$ for a suitable index "j", and this set G_{k+1} satisfies (7.43). This completes the proof of Proposition 7.28, then the proof of Proposition 7.30, and finally the proof of Theorem 7.5.

7.7. **Deriving Theorem 1.11 from Theorem 7.5.** It clearly suffices to prove Theorem 1.11 for all $\kappa \in (0, c]$, where $c > 0$ is a small absolute constant to be determined later. Fix $\kappa \in (0, c]$, and let $\epsilon = \epsilon(\kappa, s) > 0$ be so small that Theorem 7.5 holds with constants κ, s.

Let $\Omega = \{(p, v) : p \in P \text{ and } v \in E(p)\}$ be a (δ, s, C)-configuration, as in Theorem 1.11. There is no *a priori* assumption in Theorem 1.11 that the sets $P, E(p)$ are δ-separated, but it is easy to reduce matters to that case; we leave this to the reader, and in fact we assume that $P \subset \mathcal{D}_\delta$ and $E(p) \subset \mathcal{S}_\delta(p)$ for all $p \in P$.

To prove Theorem 1.11, we need to find a subset $G \subset \Omega$ satisfying $|G| \geqslant \delta^\kappa |\Omega|$, and

$$m_\delta(w \mid G) \lessapprox_\delta \delta^{-\kappa}, \qquad w \in \mathbb{R}^2. \tag{7.98}$$

We start by applying Theorem 7.5 to Ω to find the subset $G \subset \Omega$ of cardinality $|G| \geqslant \delta^\kappa |\Omega|$. By the choice of "$\epsilon$" above, we then have

$$m_{\delta,\lambda,t}^{\delta^{-\epsilon},\kappa^{-1}}(\omega \mid G) \leqslant \delta^{-\kappa}, \qquad \omega \in G. \tag{7.99}$$

We claim that if the absolute constant "$c > 0$" is chosen small enough (thus $\kappa^{-1} \geqslant c^{-1} > 0$ is sufficiently large), then (7.99) implies that

$$|\{(p', v') \in G : v' \in B(v, 2\delta)\}| = m_{2\delta}((p, v) \mid G) \lessapprox_\delta \delta^{-\kappa}, \qquad (p, v) \in G. \tag{7.100}$$

Let us quickly check that this implies (7.98) for all $w \in \mathbb{R}^2$. Indeed, if $m_\delta(w \mid G) > 0$, then there exists at least one pair $(p, v) \in G$ such that $w \in B(v, \delta)$. Now, it is easy to see from the definitions that $m_\delta(w \mid G) \leqslant m_{2\delta}((p, v) \mid G)$.

The idea for proving (7.100) is to bound the total multiplicity function $m_{2\delta}$ from above by a suitably chosen partial multiplicity function $m_{\delta,\lambda,t}$. Fix $(p, v) = \omega \in G$. Then,

$$m_{2\delta}(\omega \mid G) \leqslant \sum_{\lambda \leqslant t} m_{2\delta}(\omega \mid G_{\lambda,t}^{\delta^{-\epsilon}}(\omega)),$$

where $G_{\lambda,t}^{\delta^{-\epsilon}}(\omega) = \{(p', v') \in G : \delta^\epsilon \lambda \leqslant \Delta(p, p') \leqslant \delta^{-\epsilon}\lambda \text{ and } \delta^\epsilon t \leqslant |p - p'| \leqslant \delta^{-\epsilon} t\}$ as in Definition 7.1, and the sum runs over some multiplicatively $\delta^{-\epsilon}$-dense sequences of $\delta \leqslant \lambda \leqslant t \leqslant 1$ (or even all dyadic values, this is not important here). In particular, there exists a fixed pair (λ, t), depending on ω, such that

$$m_{2\delta}(\omega \mid G) \lessapprox_\delta m_{2\delta}(\omega \mid G_{\lambda,t}^{\delta^{-\epsilon}}(\omega)) = |\{(p', v') \in G_{\lambda,t}^{\delta^{-\epsilon}}(\omega) : v' \in B(v, 2\delta)\}|.$$

Let $\{\omega_j\}_{j=1}^N = \{(p_j, v_j)\}_{j=1}^N \subset G_{\lambda,t}^{\delta-\epsilon}(\omega)$ be an enumeration of the pairs on the right hand side. The points $\{p_1, \ldots, p_N\}$ may not all be distinct. However, note that if p_i is fixed, there are $\lesssim 1$ options $v' \in \mathcal{S}_\delta(p_i)$ such that $v' \in B(v, 2\delta)$ (since v is fixed). Therefore, there is a subset of $\sim N$ pairs among $\{(p_j, v_j)\}$ such that the points p_j are all distinct. Restricting attention to this subset if necessary, we assume that all the points p_j are distinct.

Write $\sigma := \delta/\sqrt{\lambda t}$. For every index $j \in \{1, \ldots, N\}$, choose $(p_j, \mathbf{v}_j) \in G_\sigma^\delta$ (the (δ, σ)-skeleton of G) such that $(p_j, v_j) \prec (p_j, \mathbf{v}_j)$. Automatically

$$(p_j, \mathbf{v}_j) \in (G_\sigma^\delta)_{\lambda,t}^{\delta-\epsilon}(\omega), \qquad 1 \leqslant j \leqslant N,$$

since the point "p_j" remained unchanged. We also note that $|v_j - \mathbf{v}_j| \lesssim \sigma$, and the pairs $(p_j, \mathbf{v}_j)$ are distinct because the points p_j are. We claim that

$$\kappa^{-1} R_\sigma^\delta(p_j, \mathbf{v}_j) \cap \kappa^{-1} R_\sigma^\delta(\omega) \neq \emptyset, \qquad 1 \leqslant j \leqslant N, \tag{7.101}$$

provided that $\kappa \leqslant c$, and $c > 0$ is sufficiently small. Indeed, fix $1 \leqslant j \leqslant N$, and recall that $v_j \in B(v, 2\delta)$. This immediately shows that $v_j \in 2R_\sigma^\delta(p, v) = 2R_\sigma^\delta(\omega)$, since $\sigma \geqslant \delta$. On the other hand, $v_j \in S(p_j)$, and $|v_j - \mathbf{v}_j| \lesssim \sigma$, so also $v_j \in CR_\sigma^\delta(p_j, \mathbf{v}_j)$ for some absolute constant $C \geqslant 1$. Now, (7.101) holds for all $\kappa^{-1} \geqslant c^{-1} \geqslant \max\{2, C\}$.

We have now shown that

$$m_{2\delta}(\omega \mid G) \lessapprox_\delta N \leqslant |\{\omega' \in (G_\sigma^\delta)_{\lambda,t}^{\delta-\epsilon}(\omega) : \kappa^{-1} R_\sigma^\delta(\omega') \cap \kappa^{-1} R_\sigma^\delta(\omega) \neq \emptyset\}| = m_{\delta,\lambda,t}^{\delta-\epsilon,\kappa^{-1}}(\omega \mid G).$$

Recalling (7.99), this proves (7.100), and consequently Theorem 1.11.

Appendix A. Proof of Proposition 4.12

We complete the proof of Proposition 4.12 in this appendix. For the reader's convenience, we recall the statement of Proposition 4.12 here.

Proposition 4.12. *Let $A \geqslant 100$ and $\delta \leqslant \sigma \leqslant 1$, and let $\mathcal{R}$ be a family of pairwise 100-incomparable (δ, σ)-rectangles. Suppose also that there exists a fixed (δ, σ)-rectangle $\mathbf{R}$ such that the union of the rectangles in $\mathcal{R}$ is contained in $A\mathbf{R}$. Then, $|\mathcal{R}| \lesssim A^{10}$.*

As mentioned in Section 4.2, we first need several auxiliary definitions and lemmas.

Definition A.1. We denote by $\pi_L \colon \mathbb{R}^2 \to L$ the orthogonal projection onto a 1-dimensional subspace L in $\mathbb{R}^2$. If $\mathbf{I} \subset L$ is a fixed segment, $p \in \mathbb{R}^3$ and $v \in S(p)$ are such that $\pi_L(v) \in \mathbf{I}$, then we denote by $\Gamma_{\mathbf{I},p,v}$ the connected component of $\pi_L^{-1}(\mathbf{I}) \cap S(p)$ containing v.

The set $\Gamma_{\mathbf{I},p,v}$ need not be a graph over $\mathbf{I}$ in general. However, given a rectangle $\mathbf{R}$ and a family $\mathcal{R}$ of (δ, σ)-rectangles as in Proposition 4.12 with a suitable upper bound on σ, we now show how to select a subfamily $\mathcal{R}_* \subset \mathcal{R}$ with $|\mathcal{R}^*| \geqslant |\mathcal{R}|/2$ such that both $A\mathbf{R}$, and the rectangles in $\mathcal{R}_*$, look like neighborhoods of 2-Lipschitz graphs over a fixed line L. By a "2-Lipschitz graph over L" we mean the graph of a 2-Lipschitz function defined on a subset of L. In the argument below, we abbreviate $R_\sigma^\delta(p, v) =: R(p, v)$.

Lemma A.2. *Let $A \geqslant 1$, $\delta \leqslant \sigma \leqslant A\sigma \leqslant \sigma_0 := 1/600$. Assume that $\mathcal{R}$ is a finite family of (δ, σ)-rectangles, all contained in $A\mathbf{R}$, where $\mathbf{R} = R(\mathbf{p}, \mathbf{v})$ is another (δ, σ)-rectangle. Then there exists a 1-dimensional subspace $L \subset \mathbb{R}^2$, an interval $\mathbf{I} \subset L$ and a subfamily $\mathcal{R}_* \subset \mathcal{R}$ with $|\mathcal{R}_*| \geqslant |\mathcal{R}|/2$ such that*

(1) *$\pi_L(A\mathbf{R}) \subset \mathbf{I}$ and $\Gamma_{\mathbf{I},\mathbf{p},\mathbf{v}}$ is a 2-Lipschitz graph over $\mathbf{I}$;*

(2) *for each* $R(p,v) \in \mathcal{R}_*$:
$\pi_L(R) \subset \mathbf{I}$ *and* $\Gamma_{\mathbf{I},p,v}$ *is a 2-Lipschitz graph over* $\mathbf{I}$.

Proof. First, we find the subspace L and the subfamily $\mathcal{R}_* \subset \mathcal{R}$. Let

$$J(\mathbf{p},\mathbf{v}) := S(\mathbf{p}) \cap B(\mathbf{v}, \tfrac{1}{100}) \quad \text{and} \quad J(p,v) := S(p) \cap B(v, \tfrac{1}{100}).$$

These are arcs on the circles $S(\mathbf{p})$ and $S(p)$ which contain the "core arcs" $S(\mathbf{p}) \cap B(\mathbf{v}, A\sigma)$ and $S(p) \cap B(v,\sigma)$ of the rectangles $A\mathbf{R}$ and R respectively. We claim that L can be chosen from one of the three lines

$$L_1 = \operatorname{span}(1,0), \quad L_2 = \operatorname{span}(1,\sqrt{3}), \quad L_3 = \operatorname{span}(-1,\sqrt{3})$$

such that $J(\mathbf{p},\mathbf{v})$ and $J(p,v)$ are 2-Lipschitz graphs over L for at least half of the rectangles $R(p,v) \in \mathcal{R}$. The idea is that the arc $J(\mathbf{p},\mathbf{v})$ (resp. $J(p,v)$) is individually a 2-Lipschitz graph over any line which is sufficiently far from perpendicular to (any tangent of) that arc. For $J(\mathbf{p},\mathbf{v})$ (resp. $J(p,v)$), this is true for at least two of the lines among $\{L_1, L_2, L_3\}$. We give some details to justify this claim.

For every circle $S(x,r)$ and every line L, there exists a segment I of length $4r/\sqrt{5}$, centered at $\pi_L(z)$, such that the two components of $\pi_L^{-1}(I) \cap S(x,r)$ are 2-Lipschitz graphs over I; see the explanation around (A.6). The constant "$1/100$" in the definition of $J(p,v)$ has been chosen so small that for each $v \in S(p)$, there are two choices of lines L_i such that $\pi_{L_i}(J(p,v))$ is contained in the segment on L_i over which the corresponding arc of $S(p)$ is a 2-Lipschitz graph. This also uses the fact that we are only considering parameters $p = (x,r) \in \mathbf{D}$, so that $r \geqslant 1/2$.

For instance, if $p = ((0,0), r)$ and $v = (-\frac{\sqrt{3}}{2}r, \frac{1}{2}r)$, then $J(p,v)$ is clearly a 2-Lipschitz graph over the line L_2, which is perpendicular to the direction of v, but $J(p,v)$ is also a 2-Lipschitz graph over the horizontal line L_1 since

$$\pi_{L_1}(J(p,v)) \subset \pi_{L_1}(B(v,1/100)) = [-\tfrac{\sqrt{3}}{2}r - \tfrac{1}{100}, -\tfrac{\sqrt{3}}{2}r + \tfrac{1}{100}] \subset [-\tfrac{2}{\sqrt{5}}r, \tfrac{2}{\sqrt{5}}r].$$

(By the same argument $J(p,v)$ is a 2-Lipschitz graph over L_1 for any $v = (r\cos\varphi, r\sin\varphi)$ with $\varphi \in [\pi/6, 5\pi/6]$).

Without loss of generality, we assume in the following that $J(\mathbf{p},\mathbf{v})$ is a 2-Lipschitz graph over L_1 and L_2. For $1 \leqslant i \leqslant 2$, define

$$\mathcal{R}_i := \{R(p,v) \in \mathcal{R} : J(p,v) \text{ is a 2-Lipschitz graph over } L_i\}.$$

We have $|\mathcal{R}| \leqslant |\mathcal{R}_1| + |\mathcal{R}_2|$. Hence if $|\mathcal{R}_1| \geqslant |\mathcal{R}|/2$, we choose $L = L_1$ and $\mathcal{R}_* = \mathcal{R}_1$. Otherwise, we choose $L = L_2$ and $\mathcal{R}_* = \mathcal{R}_2$. For an illustration, see Figure 5. We assume with no loss of generality that $L = L_1 = \operatorname{span}(1,0)$ and $\mathcal{R}_* = \mathcal{R}_1$. We abbreviate $\pi := \pi_L$ and identify L with $\mathbb{R}$ via $(x_1, 0) \mapsto x_1$. Next, we note that

$$\mathbf{I} := [\pi(\mathbf{v}) - \tfrac{1}{600}, \pi(\mathbf{v}) + \tfrac{1}{600}] \subset \pi(J(\mathbf{p},\mathbf{v})) \cap \bigcap_{R(p,v) \in \mathcal{R}_*} \pi(J(p,v)). \tag{A.3}$$

This follows easily from the fact that $|v - \mathbf{v}| \leqslant 1/600$ and that π restricted to $J(\mathbf{p},\mathbf{v})$ and $J(p,v)$ is 2-Lipschitz; we omit the details. Since $J(\mathbf{p},\mathbf{v}), J(p,v)$ are 2-Lipschitz graphs over L, the inclusion (A.3) shows that $\Gamma_{\mathbf{I},\mathbf{p},\mathbf{v}}, \Gamma_{\mathbf{I},p,v}$ are 2-Lipschitz graphs over the segment $\mathbf{I}$. Moreover, it is clear that

$$\pi(R) \subset \pi(A\mathbf{R}) \subset \pi(B(\mathbf{v}, \tfrac{1}{600})) = \mathbf{I}, \qquad R \in \mathcal{R}_*.$$

This completes the proof of the lemma. $\square$

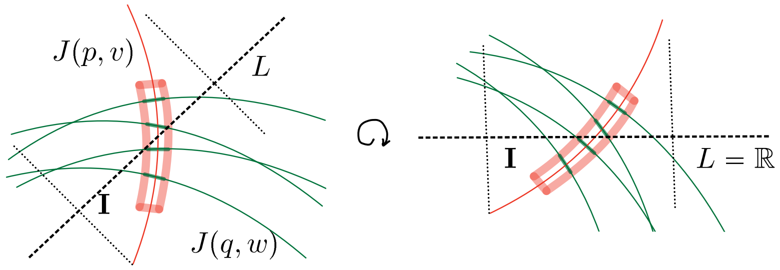

FIGURE 5. Finding the line L. The fat red rectangle represents $A\mathbf{R}$, and the smaller green rectangles inside $A\mathbf{R}$ represent the rectangles $R' \in \mathcal{R}$.

We will apply (a corollary of) Lemma A.2 to the rectangle $\mathbf{R}$ in Proposition 4.12. For that purpose we may assume without loss of generality that the line L given by Lemma A.2 is $\mathrm{span}(1,0)$, and we restrict the following discussion to this case. This convention leaves for each graph $\Gamma_{\mathbf{I},p,v}$ two possibilities: it is contained either on an 'upper' or on a 'lower' half-circle. For $p = (x,r) = (x_1,x_2,r) \in \mathbb{R}^2 \times (0,\infty)$, we write the circle $S(p)$ as the union of two graphs over L as follows

$$S(p) = S(x,r) = \{(y_1,y_2) \in \mathbb{R}^2 : (y_1-x_1)^2 + (y_2-x_2)^2 = r^2\} = S_+(p) \cup S_-(p),$$

where

$$S_\pm(p) = \left\{(y_1, \pm\sqrt{r^2-(y_1-x_1)^2} + x_2) : y_1 \in [x_1-r, x_1+r]\right\}.$$

Now for $p = (x_1,x_2,r) \in \mathbb{R}^2 \times (0,\infty)$, we define

$$f_{p,\pm}(\theta) := \pm\sqrt{r^2-(\theta-x_1)^2} + x_2, \quad \theta \in [x_1-r, x_1+r]. \tag{A.4}$$

We record for any $\theta \in (x_1-r, x_1+r)$,

$$f'_{p,\pm}(\theta) = \mp\frac{\theta-x_1}{\sqrt{r^2-(\theta-x_1)^2}} \quad \text{and} \quad f''_{p,\pm}(\theta) = \mp\frac{r^2}{[r^2-(\theta-x_1)^2]^{3/2}}. \tag{A.5}$$

The functions $f_{p,\pm}$ are 2-Lipschitz on $[x_1 - \frac{2}{\sqrt{5}}r, x_1 + \frac{2}{\sqrt{5}}r]$, and this is the largest interval with that property. At the endpoints of it, the corresponding function values are

$$f_{p,\pm}(x_1 - \tfrac{2r}{\sqrt{5}}) = f_{p,\pm}(x_1 + \tfrac{2r}{\sqrt{5}}) = x_2 \pm \tfrac{r}{\sqrt{5}}. \tag{A.6}$$

The tangents to $S(p)$ in the respective points on $S(p)$ have precise slopes $+2$ or -2.

For simplicity, we denote $\pi = \pi_L : (y_1,y_2) \mapsto y_1$. Assume that $\mathbf{I} \subset L$ is an interval and consider $p \in \mathbf{D}$ and $v \in S(p)$. If the arc $\Gamma_{\mathbf{I},p,v}$ introduced in Definition A.1 is a graph over L, then

$$\text{either} \quad \Gamma_{\mathbf{I},p,v} = \pi^{-1}(\mathbf{I}) \cap S_+(p) \quad \text{or} \quad \Gamma_{\mathbf{I},p,v} = \pi^{-1}(\mathbf{I}) \cap S_-(p) \tag{A.7}$$

and $\Gamma_{\mathbf{I},p,v}$ is the graph of $f_{p,+}|_{\mathbf{I}}$ or $f_{p,-}|_{\mathbf{I}}$, respectively. We may in the following assume that the rectangles $R(p,v) \in \mathcal{R}_*$ given by Lemma A.2 all yield functions of the same type, either all associated to upper half-circles, or all associated to lower half-circles. This type may not however be the same as for the rectangle $\mathbf{R}$, cf. Figure 5.

Lemma A.8. *Under the assumptions of Lemma A.2 (with $L = \mathrm{span}(1,0)$), there exists a subset $\mathcal{R}_* \subset \mathcal{R}$ with $|\mathcal{R}_*| \geqslant |\mathcal{R}|/4$ such that the conclusions (1)-(2) hold and additionally, either $\Gamma_{\mathbf{I},p,v} = \pi_1^{-1}(\mathbf{I}) \cap S_+(p)$ for all $R(p,v) \in \mathcal{R}_*$, or $\Gamma_{\mathbf{I},p,v} = \pi_1^{-1}(\mathbf{I}) \cap S_-(p)$ for all $R(p,v) \in \mathcal{R}_*$.*

Proof. Observation (A.7) shows that the additional property can be arranged by discarding at most half of the elements in the original family $\mathcal{R}_*$ given by Lemma A.2. □

Even with this additional assumption in place, the family $\mathcal{R}_*$ is not quite of the same form as the families of graph neighborhoods considered in [17], but it is also not too different. For arbitrary $\eta > 0$ and subinterval $I \subset \mathbf{I}$ in the domain of $f_{p,\pm}$, we define the *vertical η-neighborhood*

$$f^{\eta}_{p,\pm}(I) := \{(y_1, y_2) \in I \times \mathbb{R} : f_{p,\pm}(y_1) - \eta \leqslant y_2 \leqslant f_{p,\pm}(y_1) + \eta\}.$$

Moreover, for any $\eta \in (0, 1/200]$, if $f_{p,\pm} : \mathbf{I} \to \mathbb{R}$ is 2-Lipschitz, then

$$\pi^{-1}(I) \cap S^{\eta}(p) \subset f^{4\eta}_{p,+}(I) \cup f^{4\eta}_{p,-}(I). \tag{A.9}$$

Here, the upper bound on η ensures that the points on $S(p)$ which are η-close to points in $\pi^{-1}(I) \cap S^{\eta}(p)$ lie in the part of the graph of $f_{p,\pm}$ where the Lipschitz constant is small enough for the inclusion (A.9) to hold. In particular, if $R = R(p, v)$ is a rectangle with $I = \pi(R) \subset \mathbf{I}$ and $\eta = \delta < 1/200$, and if $f_p \in \{f_{p,+}, f_{p,-}\}$ is such that $\Gamma_{\mathbf{I},p,v}$ is the graph of f_p, then the inclusion in (A.9) yields

$$R \subset f^{4\delta}_p(\pi(R)) \tag{A.10}$$

since $R \subset \pi^{-1}(\pi(R)) \cap S^{\delta}(p)$. A priori, (A.9) only yields $R \subset f^{4\delta}_{p,+}(\pi(R)) \cup f^{4\delta}_{p,-}(\pi(R))$, but the conditions $\delta < 1/200$, $\pi(R) \subset \mathbf{I}$ and the assumptions on $\Gamma_{\mathbf{I},p,v}$ ensure that either $R \subset f^{4\delta}_{p,+}(\pi(R))$ or $R \subset f^{4\delta}_{p,-}(\pi(R))$.

We will also need an opposite inclusion for enlarged rectangles. Let $\delta \leqslant \sigma$, $R = R(p, v)$ with $\pi(R) \subset \mathbf{I}$ and $f_p : \mathbf{I} \to \mathbb{R}$ be 2-Lipschitz with graph equal to $\Gamma_{\mathbf{I},p,v}$. Then, for any $C \geqslant 1$, if $I \subset \mathbf{I}$ is an interval with $|I| \leqslant C\sigma$ and such that $\pi(R) \subset I$, then

$$f^{C\delta}_p(I) \subset 4CR. \tag{A.11}$$

The inclusion $f^{C\delta}_p(I) \subset S^{4C\delta}(p)$ is clear. To prove that also $f^{C\delta}_p(I) \subset B(v, 4C\sigma)$, consider an arbitrary point $y = (y_1, y_2) \in f^{C\delta}_p(I)$. Since $\pi(R) \subset I \subset \mathbf{I}$ and $\Gamma_{\mathbf{I},p,v}$ is the graph of f_p, there exists $\theta \in I$ such that $v = (\theta, f_p(\theta))$ and using the 2-Lipschitz continuity of f_p on $\mathbf{I} \supset I$, we can estimate

$$|y - v| \leqslant |y_2 - f_p(y_1)| + |(y_1, f_p(y_1)) - (\theta, f_p(\theta))| \leqslant C\delta + \sqrt{5}|y_1 - \theta| \leqslant C\delta + 3|I| \leqslant 4C\sigma,$$

concluding the proof of (A.11). In order to apply arguments that were stated in [17] for certain C^2 functions, we need a preliminary result about the behavior of $p \mapsto f_{p,\pm}$ with respect to the $C^2(\mathbf{I})$-norm.

Lemma A.12. *There exists an absolute constant $K \geqslant 1$ such that for all $p, p' \in \mathbf{D}$, if $\mathbf{I} \subset \mathbb{R}$ is an interval so that $f_{p,+}, f_{p',+} : \mathbf{I} \to \mathbb{R}$ are 2-Lipschitz, then*

$$\|f_{p,+} - f_{p',+}\|_{C^2(\mathbf{I})} \leqslant K|p - p'|. \tag{A.13}$$

The corresponding result for the pair $(f_{p,-}, f_{p',-})$ is also true, but not needed.

Proof. We abbreviate $f_p = f_{p,+}$ for $p = (x_1, x_2, r)$. The norm $\|f_p\|_{C^2(\mathbf{I})}$ is uniformly bounded for all p and $\mathbf{I}$ as in the statement of the lemma. Indeed, since f_p is assumed to be 2-Lipschitz on $\mathbf{I}$, we have $f_p(\theta) \in [x_2 + \frac{r}{\sqrt{5}}, x_2 + r]$, $\theta \in \mathbf{I}$, by the discussion around (A.6) and hence

$$\tfrac{r}{\sqrt{5}} \leqslant \sqrt{r^2 - (\theta - x_1)^2} \leqslant r \tag{A.14}$$

for all $\theta \in \mathbf{I}$. Since $p \in \mathbf{D}$, this yields a uniform upper bound for $\|f_p\|_{C^2(\mathbf{I})}$, recalling the expressions stated in (A.4)–(A.5) for f_p and its derivatives. Thus it suffices to prove (A.13) under the assumption that $|p - p'| \leqslant 1/400$.

For arbitrary $p, p' \in \mathbb{R}^2 \times (0, \infty)$, we have

$$S(p') \subset S^{2|p-p'|}(p). \tag{A.15}$$

In particular,

$$(\theta, f_{p'}(\theta)) \in \pi^{-1}(\mathbf{I}) \cap S^{2|p-p'|}(p) \overset{(A.9)}{\subset} f_{p,+}^{8|p-p'|}(\mathbf{I}) \cup f_{p,-}^{8|p-p'|}(\mathbf{I}), \quad \theta \in \mathbf{I}.$$

Our upper bound $|p - p'| \leqslant 1/400$ and the assumption $p, p' \in \mathbf{D}$ rule out the possibility that $(\theta, f_{p'}(\theta)) \in f_{p,-}^{8|p-p'|}(\mathbf{I})$. Indeed, by (A.6), we know on the one hand that

$$f_{p'}(\theta) \in [x_2' + \tfrac{r'}{5}, x_2' + r'].$$

On the other hand, again by (A.6), if $(\theta, f_{p'}(\theta)) \in f_{p,-}^{8|p-p'|}(\mathbf{I})$, then necessarily

$$f_{p'}(\theta) \in \left[x_2 - r - 8|p - p'|, x_2 - \tfrac{r}{\sqrt{5}} + 8|p - p'|\right].$$

The two inclusions are compatible only if

$$x_2' + \tfrac{r'}{5} \leqslant x_2 - \tfrac{r}{\sqrt{5}} + 8|p - p'|,$$

or in other words, if

$$8|p - p'| \geqslant x_2' - x_2 + \tfrac{r + r'}{\sqrt{5}}.$$

Since this implies that $9|p - p'| \geqslant 1/\sqrt{5}$, it is impossible. Thus we conclude that

$$(\theta, f_{p'}(\theta)) \in f_p^{8|p-p'|}(\mathbf{I}), \quad \theta \in \mathbf{I}.$$

In particular, it follows

$$\|f_{p'} - f_p\|_\infty := \sup_{\theta \in \mathbf{I}} |f_{p'}(\theta) - f_p(\theta)| \leqslant 8|p - p'|. \tag{A.16}$$

We write again in coordinates $p = (x_1, x_2, r)$. The estimate (A.14), established at the beginning of the proof, combined with (A.16), the assumption $p, p' \in \mathbf{D}$, and a direct computation gives

$$\|f_{p'}' - f_p'\|_\infty \lesssim |p - p'|, \quad \|f_{p'}'' - f_p''\|_\infty \lesssim |p - p'|,$$

with uniform implicit constants. Together with (A.16), this concludes the proof. □

To prove Proposition 4.12, we have to deal with rectangles that are 100-incomparable in the sense of Definition 4.8. We now record a simple consequence of this property that will be easier to apply when working with the 'graph neighborhood rectangles'.

Lemma A.17. *Let $0 < \delta \leqslant \sigma \leqslant 1/200$ and assume that $R = R(p, v), R' = R(p', v')$ are 100-incomparable (δ, σ)-rectangles with $p, p' \in \mathbf{D}$. Suppose further that there exists an interval $\mathbf{I}$ such that $\Gamma_{\mathbf{I},p,v} \subset S_+(p)$, $\Gamma_{\mathbf{I},p',v'} \subset S_+(p')$, $\pi(R) \cup \pi(R') \subset \mathbf{I}$ and so that $f_{p,+}$ and $f_{p',+}$ are 2-Lipschitz on $\mathbf{I}$.*

Then, if $R(p, v) \cap R(p', v') \neq \emptyset$, there exists a point $\theta \in \pi(R(p, v) \cup R(p', v'))$ such that

$$|f_{p,+}(\theta) - f_{p',+}(\theta)| > 20\delta.$$

Proof. We denote $I := \pi(R(p,v) \cup R(p',v'))$ and observe that this is an interval since $R(p,v) \cap R(p',v') \neq \varnothing$. By assumption $I \subset \mathbf{I}$. To prove the lemma, we argue by contradiction and assume for all $\theta \in I$,

$$|f_{p,+}(\theta) - f_{p',+}(\theta)| \leqslant 20\delta. \tag{A.18}$$

This implies that

$$R(p',v') \overset{(A.10)}{\subset} f^{4\delta}_{p',+}(I) \overset{(A.18)}{\subset} f^{24\delta}_{p,+}(I). \tag{A.19}$$

Since

$$|I| = |\pi(R(p,v) \cup R(p',v'))| \leqslant |\pi(B(v,\sigma))| + |\pi(B(v',\sigma))| \leqslant 4\sigma,$$

we can use (A.11) to conclude from (A.19) that $R(p',v') \subset 100R(p,v)$, contradicting the 100-incomparability of $R(p,v)$ and $R(p',v')$. This concludes the proof. □

We are now in a position to show Proposition 4.12:

Proof of Proposition 4.12. Let $\mathbf{R} = R(\mathbf{p},\mathbf{v})$ be a fixed (δ,σ)-rectangle as in the statement of the proposition. Since every (δ,σ)-rectangle $R = R(p,v) \subset A\mathbf{R}$ is contained in $S^{A\delta}(\mathbf{p},\mathbf{v}) \cap S^{A\delta}(p,v)$, and since $S^{A\delta}(\mathbf{p},\mathbf{v}) \cap S^{A\delta}(p,v)$ can be covered by boundedly many $(A\delta, \sqrt{A\delta/|p-\mathbf{p}|})$-rectangles according to Lemma 4.3, it follows that

$$\sigma \lesssim \sqrt{\tfrac{A\delta}{|p-\mathbf{p}|}}.$$

This holds in particular for all $R = R(p,v) \in \mathcal{R}$. Hence defining

$$P_{\mathbf{R}} := \{p \in \mathbf{D} \colon R(p,v) \in \mathcal{R} \text{ for some } v \in S(p)\},$$

we know that there exists a universal constant $\mathbf{C} > 0$ such that

$$P_{\mathbf{R}} \subset B\big(\mathbf{p}, \mathbf{C}\tfrac{A\delta}{\sigma^2}\big) \subset \mathbb{R}^3. \tag{A.20}$$

We make one more observation about the family $\mathcal{R}$, which will show in particular that it is finite. Namely, if $R(p,v) \in \mathcal{R}$, then

$$|\{R(p',v') \in \mathcal{R} \colon |p-p'| \leqslant \delta\}| \lesssim A. \tag{A.21}$$

Indeed, let $R(p_1,v_1),\ldots,R(p_n,v_n)$ be a listing of the rectangles on the left. Then $v_i \in A\mathbf{R} \cap S^{3\delta}(p)$ for all $1 \leqslant i \leqslant n$. Note that $\operatorname{diam}(A\mathbf{R}) \sim A\sigma$. Now, if $n \geqslant CA$ for a suitable absolute constant $C \geqslant 1$, we may find two elements v_i, v_j with $|v_i - v_j| \leqslant 10\sigma$. But since $|p_i - p_j| \leqslant 2\delta$, it would follow that the rectangles $R(p_i,v_i)$ and $R(p_j,v_j)$ are 100-comparable, contrary to our assumption. This proves (A.21).

We divide the remaining proof into two cases according to the size of σ, using the threshold $\sigma_0 = 1/600$ from Lemma A.2. The first case, where σ is close to 1, will follow roughly speaking because the rectangles in $\mathcal{R}$ are so curvy that their containment in a common rectangle $A\mathbf{R}$ forces $P_{\mathbf{R}}$ to be contained in a $\sim_A \delta$ ball. The second case falls under the regime where the assumptions of Lemmas A.2 and A.8 are satisfied, and we can work with rectangles that are essentially neighborhoods of graphs over a fixed line.

Case 1 ($A^{-1}\sigma_0 < \sigma \leqslant 1$). Inserting the lower bound for σ into (A.20), we find that there exists a universal constant $\mathbf{C} > 0$ (possibly larger than before) such that

$$P_{\mathbf{R}} \subset B(\mathbf{p}, \mathbf{C}A^3\delta).$$

Hence, $P_{\mathbf{R}}$ can be covered by $N \lesssim (\mathbf{C}A^3)^3$ balls $B_1, \ldots, B_N$ of radius $\delta/2$. By (A.21), for every $i = 1, \ldots, N$, there are $\lesssim A$ rectangles $R(p,v) \in \mathcal{R}$ with $p \in B_i$. We deduce that

$$|\mathcal{R}| \lesssim (\mathbf{C}A^3)^3 \max_{i \in \{1,\ldots,N\}} |\{R(p,v) \in \mathcal{R} \colon p \in P_{\mathbf{R}} \cap B_i\}| \lesssim (\mathbf{C}A^3)^3 \cdot A \sim A^{10}.$$

Case 2 ($\sigma \leqslant A^{-1}\sigma_0$). Let now $\mathcal{R}_* \subset \mathcal{R}$ be the subfamily given by Lemma A.8. Without loss of generality we may assume that for every $R(p,v) \in \mathcal{R}_*$, we have $\Gamma_{\mathbf{I},p,v} \subset S_+(p)$. To implement the approach from the proof of [17, Lemma 3.15], we need one more reduction to ensure that the rectangles $R(p,v)$ we consider give rise to functions $f_{p,+}$ that are sufficiently close to each other in $C^2(\mathbf{I})$-norm. Using Lemma A.12, this can be ensured if the parameters p are sufficiently close in $\mathbf{D}$. By (A.20), and recalling $\operatorname{diam} \mathbf{D} \leqslant 2$, we know already that

$$P_{\mathbf{R},*} := \{p \in \mathbf{D} \colon \text{ there is } v \in S(p) \text{ with } R(p,v) \in \mathcal{R}_*\} \subset B(\mathbf{p}, \mathbf{A}t), \tag{A.22}$$

where $\mathbf{A} \lesssim A$ and

$$t := \min\{\delta/\sigma^2, 2\}.$$

On the other hand, by (A.21), we also know that for each $p \in P_{\mathbf{R},*}$, there are at most $\lesssim A$ many $v \in S(p)$ such that $R(p,v) \in \mathcal{R}_*$. As a result,

$$|P_{\mathbf{R},*}| \gtrsim A^{-1}|\mathcal{R}_*|. \tag{A.23}$$

Combining (A.22) and (A.23), we may choose a ball

$$B_0 \subset B(\mathbf{p}, \mathbf{A}t) \tag{A.24}$$

of radius $\frac{t}{2K}$, where $K \geqslant 1$ is the constant from Lemma A.12, such that

$$|P_{\mathbf{R},*} \cap B_0| \gtrsim \mathbf{A}^{-3}|P_{\mathbf{R},*}|. \tag{A.25}$$

We define a further subfamily

$$\mathcal{R}_*^\circ := \{R(p,v) \in \mathcal{R}_* : p \in P_{\mathbf{R},*} \cap B_0\}.$$

Hence by (A.23) and (A.25)

$$|\mathcal{R}_*^\circ| \geqslant |P_{\mathbf{R},*} \cap B_0| \gtrsim A^{-4}|\mathcal{R}_*|.$$

Thus if we manage to show that $|\mathcal{R}_*^\circ| \lesssim A^3$, we can deduce that

$$A^{-4}|\mathcal{R}_*| \lesssim |\mathcal{R}_*^\circ| \lesssim A^3 \quad \Longrightarrow \quad |\mathcal{R}_*| \lesssim A^4A^3 \lesssim A^{10}.$$

This will conclude the proof since $|\mathcal{R}_*| \sim |\mathcal{R}|$ by Lemma A.8.

It remains to prove that $|\mathcal{R}_*^\circ| \lesssim A^3$. Applying Corollary A.12, we deduce that

$$\|f_i - f_j\|_{C^2(\mathbf{I})} \leqslant t \qquad p_i, p_j \in B_0, \tag{A.26}$$

where $f_i := f_{p_i,+}$ and $f_j := f_{p_j,+}$. Following the argument in [17, Lemma 3.15], we will show that

$$|\{R \in \mathcal{R}_*^\circ : z \in R\}| \lesssim A, \qquad z \in \mathbb{R}^2. \tag{A.27}$$

This will give

$$|\mathcal{R}_*^\circ| \cdot \delta\sigma \lesssim \int_{A\mathbf{R}} \sum_{R \in \mathcal{R}_*^\circ} \mathbf{1}_R \lesssim A \cdot \mathrm{Leb}(A\mathbf{R}) \lesssim A^3\delta\sigma,$$

as desired.

To prove (A.27), fix $z = (\theta_0, y_0) \in \mathbb{R}^2$ which is contained in, say, N pairwise 100-incomparable (δ,σ)-rectangles $R_j \in \mathcal{R}_*^\circ$, for $1 \leqslant j \leqslant N$. The claim is that $N \lesssim A$.

Note that $\pi(R_j)$ necessarily contains the point $\theta_0 + \sigma/3$ or $\theta_0 - \sigma/3$, and we can bound individually the cardinality of the two subfamilies of $\{R_j : j = 1, \ldots, N\}$ where one of the two options occur. Thus let us assume in the following without loss of generality that $\theta_0 + \sigma/3 \in \pi(R_j)$ for all j.

To show our claim, it suffices to establish the following two inequalities:

$$|f_i'(\theta_0) - f_j'(\theta_0)| \leqslant 100A \cdot (\delta/\sigma), \qquad 1 \leqslant i,j \leqslant N, \tag{A.28}$$

and

$$|f_i'(\theta_0) - f_j'(\theta_0)| \geqslant \delta/\sigma, \qquad 1 \leqslant i \neq j \leqslant N. \tag{A.29}$$

The first inequality will be based on the assumption that the rectangles in $\mathcal{R}$ are contained in $A\mathbf{R}$, and the second inequality uses the 100-incomparability of the rectangles in $\mathcal{R}_*^\circ$.

We give one argument that takes care both of the short rectangles ($\sigma \leqslant \sqrt{\delta}$), and the long rectangles ($\sigma \geqslant \sqrt{\delta}$) treated in [17]. Recalling the $C^2(\mathbf{I})$ bound (A.26), we have

$$\|f_i - f_j\|_{C^2(\mathbf{I})} \leqslant t = \min\{\delta/\sigma^2, 2\}. \tag{A.30}$$

We apply this to prove (A.28). Let us denote $h := f_i - f_j$, and let us assume to the contrary that $|h'(\theta_0)| > 100A \cdot (\delta/\sigma)$. Then, using (A.30), for all $\theta \in \pi(R_i) \cup \pi(R_j)$ with $|\theta - \theta_0| \leqslant \sigma$, we have

$$|h'(\theta)| \geqslant |h'(\theta_0)| - \|h''\|_\infty |\theta - \theta_0| \geqslant 100A \cdot (\delta/\sigma) - \min\{\delta/\sigma^2, 2\}\sigma > 99A \cdot (\delta/\sigma),$$

using $A \geqslant 1$. By (A.10) and the assumption that the rectangles R_j all intersect at (θ_0, y_0) and $\theta_0 \in \pi(R_j) \subset \mathbf{I}$, we have $|h(\theta_0)| \leqslant 8\delta$. We will combine this information with the lower bound for $|h'|$ on the interval $\pi(R_i) \cup \pi(R_j)$ to reach a contradiction with the assumption that $R_i \cup R_j \subset A\mathbf{R}$. Recall that $\theta_0 + \sigma/3 \in \pi(R_i) \cap \pi(R_j)$. Then,

$$|h(\theta_0 + \sigma/3)| \geqslant |h(\theta_0 + \sigma/3) - h(\theta_0)| - 8\delta \geqslant 99A \cdot (\delta/\sigma) \cdot \sigma/3 - 8\delta \geqslant 33A \cdot \delta - 8\delta > 25A\delta.$$

But this is not consistent with the assumption that

$$\{(\theta_0 + \sigma/3, f_i(\theta_0 + \sigma/3)), (\theta_0 + \sigma/3, f_j(\theta_0 + \sigma/3))\} \subset R_i \cup R_j \subset A\mathbf{R},$$

noting that the "vertical" thickness of $A\mathbf{R}$ is at most $8A\delta$ since $A\mathbf{R} \subset f_{\mathbf{p},+}^{4A\delta}(\pi(A\mathbf{R}))$ or $A\mathbf{R} \subset f_{\mathbf{p},-}^{4A\delta}(\pi(A\mathbf{R}))$ according to (A.10).

The proof of (A.29) is similar. This time we make the counter assumption that $|h'(\theta_0)| < \delta/\sigma$. The assumption $\theta_0 \in \pi(R_i \cap R_j)$ implies that $\pi(R_i \cup R_j)$ is an interval contained in $[\theta_0 - 2\sigma, \theta_0 + 2\sigma]$. Using (A.30), as above, this leads to the following estimate

$$|h'(\theta)| \leqslant |h'(\theta_0)| + \|h'\|_\infty |\theta - \theta_0| < \tfrac{\delta}{\sigma} + \min\left\{\tfrac{\delta}{\sigma^2}, 2\right\} 2\sigma \leqslant 3\delta/\sigma, \qquad \theta \in \pi(R_i \cup R_j).$$

Finally, since $|h(\theta_0)| \leqslant 8\delta$, we deduce from the preceding estimate that

$$|h(\theta)| \leqslant 8\delta + (3\delta/\sigma) \cdot 2\sigma = 14\delta, \qquad \theta \in \pi(R_i \cup R_j).$$

This inequality contradicts Lemma A.17 and shows that the counter-assumption cannot hold. This completes the proof of (A.29), and thus the proof of Proposition 4.12. $\square$

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