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Volume growth, capacity estimates, p-parabolicity and sharp integrability properties of p-harmonic Green functions

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Abstract. In a complete metric space equipped with a doubling measure supporting a p-Poincaré inequality, we prove sharp growth and integrability results for p-harmonic Green functions and their minimal p-weak upper gradients. We show that these properties are determined by the growth of the underlying measure near the singularity. Corresponding results are obtained also for more general p-harmonic functions with poles, as well as for singular solutions of elliptic differential equations in divergence form on weighted $\mathbf{R}^n$ and on manifolds.

The proofs are based on a new general capacity estimate for annuli, which implies precise pointwise estimates for p-harmonic Green functions. The capacity estimate is valid under considerably milder assumptions than above. We also use it, under these milder assumptions, to characterize singletons of zero capacity and the p-parabolicity of the space. This generalizes and improves earlier results that have been important especially in the context of Riemannian manifolds.

Key words and phrases: capacity, doubling measure, Green function, integrability, metric space, p-harmonic function, p-hyperbolic space, Poincaré inequality, p-parabolic space, singular function, volume growth, weak upper gradient.

Mathematics Subject Classification (2020): Primary: 31C45; Secondary: 30L99, 31C12, 31C15, 31E05, 35J08, 35J92, 46E36, 49Q20.

1. Introduction

In this paper we study the growth and L^τ-integrability of p-harmonic Green (and singular) functions in metric measure spaces, as well as L^t-integrability of their minimal p-weak upper gradients, with $1 < p < \infty$. We show that these properties are determined by the growth of the measure near the singularity. We also obtain corresponding results for more general p-harmonic functions with poles, as well as for singular solutions of elliptic differential equations in divergence form on weighted $\mathbf{R}^n$ and on manifolds.

Recall that u is a *p-harmonic Green function* in a bounded domain $\Omega \subset \mathbf{R}^n$ with singularity at $x_0 \in \Omega$ if

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\delta_{x_0} \quad \text{in } \Omega \tag{1.1}$$

with zero boundary values on $\partial\Omega$ (in Sobolev sense), where δ_{x_0} is the Dirac measure at x_0. Such a function u is p-harmonic (i.e. $\Delta_p u = 0$) in $\Omega \setminus \{x_0\}$ and p-superharmonic (i.e. $\Delta_p u \le 0$) in the whole domain Ω. If $1 < p \le n$, then also $\lim_{x\to x_0} u(x) = \infty$.

In a metric measure space $X = (X, d, \mu)$ there is (a priori) no equation available for defining p-harmonic functions, and they are instead defined as local minimizers of the p-energy integral

$$\int g_u^p \, d\mu,$$

where g_u is the minimal p-weak upper gradient of u. For example, on $\mathbf{R}^n$ we have $g_u = |\nabla u|$ and these definitions of p-harmonic functions and p-harmonic Green functions are equivalent to the definitions using the p-Laplace operator $\Delta_p u$.

Let $\Omega \subset X$ be a bounded domain and assume that $x_0 \in \Omega$ with p-capacity $C_p(\{x_0\}) = 0$. Following our earlier paper [11], we say that u is a *singular function* in Ω with singularity at x_0 if u is p-harmonic in $\Omega \setminus \{x_0\}$ and p-superharmonic in Ω, $u = 0$ on $\partial\Omega$ in the Sobolev sense and $\lim_{x\to x_0} u(x) = \infty$. A *Green function* is then a precisely scaled singular function. See Definition 6.3 for exact definitions. Earlier definitions are due to Holopainen [31] for manifolds, Heinonen–Kilpeläinen–Martio [28, Section 7.38] for weighted $\mathbf{R}^n$ and Holopainen–Shanmugalingam [35] for metric spaces.

Throughout the paper, we fix $1 < p < \infty$ and a point $x_0 \in X$ and write $B_r = B(x_0, r)$. For the rest of the introduction, we also assume that X is a complete metric space equipped with a doubling measure μ that supports a p-Poincaré inequality.

The following result summarizes some of the main results in this paper, many of which are new also in weighted $\mathbf{R}^n$ and on manifolds. Clearly, it contains the known sharp results for unweighed $\mathbf{R}^n$ and $1 < p \le n$ as special cases (with $\bar{s}_0 = n$). Let

$$\bar{s}_0 = \inf\{s > 0 : \text{there is } C_s > 0 \text{ so that } \mu(B(x_0, r)) \ge C_s r^s \text{ for } 0 < r \le 1\},$$

$$\tau_p = \begin{cases} \dfrac{\bar{s}_0(p-1)}{\bar{s}_0 - p}, & \text{if } p < \bar{s}_0, \\ \infty, & \text{if } p = \bar{s}_0, \end{cases} \qquad \text{and} \qquad t_p = \frac{\bar{s}_0(p-1)}{\bar{s}_0 - 1}. \tag{1.2}$$

Theorem 1.1. *Let $\Omega \subset X$ be a bounded domain containing x_0, and u be a singular or Green function in Ω with singularity at x_0. Assume that $C_p(\{x_0\}) = 0$. Then the following are true*:

(a) *$p \le \bar{s}_0$ and u is unbounded*;
(b) *$u \in L^\tau(\Omega)$ for all $0 < \tau < \tau_p$*;
(c) *$u \notin L^\tau(\Omega)$ if $\tau > \tau_p$*;
(d) *$g_u \in L^t(\Omega)$ for all $0 < t < t_p$*;
(e) *if $p = \bar{s}_0$, then $g_u \in L^t(\Omega)$ if and only if $0 < t < p$*;
(f) *if $p < \bar{s}_0$, then $g_u \notin L^t(\Omega)$ whenever μ supports a t-Poincaré inequality* (*at x_0 and for small radii*), *$t > t_p$ and $t \ge 1$.*

The case $C_p(\{x_0\}) > 0$ is not of interest here, since in this case every singular (and Green) function u in Ω with singularity at x_0 is bounded and $g_u \in L^p(\Omega)$; see

Theorem 6.6, which also shows that the Green function (with singularity at x_0) is unbounded if and only if

$$\int_0^1 \left(\frac{\rho}{\mu(B_\rho)}\right)^{1/(p-1)} d\rho = \infty. \tag{1.3}$$

In the borderline case $\tau = \tau_p$ we completely characterize when $u \in L^{\tau_p}(\Omega)$ in terms of integrals similar to the one in (1.3), see Theorem 9.5. We also provide sharp results on the integrability of the minimal p-weak upper gradient g_u in the borderline case $t = t_p$, see Theorem 10.3. In the locally pointwise Ahlfors Q-regular case we obtain the following complete characterization. In particular, it applies to Riemannian manifolds with nonnegative Ricci curvature and to Carnot groups.

Theorem 1.2. *Let $\Omega \subset X$ be a bounded domain containing x_0, and u be a singular or Green function in Ω with singularity at x_0. Assume that $Q \geq p$ and that μ is Ahlfors Q-regular around x_0 for small radii, i.e.*

$$\mu(B(x_0, r)) \simeq r^Q, \quad \text{if } 0 < r \leq 1.$$

Then in a neighbourhood of x_0,

$$u(x) \simeq \begin{cases} d(x,x_0)^{(p-Q)/(p-1)}, & \text{if } p < Q, \\ -\log d(x,x_0), & \text{if } p = Q, \end{cases} \tag{1.4}$$

and the following are true:
- (a) *$u \in L^\tau(\Omega)$ if and only if $0 < \tau < \tau_p := Q(p-1)/(Q-p)$ (where $\tau_Q = \infty$);*
- (b) *$g_u \in L^t(\Omega)$ for all $0 < t < t_p := Q(p-1)/(Q-1)$;*
- (c) *$g_u \notin L^t(\Omega)$ if $t \geq \max\{1, t_p\}$ and X supports a t-Poincaré inequality (at x_0 and for small radii).*

Using the flexibility of our definition of singular functions, with no a priori superlevel set requirements, Theorem 1.1 implies the following very similar growth properties for general p-harmonic functions with poles.

Theorem 1.3. *Let $\Omega \subset X$ be an open set containing x_0. Assume that $u \geq 0$ is a p-harmonic function in $\Omega \setminus \{x_0\}$ such that $\lim_{x\to x_0} u(x) = \infty$. Then $C_p(\{x_0\}) = 0$ and the statements (a)–(f) in Theorem 1.1 about (non)integrability hold true with $L^\tau_{\mathrm{loc}}(\Omega)$ and $L^t_{\mathrm{loc}}(\Omega)$ instead of $L^\tau(\Omega)$ and $L^t(\Omega)$.*

Moreover, there exists $R > 0$ such that in a neighbourhood of x_0,

$$u(x) \simeq \inf_{B_R} u + \int_{d(x,x_0)}^R \left(\frac{\rho}{\mu(B_\rho)}\right)^{1/(p-1)} d\rho. \tag{1.5}$$

For Green and singular functions, $\inf_{B_R} u$ can be replaced by 0.

See also Theorem 13.1 for corresponding integrability properties of singular functions for elliptic differential equations in divergence form on weighted $\mathbf{R}^n$. The comparison (1.5) can be seen as a Wolff potential estimate (12.8) in terms of the Dirac measure δ_{x_0}, cf. Remark 12.3. This is natural in view of (1.1) even though there need not be such an equation in the metric setting.

Existence of singular and Green functions in metric spaces was proved in [11, Theorem 1.3] for bounded open sets Ω with $C_p(X \setminus \Omega) > 0$. It was also shown therein that any two Green functions in Ω with the same singularity are comparable to each other and thus have the same growth behaviour near the singularity; see Theorem 6.4. More explicitly, for a Green function u in Ω with singularity at x_0, we have by Theorem 7.1 that

$$u(x) \simeq \mathrm{cap}_p(B_r, \Omega)^{1/(1-p)}, \quad \text{if } 0 < d(x,x_0) = r < R, \tag{1.6}$$

where R depends only on Ω. In many cases, estimate (1.6) can be expressed in terms of r and $\mu(B_r)$ by using the critical exponents and exponent sets for the volume growth, studied in [10]; see Section 3 and Corollary 7.2.

The pointwise estimates and integrability properties of Green functions and their minimal p-weak upper gradients in Theorems 1.1–1.3 are based on (1.6) and the following new general capacity estimate,

$$\operatorname{cap}_p(B_r, B_R) \simeq \biggl(\int_r^R \biggl(\frac{\rho}{\mu(B_\rho)}\biggr)^{1/(p-1)} d\rho\biggr)^{1-p} \tag{1.7}$$

for $0 < 2r \le R \le \frac{1}{4} \operatorname{diam} X$. We prove (1.7) in Theorem 4.2 only assuming that the Poincaré inequality and the doubling (and reverse-doubling) condition for μ hold for balls centred at x_0. In Propositions 5.3 and 5.5 we characterize when singletons have zero capacity and when X is p-parabolic, by letting $r \to 0$ and $R \to \infty$ in (1.7), respectively.

In the weighted linear case on $\mathbf{R}^n$ (with $p = 2$), Fabes–Jerison–Kenig [20, Lemma 3.1 and Theorem 3.3] obtained (1.6) and (1.7) already in 1982. Integrability of the Green functions and their gradients was proved in Chanillo–Wheeden [17, Theorem 1.3] for linear equations in A_2-weighted $\mathbf{R}^n$ for exponents smaller than the global analogues of our τ_2 and t_2, respectively (i.e. as in Theorem 8.1 below with $p = 2$). Such integrability estimates are useful in integral representation formulas for solutions of partial differential equations.

Nevertheless, as far as we know, even in this case the sharp integrability results with pointwise exponents as in (1.2) and Theorem 1.1 do not appear in the existing literature. Neither do the nonintegrability results as in (c) and (f), although they are well known for unweighted $\mathbf{R}^n$ and many linear and nonlinear equations with explicit pointwise estimates for Green functions; (f) follows from (c) and the Sobolev inequality, see also (0.8) in Kichenassamy–Véron [38] for $\Delta_p u = 0$ and $p > 1$.

For rather general nonlinear equations in unweighted $\mathbf{R}^n$, pointwise estimates of the type (1.4) for solutions with isolated singularities (including Green functions) were obtained by Serrin [54, Theorem 12]. Pointwise estimates for capacitary potentials associated with such equations were obtained by Maz′ya [50, Lemmas 3 and 4] (and were used therein in the proof of the sufficiency part of the Wiener criterion for such equations). These estimates were later extended to weighted equations in Heinonen–Kilpeläinen–Martio [28]. Since the scaled truncations $\min\{u/m, 1\}$, $m > 0$, of singular functions are essentially the capacitary potentials of the superlevel sets $\{u \ge m\}$, such potential estimates (and their metric space version from [15]) lie behind (1.6) and the construction of singular functions in [11].

Our general assumptions on doubling and p-Poincaré inequality are fulfilled on weighted $\mathbf{R}^n$ equipped with a p-admissible weight as in [28], on Riemannian manifolds with nonnegative Ricci curvature, on Carnot groups, and for vector fields satisfying the Hörmander condition, as well as in many other situations. Thus, our results hold for p-harmonic functions and corresponding subelliptic equations in all these settings, see Hajłasz–Koskela [25, Sections 10–13] for further details. Moreover, as in [11, Section 11] the assumptions can be relaxed to similar local assumptions.

The exponents in Theorem 1.1 are often better than in the integrability results for general p-superharmonic functions and their gradients from Heinonen–Kilpeläinen–Martio [28, Theorem 7.46] (on weighted $\mathbf{R}^n$) and Kinnunen–Martio [42, Section 5] (on metric spaces). This happens e.g. if the local dimension $\bar{s}_0$ at x_0 is smaller than the global dimension of the space, provided by the doubling property of μ. For example, the 1-admissible weight $w(x) = |x|^{-\alpha}$ on $\mathbf{R}^n$, with $0 < \alpha < n$, has $\bar{s}_0 = n - \alpha$ at x_0, while the general integrability for p-superharmonic functions is dictated by the dimension n (see also Example 3.1). In the globally Ahlfors

Q-regular case, i.e. when

$$\mu(B(x,r)) \simeq r^Q \quad \text{for all } x \in X \text{ and } r > 0,$$

the integrability conditions in Theorem 1.2 follow from the general integrability results in Kinnunen–Martio [42], but even in this case the nonintegrability conditions in Theorem 1.2 seem to be new. For corresponding singular solutions in Carnot–Carathéodory spaces, the nonintegrability in Theorem 1.1 (c) follows from Capogna–Danielli–Garofalo [16, Corollary 6.1]. See also [16, Theorem 7.1] for pointwise estimates related to (1.6).

Danielli–Garofalo–Marola [19, Corollary 5.4] obtained positive integrability results as in Theorem 1.1 (b) and (d) for singular functions defined as in Holopainen–Shanmugalingam [35], under some additional assumptions on X. However they had smaller ranges of p, τ and t, see Remark 9.2 and the comments after Theorem 10.1. On the other hand, as shown by (c), (e) and (f), the ranges in Theorem 1.1 are optimal up to the borderline cases.

Estimates (1.6) and (1.7) generalize and improve many results obtained earlier mainly in the setting of Riemannian manifolds and under additional geometric and curvature assumptions. For example, estimates using chains of balls along geodesics were obtained in Holopainen–Koskela [34], while in Coulhon–Holopainen–Saloff-Coste [18, Theorem 3.5], capacity was estimated by means of the p-isometric profile. See also Holopainen [33, p. 329]. Different but equivalent formulas for A_p-weighted $\mathbf{R}^n$ were given in Heinonen–Kilpeläinen–Martio [28, Theorems 2.18 and 2.19].

Even though we only deal with Green and singular functions on bounded domains, the capacity estimate (1.7) has consequences for the existence of global Green functions as well. More precisely, a complete noncompact (sub)Riemannian manifold is called *p-parabolic* if it does *not* carry a global Green function. The property of p-parabolicity has applications for quasiconformal mappings and Picard theorems, and has been extensively studied in e.g. Coulhon–Holopainen–Saloff-Coste [18], Grigor′yan [24] ($p = 2$) and Holopainen [31] and [33]. In the manifold setting, it is known that p-parabolicity is implied by the condition

$$\int_{r_0}^{\infty} \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(p-1)} d\rho = \infty, \tag{1.8}$$

see e.g. [18, Corollary 3.2], [24, Theorem 7.3] ($p = 2$), [33, Proposition 1.7] and Kesel′man–Zorich [37] ($p = n$). The converse is in general not true (by [33, p. 322] or Varopoulos [58]), but has been proved in some (sub)Riemannian manifolds. In particular, p-parabolicity and (1.8) are equivalent in Riemannian manifolds with nonnegative Ricci curvature or, more generally, satisfying a global doubling condition and a global Poincaré inequality, see [18, Proposition 3.4], [33, Corollary 4.12] and [34, Theorem 1.7] for more details.

One of the well-known equivalent characterizations of p-parabolicity of Riemannian manifolds is that all balls have global variational p-capacity zero, see [24, Theorem 5.1] ($p = 2$), Holopainen [31, Theorem 3.27] and [33, p. 322]. This property is used as the definition of p-parabolicity in metric spaces by Holopainen–Koskela [34, p. 3428] and Holopainen–Shanmugalingam [35, Definition 3.13]. Following the same definition, we show in Theorem 5.5 that under rather mild assumptions, an unbounded metric space is p-parabolic if and only if (1.8) holds. This recovers and complements the sufficient condition proved in [34, Proposition 2.3] and generalizes several of the above results. Recall also that an unbounded space is said to be *p-hyperbolic* if it is not p-parabolic.

The outline of the paper is as follows. In Section 2 we recall basic definitions and assumptions related to the analysis on metric spaces and in Section 3 we introduce the pointwise exponent sets, which govern the volume growth near x_0. The capacity estimate (1.7) is proved in Section 4, where we also study some of its consequences, while the applications of (1.7) for the p-parabolicity and capacity of singletons are discussed in Section 5.

Background material on p-(super)harmonic functions as well as the definitions of singular and Green functions are given in Section 6. Note that in the rest of the paper, we suppress the dependence on p and write "superharmonic" instead of "p-superharmonic", but keep the term "p-harmonic". The pointwise behaviour of Green (and singular) functions near the singularity is studied in Section 7, where we also show comparability of Green functions (having the same singularity x_0) on comparable open sets with comparable measures.

General integrability properties of superharmonic functions, recalling and extending the results in Kinnunen–Martio [42], are reviewed in Section 8. Sections 9 and 10 contain our main results concerning the (non)integrability of Green functions and their minimal p-weak upper gradients, respectively. In particular, Theorems 1.1 and 1.2 are proved at the end of Section 10. Some examples, based on radial weights on $\mathbf{R}^n$ and complementing the general integrability results, are given in Section 11. In Section 12 we generalize the growth and integrability results to p-harmonic functions having a pole at x_0. Theorem 1.3 is proved at the end of Section 12. Finally, in Section 13 we discuss how the (non)integrability results for Green functions can be extended to singular functions for elliptic differential equations in divergence form on weighted $\mathbf{R}^n$ and on Riemannian manifolds.

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2. Preliminaries

We assume throughout the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. Under these assumptions, X is separable. The σ-algebra on which μ is defined is obtained by the completion of the Borel σ-algebra. To avoid pathological situations we assume that X contains at least two points. We also write $B(x, r) = \{y \in X : d(x, y) < r\}$.

Next we are going to introduce the necessary background on Sobolev spaces and capacities in metric spaces. Proofs of most of the results mentioned here can be found in the monographs Björn–Björn [5] and Heinonen–Koskela–Shanmugalingam–Tyson [30].

A *curve* is a continuous mapping from an interval; it is *rectifiable* if it has finite length, in which case it can be parameterized by its arc length ds. A property holds for *p-almost every curve* if it fails only for a curve family Γ with zero p-modulus, i.e. there is $\rho \in L^p_{\mathrm{loc}}(X)$ such that $\int_\gamma \rho \, ds = \infty$ for every $\gamma \in \Gamma$.

A measurable function $g \colon X \to [0, \infty]$ is a *p-weak upper gradient* of $u \colon X \to [-\infty, \infty]$ if for p-almost every nonconstant compact rectifiable curve $\gamma \colon [0, l_\gamma] \to X$,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g \, ds, \tag{2.1}$$

where we follow the convention that the left-hand side is ∞ whenever at least one of the terms therein is $\pm\infty$. Weak upper gradients were introduced by Koskela–MacManus [45], see also Heinonen–Koskela [29]. If u has a p-weak upper gradient in $L^p_{\mathrm{loc}}(X)$, then it has an a.e. unique *minimal p-weak upper gradient* $g_u \in L^p_{\mathrm{loc}}(X)$ in the sense that $g_u \le g$ a.e. for every p-weak upper gradient $g \in L^p_{\mathrm{loc}}(X)$ of u.

Following Shanmugalingam [55], we define a version of Sobolev spaces on X. For a measurable function $u\colon X \to [-\infty,\infty]$, let

$$\|u\|_{N^{1,p}(X)} = \biggl(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu\biggr)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The *Newtonian space* on X is

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}.$$

The space $N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u-v\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere, not just up to an equivalence class in the corresponding function space. This is needed for (2.1) in the definition of p-weak upper gradients to make sense. For an open set $\Omega \subset X$, the Newtonian space $N^{1,p}(\Omega)$ is defined by considering $(\Omega, d|_\Omega, \mu|_\Omega)$ as a metric space in its own right. Moreover, $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if for every $x \in \Omega$ there exists $r_x > 0$ such that $B(x,r_x) \subset \Omega$ and $u \in N^{1,p}(B(x,r_x))$. The space $L^p_{\mathrm{loc}}(\Omega)$ is defined similarly. If $u,v \in N^{1,p}_{\mathrm{loc}}(X)$, then $g_u = g_v$ a.e. in $\{x \in X : u(x) = v(x)\}$. In particular $g_{\min\{u,c\}} = g_u \chi_{\{u<c\}}$ for $c \in \mathbf{R}$.

The *Sobolev capacity* of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}_{\mathrm{loc}}(X)$, then $u \sim v$ if and only if $u = v$ q.e. (quasieverywhere), that is $C_p(\{x : u(x) \ne v(x)\}) = 0$. Moreover, if $u,v \in N^{1,p}_{\mathrm{loc}}(X)$ and $u = v$ a.e., then $u = v$ q.e. Both the Sobolev and the following variational capacity are countably subadditive.

For an open set $\Omega \subset X$, let

$$N^{1,p}_0(\Omega) = \{u|_\Omega : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus \Omega\}.$$

The *variational capacity* of $E \subset \Omega$ with respect to Ω is

$$\operatorname{cap}_p(E,\Omega) = \inf_u \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}_0(\Omega)$ such that $u \ge 1$ in E. One can equivalently take the above infimum over all $u \in N^{1,p}(X)$ such that $u = 1$ on E and $u = 0$ on $X \setminus \Omega$; we call such u *admissible* for the capacity $\operatorname{cap}_p(E,\Omega)$. Similarly, whenever convenient, $u \in N^{1,p}_0(\Omega)$ will be regarded as extended by 0 outside Ω.

The measure μ is (*globally*) *doubling* if there is a constant $C > 0$ such that for all balls $B \subset X$ we have

$$\mu(2B) \le C\mu(B),$$

where $\lambda B(x,r) = B(x,\lambda r)$ for $\lambda > 0$. If X is complete and μ is doubling, then X is also *proper*, i.e. sets which are closed and bounded are compact.

The space X (or the measure μ) supports a (*global*) p-*Poincaré inequality* if there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B = B(x,r) \subset X$, all integrable functions u on X, and all p-weak upper gradients g of u,

$$⨍_B |u - u_B| \, d\mu \le Cr \biggl(⨍_{\lambda B} g^p \, d\mu\biggr)^{1/p}, \tag{2.2}$$

where $u_B := ⨍_B u \, d\mu := \int_B u \, d\mu/\mu(B)$. If X supports a Poincaré inequality, then X is connected.

If $X = \mathbf{R}^n$ is equipped with $d\mu = w\,dx$, then $w \ge 0$ is a p-admissible weight in the sense of Heinonen–Kilpeläinen–Martio [28] if and only if μ is a doubling measure which supports a p-Poincaré inequality, see Corollary 20.9 in [28] (which is only in the second edition) and Proposition A.17 in [5]. In this case, $N^{1,p}(\mathbf{R}^n)$ and $N^{1,p}(\Omega)$ are the refined Sobolev spaces defined in [28, p. 96], and moreover the above Sobolev and variational capacities coincide with those in [28]; see Björn–Björn [5, Theorem 6.7 (ix) and Appendix A.2] and [7, Theorem 5.1]. The situation is similar on Riemannian manifolds with nonnegative Ricci curvature and on Carnot groups equipped with their natural measures; see Hajłasz–Koskela [25, Sections 10 and 11] for further details.

Throughout the paper, we write $Y \lesssim Z$ if there is an implicit constant $C > 0$ such that $Y \le CZ$. We also write $Y \gtrsim Z$ if $Z \lesssim Y$, and $Y \simeq Z$ if $Y \lesssim Z \lesssim Y$. Unless otherwise stated, we always allow the implicit comparison constants to depend on the standard parameters, such as p, the doubling constant and the constants in the Poincaré inequality.

3. Exponent sets

If X is connected (which in particular holds if it supports a Poincaré inequality) and μ is doubling, then there are positive constants $\underline{\theta} \le \bar{\theta}$ and C such that

$$\frac{1}{C}\Big(\frac{r}{R}\Big)^{\bar{\theta}} \le \frac{\mu(B(x,r))}{\mu(B(x,R))} \le C\Big(\frac{r}{R}\Big)^{\underline{\theta}}$$

whenever $x \in X$ and $0 < r \le R < 2 \operatorname{diam} X$. It is easy to see that this condition is equivalent to the corresponding noncentred conditions (3.1) and (3.2) in [5], provided that μ is doubling. Example 3.1 below shows that $\underline{\theta}$ may need to be close to 0. On the other hand, if X is connected, then $\bar{\theta} \ge 1$, see Proposition 3.2.

The exponent $\bar{\theta}$ plays a crucial role in various results in the nonlinear potential theory, such as in optimal exponents in Sobolev and (q,p)-Poincaré inequalities, see Theorem 5.1 in Hajłasz–Koskela [25] or [5, Section 4.4]. It also plays a prominent role in the integrability results for superharmonic functions by Kinnunen–Martio [42], see Section 8.

There may or may not be optimal values for $\bar{\theta}$ and $\underline{\theta}$ and it will therefore be useful to introduce the *exponent sets*

$$\begin{aligned}
\underline{\Theta} = \bigg\{ \underline{\theta} > 0 : \text{there is } C_{\underline{\theta}} > 0 \text{ so that } \frac{\mu(B(x,r))}{\mu(B(x,R))} \le C_{\underline{\theta}}\Big(\frac{r}{R}\Big)^{\underline{\theta}} \\
\text{for all } x \in X \text{ and } 0 < r \le R < 2 \operatorname{diam} X \bigg\}, \\
\overline{\Theta} = \bigg\{ \bar{\theta} > 0 : \text{there is } C_{\bar{\theta}} > 0 \text{ so that } \frac{\mu(B(x,r))}{\mu(B(x,R))} \ge C_{\bar{\theta}}\Big(\frac{r}{R}\Big)^{\bar{\theta}} \\
\text{for all } x \in X \text{ and } 0 < r \le R < 2 \operatorname{diam} X \bigg\}.
\end{aligned}$$

Also the following pointwise *exponent sets* at the fixed $x_0 \in X$, introduced in Björn–Björn–Lehrbäck [10], will be crucial in this paper. Recall from the introduction that $B_r = B(x_0, r)$.

$$
\begin{aligned}
\underline{Q}_0 &= \biggl\{ q > 0 : \text{there is } C_q > 0 \text{ so that } \frac{\mu(B_r)}{\mu(B_R)} \le C_q \Bigl(\frac{r}{R}\Bigr)^q \text{ for } 0 < r < R \le 1 \biggr\}, \\
\underline{S}_0 &= \{ s > 0 : \text{there is } C_s > 0 \text{ so that } \mu(B_r) \le C_s r^s \text{ for } 0 < r \le 1 \}, \\
\bar{S}_0 &= \{ s > 0 : \text{there is } C_s > 0 \text{ so that } \mu(B_r) \ge C_s r^s \text{ for } 0 < r \le 1 \}, \\
\bar{Q}_0 &= \biggl\{ q > 0 : \text{there is } C_q > 0 \text{ so that } \frac{\mu(B_r)}{\mu(B_R)} \ge C_q \Bigl(\frac{r}{R}\Bigr)^q \text{ for } 0 < r < R \le 1 \biggr\}.
\end{aligned}
$$

The subscript 0 in the above definitions stands for the fact that the inequalities are required to hold for small radii. All these sets are intervals and the reason for introducing them as sets is that they may or may not contain their endpoints

$$
\underline{\theta}_0 = \sup \underline{\Theta}, \quad \underline{q}_0 = \sup \underline{Q}_0, \quad \underline{s}_0 = \sup \underline{S}_0, \quad \bar{s}_0 = \inf \bar{S}_0, \quad \bar{q}_0 = \inf \bar{Q}_0, \quad \bar{\theta}_0 = \inf \overline{\Theta},
$$

respectively. Nevertheless, it is always true that

$$
\underline{\Theta} \subset \underline{Q}_0 \subset \underline{S}_0, \quad \overline{\Theta} \subset \bar{Q}_0 \subset \bar{S}_0 \quad \text{and} \quad \underline{\theta}_0 \le \underline{q}_0 \le \underline{s}_0 \le \bar{s}_0 \le \bar{q}_0 \le \bar{\theta}_0.
$$

It was shown in [10, Lemmas 2.4 and 2.5] that the ranges $0 < r < R \le 1$ and $0 < r \le 1$, in $\underline{Q}_0$, $\underline{S}_0$, $\bar{S}_0$ and $\bar{Q}_0$, can equivalently be replaced by $0 < r < R \le R_0$ and $0 < r \le R_0$ for any fixed $R_0 > 0$ without changing the resulting exponent sets. The constants C_q and C_s may however change. By Remark 4.10 in [10], the capacity estimates in that paper hold for the exponent sets defined above, under appropriate restrictions of the radii. We will use these facts without further ado.

The following example shows that it is possible to have $\bar{q}_0 < 1$, while as already mentioned we always have $\bar{\theta}_0 \ge 1$ provided that X is connected, see Proposition 3.2 below.

Example 3.1. Let $X = \mathbf{R}^n$, $n \ge 2$, and $0 < \alpha < n$. Then it is well known that $w(x) = |x|^{-\alpha}$ is a Muckenhoupt A_1-weight and is thus 1-admissible, by Theorem 4 in Björn [14]. For $x_0 = 0$, it is easily verified that $\mu(B_r) \simeq r^{n-\alpha}$ and thus

$$
\underline{Q}_0 = \underline{S}_0 = (0, n-\alpha] \quad \text{and} \quad \bar{S}_0 = \bar{Q}_0 = [n-\alpha, \infty).
$$

In particular, if $n-1 < \alpha < n$, then $\underline{q}_0 = \underline{s}_0 = \bar{s}_0 = \bar{q}_0 = n - \alpha < 1$. Moreover, $\underline{\theta}_0 = n - \alpha$ and $\bar{\theta}_0 = n$.

For other examples with the exponent sets $\underline{Q}_0$, $\underline{S}_0$, $\bar{S}_0$ and $\bar{Q}_0$ having various properties, see [10, Section 3], H. Svensson [56] and S. Svensson [57].

Proposition 3.2. *If X is connected, then $\bar{\theta}_0 \ge 1$.*

Proof. Let $\bar{\theta} \in \overline{\Theta}$, $0 < R < \frac{1}{4} \operatorname{diam} X$ and $x \in X$. Then $X \setminus B(x, 2R)$ is nonempty. As X is connected, for each $0 < \rho < 2R$ there is x_ρ such that $d(x, x_\rho) = \rho$. Let $N \ge 2$ be an integer.

Then the balls $B^j := B(x_{jR/N}, R/2N)$, $j = 1, \dots, N-1$, are pairwise disjoint and contained in $B(x, R)$, and hence there is some $1 \le k \le N-1$ so that

$$
\frac{\mu(B^k)}{\mu(B(x,R))} \le \frac{1}{N-1}.
$$

Thus, as $\bar{\theta} \in \overline{\Theta}$,

$$
\frac{1}{N-1} \ge \frac{\mu(B^k)}{\mu(B(x,R))} \ge \frac{\mu(B^k)}{\mu(B(x_{kR/N}, 2R))} \ge \frac{1}{C} \Bigl(\frac{1}{4N}\Bigr)^{\bar{\theta}},
$$

where C is the constant dictated by $\bar{\theta}$. As N was arbitrary this is possible only if $\bar{\theta} \ge 1$. Thus also $\bar{\theta}_0 \ge 1$. $\square$

4. General capacity estimates for annuli

Before going on to the core of this paper – the study of p-harmonic Green functions – we establish precise general estimates for the capacities of annuli. These results will play an important role for instance in the pointwise estimates for Green functions, see Theorem 7.1. Unlike in most of this paper, the estimates in this section hold under rather weak assumptions. We will consider the following pointwise properties. Some of these conditions were introduced in [10], but here it will be enough to have them for certain radii. Recall that $x_0 \in X$ is fixed.

Definition 4.1. We say that μ is *doubling at* x_0 if

$$\mu(B(x_0, 2r)) \lesssim \mu(B(x_0, r))$$

for all radii $r > 0$. Similarly, μ supports a *p-Poincaré inequality at* x_0 if (2.2) holds for all balls $B = B(x_0, r)$. Moreover, μ is *reverse-doubling at* x_0 if there are constants $\xi, C > 1$ such that

$$\mu(B(x_0, \xi r)) \ge C\mu(B(x_0, r))$$

for all $0 < r \le \operatorname{diam} X/2\xi$.

We also say that a property, as above, holds at x_0 *for radii up to* R_0 if it holds for all $0 < r \le R_0$. (Here we allow for $R_0 = \infty$, while r is always finite, i.e. $r < R_0$ if $R_0 = \infty$.) Finally, a property holds *for small* (*resp. large*) *radii*, if there is some $0 < R_0 < \infty$ such that the property holds at x_0 for all $0 < r \le R_0$ (resp. all $R_0 \le r < \infty$).

Note that if X is bounded and μ is reverse-doubling at x_0 for radii up to R_0, then necessarily $R_0 \le \operatorname{diam} X$. (Letting $X = B(0, 2) \subset \mathbf{R}^n$, equipped with the metric $d(x, y) = \min\{|x-y|, 1\}$ and the Lebesgue measure, shows that it is possible to satisfy the reverse-doubling condition with $R_0 = \operatorname{diam} X$.) It is easy to see by iteration that if μ is doubling at x_0 for small radii, then $\overline{q}_0 < \infty$, and if μ is reverse-doubling at x_0 for small radii, then $\underline{q}_0 > 0$.

If X is connected (which in particular holds if μ supports a global Poincaré inequality) and μ is globally doubling, then μ is reverse-doubling at every x with constants $C > 1$ and $\xi = 2$ independent of x, see Lemma 3.7 in [5] (θ therein corresponds to ξ here). On the other hand, a reverse-doubling measure is not necessarily doubling, and if μ is doubling at some x_0, then μ is not necessarily reverse-doubling at x_0 (even if X is connected and satisfies the 1-Poincaré inequality). This is illustrated by $X = [0, \infty)$, equipped with the weights $\min\{1, e^{1/x}/x^2\}$ and $\min\{1, 1/x\}$, respectively, see [9, Example 6.2] and Example 7.4 in the preprint version of [9] in `arXiv:1512.06577`.

It is straightforward to show that if μ is doubling or reverse-doubling for large radii at one point x_0, then the same property holds at any other point, although the constants and radial bounds may change from point to point. Similarly, if μ supports a p-Poincaré inequality at x_0 for large radii, and μ is doubling at x_0 for large radii, then μ supports a p-Poincaré inequality at any point for large radii.

Our main result in this section is the following estimate.

Theorem 4.2. *Let $0 < R_0 \le \infty$. Assume that*

(i) *μ is reverse-doubling at x_0 for radii up to R_0, with constant ξ,*
(ii) *μ is doubling at x_0 for radii up to $\max\{1, \frac{1}{2}\xi\}R_0$, and*
(iii) *μ supports a p_0-Poincaré inequality at x_0 for radii up to $\max\{2, \xi\}R_0$ for some $1 \le p_0 < p$.*

Then for all $0 < 2r \le R \le R_0$,

$$\operatorname{cap}_p(B_r, B_R) \simeq \left(\int_r^R \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(p-1)} d\rho \right)^{1-p}, \tag{4.1}$$

where the implicit comparison constants depend on p, p_0 *and the doubling, reverse-doubling and Poincaré constants from* (i)–(iii)*, but not on* R_0.

As above, when $R_0 = \infty$, we still require R to be finite.

Remark 4.3. The proofs in this section reveal that the assumptions in Theorem 4.2, Lemmas 4.7 and 4.8 and Corollary 4.10 about the (reverse) doubling and the Poincaré inequality can be further restricted to radii $\ge r$.

Theorem 4.2 gives an estimate of $\operatorname{cap}_p(B_r, B_R)$ for a large class of measures. This generalizes many of the estimates in [10], which in turn were generalizations and improvements of earlier results in Adamowicz–Shanmugalingam [1] and Garofalo–Marola [21]. On the other hand, the assumption of p_0-Poincaré inequality at x_0 for some $1 \le p_0 < p$ is stronger than in [10, Section 6].

If X is complete and μ is globally doubling and supports a global p-Poincaré inequality, then by Keith–Zhong [36, Theorem 1.0.1] there is $1 \le p_0 < p$ such that X supports a global p_0-Poincaré inequality. Under these assumptions μ is also reverse-doubling with uniform constants $C > 1$ and $\xi = 2$, and hence the capacity estimate in Theorem 4.2 holds with uniform constants for all $x_0 \in X$ with $R_0 = \frac{1}{4} \operatorname{diam} X$.

Theorem 4.2 can be used, for instance, to characterize when singletons have zero capacity and when X is p-parabolic, see Propositions 5.3 and 5.5. Some one-sided estimates for capacities in terms of the volume growth, as in (4.1), were given in Coulhon–Holopainen–Saloff-Coste [18, pp. 1151 and 1162], Holopainen [33, p. 329] and Holopainen–Koskela [34] mainly in the setting of Riemannian manifolds.

Other useful applications of Theorem 4.2 are the pointwise estimates for Green and singular functions in Section 7, as well as the (non)integrability results for these functions and their minimal p-weak upper gradients in Sections 9 and 10.

Example 4.4. If $w(x) = w(|x|)$ is a radial weight on $\mathbf{R}^n$, $n \ge 2$, such that the measure $d\mu = w\,dx$ supports a p-Poincaré inequality at $x_0 = 0$, then Proposition 10.8 in [10] shows that for all $0 < r < R$,

$$\operatorname{cap}_{p,\mu}(B_r, B_R) = \left(\int_r^R f'(\rho)^{1/(1-p)}\, d\rho \right)^{1-p},$$

where $f(\rho) = \mu(B_\rho)$. Thus, (4.1) can be seen as a generalization of this formula for annuli that are not too thin. Note that $f'(\rho) \simeq f(\rho)/\rho$ in many cases.

The following example shows that the assumption (iii) of a Poincaré inequality cannot be dropped from Theorem 4.2.

Example 4.5. Let $X = \overline{B(0,1)} \cup \{(x_1, x_2) : x_1 \ge 2\} \subset \mathbf{R}^2$ equipped with the Lebesgue measure μ. Then X is complete and μ is globally doubling and globally reverse-doubling. However, if $x_0 = 0$, $0 < r < 2$ and $R > 1$, then $\operatorname{cap}_p(B_r, B_R) = 0$.

A similar connected example is the bow-tie

$$X = \{(x_1, x_2) : |x_2| \le |x_1| \text{ and } x_1 \ge -1\} \subset \mathbf{R}^2$$

equipped with the Lebesgue measure μ. Again X is complete and μ is globally doubling and globally reverse-doubling. However, if $x_0 = (-1, 0)$, $0 < r \le 1 < R$ and $1 < p \le 2$, then $\operatorname{cap}_p(B_r, B_R) = 0$.

Lemma 2.6 in Heinonen–Kilpeläinen–Martio [28] (or Lemma 2.1 in Holopainen–Koskela [34]) implies that for $R = 2^{k_0} r$,

$$\operatorname{cap}_p(B_r, B_R) \le \biggl(\sum_{k=1}^{k_0} \operatorname{cap}_p(B^{k-1}, B^k)^{1/(1-p)} \biggr)^{1-p}, \tag{4.2}$$

where $B^k = 2^k B_r$, $k = 0, 1, \dots, k_0$. In fact, in [28] and [34], (4.2) is formulated for more general condensers, but we are only interested in dyadic sequences of concentric balls. The proof therein only uses suitable convex combinations of test functions and does not require any assumptions about the measure μ.

Reformulating (4.2) as follows gives us the upper bound of Theorem 4.2. The first inequality, with no doubling assumption, will be useful when deducing Lemma 5.2, while the second inequality is convenient when μ is doubling.

Proposition 4.6. *For all $0 < r \le \frac{1}{2} R$,*

$$\operatorname{cap}_p(B_r, B_R) \lesssim \biggl(\int_{2r}^R \biggl(\frac{\rho}{\mu(B_\rho)} \biggr)^{1/(p-1)} d\rho \biggr)^{1-p} \tag{4.3}$$

and

$$\operatorname{cap}_p(B_r, B_R) \lesssim \frac{\mu(B_{2r})}{\mu(B_r)} \biggl(\int_{r}^R \biggl(\frac{\rho}{\mu(B_\rho)} \biggr)^{1/(p-1)} d\rho \biggr)^{1-p}. \tag{4.4}$$

Proof. Write $r_k = 2^k r$ and $B^k = B_{r_k}$, $k = 0, 1, \dots$, and find an integer k_0 such that $r_{k_0} \le R < r_{k_0+1}$. Proposition 5.1 in [10] shows that $\operatorname{cap}_p(B^{k-1}, B^k) \lesssim \mu(B^k)/r_k^p$. (Note that the proof of this part of Proposition 5.1 in [10] does not use any doubling property.) Inserting this estimate into (4.2) yields

$$\begin{aligned} \operatorname{cap}_p(B_r, B_R) &\le \operatorname{cap}_p(B^0, B^{k_0}) \\ &\lesssim \biggl(\sum_{k=1}^{k_0} \biggl(\frac{r_k^p}{\mu(B^k)} \biggr)^{1/(p-1)} \biggr)^{1-p} \lesssim \biggl(\int_{2r}^R \biggl(\frac{\rho}{\mu(B_\rho)} \biggr)^{1/(p-1)} d\rho \biggr)^{1-p}. \end{aligned} \tag{4.5}$$

Finally,

$$\int_r^{2r} \biggl(\frac{\rho}{\mu(B_\rho)} \biggr)^{1/(p-1)} d\rho \le \biggl(\frac{2r}{\mu(B_r)} \biggr)^{1/(p-1)} r = \frac{1}{2} \biggl(\frac{\mu(B_{2r})}{\mu(B_r)} \frac{r_1^p}{\mu(B^1)} \biggr)^{1/(p-1)},$$

which together with the last inequality in (4.5) gives (4.4). □

For the proof of the lower bound in Theorem 4.2, we recall the following version of the well-known "telescoping argument".

Lemma 4.7. ([10, Lemma 4.9]) *Let $R_0 \in (0, \infty]$. Assume that $1 \le p_0 < \infty$ and that*

(i) *μ is reverse-doubling at x_0 for radii up to R_0, with constant ξ,*
(ii) *μ is doubling at x_0 for radii up to $\max\{1, \frac{1}{2}\xi\} R_0$, and*
(iii) *μ supports a p_0-Poincaré inequality at x_0 for radii up to $\max\{2, \xi\} R_0$.*

For $0 < 2r \le R_0$, write $r_k = 2^k r$ and $B^k = B_{r_k}$, $k = 0, 1, \dots$, and let $k_0 \ge 1$ be such that $r_{k_0} \le R_0$. Then we have for every $u \in N_0^{1,p_0}(B^{k_0})$ that

$$|u_{B_r}| \lesssim \sum_{k=1}^{k_0} r_k \biggl(\fint_{\lambda B^k} g_u^{p_0} \, d\mu \biggr)^{1/p_0},$$

where λ is the dilation constant in the p_0-Poincaré inequality at x_0.

The assumptions in Lemma 4.7 are slightly weaker than in [10, Lemma 4.9]. However, a careful check of the proof therein shows that only assumptions (i)–(iii) are needed. In particular, (i) and (ii) are enough to guarantee the comparability of the measures in the second displayed formula in the proof in [10], while the Poincaré inequality is only used for the radii assumed in (iii).

To make use of the above lemma we shall exploit the following general estimate that may be of independent interest since the assumption is very mild and the first factor on the right-hand side of (4.6) is strongly related to the right-hand side of (4.2), cf. the proof of Proposition 4.6.

Lemma 4.8. *Let $0 < R_0 \le R_0' \le \infty$ and assume that μ is reverse-doubling at x_0 for radii up to R_0. For $0 < 2r \le R_0'$, write $r_k = 2^k r$ and $B^k = B_{r_k}$, $k = 0, 1, \dots$ and let $k_0 \ge 1$ be such that $r_{k_0} \le R_0'$. Also let $1 \le p_0 < p$. Then we have for every $g \in L^p(B^{k_0})$,*

$$\sum_{k=1}^{k_0} r_k \left(\fint_{B^k} g^{p_0} \, d\mu \right)^{1/p_0} \lesssim \left(\sum_{k=1}^{k_0} \left(\frac{r_k^p}{\mu(B^k)} \right)^{1/(p-1)} \right)^{1-1/p} \left(\int_{B^{k_0}} g^p \, d\mu \right)^{1/p}, \tag{4.6}$$

where the implicit comparison constant depends on p, p_0, the reverse-doubling constants and R_0'/R_0.

Before proving Lemma 4.8 we show how it leads to the lower bound in Theorem 4.2.

Proof of Theorem 4.2. The upper bound follows from Proposition 4.6.

Conversely, let $0 \le u \in N^{1,p}(X)$ be admissible for $\operatorname{cap}_p(B_r, B_R)$. Write $r_k = 2^k r$ and $B^k = B_{r_k}$, $k = 0, 1, \dots$, and find an integer k_0 such that $r_{k_0 - 1} < R \le r_{k_0}$. The telescoping Lemma 4.7, followed by Lemma 4.8 applied to the balls λB^k in place of B^k and $R_0' = \lambda R_0$, gives

$$1 \lesssim \sum_{k=1}^{k_0} r_k \left(\fint_{\lambda B^k} g_u^{p_0} \, d\mu \right)^{1/p_0} \lesssim \left(\sum_{k=1}^{k_0} \left(\frac{r_k^p}{\mu(B^k)} \right)^{1/(p-1)} \right)^{1-1/p} \left(\int_{B_R} g_u^p \, d\mu \right)^{1/p}.$$

Taking infimum over all u admissible for $\operatorname{cap}_p(B_r, B_R)$ and replacing the sum on the right-hand side by the corresponding integral yields the lower bound in (4.1). □

Remark 4.9. Under the assumptions in Theorem 4.2, by Propositions 5.1 and 6.2 in [10] we have

$$\operatorname{cap}_p(B^{k-1}, B^k) \simeq \frac{\mu(B^k)}{r_k^p}$$

and hence the proof of Theorem 4.2 shows that the lower bound in (4.1) can also be written as

$$\operatorname{cap}_p(B_r, B_R) \gtrsim \left(\sum_{k=1}^{k_0} \operatorname{cap}_p(B^{k-1}, B^k)^{1/(1-p)} \right)^{1-p},$$

where $R = 2^{k_0} r$. Thus, (4.2) is essentially sharp.

Proof of Lemma 4.8. Write the left-hand side S in (4.6) as

$$S = \sum_{k=1}^{k_0} \frac{r_k}{\mu(B^k)^{1/p_0}} \left(\int_{B^k} g^{p_0} \, d\mu \right)^{1/p_0}. \tag{4.7}$$

We split the integral over B^k into integrals over $A_0 := B^0$ and the annuli $A_j = B^j \setminus B^{j-1}$, $j = 1, 2, \dots, k$, apply Hölder's inequality (with p/p_0 and $p_1 := p/(p-p_0)$) to each of these integrals, and obtain

$$\int_{B^k} g^{p_0}\, d\mu \le \sum_{j=0}^{k} \biggl(\int_{A_j} g^p \, d\mu \biggr)^{p_0/p} \mu(A_j)^{1/p_1}. \tag{4.8}$$

The reverse-doubling property at x_0 implies that for some $\beta > 0$, we can estimate $\mu(A_j)^{1/p_1}$ as

$$\mu(A_j)^{1/p_1} \le \mu(B^j)^{1/p_1} \lesssim \mu(B^k)^{1/p_1} \biggl(\frac{r_j}{r_k}\biggr)^{2\beta},$$

where the comparison constant depends on R_0'/R_0. Since $1/p_1 = 1 - p_0/p$, inserting this first into (4.8) and then into (4.7) gives

$$S \lesssim \sum_{k=1}^{k_0} \frac{r_k}{\mu(B^k)^{1/p_0}} \frac{\mu(B^k)^{1/p_0 - 1/p}}{r_k^{2\beta/p_0}} \biggl(\sum_{j=0}^{k} r_j^{2\beta} \biggl(\int_{A_j} g^p \, d\mu \biggr)^{p_0/p} \biggr)^{1/p_0}. \tag{4.9}$$

The last sum in (4.9) is now estimated using Hölder's inequality for sums (with $p_1 = p/(p-p_0)$ and p/p_0 again) as follows

$$\begin{aligned}\sum_{j=0}^{k} r_j^{\beta} r_j^{\beta} \biggl(\int_{A_j} g^p \, d\mu \biggr)^{p_0/p} &\le \biggl(\sum_{j=0}^{k} r_j^{\beta p_1} \biggr)^{1/p_1} \biggl(\sum_{j=0}^{k} r_j^{\beta p/p_0} \int_{A_j} g^p \, d\mu \biggr)^{p_0/p} \\ &\simeq r_k^{\beta} \biggl(\sum_{j=0}^{k} r_j^{\beta p/p_0} \int_{A_j} g^p \, d\mu \biggr)^{p_0/p}.\end{aligned}$$

Inserting this into (4.9) yields

$$S \lesssim \sum_{k=1}^{k_0} \biggl(\frac{r_k^p}{\mu(B^k)} \biggr)^{1/p} r_k^{-\beta/p_0} \biggl(\sum_{j=0}^{k} r_j^{\beta p/p_0} \int_{A_j} g^p \, d\mu \biggr)^{1/p}.$$

Another use of Hölder's inequality for sums (with $p/(p-1)$ and p) implies

$$S \lesssim \biggl(\sum_{k=1}^{k_0} \biggl(\frac{r_k^p}{\mu(B^k)} \biggr)^{1/(p-1)} \biggr)^{1-1/p} \biggl(\sum_{k=1}^{k_0} r_k^{-\beta p/p_0} \sum_{j=0}^{k} r_j^{\beta p/p_0} \int_{A_j} g^p \, d\mu \biggr)^{1/p}.$$

The last factor is estimated by changing the order of summation as

$$\biggl(\sum_{j=0}^{k_0} r_j^{\beta p/p_0} \int_{A_j} g^p \, d\mu \sum_{k=\max\{1,j\}}^{k_0} r_k^{-\beta p/p_0} \biggr)^{1/p} \simeq \biggl(\sum_{j=0}^{k_0} \int_{A_j} g^p \, d\mu \biggr)^{1/p},$$

since the geometric sum is comparable to $r_j^{-\beta p/p_0}$. We can thus conclude that

$$S \lesssim \biggl(\sum_{k=1}^{k_0} \biggl(\frac{r_k^p}{\mu(B^k)} \biggr)^{1/(p-1)} \biggr)^{1-1/p} \biggl(\sum_{j=0}^{k_0} \int_{A_j} g^p \, d\mu \biggr)^{1/p},$$

and the claim follows. □

As a consequence of Theorem 4.2 we obtain the following estimates for different capacities, which will be important when deducing Theorem 10.3.

Corollary 4.10. *Let $1 < q < t < p$, $\alpha = (p-t)/(p-q)$ and $R_0 \in (0, \infty]$. Assume that*

(i) *μ is reverse-doubling at x_0 for radii up to R_0, with constant ξ,*
(ii) *μ is doubling at x_0 for radii up to $\max\{1, \frac{1}{2}\xi\}R_0$, and*
(iii) *μ supports a t_0-Poincaré inequality at x_0 for radii up to $\max\{2, \xi\}R_0$ for some $1 \le t_0 < t$.*

Let $0 < 2r \le R \le R_0$. Then the following are true, with the implicit comparison constants depending on p, q, t, the (reverse) doubling and Poincaré constants from (i)–(iii), and in (b) and (c) also on R_0.

(a) *In general,*

$$\operatorname{cap}_t(B_r, B_R) \gtrsim \operatorname{cap}_p(B_r, B_R)^{1-\alpha} \left(\int_r^R \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(q-1)} d\rho \right)^{\alpha(1-q)}$$
$$\gtrsim \operatorname{cap}_p(B_r, B_R)^{1-\alpha} \operatorname{cap}_q(B_r, B_R)^{\alpha}.$$

(b) *If $q < \underline{q}_0$ and $R_0 < \infty$, then*

$$\operatorname{cap}_t(B_r, B_R) \gtrsim \operatorname{cap}_p(B_r, B_R)^{1-\alpha} \left(\frac{\mu(B_r)}{r^q} \right)^{\alpha}.$$

(c) *If $q = \underline{q}_0 \in \underline{Q}_0$ and $R_0 < \infty$, then*

$$\operatorname{cap}_t(B_r, B_R) \gtrsim \operatorname{cap}_p(B_r, B_R)^{1-\alpha} \left(\frac{\mu(B_r)}{r^q} \right)^{\alpha} \left(\log \frac{R}{r} \right)^{\alpha(1-q)}.$$

Note that since $p > t$, (4.1) holds for cap_p, which can therefore be replaced in Corollary 4.10 by a corresponding integral. Also observe that (b) and (c) do not follow from (a) together with the estimates in [10] since we here assume a weaker Poincaré inequality.

Proof. (a) Let $\beta = 1 - \alpha = (t-q)/(p-q)$ and note that

$$\frac{\alpha(q-1)}{t-1} + \frac{\beta(p-1)}{t-1} = 1 \quad \text{and} \quad \alpha q + \beta p = t.$$

Hölder's inequality then implies that

$$\int_r^R \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(t-1)} d\rho = \int_r^R \left(\frac{\rho}{\mu(B_\rho)} \right)^{\alpha/(t-1)+\beta/(t-1)} d\rho$$
$$\le \left(\int_r^R \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(q-1)} d\rho \right)^{\alpha(q-1)/(t-1)}$$
$$\times \left(\int_r^R \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(p-1)} d\rho \right)^{\beta(p-1)/(t-1)}.$$

Raising both sides to the power $1-t$ and applying Theorem 4.2 to cap_t, and (4.4) to cap_p and cap_q, now yields (a).

(b) Let $q < q' < \underline{q}_0$ and $\gamma = (q'-1)/(q-1) > 1$. Then

$$\frac{\rho^{q'}}{\mu(B_\rho)} \lesssim \frac{r^{q'}}{\mu(B_r)} \quad \text{for } r < \rho \le R_0,$$

and so

$$\int_r^R \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(q-1)} d\rho = \int_r^R \left(\frac{\rho^{q'}}{\mu(B_\rho)} \right)^{1/(q-1)} \frac{d\rho}{\rho^\gamma}$$
$$\lesssim \int_r^R \left(\frac{r^{q'}}{\mu(B_r)} \right)^{1/(q-1)} \frac{d\rho}{\rho^\gamma} \lesssim \left(\frac{r^{q'}}{\mu(B_r)} \right)^{1/(q-1)} r^{1-\gamma}.$$

Hence,

$$\biggl(\int_r^R \biggl(\frac{\rho}{\mu(B_\rho)}\biggr)^{1/(q-1)} d\rho\biggr)^{1-q} \gtrsim \frac{\mu(B_r)}{r^{q'}} r^{(\gamma-1)(q-1)} = \frac{\mu(B_r)}{r^q},$$

and inserting this into (a) gives (b).

(c) Proceeding as in (b), with $q' = q$, we see that

$$\int_r^R \biggl(\frac{\rho}{\mu(B_\rho)}\biggr)^{1/(q-1)} d\rho \lesssim \biggl(\frac{r^q}{\mu(B_r)}\biggr)^{1/(q-1)} \int_r^R \frac{d\rho}{\rho} = \biggl(\frac{r^q}{\mu(B_r)}\biggr)^{1/(q-1)} \log\frac{R}{r}.$$

Inserting this into (a) gives (c). □

The dependence of the implicit constants in (b) and (c) on R_0 is through the constant C_q appearing in the definition of $\underline{Q}_0$ for $0 < r < R \le R_0$. It therefore also depends on the particular choice of $q' > q$ in the proof of (b). If $R_0 = \infty$, then $\operatorname{diam} X = \infty$ and (b) and (c) hold for $q < \underline{q} := \sup \underline{Q}$ and $q = \underline{q} \in \underline{Q}$, respectively, where

$$\underline{Q} = \biggl\{q > 0 : \frac{\mu(B_r)}{\mu(B_R)} \lesssim \Bigl(\frac{r}{R}\Bigr)^q \text{ for all } 0 < r < R < \infty\biggr\}.$$

Remark 4.11. Choosing $q < \min\{t, \underline{q}_0\}$ and $p > \max\{t, \overline{q}_0\}$ in Corollary 4.10 (b), together with Proposition 6.1 (b) in [10] and a direct calculation, yields for $0 < 2r < R \le \operatorname{diam} X/2\xi$ that

$$\operatorname{cap}_t(B_r, B_R) \gtrsim \biggl(\frac{\mu(B_r)}{r^t}\biggr)^{(p-t)/(p-q)} \biggl(\frac{\mu(B_R)}{R^t}\biggr)^{(t-q)/(p-q)} \Bigl(\frac{r}{R}\Bigr)^{(p-t)(t-q)/(p-q)}. \tag{4.10}$$

Since

$$\frac{\mu(B_r)}{r^t} \gtrsim \operatorname{cap}_t(B_r, B_R) \quad \text{and} \quad \frac{\mu(B_R)}{R^t} \gtrsim \operatorname{cap}_t(B_r, B_R),$$

this implies and improves the estimates in [10, Proposition 6.2], which use only one of the balls B_r and B_R. The borderline cases $q = \underline{q}_0 \in \underline{Q}_0$ and $p = \overline{q}_0 \in \overline{Q}_0$ which are allowed in [10, Proposition 6.2], are however not included in (4.10), and the Poincaré assumption is slightly stronger here.

Note that the product of the estimates in (a) and (b) of [10, Proposition 6.2] gives an estimate similar to (4.10), but with twice as large exponent at r/R. Moreover, [10, Proposition 5.1] implies that

$$\operatorname{cap}_t(B_r, B_R) \lesssim \biggl(\frac{\mu(B_r)}{r^t}\biggr)^{(p-t)/(p-q)} \biggl(\frac{\mu(B_R)}{R^t}\biggr)^{(t-q)/(p-q)}.$$

If $q = \underline{q}_0 \in \underline{Q}_0$ and $p = \overline{q}_0 \in \overline{Q}_0$, one can combine Corollary 4.10 with [10, Proposition 7.1] to obtain more explicit lower bounds for cap_t, also containing $\log(R/r)$.

5. Capacity of singletons and p-parabolicity

By letting $r \to 0$ or $R \to \infty$ in Theorem 4.2, we will in this section characterize points of zero capacity and p-parabolic metric spaces in terms of integrals of the type (4.1). This gives more precise descriptions than some earlier conditions based on dimensions and exponent sets, as in the following result from [10].

Proposition 5.1. (Proposition 8.2 in [10])

(a) *If $p < \bar{s}_0$ or $p = \bar{s}_0 \notin \overline{S}_0 \setminus \underline{S}_0$, then $C_p(\{x_0\}) = 0$.*

(b) *If $p > \bar{s}_0$, μ is doubling and reverse-doubling at x_0 and supports a p-Poincaré inequality at x_0, all three properties holding for small radii, and x_0 has a locally compact neighbourhood, then $C_p(\{x_0\}) > 0$.*

A careful check of the proofs in [10] shows that no doubling assumption is needed for Proposition 5.1 (a). Already Holopainen–Shanmugalingam [35], in the comment following the proof of Lemma 3.6 therein, pointed out that if $p \in \underline{S}_0$ and X is locally compact, then $\operatorname{cap}_p(\{x_0\}, \Omega) = 0$ whenever $\Omega \ni x_0$ is open, from which it easily follows that $C_p(\{x_0\}) = 0$ (cf. (5.3) below).

If $p = \bar{s}_0 \in \bar{S}_0 \setminus \underline{S}_0$, then the exponent sets are not fine enough to capture when x_0 has zero capacity, see Example 9.4 in [10]. The following results, which are based on the general capacity estimates from Section 4, are therefore of interest.

Lemma 5.2. *Let $\Omega \subset X$ be a bounded open set with $x_0 \in \Omega$. If*

$$\int_0^\delta \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(p-1)} d\rho = \infty \quad \textit{for some } \delta > 0, \tag{5.1}$$

or equivalently

$$\sum_{k=k_0}^{\infty} \left(\frac{2^{-kp}}{\mu(B_{2^{-k}})} \right)^{1/(p-1)} = \infty \quad \textit{for some integer } k_0 \geq 0, \tag{5.2}$$

then $\operatorname{cap}_p(\{x_0\}, \Omega) = C_p(\{x_0\}) = 0$.

Note that the conditions in (5.1) and (5.2) can be equivalently required for all $\delta > 0$ and all integers $k_0 \geq 0$, respectively.

Proof. We may assume that $B(x_0, \delta) \subset \Omega$. Hence it follows from (4.3) and (5.1) that

$$\operatorname{cap}_p(\{x_0\}, \Omega) \leq \lim_{r \to 0} \operatorname{cap}_p(B_r, B_\delta) \lesssim \lim_{r \to 0} \left(\int_{2r}^{\delta} \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(p-1)} d\rho \right)^{1-p} = 0.$$

For the second part, we have $\mu(\{x_0\}) = 0$ since the integral in (5.1) diverges, and thus

$$C_p(\{x_0\}) \leq \lim_{\delta \to 0} \bigl(\mu(B(x_0, \delta)) + \operatorname{cap}_p(\{x_0\}, B(x_0, \delta))\bigr) = \lim_{\delta \to 0} \mu(B(x_0, \delta)) = 0, \tag{5.3}$$

by the regularity of the Borel regular measure μ. □

To obtain also the converse direction, we need stronger (pointwise) assumptions as follows.

Proposition 5.3. *Assume that*

(i) *μ is reverse-doubling at x_0 for small radii,*
(ii) *μ is doubling at x_0 for small radii, and*
(iii) *μ supports a p_0-Poincaré inequality at x_0 for small radii and some $1 \leq p_0 < p$.*

Let $\Omega \subset X$ be a bounded open set with $x_0 \in \Omega$, and assume that x_0 has a locally compact neighbourhood.

(a) *Then $C_p(\{x_0\}) = 0$ if and only if (5.1) holds.*
(b) *If μ supports a p-Poincaré inequality at x_0, then $\operatorname{cap}_p(\{x_0\}, \Omega) = 0$ if and only if either (5.1) holds or $C_p(X \setminus \Omega) = 0$.*

Instead of (5.1) one can equivalently require that (5.2) holds. For $n \ge 2$, consider the following union of concentric layers

$$X = \{x \in \mathbf{R}^n : |x| \notin E\}, \quad \text{where } E = \bigcup_{j=1}^{\infty} (2^{-2j}, 2^{1-2j}) \subset \mathbf{R},$$

equipped with the Lebesgue measure. In this case $C_p(\{0\}) = 0$ for all $1 < p < \infty$, but (5.1) fails if $p > n$ and $x_0 = 0$. Thus the assumption (iii) cannot be dropped, even if μ is assumed to be globally doubling and globally reverse-doubling. That the extra Poincaré assumption in (b) cannot be dropped is easily seen by considering e.g. $X = \overline{B(0,1)} \cup \overline{B(3,1)}$ in $\mathbf{R}^n$, with $x_0 = 0$, $\Omega = \overline{B(0,1)}$ and $p > n$.

Proof. (a) Assume first that $C_p(\{x_0\}) = 0$, and let $\varepsilon, \delta > 0$. By Proposition 4.7 in [10], there is $0 < r < \delta$ such that $\operatorname{cap}_p(B_r, B_\delta) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain from Theorem 4.2 that (5.1) holds. The converse implication follows directly from Lemma 5.2.

(b) If $\operatorname{cap}_p(\{x_0\}, \Omega) = 0$ and $C_p(X \setminus \Omega) > 0$, then [10, Proposition 4.6] shows that $C_p(\{x_0\}) = 0$, and thus (5.1) holds by part (a). The converse implication follows from Lemma 5.2 and the fact that if $C_p(X \setminus \Omega) = 0$ then $u \equiv 1$ is admissible in the definition of $\operatorname{cap}_p(\{x_0\}, \Omega)$, which is thus zero. $\square$

Definition 5.4. An unbounded space X is *p-parabolic* if $\operatorname{cap}_p(B, X) = 0$ for all balls $B \subset X$, otherwise it is *p-hyperbolic*.

On (sub)Riemannian manifolds, p-parabolicity is often defined as the *nonexistence of global p-harmonic Green functions*, which in those situations is known to be equivalent to the above requirement that $\operatorname{cap}_p(B, X) = 0$ for all balls $B \subset X$. In the generality of this section, there is no available theory for p-harmonic functions. Even in the standard setting of complete metric spaces with a doubling measure supporting a Poincaré inequality, global p-harmonic Green functions are little studied.

In Holopainen [33, p. 322], Holopainen–Koskela [34, p. 3428] and Holopainen–Shanmugalingam [35, Definition 3.13], p-parabolicity was defined by requiring that $\operatorname{cap}_p(K, X) = 0$ for all compact sets K. This is equivalent to Definition 5.4 provided that X is proper, but in nonproper spaces our definition seems to be more relevant.

Sufficient and/or necessary conditions for p-parabolicity using the integral (5.4) below have been obtained under various assumptions in a number of papers, see e.g. [18], [31]–[35] and the end of the introduction for more details. In [10, Proposition 8.6 and Remark 8.7] we gave simple conditions for $\operatorname{cap}_p(B, X) = 0$ (and thus p-parabolicity) in terms of exponent sets defined for large radii similarly to $\underline{S}_0$ and $\overline{S}_0$. As in Proposition 5.1, also here there are cases in which these exponent sets are not fine enough to capture when p-parabolicity holds, but using the estimate in Theorem 4.2 we are now able to give a precise characterization, under mild assumptions.

Theorem 5.5. *Let X be unbounded and $r_0 > 0$. Then X is p-parabolic if*

$$\int_{r_0}^{\infty} \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(p-1)} d\rho = \infty. \tag{5.4}$$

If moreover μ is doubling and reverse-doubling at x_0 and supports a p_0-Poincaré inequality at x_0 for some $1 \le p_0 < p$, all three properties holding for large radii, then X is p-parabolic if and only if (5.4) *holds.*

Condition (5.4) is clearly independent of the choices of r_0 and x_0. The first part recovers Proposition 2.3 in Holopainen–Koskela [34].

Proof. For the first part, let $B \subset X$ be a ball. Then $B \subset B_r$ for some $r \geq r_0$, and for $R \geq 2r$ we have by Proposition 4.6 that

$$\operatorname{cap}_p(B, X) \leq \operatorname{cap}_p(B_r, B_R) \lesssim \biggl(\int_{2r}^{R} \biggl(\frac{\rho}{\mu(B_\rho)} \biggr)^{1/(p-1)} d\rho \biggr)^{1-p} \to 0, \quad \text{as } R \to \infty,$$

whenever (5.4) holds.

For the converse implication in the second part we may assume that the doubling and reverse-doubling conditions and the Poincaré inequality hold for balls $B_r = B(x_0, r)$ with $r \geq r_0$. If $\operatorname{cap}_p(B_r, X) = 0$ for some $r \geq r_0$, then let $\varepsilon > 0$ be arbitrary and find $u \in N^{1,p}(X)$ so that $u \geq 1$ on B_r and $\int_X g_u^p \, d\mu < \varepsilon$. For $R \geq 2r + 1$, let $\eta(x) = \min\{(R - d(x, x_0))_+, 1\}$ be a cut-off function. Then $u\eta$ is admissible for $\operatorname{cap}_p(B_r, B_R)$ and hence, by Theorem 4.2 and Remark 4.3,

$$\begin{aligned} \biggl(\int_{r}^{R} \biggl(\frac{\rho}{\mu(B_\rho)} \biggr)^{1/(p-1)} d\rho \biggr)^{1-p} &\simeq \operatorname{cap}_p(B_r, B_R) \leq \int_X g_{u\eta}^p \, d\mu \\ &\leq 2^p \biggl(\int_X g_u^p \, d\mu + \int_{B_R \setminus B_{R-1}} |u|^p \, d\mu \biggr) \to 2^p \int_X g_u^p \, d\mu < 2^p \varepsilon, \end{aligned}$$

as $R \to \infty$, since $u \in L^p(X)$. As ε was arbitrary, we conclude that (5.4) holds (r_0 therein can clearly be replaced by r). $\square$

The following example shows that the assumption of a Poincaré inequality cannot be dropped from the second part of Theorem 5.5.

Example 5.6. For $n \geq 2$, consider the following union of concentric layers

$$X = \{x \in \mathbf{R}^n : |x| \notin E\}, \quad \text{where } E = \bigcup_{j=1}^{\infty} (2^{2j-1}, 2^{2j}) \subset \mathbf{R},$$

equipped with the Lebesgue measure. In this case X is p-parabolic for all $1 < p < \infty$, but (5.4) fails if $1 < p < n$. Thus the Poincaré assumption cannot be dropped, even if μ is assumed to be globally doubling and globally reverse-doubling.

Similar connected examples are given by "the infinite chessboard"

$$X = \mathbf{R}^2 \setminus \bigcup_{j,k=-\infty}^{\infty} \bigl((2j, 2j+1) \times (2k, 2k+1)\bigr) \cup \bigl((2j-1, 2j) \times (2k-1, 2k)\bigr)$$

and its generalizations to $\mathbf{R}^n$, $n \geq 3$.

6. p-harmonic, singular and Green functions

From now on, we assume that X is complete, that μ is doubling and supports a p-Poincaré inequality, and that $\Omega \subset X$ is a nonempty open set with $x_0 \in \Omega$. As always in this paper, $1 < p < \infty$.

In this section we first recall the definitions of p-harmonic and superharmonic functions and present some of their important properties that will be needed later. After that we recall results from Björn–Björn–Lehrbäck [11] on the existence and properties of singular and Green functions.

It follows from the assumptions that X is proper and quasiconvex and thus also connected and locally connected. Moreover, μ is reverse-doubling with $\xi = 2$. These facts will be important to keep in mind. Recall also that by Keith–Zhong [36, Theorem 1.0.1], X supports a p_0-Poincaré inequality for some $1 \leq p_0 < p$. This is assumed explicitly in some of the papers we refer to below.

The results in the rest of the paper also hold if X is a proper metric space equipped with a locally doubling measure μ that supports a local p-Poincaré inequality, as defined in Björn–Björn [8], see [11, Section 11]. The dependence on the constants will then be affected in a natural way.

Definition 6.1. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *(super)minimizer* in Ω if

$$\int_{\varphi \neq 0} g_u^p \, d\mu \le \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu \quad \text{for all (nonnegative) } \varphi \in N^{1,p}_0(\Omega).$$

A *p-harmonic function* is a continuous minimizer (by which we mean real-valued continuous in this paper).

For various characterizations of minimizers and superminimizers see Björn [3]. It was shown in Kinnunen–Shanmugalingam [43] that under our standing assumptions, a minimizer can be modified on a set of zero (Sobolev) capacity to obtain a p-harmonic function. For a superminimizer u, it was shown by Kinnunen–Martio [41] that its *lsc-regularization*

$$u^*(x) := \operatorname*{ess\,lim\,inf}_{y \to x} u(y) = \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u$$

is also a superminimizer and $u^* = u$ q.e.

Definition 6.2. A function $u \colon \Omega \to (-\infty, \infty]$ is *superharmonic* in Ω if

(i) u is lower semicontinuous;
(ii) u is not identically ∞ in any component of Ω;
(iii) for every nonempty open set $G \Subset \Omega$ with $C_p(X \setminus G) > 0$, and all functions $v \in C(\overline{G})$ such that v is p-harmonic in G we have $v \le u$ in G whenever $v \le u$ on ∂G.

As usual, by $G \Subset \Omega$ we mean that $\overline{G}$ is a compact subset of Ω. By Theorem 6.1 in Björn [2] (or [5, Theorem 14.10]), this definition of superharmonicity is equivalent to the definition usually used in metric spaces, e.g. in [5] and [11]. It also coincides with the definitions used in $\mathbf{R}^n$ and on Riemannian manifolds. Superharmonic functions are always lsc-regularized (i.e. $u^* = u$). Any lsc-regularized superminimizer is superharmonic, and conversely any *bounded* superharmonic function is an lsc-regularized superminimizer and thus belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$.

The following definition of singular and Green functions in metric spaces was given in [11, Definition 1.1]. Recall that a *domain* is a nonempty open connected set.

Definition 6.3. Let $\Omega \subset X$ be a bounded domain. A positive function $u \colon \Omega \to (0, \infty]$ is a *singular function* in Ω with singularity at $x_0 \in \Omega$ if it satisfies the following properties:

(S1) u is superharmonic in Ω;
(S2) u is p-harmonic in $\Omega \setminus \{x_0\}$;
(S3) $u(x_0) = \sup_\Omega u$;
(S4) $\inf_\Omega u = 0$;
(S5) $\tilde{u} \in N^{1,p}_{\mathrm{loc}}(X \setminus \{x_0\})$, where

$$\tilde{u} = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

A *Green function* is a singular function which satisfies

$$\operatorname{cap}_p(\Omega^b, \Omega) = b^{1-p}, \quad \text{when } 0 < b < u(x_0), \tag{6.1}$$

where $\Omega^b = \{x \in \Omega : u(x) \ge b\}$.

An earlier definition of singular functions along similar lines is due to Holopainen–Shanmugalingam [35]. Under our assumptions and with natural interpretation for the values at x_0 and in $X \setminus \Omega$, a Green function as in Definition 6.3 is a singular function in the sense of [35], while singular functions in [35] are special cases of our singular functions. See [11, Section 12] for a more precise comparison of these definitions.

The existence of singular and Green functions in bounded domains was studied in detail in [11], and the following is one of the main results therein. Note that the condition $C_p(X \setminus \Omega) > 0$ below is always true if X is unbounded. In fact, under the assumption $C_p(X \setminus \Omega) > 0$ conditions (S3) and (S4) are superfluous in Definition 6.3 by [11, Theorem 1.6].

Theorem 6.4. (Theorem 1.3 in [11]) *Let $\Omega \subset X$ be a bounded domain and let $x_0 \in \Omega$. Then there exists a Green function in Ω with singularity at x_0 if and only if $C_p(X \setminus \Omega) > 0$. Moreover, if u is a singular function in Ω with singularity at x_0, then there is a unique $\alpha > 0$ such that αu is a Green function.*

Remark 6.5. Let u be a singular function in a bounded domain Ω. If $C_p(\{x_0\}) > 0$, then $g_u \in L^p(\Omega)$ by [11, Theorem 8.6]. Assume instead that $C_p(\{x_0\}) = 0$. As u is p-harmonic in $\Omega \setminus \{x_0\}$ it belongs to $N^{1,p}_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ and thus has a minimal p-weak upper gradient $g_u \in L^p_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ in $\Omega \setminus \{x_0\}$. Since $C_p(\{x_0\}) = 0$, Proposition 1.48 in [5] shows that g_u is also a p-weak upper gradient of u within Ω, even though $g_u \notin L^p_{\mathrm{loc}}(\Omega)$, because of Theorem 6.6 below together with (S5). As $\min\{u, k\}$ is a superminimizer in Ω (and thus belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$), the function

$$G_u := \lim_{k\to\infty} g_{\min\{u,k\}} \tag{6.2}$$

is a p-weak upper gradient of u which is minimal in a certain sense, see Kinnunen–Martio [42, Section 5] (or [5, Section 2.8]) for further details. As $u \in N^{1,p}_{\mathrm{loc}}(\Omega \setminus \{x_0\})$, we have $G_u = g_u$ a.e. in $\Omega \setminus \{x_0\}$ and thus a.e. in Ω. For singular functions u, we will therefore denote the minimal p-weak upper gradient by g_u even within Ω.

Theorem 6.6. *Let u be a Green function in a bounded domain Ω with singularity at x_0. Then the following are equivalent*:

(a) $u(x_0) = \infty$;
(b) *u is unbounded*;
(c) $u \notin N^{1,p}(\Omega)$;
(d) $g_u \notin L^p(\Omega)$;
(e) $C_p(\{x_0\}) = 0$;
(f) $\displaystyle\int_0^\delta \biggl(\frac{\rho}{\mu(B_\rho)}\biggr)^{1/(p-1)} d\rho = \infty$ *for some (or equivalently all) $\delta > 0$.*

In particular, all of these statements are true if $p < \bar{s}_0$ and false if $p > \bar{s}_0$.

Proof. The statements (a)–(e) were shown to be equivalent in [11, Theorem 8.6]. The equivalence (e) $\Leftrightarrow$ (f) follows from Proposition 5.3 (a) and the self-improvement of the p-Poincaré inequality. The last part follows from Proposition 5.1. $\square$

7. Pointwise estimates for Green functions

Recall the general assumptions from the beginning of Section 6.

From Theorem 4.2 and [11, Theorem 1.5] we obtain the following pointwise estimates for Green functions. Recall that by Theorem 6.4, all estimates for Green functions in this section also hold for singular functions u, but with comparison constants also depending on u. For global Green functions on Riemannian manifolds, estimates similar to (7.3) were under various assumptions obtained in Holopainen [33, Section 5].

Theorem 7.1. *Let $0 < R_1 \le R_2 < \frac{1}{4} \operatorname{diam} X$ be fixed. Assume that u is a Green function with singularity at x_0 in a domain Ω such that*

$$B_{R_1} \subset \Omega \subset B_{R_2}.$$

If $r := d(x, x_0) < R_1/50\lambda$, where λ is the dilation constant in the Poincaré inequality, then

$$u(x) \simeq \operatorname{cap}_p(B_r, \Omega)^{1/(1-p)} \gtrsim \biggl(\frac{r^p}{\mu(B_r)}\biggr)^{1/(p-1)} \tag{7.1}$$

and

$$u(x) \simeq \operatorname{cap}_p(B_r, B_{R_1})^{1/(1-p)} \simeq \operatorname{cap}_p(B_r, B_{R_2})^{1/(1-p)} \tag{7.2}$$

$$\simeq \int_r^{R_1} \biggl(\frac{\rho}{\mu(B_\rho)}\biggr)^{1/(p-1)} d\rho \simeq \int_r^{R_2} \biggl(\frac{\rho}{\mu(B_\rho)}\biggr)^{1/(p-1)} d\rho, \tag{7.3}$$

with comparison constants depending only on p, the doubling constant of μ, the constants in the Poincaré inequality, and in (7.2)–(7.3) also on the quotient R_2/R_1.

From now on we let $S_r = \{x : d(x, x_0) = r\}$.

Proof. The first comparison in (7.1) follows from [11, Theorem 1.5] when $x \in \partial B_r$. To extend it to all $x \in S_r$, we use [11, Proposition 4.4] with $r < \rho < \min\{2r, R_1/50\lambda\}$ and $K = \overline{B}_\rho \setminus \frac{1}{2}B_\rho$ as follows:

$$\max_{\partial B_r} u \le \max_{S_r} u \le \max_K u \le A \min_K u \le A \min_{S_r} u \le A \min_{\partial B_r} u \le A \max_{\partial B_r} u,$$

where A depends only on p, the doubling constant of μ and the constants in the Poincaré inequality. The inequality in (7.1) then follows from [10, Proposition 5.1]. Next, Lemma 11.22 in [5] yields

$$\operatorname{cap}_p(B_r, B_{R_2}) \le \operatorname{cap}_p(B_r, \Omega) \le \operatorname{cap}_p(B_r, B_{R_1}) \lesssim \operatorname{cap}_p(B_r, B_{R_2}),$$

and hence (7.2) holds, while (7.3) follows directly from Theorem 4.2. $\square$

If μ is Ahlfors Q-regular around x_0 as in Theorem 1.2, then, by (7.3), $u(x) \simeq r^{(p-Q)/(p-1)}$ if $p < Q$ and $u(x) \simeq \log(R_1/r)$ if $p = Q$, while $u(x)$ is bounded if $p > Q$. The estimates (7.2) can also be combined with the capacity estimates in [10] to describe the pointwise behaviour of Green functions as follows.

Corollary 7.2. *Assume that the assumptions in Theorem 7.1 are satisfied, and in particular that $r := d(x, x_0) < R_1/50\lambda$. Then the following are true, with comparison constants independent of u, x, r and Ω:*

(a) *If $p < \underline{q}_0$, then*

$$u(x) \simeq \biggl(\frac{r^p}{\mu(B_r)}\biggr)^{1/(p-1)}. \tag{7.4}$$

(b) *If $p = \underline{q}_0 \in \underline{Q}_0$, then*

$$\max\biggl\{\biggl(\frac{R_1^p}{\mu(B_{R_1})}\biggr)^{1/(p-1)} \log\frac{R_1}{r}, \biggl(\frac{r^p}{\mu(B_r)}\biggr)^{1/(p-1)}\biggr\} \lesssim u(x) \lesssim \biggl(\frac{r^p}{\mu(B_r)}\biggr)^{1/(p-1)} \log\frac{R_1}{r}. \tag{7.5}$$

(c) *If* $p > q \in \underline{Q}_0$, *then*

$$\biggl(\frac{r^p}{\mu(B_r)}\biggr)^{1/(p-1)} \lesssim u(x) \lesssim \biggl(\frac{r^q}{\mu(B_r)}\biggr)^{1/(p-1)}.$$

(d) *If* $p < s \in \bar{S}_0$, *then*

$$1 \lesssim u(x) \lesssim r^{(p-s)/(p-1)}.$$

Proof. This follows directly from Theorem 7.1 together with [10, Theorems 1.1 and 1.2 and Propositions 6.2, 8.1 and 8.3]. □

Unweighted $\mathbf{R}^n$ with $p = n$ shows sharpness of both the lower and upper bounds in (b), and with $p < n = s$ of the upper bound in (d).

Earlier, using the definition of singular functions in Holopainen–Shanmugalingam [35], and especially the a priori superlevel set property therein, Danielli–Garofalo–Marola [19, Theorem 5.2] established (7.4) for such functions. In the borderline case $p = \underline{q}_0 \in \underline{Q}_0$, they also gave an estimate which is essentially equivalent to (7.5), since the constant in [19, Theorem 5.2] in this case depends on $r^p/\mu(B_r)$; cf. [19, Theorem 3.1].

Another consequence of (7.3) in Theorem 7.1 is that *all* Green functions with respect to comparable open sets and comparable measures are comparable near the singularity in the following sense.

Theorem 7.3. *Let* $0 < R_1 \le R_2 < \frac{1}{4} \operatorname{diam} X$ *be fixed. Assume that* Ω_1 *and* Ω_2 *are domains such that*

$$B_{R_1} \subset \Omega_j \subset B_{R_2}, \quad j = 1, 2.$$

Let μ_1 *and* μ_2 *be doubling measures supporting* p*-Poincaré inequalities on* X*. Assume in addition that*

$$\mu_1(B_\rho) \lesssim \mu_2(B_\rho) \quad \text{for all } 0 < \rho \le R_2. \tag{7.6}$$

Also let u_j *be a Green function, with respect to* μ_j*, in* Ω_j *with singularity at* x_0*,* $j = 1, 2$*. Then*

$$u_1(x) \gtrsim u_2(x) \quad \text{for all } x \in B_{R_1/50\lambda}, \tag{7.7}$$

with comparison constant depending only on p*,* R_1*,* R_2*, the doubling and Poincaré constants, and the comparison constant in* (7.6).

Moreover, if $K \subset \Omega_1 \cap \Omega_2$ *is compact then*

$$u_1(x) \gtrsim u_2(x) \quad \text{for all } x \in K,$$

with comparison constant also depending on K.

Proof. By Theorem 7.1,

$$u_j(x) \simeq \int_{d(x,x_0)}^{R_1} \biggl(\frac{\rho}{\mu_j(B_\rho)}\biggr)^{1/(p-1)} d\rho, \ j = 1, 2, \quad \text{for all } x \in B_{R_1/50\lambda}.$$

Thus (7.7) follows from (7.6).

For the last part, let $\Omega = \Omega_1 \cap \Omega_2 \supset B_{R_1}$ and $0 < R < R_1/50\lambda$. Since we have already shown (7.7) we may replace K by $(K \setminus B_R) \cup \partial B_R$ and assume that $\partial B_R \subset K \subset \Omega \setminus B_R$. As Ω_1 is connected, every component of $\Omega_1 \setminus \{x_0\}$ that intersects K must contain a point in ∂B_R, and since ∂B_R is compact there are only finitely many such components $G_1, \dots, G_m$ of $\Omega_1 \setminus \{x_0\}$. By Harnack's inequality, as in Corollary 8.19 in [5], there are constants C_j such that

$$\sup_{K\cap G_j} v \le C_j \inf_{K\cap G_j} v \quad \text{for every positive } p\text{-harmonic function } v \text{ in } G_j.$$

Together with [11, Proposition 4.4], this shows that $\sup_K u_1 \lesssim \inf_K u_1$. Similarly $\sup_K u_2 \lesssim \inf_K u_2$, which together with (7.7) shows that $u_1 \gtrsim u_2$ on K. □

Theorem 7.3 can be seen as a generalization of the well-known comparisons for the classical Green functions in Euclidean domains for various linear and nonlinear elliptic equations, cf. Littman–Stampacchia–Weinberger [49, Theorem 7.1] and Serrin [54, Theorem 12]. Those estimates state that near the singularity, Green functions have certain predetermined growth and are thus comparable to the fundamental solution for the Laplace or p-Laplace equation, see also Section 13 below. Such estimates have proved to be of great importance for both the interior and boundary regularity of such equations. Theorem 7.3 and the other results in this section provide us with similar comparisons for Green functions associated with the energy functionals $\int |\nabla u|^p \, d\mu$ for large classes of comparable measures.

On the contrary, the so-called quasiminimizers, introduced by Giaquinta and Giusti [22], [23] as a natural unification of differential equations with various ellipticities, can (even in unweighted $\mathbf{R}^n$) have singularities of arbitrary order, depending on the quasiminimizing constant, see Björn–Björn [6]. In particular, quasiminimizers are not always solutions to partial differential equations of p-Laplacian type, since all such solutions have comparable behaviour near their singularity, by Serrin [54, Theorem 12], and thus also the same integrability.

Remark 7.4. Assume that Ω is a bounded domain and $B_R \subset \Omega$. Let u be a Green function in Ω with singularity at x_0. If $C_p(\{x_0\}) > 0$, then u is bounded by Theorem 6.6. On the other hand, if $C_p(\{x_0\}) = 0$ then (7.1) implies that for all $0 < r < R/50\lambda$ and $x \in S_r$,

$$C_1 \operatorname{cap}_p(B_r, \Omega)^{1/(1-p)} \le u(x) \le C_2 \operatorname{cap}_p(B_r, \Omega)^{1/(1-p)},$$

where $C_1, C_2 > 0$ depend only on p, the doubling constant of μ and the constants in the Poincaré inequality, but not on u, x, r or Ω. In particular, letting $r \nearrow R_2 := R/50\lambda$, we see that

$$k := \max_{\partial B_{R_2}} u \le C_2 \operatorname{cap}_p(B_{R_2}, \Omega)^{1/(1-p)}$$

and $u - k$ is a Green function in

$$\Omega_k = \{x \in \Omega : u(x) > k\} \subset B_{R_2}.$$

As $\operatorname{cap}_p(\{x_0\}, \Omega) = 0$ and cap_p is an outer capacity (by [5, Theorem 6.19 (vii)]), we can now find $R_1 > 0$ such that

$$\operatorname{cap}_p(B_{R_1}, \Omega)^{1/(1-p)} > \frac{C_2}{C_1} \operatorname{cap}_p(B_{R_2}, \Omega)^{1/(1-p)} \ge \frac{k}{C_1}.$$

It follows that $\min_{\partial B_{R_1}} u \ge k$ and hence $B_{R_1} \subset \Omega_k \subset B_{R_2}$. Note that R_1 depends only on R_2, C_1, C_2 and Ω, but not on u. Since $B_{50\lambda R_2} = B_R \subset \Omega \ne X$ by Theorem 6.4, we also have that $50\lambda R_2 \le \operatorname{diam} X$ and hence $R_2 < \frac{1}{4} \operatorname{diam} X$.

Applying Theorem 7.1 to $u - k$, $\Omega_k \subset B_{R_2}$ and $r < R_1/50\lambda$, we thus see that all the estimates in this section hold near the singularity for Green functions in arbitrary bounded domains Ω, even if Ω is not contained in some B_{R_2} with $R_2 < \frac{1}{4} \operatorname{diam} X$. Note that $C_p(X \setminus \Omega) > 0$, by Theorem 6.4, since the Green function is presumed to exist.

8. Integrability of superharmonic functions

Recall the general assumptions from the beginning of Section 6.

Green functions are particular examples of superharmonic functions. Before studying special integrability properties of Green functions and their minimal p-weak upper gradients we therefore recall general integrability results for superharmonic functions, due to Kinnunen–Martio [42, Theorems 5.1 and 5.6]. (On unweighted $\mathbf{R}^n$ (where $\bar{\theta}_0 = n$), these results are due to Lindqvist [48, Theorems 1.4 and 4.2] while on weighted $\mathbf{R}^n$, with a p-admissible weight, they were obtained by Heinonen–Kilpeläinen–Martio [28, Theorem 7.46].)

Later in this section we will deduce a general result on integrability of the minimal p-weak upper gradients of superharmonic functions, which will play a crucial role in Section 10. If u is superharmonic, we define G_u as in (6.2). That a superharmonic function fails to belong to $N^{1,p}_{\mathrm{loc}}(\Omega)$ only if it is too large is a consequence of Proposition 7.4 in Björn–Björn–Parviainen [13] (or [5, Corollary 9.6]). Recall from Section 3 that

$$\bar{\theta}_0 = \inf\biggl\{\bar{\theta} > 0 : \frac{\mu(B(x,r))}{\mu(B(x,R))} \gtrsim \Bigl(\frac{r}{R}\Bigr)^{\bar{\theta}} \text{for all } x \in X \text{ and } 0 < r \le R < 2 \operatorname{diam} X\biggr\}$$

and that $\bar{\theta}_0 \ge 1$, by Proposition 3.2.

Theorem 8.1. ([42, Theorems 5.1 and 5.6]) *Let u be a superharmonic function in Ω.*

(a) *If $p \le \bar{\theta}_0$, then $u \in L^\tau_{\mathrm{loc}}(\Omega)$ and $G_u \in L^t_{\mathrm{loc}}(\Omega)$ whenever*

$$0 < \tau < \begin{cases} \dfrac{\bar{\theta}_0(p-1)}{\bar{\theta}_0 - p}, & \text{if } p < \bar{\theta}_0, \\ \infty, & \text{if } p = \bar{\theta}_0, \end{cases} \qquad \text{and} \qquad 0 < t < \frac{\bar{\theta}_0(p-1)}{\bar{\theta}_0 - 1},$$

respectively.

(b) *If $p > \bar{\theta}_0$, then u is continuous and thus locally bounded. Moreover, $G_u \in L^p_{\mathrm{loc}}(\Omega)$ and $G_u = g_u$.*

In particular, it is always true that $u, G_u \in L^{p-1}_{\mathrm{loc}}(\Omega)$.

Proof. In case (a), it follows from Theorem 5.1 in Hajłasz–Koskela [25] (or [5, Theorem 4.21]) that so-called (q,p)-Poincaré inequalities hold for every $1 \le q < \bar{\theta}_0 p/(\bar{\theta}_0 - p)$ (every $1 \le q < \infty$ if $p = \bar{\theta}_0$). Thus this part follows directly from Theorems 5.1 and 5.6 in [42] (or [5, Theorems 9.53 and 9.54]), upon letting $\bar{\theta} \to \bar{\theta}_0$.

In case (b), $u_k := \min\{u, k\}$ is a superminimizer and thus belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$. It then follows from Corollary 5.39 in [5] that u_k is continuous and that all points have positive capacity. In particular, considering u_k with

$$\liminf_{x \to x_0} u(x) < k < \limsup_{x \to x_0} u(x)$$

leads to a contradiction and hence u is an $(-\infty, \infty]$-valued continuous function.

On the other hand, by Proposition 2.2 in Kinnunen–Shanmugalingam [44] (or Corollary 9.51 in [5]), the set $\{x \in \Omega : u(x) = \infty\}$ has zero p-capacity, and thus must be empty. Hence u is real-valued continuous, and therefore locally bounded. Corollary 7.8 in Kinnunen–Martio [41] (or [5, Corollary 9.6]) then shows that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, and in particular, $g_u = G_u \in L^p_{\mathrm{loc}}(\Omega)$. □

Using the ideas from the proof of Kinnunen–Martio [42, Theorem 5.6], we obtain the following generalization of Theorem 8.1, which will be important when studying Green functions.

Theorem 8.2. *Let u be a superharmonic function in Ω.*

(a) *If $u \in L^\tau_{\mathrm{loc}}(\Omega)$ for all $0 < \tau < \tau_0$, then $G_u \in L^t_{\mathrm{loc}}(\Omega)$ whenever*

$$0 < t < \begin{cases} \dfrac{p\tau_0}{\tau_0 + 1}, & \textit{if } \tau_0 < \infty, \\ p, & \textit{if } \tau_0 = \infty. \end{cases}$$

(b) *If u is locally bounded, then $G_u \in L^p_{\mathrm{loc}}(\Omega)$.*

It follows from Example 11.2 below that even if $u \in L^{\tau_0}_{\mathrm{loc}}(\Omega)$ then it can happen that $G_u \notin L^{p\tau_0/(\tau_0+1)}_{\mathrm{loc}}(\Omega)$, which in particular shows that (a) is sharp if $\tau_0 < \infty$. That it is sharp also when $\tau_0 = \infty$ follows from unweighted $\mathbf{R}^n$ with $p = n$.

Proof. (b) If u is locally bounded then $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, by Corollary 7.8 in Kinnunen–Martio [41] (or [5, Corollary 9.6]). In particular, $g_u = G_u \in L^p_{\mathrm{loc}}(\Omega)$.

(a) It suffices to consider $\tau_0 < \infty$, since the infinite case is obtained by letting $\tau_0 \to \infty$. Let $B \subset 2B \Subset \Omega$ be a ball and $0 < \varepsilon < \tau_0(p-t)/t - 1$. Let

$$m = \min_{2B} u > -\infty \quad \text{and} \quad u_k = \min\{u - m, k\} + 1, \quad k = 1, 2, \dots, \tag{8.1}$$

which is a positive superminimizer in $2B$. Then, using the Caccioppoli inequality for superminimizers (Lemma 3.1 in Kinnunen–Martio [42] or [5, Proposition 8.8]) with a suitable cut-off function $\eta \in \mathrm{Lip}_c(2B)$, we obtain

$$\begin{aligned} \int_B G_u^t \, d\mu &= \lim_{k\to\infty} \int_B g_{u_k}^t u_k^{-(1+\varepsilon)t/p} u_k^{(1+\varepsilon)t/p} \, d\mu \\ &\le \lim_{k\to\infty} \biggl(\int_B g_{u_k}^p u_k^{-(1+\varepsilon)} \, d\mu \biggr)^{t/p} \biggl(\int_B u_k^{(1+\varepsilon)t/(p-t)} \, d\mu \biggr)^{1-t/p} \\ &\le C \biggl(\int_{2B} (u - m + 1)^{p-(1+\varepsilon)} \, d\mu \biggr)^{t/p} \biggl(\int_B (u - m + 1)^{(1+\varepsilon)t/(p-t)} \, d\mu \biggr)^{1-t/p} \\ &< \infty, \end{aligned}$$

where the first integral on the right-hand side is finite by the last part of Theorem 8.1, while the last integral is finite by assumption. □

9. Integrability of Green functions

In addition to the general assumptions from the beginning of Section 6, we assume in this section that $\Omega \subset X$ is a bounded domain.

In this section we study L^τ-integrability (and nonintegrability) of Green functions. In this case, we can improve upon the general integrability results for superharmonic functions in Theorem 8.1. We state the results here and in the next Section 10 for Green functions, but by Theorem 6.4 the same conclusions hold for singular functions as well. See also Section 12 for similar results for general p-harmonic functions with poles.

Note that, due to Theorem 7.3 and Remark 7.4, all Green functions with the same singularity x_0 belong to the same L^τ spaces. If $p > \bar{s}_0$, then by Proposition 5.1 (b) and Theorem 6.6 every Green function is bounded. We therefore omit this case in the rest of this section. For $p \le \bar{s}_0$ we have the *critical exponent*

$$\tau_p = \begin{cases} \dfrac{\bar{s}_0(p-1)}{\bar{s}_0 - p}, & \text{if } p < \bar{s}_0, \\ \infty, & \text{if } p = \bar{s}_0. \end{cases} \tag{9.1}$$

Theorem 9.1. *Let u be a Green function in Ω with singularity at x_0. If $p \le \bar{s}_0$ then $u \in L^\tau(\Omega)$ for all $0 < \tau < \tau_p$.*

Since $\bar{s}_0 \le \bar{\theta}_0$, we have for all $p \le \bar{s}_0$ that $\tau_p \ge \bar{\theta}_0(p-1)/(\bar{\theta}_0 - p)$, where the right-hand side is the borderline exponent for τ in Theorem 8.1 and the inequality is strict when $\bar{s}_0 < \bar{\theta}_0$. Hence Green functions have higher integrability than what is known for general superharmonic functions when $\bar{s}_0 < \bar{\theta}_0$.

Remark 9.2. Danielli–Garofalo–Marola [19, Corollary 5.4] showed for singular functions u, as defined in Holopainen–Shanmugalingam [35], that $u \in L^\tau(\Omega)$ whenever

$$p < \underline{q} := \sup \underline{Q} \quad \text{and} \quad \tau < \frac{\underline{q}(p-1)}{\tilde{\theta} - p}, \quad \text{where } \tilde{\theta} = \log_2 C_\mu \in \overline{\Theta},$$

C_μ is the doubling constant of μ and

$$\underline{Q} = \biggl\{ q > 0 : \text{there is } C \text{ so that } \frac{\mu(B_r)}{\mu(B_R)} \le C \Bigl(\frac{r}{R}\Bigr)^q \text{ for } 0 < r < R < \infty \biggr\}. \tag{9.2}$$

Note that $\underline{q} \le \underline{q}_0$, where the inequality can be strict as the range in (9.2) is $0 < r < R < \infty$. They however also implicitly assumed that $\underline{q} \in \underline{Q}$, see [19, eq. (2.2)], and that X is linearly locally connected (LLC), through their use (at the bottom of p. 354) of Lemma 5.3 in Björn–MacManus–Shanmugalingam [15].

Note that $\underline{q}$ in [19] can be much smaller than our $\bar{s}_0$, which in turn can be much smaller than $\tilde{\theta}$. Thus, both the numerator and the denominator are in general worse in [19] than in the critical exponent (9.1), and the range of possible exponents p is smaller than here. Thus Theorem 9.1 is a substantial improvement upon the results in [19]. Moreover, Theorem 9.1 is sharp (up to certain borderline cases), by Theorem 9.3.

Proof of Theorem 9.1. We may assume that $B_{R_1} \subset \Omega \subset B_{R_2}$, where $0 < R_1 \le R_2 < \frac{1}{4} \operatorname{diam} X$, see Remark 7.4. Let $r_k = 2^{-k} R_1/50\lambda$ and $B^k = B_{r_k}$, $k = 1, 2, \ldots$. By the assumptions on τ, we find $\bar{s} > \bar{s}_0$ such that $\tau < \bar{s}(p-1)/(\bar{s} - p) =: \beta$. In particular, $\bar{s} \in \bar{S}_0$.

Consider x such that $r_{k+1} \le r := d(x, x_0) < r_k$. Then, by Theorem 7.1 and since $\bar{s} \in \bar{S}_0$,

$$\begin{aligned} u(x) &\simeq \int_r^{R_1} \biggl(\frac{\rho^p}{\mu(B_\rho)^{1-p/\bar{s}} \mu(B_\rho)^{p/\bar{s}}} \biggr)^{1/(p-1)} \frac{d\rho}{\rho} \\ &\lesssim \biggl(\frac{1}{\mu(B_r)^{1-p/\bar{s}}} \biggr)^{1/(p-1)} \int_r^{R_1} \frac{d\rho}{\rho} = \frac{\log(R_1/r)}{\mu(B_r)^{1/\beta}} \lesssim \frac{k}{\mu(B^k)^{1/\beta}}. \end{aligned}$$

As $\tau/\beta < 1$, we have for any $\underline{s} \in \underline{S}_0$ that

$$\int_{B^1} u^\tau \, d\mu \lesssim \sum_{k=1}^\infty k^\tau \mu(B^k)^{1-\tau/\beta} \lesssim \sum_{k=1}^\infty k^\tau 2^{-k\underline{s}(1-\tau/\beta)} < \infty,$$

where we recall that $\underline{S}_0 \ne \varnothing$ under our assumptions. Hence $u \in L^\tau(\Omega)$. $\square$

Next we consider nonintegrability of u.

Theorem 9.3. *Let u be a Green function in Ω with singularity at x_0 and assume that $p \le \bar{s}_0$.*

(a) *If $\tau > \tau_p$ then $u \notin L^\tau(\Omega)$.*
(b) *If $\bar{s}_0 \notin \bar{S}_0 \setminus \underline{S}_0$, then $u \notin L^{\tau_p}(\Omega)$.*

Proof. By Remark 7.4, we may assume that $B_{R_1} \subset \Omega \subset B_{R_2}$, where $0 < R_1 \le R_2 < \frac{1}{4} \operatorname{diam} X$. Let $r_k = 2^{-k} R_1/50\lambda$ and $B^k = B_{r_k}$, $k = 0, 1, \dots$.

If $p = \bar{s}_0$, then $\tau_p = \infty$ and there is nothing to prove in (a), while Proposition 5.1 and Theorem 6.6 show that $u \notin L^\infty(\Omega)$ in (b).

Assume now that $p < \bar{s}_0$. In case (a), let $s > p$ be such that $\tau \ge s(p-1)/(s-p)$ and $s \notin \overline{S}_0$. In case (b), we instead take $s = \bar{s}_0$ and $\tau = \tau_p$. In both cases, there is a sequence $k_j \nearrow \infty$ such that $\mu(B^{k_j}) \lesssim r_{k_j}^s$. As $\tau \ge \tau_p > p-1$, we therefore obtain from (7.1) and the reverse-doubling property of μ that

$$\int_{B^0} u^\tau \, d\mu \gtrsim \sum_{k=0}^\infty \biggl(\frac{r_k^p}{\mu(B^k)} \biggr)^{\tau/(p-1)} \mu(B^k) \gtrsim \sum_{j=0}^\infty r_{k_j}^{p\tau/(p-1)+s(1-\tau/(p-1))} = \infty,$$

since the last exponent is nonpositive. □

Thus, when $p < \bar{s}_0$ we know exactly which L^τ-integrability u has, apart from the borderline case $\tau = \tau_p$. When $p = \bar{s}_0$ we also lack a complete characterization of when u is bounded. This is however natural, as knowing $\underline{S}_0$ and $\overline{S}_0$ is not enough to determine integrability and boundedness in these cases, see the last part with $p = s$ in Example 9.7 below and the comment before Lemma 5.2. At the same time, under the assumption $\bar{s}_0 \notin \overline{S}_0 \setminus \underline{S}_0$ we have a complete characterization, as follows.

Corollary 9.4. *Let u be a Green function in Ω with singularity at x_0 and assume that $p \le \bar{s}_0 \notin \overline{S}_0 \setminus \underline{S}_0$. Then u is unbounded, and $u \in L^\tau(\Omega)$ if and only if $\tau < \tau_p$.*

Proof. By Proposition 5.1, $C_p(\{x_0\}) = 0$ and hence u is unbounded by Theorem 6.6. The rest of the conclusion follows directly from Theorems 9.1 and 9.3. □

The following more general characterizations can be used also when the critical case is not captured by the S-sets, and hence they complement Corollary 9.4 when $\bar{s}_0 \in \overline{S}_0 \setminus \underline{S}_0$.

Theorem 9.5. *Let u be a Green function in Ω with singularity at x_0.*

(a) *If $p < \bar{s}_0$, then $u \in L^{\tau_p}(\Omega)$ if and only if*

$$\sum_{k=1}^\infty \biggl(\frac{2^{-k\bar{s}_0}}{\mu(B_{2^{-k}})} \biggr)^{p/(\bar{s}_0-p)} < \infty, \tag{9.3}$$

or equivalently

$$\int_0^1 \biggl(\frac{\rho^{\bar{s}_0}}{\mu(B_\rho)} \biggr)^{p/(\bar{s}_0-p)} \frac{d\rho}{\rho} < \infty.$$

(b) *If $p = \bar{s}_0$, then u is bounded (i.e. $u \in L^{\tau_p}(\Omega)$) if and only if*

$$\sum_{k=1}^\infty \biggl(\frac{2^{-k\bar{s}_0}}{\mu(B_{2^{-k}})} \biggr)^{1/(\bar{s}_0-1)} < \infty, \tag{9.4}$$

or equivalently

$$\int_0^1 \biggl(\frac{\rho^{\bar{s}_0}}{\mu(B_\rho)} \biggr)^{1/(\bar{s}_0-1)} \frac{d\rho}{\rho} = \int_0^1 \biggl(\frac{\rho}{\mu(B_\rho)} \biggr)^{1/(\bar{s}_0-1)} d\rho < \infty.$$

Perhaps surprisingly, condition (9.4) is not the limit of (9.3), as $p \to \bar{s}_0$, but coincides with (9.3) for $p = 1$ (which however is not allowed in (a)).

Proof. The equivalences between the sums and integrals are obvious. Part (b) follows from Proposition 5.3 and Theorem 6.6, so we turn to (a).

By Remark 7.4, we may assume that $B_{R_1} \subset \Omega \subset B_{R_2}$, where $0 < R_1 \le R_2 < \frac{1}{4} \operatorname{diam} X$. Let $r_k = 2^{-k} R_1/50\lambda$ and $B^k = B_{r_k}$, $k = 0, 1, \dots$. Consider x such that $r_{k+1} \le r := d(x, x_0) < r_k$. By (7.1),

$$u(x) \gtrsim \left(\frac{r_k^p}{\mu(B^k)} \right)^{1/(p-1)}.$$

Note from (9.1) that

$$\frac{\tau_p}{p-1} = \frac{\bar{s}_0}{\bar{s}_0 - p} \quad \text{and} \quad \frac{\tau_p}{p-1} - 1 = \frac{p}{\bar{s}_0 - p}. \tag{9.5}$$

Hence, using that μ is reverse-doubling with $\xi = 2$,

$$\int_{B^1} u^{\tau_p} \, d\mu \gtrsim \sum_{k=1}^{\infty} \left(\frac{r_k^p}{\mu(B^k)} \right)^{\tau_p/(p-1)} \mu(B^k \setminus B^{k+1}) \simeq \sum_{k=1}^{\infty} \left(\frac{r_k^{\bar{s}_0}}{\mu(B^k)} \right)^{p/(\bar{s}_0 - p)}, \tag{9.6}$$

which diverges if and only if the sum in (9.3) diverges.

Conversely, we apply (7.3) to $u(x)$ and obtain

$$\int_{B^1} u^{\tau_p} \, d\mu \lesssim \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \left(\frac{r_j^p}{\mu(B^j)} \right)^{1/(p-1)} \right)^{\tau_p} \mu(B^k \setminus B^{k+1}). \tag{9.7}$$

We distinguish two cases. If $\tau_p \le 1$, then (9.7) gives

$$\begin{aligned} \int_{B^1} u^{\tau_p} \, d\mu &\lesssim \sum_{k=1}^{\infty} \sum_{j=1}^{k} \left(\frac{r_j^p}{\mu(B^j)} \right)^{\tau_p/(p-1)} \mu(B^k \setminus B^{k+1}) \\ &= \sum_{j=1}^{\infty} \left(\frac{r_j^p}{\mu(B^j)} \right)^{\tau_p/(p-1)} \sum_{k=j}^{\infty} \mu(B^k \setminus B^{k+1}) = \sum_{j=1}^{\infty} \left(\frac{r_j^p}{\mu(B^j)} \right)^{\tau_p/(p-1)} \mu(B^j), \end{aligned}$$

which, by (9.5), is the same as the last sum in (9.6).

If $\tau_p > 1$, we rewrite (9.7) as

$$\int_{B^1} u^{\tau_p} \, d\mu \lesssim \sum_{k=1}^{\infty} r_k^{\varepsilon} \left(\sum_{j=1}^{k} \frac{r_j^{p/(p-1) - \varepsilon/\tau_p}}{\mu(B^j)^{1/(p-1) - 1/\tau_p}} \left(\frac{r_j^{\varepsilon} \mu(B^k)}{r_k^{\varepsilon} \mu(B^j)} \right)^{1/\tau_p} \right)^{\tau_p}, \tag{9.8}$$

where $\varepsilon > 0$. Now, for any $0 < q \in \underline{Q}_0$, we have

$$\frac{r_j^{\varepsilon} \mu(B^k)}{r_k^{\varepsilon} \mu(B^j)} \lesssim \left(\frac{r_k}{r_j} \right)^{q - \varepsilon} = 2^{(j-k)(q-\varepsilon)}.$$

The inner sum on the right-hand side of (9.8) is then estimated using (9.5) and Hölder's inequality with exponents τ_p and $\tau_p/(\tau_p - 1)$, as follows

$$\sum_{j=1}^{k} \lesssim \left(\sum_{j=1}^{k} \frac{r_j^{\bar{s}_0 p/(\bar{s}_0 - p) - \varepsilon}}{\mu(B^j)^{p/(\bar{s}_0 - p)}} \right)^{1/\tau_p} \left(\sum_{j=1}^{k} 2^{(j-k)(q-\varepsilon)/(\tau_p - 1)} \right)^{1 - 1/\tau_p}.$$

For $q > \varepsilon$, the last sum is bounded from above by a constant, independent of k. Hence, by inserting the previous estimate into (9.8) and changing the order of summation, we obtain

$$\begin{aligned} \int_{B^1} u^{\tau_p} \, d\mu &\lesssim \sum_{k=1}^{\infty} r_k^{\varepsilon} \sum_{j=1}^{k} \left(\frac{r_j^{\bar{s}_0}}{\mu(B^j)} \right)^{p/(\bar{s}_0 - p)} r_j^{-\varepsilon} \\ &= \sum_{j=1}^{\infty} r_j^{-\varepsilon} \left(\frac{r_j^{\bar{s}_0}}{\mu(B^j)} \right)^{p/(\bar{s}_0 - p)} \sum_{k=j}^{\infty} r_k^{\varepsilon} \simeq \sum_{j=1}^{\infty} \left(\frac{r_j^{\bar{s}_0}}{\mu(B^j)} \right)^{p/(\bar{s}_0 - p)}. \end{aligned} \qquad \square$$

Next we compare Green functions for different parameters p under a natural assumption on the validity of Poincaré inequalities.

Corollary 9.6. *Assume that μ supports a p_0-Poincaré inequality for some $1 \le p_0 < \bar{s}_0$. For each $p_0 < p \le \bar{s}_0$, let u_p be a Green function in Ω with singularity at x_0 with respect to p. Then the following are true:*

(a) *If $p_0 < p_1 < p_2 < \bar{s}_0$ and $u_{p_1} \in L^{\tau_{p_1}}(\Omega)$, then $u_{p_2} \in L^{\tau_{p_2}}(\Omega)$.*

(b) *If $u_{\bar{s}_0}$ is bounded, then $u_p \in L^{\tau_p}(\Omega)$ for all $p_0 < p < \bar{s}_0$.*

Note that $\tau_{p_1} < \tau_{p_2} < \infty$ in (a). Example 9.7 below shows that the converse implications can fail, and that it is possible to have $u_p \in L^{\tau_p}(\Omega)$ for all $1 < p < \bar{s}_0$ but not for $p = \bar{s}_0$.

Proof. (a) Let $a_k = 2^{-k\bar{s}_0}/\mu(B_{2^{-k}})$. Then by Theorem 9.5, $u_{p_j} \in L^{\tau_{p_j}}(\Omega)$ if and only if

$$\sum_{k=1}^{\infty} a_k^{p_j/(\bar{s}_0 - p_j)} < \infty.$$

Hence, if $u_{p_1} \in L^{\tau_{p_1}}(\Omega)$, then

$$\sum_{k=1}^{\infty} a_k^{p_2/(\bar{s}_0 - p_2)} \le \biggl(\sum_{k=1}^{\infty} a_k^{p_1/(\bar{s}_0 - p_1)} \biggr)^{\beta_2/\beta_1} < \infty,$$

where $\beta_2 := p_2/(\bar{s}_0 - p_2) > p_1/(\bar{s}_0 - p_1) =: \beta_1$. The proof of (b) is similar upon noting that condition (9.4) is condition (9.3) with $p = 1$ in Theorem 9.5. $\square$

Example 9.7. Let $s > 1$, $\beta > 0$ and $n \ge 2$. Consider $\mathbf{R}^n$ equipped with the measure $d\mu = w\,dx$, where (by abuse of notation) $w(x) = w(|x|)$ and

$$w(\rho) = \begin{cases} \rho^{s-n} |\log \rho|^{\beta}, & \text{if } 0 < \rho \le 1/e, \\ \rho^{s-n}, & \text{otherwise.} \end{cases}$$

Let u be a Green function with singularity at $x_0 = 0$ in a bounded domain $\Omega \ni x_0$. By [10, Proposition 10.5 and Remark 10.6], μ is doubling and supports a 1-Poincaré inequality, i.e. w is 1-admissible on $\mathbf{R}^n$. Moreover, by Example 3.1 in [10],

$$\underline{S}_0 = \underline{Q}_0 = (0, s) \quad \text{and} \quad \bar{S}_0 = \bar{Q}_0 = [s, \infty).$$

In particular, $\underline{s}_0 = \underline{q}_0 = \bar{s}_0 = \bar{q}_0 = s$ and $\bar{s}_0 \in \bar{S}_0 \setminus \underline{S}_0$, so the assumption on $\bar{s}_0$ in Theorem 9.3 (b) and Corollary 9.4 fails.

It was also observed in Example 3.1 in [10] that $\mu(B_r) \simeq r^s |\log r|^{\beta}$ for $r \le 1/e$. Thus

$$\frac{2^{-ks}}{\mu(B_{2^{-k}})} \simeq \frac{1}{k^{\beta}}. \tag{9.9}$$

Hence, by Theorem 9.5 (a), for $p < s$,

$$u \in L^{\tau_p}(\Omega, w) \quad \text{if and only if} \quad \frac{\beta p}{s - p} > 1, \text{ i.e. } \frac{s}{1 + \beta} < p < s,$$

showing that the sets $\bar{S}_0$ and $\underline{S}_0$ themselves are not fine enough to determine the borderline L^{τ_p}-integrability of Green functions. In particular, if $\beta \ge s - 1$, then $u \in L^{\tau_p}(\Omega, w)$ for all $1 < p < s$.

For $p = s$, Theorem 9.5 (b) and (9.9) show that u is bounded if and only if $\beta > s - 1$. In particular, if $\beta = s - 1$, then $u \in L^{\tau_p}(\Omega, w)$ for all $1 < p < s$, but $u \notin L^{\tau_p}(\Omega, w)$ when $p = s$.

10. Integrability of gradients of Green functions

In addition to the general assumptions from the beginning of Section 6, we assume in this section that $\Omega \subset X$ is a bounded domain.

In this section we turn to the L^t-integrability of the minimal p-weak upper gradient g_u of a Green function u. See Remark 6.5 for how to interpret g_u. If $p > \bar{s}_0$ then $g_u \in L^p(\Omega)$, by Proposition 5.1 (b) and Theorem 6.6, and we therefore omit this case in the rest of this section. Thus, in particular, we assume that $\bar{s}_0 > 1$. The exponent

$$t_p = \frac{\bar{s}_0(p-1)}{\bar{s}_0 - 1}$$

will be critical. Note that

$$t_p = \frac{p\tau_p}{\tau_p + 1} < \tau_p \quad \text{if } p < \bar{s}_0 \tag{10.1}$$

and that $t_p \geq 1$ if and only if $p \geq 2 - 1/\bar{s}_0$.

Theorem 10.1. *Let u be a Green function in Ω with singularity at x_0 and assume that $p \leq \bar{s}_0$. Then $g_u \in L^t(\Omega)$ whenever $0 < t < t_p$.*

Moreover, $u \in N^{1,t}(\Omega)$ whenever $1 \leq t < t_p$.

Since $\bar{s}_0 \leq \bar{\theta}_0$, we have for all $p \leq \bar{s}_0$ that $t_p \geq \bar{\theta}_0(p-1)/(\bar{\theta}_0 - 1)$, where the right-hand side is the borderline exponent t in Theorem 8.1 and the inequality is strict when $\bar{s}_0 < \bar{\theta}_0$. Hence the minimal p-weak upper gradients of Green functions have higher integrability than what is known for the minimal p-weak upper gradients of general superharmonic functions when $\bar{s}_0 < \bar{\theta}_0$.

It follows from Danielli–Garofalo–Marola [19, Corollary 5.4] that $g_u \in L^t(\Omega)$ if $p < \underline{q}$ and $t < \underline{q}(p-1)/(\tilde{\theta} - 1)$, where $\underline{q}$, $\tilde{\theta}$ and X are as in Remark 9.2. Thus Theorem 10.1 is a substantial improvement upon the results in [19]. Moreover, Theorem 10.1 is sharp by Theorem 10.3, at least when $t_p \geq 1$ and X supports a t_p-Poincaré inequality at x_0 for small radii.

Proof of Theorem 10.1. The first part follows directly from Theorem 9.1 together with Theorem 8.2, (10.1) and (S5) in Definition 6.3. If $1 \leq t < t_p$, then g_u is also a t-weak upper gradient of u (although not necessarily minimal). Moreover, $u \in L^t(\Omega)$ by Theorem 9.1 and thus $u \in N^{1,t}(\Omega)$. □

For $p = \bar{s}_0$ we get the following consequences, when combining Theorem 10.1 with Theorem 6.6.

Corollary 10.2. *Assume that $p = \bar{s}_0$.*
(a) *If $C_p(\{x_0\}) = 0$, then $g_u \in L^t(\Omega)$ if and only if $0 < t < p$.*
(b) *If $C_p(\{x_0\}) > 0$, then $g_u \in L^p(\Omega)$.*

If μ supports a suitable global Poincaré inequality, then nonintegrability results for g_u can be obtained by combining Theorem 9.3 and the Sobolev inequality (see [5, Corollary 4.23]) determined by the global exponent set $\overline{\Theta}$. For instance, in the case when $p < Q$ and μ is globally Ahlfors Q-regular, so that $\bar{s}_0 = Q \in \overline{\Theta}$, and μ supports a global t-Poincaré inequality for $t \geq \max\{1, t_p\}$, this already leads to the nonintegrability results in Theorem 1.2 (c), since otherwise the Sobolev inequality together with $g_u \in L^t(\Omega)$ would imply that u is bounded (when $t \geq Q$) or that

$$u \in L^{Qt/(Q-t)}(\Omega) \subset L^{\tau_p}(\Omega) \quad (\text{when } t_p \leq t < Q).$$

When $t = t_p$, this is sharp by Theorem 1.2 (b). Using the Sobolev inequality from [8, Theorem 5.1] makes it possible to instead use a local t-Poincaré inequality and a local exponent set (defined similarly to the global exponent set $\overline{\Theta}$). However, considering the 1-admissible power weight $w(x) = |x|^{-\alpha}$ on $\mathbf{R}^n$, as in Example 3.1, with $0 < \alpha < n-1$ and $x_0 = 0$, shows that such a direct application of Sobolev inequalities does not lead to the results in Theorems 1.1 (f) and 1.2 (c), which use the pointwise exponent set $\overline{S}_0$ (and only assume a pointwise t-Poincaré inequality). Indeed, in this case $\bar{s}_0 = n - \alpha < n$ while the infimum of the local exponent set is n. (If $n-1 \le \alpha < n$, then $\bar{s}_0 \le 1 < p$ and the Green function u is bounded and $g_u \in L^p(\Omega)$.)

More generally, under our weaker assumptions, for the nonintegrability of g_u it is convenient to use one more exponent,

$$\hat{q} = \underline{q}_0 + 1 - \frac{\underline{q}_0}{\bar{s}_0}.$$

The reason for introducing $\hat{q}$ is that $t_p < \underline{q}_0$ if and only if $p < \hat{q}$, which will be important below. Note that $\hat{q} \le \bar{s}_0$, with equality if and only if $\underline{q}_0 = \bar{s}_0$, since $\bar{s}_0 > 1$.

Theorem 10.3. *Let u be a Green function in Ω, with singularity at x_0.*

(a) *If $p < \bar{s}_0$, then $g_u \notin L^t(\Omega)$ whenever μ supports a t-Poincaré inequality at x_0 for small radii, $t > t_p$ and $t \ge 1$.*
(b) *If $p < \hat{q}$, $t_p \ge 1$ and $\bar{s}_0 \notin \overline{S}_0 \setminus \underline{S}_0$, then $g_u \notin L^{t_p}(\Omega)$ whenever μ supports a t_p-Poincaré inequality at x_0 for small radii.*
(c) *If $\hat{q} \le p < \bar{s}_0 \notin \overline{S}_0 \setminus \underline{S}_0$ and $\underline{q}_0 > 1$, then $g_u \notin L^{t_p}(\Omega)$ whenever μ supports a t_0-Poincaré inequality at x_0 for small radii and some $1 \le t_0 < t_p$.*

Note that $\underline{q}_0 > t_p \ge 1$ in (b), while $\underline{q}_0 > 1$ needs to be assumed explicitly in (c).

Proof. By Remark 7.4, we may assume that $B_{R_1} \subset \Omega \subset B_{R_2}$, where $0 < R_1 \le R_2 < \frac{1}{4} \operatorname{diam} X$. The strong minimum principle for superharmonic functions (Theorem 9.13 in [5]) and (7.1) imply that for $0 < r < R_1/50\lambda$,

$$m_r := \inf_{B_r} u = \min_{\partial B_r} u \gtrsim \left(\frac{\mu(B_r)}{r^p} \right)^{1/(1-p)}. \tag{10.2}$$

(a) Note that $t_p < p < \bar{s}_0$. Let $q > \max\{t, \bar{s}_0\}$. Since $\min\{1, u/m_r\}$ is admissible for $\operatorname{cap}_t(B_r, \Omega)$, we obtain by [10, Proposition 8.3] that

$$\int_{\Omega \setminus B_r} g_u^t \, d\mu \ge m_r^t \operatorname{cap}_t(B_r, \Omega) \ge m_r^t \operatorname{cap}_t(B_r, B_{R_2}) \gtrsim \left(\frac{\mu(B_r)}{r^p} \right)^{t/(1-p)} r^{q-t}. \tag{10.3}$$

If $s < \bar{s}_0$, then there is a sequence $r_j \searrow 0$ such that $\mu(B_{r_j}) \lesssim r_j^s$. For this sequence, (10.3) becomes

$$\int_{\Omega \setminus B_{r_j}} g_u^t \, d\mu \gtrsim r_j^{(s-p)t/(1-p)+q-t} \to \infty, \quad \text{as } j \to 0,$$

provided that the exponent $(s-p)t/(1-p) + q - t < 0$, which is equivalent to

$$q < \frac{t(s-1)}{p-1}.$$

Clearly, for every $t > t_p$, this is satisfied for some $q > \max\{t, \bar{s}_0\}$ and $s < \bar{s}_0$, and (a) follows.

(b) Since $\bar{s}_0 \in \underline{S}_0$ or $\bar{s}_0 \notin \overline{S}_0$, there is a sequence $r_k \searrow 0$ such that $\mu(B_{r_k}) \lesssim r_k^{\bar{s}_0}$, $k = 0, 1, \ldots$. Let $B^k = B_{r_k}$ and write $a_k = m_{r_k}$. Because $\lim_{r \to 0} m_r = \infty$, we can also assume that

$$\tfrac{1}{2} a_{k+1} \ge A_k := \max_{\partial B^k} u,$$

$r_{k+1} < \frac{1}{2} r_k$ and that $r_0 < R_1/50\lambda$ is so small that the t_p-Poincaré inequality at x_0 holds for radii up to $2r_0$. This will be important below when we use the capacity estimates from [10]. Then for all $k = 1, 2, \dots$,

$$v_k = \frac{(\min\{u, a_k\} - A_{k-1})_+}{a_k - A_{k-1}}$$

is admissible for $\operatorname{cap}_{t_p}(B^k, B^{k-1})$. Moreover, $a_k - A_{k-1} \geq \frac{1}{2} a_k$. As g_u is also a t_p-weak upper gradient of u (although not necessarily minimal), we have

$$\int_{B^{k-1}\setminus B^k} g_u^{t_p}\, d\mu \geq (a_k - A_{k-1})^{t_p} \int_{B^{k-1}\setminus B^k} g_{v_k}^{t_p}\, d\mu \gtrsim a_k^{t_p} \operatorname{cap}_{t_p}(B^k, B^{k-1}). \tag{10.4}$$

Since $p < \hat{q}$, we see that $t_p < \underline{q}_0$. As μ supports a t_p-Poincaré inequality at x_0 for radii up to $2r_0$, we get by [10, Proposition 6.1], using also (10.2) and $t_p \geq p-1$, that

$$\begin{aligned}\int_{B^1} g_u^{t_p}\, d\mu &\gtrsim \sum_{k=1}^{\infty} a_k^{t_p} \operatorname{cap}_{t_p}(B^k, B^{k-1}) \gtrsim \sum_{k=1}^{\infty} \biggl(\frac{r_k^p}{\mu(B^k)}\biggr)^{t_p/(p-1)} \frac{\mu(B^k)}{r_k^{t_p}} \\ &= \sum_{k=1}^{\infty} r_k^{t_p/(p-1)} \mu(B^k)^{1-t_p/(p-1)} \gtrsim \sum_{k=1}^{\infty} r_k^{\beta},\end{aligned}$$

where

$$\beta = \frac{t_p}{p-1} + \bar{s}_0 \biggl(1 - \frac{t_p}{p-1}\biggr) = \bar{s}_0 - (\bar{s}_0 - 1)\frac{t_p}{p-1} = \bar{s}_0 - \bar{s}_0 = 0.$$

Thus the series $\sum_{k=1}^{\infty} r_k^{\beta}$ diverges and hence $g_u \notin L^{t_p}(\Omega)$.

(c) We proceed as in (b) up to (10.4), but this time assuming a t_0-Poincaré inequality at x_0 for radii up to $2r_0$. As $p \geq \hat{q}$, we see that $\underline{q}_0 \leq t_p < p$. Let

$$1 < q < \underline{q}_0 \quad \text{and} \quad \alpha = \frac{p - t_p}{p - q} = \frac{\bar{s}_0 - p}{(\bar{s}_0 - 1)(p - q)}.$$

Theorem 7.1 shows that

$$a_k \simeq \operatorname{cap}_p(B^k, \Omega)^{1/(1-p)} \geq \operatorname{cap}_p(B^k, B^{k-1})^{1/(1-p)},$$

and Corollary 4.10 (b) with $t = t_q$ then implies that

$$\begin{aligned}a_k^{t_p} \operatorname{cap}_{t_p}(B^k, B^{k-1}) &\gtrsim \operatorname{cap}_p(B^k, B^{k-1})^{t_p/(1-p)} \operatorname{cap}_{t_p}(B^k, B^{k-1}) \\ &\gtrsim \operatorname{cap}_p(B^k, B^{k-1})^{t_p/(1-p)+1-\alpha} \biggl(\frac{\mu(B^k)}{r_k^q}\biggr)^{\alpha}.\end{aligned} \tag{10.5}$$

Note that

$$\frac{t_p}{1-p} + 1 = \frac{\bar{s}_0}{1 - \bar{s}_0} + 1 = \frac{1}{1 - \bar{s}_0} < 0.$$

Hence also $t_p/(1-p) + 1 - \alpha < 0$ and [10, Proposition 5.1] gives

$$\operatorname{cap}_p(B^k, B^{k-1})^{t_p/(1-p)+1-\alpha} \gtrsim \biggl(\frac{\mu(B^k)}{r_k^p}\biggr)^{t_p/(1-p)+1-\alpha} = \biggl(\frac{\mu(B^k)}{r_k^p}\biggr)^{1/(1-\bar{s}_0)-\alpha}. \tag{10.6}$$

Since $\alpha(p-q)(1-\bar{s}_0) = p - \bar{s}_0$, we obtain from (10.5) and (10.6) that

$$\begin{aligned}a_k^{t_p} \operatorname{cap}_{t_p}(B^k, B^{k-1}) &\gtrsim \biggl(\frac{\mu(B^k)}{r_k^p}\biggr)^{1/(1-\bar{s}_0)-\alpha} \biggl(\frac{\mu(B^k)}{r_k^q}\biggr)^{\alpha} \\ &= \biggl(\frac{\mu(B^k)}{r_k^{p-\alpha(p-q)(1-\bar{s}_0)}}\biggr)^{1/(1-\bar{s}_0)} = \biggl(\frac{\mu(B^k)}{r_k^{\bar{s}_0}}\biggr)^{1/(1-\bar{s}_0)} \gtrsim 1,\end{aligned}$$

by the choice of r_k. Hence, by applying (10.4) and summing up the estimates, we conclude that $g_u \notin L^{t_p}(\Omega)$. $\square$

In the following case we get a complete characterization.

Corollary 10.4. *Let u be a Green function in Ω, with singularity at x_0. Assume that $1 < 2 - 1/\bar{s}_0 \le p \le \bar{s}_0 \notin \bar{S}_0 \setminus \underline{S}_0$ and that one of the following conditions holds:*
(a) *μ supports a t_0-Poincaré inequality at x_0 for small radii and some $t_0 < t_p$;*
(b) *$p < \hat{q}$ and μ supports a t_p-Poincaré inequality at x_0 for small radii;*
(c) *$p = \bar{s}_0$.*
Then $g_u \in L^t(\Omega)$ if and only if $0 < t < t_p$.

In particular, Corollary 10.4 applies if $1 < 2 - 1/Q \le p \le Q$ and μ is locally Ahlfors Q-regular at x_0, as in Theorem 1.2, and supports a t_p-Poincaré inequality at x_0 for small radii, since in this case $\hat{q} = \underline{q}_0 = \bar{s}_0 = Q \in \underline{S}_0$. Under the assumptions in Corollary 10.4, it follows from Corollary 9.4 that $u \in L^\tau(\Omega)$ if and only if $\tau < \tau_p$, i.e. we have a full characterization for the integrability of both u and g_u.

Proof. Since $2 - 1/\bar{s}_0 \le p \le \bar{s}_0$, we have $1 \le t_p \le p$. Parts (a) and (b) thus follow from Theorems 10.1 and 10.3 (b)–(c), while part (c) follows from Proposition 5.1 (a) and Corollary 10.2 (a). $\square$

We have now also completed the proofs of Theorems 1.1 and 1.2. More precisely, they are deduced as follows.

Proof of Theorem 1.1. Part (a) follows from Proposition 5.1 (b) and Theorem 6.6, while parts (b)–(f) follow directly from Theorems 9.1, 9.3 (a), 10.1, Corollary 10.2 (a) and Theorem 10.3 (a), respectively. $\square$

Proof of Theorem 1.2. Formula (1.4) follows from (7.3), while (a)–(c) follow from Theorem 1.1, together with Theorem 9.3 (b) for (a) and Theorem 10.3 (b) for (c). Note that in this case, $\hat{q} = \underline{q}_0 = \bar{s}_0$. $\square$

11. Radial weights

Radial weights are useful for creating examples with various properties related to the exponent sets $\underline{Q}_0$, $\underline{S}_0$, $\bar{S}_0$ and $\bar{Q}_0$, see Example 9.7, [10, Section 3], H. Svensson [56] and S. Svensson [57].

In this section we take a more detailed look at the integrability of Green functions for radial weights. Throughout this section we consider $\mathbf{R}^n$, $n \ge 2$, equipped with a radial p-admissible measure $d\mu = w(|x|)\,dx$.

We also let $x_0 = 0$ and $\Omega = B_1$, and define for $x \in B_1$,

$$u(x) = \int_{|x|}^1 \bigl(w(\rho)\rho^{n-1}\bigr)^{1/(1-p)}\,d\rho. \tag{11.1}$$

Then u is p-harmonic in $B_1 \setminus \{0\}$ and (6.1) holds, see [10, Proposition 10.8 and its proof]. Thus, u is a Green function by [11, Theorem 8.5]. The proof of Proposition 10.8 in [10] also shows that

$$g_u(x) = \bigl(w(|x|)|x|^{n-1}\bigr)^{1/(1-p)}$$

and thus

$$\begin{aligned}\int_{B_1} g_u^t\,d\mu &= \omega_{n-1}\int_0^1 \bigl(w(\rho)\rho^{n-1}\bigr)^{-t/(p-1)} w(\rho)\rho^{n-1}\,d\rho \\ &= \omega_{n-1}\int_0^1 \bigl(w(\rho)\rho^{n-1}\bigr)^{1-t/(p-1)}\,d\rho,\end{aligned}$$

where ω_{n-1} is the surface area of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}$.

In particular,

$$\int_{B_1} g_u^{t_p}\,d\mu = \omega_{n-1}\int_0^1 \bigl(w(\rho)\rho^{n-1}\bigr)^{1/(1-\bar{s}_0)}\,d\rho \tag{11.2}$$

which is independent of p. (We do not know if in other situations the integrability in the borderline case t_p can depend on p.) Note that $g_u = |\nabla u|$ a.e. in Ω, by [5, Proposition A.13], where ∇u is the gradient of u in the weighted Sobolev space $H^{1,p}_{\mathrm{loc}}(\Omega\setminus\{x_0\},w)$ as in Heinonen–Kilpeläinen–Martio [28, Section 2]. As a consequence we get the following improvement of Corollary 10.4 for radial weights, covering also the case $1<p<2-1/\bar{s}_0$, i.e. when $t_p<1$ and thus $g_u\notin L^1(\Omega,w)$.

Proposition 11.1. *Assume that $p\le \bar{s}_0\notin \bar{S}_0\setminus \underline{S}_0$, $d\mu = w(|x|)\,dx$ on $\mathbf{R}^n$ and that μ supports a q-Poincaré inequality at x_0 for small radii and some $q<\underline{q}_0$. Let u be the radially symmetric Green function as in (11.1). Then $g_u\in L^t(\Omega,w)$ if and only if $0<t<t_p$.*

Note that u of course depends on p. Moreover, the assumption that w is p-admissible may implicitly further limit the range of p in Proposition 11.1.

Proof. By continuity we find $1<p_0<\bar{s}_0$ so that $t_{p_0}=q$. Since $q<\underline{q}_0$, it follows that $p_0<\hat{q}$. Thus Theorem 10.3 (b) shows that $g_u\notin L^{t_{p_0}}(\Omega,w)$ for $p=p_0$. By (11.2), we get that $g_u\notin L^{t_p}(\Omega,w)$ is true for all $1<p\le\bar{s}_0$. The integrability for $t<t_p$ follows from Theorem 10.1. $\square$

Example 11.2. Consider the weight w as in Example 9.7, i.e.

$$w(\rho)=\begin{cases}\rho^{s-n}|\log\rho|^\beta, & \text{if } 0<\rho\le 1/e,\\ \rho^{s-n}, & \text{otherwise,}\end{cases}$$

where $s>1$ and $\beta>0$. As observed in Example 9.7, w is 1-admissible on $\mathbf{R}^n$,

$$\underline{s}_0=\underline{q}_0=\bar{s}_0=\bar{q}_0=s,\quad \underline{S}_0=\underline{Q}_0=(0,s)\quad\text{and}\quad \bar{S}_0=\bar{Q}_0=[s,\infty).$$

Then the case $t=t_p$ is not covered by Theorem 10.3, but all other exponents are covered by either Theorem 10.1 or 10.3 (a), when $p<s$. For $t=t_p$, (11.2) simplifies to

$$\int_{B_1} g_u^{t_p}\,d\mu \simeq \int_0^{1/e}\frac{d\rho}{\rho|\log\rho|^{\beta/(s-1)}},$$

which converges if and only if $\beta>s-1$. So if $0<\beta\le s-1$, then $g_u\notin L^{t_p}(\Omega,w)$, while $u\in L^{\tau_p}(\Omega,w)$ if and only if $s/(1+\beta)<p<s$, by Example 9.7. Since $t_p=p\tau_p/(\tau_p+1)$, this shows that the strict inequalities in Theorem 8.2 cannot be replaced by nonstrict ones, when $\tau_p<\infty$.

On the other hand, for $p=s$, Theorem 6.6 shows that $u\in L^{\tau_p}(\Omega,w)$ (i.e. u is bounded) if and only if $g_u\in L^p(\Omega,w)$. According to Example 9.7, this is equivalent to $\beta>s-1$.

12. General p-harmonic functions with poles

Recall the general assumptions from the beginning of Section 6.

In [11, Theorem 10.1] it was shown that any p-harmonic function in $\Omega\setminus\{x_0\}$ with a pole at x_0 has similar growth properties near the pole as Green functions with singularity at x_0. Hence the results in the previous sections can be extended to such functions, in a local sense.

Theorem 12.1. *Assume that u is a p-harmonic function in $\Omega \setminus \{x_0\}$ such that $u(x_0) := \lim_{x \to x_0} u(x) = \infty$. Then the following are true:*

(a) $C_p(\{x_0\}) = 0$;
(b) $p \le \bar{s}_0$;
(c) $u \in L^\tau_{\mathrm{loc}}(\Omega)$ *for all* $0 < \tau < \tau_p$;
(d) $u \notin L^\tau_{\mathrm{loc}}(\Omega)$ *if* $\tau > \tau_p$;
(e) *if* $\bar{s}_0 \notin \bar{S}_0 \setminus \underline{S}_0$, *then* $u \notin L^{\tau_p}_{\mathrm{loc}}(\Omega)$;
(f) $u \in L^{\tau_p}_{\mathrm{loc}}(\Omega)$ *if and only if* $p < \bar{s}_0$ *and* (9.3) *holds, or* $p = \bar{s}_0$ *and* (9.4) *holds*;
(g) $g_u \in L^t_{\mathrm{loc}}(\Omega)$ *for all* $0 < t < t_p$;
(h) *if* $p = \bar{s}_0$, *then* $g_u \in L^t_{\mathrm{loc}}(\Omega)$ *if and only if* $0 < t < p$;
(i) *if* $p < \bar{s}_0$, *then* $g_u \notin L^t_{\mathrm{loc}}(\Omega)$ *whenever* μ *supports a* t*-Poincaré inequality at* x_0 *for small radii,* $t > t_p$ *and* $t \ge 1$;
(j) *if* $p < \hat{q}$, $t_p \ge 1$ *and* $\bar{s}_0 \notin \bar{S}_0 \setminus \underline{S}_0$, *then* $g_u \notin L^{t_p}_{\mathrm{loc}}(\Omega)$ *whenever* μ *supports a* t_p*-Poincaré inequality at* x_0 *for small radii*;
(k) *if* $\hat{q} \le p < \bar{s}_0 \notin \bar{S}_0 \setminus \underline{S}_0$ *and* $\underline{q}_0 > 1$, *then* $g_u \notin L^{t_p}_{\mathrm{loc}}(\Omega)$ *whenever* μ *supports a* t_0*-Poincaré inequality at* x_0 *for small radii and some* $1 \le t_0 < t_p$.

Proof. By Theorems 1.3 (b) and 10.1 (b) in [11], there are $a > 0$, $b \in \mathbf{R}$ and a bounded domain $U \subset \Omega$ such that $x_0 \in U$ and $v := au + b$ is a Green function in U with singularity at x_0.

As u is p-harmonic in $\Omega \setminus \{x_0\}$ it is locally bounded therein and also $g_u \in L^p_{\mathrm{loc}}(\Omega \setminus \{x_0\})$. It is therefore enough to consider the integrability and nonintegrability conditions on U (since we only consider L^t_{loc}-integrability of g_u for $t \le p$). Note that $g_v = a g_u$.

(a) This follows from Theorem 10.1 (a) in [11].
(b) This follows from (a) and Proposition 5.1 (b).
(c) This follows from Theorem 9.1.
(d) and (e) These statements follow from Theorem 9.3.
(f) This follows from Theorem 9.5.
(g) This follows from Theorem 10.1.
(h) This follows from Corollary 10.2.
(i)–(k) These statements follow from Theorem 10.3. □

Theorem 12.2. *Assume that $u \ge 0$ is a p-harmonic function in $B_{50\lambda R} \setminus \{x_0\}$ such that $u(x_0) := \lim_{x \to x_0} u(x) = \infty$. Then for all $0 < r < R/50\lambda$ and $x \in S_r$,*

$$u(x) \simeq A\biggl(\inf_{B_R} u + \int_r^R \biggl(\frac{\rho}{\mu(B_\rho)}\biggr)^{1/(p-1)} d\rho\biggr), \tag{12.1}$$

where the implicit comparison constants are independent of u, while A depends on u only as follows,

$$A = \biggl(\int_{a<u<a+1} g_u^p \, d\mu\biggr)^{1/(p-1)} \quad \textit{for any } a \ge \max_{\partial B_R} u. \tag{12.2}$$

Note that $A = \operatorname{cap}_p(G_{a+1}, G_a)^{1/(p-1)}$, where $G_a = \{y \in B_R : u(y) > a\}$. The proof below shows that the integral in (12.1) can be replaced by $\operatorname{cap}_p(B_r, \Omega_0)^{1/(1-p)}$, where

$$\Omega_0 = \{x \in B_R : u(x) > \max_{\partial B_R} u\}.$$

This integral can thus be estimated by $\operatorname{cap}_p(B_r, B_R)^{1/(1-p)}$ and $\operatorname{cap}_p(B_r, B_{R_1})^{1/(1-p)}$ from above and below, respectively, whenever $B_{R_1} \subset \Omega_0$. In particular, Theorem 12.2 generalizes the estimates obtained for p-harmonic Green functions in Danielli–Garofalo–Marola [19, Lemma 5.1].

Proof of Theorem 12.2. Proposition 4.4 in [11] implies that

$$k_0 := \max_{\partial B_R} u \simeq \min_{\partial B_R} u = \inf_{B_R} u, \tag{12.3}$$

with the implicit comparison constants depending only on p, the doubling constant of μ and the constants in the Poincaré inequality. By [11, Theorem 1.6], $u - k_0$ is a singular function in $\Omega_0 = \{x \in B_R : u(x) > k_0\}$, and thus $\operatorname{cap}_p(\{x_0\}, \Omega) = C_p(\{x_0\}) = 0$, by Theorem 6.6. Hence, [11, Theorem 9.3] implies that $\tilde{u} = (u - k_0)/A$ is a Green function in Ω_0, where $A > 0$ is given by (12.2). Let $B_{R_1} \subset \Omega_0$. Applying (7.1) to $\tilde{u}$ and Ω_0, instead of u and Ω, shows that for all $0 < r < R_1/50\lambda$ and $x \in S_r$,

$$k_0 + C_1 A \operatorname{cap}_p(B_r, \Omega_0)^{1/(1-p)} \le u(x) \le k_0 + C_2 A \operatorname{cap}_p(B_r, \Omega_0)^{1/(1-p)}, \tag{12.4}$$

where $C_1, C_2 > 0$ depend only on p, the doubling constant of μ and the constants in the Poincaré inequality, but not on R_1, u, x or Ω_0.

By Theorem 6.3 in Björn [4] (or [11, Lemma 4.3]), u is superharmonic in $B_{50\lambda R}$. As u is nonconstant, [5, Corollary 9.14] shows that $X \ne B_{50\lambda R}$, and thus $50\lambda R \le \operatorname{diam} X$. Since $B_{R_1} \subset \Omega_0 \subset B_R$, we therefore conclude from Theorem 4.2 that

$$\begin{aligned} \operatorname{cap}_p(B_r, \Omega_0)^{1/(1-p)} &\le \operatorname{cap}_p(B_r, B_R)^{1/(1-p)} \simeq \int_r^R \psi(\rho)\, d\rho, \\ \operatorname{cap}_p(B_r, \Omega_0)^{1/(1-p)} &\ge \operatorname{cap}_p(B_r, B_{R_1})^{1/(1-p)} \simeq \int_r^{R_1} \psi(\rho)\, d\rho, \end{aligned} \tag{12.5}$$

where

$$\psi(\rho) = \left(\frac{\rho}{\mu(B_\rho)} \right)^{1/(p-1)}.$$

Moreover, as $\operatorname{cap}_p(\{x_0\}, \Omega_0) = 0$, Theorem 4.2 also implies that $\int_0^{R_1} \psi(\rho)\, d\rho = \infty$. Thus, there exists $r_1 < R_1/50\lambda$ such that for all $0 < r \le r_1$, the integral in (12.5) satisfies

$$\int_r^{R_1} \psi(\rho)\, d\rho = \int_r^R \psi(\rho)\, d\rho - \int_{R_1}^R \psi(\rho)\, d\rho \ge \frac{1}{2} \int_r^R \psi(\rho)\, d\rho.$$

Inserting this into (12.5) and (12.4), together with (12.3), proves (12.1) for $r \le r_1$.

In order to obtain (12.1) for $r < R/50\lambda$, let v be a Green function in B_R with singularity at x_0, and let $v = 0$ outside B_R. In particular, $v \in N^{1,p}_{\mathrm{loc}}(X \setminus \{x_0\})$. Then by (7.3), applied to v and $\Omega = B_R$, we have

$$v(x) \simeq \int_r^R \psi(\rho)\, d\rho, \quad \text{where } r = d(x, x_0) < R/50\lambda, \tag{12.6}$$

with the implicit constants depending only on p, the doubling constant of μ and the constants in the Poincaré inequality. In particular, together with the already proved (12.1) for $r \le r_1$, we have

$$Av \simeq u - k_0 \le u - m \quad \text{on } \partial B_{r_1}, \quad \text{where } m := \min_{\partial B_R} u \le k_0.$$

Since $u - k_0 \le 0 = Av \le u - m$ on ∂B_R, the comparison principle for p-harmonic functions with Sobolev boundary values ([5, Lemma 8.32]) implies that $u - k_0 \lesssim Av \lesssim u - m$ in $B_R \setminus B_{r_1}$. Thus, (12.6) and (12.3) show that (12.1) holds for all $r < R/50\lambda$. $\square$

Remark 12.3. The estimate in Theorem 12.2 is related to the so-called Wolff potential

$$W_{1,p}^{\nu}(x,R) = \int_0^R \left(\frac{\nu(B(x,\rho))}{\rho^{n-p}}\right)^{1/(p-1)} \frac{d\rho}{\rho}, \tag{12.7}$$

which was (together with another nonlinear analogue of the Riesz potential, namely the Maz′ya–Havin potential) introduced and studied in Maz′ya–Havin [51], [52]. The potential (12.7) was used in Kilpeläinen–Malý [40, Theorem 1.6] to estimate p-superharmonic functions $u \geq 0$ in $B(x,3R) \subset \mathbf{R}^n$ (unweighted) as

$$W_{1,p}^{\nu}(x,R) \lesssim u(x) \lesssim \inf_{B(x,R)} u + W_{1,p}^{\nu}(x,2R), \tag{12.8}$$

where $\nu = -\Delta_p u$ is the Riesz measure of u.

In the case of Green functions, $\nu = \delta_{x_0}$ is the Dirac measure at x_0 and hence

$$W_{1,p}^{\delta_{x_0}}(x,R) = \int_r^R \rho^{(p-n)/(p-1)-1}\, d\rho \quad \text{when } r = |x-x_0| < R,$$

which is comparable to the integral in (12.1), since the Lebesgue measure of $B(x,\rho)$ is a multiple of ρ^n. Thus, our Theorem 12.2 extends (12.8) for the Dirac measure to p-harmonic functions with poles in metric spaces, even in the case when there is no p-harmonic equation.

Proof of Theorem 1.3. This follows directly from Theorems 12.1, 12.2 and 7.1. $\square$

13. Elliptic equations in divergence form

Estimates similar to (12.8) also hold for elliptic differential equations in divergence form of the type

$$\operatorname{div} A(x,\nabla u) = 0, \tag{13.1}$$

including weighted and vectorial ones, see Kilpeläinen–Malý [39], [40], Mikkonen [53], Hara [26], the monograph Heinonen–Kilpeläinen–Martio [28, Theorem 21.21] and the expository papers Kuusi–Mingione [46], [47]. On metric spaces, such estimates have been obtained for Cheeger p-harmonic functions in Björn–MacManus–Shanmugalingam [15] and Hara [27].

In this section we explain how to extend our (non)integrability results from energy minimizers to Green functions for (13.1), i.e. functions u satisfying

$$\operatorname{div} A(x,\nabla u) = -\delta_{x_0}$$

in Ω with zero boundary values on $\partial\Omega$ (in Sobolev sense). Here, ∇u stands for one of the following gradients:

- Usual (distributional) gradient in unweighted $\mathbf{R}^n$ or the gradient defined for the weighted Sobolev space $H^{1,p}_{\mathrm{loc}}(\Omega,w)$ as in Heinonen–Kilpeläinen–Martio [28, Section 1.9].
- Natural gradient ∇u on Riemannian manifolds with nonnegative Ricci curvature as in Holopainen [31], [32] and [33].

The vector-valued function $A(x,\xi)$ is for a.e. x and all ξ assumed to satisfy the usual ellipticity conditions as in [28, (3.3)–(3.7)], Fabes–Jerison–Kenig [20] or Holopainen [32, (2.9)–(2.12)]. Subelliptic equations associated with left-invariant vector fields

$$Xu = (X_1 u, \dots, X_k u)$$

in Heisenberg or Carnot groups, as in Hajłasz–Koskela [25, Sections 11.3 and 11.4] or Capogna–Danielli–Garofalo [16], could also be considered with obvious interpretations, whenever the main ingredients, specified below, are satisfied.

For the integrability of the Green functions for $\operatorname{div} A(x, \nabla u) = 0$, it is sufficient to invoke the capacitary estimate

$$u(x) \simeq \operatorname{cap}_{p,\mu}(B_r, \Omega)^{1/(1-p)} \quad \text{for sufficiently small } r := d(x, x_0) > 0, \tag{13.2}$$

which has been proved in weighted $\mathbf{R}^n$ in [20, Lemma 3.1] (for $p = 2$) and in [28, Theorem 7.41] (for balls and $1 < p < \infty$), while on manifolds it follows from the proof of Theorem 3.19 in [31]. On unweighted $\mathbf{R}^n$ it appears in Serrin [54, Section 12]. As mentioned before, estimates for capacitary potentials similar to those from Maz′ya [50, Lemmas 3 and 4] are a useful tool for estimates like (13.2).

In view of (7.1), estimate (13.2) implies that u is in a neighbourhood of x_0 comparable to the Green function considered in this paper and thus has the same (non)integrability properties.

We can also obtain (non)integrability results for the gradient ∇u. In addition to (13.2) and our results in the previous sections, the only required tools are the minimum principle and the Caccioppoli inequality

$$\int_B |\nabla v|^p v^{-(1+\varepsilon)} \, d\mu \le C \int_{2B} v^{p-(1+\varepsilon)} \, d\mu \tag{13.3}$$

for positive supersolutions v of (13.1) in $2B$ and for $\varepsilon > 0$, with C independent of v; this will be applied to truncations of u. Such inequalities are well known in the cases considered above, see e.g. [28, Lemma 3.57] and [32, Lemma 3.1]. (In [32, Lemma 3.1], this is proved for solutions of (13.1), but the proof goes through verbatim also for supersolutions when $q = p - (1 + \varepsilon)$.)

For simplicity, we formulate and prove the next result only in weighted $\mathbf{R}^n$ with a p-admissible weight as in [28]. The case of Riemannian manifolds with nonnegative Ricci curvature (so that the doubling property and the p-Poincaré inequality hold, see Hajłasz–Koskela [25, Chapter 10.1]) is similar with obvious modifications. For the definition and existence of singular and Green functions in these settings we refer to [28, Section 7.38] and [31, Definition 3.9 and Theorem 3.19]. Since the capacitary estimate (13.2) is in [28, Theorem 7.41] proved only when Ω is a ball, we restrict ourselves to this case. (In [28, Theorem 7.41] the assumption that h is nonnegative should be added.)

Theorem 13.1. *Let w be a p-admissible weight on $\mathbf{R}^n$, $d\mu = w\,dx$ and assume that $A\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the ellipticity conditions* (3.3)–(3.7) *in* [28]. *Assume that the capacity $C_{p,\mu}(\{x_0\}) = 0$ and let $u \in H^{1,p}_{\mathrm{loc}}(B_R \setminus \{x_0\}, w)$ be a continuous weak solution of* (13.1) *in $B_R \setminus \{x_0\}$ such that*

$$u(x_0) = \lim_{x \to x_0} u(x) = \infty. \tag{13.4}$$

Assume that u has zero boundary values on ∂B_R, either in the Sobolev sense (S5) *or as the limit*

$$\lim_{\Omega \ni y \to x} u(x) = 0 \quad \textit{for all } x \in \partial B_R. \tag{13.5}$$

Then the conclusions in Section 9 hold for u in $\Omega := B_R$. Also Theorems 10.1 and 10.3 hold for $g_u = |\nabla u|$.

Remark 13.2. In fact, the assumptions (S5) and (13.5) in Theorem 13.1 are equivalent. Indeed, [28, Theorem 6.31] (applied to $B_R \setminus \overline{B}_{R/2}$) shows that (S5) implies (13.5). The converse implication follows from the uniqueness Theorem 3.12 in

Björn–Björn–Mwasa [12], together with [28, Corollary 9.29] applied to the boundary data $f := u\eta \in H^{1,p}(B_R \setminus \overline{B}_{R/2}, w)$ with a suitable cut-off function $\eta \in C_0^\infty(B_R)$.

Moreover, Harnack's inequality on spheres S_r implies that for nonnegative u the limit in (13.4) can be replaced by $\limsup$. The removability Theorem 7.35 in [28] shows that u is A-superharmonic in the ball B_R. Hence, by Theorem 2.2 in Mikkonen [53] (or [28, Theorem 21.2]), there exists a Riesz measure μ such that $\operatorname{div} A(x, \nabla u) = -\mu$. As u is A-harmonic in $B_R \setminus \{x_0\}$ the Riesz measure must be concentrated to $\{x_0\}$. Therefore a multiple of u is a Green function for (13.1), as defined in the beginning of this section.

Proof of Theorem 13.1. Proposition A.17 in [5] implies that $X = (\mathbf{R}^n, \mu)$ satisfies the general assumptions from the beginning of Section 6. Theorem 7.41 in [28] shows that (13.2), and hence also (7.1), holds. Thus u has the same (non)integrability properties near x_0 as the p-harmonic Green functions studied in this paper for bounded domains in X. Moreover, u is locally bounded in $B_R \setminus \{x_0\}$. Together with (13.5), this proves the results from Section 9 for u.

To obtain the integrability of ∇u, note that the proof of Theorem 8.2 shows that statement (a) therein holds for u, since its shifted truncations u_k in (8.1) are supersolutions and thus satisfy the Caccioppoli inequality (13.3) whenever $2B \Subset \Omega$. Combined with the integrability of u itself, we conclude that $\nabla u \in L^t_{\rm loc}(\Omega)$ when $0 < t < t_p$, as in Theorem 10.1. As $u \in H^{1,p}(B_R \setminus \overline{B}_{R/2}, w)$ by (S5), we conclude that $\nabla u \in L^t(\Omega)$ when $0 < t < t_p$.

For the nonintegrability of ∇u, we follow the proof of Theorem 10.3. The functions $\min\{1, u/m_r\}$ and v_k, considered therein, are admissible for $\operatorname{cap}_{t,\mu}(B_r, \Omega)$ and $\operatorname{cap}_{t_p,\mu}(B^k, B^{k-1})$ also in this case. The minimum principle holds for u by [28, Theorem 7.12]. Since the rest of the proof depends only on the capacity estimates (10.2) and (10.5), provided in this case by (13.2), the conclusions of Theorem 10.3 follow. □

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