AN INVERSE PROBLEM FOR SEMILINEAR EQUATIONS INVOLVING THE FRACTIONAL LAPLACIAN

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ABSTRACT. Our work concerns the study of inverse problems of heat and wave equations involving the fractional Laplacian operator with zeroth order nonlinear perturbations. We recover nonlinear terms in the semilinear equations from the knowledge of the fractional Dirichlet-to-Neumann type map combined with the Runge approximation and the unique continuation property of the fractional Laplacian.

1. INTRODUCTION AND MAIN RESULTS

We investigate inverse problems for heat and wave equations involving the fractional Laplacian operator with zeroth order nonlinear perturbations. The study of inverse problems involving the fractional Laplace began with the work [GSU20] by Ghosh, Salo and Uhlmann. In [GSU20], they proposed and proved a Calderón type inverse problem for a linear fractional Laplace operator. The Calderón problem was initiated by Calderón in his work [Cal06] for non-fractional Laplace equations. There is ample amount of literature available on the non-fractional Calderón problem and we refer the readers to the survey [Uhl09]. The key tool for studying fractional type of inverse problems is the Runge approximation property, which is a consequence of the fractional unique continuation property (fUCP), i.e. if $u = (-\Delta)^s u = 0$ in certain open set, then $u = 0$ everywhere. Utilizing these tools, inverse problems involving fractional operators have been greatly investigated by numerous authors in recent years. We refer readers to [GRSU20, LL22a, LO22, Li21, Lin20, LL22b] for some recent works involving inverse problems for fractional semilinear elliptic equations.

Compared to the study of inverse problems involving fractional order operators, the study of inverse problems involving nonlinear terms goes back to Isakov [Isa01] and has been under extensive study in the literature. In [Isa01] he studied the nonlinear inverse problems for elliptic and parabolic equations using first order linearization techniques. In [LLLS21] the authors successfully implemented *higher order linearization techniques* to solve inverse problems for elliptic equations involving power type nonlinearity. In the higher order linearization, the idea is to use product of the solutions of "*free equation*" $\Delta u = 0$ (*i.e.* only principal operator, no lower order term is attached). It was observed that using non-linearity as a tool one can solve certain inverse problems which are not available for linear case. The method was also used to solve several nonlinear inverse problems including partial data [KU20b, KU20a, HL22] and Riemannian manifolds [FO20, FLL21, LLST22]. Inverse problems related to more general nonlinearities we refer [CFK+21, MU20] and the references cited there.

The study of inverse problems related to semilinear wave equations with quadratic nonlinearity started with the fundamental work [KLU18] by Kurylev, Lassas and Uhlmann. In [KLU18] the authors used propagation of non-linear interaction of non smooth plane waves having conormal singularities. Then in [FO22] authors used wave packet (sometimes

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2020 *Mathematics Subject Classification.* 35R11, 35R30, 46T20.

Key words and phrases. fractional Laplacian, fractional Calderón problem, nonlocal semilinear equations, fractional diffusion equation, fractional wave equation, Runge approximation.

it is also called quasimode construction) construction to solve certain non-linear hyperbolic inverse problems. This helps to avoid the need to use microlocal analysis techniques. For a comparison between these two methods mentioned above we refer [HUZ21]. Inverse problems for nonlinear parabolic equations have been well studied. We refer [LOST22, FKU22] and the references therein for more results.

Motivated by the works mentioned above, in this article we consider an inverse problem for nonlinear fractional parabolic equations. Fractional parabolic equations have applications in random processes [BBCK09]. We study the fractional type heat equations as well as the fractional type wave equations, and we start with the heat equation first.

Let $n \geq 1$ be a non-negative integer and $0 < s < 1$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $\Omega^e := \mathbb{R}^n \setminus \overline{\Omega}$. Let W be any bounded Lipschitz domain in Ω^e. Let $u = u(t,x)$ satisfy the following fractional diffusion equation with nonlinear term $q = q(t,x,z)$:

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^s u(t,x) + q(t,x,u(t,x)) = 0 & \text{in } \Omega_T \equiv (0,T)\times\Omega, \\ u(t,x) = f(t,x) & \text{in } \Omega_T^e \equiv (0,T)\times\Omega^e, \\ u(0,x) = 0 & \forall x \in \Omega, \end{cases} \tag{1.1}$$

for certain appropriate exterior data $f = f(t,x) \in \mathcal{C}_c^\infty(W_T)$, where $W_T := (0,T)\times W$ and $\mathcal{C}_c^\infty(\cdot)$ denotes the space of smooth compactly supported functions on their domain of definition. Here, the fractional Laplacian $(-\Delta)^s$ is defined via the Fourier transform: $\mathscr{F}((-\Delta)^s v)(\xi) := |\xi|^{2s}\hat{v}(\xi)$ for all $\xi \in \mathbb{R}^n$, where $\hat{v} = \mathscr{F}v$ is the Fourier transform of distribution v. Given any open sets V and W in Ω^e, we define the DN-map corresponding to (1.1) as follows:

$$\Lambda_q^{\text{heat}}(f) := (-\Delta)^s u\big|_{V_T} \quad \text{for all “sufficiently small” } f \in \mathcal{C}_c^\infty(W_T), \tag{1.2}$$

where u is the unique solution of (1.1), see Proposition 2.10. We now state the assumptions on the coefficient under which we state and prove our main results.

Assumptions 1.1. *Let $C^k(\cdot)$ be the space of k-times continuously differentiable functions for all integers $k \geq 0$. Assume that the function $q(t,x,z)$ satisfies following conditions.*

(Q.1) *For each $(t,x) \in (0,T)\times\Omega$, the mapping $z \mapsto q(t,x,z)$ is in $\mathcal{C}^{m+1}((-\delta,\delta))$.*
(Q.2) *$q(t,x,0) = 0$ for all $(t,x) \in \Omega_T$.*
(Q.3) *There exists a non-decreasing function $\Phi : (-\delta,\delta) \to \mathbb{R}_+$ such that*

$$\sup_{(t,x)\in\Omega_T, |z|\leq\epsilon} |\partial_z q(t,x,z)| \leq \Phi(\epsilon)$$

for all $0 < \epsilon < \delta$ and $\lim_{\epsilon\to 0}\Phi(\epsilon) = 0$.
(Q.4) *Given any $k = 2,3,\cdots,m+1$, there exists M_k (depending on k) such that*

$$\sup_{(t,x)\in\Omega_T, |z|\leq\delta} |\partial_z^k q(t,x,z)| \leq M_k. \tag{1.3}$$

With these assumptions on the coefficient, the following is our first main result:

Theorem 1.1 (Global uniqueness from DN-map)**.** *Choose any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $W, V \subset \Omega^e$ be any open sets, both with Lipschitz boundary, satisfying $\overline{V}\cap\overline{\Omega} = \emptyset$ and $\overline{W}\cap\overline{\Omega} = \emptyset$. Fix an integer $m \geq 2$ and a positive number $\delta > 0$. Assume that each q_j ($j = 1,2$) satisfies (Q.1)-(Q.4). Then there exists a constant $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n,s,\Omega,T,\delta)$ such that, if*

$$\Lambda_{q_1}^{\text{heat}}(f) = \Lambda_{q_2}^{\text{heat}}(f) \quad \text{for all } f \in \mathcal{C}_c^\infty(W_T) \text{ satisfying } \|f\|_{\text{ext}} \leq \tilde{\epsilon}_0,$$

where the norm $\|\cdot\|_{\text{ext}}$ is defined in (2.2) below, then we have

$$\partial_z^k q_1(t,x,0) = \partial_z^k q_2(t,x,0) \quad \forall (t,x) \in \Omega_T, \quad k = 0,1,2,\cdots,m. \tag{1.4}$$

Additionally, if we assume $z \longmapsto q(t,x,z)$ is analytic for $(t,x) \in \Omega_T$, then we have

$$q_1(t,x,z) = q_2(t,x,z) \quad \forall (t,x) \in \Omega_T, \ \forall z \in I.$$

Following the ideas from [GRSU20], one can strengthen above result and recover the coefficients based on a finite dimensional data set. Our next corollary is related to a single measurement result for linear fractional Laplace equation, which can be proved by examining carefully the proof of Theorem 1.1.

Corollary 1.2 (Recovery of m-jet from m-dimensional measurements)**.** *Suppose the assumptions in Theorem 1.1 hold. We further assume that for $j=1,2$*

$$\partial_z^k q_j(\cdot,0) \in \mathcal{C}^0(\overline{\Omega}) \text{ is independent of time variable } t.$$

Fix any $g_1, \cdots, g_m \in \mathcal{C}_c^\infty(W_T)$ such that $g_1(t_0,\cdot), \cdots, g_m(t_0,\cdot) \not\equiv 0$ for some $t_0 \in (0,T)$. Then $\Lambda_{q_1}^{\mathrm{heat}}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m) = \Lambda_{q_2}^{\mathrm{heat}}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m)$, for all sufficiently small $\epsilon_j > 0$ $(j = 1, \cdots, m)$, implies (1.4).

In this article, we also take into consideration a nonlinear inverse problem for fractional wave equations in one spatial dimension. Let $u = u(t,x)$ satisfy

$$\begin{cases} \partial_t^2 u(t,x) + (-\Delta)^s u(t,x) + q(t,x,u(t,x)) = 0 & \text{in } \Omega_T, \\ u(t,x) = f(t,x) & \text{in } \Omega_T^e, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{for all } x \in \Omega, \end{cases} \tag{1.5}$$

for certain appropriate exterior data. We can define the following hyperbolic DN-map corresponding to (1.5) as follows:

$$\Lambda_q^{\mathrm{wave}}(f) := (-\Delta)^s u\big|_{V_T} \quad \text{for all “sufficiently small” } f \in \mathcal{C}_c^\infty(W_T),$$

where u is the unique solution of (1.5), see Proposition 5.4 for the well-posedness. The following result can be proved adapting the similar ideas:

Theorem 1.3 (Global uniqueness from DN-map)**.** *Let $n = 1$ and $1/2 < s < 1$. Let $\Omega \subset \mathbb{R}$ be a bounded open set, let $W, V \subset \Omega^e$ be any open sets satisfying $\overline{V} \cap \overline{\Omega} = \emptyset$ and $\overline{W} \cap \overline{\Omega} = \emptyset$. Fix any integer $m \geq 2$ and a positive number $\delta > 0$. Assume that q_j $(j = 1,2)$ satisfy (Q.1)–(Q.4). Then there exists a constant $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(s,\Omega,T,\delta)$ such that, if $\Lambda_{q_1}^{\mathrm{wave}}(f) = \Lambda_{q_2}^{\mathrm{wave}}(f)$ for all $f \in \mathcal{C}_c^\infty(W_T)$ satisfying (2.2), then we have (1.4). Additionally, if we assume $z :\to q(t,x,z)$ is analytic for $(t,x) \in \Omega_T$ then we have*

$$q_1(t,x,z) = q_2(t,x,z) \quad \forall (t,x) \in \Omega_T, \ \forall z \in I.$$

The next corollary is analogous to Corollary 1.2.

Corollary 1.4 (Recovery of m-jet from m-dimensional measurements)**.** *Suppose the assumptions in Theorem 1.3 hold. We further assume that*

$$\partial_z^k q_j(\cdot,0) \in \mathcal{C}^0(\overline{\Omega}) \text{ is independent of time variable } t.$$

Fix any $g_1, \cdots, g_m \in \mathcal{C}_c^\infty(W_T)$ such that $g_1(t_0,\cdot), \cdots, g_m(t_0,\cdot) \not\equiv 0$ for some $t_0 \in (0,T)$. If $\Lambda_{q_1}^{\mathrm{wave}}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m) = \Lambda_{q_2}^{\mathrm{wave}}(\epsilon_1 g_1 + \cdots + \epsilon_m g_m)$ for all sufficiently small $\epsilon_j > 0$ $(j = 1, \cdots, m)$, then we conclude (1.4).

There are only a few work available in the literature about the inverse problems for fractional heat equations as well as fractional wave equations. To motivate our work, we mention several closely related ones. In [Li21], the author solved certain inverse problems for fractional type heat operators, however the assumptions on the nonlinear term in [Li21] are different from ours. Then in [KLW21], the authors studied an inverse problem involving fractional wave equation, while in [LLL21], the authors solved an inverse problem for hyperbolic systems.

The rest of the paper is organized as follows. We discuss the forward problem of the fractional diffusion equation in Section 2. We prove a Runge approximation for the fractional diffusion equation in Section 3. With these tools at hand, Section 4 is dedicated to the proof of Theorem 1.1. Finally, we investigate Theorem 1.3 in Section 5. To make our paper self-contained, we also present the proof of the well-posedness of the linear fractional diffusion equation (Proposition 2.2) in Appendix A. Then in Appendix B we discuss the issue of considering Theorem 1.3 in one spatial dimension.

2. The forward problem for the fractional diffusion equation

In this section, we prove several preliminaries that will be useful in this work.

2.1. **Fractional Sobolev spaces.**

We use notations for fractional Sobolev spaces as in [KLW21]. To make the paper self-contained, we give brief introductions to them. For $\alpha \in \mathbb{R}$, denote as $H^\alpha(\mathbb{R}^n)$ the standard L^2-based fractional Sobolev spaces, which is defined via Fourier transform [DNPV12, Kwa17, Ste16]. For $s \in (0,1)$, in fact

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \,\Big|\, \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\} \quad \text{(as sets)}$$

with equivalent norm: $\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + [u]^2_{\dot{H}^s(\mathbb{R}^n)}$, where

$$[u]^2_{\dot{H}^s(\mathbb{R}^n)} = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y. \tag{2.1}$$

Here, (2.1) is called the Aronszajn-Gagliardo-Slobodeckij seminorm, see [DNPV12, equation (2.2)] for reference.

Let $\mathscr{O}$ be any open set in $\mathbb{R}^n$, and let $\alpha \in \mathbb{R}$. We define the following Sobolev spaces:

$$H^\alpha(\mathscr{O}) := \{u|_{\mathscr{O}} \,|\, u \in H^\alpha(\mathbb{R}^n)\}, \ \tilde{H}^\alpha(\mathscr{O}) := \text{closure of } \mathcal{C}_c^\infty(\mathscr{O}) \text{ in } H^\alpha(\mathbb{R}^n)$$
$$H_0^\alpha(\mathscr{O}) := \text{closure of } \mathcal{C}_c^\infty(\mathscr{O}) \text{ in } H^\alpha(\mathscr{O}), \ H^\alpha_{\overline{\mathscr{O}}} := \{u \in H^\alpha(\mathbb{R}^n) \,|\, \mathrm{supp}\,(u) \subset \overline{\mathscr{O}}\}.$$

The Sobolev space $H^\alpha(\mathscr{O})$ is complete under the quotient norm

$$\|u\|_{H^\alpha(\mathscr{O})} := \inf\left\{ \|v\|_{H^\alpha(\mathbb{R}^n)} \,\Big|\, v \in H^\alpha(\mathbb{R}^n) \text{ and } v|_{\mathscr{O}} = u \right\}.$$

It is easy to see that $\tilde{H}^\alpha(\mathscr{O}) \subset H_0^\alpha(\mathscr{O})$, and that $H^\alpha_{\overline{\mathscr{O}}}$ is a closed subspace of $H^\alpha(\mathbb{R}^n)$. If Ω is a bounded Lipschitz domain, then we also have following identifications (with equivalent norms):

$$\begin{cases} \tilde{H}^\alpha(\Omega) = H^\alpha_{\overline{\Omega}}, \quad (H^\alpha_{\overline{\Omega}})' = H^{-\alpha}(\Omega) \text{ and } (H^\alpha(\Omega))' = H^{-\alpha}_{\overline{\Omega}} \quad \forall \alpha \in \mathbb{R}, \\ H^s(\Omega) = H^s_{\overline{\Omega}} = H^s_0(\Omega) \quad \forall -1/2 < s < 1/2, \end{cases}$$

see e.g. [GSU20, Section 2A], [McL00, Chapter 3], and [Tri02]. Next following [Eva10, Chapter 5], we define time dependent fractional Sobolev space for all integers $p \geq 1$ denoted by $L^p((0,T);H^s)$. Then the exterior norm of $f \in \mathcal{C}_c^\infty(W_T)$ is given by

$$\|f\|^2_{\mathrm{ext}} := \|f\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} + \|(-\Delta)^s f\|^2_{L^2(\Omega_T)}. \tag{2.2}$$

Moreover, for any measurable set $A \subset \mathbb{R}^n$ we use the following notations:

$$(f,g)_{L^2(A)} := \int_A fg \,\mathrm{d}x, \quad (F,G)_{L^2(A_T)} := \int_0^T \int_A FG \,\mathrm{d}x\,\mathrm{d}t.$$

2.2. **Well-posedness for the linear equation.** We state the well-posedness of the *linear* fractional diffusion equation. Let $T > 0$, $s \in (0,1)$, and $a = a(t,x) \in L^\infty(\Omega_T)$, and we consider the following initial-exterior value problem:

$$\begin{cases} (\partial_t + (-\Delta)^s + a)u = F & \text{in } \Omega_T, \\ u = f & \text{in } \Omega_T^e, \\ u = \varphi & \text{in } \{0\} \times \mathbb{R}^n, \end{cases} \tag{2.3}$$

where $f \in \mathcal{C}_c^\infty(W_T)$ for some open set with Lipschitz boundary $W \subset \Omega_e$ satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$, and $\varphi \in \tilde{H}^0(\Omega) = \left\{ \varphi \in L^2(\mathbb{R}^n) \,\middle|\, \operatorname{supp}\varphi \subset \overline{\Omega} \right\}$. Setting $v := u - f$, we then consider the following linear equation with zero exterior data:

$$\begin{cases} (\partial_t + (-\Delta)^s + a)v = \tilde{F} & \text{in } \Omega_T, \\ v = 0 & \text{in } \Omega_T^e, \\ v = \varphi & \text{in } \{0\} \times \mathbb{R}^n, \end{cases} \tag{2.4}$$

where $\tilde{F} = F - (-\Delta)^s f$. Now it suffices to study the well-posedness of (2.4).

Define functions $\boldsymbol{v} : [0,T] \to \tilde{H}^s(\Omega)$ and $\tilde{\boldsymbol{F}} : [0,T] \to L^2(\Omega)$ by

$$[\boldsymbol{v}(t)](x) := v(t,x), \quad [\tilde{\boldsymbol{F}}(t)](x) := \tilde{F}(t,x) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^n. \tag{2.5}$$

Let $\langle \cdot, \cdot \rangle$ be the duality pairing on $H^{-s}(\Omega) \oplus \tilde{H}^s(\Omega)$. Multiplying (2.4) by any $\phi \in \tilde{H}^s(\Omega)$ gives

$$\langle \boldsymbol{v}'(t), \phi \rangle + \mathcal{B}[\boldsymbol{v}, \phi; t] = (\tilde{\boldsymbol{F}}(t), \phi)_{L^2(\Omega)} \quad \text{for } 0 \le t \le T,$$

where $\mathcal{B}[\boldsymbol{v}, \phi; t]$ is the bilinear form given by

$$\mathcal{B}[\boldsymbol{v}, \phi; t] := \int_{\mathbb{R}^n} (-\Delta)^{s/2} \boldsymbol{v}(t) (-\Delta)^{s/2} \phi \, \mathsf{d}x + \int_\Omega a(t,\cdot) \boldsymbol{v}(t) \phi \, \mathsf{d}x.$$

Definition 2.1 (Weak solutions)**.** We say that v is a weak solution of (2.4), if

(a) $v \in L^2(0,T; \tilde{H}^s(\Omega))$ and $v' \in L^2(0,T; H^{-s}(\Omega))$;
(b) $\langle \boldsymbol{v}'(t), \phi \rangle + \mathcal{B}[\boldsymbol{v}, \phi; t] = (\tilde{\boldsymbol{F}}(t), \phi)_{L^2(\Omega)}$ for all $\phi \in \tilde{H}^s(\Omega)$ for (almost) all $0 \le t \le T$;
(c) $\boldsymbol{v}(0) = \varphi$,

where $\boldsymbol{v}$ and $\tilde{\boldsymbol{F}}$ are defined according to (2.5).

Proposition 2.2 (Well-posedness)**.** *Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $a \in L^\infty(\Omega_T)$. For any $\tilde{F} \in L^2(\Omega_T)$ and $\varphi \in \tilde{H}^0(\Omega)$, there exists a unique weak solution v of (2.4) and satisfies the following estimate:*

$$\|v\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|v\|_{L^2(0,T;\tilde{H}^s(\Omega))}^2 + \|\partial_t v\|_{L^2(0,T;H^{-s}(\Omega))}^2 \le C(\|\varphi\|_{L^2(\Omega)}^2 + \|\tilde{F}\|_{L^2(\Omega_T)}^2) \tag{2.6}$$

for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)})$. If we further assume $\varphi \in \tilde{H}^s(\Omega)$, then $v \in L^\infty(0,T;\tilde{H}^s(\Omega))$ and $\partial_t v \in L^2(\Omega_T)$. In this case, the unique weak solution also satisfies the following estimate:

$$\|v\|_{L^\infty(0,T;\tilde{H}^s(\Omega))}^2 + \|\partial_t v\|_{L^2(\Omega_T)}^2 \le C(\|\varphi\|_{\tilde{H}^s(\Omega)}^2 + \|\tilde{F}\|_{L^2(\Omega_T)}^2) \tag{2.7}$$

for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)})$.

The proof of Proposition 2.2 is analogous to the standard well-posedness proof of the classical diffusion equation. However, for completeness, we present a proof in Appendix A.

Corollary 2.3. *Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and $W \subset \Omega^e$ be any open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$.*

Let $a \in L^\infty(\Omega_T)$. Then for any $\tilde{F} \in L^2(\Omega_T)$, $\varphi \in \tilde{H}^0(\Omega)$, and $f \in C_c^\infty(W_T)$, there exists a unique weak solution $u = v + f$ of (2.3) satisfying

$$\begin{aligned}&\|u-f\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u-f\|^2_{L^2(0,T;\tilde{H}^s(\Omega))} + \|\partial_t(u-f)\|^2_{L^2(0,T;H^{-s}(\Omega))}\\ &\le C(\|\varphi\|^2_{L^2(\Omega)} + \|F-(-\Delta)^s f\|^2_{L^2(\Omega_T)})\end{aligned}$$

for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)})$. If we further assume $\varphi \in \tilde{H}^s(\Omega)$, then the unique weak solution u also satisfies the following estimate:

$$\|u-f\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \|\partial_t u\|^2_{L^2(\Omega_T)} \le C(\|\varphi\|^2_{\tilde{H}^s(\Omega)} + \|F-(-\Delta)^s f\|^2_{L^2(\Omega_T)}) \tag{2.8}$$

for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)})$.

We skip the proof of Corollary 2.3 as it is a straightforward consequence of Proposition 2.2.

2.3. **Maximum principle for the linear equation.** Modifying the ideas in [LL19, Proposition 3.1] or [RO16, Proposition 4.1], we can obtain the following proposition:

Proposition 2.4 (Maximum principle)**.** *Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $a \in L^\infty(\Omega_T)$. Suppose that $u \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$ is a weak solution of (2.3). If $F \ge 0$ in Ω_T, $f \ge 0$ in Ω_T^e, $\varphi \ge 0$ in $\mathbb{R}^n$, then $u \ge 0$ in Ω_T.*

Proof. Let M be a real number which shall be determined later. We define

$$\begin{aligned}&u_M(t,x) := e^{-Mt}u(t,x),\ a_M(t,x) := a(t,x)+M,\ F_M(t,x) := e^{-Mt}F(t,x) \text{ in } \Omega_T,\\ &f_M(t,x) := e^{-Mt}f(t,x) \text{ in } \Omega_T^e.\end{aligned} \tag{2.9}$$

We see that u_M satisfies

$$\begin{cases}(\partial_t + (-\Delta)^s + a_M)u_M = F_M & \text{in } \Omega_T,\\ u_M = f_M & \text{in } \Omega_T^e,\\ u_M = \varphi & \text{on } \{0\}\times\mathbb{R}^n.\end{cases} \tag{2.10}$$

We choose $M = \|a\|_{L^\infty(\Omega_T)}$, then $a_M \ge 0$ in Ω_T. Next we write $u_M = u_M^+ - u_M^-$, where $u_M^+ = \max\{u_M, 0\}$ and $u_M^- = \max\{-u_M, 0\}$. Since $u_M \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$, then $u_M^\pm \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$ and that

$$\partial_t(u_M^-) = \begin{cases}-\partial_t u_M & \text{in } \{u_M < 0\},\\ 0 & \text{in } \{u_M \ge 0\}.\end{cases}$$

Since $u_M = f_M \ge 0$ in Ω_T^e, hence $u_M^- = 0$ in Ω_T^e, which implies $u_M^- \in L^2(0,T;\tilde{H}^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Testing the first equation of (2.10) by u_M^-, we have

$$\begin{aligned}0 &\le (\boldsymbol{F}_M(t), \boldsymbol{u}_M^-(t))_{L^2(\Omega)} \quad (\text{because } F_M \ge 0 \text{ and } u_M^- \ge 0 \text{ in } \Omega_T)\\ &= \int_\Omega (\partial_t \boldsymbol{u}_M(t))\boldsymbol{u}_M^-(t)\,\mathsf{d}x + \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}\boldsymbol{u}_M(t)(-\Delta)^{\frac{s}{2}}\boldsymbol{u}_M^-(t)\,\mathsf{d}x + \int_\Omega a_M(t,\cdot)\boldsymbol{u}_M\boldsymbol{u}_M^-\,\mathsf{d}x\\ &= -\frac{\mathsf{d}}{\mathsf{d}t}\Big(\frac{1}{2}\int_\Omega |\boldsymbol{u}_M^-(t)|^2\,\mathsf{d}x\Big) + \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}\boldsymbol{u}_M(t)(-\Delta)^{\frac{s}{2}}\boldsymbol{u}_M^-(t)\,\mathsf{d}x - \int_\Omega a_M(t,\cdot)|\boldsymbol{u}_M^-|^2\,\mathsf{d}x\end{aligned}$$

for all $0 < t < T$. In [LL19, Proposition 3.1] or [RO16, Proposition 4.1], they showed that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}\boldsymbol{u}_M(t)(-\Delta)^{\frac{s}{2}}\boldsymbol{u}_M^-(t)\,\mathsf{d}x \le 0 \quad \text{for all } 0 < t < T.$$

Combining the preceding two inequalities, we then conclude $\frac{\mathsf{d}}{\mathsf{d}t}\big(\int_\Omega |\boldsymbol{u}_M^-(t)|^2\,\mathsf{d}x\big) \le 0$ holds true for all $0<t<T$. Since $u_M^- = 0$ on $\mathbb{R}^n \times \{0\}$ (because $\varphi \ge 0$ in $\mathbb{R}^n$), then we conclude $\int_\Omega |\boldsymbol{u}_M^-(t)|^2\,\mathsf{d}x = 0$ for all $0<t<T$, which completes our proof. □

Corollary 2.5 (Comparison principle)**.** *Given any $n \in \mathbb{N}$ and $0<s<1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $a \in L^\infty(\Omega_T)$. Let u_1 and u_2 be weak solutions of*

$$\begin{cases} (\partial_t + (-\Delta)^s + a)u_j = F_j & \text{in } \Omega_T, \\ u_j = f_j & \text{in } \Omega_T^e, \\ u_j = \varphi_j & \text{on } \{0\}\times\mathbb{R}^n, \end{cases}$$

for $j=1,2$. If $F_1 \ge F_2$ in Ω_T, $f_1 \ge f_2$ in Ω_T^e, $\varphi_1 \ge \varphi_2$ in $\mathbb{R}^n$, then $u_1 \ge u_2$ in Ω_T.

Proof. By applying Proposition 2.4 with $u = u_1 - u_2$, this can be proved immediately. □

Remark 2.1. Proposition 2.4 as well as Corollary 2.5 also imply the uniqueness part of Proposition 2.2 and Corollary 2.3.

2.4. **L^∞-bounds of solutions of the linear equation.** For our purposes, we require the following L^∞-bound estimate, which can be found in [Li22, Proposition 3.3]:

Proposition 2.6. *Given any $n \in \mathbb{N}$ and $0<s<1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $a \in L^\infty(\Omega_T)$. Suppose that $u \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$ is a weak solution of*

$$\begin{cases} (\partial_t + (-\Delta)^s + a)u = F & \text{in } \Omega_T, \\ u = f & \text{in } \Omega_T^e, \\ u = 0 & \text{on } \{0\}\times\mathbb{R}^n, \end{cases}$$

with $F \in L^\infty(\Omega_T)$ and $f \in L^\infty(\Omega_T^e)$. Then $\|u\|_{L^\infty(\Omega_T)} \le C(\|f\|_{L^\infty(\Omega_T^e)} + \|F\|_{L^\infty(\Omega_T)})$, for some constant $C = C(n,s,T,\Omega,\|a\|_{L^\infty(\Omega_T)})$.

To make our paper more self-contained, here we sketch the proof of Proposition 2.6. The following lemma can be found in [LL19, Lemma 3.4] (with $a \equiv 0$) or [RO16, Lemma 5.1].

Lemma 2.7 (Elliptic barrier)**.** *Given any $n \in \mathbb{N}$ and $0<s<1$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists a function $\phi = \phi(x) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that*

$$(-\Delta)^s\phi \ge 1 \quad \text{in } \Omega, \quad \phi \ge 0 \quad \text{in } \mathbb{R}^n, \quad \phi \le C \quad \text{in } \Omega, \quad \text{for some constant } C = C(n,s,\Omega).$$

If we define $\Phi(t,x) := e^t\phi(x)$, we immediately obtain the following corollary:

Corollary 2.8 (Parabolic barrier)**.** *Given any $n \in \mathbb{N}$ and $0<s<1$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists a function $\Phi \in \mathcal{C}_c^\infty([0,T]\times\mathbb{R}^n)$ such that*

$$(\partial_t + (-\Delta)^s)\Phi \ge 1 \quad \text{in } \Omega_T \quad \Phi \ge 0 \quad \text{in } [0,T)\times\mathbb{R}^n, \quad \Phi \le C \quad \text{in } \Omega_T,$$

for some constant $C = C(n,s,T,\Omega)$.

Using the barrier in Corollary 2.8, we now can obtain the following L^∞-bound for the solution of (2.3).

Proof of Proposition 2.6. Using the functions given in (2.9) with $M = \|a\|_{L^\infty(\Omega_T)}$, we know that

$$\begin{cases} (\partial_t + (-\Delta)^s + a_M)u_M = F_M & \text{in } \Omega_T, \\ u_M = f_M & \text{in } \Omega_T^e, \\ u_M = 0 & \text{on } \{0\}\times\mathbb{R}^n, \end{cases}$$

with $a_M \geq 0$. Let $v(t,x) := \|f_M\|_{L^\infty(\Omega_T^e)} + \|F_M\|_{L^\infty(\Omega_T)}\Phi(t,x) \geq 0$ in $[0,T)\times\Omega$, where Φ is the barrier given in Corollary 2.8. We see that

$$\begin{aligned}(\partial_t + (-\Delta)^s + a_M)v \geq (\partial_t + (-\Delta)^s)v = \|F_M\|_{L^\infty(\Omega_T)}(\partial_t + (-\Delta)^s)\Phi \geq \|F_M\|_{L^\infty(\Omega_T)} \\ \geq \mp F_M = \mp(\partial_t + (-\Delta)^s + a_M)u_M \quad \text{in} \quad \Omega_T.\end{aligned}$$

Moreover, we also have

$$\begin{aligned}&(\partial_t + (-\Delta)^s + a_M)(v \pm u_M) \geq 0 && \text{in } \Omega_T \\ &v \pm u_M = v \pm f_M \geq \|f_M\|_{L^\infty(\Omega_T^e)} \pm f_M \geq 0 && \text{in } \Omega_T^e \\ &v \pm u_M = v \geq 0 && \text{on } \{0\}\times\mathbb{R}^n.\end{aligned} \tag{2.11}$$

Combining relations in (2.11), and Proposition 2.4, we see that $v \geq \pm u_M$ in Ω_T, which further implies that $\|u_M\|_{L^\infty(\Omega_T)} \leq \|v\|_{L^\infty(\Omega_T)} \leq \|f_M\|_{L^\infty(\Omega_T^e)} + C\|F_M\|_{L^\infty(\Omega_T)}$, where $C = C(n,s,T,\Omega)$ is the constant given in the Corollary 2.8. Finally, utilizing

$$\begin{aligned}|u(t,x)| = e^{Mt}|u_M(t,x)| \leq e^{T\|a\|_{L^\infty(\Omega_T)}}\|u_M\|_{L^\infty(\Omega_T)} \quad &\text{in } \Omega_T, \\ |F_M(t,x)| = e^{-Mt}|F(t,x)| \leq \|F\|_{L^\infty(\Omega_T)} \quad &\text{in } \Omega_T, \\ |f_M(t,x)| = e^{-Mt}|f(t,x)| \leq \|f\|_{L^\infty(\Omega_T^e)} \quad &\text{in } \Omega_T^e,\end{aligned}$$

we conclude the proof. □

We skip the proof of the following well-posedness result as it follows from combining Corollary 2.3 and Proposition 2.6.

Proposition 2.9. *Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, let $W \subset \Omega^e$ be any open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Then for any $\tilde{F} \in L^\infty(\Omega_T)$ and $f \in \mathcal{C}_c^\infty(W_T)$, there exists a unique weak solution u of*

$$\begin{cases}(\partial_t + (-\Delta)^s + a)u = F & \text{in } \Omega_T, \\ u = f & \text{in } \Omega_T^e, \\ u = 0 & \text{in } \{0\}\times\mathbb{R}^n,\end{cases}$$

satisfying

$$\begin{aligned}&\|u\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} + \|\partial_t u\|^2_{L^2(\Omega_T)} \\ &\quad \leq C\big(\|F\|^2_{L^\infty(\Omega_T)} + \|f\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} + \|(-\Delta)^s f\|^2_{L^2(\Omega_T)}\big)\end{aligned}$$

for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)},\Omega)$.

2.5. **Well-posedness for the nonlinear equation.** We now state the well-posedness of (1.1) for small exterior data:

Proposition 2.10. *Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and $W \subset \Omega^e$ be any open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Fixing any parameter $\delta > 0$. Assume that q satisfies (Q.1)–(Q.3). Then there exists a sufficiently small parameter $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n,s,\Omega,T,\delta) > 0$ such that the following statement holds: Given any $f \in \mathcal{C}_c^\infty(W_T)$ with $\|f\|_{\mathrm{ext}} \leq \tilde{\epsilon}_0$, there exists a unique solution $u \in L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)$ of (1.1) with*

$$\|u\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} \leq C\|f\|_{\mathrm{ext}} \tag{2.12}$$

for certain constant $C = C(n,s,T,\Omega)$.

Remark 2.2. In order to prove Proposition 2.10, we only need q to be $\mathcal{C}^1$-smooth in z variable. However to recover m-th jet of q we need to assume (Q.1).

Remark 2.3. In [MBRS16, Theorem 11.2], they showed that there exist infinitely many solutions w_j to

$$\begin{cases}(-\Delta)^s w_j + q(x,w_j) + h(x) = 0 & \text{in } \Omega,\\ w_j = 0 & \text{in } \Omega^e,\end{cases}$$

such that $\|w_j\|_{H^s(\mathbb{R}^n)} \to \infty$ as $j \to \infty$. Therefore, the smallness assumption on f seems to be necessary to ensure the uniqueness of the solution to (1.1).

Proof of Proposition 2.10. **Step 1: Initialization.** Given any $f \in \mathcal{C}_c^\infty(\Omega_T)$, from Proposition 2.9, there exists a unique solution $u_0 = u_0(t,x)$ of

$$\begin{cases}(\partial_t + (-\Delta)^s)u = 0 & \text{in } \Omega_T,\\ u_0 = f & \text{in } \Omega_T^e,\\ u_0 = 0 & \text{in } \{0\}\times\mathbb{R}^n,\end{cases}$$

with

$$\|u_0\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} \leq C\|f\|_{\mathrm{ext}}, \tag{2.13}$$

for some constant $C = C(n,s,T,\Omega)$. If u is a solution of (1.1), then the remainder function $v \equiv u - u_0$ satisfies

$$\begin{cases}(\partial_t + (-\Delta)^s)v = \mathcal{F}(v) \equiv -q(t,x,(v+u_0)(t,x)) & \text{in } \Omega_T,\\ v = 0 & \text{in } \Omega_T^e,\\ v = 0 & \text{in } \{0\}\times\mathbb{R}^n.\end{cases} \tag{2.14}$$

Again, using Proposition 2.9, given any $F = F(t,x) \in L^\infty(\Omega_T)$, there exists a unique solution $\mathcal{S}F \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)$ of

$$\begin{cases}(\partial_t + (-\Delta)^s)\mathcal{S}F = F & \text{in } \Omega_T,\\ \mathcal{S}F = 0 & \text{in } \Omega_T^e,\\ \mathcal{S}F = 0 & \text{in } \{0\}\times\mathbb{R}^n,\end{cases} \tag{2.15}$$

with $\|\mathcal{S}F\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \leq C\|F\|_{L^\infty(\Omega_T)}$for some constant $C = C(n,s,T,\Omega)$. In other words, the solution operator

$$\mathcal{S} : L^\infty(\Omega_T) \to L^\infty(0,T;\tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T) \tag{2.16}$$

of (2.15) is a bounded linear operator.

Step 2: Contraction. Let $\epsilon = \|f\|_{\mathrm{ext}}$, and we define

$$X_\epsilon := \left\{ v \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T) \,\middle|\, \|v\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \leq \epsilon \right\}.$$

We first show that

$$\mathcal{S}\circ\mathcal{F}(v) \in X_\epsilon \quad \text{for all } v \in X_\epsilon. \tag{2.17}$$

Given any $v \in X_\epsilon$, using (2.13), we know

$$\|u_0 + v\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \leq C\epsilon. \tag{2.18}$$

By choosing $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n,s,\Omega,T,\delta) > 0$ to be sufficiently small, we can guarantee that $2C\epsilon \leq 2C\tilde{\epsilon}_0 < \delta$. From (Q.1) and (Q.2), by using the mean value theorem, we can find a function $0 \leq \zeta(t,x) \leq 1$ such that

$$\begin{aligned}\mathcal{F}(v)(t,x) &= q(t,x,(u_0+v)(t,x)) - q(t,x,0)\\ &= \partial_z q\big(t,x,(\zeta(u_0+v))(t,x)\big)(u_0+v)(t,x), \quad \text{for all } x \in \Omega.\end{aligned} \tag{2.19}$$

Therefore, using (Q.3), combining (2.18) and (2.19), we obtain $\|\mathcal{F}(v)\|_{L^\infty(\Omega_T)} \leq \Phi(C\epsilon)\|u_0 + v\|_{L^\infty(\mathbb{R}^n_T)} \leq C\Phi(C\epsilon)\epsilon$. Using (2.16), we then obtain $\|\mathcal{S}\circ\mathcal{F}(v)\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \leq$

$\tilde{C}\,\Phi(C\epsilon)\,\epsilon$. Since Φ is non-decreasing, using the assumption (Q.3), and by choosing a smaller $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$, we can assure that $\Phi(C\epsilon) \le \Phi(C\tilde{\epsilon}_0) \le \tilde{C}^{-1}$, and can obtain

$$\|\mathcal{S} \circ \mathcal{F}(v)\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \le \epsilon, \tag{2.20}$$

which concludes (2.17).

We next show that

$$\mathcal{S} \circ \mathcal{F} \text{ is a contraction on } X_\epsilon. \tag{2.21}$$

Let $v_1, v_2 \in X_\epsilon$, similar to (2.18), we have

$$\|u_0 + v_j\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \le C\epsilon \quad \text{for all } \; j = 1, 2. \tag{2.22}$$

From (Q.1), by using the mean value theorem, we can find a function $0 \le \zeta(t, x) \le 1$ such that

$$\begin{aligned} \mathcal{F}(v_1)(t,x) - \mathcal{F}(v_2)(t,x) &= q(t, x, (u_0 + v_1)(t,x)) - q(t, x, (u_0 + v_2)(t,x)) \\ &= \partial_z q\big(t, x, (\zeta(u_0 + v_1) + (1-\zeta)(u_0 + v_2))(t,x)\big)(v_1 - v_2)(t,x). \end{aligned} \tag{2.23}$$

Using (2.22), we know that

$$\|\zeta(u_0 + v_1) + (1-\zeta)(u_0 + v_2)\|_{L^\infty(\mathbb{R}^n_T)} \le C\epsilon \le C\tilde{\epsilon}_0. \tag{2.24}$$

Since $C\tilde{\epsilon}_0 < \delta$, using (Q.3), combining (2.23), we have

$$\begin{aligned} \|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^\infty(\Omega_T)} &\le \Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(\Omega_T)} \\ &\le \Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)}. \end{aligned}$$

Using (2.16), we then obtain

$$\|\mathcal{S} \circ \mathcal{F}(v_1) - \mathcal{S} \circ \mathcal{F}(v_2)\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \le \tilde{C}\Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)}.$$

Since Φ is non-decreasing, using (Q.3), possibly choosing a smaller $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$, we can assure that $\Phi(C\tilde{\epsilon}_0) \le \frac{1}{2}\tilde{C}^{-1}$, and we obtain

$$\|\mathcal{S} \circ \mathcal{F}(v_1) - \mathcal{S} \circ \mathcal{F}(v_2)\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)} \le \frac{1}{2}\|v_1 - v_2\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)},$$

which concludes (2.21).

Step 3: Conclusion. From (2.17) and (2.21), by using the Banach fixed point theorem, there exists a unique $v \in X_\epsilon$ such that $v = \mathcal{S} \circ \mathcal{F}(v)$, that is, there exists a unique $v \in X_\epsilon$ satisfying (2.14). Hence we know that $u \equiv v + u_0 \in L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)$ is the unique solution of (1.1). Moreover, from (2.13) and (2.20), we can conclude (2.12). □

3. The Runge approximation for the fractional diffusion equation

The following unique continuation property for $(-\Delta)^s$ (see [GSU20]) is crucial for our work.

Lemma 3.1 (Antilocality)**.** *Suppose $u = (-\Delta)^s u = 0$ in $\mathcal{O}_T$, for some open set $\mathcal{O} \subset \mathbb{R}^n$, then $u \equiv 0$ in $\mathbb{R}^n_T$.*

The following Runge approximation property for the diffusion equation can be found in [Li22, Proposition 2.4]. To make our paper self-contained, here we still present the proof.

Proposition 3.2. *Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, let $W \subset \Omega^e$ be any open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Fixing any $a \in L^\infty(\Omega_T)$. For each $f \in \mathcal{C}_c^\infty(W_T)$, let $\mathcal{P}_a f \in L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)$ be the unique solution (see Proposition 2.9) of*

$$\begin{cases} (\partial_t + (-\Delta)^s + a(t,x))\mathcal{P}_a f = 0 & \text{in } \Omega_T, \\ \mathcal{P}_a f = f & \text{in } \Omega_T^e, \\ \mathcal{P}_a f = 0 & \text{on } \{0\} \times \mathbb{R}^n. \end{cases}$$

Then the set $\mathcal{D} := \{\mathcal{P}_a f|_{\Omega_T} \mid f \in \mathcal{C}_c^\infty(W_T)\}$ is dense in $L^2(\Omega_T)$.

Proof. Using the Hahn-Banach theorem (see e.g. [Bre11, Corollary 1.8]), we only need to show the following: if $v \in L^2(\Omega_T)$ satisfies

$$(\mathcal{P}_a f, v)_{L^2(\Omega_T)} = 0 \quad \text{for all } f \in \mathcal{C}_c^\infty(W_T),$$

then $v = 0$ in Ω_T. By Proposition 2.2, there exists a unique $\tilde{w} \in L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)$ such that

$$\begin{cases} (\partial_t + (-\Delta)^s + a(T-t,x))\tilde{w} = v(T-t,x) & \text{in } \Omega_T, \\ \tilde{w} = 0 & \text{in } \Omega_T^e, \\ \tilde{w} = 0 & \text{on } \{0\} \times \mathbb{R}^n. \end{cases}$$

Define $w(t,x) := \tilde{w}(T-t,x)$, then

$$\begin{cases} (-\partial_t + (-\Delta)^s + a(t,x))w = v(t,x) & \text{in } \Omega_T, \\ w = 0 & \text{in } \Omega_T^e, \\ w = 0 & \text{on } \{T\} \times \mathbb{R}^n. \end{cases} \tag{3.1}$$

We note that

$$\begin{aligned} (\mathcal{P}_a f, v)_{L^2(\Omega_T)} &= (\mathcal{P}_a f - f, v)_{L^2(\Omega_T)} \qquad (\text{because } \operatorname{supp}(f) \cap \overline{\Omega_T} = \emptyset) \\ &= (\mathcal{P}_a f - f, (-\partial_t + (-\Delta)^s + a(t,x))w)_{L^2(\Omega_T)} \\ &= -(\mathcal{P}_a f, \partial_t w)_{L^2(\Omega_T)} + (\mathcal{P}_a f, aw)_{L^2(\Omega_T)} \qquad (\text{because } \operatorname{supp}(f) \cap \overline{\Omega_T} = \emptyset) \\ &\quad + (\mathcal{P}_a f - f, (-\Delta)^s w)_{L^2(\mathbb{R}^n)} \qquad (\text{because } \operatorname{supp}(\mathcal{P}_a f - f) \subset \overline{\Omega_T}) \\ &= ((\partial_t + (-\Delta)^s + a)\mathcal{P}_a f, w)_{L^2(\Omega_T)} - (f, (-\Delta)^s w)_{L^2(\mathbb{R}^n)} \\ &= -(f, (-\Delta)^s w)_{L^2(W_T)}. \end{aligned}$$

Combining this equality with $(\mathcal{P}_a f, v)_{L^2(\Omega_T)} = 0$, we obtain $(f, (-\Delta)^s w)_{L^2(W_T)} = 0$ for all $f \in \mathcal{C}_c^\infty(W_T)$, which implies $(-\Delta)^s w = 0$ in W_T. Since $w = 0$ in W_T, using Lemma 3.1, we conclude $w \equiv 0$ in $\mathbb{R}^n_T$, and hence from (3.1), we conclude that $v = 0$ in Ω_T. □

4. The inverse problems for the fractional diffusion equation

In this section we perform higher order linearizations to the nonlinear fractional diffusion equation (1.1) as well as the DN map (1.2), which is also nonlinear. For each linearization step we derive certain identities and combine them with the Runge approximation to recover partial derivatives of q. We start with the zeroth order linearization.

4.1. **Zeroth order linearization.** Let $u_j^{\boldsymbol{\epsilon}}$ be the unique solution of

$$\begin{cases} \partial_t u_j^{\boldsymbol{\epsilon}} + (-\Delta)^s u_j^{\boldsymbol{\epsilon}} + q_j(\cdot, u_j^{\boldsymbol{\epsilon}}) = 0 & \text{in } \Omega_T, \\ u_j^{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon} \cdot \boldsymbol{g} = \epsilon_1 g_1 + \cdots + \epsilon_m g_m & \text{in } \Omega_T^e, \\ u_j^{\boldsymbol{\epsilon}} = 0 & \text{on } \{0\} \times \mathbb{R}, \end{cases} \tag{4.1}$$

where $\boldsymbol{g} = (g_1, \cdots, g_m) \in (\mathcal{C}_c^\infty(W_T))^m$. Since q_j $(j = 1, 2)$ satisfies (Q.1)–(Q.3), there exists a constant $\epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, \boldsymbol{g}) > 0$ with $\epsilon_0 \le \tilde{\epsilon}_0$, where $\tilde{\epsilon}_0$ is the constant given in Proposition 2.10, such that the following statement holds: Given any $\boldsymbol{\epsilon}$ with $|\boldsymbol{\epsilon}| = \max_{1\le k\le m} |\epsilon_k| < \epsilon_0$, there exists a unique solution $u_j^{\boldsymbol{\epsilon}} \in L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)$ of (4.1) with

$$\|u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} \le C(n, s, \Omega, T, \boldsymbol{g}, m)|\boldsymbol{\epsilon}|. \tag{4.2}$$

Therefore, the corresponding DN-map is $\Lambda_{q_j}(\boldsymbol{\epsilon}\cdot\boldsymbol{g}) = (-\Delta)^s u_j^{\boldsymbol{\epsilon}}\big|_{V_T}$ for all $0 \le |\boldsymbol{\epsilon}| < \epsilon_0$. We now show that $\boldsymbol{\epsilon} \to u_j^{\boldsymbol{\epsilon}}$ is continuous in the following sense:

Lemma 4.1. *The mapping $\boldsymbol{\epsilon} \to u_j^{\boldsymbol{\epsilon}}$ is continuous in $L^\infty(0, T; \tilde{H}^s(\Omega))$, that is,*

$$\lim_{|\boldsymbol{\theta}|\to 0} \|u_j^{\boldsymbol{\epsilon}+\boldsymbol{\theta}} - u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} = 0 \quad \textit{for each } |\boldsymbol{\epsilon}| < \epsilon_0. \tag{4.3}$$

Proof. Let $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_m) \in \mathbb{R}^m$ with $|\boldsymbol{\theta}| \le |\boldsymbol{\epsilon}|$ and $|\boldsymbol{\epsilon}| + |\boldsymbol{\theta}| < \epsilon_0$. We define $\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}} = u_j^{\boldsymbol{\epsilon}+\boldsymbol{\theta}} - u_j^{\boldsymbol{\epsilon}}$, and observe that

$$\begin{cases} (\partial_t + (-\Delta)^s)\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}} = \mathcal{G} & \text{in } \Omega_T, \\ \delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}} = \boldsymbol{\theta}\cdot\boldsymbol{g} & \text{in } \Omega_T^e, \\ \delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}} = 0 & \text{on } \{0\}\times\mathbb{R}^n, \end{cases}$$

where $\mathcal{G} = -q_j(\cdot, u_j^{\boldsymbol{\epsilon}+\boldsymbol{\theta}}) + q_j(\cdot, u_j^{\boldsymbol{\epsilon}})$. From Proposition 2.9, we know that

$$\|\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} \le C\big(\|\mathcal{G}\|_{L^\infty(\Omega_T)} + |\boldsymbol{\theta}|\big). \tag{4.4}$$

Using mean value theorem on the z variable of q, there exists $0 \le \zeta(t, x) \le 1$ such that $\mathcal{G} = -\partial_z q_j(\cdot, \zeta u_j^{\boldsymbol{\epsilon}+\boldsymbol{\theta}} + (1-\zeta)u_j^{\boldsymbol{\epsilon}})\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}}$ in Ω_T. From (4.2), we know that

$$\|\zeta u_j^{\boldsymbol{\epsilon}+\boldsymbol{\theta}} + (1-\zeta)u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} \le C|\boldsymbol{\epsilon}| \quad (\text{because } |\boldsymbol{\theta}| \le |\boldsymbol{\epsilon}|).$$

Using (Q.3), we see that

$$\|\mathcal{G}\|_{L^\infty(\Omega_T)} \le \Phi(C|\boldsymbol{\epsilon}|)\|\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(\Omega_T)} \le \Phi(C|\boldsymbol{\epsilon}|)\|\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)}.$$

Substituting this inequality into (4.4), we obtain

$$\|\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} \le \tilde{C}\Phi(C|\boldsymbol{\epsilon}|)\|\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} + C|\boldsymbol{\theta}|.$$

Since Φ is non-decreasing, using the assumption (Q.3), and possibly by choosing a smaller constant $\epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, \boldsymbol{g}, m) > 0$, we can assure $\Phi(C|\boldsymbol{\epsilon}|) \le \Phi(C\epsilon_0) \le \frac{1}{2}\tilde{C}^{-1}$, thus $\|\delta_{\boldsymbol{\theta}} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)} \le 2C|\boldsymbol{\theta}|$, which implies (4.3). This completes the proof. □

By setting $\boldsymbol{\epsilon} = 0$ in (4.1), we obtain

$$\begin{aligned} &\partial_t u_j^0 + (-\Delta)^s u_j^0 + q_j(\cdot, u_j^0) = 0 \quad \text{in } \Omega_T, \\ &u_j^0 = 0 \quad \text{in } \Omega_T^e, \quad u_j^0 = 0 \quad \text{on } \{0\}\times\mathbb{R}. \end{aligned}$$

From (4.2), we know that $u_j^0 \equiv 0$ in $\mathbb{R}^n_T$.

4.2. **First order linearization.** Assuming the derivative ∂_{ϵ_1} to (4.1) is well-defined, we obtain

$$\begin{cases} (\partial_t + (-\Delta)^s + \partial_z q_j(\cdot, u_j^{\boldsymbol{\epsilon}}))(\partial_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}) = 0 & \text{in } \Omega_T, \\ \partial_{\epsilon_1} u_j^{\boldsymbol{\epsilon}} = g_1 & \text{in } \Omega_T^e, \\ \partial_{\epsilon_1} u_j^{\boldsymbol{\epsilon}} = 0 & \text{on } \{0\} \times \mathbb{R}^n. \end{cases} \tag{4.5}$$

Using (Q.3), we know that $\|\partial_z q_j(\cdot, u_j^{\boldsymbol{\epsilon}})\|_{L^\infty(\Omega_T)} \le \Phi(C\epsilon_0) \le 1$. Therefore, using Proposition 2.9, given any $\boldsymbol{\epsilon}$ with $|\boldsymbol{\epsilon}| < \epsilon_0$, there exists a unique solution $v_j^{\boldsymbol{\epsilon}} \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)$ to (4.5) with

$$\|v_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} \le C\|g_1\|_{\text{ext}}. \tag{4.6}$$

Here, $v_j^{\boldsymbol{\epsilon}}$ is just a intermediate function which will be dropped after showing that $\partial_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}$ is well-defined.

Lemma 4.2. *There exists a constant $\epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, \boldsymbol{g}, m) > 0$ with $0 < \epsilon_0 < \tilde{\epsilon}_0$, where $\tilde{\epsilon}_0$ is given in Proposition 2.10, such that for each $\boldsymbol{\epsilon}$ with $|\boldsymbol{\epsilon}| < \epsilon_0$, we have*

$$\lim_{\epsilon_1 \to 0} \|v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} = 0,$$

where $\delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}} = \frac{u_j^{\boldsymbol{\epsilon}+\epsilon_1 \boldsymbol{e}_1} - u_j^{\boldsymbol{\epsilon}}}{\epsilon_1}$ *for all* $(t,x) \in \Omega_T$, *provided that* $|\boldsymbol{\epsilon}| + |\epsilon_1| < \epsilon_0$.

Proof. Let ϵ_1 satisfies $|\epsilon_1| \le |\boldsymbol{\epsilon}|$ and $|\boldsymbol{\epsilon}| + |\epsilon_1| < \epsilon_0$. Note that

$$\begin{cases} (\partial_t + (-\Delta)^s)(v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}) = \mathcal{G}_1 & \text{in } \Omega_T, \\ (v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}) = 0 & \text{in } \Omega_T^e \text{ and on } \{0\} \times \mathbb{R}^n, \end{cases}$$

with $-\mathcal{G}_1 = \partial_z q(\cdot, u_j^{\boldsymbol{\epsilon}}) v_j^{\boldsymbol{\epsilon}} - \frac{q_j(\cdot, u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1}) - q_j(\cdot, u_j^{\boldsymbol{\epsilon}})}{\epsilon_1}$. From Proposition 2.9, we know that

$$\|v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;\tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)} \le C\|\mathcal{G}_1\|_{L^\infty(\Omega_T)}. \tag{4.7}$$

Using the mean value theorem on the z variable of q, there exists $0 \le \zeta(t,x) \le 1$ such that

$$\begin{aligned} -\mathcal{G}_1 = &\big[\partial_z q(\cdot, u_j^{\boldsymbol{\epsilon}}) - \partial_z q_j(\cdot, \zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}})\big] v_j^{\boldsymbol{\epsilon}} \\ &+ \partial_z q_j(\cdot, \zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}}) \big[v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}\big]. \end{aligned}$$

Using mean value theorem on the z variable of $\partial_z q$, there exists $0 \le \eta(t,x) \le 1$ such that

$$\begin{aligned} -\mathcal{G}_1 = &-\zeta \partial_z^2 q\big(\eta u_j^{\boldsymbol{\epsilon}} - (1-\eta)(\zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}})\big)(u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} - u_j^{\boldsymbol{\epsilon}}) v_j^{\boldsymbol{\epsilon}} \\ &+ \partial_z q_j(\cdot, \zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}}) \big[v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}\big] \end{aligned}$$

From (4.2) and $|\epsilon_1| \le |\boldsymbol{\epsilon}|$, we have

$$\begin{aligned} \|\eta u_j^{\boldsymbol{\epsilon}} - (1-\eta)(\zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}})\|_{L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} &\le C|\boldsymbol{\epsilon}|, \\ \|\zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} &\le C|\boldsymbol{\epsilon}|. \end{aligned}$$

Hence, by (Q.3) and (Q.4), we know that

$$\begin{aligned} \big\|\zeta \partial_z^2 q\big(\eta u_j^{\boldsymbol{\epsilon}} - (1-\eta)(\zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}})\big)\big\|_{L^\infty(\Omega_T)} &\le M_2, \\ \|\partial_z q_j(\cdot, \zeta u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} + (1-\zeta) u_j^{\boldsymbol{\epsilon}})\|_{L^\infty(\Omega_T)} &\le \Phi(C|\boldsymbol{\epsilon}|). \end{aligned}$$

Hence, by using (4.6) we know that

$$\|\mathcal{G}_1\|_{L^\infty(\Omega_T)} \le CM_2 \|g_1\|_{\text{ext}} \|u_j^{\boldsymbol{\epsilon}+\epsilon_1\boldsymbol{e}_1} - u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(\Omega_T)} + \Phi(C|\boldsymbol{\epsilon}|)\|v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_1} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(\Omega_T)}.$$

Combining this with (4.7), we have

$$\begin{aligned}&\|v_j^{\boldsymbol\epsilon}-\delta_{\epsilon_1}u_j^{\boldsymbol\epsilon}\|_{L^\infty(0,T;\tilde H^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)}\\ &\le \tilde C M_2\|g_1\|_{\mathrm{ext}}\|u_j^{\boldsymbol\epsilon+\epsilon_1\boldsymbol e_1}-u_j^{\boldsymbol\epsilon}\|_{L^\infty(\Omega_T)}+\Phi(C|\boldsymbol\epsilon|)\|v_j^{\boldsymbol\epsilon}-\delta_{\epsilon_1}u_j^{\boldsymbol\epsilon}\|_{L^\infty(0,T;\tilde H^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)}.\end{aligned}$$

Since Φ is non-decreasing, using the limiting assumption of Φ in (Q.3), possibly choosing a smaller $\epsilon_0=\epsilon_0(n,s,\Omega,T,\delta,\boldsymbol g,m)>0$, we can assure that $\Phi(C|\boldsymbol\epsilon|)\le\frac12$, and hence

$$\|v_j^{\boldsymbol\epsilon}-\delta_{\epsilon_1}u_j^{\boldsymbol\epsilon}\|_{L^\infty(0,T;\tilde H^s(\Omega))\cap L^\infty(\mathbb{R}^n_T)}\le \tilde C M_2\|g_1\|_{\mathrm{ext}}\|u_j^{\boldsymbol\epsilon+\epsilon_1\boldsymbol e_1}-u_j^{\boldsymbol\epsilon}\|_{L^\infty(\Omega_T)}.$$

Finally, using Lemma 4.1, we conclude Lemma 4.2. □

Using (Q.4), we also see that $\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}$ satisfies

$$\begin{cases}(\partial_t+(-\Delta)^s)(\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0})=0 & \text{in } \Omega_T,\\ \partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}=g_1 & \text{in } \Omega_T^e,\\ \partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}=0 & \text{on } \{0\}\times\mathbb{R}^n.\end{cases}\tag{4.8}$$

By uniqueness of solutions (see Proposition 2.9), we know that $\partial_{\epsilon_1}u_1^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}=\partial_{\epsilon_1}u_2^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}\text{in } \Omega_T$. For later convenience, we simply denote

$$\partial_{\epsilon_1}u^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}=\partial_{\epsilon_1}u_1^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}=\partial_{\epsilon_1}u_2^{\boldsymbol\epsilon}|_{\boldsymbol\epsilon=0}\text{in } \Omega_T.\tag{4.9}$$

In the next lemma we show that the information from DN-map can be passed to the first-order linearized DN-map:

Lemma 4.3. *If $\Lambda_{q_1}(f)=\Lambda_{q_1}(f)$ for all $f\in C_c^\infty(W_T)$ with $\|f\|_{\mathrm{ext}}\le\tilde\epsilon_0$, where $\tilde\epsilon_0$ is the constant given in Proposition 2.10, then there exists a constant $\epsilon_0=\epsilon_0(n,s,\Omega,T,\delta,\boldsymbol g,m)>0$ with $0<\epsilon_0<\tilde\epsilon_0$ such that*

$$(-\Delta)^s\partial_{\epsilon_1}u_1^{\boldsymbol\epsilon}\big|_{V_T}=(-\Delta)^s\partial_{\epsilon_1}u_2^{\boldsymbol\epsilon}\big|_{V_T}\quad \textit{for all}\quad \epsilon\quad \textit{with}\quad |\boldsymbol\epsilon|\le\epsilon_0.\tag{4.10}$$

Proof. We have

$$\begin{aligned}&\Big\|(-\Delta)^s\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}-\frac{\Lambda_{q_j}((\boldsymbol\epsilon+\epsilon_1\boldsymbol e_1)\cdot\boldsymbol g)-\Lambda_{q_j}(\boldsymbol\epsilon\cdot\boldsymbol g)}{\epsilon_1}\Big\|_{L^\infty(0,T;H^{-s}(V))}\\ =\;&\Big\|(-\Delta)^s\Big(\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}-\frac{u_j^{\boldsymbol\epsilon+\epsilon_1\boldsymbol e_1}-u_j^{\boldsymbol\epsilon}}{\epsilon_1}\Big)\Big\|_{L^\infty(0,T;H^{-s}(V))}\\ \le\;&\Big\|(-\Delta)^s\Big(\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}-\frac{u_j^{\boldsymbol\epsilon+\epsilon_1\boldsymbol e_1}-u_j^{\boldsymbol\epsilon}}{\epsilon_1}\Big)\Big\|_{L^\infty(0,T;H^{-s}(\mathbb{R}^n))}\\ \le\;& C\Big\|\Big(\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}-\frac{u_j^{\boldsymbol\epsilon+\epsilon_1\boldsymbol e_1}-u_j^{\boldsymbol\epsilon}}{\epsilon_1}\Big)\Big\|_{L^\infty(0,T;H^{s}(\mathbb{R}^n))}.\end{aligned}$$

From Lemma 4.2, we have

$$\lim_{\epsilon_1\to0}\Big\|(-\Delta)^s\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon}-\frac{\Lambda_{q_j}((\boldsymbol\epsilon+\epsilon_1\boldsymbol e_1)\cdot\boldsymbol g)-\Lambda_{q_j}(\boldsymbol\epsilon\cdot\boldsymbol g)}{\epsilon_1}\Big\|_{L^\infty(0,T;H^{-s}(V))}=0.$$

Combining this equality with the assumption $\Lambda_{q_1}=\Lambda_{q_2}$, we conclude (4.10). □

4.3. **Second order linearization.** First of all, we recall (4.6):

$$\|\partial_{\epsilon_p}u_j^{\boldsymbol\epsilon}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^n_T)}\le C\|g_p\|_{\mathrm{ext}}\quad \text{for } p=1,2.\tag{4.11}$$

see Lemma 4.2. Acting a formal differential operator ∂_{ϵ_2} on (4.5), we obtain

$$\begin{cases}\begin{aligned}&(\partial_t+(-\Delta)^s+\partial_z q_j(\cdot,u_j^{\boldsymbol\epsilon}))(\partial_{\epsilon_1\epsilon_2}u_j^{\boldsymbol\epsilon})\\ &\quad+\partial_z^2 q_j(\cdot,u_j^{\boldsymbol\epsilon})(\partial_{\epsilon_1}u_j^{\boldsymbol\epsilon})(\partial_{\epsilon_2}u_j^{\boldsymbol\epsilon})=0\end{aligned} & \text{in } \Omega_T,\\ \partial_{\epsilon_1\epsilon_2}u_j^{\boldsymbol\epsilon}=0 & \text{in } \Omega_T^e \text{ and on } \{0\}\times\mathbb{R}^n.\end{cases}\tag{4.12}$$

Since the term $\partial_z^2 q_j(\cdot,u_j^{\boldsymbol{\epsilon}})(\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}})(\partial_{\epsilon_2}u_j^{\boldsymbol{\epsilon}})$ is bounded in Ω_T, using Proposition 2.9, there exists a unique solution $v_j^{\boldsymbol{\epsilon}} \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}_T^n)$ to (4.12) with

$$\begin{aligned}\|v_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;\tilde{H}^s(\Omega))\cap L^\infty(\mathbb{R}_T^n)} &\le C\|\partial_z^2 q_j(\cdot,u_j^{\boldsymbol{\epsilon}})(\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}})(\partial_{\epsilon_2}u_j^{\boldsymbol{\epsilon}})\|_{L^\infty(\Omega_T)} \\ &\le CM_2\|g_1\|_{\mathrm{ext}}\|g_2\|_{\mathrm{ext}}. \qquad \text{(using (Q.4) and (4.11))}\end{aligned} \tag{4.13}$$

Again, $v_j^{\boldsymbol{\epsilon}}$ is temporary notation, which will be dropped after showing $\partial_{\epsilon_1\epsilon_2}u_j^{\boldsymbol{\epsilon}}$ is well-defined. We emphasize that we have already dropped $v_j^{\boldsymbol{\epsilon}}$, so this will not conflict with the one used in Section 4.2.

Lemma 4.4. *There exists a constant $\epsilon_0 = \epsilon_0(n,s,\Omega,T,\delta,\boldsymbol{g},m) > 0$ with $0 < \epsilon_0 < \tilde{\epsilon}_0$, where $\tilde{\epsilon}_0$ is given in Proposition 2.10, such that for each $\boldsymbol{\epsilon}$ with $|\boldsymbol{\epsilon}| < \epsilon_0$, we have*

$$\lim_{\epsilon_2\to 0}\|v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_2}\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}_T^n)} = 0, \tag{4.14}$$

where $\delta_{\epsilon_2}\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}} = \frac{\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}+\epsilon_2\boldsymbol{e}_2}-\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}}}{\epsilon_2}$ in Ω_T, provided $|\boldsymbol{\epsilon}|+|\epsilon_2| < \epsilon_0$.

Proof. Let ϵ_2 satisfies $|\epsilon_2| \le |\boldsymbol{\epsilon}|$ and $|\boldsymbol{\epsilon}|+|\epsilon_2| < \epsilon_0$. Note that

$$\begin{cases}(\partial_t + (-\Delta)^s)(v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_2}\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}}) = \mathcal{G}_2 & \text{in } \Omega_T,\\ v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_2}\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}} = 0 & \text{in } \Omega_T^e \text{ and on } \{0\}\times\mathbb{R}^n,\end{cases}$$

where

$$\begin{aligned}-\mathcal{G}_2 &= \partial_z q_j(\cdot,u_j^{\boldsymbol{\epsilon}})v_j^{\boldsymbol{\epsilon}} + \partial_z^2 q_j(\cdot,u_j^{\boldsymbol{\epsilon}})(\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}})(\partial_{\epsilon_2}u_j^{\boldsymbol{\epsilon}}) \\ &\quad - \frac{\partial_z q_j(\cdot,u_j^{\boldsymbol{\epsilon}+\epsilon_2\boldsymbol{e}_2})\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}+\epsilon_2\boldsymbol{e}_2} - \partial_z q_j(\cdot,u_j^{\boldsymbol{\epsilon}})\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}}}{\epsilon_2}.\end{aligned}$$

After some computation we can write $-\mathcal{G}_2 = \mathcal{G}_{21}+\mathcal{G}_{22}+\mathcal{G}_{23}$, where

$$\begin{cases}\mathcal{G}_{21} = \partial_z q_j(\cdot,u_j^{\boldsymbol{\epsilon}})\big[v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_2}\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}}\big],\\ \mathcal{G}_{22} = \big[\partial_z^2 q_j(\cdot,u_j^{\boldsymbol{\epsilon}})(\partial_{\epsilon_2}u_j^{\boldsymbol{\epsilon}}) - \frac{\partial_z q_j(\cdot,u_j^{\boldsymbol{\epsilon}+\epsilon_2\boldsymbol{e}_2})-\partial_z q_j(\cdot,u_j^{\boldsymbol{\epsilon}})}{\epsilon_2}\big](\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}+\epsilon_2\boldsymbol{e}_2}),\\ \mathcal{G}_{23} = \partial_z^2 q_j(\cdot,u_j^{\boldsymbol{\epsilon}})\partial_{\epsilon_2}u_j^{\boldsymbol{\epsilon}}\big[\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}} - \partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}+\epsilon_2\boldsymbol{e}_2}\big],\end{cases}$$

Note that $\|v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_2}\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}_T^n)} \le C\|\mathcal{G}_2\|_{L^\infty(\Omega_T)}$. Possibly choosing a smaller ϵ_0, we have $\|\mathcal{G}_{21}\|_{L^\infty(\Omega_T)} \le \frac{1}{2}\|v_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_2}\partial_{\epsilon_1}u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}_T^n)}$.Using the mean value theorem and Lemma 4.2, we know that $\lim_{\epsilon_2\to 0}(\|\mathcal{G}_{22}\|_{L^\infty(\Omega_T)} + \|\mathcal{G}_{23}\|_{L^\infty(\Omega_T)}) = 0$. We then conclude (4.14) by using arguments similar to Lemma 4.2. □

Akin to Lemma 4.3, we next demonstrate the following lemma.

Lemma 4.5. *If $\Lambda_{q_1}(f) = \Lambda_{q_1}(f)$ for all $f \in \mathcal{C}_c^\infty(W_T)$ with $\|f\|_{\mathrm{ext}} \le \tilde{\epsilon}_0$, where $\tilde{\epsilon}_0$ is the constant given in Proposition 2.10, then there exists a constant $\epsilon_0 = \epsilon_0(n,s,\Omega,T,\delta,\boldsymbol{g},m) > 0$ with $0 < \epsilon_0 < \tilde{\epsilon}_0$ such that*

$$(-\Delta)^s\partial_{\epsilon_1\epsilon_2}u_1^{\boldsymbol{\epsilon}}\big|_{V_T} = (-\Delta)^s\partial_{\epsilon_1\epsilon_2}u_2^{\boldsymbol{\epsilon}}\big|_{V_T} \quad \forall\, \boldsymbol{\epsilon} \text{ with } |\boldsymbol{\epsilon}| \le \epsilon_0. \tag{4.15}$$

Proof. Using similar arguments as in Lemma 4.3 (with Lemma 4.4), we can show that (4.10) implies (4.15). Then we conclude the lemma by Lemma 4.3. □

Proof of Theorem 1.1 for $m = 2$. Using (Q.3) and (4.9), we know that $\partial_{\epsilon_1\epsilon_2}u_j^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}$ satisfies

$$\begin{cases}\begin{aligned}&(\partial_t + (-\Delta)^s)(\partial_{\epsilon_1\epsilon_2}u_j^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0})\\ &\quad + \partial_z^2 q_j(\cdot,0)(\partial_{\epsilon_1}u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0})(\partial_{\epsilon_2}u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}) = 0\end{aligned} & \text{in } \Omega_T,\\ \partial_{\epsilon_1\epsilon_2}u_j^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0} = 0 & \text{in } \Omega_T^e \text{ and on } \{0\}\times\mathbb{R}^n.\end{cases}$$

Hence, we know that $v := \partial_{\epsilon_1\epsilon_2} u_1^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0} - \partial_{\epsilon_1\epsilon_2} u_2^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}$ satisfies

$$\begin{cases} (\partial_t + (-\Delta)^s)v + (\partial_z^2 q_1(\cdot,0) - \partial_z^2 q_2(\cdot,0))(\partial_{\epsilon_1} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0})(\partial_{\epsilon_2} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}) = 0 & \text{in } \Omega_T, \\ v = 0 \quad \text{in } \Omega_T^e \text{ and on } \{0\}\times\mathbb{R}^n. \end{cases}$$

From Lemma 4.5, we know that $(-\Delta)^s v\big|_{V_T} = 0$. Since $v = 0$ in V_T, using the unique continuation property of the fractional Laplacian in Lemma 3.1, we conclude that $v \equiv 0$. Therefore, we know that

$$\big(\partial_z^2 q_1(\cdot,0) - \partial_z^2 q_2(\cdot,0)\big)(\partial_{\epsilon_1} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0})(\partial_{\epsilon_2} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}) = 0. \tag{4.16}$$

Since $g_1, g_2 \in \mathcal{C}_c^\infty(W_T)$ are arbitrary, using (4.8) and the Runge approximation for fractional diffusion equation in Proposition 3.2, we conclude $\partial_z^2 q_1(\cdot,0) - \partial_z^2 q_2(\cdot,0) = 0$ in Ω_T, which proves Theorem 1.1 for $m = 2$. □

Proof of Corollary 1.2 for $m = 2$. Using the same argument as above, we reach (4.16):

$$(\partial_z^2 q_1(\cdot,0) - \partial_z^2 q_2(\cdot,0))(\partial_{\epsilon_1} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0})(\partial_{\epsilon_2} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}) = 0. \tag{4.17}$$

Using (4.8), the unique continuation property of the fractional Laplacian in Lemma 3.1 and a simple contradiction argument, for each $x_0 \in \Omega$, we can find a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$ with $x_k \to x_0$ and a sequence $\{t_k\}_{k\in\mathbb{N}} \subset (0,T)$ such that

$$\partial_{\epsilon_1} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}(t_k, x_k) \neq 0 \quad \text{and} \quad \partial_{\epsilon_2} u^{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0}(t_k, x_k) \neq 0.$$

Therefore from (4.17) we know that $\partial_z^2 q_1(x_k,0) = \partial_z^2 q_2(x_k,0)$ (here we have assumed that $\partial_z^2 q_1(\cdot,0)$ and $\partial_z^2 q_2(\cdot,0)$ are independent of t). Hence by continuity of $\partial_z^2 q_1(\cdot,0)$, $\partial_z^2 q_2(\cdot,0)$ and the arbitrariness of $x_0 \in \Omega$, we conclude $\partial_z^2 q_1(\cdot,0) - \partial_z^2 q_2(\cdot,0) = 0$ in Ω, which proves Corollary 1.2 for $m = 2$. □

4.4. **Higher order linearization.** For each $2 \le p \le m$, we denote $\partial_{(p)} = \partial_{\epsilon_1}\cdots\partial_{\epsilon_p}$. By repeating formal differentiations to the equation (4.12), we obtain the following p-th order linearization

$$\begin{cases} (\partial_t + (-\Delta)^s)\partial_{(p)} u_j^{\boldsymbol{\epsilon}} + \partial_{(p)} q_j(\cdot, u_j^{\boldsymbol{\epsilon}}) = 0 & \text{in } \Omega_T, \\ \partial_{(p)} u_j^{\boldsymbol{\epsilon}} = 0 & \text{in } \Omega_T^e \text{ and on } \{0\}\times\mathbb{R}^n. \end{cases}$$

where we simply denote $\partial_{(p)} = \partial_{\epsilon_1}\cdots\partial_{\epsilon_p}$. By induction, we can verify

$$\partial_{(p)} q_j(\cdot, u_j^{\boldsymbol{\epsilon}}) = \partial_z q_j(\cdot, u_j^{\boldsymbol{\epsilon}})\partial_{(p)} u_j^{\boldsymbol{\epsilon}} + \sum_{\ell=2}^{p-1} \partial_z^\ell q_j(\cdot, u_j^{\boldsymbol{\epsilon}}) \mathcal{T}_p^\ell(u_j^{\boldsymbol{\epsilon}}) + \partial_z^p q_j(\cdot, u_j^{\boldsymbol{\epsilon}}) \prod_{\ell=1}^{p} \partial_{\epsilon_\ell} u_j^{\boldsymbol{\epsilon}},$$

where $\mathcal{T}_p^\ell(u_j^{\boldsymbol{\epsilon}})$ is a generic notation (in order p linearization) signifying a combination of the terms $\partial_{\boldsymbol{\epsilon}}^{\boldsymbol{\alpha}} u_j^{\boldsymbol{\epsilon}}$ with multi-index $\boldsymbol{\alpha}$ satisfying $1 \le |\boldsymbol{\alpha}| \le p-1$. The following facts can be proved using strong induction on m:

(1) Functions $\partial_{(p)} u_j^{\boldsymbol{\epsilon}} \in L^\infty(0,T;H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}_T^n)$ are well-defined for each $2 \le p \le m$.

(2) There exists $\epsilon_0 = \epsilon_0(n,s,\Omega,T,\delta,\boldsymbol{g},m) > 0$ with $0 < \epsilon_0 < \tilde{\epsilon}_0$, where $\tilde{\epsilon}_0$ is the constant given in Proposition 2.10, such that

$$\lim_{\epsilon_p\to 0} \|\partial_{(p)} u_j^{\boldsymbol{\epsilon}} - \delta_{\epsilon_p}\partial_{(p-1)} u_j^{\boldsymbol{\epsilon}}\|_{L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^\infty(\mathbb{R}_T^n)} = 0 \quad \text{for all } 2 \le p \le m,$$

where $\delta_{\epsilon_p}\partial_{(p-1)} u_j^{\boldsymbol{\epsilon}} = \frac{\partial_{(p-1)} u_j^{\boldsymbol{\epsilon}+\epsilon_p \boldsymbol{e}_p} - \partial_{(p-1)} u_j^{\boldsymbol{\epsilon}}}{\epsilon_p}$ in Ω_T, provided $|\boldsymbol{\epsilon}| + |\epsilon_p| < \epsilon_0$.

(3) Moreover, if $\Lambda_{q_1}(f) = \Lambda_{q_1}(f)$ for all $f \in \mathcal{C}_c^\infty(W_T)$ with $\|f\|_{\mathrm{ext}} \le \tilde{\epsilon}_0$, then we have

$$(-\Delta)^s \partial_{(p)} u_1^{\boldsymbol{\epsilon}}\big|_{V_T} = (-\Delta)^s \partial_{(p)} u_2^{\boldsymbol{\epsilon}}\big|_{V_T} \quad \text{for all } 2 \le p \le m. \tag{4.18}$$

Using the observations above, we are now ready to prove our main result.

Proof of Theorem 1.1. We prove by induction on m. We assume the following hypothesis:

$$\partial_z^p q_1(\cdot,0) = \partial_z^p q_1(\cdot,0) \quad \text{for all } \ 2 \le p \le m-1. \tag{4.19}$$

Using (4.9), we see that

$$\partial_{(m)} q_j(\cdot,u_j^\epsilon)|_{\epsilon=0} = \partial_z q_j(\cdot,0)\partial_{(m)} u_j^\epsilon|_{\epsilon=0} + \sum_{\ell=2}^{m-1} \partial_z^\ell q_j(\cdot,0)\mathcal{T}_m^\ell(u_j^\epsilon)\big|_{\epsilon=0} + \partial_z^m q_j(\cdot,0)\prod_{\ell=1}^{m} \partial_{\epsilon_\ell} u^\epsilon|_{\epsilon=0}.$$

Using (4.19), we see that $\sum_{\ell=2}^{m-1} \partial_z^\ell q_j(\cdot,0)\mathcal{T}_m^\ell(u_1^\epsilon)\big|_{\epsilon=0} = \sum_{\ell=2}^{m-1} \partial_z^\ell q_j(\cdot,0)\mathcal{T}_m^\ell(u_2^\epsilon)\big|_{\epsilon=0}$. Hence, we know that $v := \partial_{(m)} u_1^\epsilon|_{\epsilon=0} - \partial_{(m)} u_2^\epsilon|_{\epsilon=0}$ satisfies

$$\begin{cases} (\partial_t + (-\Delta)^s)v + (\partial_z^m q_1(\cdot,0) - \partial_z^m q_2(\cdot,0))\prod_{\ell=1}^m \partial_{\epsilon_\ell} u^\epsilon|_{\epsilon=0} = 0 & \text{in } \ \Omega_T \\ v = 0 & \text{in } \ \Omega_T^e \ \text{ and on } \ \{0\}\times\mathbb{R}^n. \end{cases}$$

Using (4.18), we know that $(-\Delta)^s v|_{V_T} = 0$. Since $v = 0$ in V_T, by using the unique continuation principle of the fractional Laplacian (see Lemma 3.1), we conclude that $v \equiv 0$. Hence, we know that $(\partial_z^m q_1(\cdot,0) - \partial_z^m q_2(\cdot,0))\prod_{\ell=1}^m \partial_{\epsilon_\ell} u^\epsilon|_{\epsilon=0} = 0$ in Ω_T. Since $g_1,\cdots,g_m \in \mathcal{C}_c^\infty(W_T)$ are arbitrary, using (4.8) and the Runge approximation for the fractional diffusion equation proved in Proposition 3.2, we conclude $\partial_z^m q_1(\cdot,0) = \partial_z^m q_2(\cdot,0)$ in Ω_T. This completes the proof of Theorem 1.1. □

5. Analogous result for the fractional wave equation

The following results can be proved by modifying the ideas in [KLW21, Corollary 2.2] (or [Eva10, Chapter 7]), which we use later to prove the well-posedness of (1.5) with small exterior data and to solve inverse problem as well.

Lemma 5.1. *Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, let $W \subset \Omega^e$ be any open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $a \in L^\infty(\Omega_T)$. Then for any $F \in L^2(\Omega_T)$, $f \in \mathcal{C}_c^\infty(W_T)$, $\psi \in \tilde{H}^0(\Omega)$, $\varphi \in \tilde{H}^s(\Omega)$, there exists a unique solution u of*

$$\begin{cases} (\partial_t^2 + (-\Delta)^s + a)u = F & \text{in } \ \Omega_T, \\ u = f & \text{in } \ \Omega_T^e, \\ u = \varphi, \quad \partial_t u = \psi & \text{on } \ \{0\}\times\mathbb{R}^n. \end{cases} \tag{5.1}$$

satisfying

$$\begin{aligned} &\|u - f\|_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le C(\|\varphi\|_{\tilde{H}^s(\Omega)} + \|\psi\|_{L^2(\Omega)} + \|F - (-\Delta)^s f\|_{L^2(\Omega_T)}) \end{aligned} \tag{5.2}$$

for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)})$.

Remark 5.1. It is interesting to compare (5.2) with (2.8): both solutions (wave and diffusion) have regularity $L^\infty(0,T;H^s(\mathbb{R}^n))$. In [KLW21, Corollary 2.2], they only consider the case when a is independent of time t. The existence of solutions can be proved using exactly the same argument, but the proof of uniqueness result need extra care. In contrast to [Eva10, Chapter 7], here we do not assume the $W^{1,\infty}(0,T;L^\infty(\Omega))$ regularity for the coefficient a.

Proof of uniqueness result of Lemma 5.1. Let $u \in L^2(0,T;H^s(\mathbb{R}^n)) \cap H^1(0,T;L^2(\Omega))$ be the solution of

$$\begin{cases} (\partial_t^2 + (-\Delta)^s + a)u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{in } \Omega_T^e, \\ u = \partial_t u = 0 & \text{on } \{0\}\times\mathbb{R}^n. \end{cases} \tag{5.3}$$

We want to show that $u \equiv 0$. Fix $0 \le \eta \le T$ and set

$$v(t,\cdot) := \begin{cases} \int_t^\eta u(\tau,\cdot)\,\mathrm{d}\tau & \text{if } 0 \le t \le \eta, \\ 0 & \text{if } \eta \le t \le T. \end{cases}$$

Then $v(t,\cdot) \in \tilde{H}^s(\Omega)$ for each $0 \le t \le T$, and so

$$\int_\Omega \int_0^\eta (\partial_t^2 u) v \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}^n} \int_0^\eta (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v \,\mathrm{d}t\,\mathrm{d}x + \int_\Omega \int_0^\eta auv \,\mathrm{d}t\,\mathrm{d}x = 0. \tag{5.4}$$

Since $\partial_t u(0) = v(\eta) = 0$ and $\partial_t v = -u$ for all $0 \le t \le \eta$, we see that

$$\begin{aligned} \int_\Omega \int_0^\eta (\partial_t^2 u) v \,\mathrm{d}t\,\mathrm{d}x &= -\int_\Omega \int_0^\eta (\partial_t u)(\partial_t v)\,\mathrm{d}t\,\mathrm{d}x = \int_\Omega \int_0^\eta (\partial_t u) u \,\mathrm{d}t\,\mathrm{d}x \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \int_0^\eta |u|^2 \,\mathrm{d}t\,\mathrm{d}x = \frac{1}{2}\|u(\eta,\cdot)\|_{L^2(\Omega)}^2. \end{aligned} \tag{5.5}$$

Using the fact $\partial_t v = -u$ for all $0 \le t \le \eta$, we also have

$$\begin{aligned} \int_{\mathbb{R}^n} \int_0^\eta (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v \,\mathrm{d}t\,\mathrm{d}x &= -\int_{\mathbb{R}^n} \int_0^\eta \partial_t\big((-\Delta)^{\frac{s}{2}} v\big)(-\Delta)^{\frac{s}{2}} v \,\mathrm{d}t\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\mathbb{R}^n} \int_0^\eta \frac{d}{dt}|(-\Delta)^{\frac{s}{2}} v|^2 \,\mathrm{d}t\,\mathrm{d}x = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} v(0,\cdot)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \tag{5.6}$$

Since $a \in L^\infty(\Omega_T)$, combining (5.4), (5.5) and (5.6), together with the Hardy-Littlewood-Sobolev inequality (A.5), we obtain

$$\|v(0,\cdot)\|_{\tilde{H}^s(\Omega)}^2 + \|u(\eta,\cdot)\|_{L^2(\Omega)}^2 \le C \int_0^\eta \big(\|v(t,\cdot)\|_{L^2(\Omega)}^2 + \|u(t,\cdot)\|_{L^2(\Omega)}^2\big)\,\mathrm{d}t. \tag{5.7}$$

Let us write $w(t,\cdot) := \int_0^t u(\tau,\cdot)\,\mathrm{d}\tau$. Since $v(0,\cdot) = w(\eta,\cdot)$ and $v(t,\cdot) = w(\eta,\cdot) - w(t,\cdot)$, from (5.7) we know that

$$\begin{aligned} \|w(\eta,\cdot)\|_{\tilde{H}^s(\Omega)}^2 + \|u(\eta,\cdot)\|_{L^2(\Omega)}^2 &\lesssim \int_0^\eta \big(\|w(\eta,\cdot) - w(t,\cdot)\|_{L^2(\Omega)}^2 + \|u(t,\cdot)\|_{L^2(\Omega)}^2\big)\,\mathrm{d}t \\ &\lesssim \eta\|w(\eta,\cdot)\|_{L^2(\Omega)} + \int_0^\eta \big(\|w(t,\cdot)\|_{L^2(\Omega)}^2 + \|u(t,\cdot)\|_{L^2(\Omega)}^2\big)\,\mathrm{d}t. \end{aligned}$$

Therefore, we can choose T_1, which is independent of η, such that

$$\|w(\eta,\cdot)\|_{\tilde{H}^s(\Omega)}^2 + \|u(\eta,\cdot)\|_{L^2(\Omega)}^2 \le C \int_0^\eta \big(\|w(t,\cdot)\|_{L^2(\Omega)}^2 + \|u(t,\cdot)\|_{L^2(\Omega)}^2\big)\,\mathrm{d}t,$$

for all $0 \le \eta \le T_1$. Using the Grönwall's inequality in [Eva10, Section B.2], we know that $u(t,\cdot) = 0$ for all $t \in [0, T_1]$. Applying the same argument on the intervals $[T_1, 2T_1]$, $[2T_1, 3T_1]$, etc., we conclude that $u \equiv 0$. □

We need the following Sobolev embedding to obtain $L^\infty(\Omega_T)$-regularity of the solution, which is a special case of [DNPV12, Theorem 8.2]:

Lemma 5.2 ([DNPV12]). *Let $n = 1$ and $1/2 < s < 1$. There exists a constant $C = C(s,\Omega)$ such that $\|f\|_{\mathcal{C}^{0,\alpha}(\Omega)} \le C(\|f\|_{L^2(\Omega)}^2 + [f]_{H^s(\Omega)}^2) \le C\|f\|_{H^s(\mathbb{R})}^2$, for any $f \in L^2(\Omega)$ with $\alpha = (2s-1)/2$.*

Therefore, Lemma 5.1 implies the following result.

Proposition 5.3. *Let $n = 1$ and $1/2 < s < 1$. Let $\Omega \subset \mathbb{R}$ be a bounded open set in $\mathbb{R}^n$, let $W \subset \Omega^e$ be any open set satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $a \in L^\infty(\Omega_T)$. Then for any $\tilde{F} \in L^\infty(\Omega_T)$ and $f \in C_c^\infty(W_T)$, there exists a unique weak solution u of (5.1) satisfying*

$$\begin{aligned}&\|u\|_{L^\infty(0,T;H^s(\mathbb{R}^1))\cap L^\infty(\mathbb{R}^1_T)} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))}\\ &\le C\big(\|\varphi\|_{\tilde{H}^s(\Omega)} + \|\psi\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega_T)}\\ &\quad + \|f\|_{L^\infty(0,T;H^s(\mathbb{R}^1))\cap L^\infty(\mathbb{R}^1_T)} + \|(-\Delta)^s f\|_{L^2(\Omega_T)}\big)\end{aligned}$$

for certain constant $C = C(s, T, \|a\|_{L^\infty(\Omega_T)}, \Omega)$.

Using the same argument as in Proposition 2.10, we also can prove the well-posedness of (1.5) for small exterior data:

Proposition 5.4. *Let $n = 1$ and $\frac{1}{2} < s < 1$. Let $\Omega \subset \mathbb{R}$ be a bounded open set, let $W \subset \Omega^e$ be any open set satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Fix any parameter $\delta > 0$. Assume q satisfies (Q.1)–(Q.3). There exists a sufficiently small parameter $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(s, \Omega, T, \delta) > 0$ such that the following statement holds: Given any $f \in C_c^\infty(W_T)$ with $\|f\|_{\mathrm{ext}} \le \tilde{\epsilon}_0$, there exists a unique solution $u \in L^\infty(0, T; H^s(\mathbb{R}^1)) \cap L^\infty(\mathbb{R}^1_T)$ of (1.5) with*

$$\|u\|_{L^\infty(0,T;H^s(\mathbb{R}^1))\cap L^\infty(\mathbb{R}^1_T)} \le C\|f\|_{\mathrm{ext}} \tag{5.8}$$

for certain constant $C = C(s, T, \Omega)$.

Finally, the inverse problem for the nonlinear fractional wave equation (1.5), i.e. Theorem 1.3 and Corollary 1.4, can be proved using exactly the same idea as in Theorem 1.1 and Corollary 1.2, respectively, see Section 4.

Appendix A. Well-posedness of the linear fractional diffusion equation

A.1. **Uniqueness of weak solution.** We first prove the uniqueness of weak solution of (2.3) as well as (2.4). It suffices to prove the following statement: If u a weak solution of

$$\begin{cases}(\partial_t + (-\Delta)^s + a)v = 0 & \text{in } \Omega_T,\\ v = 0 & \text{in } \Omega_T^e \text{ and on } \{0\} \times \mathbb{R}^n.\end{cases} \tag{A.1}$$

then $u \equiv 0$. Multiplying the first equation of (A.1) by v, we obtain

$$\begin{aligned}0 = \langle \boldsymbol{v}', \boldsymbol{v}\rangle + \mathcal{B}[\boldsymbol{v}, \boldsymbol{v}; t] &= \frac{\mathsf{d}}{\mathsf{d}t}\Big(\frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)}\Big) + \mathcal{B}[\boldsymbol{v}, \boldsymbol{v}; t]\\ &\ge \frac{\mathsf{d}}{\mathsf{d}t}\Big(\frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)}\Big) - \|a\|_{L^\infty(\Omega_T)}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)},\end{aligned}$$

that is, $\frac{\mathsf{d}}{\mathsf{d}t}\big(\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)}\big) \le 2\|a\|_{L^\infty(\Omega_T)}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)}$. Using the Grönwall's inequality in [Eva10, Section B.2], we conclude $\|\boldsymbol{v}(t)\|^2_{L^2(\Omega)} = 0$ for all $0 \le t \le T$, hence $u \equiv 0$. The uniqueness is proved.

A.2. **Existence of weak solution.** Now it suffices prove that there exists a weak solution of (2.4).

Step 1: Galerkin approximation. We now set up the Galerkin approximation for (2.4). Similar to [KLW21, Appendix A], we consider an eigenbasis $\{w_k\}_{k\in\mathbb{N}}$ associated with the Dirichlet fractional Laplacian in a bounded domain Ω. We normalize these eigenfunctions so that

$$\begin{aligned}&\{w_k\}_{k\in\mathbb{N}} \text{ be an orthogonal basis in } \tilde{H}^s(\Omega),\\ &\{w_k\}_{k\in\mathbb{N}} \text{ be an orthonormal basis in } L^2(\Omega).\end{aligned}$$

Given any fixed integer $m \in \mathbb{N}$, we consider the following ansatz:

$$\boldsymbol{v}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k. \tag{A.2}$$

Plugging the ansatz (A.2) into Definition 2.1(b), we obtain

$$\begin{cases} (\boldsymbol{v}_m'(t), w_k)_{L^2(\Omega)} + \mathcal{B}[\boldsymbol{v}_m, w_k; t] = (\tilde{\boldsymbol{F}}(t), w_k)_{L^2(\Omega)} \quad \text{for all } \ 0 \le t \le T, \\ d_m^k(0) = (\tilde{\varphi}, w_k)_{L^2(\Omega)}. \end{cases} \tag{A.3}$$

Note that $(\boldsymbol{v}_m', w_k)_{L^2(\Omega)} = (d_m^k)'(t)$, $\mathcal{B}[\boldsymbol{v}_m, w_k; t] = \sum_{\ell=1}^m e^{k\ell}(t) d_m^k(t)$ with the coefficients $e^{k\ell}(t) := \mathcal{B}[w_\ell, w_k; t]$. This shows that $d_m^k(t)$ satisfies the following linear system of ordinary differential equation (ODE):

$$\begin{cases} (d_m^k)'(t) + \sum_{\ell=1}^m e^{k\ell}(t) d_m^k(t) = (\tilde{\boldsymbol{F}}(t), w_k)_{L^2(\Omega)} \quad \text{for all } \ 0 \le t \le T, \\ d_m^k(0) = (\tilde{\varphi}, w_k)_{L^2(\Omega)}. \end{cases}$$

Therefore, the standard ODE theory guarantees the existence and uniqueness of such $d_m^k(t)$, and thus (A.2) is a valid discretization of (2.4).

Step 2: Energy estimate. Multiplying (A.3) by $d_m^k(t)$, and summing over index $k = 1, \cdots, m$, we have

$$(\boldsymbol{v}_m', \boldsymbol{v}_m)_{L^2(\Omega)} + \mathcal{B}[\boldsymbol{v}_m, \boldsymbol{v}_m; t] = (\tilde{\boldsymbol{F}}, \boldsymbol{v}_m)_{L^2(\Omega)}. \tag{A.4}$$

The following Hardy-Littlewood-Sobolev inequality can be found in [Pon16, Prop. 15.5] or in [KLW21, equation (A.11)]:

$$\|\boldsymbol{v}_m\|_{L^2(\mathbb{R}^1)} = \|\boldsymbol{v}_m\|_{L^2(\Omega)} \le C(n,s)\|\phi\|_{L^{\frac{2n}{n-s}}(\mathbb{R}^n)} \le C(n,s)\|(-\Delta)^{\frac{s}{2}}\phi\|_{L^2(\mathbb{R}^1)} \tag{A.5}$$

for $n = 1$ and for all $\phi \in \tilde{H}^s(\Omega)$. On the other hand, we observe that $(\boldsymbol{v}_m', \boldsymbol{v}_m)_{L^2(\Omega)} = \frac{\mathsf{d}}{\mathsf{d}t}\big(\frac{1}{2}\|\boldsymbol{v}_m\|_{L^2(\Omega)}^2\big)$. Hence, from (A.4) we have

$$\frac{\mathsf{d}}{\mathsf{d}t}\Big(\frac{1}{2}\|\boldsymbol{v}_m\|_{L^2(\Omega)}^2\Big) + \|\boldsymbol{v}_m\|_{\tilde{H}^s(\Omega)}^2 \le C(n,s,\|a\|_{L^\infty(\Omega_T)})\big(\|\boldsymbol{v}_m\|_{L^2(\Omega)}^2 + \|\tilde{\boldsymbol{F}}\|_{L^2(\Omega)}^2\big) \tag{A.6}$$

for all $0 \le t \le T$. Using the Grönwall's inequality in [Eva10, Section B.2], we have

$$\|\boldsymbol{v}_m(t)\|_{L^2(\Omega)}^2 \le e^{Ct}\Big(\|\boldsymbol{v}_m(0)\|_{L^2(\Omega)}^2 + C\int_0^t \|\tilde{\boldsymbol{F}}(s)\|_{L^2(\Omega)}^2 \,\mathsf{d}s\Big) \quad \text{for all } \ 0 \le t \le T.$$

Since $\|\boldsymbol{v}_m(0)\|_{L^2(\Omega)}^2 = \sum_{k=1}^m |(\tilde{\varphi}, w_k)_{L^2(\Omega)}|^2 \le \sum_{k=1}^\infty |(\tilde{\varphi}, w_k)_{L^2(\Omega)}|^2 = \|\varphi\|_{L^2(\Omega)}^2$, then we have

$$\sup_{0 \le t \le T} \|\boldsymbol{v}_m(t)\|_{L^2(\Omega)}^2 \le C_{s,T,\|a\|_\infty}\big(\|\varphi\|_{L^2(\Omega)}^2 + \|\tilde{\boldsymbol{F}}\|_{L^2(\Omega_T)}^2\big). \tag{A.7}$$

Integrating (A.6) on $t \in [0,T]$, we obtain

$$\|\boldsymbol{v}_m\|_{L^2(0,T;\tilde{H}^s(\Omega))}^2 \le C_{s,\|a\|_\infty}\big(\|\boldsymbol{v}_m\|_{L^2(\Omega_T)}^2 + \|\tilde{\boldsymbol{F}}\|_{L^2(\Omega_T)}^2\big). \tag{A.8}$$

Combining (A.7) and (A.8), we obtain the following energy estimate:

$$\sup_{0 \le t \le T} \|\boldsymbol{v}_m(t)\|_{L^2(\Omega)}^2 + \|\boldsymbol{v}_m\|_{L^2(0,T;\tilde{H}^s(\Omega))}^2 \le C(n,s,T,\|a\|_{L^\infty(\Omega_T)})(\|\varphi\|_{L^2(\Omega)}^2 + \|\tilde{\boldsymbol{F}}\|_{L^2(\Omega_T)}^2). \tag{A.9}$$

Fixing any $\phi \in \tilde{H}^s(\Omega)$ with $\|\phi\|_{\tilde{H}^s(\Omega)} \le 1$, we write $\phi = \phi_1 + \phi_2$, where $\phi_1 \in \operatorname{span}\{w_k\}_{k=1}^m$ and $(\phi_2, w_k)_{L^2(\Omega)} = 0$ for $k = 1, \cdots, m$. Using (A.3), we see that

$$(\boldsymbol{v}_m'(t), \phi)_{L^2(\Omega)} = (\boldsymbol{v}_m'(t), \phi_1)_{L^2(\Omega)} = (\tilde{\boldsymbol{F}}, \phi_1) - \mathcal{B}[\boldsymbol{v}_m, \phi_1; t].$$

Since $\|\phi_1\|_{\tilde{H}^s(\Omega)} \leq 1$, this implies $|(\boldsymbol{v}_m'(t),\phi)_{L^2(\Omega)}| \leq C(\|\tilde{\boldsymbol{F}}(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_m\|^2_{\tilde{H}^s(\Omega)})$. Hence we know that

$$\|\boldsymbol{v}_m'(t)\|^2_{H^{-s}(\Omega)} := \sup_{\|\phi\|_{\tilde{H}^s(\Omega)}\leq 1} |(\boldsymbol{v}_m'(t),\phi)_{L^2(\Omega)}| \leq C_{\|a\|_\infty}(\|\tilde{\boldsymbol{F}}(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_m\|^2_{\tilde{H}^s(\Omega)}).$$

Integrating the inequality above on $t\in[0,T]$, and combining the result with (A.9), we obtain

$$\begin{aligned}&\sup_{0\leq t\leq T}\|\boldsymbol{v}_m(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_m\|^2_{L^2(0,T;\tilde{H}^s(\Omega))} + \|\boldsymbol{v}_m'\|^2_{L^2(0,T;H^{-s}(\Omega))}\\ &\quad\leq C_{s,T,\|a\|_\infty}(\|\varphi\|^2_{L^2(\Omega)} + \|\tilde{\boldsymbol{F}}\|^2_{L^2(\Omega_T)}).\end{aligned} \tag{A.10}$$

Step 3: Passing to the limit. By (A.10), we can extract a subsequence of $\{\boldsymbol{v}_m\}_{m\in\mathbb{N}}$, still denoted by $\{\boldsymbol{v}_m\}_{m\in\mathbb{N}}$ (for simplicity), such that

$$\begin{cases}\boldsymbol{v}_m \rightharpoonup \boldsymbol{v} & \text{weakly in } L^2(0,T;\tilde{H}^s(\Omega)),\\ \boldsymbol{v}_m' \rightharpoonup \boldsymbol{v}' & \text{weakly in } L^2(0,T;H^{-s}(\Omega)).\end{cases} \tag{A.11}$$

Given any fixed integer N, we write $\tilde{\boldsymbol{v}}(t) := \sum_{k=1}^N d^k(t)w_k$, where $d^k(t)$ $(k=1,\cdots,N)$ are arbitrary smooth functions (not the one in (A.2)). Choosing $m\geq N$, multiplying (A.3) by $d^k(t)$, and summing over $k=1,\cdots,N$, we obtain

$$\int_0^T \big((\boldsymbol{v}_m'(t),\tilde{\boldsymbol{v}}(t))_{L^2(\Omega)} + \mathcal{B}[\boldsymbol{v}_m,\tilde{\boldsymbol{v}};t]\big)\,\mathrm{d}t = \int_0^T (\tilde{\boldsymbol{F}}(t),\tilde{\boldsymbol{v}}(t))_{L^2(\Omega)}\,\mathrm{d}t. \tag{A.12}$$

Taking $m\to+\infty$ in (A.12), and from (A.11), we know that

$$\int_0^T \big(\langle\boldsymbol{v}'(t),\tilde{\boldsymbol{v}}(t)\rangle + \mathcal{B}[\boldsymbol{v},\tilde{\boldsymbol{v}};t]\big)\,\mathrm{d}t = \int_0^T (\tilde{\boldsymbol{F}}(t),\tilde{\boldsymbol{v}}(t))_{L^2(\Omega)}\,\mathrm{d}t. \tag{A.13}$$

Due to the arbitrariness of N and $\{d^k\}_{k=1}^N$, we have

$$\langle\boldsymbol{v}',\phi\rangle + \mathcal{B}[\boldsymbol{v},\phi;t] = (\tilde{\boldsymbol{F}}(t),\phi)_{L^2(\Omega)} \quad \text{for all } \phi\in\tilde{H}^s(\Omega).$$

This together with (A.10) verifies Definition 2.1(a)(b).

It remains to show $\boldsymbol{v}$ verifies Definition 2.1(c). To that end, let us choose any $\tilde{\boldsymbol{v}}\in\mathcal{C}^1(0,T;\tilde{H}^s(\Omega))$ with $\tilde{\boldsymbol{v}}(T)=0$. From (A.13), we have

$$\int_0^T \big((\tilde{\boldsymbol{v}}'(t),\boldsymbol{v}(t))_{L^2(\Omega)} + \mathcal{B}[\boldsymbol{v},\tilde{\boldsymbol{v}};t]\big)\,\mathrm{d}t = \int_0^T (\tilde{\boldsymbol{F}}(t),\tilde{\boldsymbol{v}}(t))_{L^2(\Omega)}\,\mathrm{d}t + (\tilde{\boldsymbol{v}}'(0),\boldsymbol{v}(0))_{L^2(\Omega)}. \tag{A.14}$$

Similarly, from (A.12), we have

$$\int_0^T \big((\tilde{\boldsymbol{v}}'(t),\boldsymbol{v}_m(t))_{L^2(\Omega)} + \mathcal{B}[\boldsymbol{v}_m,\tilde{\boldsymbol{v}};t]\big)\,\mathrm{d}t = \int_0^T (\tilde{\boldsymbol{F}}(t),\tilde{\boldsymbol{v}}(t))_{L^2(\Omega)}\,\mathrm{d}t + (\tilde{\boldsymbol{v}}'(0),\boldsymbol{v}_m(0))_{L^2(\Omega)}. \tag{A.15}$$

Combining (A.11) and (A.15), we obtain

$$\int_0^T \big((\tilde{\boldsymbol{v}}'(t),\boldsymbol{v}(t))_{L^2(\Omega)} + \mathcal{B}[\boldsymbol{v},\tilde{\boldsymbol{v}};t]\big)\,\mathrm{d}t = \int_0^T (\tilde{\boldsymbol{F}}(t),\tilde{\boldsymbol{v}}(t))_{L^2(\Omega)}\,\mathrm{d}t + (\tilde{\boldsymbol{v}}'(0),\varphi)_{L^2(\Omega)}.$$

Comparing this with (A.14), we see that $(\tilde{\boldsymbol{v}}'(0),\boldsymbol{v}(0))_{L^2(\Omega)} = (\tilde{\boldsymbol{v}}'(0),\varphi)_{L^2(\Omega)}$. Due to the arbitrariness of $\tilde{\boldsymbol{v}}$, we conclude that $\boldsymbol{v}$ verifies Definition 2.1(c).

Step 4. Higher regularity. We now further assume $\varphi\in\tilde{H}^s(\Omega)$. Multiplying (A.3) by $(d_m^k)'(t)$, and summing over $k=1,\cdots,m$, we have

$$(\boldsymbol{v}_m',\boldsymbol{v}_m')_{L^2(\Omega)} + \mathcal{B}[\boldsymbol{v}_m,\boldsymbol{v}_m'] = (\tilde{\boldsymbol{F}},\boldsymbol{v}_m')_{L^2(\Omega)}. \tag{A.16}$$

Note that we have

$$\mathcal{B}[\boldsymbol{v}_m, \boldsymbol{v}_m'] = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \boldsymbol{v}_m(t) (-\Delta)^{\frac{s}{2}} \boldsymbol{v}_m'(t) \, \mathsf{d}x + \int_\Omega a(t,x) \boldsymbol{v}_m(t,x) \boldsymbol{v}_m'(t,x) \, \mathsf{d}x$$
$$= \frac{\mathsf{d}}{\mathsf{d}t} \Big(\frac{1}{2} \int_{\mathbb{R}^1} |(-\Delta)^{\frac{s}{2}} \boldsymbol{v}_m(t)|^2 \, \mathsf{d}x \Big) + \int_\Omega a(t,x) \boldsymbol{v}_m(t,x) \boldsymbol{v}_m'(t,x) \, \mathsf{d}x,$$

and $\int_\Omega a(t,x)\boldsymbol{v}_m(t)\boldsymbol{v}_m'(t)\,\mathsf{d}x \le \epsilon \|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + C\epsilon^{-1}\|\boldsymbol{v}_m(t)\|^2_{L^2(\Omega)}$ and $|(\tilde{\boldsymbol{F}}, \boldsymbol{v}_m')_{L^2(\Omega)}| \le \epsilon\|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + C\epsilon^{-1}\|\tilde{\boldsymbol{F}}(t)\|^2_{L^2(\Omega)}$. These along with (A.16) imply

$$\|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + \frac{\mathsf{d}}{\mathsf{d}t}\Big(\frac{1}{2}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\boldsymbol{v}_m(t)|^2\,\mathsf{d}x\Big)$$
$$\le 2\epsilon\|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + C\epsilon^{-1}\|\boldsymbol{v}_m(t)\|^2_{L^2(\Omega)} + C\epsilon^{-1}\|\tilde{\boldsymbol{F}}(t)\|^2_{L^2(\Omega)}.$$

Choosing $\epsilon = 1/4$, we obtain

$$\|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)} + \frac{\mathsf{d}}{\mathsf{d}t}\Big(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\boldsymbol{v}_m(t)|^2\,\mathsf{d}x\Big) \le C(\|\boldsymbol{v}_m(t)\|^2_{L^2(\Omega)} + \|\tilde{\boldsymbol{F}}(t)\|^2_{L^2(\Omega)}). \tag{A.17}$$

Given any $0 \le \tilde{t} \le T$, we integrate (A.17) on $t \in [0, \tilde{t}]$,

$$\int_0^{\tilde{t}} \|\boldsymbol{v}_m'(t)\|^2_{L^2(\Omega)}\,\mathsf{d}t + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\boldsymbol{v}_m(\tilde{t})|^2\,\mathsf{d}x - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\boldsymbol{v}_m(0)|^2\,\mathsf{d}x$$
$$\le C\Big(\int_0^{\tilde{t}} \|\boldsymbol{v}_m(t)\|^2_{L^2(\Omega)} + \int_0^{\tilde{t}} \|\tilde{\boldsymbol{F}}(t)\|^2_{L^2(\Omega)}\Big) \le C\big(\|\boldsymbol{v}_m\|^2_{L^2(\Omega_T)} + \|\tilde{F}\|^2_{L^2(\Omega_T)}\big).$$

Combining this inequality with (A.5), we obtain

$$\|\boldsymbol{v}_m'\|^2_{L^2(\Omega_T)} + \|\boldsymbol{v}_m\|^2_{L^\infty(0,T;\tilde{H}^s(\Omega))} \le C\big(\|\boldsymbol{v}_m'\|^2_{L^2(\Omega_T)} + \sup_{0\le\tilde{t}\le T}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\boldsymbol{v}_m(\tilde{t})|^2\,\mathsf{d}x\big)$$
$$\le C\Big(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}\boldsymbol{v}_m(0)|^2\,\mathsf{d}x + \|\boldsymbol{v}_m\|^2_{L^2(\Omega_T)} + \|\tilde{F}\|^2_{L^2(\Omega_T)}\Big)$$
$$\le C(\|\boldsymbol{v}_m(0)\|^2_{\tilde{H}^s(\Omega)} + \|\boldsymbol{v}_m\|^2_{L^2(\Omega_T)} + \|\tilde{F}\|^2_{L^2(\Omega_T)}). \tag{A.18}$$

Since $\|\boldsymbol{v}_m(0)\|^2_{\tilde{H}^s(\Omega)} \le \sum_{k=1}^\infty |(g, w_k)_{L^2(\Omega)}|^2\|w_k\|^2_{\tilde{H}^s(\Omega)} = \|\varphi\|^2_{\tilde{H}^s(\Omega)}$, (A.18) implies

$$\|\boldsymbol{v}_m'\|^2_{L^2(\Omega_T)} + \|\boldsymbol{v}_m\|^2_{L^\infty(0,T;\tilde{H}^s(\Omega))} \le C(\|\tilde{\varphi}\|^2_{\tilde{H}^s(\Omega)} + \|\boldsymbol{v}_m\|^2_{L^2(\Omega_T)} + \|\tilde{F}\|^2_{L^2(\Omega_T)}).$$

Therefore, combining this inequality with (A.10), we obtain

$$\|\boldsymbol{v}_m'\|^2_{L^2(\Omega_T)} + \|\boldsymbol{v}_m\|^2_{L^\infty(0,T;\tilde{H}^s(\Omega))} \le C(\|\varphi\|^2_{\tilde{H}^s(\Omega)} + \|\tilde{F}\|^2_{L^2(\Omega_T)}).$$

Finally, taking the limit $m \to \infty$, we complete our proof.

Appendix B. Some discussions

The main difficulty in proving Theorem 1.3 is the regularity of the solutions. Due to this difficulty, we are only able to prove Theorem 1.3 in one dimension. The method we used requires the $L^\infty(\Omega_T)$-regularity for the linear fractional wave equation. The $L^\infty(\Omega_T)$-regularity is required to guarantee the well-posedness of (1.5), and it is essential to prove that the linearization is well-defined as we see in Section 2. However, we are only able to obtain this regularity in the case when $n = 1$ and $\frac{1}{2} < s < 1$. If one can prove the well-posedness of (1.5) for general $n \in \mathbb{N}$ and $0 < s < 1$, then Theorem 1.3 immediately extends for general $n \in \mathbb{N}$ and $0 < s < 1$.

In view of standard elliptic regularity results (see [GT01] or [JLS17, Proposition A.1] in terms of other norms) as well as Sobolev embedding, an attempt to improve the result in

Theorem 1.3 is to try to obtain the $L^\infty(0,T;H^{2s}(\mathbb{R}^n))$ regularity for the solution. However, this idea is less likely to be feasible. Using [GSU20, Lemma 2.3], we know there exists a unique solution $w \in \tilde{H}^s(\Omega)$ of

$$\begin{cases} (-\Delta)^s w = F & \text{in } \Omega, \\ w = 0 & \text{in } \Omega^e, \end{cases} \tag{B.1}$$

for $F \in H^{-s}(\Omega)$. Choose Ω to be the unit disk and F to be a positive constant in Ω. Then the best regularity result of (B.1) we know is $\mathcal{C}^s(\mathbb{R}^n)$ [RO16, Proposition 7.2]. In fact, [RO16, Lemma 5.4] gives an explicit solution $w(x) = (1-|x|^2)_+^s \in \mathcal{C}^s(\mathbb{R}^n)$, and such w does not belong to $\mathcal{C}^{s'}(\mathbb{R}^n)$ for any $s' > s$. When $n = 1$, we have the continuous embedding $H^{2s}(\mathbb{R}) \hookrightarrow C^{2s-\frac{1}{2}}(\mathbb{R})$. Therefore, at least when $n = 1$ and $s > \frac{1}{2}$, such a solution w of (B.1) cannot be in $H^{2s}(\mathbb{R})$.

Acknowledgements

All the authors were partly supported by the Academy of Finland (Centre of Excellence in Inverse Modelling and Imaging, grant 284715) and by the European Research Council under Horizon 2020 (ERC CoG 770924).

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