DIMENSION ESTIMATES FOR THE BOUNDARY OF PLANAR SOBOLEV EXTENSION DOMAINS

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Abstract. We prove an asymptotically sharp dimension upper-bound for the boundary of bounded simply-connected planar Sobolev $W^{1,p}$-extension domains via the weak mean porosity of the boundary. The sharpness of our estimate is shown by examples.

Contents

1. Introduction

A set is porous if it has holes arbitrarily close to any point, and those holes have diameter comparable to the distance to the point. It is easy to see that porous sets in $\mathbb{R}^d$ have zero Lebesgue measure. If the porosity of the set $A \subset \mathbb{R}^d$ is stronger, in the sense that

$$\mathrm{por}(A) \coloneqq \inf_{x\in A} \liminf_{r\searrow 0} \mathrm{por}(A, x, r) > 0,$$

where we denote the maximal size of a hole of the set $A \subset \mathbb{R}^d$ at $x \in \mathbb{R}^d$ and of scale $r > 0$ by

$$\mathrm{por}(A, x, r) \coloneqq \sup\{\alpha \geq 0 : \text{there exists } y \in \mathbb{R}^d \text{ such that } B(y, \alpha r) \subset B(x, r) \setminus A\},$$

then the Hausdorff dimension of A is strictly less than d. It was shown by Mattila [8] that as $\mathrm{por}(A)$ gets closer to its maximal value $\frac{1}{2}$, the dimension upper-bound for A goes to $d-1$. The sharp asymptotic behaviour when $\mathrm{por}(A) \to \frac{1}{2}$ was then established by Salli in [11]. Later, several variants of porosity have been considered. For example, in a variant

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of porosity called k-porosity, one looks at k holes in orthogonal directions, instead of just one, see [3, 4]. For it, the dimension upper-bound approaches $d-k$ as the porosity goes to its maximal value.

In the present paper we are interested in the asymptotic behaviour of the dimension upper-bound when $\mathrm{por}(A) \to 0$. In this case, for the usual porosity defined above we have the sharp upper-bound

$$\dim_{\mathcal{H}}(A) \leq d - c\,\mathrm{por}(A)^d,$$

for some constant c depending on the dimension, see for instance [7]. However, sometimes we are in a setting where the porosity condition is not satisfied in the exact form as stated above, but almost. One such instance is the study of growth conditions on the hyperbolic metric, which imply the existence of holes only in a portion of the scales, but not all scales. Motivated by this, Koskela and Rohde introduced a version of porosity called *mean porosity* and proved a sharp dimension upper bound for mean porous sets [6] (see also the estimates by Beliaev and Smirnov [1] that deal also with a generalization of Salli's result).

Our aim in this paper is to show sharp dimension bounds for boundaries of Sobolev extension domains. For obtaining these, even the mean porosity of Koskela and Rohde is not flexible enough, because we might have many holes in a more sparse set of scales. Therefore, we use a variant of mean porosity introduced by Nieminen in [10], called *weak mean porosity* (see Section 2.1 for the definition).

Recall that a domain $\Omega \subset \mathbb{R}^d$ is called a Sobolev $W^{1,p}$-extension domain, if there exists a constant $C \in (1,\infty)$ so that for every $f \in W^{1,p}(\Omega)$ there exists $F \in W^{1,p}(\mathbb{R}^d)$ so that $F|_\Omega = f$ and $\|F\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|f\|_{W^{1,p}(\Omega)}$. When $p > 1$, the operator $f \mapsto F$ can always be assumed to be linear [2]. In [12] and [5], bounded simply-connected Sobolev extension domains $\Omega \subset \mathbb{R}^2$ were characterized by a curve condition, which for the range $1 < p < 2$ is the following: There exists a constant $C > 1$ such that for every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting z_1 and z_2 and satisfying

$$\int_\gamma \mathrm{dist}(z, \partial\Omega)^{1-p}\,\mathrm{d}s(z) \leq C\|z_1 - z_2\|^{2-p}. \tag{1.1}$$

We give an upper bound on the Hausdorff dimension $\dim_{\mathcal{H}}$ of the boundary of Ω in terms of the constant C in (1.1). This is done by showing the weak mean porosity of the boundary in Theorem 3.2 and by combining it with the dimension estimate proven by Nieminen (Theorem 2.1). The result we obtain is the following.

Theorem 1.1. *There exists a universal constant $M > 0$ such that for every bounded simply-connected domain $\Omega \subset \mathbb{R}^2$ satisfying the curve condition (1.1) with some $C \in (1,\infty)$ the following holds:*

$$\dim_{\mathcal{H}}(\partial\Omega) \leq 2 - \frac{M}{C}.$$

In Section 4, we show that Theorem 1.1 is sharp in the sense that there exists another constant $M' > 0$ so that for every $p \in (1,2)$ and $C \in (M'/(2-p),\infty)$ there exists a Jordan domain $\Omega_C \subset \mathbb{R}^2$ satisfying (1.1) with

$$\dim_{\mathcal{H}}(\partial\Omega_C) \geq 2 - \frac{M'}{(2-p)C}.$$

Notice, however, the factor $\frac{1}{2-p}$ difference between Theorem 1.1 and the examples. The curve condition (1.1) implies that $\mathbb{R}^2 \setminus \Omega$ is quasi-convex. Consequently, the domain Ω is a J-John domain [9], meaning that there exists a constant $J > 0$ and a point $x_0 \in \Omega$ so that for every $x \in \Omega$ there exists a unit speed curve $\gamma\colon [0, \ell(\gamma)] \to \Omega$ such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq Jt \qquad \text{for all } t \in [0, \ell(\gamma)]. \tag{1.2}$$

Koskela and Rohde showed that the boundary of a J-John domain $\Omega \subset \mathbb{R}^2$ has the dimension bound

$$\dim_{\mathcal{H}}(\partial\Omega) \leq 2 - cJ, \tag{1.3}$$

for some constant $c > 0$. In Section 4 we show that the bound (1.3) is also sharp.

In Section 4 we also show that from the curve condition, via the John condition and the mean porosity of Koskela and Rohde [6], it is not possible to get a better bound than

$$\dim_{\mathcal{H}}(\partial\Omega) \leq 2 - \frac{M}{((2-p)C)^{1/(2-p)}}.$$

A reason why the John condition does not give the sharper bound is that using it we consider holes only in the domain (or its complement), whereas by going from the curve condition directly to weak mean porosity, we can use holes on both sides of the boundary.

2. Preliminaries

Let us start by introducing some notation and preliminary results. By a *cube* in $\mathbb{R}^d$ we mean an open cube whose sides are parallel to the axes in $\mathbb{R}^d$. The side-length of a cube $Q \subset \mathbb{R}^d$ will be denoted by $\ell(Q)$. By a dyadic cube Q we mean that it is of the form

$$Q = (i_1 2^{-k}, (i_1+1)2^{-k}) \times (i_2 2^{-k}, (i_2+1)2^{-k}) \times \cdots \times (i_d 2^{-k}, (i_d+1)2^{-k})$$

for some $k, i_1, i_2, \ldots, i_d \in \mathbb{Z}$. We denote the set of dyadic cubes in $\mathbb{R}^d$ by $\mathcal{D}_d$. Given an open non-empty set $U \subset \mathbb{R}^d$ that is not the whole $\mathbb{R}^d$, we denote by $\mathcal{W}_U$ the *Whitney decomposition* of U, defined as

$$\mathcal{W}_U = \{Q \in \widetilde{\mathcal{W}}_U \,:\, \text{if } Q' \in \widetilde{\mathcal{W}}_U \text{ with } Q' \cap Q \neq \emptyset, \text{ then } Q' \subset Q\},$$

where

$$\widetilde{\mathcal{W}}_U = \{Q \in \mathcal{D}_d \,:\, \text{if } Q' \in \mathcal{D}_d \text{ with } \overline{Q} \cap \overline{Q}' \neq \emptyset \text{ and } \ell(Q) = \ell(Q') \text{ then } Q' \subset U\}.$$

See Figure 1 for an illustration of the Whitney decomposition. It readily follows that $\mathcal{W}_U$ is a collection of pairwise disjoint dyadic cubes Q so that $U = \bigcup_{Q \in \mathcal{W}_U} \overline{Q}$. Moreover, the following condition is satisfied by each $Q \in \mathcal{W}_U$:

$$\ell(Q) \leq \operatorname{dist}(Q, \partial U) \leq 4\operatorname{diam}(Q) = 4\sqrt{d}\,\ell(Q). \tag{2.1}$$

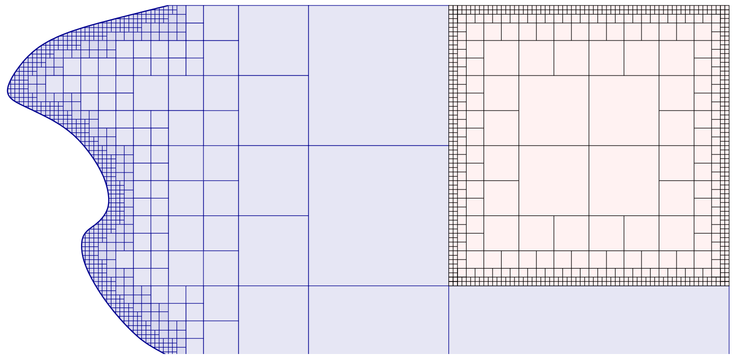

FIGURE 1. In our proof, we will use a double dyadic decomposition similar to the one used in [6]. A domain is first decomposed into its Whitney cubes. Then each Whitney cube is decomposed into its own Whitney cubes, as illustrated here only for the largest cube in the first decomposition.

Moreover, if $Q, Q' \in \mathcal{W}_U$ with $\overline{Q} \cap \overline{Q}' \neq \emptyset$, then

$$\frac{1}{2} \leq \frac{\ell(Q)}{\ell(Q')} \leq 2. \tag{2.2}$$

In the specific case where we take the Whitney decomposition of a dyadic cube $Q \in \mathcal{D}_d$, we have

$$\mathcal{W}_Q = \{Q' \subset Q \,:\, Q' \text{ dyadic cube with } \ell(Q') = \operatorname{dist}(Q', \partial Q)\}\,.$$

See again Figure 1 for an illustration. It is then easy to check that

$$\#\left\{Q' \in \mathcal{W}_Q :\, \ell(Q') = 2^{-j}\ell(Q)\right\} \geq 2^{(j-1)(d-1)} \quad \text{holds for every } j \geq 2. \tag{2.3}$$

Given any ball $B \subset \mathbb{R}^d$ and any $r > 0$, we denote by rB the ball having the same center as B and the radius r times that of B. The ball of radius $r > 0$, centered in $x \in \mathbb{R}^d$ is denoted by $B(x, r)$, while by $B(E, r)$ we denote the r-neighbourhood of a given set $E \subset \mathbb{R}^d$.

Recall that the *Hausdorff dimension* of a set $E \subset \mathbb{R}^d$ is defined by

$$\dim_{\mathcal{H}}(E) \coloneqq \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{s > 0 : \mathcal{H}^s(E) = +\infty\},$$

where $\mathcal{H}^s$ stands for s-dimensional Hausdorff measure in $\mathbb{R}^d$.

2.1. **Weakly mean porous sets.** In the present subsection, we recall the concept of weak mean porosity introduced in [10]. The weak mean porosity is a variant of mean porosity introduced in [6].

Let $E \subset \mathbb{R}^d$ be a compact set. Let $\alpha\colon]0,1[\to]0,1[$ be a continuous function such that $\alpha(t)/t$ is increasing in t, and let $\lambda\colon \mathbb{Z}^+ \to \mathbb{R}$ be a function. Let $\mathcal{D}$ be a disjointed collection of open cubes in $\mathbb{R}^d \setminus E$. Define

$$\chi_k^{\mathcal{D}}(x) = \begin{cases} 1, & \text{if there exist at least } \lambda(k) \text{ cubes } Q \in \mathcal{D} \text{ with } Q \subset A_k(x) \text{ and } \ell(Q) \geq \alpha(2^{-k}), \\ 0, & \text{otherwise,} \end{cases}$$

where $A_k(x) := B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$. Let

$$S_j^{\mathcal{D}}(x) = \sum_{k=1}^{j} \chi_k^{\mathcal{D}}(x).$$

We say that E is *weakly mean porous with parameters* (α, λ), if there exists a collection $\mathcal{D}$ and $j_0 \in \mathbb{Z}^+$ such that

$$\frac{S_j^{\mathcal{D}}(x)}{j} > \frac{1}{2}$$

for all $x \in E$ and for all $j \geq j_0$.

We will apply weak mean porosity in the case

$$\lambda(k) = c\varepsilon^{-1} \qquad \text{and} \qquad \alpha(t) = \varepsilon t, \tag{2.4}$$

for some $\varepsilon \in]0, 1[$ and a fixed constant $c > 0$. In this case, we have the following dimension estimate as a direct corollary of [10, Theorem 3.3].

Theorem 2.1. *There exists a constant $C(d, c) > 0$ such that any weakly mean porous set $E \subset \mathbb{R}^d$ with parameters (α, λ) defined in (2.4) satisfies*

$$\dim_{\mathcal{H}}(E) \leq d - C(d, c)\varepsilon^{d-1}.$$

3. Weak mean porosity of the boundary of Sobolev extension domains

In this section we will show that the boundary of a planar bounded simply-connected $W^{1,p}$-extension domain (with $1 < p < 2$) is weakly mean porous with the parameters depending on the constant C appearing in the *curve condition* (3.1) that characterises $W^{1,p}$-extension domains (cf. Theorem 3.1 below).

The following result has been proven in [5]:

Theorem 3.1. *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. Then, Ω is a $W^{1,p}$-extension domain if and only if there exists a constant $C = C(\Omega, p) > 0$ such that every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be joined by a rectifiable curve $\gamma \in \mathbb{R}^2 \setminus \Omega$ satisfying*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq C \|z_1 - z_2\|^{2-p}. \tag{3.1}$$

Now we are ready to state our main result.

Theorem 3.2. *There exist universal constants $C', C'' > 0$ so that the following holds. Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected $W^{1,p}$-extension domain. Let C be the constant from the curve condition (3.1). Then, $\partial\Omega$ is weakly mean porous with parameters (α, λ), where $\lambda(k) = C'C$ and $\alpha(t) = \frac{C''}{C}t$.*

In the proof of Theorem 3.2, we use the following result to relate the length of the curve γ in (3.1) to the diameter of cubes it intersects.

Lemma 3.3. *Let $1 < p < 2$, let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected $W^{1,p}$-extension domain and let $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$. Then there exists a curve γ connecting z_1 and z_2 in $\mathbb{R}^2 \setminus \Omega$ that minimizes*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \tag{3.2}$$

and satisfies

$$\mathcal{H}^1(\gamma \cap \overline{Q}) \leq 10\, \ell(Q)$$

for every $Q \in \mathcal{W}_{\mathbb{R}^2 \setminus \overline{\Omega}}$.

Proof. The existence of a minimizer for (3.2) is standard and has been established in the proof of [5, Lemma 2.17]. Let $Q \in \mathcal{W}_{\mathbb{R}^2 \setminus \overline{\Omega}}$. Define $t_1 = \min\{t : \gamma(t) \in \overline{Q}\}$ and $t_2 = \max\{t : \gamma(t) \in \overline{Q}\}$. Then, by (2.1) and the minimality of γ,

$$\begin{aligned}
\mathcal{H}^1(\gamma \cap \overline{Q}) \left(5\sqrt{2}\ell(Q)\right)^{1-p} &\leq \mathcal{H}^1(\gamma \cap \overline{Q})\,(\operatorname{dist}(Q, \partial\Omega) + \operatorname{diam}(Q))^{1-p} \\
&\leq \int_{\gamma \cap \overline{Q}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \\
&\leq \int_{[\gamma(t_1), \gamma(t_2)]} \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \\
&\leq \operatorname{diam}(Q) \operatorname{dist}(Q, \partial\Omega)^{1-p} \leq \sqrt{2}\ell(Q)^{2-p}.
\end{aligned}$$

Thus, the claim holds. □

Proof of Theorem 3.2. Without loss of generality, we may assume that $C \geq 1$. Let $\varepsilon := 2^{-m} \in (2^{-15}/C, 2^{-14}/C]$ with $m \in \mathbb{Z}$. We start by constructing the collection $\mathcal{D}$ of cubes in $\mathbb{R}^2 \setminus \partial\Omega$. We decompose every $Q \in \mathcal{W}_{\mathbb{R}^2 \setminus \partial\Omega}$ into $\mathcal{W}_Q$ and enumerate $\mathcal{W}_Q = \{Q_i(Q)\}_{i \in \mathbb{N}}$. We will show that the family

$$\mathcal{D} := \left\{Q_i(Q) : i \in \mathbb{N}, Q \in \mathcal{W}_{\mathbb{R}^2 \setminus \partial\Omega}\right\}$$

gives the claimed weak mean porosity of $\partial\Omega$ with the functions $\lambda(k) = \varepsilon^{-1} 2^{-10}$ and $\alpha(t) = \varepsilon t$.

Let k_0 be the smallest positive integer for which $2^{-k_0} < \operatorname{diam}(\Omega)$. It suffices to show that $\chi_k^{\mathcal{P}}(x) = 1$ for all $k \geq k_0$ and $x \in \partial\Omega$. Let us fix $k \in \mathbb{N}$, with $k \geq k_0$, and $x \in \partial\Omega$.

Case 1: First, let us suppose that the following condition holds true:

$$\text{For every } r \in \left[\frac{2}{3}2^{-k}, \frac{5}{6}2^{-k}\right] \text{ there exists } y \in \partial B(x, r) \text{ so that } B(y, \varepsilon 2^{-k+5}) \cap \partial\Omega = \emptyset. \tag{3.3}$$

Consider the set of radii

$$R := \left\{r : r = \frac{2}{3}2^{-k} + \varepsilon 2^{-k+6} i \leq \frac{5}{6}2^{-k}, i \in \mathbb{N}\right\}.$$

For each $r \in R$ we select a point $y_r \in \partial B(x,r)$ so that $B(y_r, \varepsilon 2^{-k+5}) \cap \partial\Omega = \emptyset$, as given by (3.3). Now, given any $r \in R$, the set $B(y_r, \varepsilon 2^{-k+5}) \subset \mathbb{R}^2 \setminus \partial\Omega$ contains a dyadic square Q of sidelength $\varepsilon 2^{-k+2}$ with distance at least $\varepsilon 2^{-k+2}$ to $\partial\Omega$. Thus, $\partial B(x,r) \cap Q \neq \emptyset$ for some $Q \in \mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}$ with $\ell(Q) \geq \varepsilon 2^{-k+2}$. Since $x \in \partial\Omega$, $\operatorname{diam}(\Omega) > 2^{-k}$ and Ω is bounded and simply-connected, we have

$$\partial B(x,r) \cap \partial\Omega \neq \emptyset,$$

and so also arbitrarily small cubes in $\mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}$ intersect $\partial B(x,r)$. Consequently, taking into account (2.2) there exists $Q_r \in \mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}$ with $\ell(Q_r) = \varepsilon 2^{-k+2}$ and

$$\partial B(x,r) \cap Q_r \neq \emptyset.$$

By the bound (2.3), there exists $Q_r' \in \mathcal{W}_{Q_r} \subset \mathcal{D}$ with $\ell(Q_r') = \varepsilon 2^{-k}$. Then the collection of cubes $\{Q_r' : r \in R\} \subset \mathcal{D}$ is disjointed. A simple calculation shows that we have $\#R \geq 2^{-9}/\varepsilon$. Thus, $\chi_k^{\mathcal{D}}(x) = 1$.

Case 2: If the condition (3.3) is violated, we argue as follows: Pick $r \in \left(\frac{2}{3}\,2^{-k}, \frac{5}{6}\,2^{-k}\right)$ such that for every $y \in \partial B(x,r)$ it holds that $B(y, \varepsilon 2^{-k+5}) \cap \partial\Omega \neq \emptyset$. Let $\{y_i\}_{i=1}^m$ be a maximal $\varepsilon 2^{-k+5}$-separated net of points in $\partial B(x,r)$ enumerated in a clockwise order around x. Since $B(y_i, \varepsilon 2^{-k+5}) \cap \partial\Omega \neq \emptyset$, we can select, for each i a point $w_i \in B(y_i, \varepsilon 2^{-k+5}) \setminus \Omega$. Let us denote $w_{m+1} = w_1$. We claim that for some $i \in \{1, \ldots, m\}$

$$\text{any curve connecting } w_i \text{ to } w_{i+1} \text{ in } \mathbb{R}^2 \setminus \Omega \text{ must exit } B(w_i, 2^{-k-3}). \tag{3.4}$$

Suppose this is not the case. Then we can connect w_i to w_{i+1} by a curve σ_i in $B(w_i, 2^{-k-3}) \setminus \Omega$. The concatenation σ of $\sigma_1, \ldots, \sigma_m$ is then contained in the annulus

$$B(x, r + 2^{-k-3}) \setminus B(x, r - 2^{-k-3}) \subset B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$$

and has winding number -1 around x. However, since $x \in \partial\Omega$ and $\Omega \setminus B(x, 2^{-k}) \neq \emptyset$, the curve σ then disconnects Ω, which is impossible. Thus, we have the existence of i for which (3.4) holds.

Let $\gamma\colon [0,1] \to \mathbb{R}^2 \setminus \Omega$ be a curve connecting $z_1 \coloneqq w_i$ and $z_2 \coloneqq w_{i+1}$ which minimizes (3.2). Call $A \coloneqq \{z \in \gamma : \operatorname{dist}(z, \partial\Omega) > 5\sqrt{2}\varepsilon 2^{-k+2}\}$ and note that (3.1) yields

$$\begin{aligned}(5\sqrt{2}\varepsilon 2^{-k+2})^{1-p}\,\mathcal{H}^1(\gamma \setminus A) &\leq \int_{\gamma\setminus A} \operatorname{dist}(z, \partial\Omega)^{1-p}\,\mathrm{d}s(z) \\ &\leq \int_{\gamma} \operatorname{dist}(z, \partial\Omega)^{1-p}\,\mathrm{d}s(z) \leq C(\varepsilon 2^{-k+7})^{2-p}.\end{aligned}$$

Consequently, by the choice of ε, we have that

$$\mathcal{H}^1(\gamma \setminus A) \leq 2^{5(2-p)+2}(5\sqrt{2})^{p-1}\varepsilon C 2^{-k} \leq 2^{10}\varepsilon C 2^{-k} \leq 2^{-k-4}$$

and hence

$$\mathcal{H}^1(A \cap B(w_i, 2^{-k-3})) = \mathcal{H}^1(\gamma \cap B(w_i, 2^{-k-3})) - \mathcal{H}^1(\gamma \setminus A) \geq 2^{-k-3} - 2^{-k-4} \geq 2^{-k-4}. \tag{3.5}$$

Now, notice that by the choice of the radius r, the point w_i and the factor ε, we get

$$\operatorname{dist}(\mathbb{R}^2 \setminus A_k(x), B(w_i, 2^{-k-3})) \geq \frac{1}{6}2^{-k} - \varepsilon 2^{-k+5} - 2^{-k-3} \geq 2^{-k-6}. \tag{3.6}$$

Write

$$\mathcal{Q} \coloneqq \{Q \in \mathcal{W}_{\mathbb{R}^2\setminus\overline{\Omega}} \,:\, \ell(Q) \geq \varepsilon 2^{-k+2} \text{ and } Q \cap B(w_i, 2^{-k-3}) \neq \emptyset\}.$$

Suppose first that there exists $Q \in \mathcal{Q}$ with $\ell(Q) \geq 2^{-k-7}$. Then, by the definition of the decomposition $\mathcal{W}_Q$ and by (3.6) a square $Q' \in \mathcal{W}_Q$ with $\ell(Q') = \varepsilon 2^{-k}$ that is closest to w_i satisfies

$$\begin{aligned}\operatorname{dist}(\mathbb{R}^2 \setminus A_k(x), Q') &\geq \operatorname{dist}(\mathbb{R}^2 \setminus A_k(x), B(w_i, 2^{-k-3})) - \sqrt{2}\operatorname{dist}(Q', \partial Q) - \operatorname{diam}(Q')\\ &\geq 2^{-k-6} - \sqrt{2}\ell(Q') - \sqrt{2}\ell(Q') \geq 2^{-k-6} - \varepsilon 2^{-k+2}.\end{aligned}$$

Therefore, by counting the consecutive squares of side-length $\varepsilon 2^{-k}$ in $\mathcal{W}_Q$ starting from this square, we obtain the estimate

$$\#\left\{Q' \in \mathcal{D} \,:\, Q' \in \mathcal{W}_Q, Q' \subset A_k(x) \text{ and } \ell(Q') = \varepsilon 2^{-k}\right\} \geq \frac{2^{-k-7}}{\varepsilon 2^{-k}} \geq \frac{2^{-7}}{\varepsilon}$$

and thus, $\chi_k^{\mathcal{D}}(x) = 1$.

Suppose then that for all $Q \in \mathcal{Q}$ we have $\ell(Q) \leq 2^{-k-7}$. Then, by (3.6) for all $Q \in \mathcal{Q}$ we have $Q \subset A_k(x)$. Notice that by (2.1), A is contained in the closure of the union of Whitney cubes $Q \in \mathcal{W}_{\mathbb{R}^2\setminus\overline{\Omega}}$ with $\ell(Q) \geq \varepsilon 2^{-k+2}$ and that $\mathcal{H}^1$-almost every point in $\mathbb{R}^2$ is contained in the closure of at most two $Q \in \mathcal{Q}$. Therefore, by using Lemma 3.3 and (3.5) we get

$$\sum_{Q\in\mathcal{Q}} \ell(Q) \geq \frac{1}{10}\sum_{Q\in\mathcal{Q}} \mathcal{H}^1(\gamma \cap \overline{Q}) \geq \frac{1}{20}\mathcal{H}^1(A \cap B(w_i, 2^{-k-3})) \geq 2^{-k-9}.$$

So, by (2.3)

$$\#\left\{Q' \in \mathcal{D} \,:\, Q' \subset A_k(x) \text{ and } \ell(Q') = \varepsilon 2^{-k}\right\} \geq \sum_{Q\in\mathcal{Q}} \frac{\ell(Q)}{\varepsilon 2^{-k+1}} \geq \varepsilon^{-1}2^{-10}.$$

Again, $\chi_k^{\mathcal{D}}(x) = 1$, concluding the proof. □

4. Examples

In this section we show the sharpness of our estimate between the constant in the curve condition and the dimension of the boundary. We also show that the dimension estimate via the John condition is necessarily less sharp. Let us write the conclusions from the two sets of examples we consider in the following theorem.

Theorem 4.1. *The following sets exist.*

(1) For every $J \in (0, 1/2)$ there exists a Jordan J-John domain $\Omega \subset \mathbb{R}^2$ for which

$$\dim_{\mathcal{H}}(\partial\Omega) \geq 2 - \frac{2}{\log(2)}J.$$

(2) *For every $p \in (1,2)$ and $C \in (72/(2-p),\infty)$ there exists a Jordan domain $\Omega \subset \mathbb{R}^2$ satisfying the curve condition (3.1) with the constant C and exponent p, for which*

$$\dim_{\mathcal{H}}(\partial\Omega) \geq 2 - \frac{24}{\log(2)(2-p)C}.$$

(3) *There exists a universal constant $c > 0$ so that for every $p \in (1,2)$ and $C \in (c,\infty)$ there exists a Jordan domain $\Omega \subset \mathbb{R}^2$ satisfying the curve condition (3.1) with the constant C, but failing to be J-John domain for any*

$$J \geq c\,((2-p)C)^{\frac{1}{p-2}}.$$

Recall that the quasi-convexity of the complement of a domain $\Omega \subset \mathbb{R}^2$, and thus in particular the curve condition (3.1), implies that Ω is John, [9]. However, the curve condition (3.1) does not imply that the complementary open set $\mathbb{R}^2 \setminus \overline{\Omega}$ would be even connected. In particular, the complementary domain does not have to be a John domain in the Jordan domain case.

In the rest of the section we prove the existence of the sets mentioned in Theorem 4.1.

4.1. **Cones.** The first set of examples shows the claim (3) in Theorem 4.1. We consider a fixed square and on top of it attach a cone whose width is the parameter ε that we vary in order to change the constants in the curve condition (3.1) and the John condition.

Example 4.2. Let $\varepsilon \in (0,1/2)$ and $1 < p < 2$. Let

$$\Omega := \{(x^1,x^2) : |x^1| < 1, |x^2+1| < 1\} \cup \{(x^1,x^2) : |x^1| < (1-x^2)\varepsilon, x^2 \geq 0\} \subset \mathbb{R}^2.$$

Then the following hold.

(i) For $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ the curve condition (3.1) holds with constant $C = \frac{c}{2-p}\varepsilon^{p-2}$ with some constant $c > 0$ independent of ε.

(ii) The set Ω fails to be J-John for any $J > \varepsilon$.

Proof of (i). Notice first that for $\Omega' := \Omega \cup (0,1) \times (0,1)$ there exists a constant $C > 0$ independent of ε so that Ω' satisfies (3.1) with this C. Write $z_i = (z_i^1, z_i^2)$. Thus, we may assume that $-1 \leq z_1^1 \leq 0 \leq z_2^1 \leq 1$ and $0 \leq z_1^2, z_2^2 \leq 1$.

Let us define $w_1 = (z_1^1 + z_1^2 - 1, 1)$ and $w_2 = (z_2^1 - z_2^2 + 1, 1)$. We claim that the concatenation γ of the line-segments $[z_1, w_1]$, $[w_1, w_2]$ and $[w_2, z_2]$ satisfies the curve condition with the claimed constant. See Figure 2 for an illustration of the curve. For the lengths of the line-segments we have the estimates

$$\|w_i - z_i\| = \sqrt{2}|z_i^2 - 1| \leq \frac{\sqrt{2}}{\varepsilon}|z_i^1| \leq \frac{\sqrt{2}}{\varepsilon}\|z_1 - z_2\|$$

and

$$\begin{aligned}\|w_1 - w_2\| &= |(z_1^1 + z_1^2 - 1) - (z_2^1 - z_2^2 + 1)| \\ &\leq |z_1^2 - 1| + |z_2^2 - 1| + |z_1^1 - z_2^1| \leq \frac{3}{\varepsilon}\|z_1 - z_2\|.\end{aligned}$$

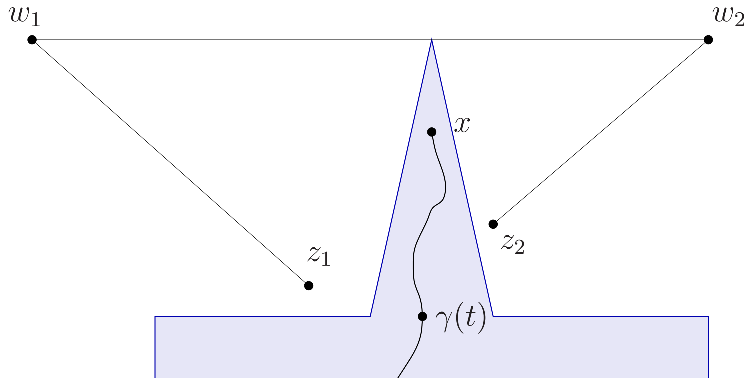

FIGURE 2. The failure of the John condition for $J > \varepsilon$ in Example 4.2 is seen by taking the point x near the tip of the cone. Then every curve γ connecting x to a John center will fail the condition at a point $\gamma(t)$. The critical case for the curve condition (3.1) is the case where z_1 and z_2 are on the opposite sides of the cone. Up to a constant, an optimal way to connect them goes through the points w_1 and w_2.

Thus, we get

$$\int_{[z_i,w_i]} \mathrm{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}s(z) \le \int_0^{\frac{\sqrt{2}}{\varepsilon}\|z_1-z_2\|} \left(\frac{t}{\sqrt{2}}\right)^{1-p} \mathrm{d}t = \frac{2^{3/2-p}}{2-p}\varepsilon^{2-p}\|z_1-z_2\|^{2-p}$$

and

$$\int_{[w_1,w_2]} \mathrm{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}s(z) \le 2\int_0^{\frac{3}{\varepsilon}\|z_1-z_2\|} \left(\frac{t}{\sqrt{2}}\right)^{1-p} \mathrm{d}t = \frac{2^{(3-p)/2}3^{2-p}}{2-p}\varepsilon^{2-p}\|z_1-z_2\|^{2-p}.$$

Combining the above estimates, the claim is proven. □

Proof of (ii). Figure 2 shows the idea of the proof. Suppose Ω is a J-John domain with the John center $x_0 = (x_0^1, x_0^2) \in \Omega$. For $x_0^2 < x^2 < 1$, consider a John curve $\gamma\colon [0,\ell(\gamma)] \to \Omega$ from $(0,x^2)$ to (x_0^1, x_0^2). Let $t \in [0,\ell(\gamma)]$ be such that $\gamma(t) \in \mathbb{R} \times \{\max(0,x_0^2)\}$. Then,

$$Jt \le \mathrm{dist}(\gamma(t),\partial\Omega) \le \varepsilon\min(1-x_0^2,1) \le \varepsilon\min\left(\frac{1-x_0^2}{x^2-x_0^2}, \frac{1}{x^2-x_0^2}\right)t \le \varepsilon\frac{1-x_0^2}{x^2-x_0^2}t.$$

Thus, by letting $x^2 \nearrow 1$, we see that $J \le \varepsilon$. □

4.2. **Koch snowflakes.** The second set of examples showing the claims (1) and (2) in Theorem 4.1 is the von Koch snowflake with varying contraction constant λ as the parameter.

Example 4.3. Let us first recall the construction of the von Koch curve K with parameter $\lambda \in [1/3, 1/2)$. It is defined as the attractor of iterated function system $\{F_1, F_2, F_3, F_4\}$, where $F_1, \ldots, F_4$ are the similitude mappings

$$F_1x = Sx, \qquad F_2x = T_{(\lambda,0)}R_\theta Sx, \qquad F_3x = T_{(1/2,h)}R_{-\theta}Sx, \qquad F_4x = T_{(1-\lambda,0)}Sx.$$

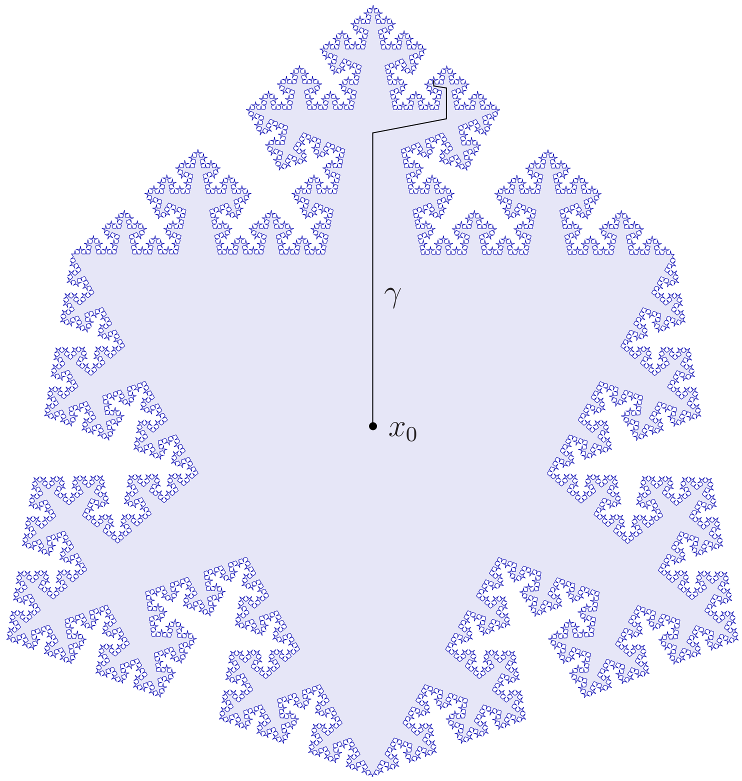

FIGURE 3. An illustration of the domain Ω bounded by three copies of a von Koch curve K together with the John center x_0 and a John curve γ.

Here $Sx = \lambda x$ is the scaling by λ, R_τ is the rotation of the plane by the angle τ, the used rotation angle θ here is defined by $\cos\theta = (\frac{1}{2} - \lambda)/\lambda$, T_a is the translation $T_a x = x + a$, and $h = \sqrt{\lambda - 1/4}$. Recall that K being the attractor means that it is the unique non-empty compact set satisfying

$$K = \bigcup_{i=1}^{4} F_i(K).$$

We define our domain Ω to be a snowflake domain whose boundary consists of three copies of K, see Figure 3. More precisely, Ω is the bounded component of the set

$$\mathbb{R}^2 \setminus \bigcup_{i=1}^{3} G_i(K),$$

where

$$G_1 x = x, \qquad G_2 x = T_{(1,0)} R_{-\frac{2\pi}{3}} x, \qquad G_3 x = T_{(1/2, -\sqrt{3}/2)} R_{\frac{2\pi}{3}} x.$$

Since the iterated function system defining K satisfies the open set condition, the Hausdorff dimension agrees with the similarity dimension, which gives

$$\dim_{\mathcal{H}}(\partial\Omega) = -\frac{\log(4)}{\log(\lambda)} \geq 2 - \frac{4}{\log(2)}\left(\frac{1}{2} - \lambda\right).$$

We claim that the following hold.

(i) The domain Ω is $\frac{\frac{1}{2}-\lambda}{\lambda}$-John, which is also optimal.
(ii) The domain Ω satisfies the curve condition (3.1) with $C = \frac{6\lambda^{2p-3}}{(2-p)(1/2-\lambda)}$.

Before proving the claims, let us introduce some additional notation for the Koch snowflake. For $k \in \{0, 1, \dots\}$, and a word $a_0 a_1 \dots a_k \in \{1, 2, 3\} \times \{1, 2, 3, 4\}^k$, we define the composed mapping

$$F_{a_0 \dots a_k} \coloneqq G_{a_0} \circ F_{a_1} \circ \cdots \circ F_{a_k}.$$

Now, we set $K_{a_0\dots a_k} \coloneqq F_{a_0\dots a_k}(K)$. Similarly, by defining $L \coloneqq [0, 1] \times \{0\}$, we set $L_{a_0\dots a_k} \coloneqq F_{a_0\dots a_k}(L)$. We also fix the following notation

$$\Delta_{a_0\dots a_k} = \mathrm{ch}(L_{a_0\dots a_k 2} \cup L_{a_0\dots a_k 3})$$
$$T_{a_0\dots a_k} = L_{a_0\dots a_k 2} \cap L_{a_0\dots a_k 3},$$

where $\mathrm{ch}(A)$ denotes the convex hull of set A.

Proof of (i). Let us first show that Ω cannot be John with a constant better than $\frac{\frac{1}{2}-\lambda}{\lambda}$. The proof is similar to the proof of (ii) in Example 4.2. Suppose that Ω is J-John with $x_0 \in \Omega$ the John center. Let $k \in \mathbb{N}$ be such that

$$x_0 \notin \mathrm{ch}(L_{12a_1\dots a_k} \cup L_{13b_1\dots b_k}) =: \Delta,$$

where $a_j = 4$ and $b_j = 1$ for all $1 \leq j \leq k$. Notice that the triangle Δ is similar to Δ_1 with both having the same top vertex $T = T_1$. Let γ be a unit speed curve connecting T to x_0 in $\Omega \cup \{T\}$. Let $x \in \partial\Omega \cap (L_{12a_1\dots a_k 4} \cup L_{13b_1\dots b_k 1})$ and $t \in [0, \ell(\gamma)]$ be such that $\mathrm{dist}(\gamma(t), \partial\Omega) = \|\gamma(t) - x\| > 0$. Then,

$$\frac{t}{\mathrm{dist}(\gamma(t), \partial\Omega)} \geq \frac{\|T - \gamma(t)\|}{\|\gamma(t) - x\|} \geq \frac{\lambda}{\frac{1}{2} - \lambda}.$$

Therefore, $J \leq \frac{\frac{1}{2}-\lambda}{\lambda}$.

Let us then show that Ω is $\frac{\frac{1}{2}-\lambda}{\lambda}$-John. Let x_0 be the barycenter of Ω, and let $x_1 \in \Omega$ be the point connected to x_0 with γ. Figure 3 shows the idea behind the following construction of the John curve γ. In the case $x_1 \in \Delta_0 \coloneqq \mathrm{ch}(L_1 \cup L_2 \cup L_3)$ the claim is clear. Assume that $x_1 \in \Delta_{a_0\dots a_k}$, $k \geq 0$, $a_0 \in \{1, 2, 3\}$, $a_j \in \{1, 2, 3, 4\}, 1 \leq j \leq k$. Let $P_{a_1\dots a_k} \in \Omega$ be the point on the line bisecting $\Delta_{a_0\dots a_k}$ through $T_{a_0\dots a_k}$, such that $\|T_{a_0\dots a_k} - P_{a_1\dots a_k}\| = \frac{\lambda^{k+1}}{2h}$, where $h = \sqrt{\lambda - 1/4}$. Now the line segment $[x_1, P_{a_0\dots a_k}]$ has length at most $\frac{\lambda^{k+1}}{2h}$ and (1.2) holds for all $x \in [x_1, P_{a_1\dots a_k}]$ with $J = \frac{\frac{1}{2}-\lambda}{\lambda}$.

By symmetry and self-similarity, the points $P_{a_0}, P_{a_0a_1}, \dots, P_{a_0a_1\dots a_k}$, where $a_0 \in \{1,2,3\}$ and $a_1,\dots,a_k \in \{1,2,3,4\}$, have the following properties:

For $x = tP_{a_0\dots a_m} + (1-t)P_{a_0\dots a_{m+1}}$, $t \in [0,1]$ and $m \geq 0$

$$\ell([P_{a_0\dots a_{m+1}}, x]) = t\frac{\lambda^{m+1}(1-\lambda)}{2h}$$

and

$$\operatorname{dist}(\partial\Omega, P_{a_0\dots a_m}) \geq \frac{(\frac{1}{2}-\lambda)\lambda^m}{2h},$$

which by the construction of Ω gives

$$\begin{aligned}\operatorname{dist}(\partial\Omega, x) &\geq (1-t)\frac{(\frac{1}{2}-\lambda)\lambda^{m+1}}{2h} + t\frac{(\frac{1}{2}-\lambda)\lambda^m}{2h}\\ &= [(1-t)\lambda + t]\frac{(\frac{1}{2}-\lambda)\lambda^m}{2h}.\end{aligned}$$

Therefore, for all $1 \leq m \leq k-1$ and $x \in [P_{a_0\dots a_{m+1}}, P_{a_0\dots a_m}]$

$$\begin{aligned}\ell(\gamma|_{x_1\to x}) &= \ell([x_1, P_{a_0\dots a_k}]) + \sum_{j=m+2}^{k} \ell([P_{a_1\dots a_{j-1}}, P_{a_1\dots a_j}]) + \ell([P_{a_1\dots a_{m+1}}, x])\\ &\leq \frac{\lambda^{k+1}}{2h} + \sum_{j=m+2}^{k}\frac{\lambda^j(1-\lambda)}{2h} + t\frac{\lambda^{m+1}(1-\lambda)}{2h}\\ &= [\lambda(1-t)+t]\frac{\lambda^{m+1}}{2h}\\ &\leq \frac{\lambda}{\frac{1}{2}-\lambda}\operatorname{dist}(\partial\Omega, x),\end{aligned}$$

where $\gamma|_{x_1\to x}$ denotes curve made of the line segments

$$[x_1, P_{a_0\dots a_k}], [P_{a_0\dots a_k}, P_{a_1\dots a_{k-1}}], \dots, [P_{a_0\dots a_{m+2}}, P_{a_0\dots a_{m+1}}], [P_{a_0\dots a_{m+1}}, x].$$

So (1.2) holds for all $x \in \gamma|_{x_1\to P_{a_1}}$ with $J = \frac{\frac{1}{2}-\lambda}{\lambda}$ and (1.2) still holds (with the same constant) when $\gamma|_{x_1\to P_{a_1}}$ is extended to x_0 with $[P_{a_1}, x_0]$. □

Proof of (ii). We will show that any two points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be connected by a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying (3.1) with $C = \frac{9\lambda^{3p-7}}{(2-p)(\frac{1}{2}-\lambda)}$. First of all, we may assume without loss of generality that $z_1, z_2 \in \partial\Omega$. Secondly, we may assume that $z_1, z_2 \in K_1$. We now divide the proof into three cases, them being case 1: $z_1 \in K_{11}$, $z_2 \in K_{12}$, case 2: $z_1 \in K_{12}$, $z_2 \in K_{13}$, and case 3: $z_1 \in K_{11}$, $z_2 \in K_{13} \cup K_{14}$. Other cases follow then by symmetry, and from self-similarity by zooming in to the construction. We treat only the case 1 in detail, giving the ideas for the other two.

Case 1: $z_1 \in K_{11}$ and $z_2 \in K_{12}$.
Let us call z_1', z_2' the orthogonal projections of z_1 and z_2 on the line-segment

$$I := T_{(\lambda,0)}R_{(\pi-\theta)/2}L$$

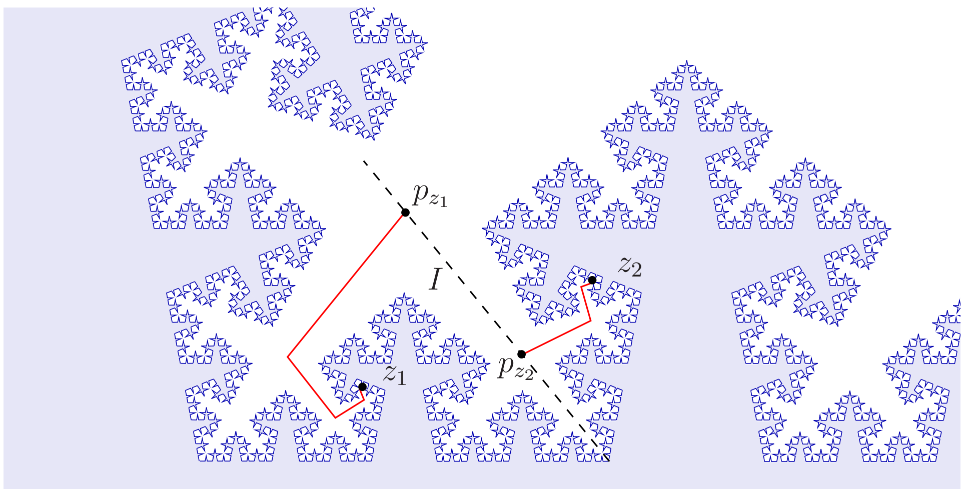

FIGURE 4. In the proof of the curve condition (3.1) we consider three critical cases. Here, in the zoomed in picture of the case 1, the points z_1 and z_2 are connected to points p_{z_1} and p_{z_2} on a line-segment I using the curves constructed in the proof of claim (i).

(a line-segment in mirroring K_{11} and K_{12}). We will define points p_{z_1} and p_{z_2} in I, that are connected to z_1 and z_2 by curves, which we will call γ_1 and γ_2. We then join the points p_{z_1} and p_{z_2} with a line-segment. See Figure 4 for an illustration.

Let us write $\{o\} := K_{11} \cap K_{12}$. If $z_1 = o$, we take $p_{z_1} = z_1$. If not, then there exists $k \geq 1$ such that $z_1 \in K_{11a_1\ldots a_k}$ with $a_i = 4$ for all $i < k$ and $a_k \neq 4$. We can make a crude estimate

$$\|z_1 - z_1'\| \geq \left(\frac{1}{2} - \lambda\right) \lambda^{k+2}. \tag{4.1}$$

Now, by the proof of the (i), z_1 can be connected to a point $p_{z_1} \in I$ by a John curve with John constant $\frac{1/2-\lambda}{\lambda}$ and length less than λ^{k-1}. Combining this with (4.1), we get

$$\begin{aligned}
\int_{\gamma_1} \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}z &\leq 2 \int_0^{\lambda^{k-1}} \left(\frac{\frac{1}{2} - \lambda}{\lambda} t\right)^{1-p} \mathrm{d}t \\
&= \frac{2}{2-p} \left(\frac{\frac{1}{2} - \lambda}{\lambda}\right)^{1-p} \left(\lambda^{k-1}\right)^{2-p} \\
&\leq \frac{2}{2-p} \frac{\lambda^{3p-7}}{\frac{1}{2} - \lambda} \|z_1 - z_1'\|^{2-p} \leq \frac{C}{3} \|z_1 - z_2\|^{2-p}.
\end{aligned} \tag{4.2}$$

By symmetry, with the same arguments we also find p_{z_2} and the curve γ_2 connecting z_2 to p_{z_2}, and get

$$\int_{\gamma_2} \operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}z \le \frac{2}{2-p}\frac{\lambda^{3p-7}}{\frac{1}{2}-\lambda}\|z_1 - z_1'\|^{2-p} \le \frac{C}{3}\|z_1 - z_2\|^{2-p}. \tag{4.3}$$

For the line-segment $[p_{z_1}, p_{z_2}]$, notice that we have

$$\begin{aligned}\|p_{z_1} - p_{z_2}\| &\le \|z_1' - z_2'\| + \|p_{z_1} - z_1'\| + \|p_{z_2} - z_2'\| \le \|z_1' - z_2'\| + 2\lambda^{k-1}\\ &\le \|z_1' - z_2'\| + \lambda^{-3}\left(\frac{1}{2}-\lambda\right)^{-1}(\|z_1 - z_1'\| + \|z_2 - z_2'\|)\\ &\le 3\lambda^{-3}\left(\frac{1}{2}-\lambda\right)^{-1}\|z_1 - z_2\|,\end{aligned}$$

and thus

$$\begin{aligned}\int_{[p_{z_1},p_{z_2}]} \operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}z &\le \int_0^{\|p_{z_1}-p_{z_2}\|}\left(\frac{\frac{1}{2}-\lambda}{\lambda}t\right)^{1-p}\mathrm{d}t\\ &\le \left(\frac{\frac{1}{2}-\lambda}{\lambda}\right)^{1-p}\frac{1}{2-p}\left(3\lambda^{-3}\left(\frac{1}{2}-\lambda\right)^{-1}\|z_1 - z_2\|\right)^{2-p}\\ &\le \frac{3^{2-p}}{2-p}\frac{\lambda^{3p-7}}{\frac{1}{2}-\lambda}\|z_1 - z_2\|^{2-p}\\ &\le \frac{C}{3}\|z_1 - z_2\|^{2-p}.\end{aligned} \tag{4.4}$$

Combining (4.2), (4.3), and (4.4), we conclude the first case.

Case 2: $z_1 \in K_{12}$ and $z_2 \in K_{13}$.
In this case, we connect z_1 and z_2 to the unique point $p \in K_{12} \cap K_{13}$ by curves γ_1 and γ_2. The estimate for γ_1 and γ_2 are exactly the same as in case 1. We connect z_1 to p_{z_1} with a John curve and then p_{z_1} to p (instead of z_1') with a line-segment.

Case 3: $z_1 \in K_{11}$ and $z_2 \in K_{13} \cup K_{14}$.
Similarly as in the second case, we can connect z_1 and z_2 to the unique point $p \in K_{12} \cap K_{13}$ obtaining the desired estimate also in this case. □

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