COAREA INEQUALITY FOR MONOTONE FUNCTIONS ON METRIC SURFACES

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Abstract. We study coarea inequalities for metric surfaces — metric spaces that are topological surfaces, without boundary, and which have locally finite Hausdorff 2-measure $\mathcal{H}^2$. For monotone Sobolev functions $u\colon X \to \mathbb{R}$, we prove the inequality

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \kappa \int_X g\rho \, d\mathcal{H}^2 \quad \text{for every Borel } g\colon X \to [0,\infty],$$

where ρ is any integrable upper gradient of u. If ρ is locally L^2-integrable, we obtain the sharp constant $\kappa = 4/\pi$. The monotonicity condition cannot be removed as we give an example of a metric surface X and a Lipschitz function $u\colon X \to \mathbb{R}$ for which the coarea inequality above fails.

1. Introduction

In this paper we prove a coarea inequality, involving upper gradients, for *monotone* Sobolev functions on *metric surfaces*. To motivate the topic, recall the classical coarea formula for Lipschitz maps $u\colon \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m \geq 1$. When $m = 1$, the only relevant case in this paper, it reads as follows.

Theorem 1.1. *If $\Omega \subset \mathbb{R}^n$ is open and $u\colon \Omega \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\mathbb{R}} \int_{u^{-1}(t)} g \, d\mathcal{H}^{n-1} dt = \int_{\Omega} g|\nabla u| \, dx \quad \text{for every Borel } g\colon \Omega \to [0,\infty]. \tag{1}$$

Throughout, $\mathcal{H}^\alpha$ stands for Hausdorff measures. Extension of this result to Sobolev class is a delicate matter, despite well-known Lipschitz approximation results.

Theorem 1.2 (Coarea formula, [MSZ03])**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ be precisely represented. Then the coarea formula (1) holds with ∇u being the weak derivative.*

We recall that continuous functions are precisely represented.

There are several equivalent approaches to Sobolev functions on metric(-measure) spaces. We use the definition based on upper gradients in the sense of Heinonen and Koskela [HK98, Sha00]. An *upper gradient* of a function $u\colon X \to \mathbb{R}$, on a metric space (X,d), is a

Research supported by the Academy of Finland, project number 308659. The second named author was also supported by the Vilho, Yrjö and Kalle Väisälä Foundation.

2020 Mathematics Subject Classification. 28A25, 28A75, 28A78, 30L10, 30L15.

Borel function $\rho\colon X \to [0,\infty]$ so that for all $x, y \in X$ and all rectifiable paths γ joining x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds.$$

Observe that for C^1-smooth functions on Euclidean domains, $\rho(x) = |\nabla u(x)|$ meets this criterion. The upper gradient approach is equivalent to the energy density approach [AGS13, ES21] and the test plan approach [AGS13]. We refer the interested reader to [ACDM15]. On Euclidean domains, the upper gradient approach leads to the classical Sobolev theory, see e.g. [HKST15].

We restrict ourselves to the class of *metric n-manifolds*. By *a metric n-manifold* (resp. with boundary) we mean a metric space homeomorphic to a topological n-manifold (resp. with boundary) with locally finite n-dimensional Hausdorff measure. Unless otherwise mentioned, a metric n-manifold is assumed to have an empty boundary. When $n = 2$, and there is no boundary, we use the term *metric surface*.

Question 1.3. (Coarea inequality) Does there exist a universal constant $C = C(n)$ such that for all metric n-manifolds X, all $u\colon X \to \mathbb{R}$ and any upper gradient $\rho\colon X \to [0,\infty]$ of u, with locally integrable u and ρ,

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^{n-1} dt \leq C \int_X g\rho \, d\mathcal{H}^n \quad \text{for every Borel } g\colon X \to [0,\infty]? \tag{2}$$

Here $\int_{\mathbb{R}}^{*}$ refers to the *upper integral* in case the integrand happens to be nonmeasurable. Question 1.3 has a positive answer for Lipschitz functions in all metric n-manifolds that support a $(1,1)$-Poincaré inequality and on which $\mathcal{H}^n$ is doubling, cf. Section 5.

In recent years the research on metric n-manifolds has been active, mainly for $n = 2$, where inequalities of the form (2) have played a prominent role in the uniformization results of metric surfaces by the third named author [Raj17] and more recently by Ntalampekos and Romney [NR22], see also [LW17, MW21]. The particular formulation of (2) is motivated by a related well-known inequality from geometric measure theory.

Theorem 1.4 (Eilenberg's Inequality, [Fed69, EH21])**.** *Let (X, d) be any metric space and fix $n \geq 1$, not necessarily an integer, and suppose that X has a locally finite $\mathcal{H}^n$-measure. Then for any Lipschitz function $u\colon X \to \mathbb{R}$,*

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^{n-1} dt \leq \frac{2\omega_{n-1}}{\omega_n} \int_X g \operatorname{lip}(u) \, d\mathcal{H}^n, \quad \textit{for every Borel } g\colon X \to [0,\infty], \tag{3}$$

where

$$\operatorname{lip}(u)(x) := \limsup_{x \neq y \to x} \frac{|u(y) - u(x)|}{d(y,x)}. \tag{4}$$

Here ω_i are normalization constants involved in the definition of the Hausdorff measure. In particular, when n is an integer, ω_n is the volume of the n-dimensional Euclidean unit ball. For example, $\omega_1 = 2$ and $\omega_2 = \pi$, so $2\omega_1/\omega_2 = 4/\pi$.

Eilenberg's inequality is often stated using the global Lipschitz constant instead of lip, but a localization argument leads to (3). See Section 5, in particular Lemma 5.2, for a proof of this folklore result. The Eilenberg inequality is closely related to the perimeter of sublevel sets of Lipschitz functions, cf. [Mir03]. However, without strong geometric assumptions on X, see e.g. [Amb01], the connection between the Hausdorff measure and perimeter measure is unclear.

Theorem 1.4 implies, indirectly, that the Sobolev theory on metric surfaces is rich, cf. Section 1.1. In particular, it is possible to construct 2-harmonic functions for suitable boundary value problems on X as was done by the third named author in [Raj17, Section 3]. The results therein generalize to $p \in (1, \infty)$ and also to related problems in potential analysis. We answer Question 1.3 in two-dimensions for a class of functions that contains such functions and other functions arising from energy minimization problems. First, a definition.

Definition 1.5. A function $u \colon X \to \mathbb{R}$ is *monotone* if u is continuous and satisfies the *maximum principle*: for every open set U, compactly contained in X,

$$\sup_{\overline{U}} u = \sup_{\partial U} u \quad \text{and} \quad \inf_{\overline{U}} u = \inf_{\partial U} u.$$

Theorem 1.6. *Let X be a metric surface and $\infty \geq p \geq 1$. If $u \colon X \to \mathbb{R}$ is a monotone function with a locally p-integrable upper gradient ρ, then for $\kappa = (4/\pi) \cdot 200$,*

$$\int_{\mathbb{R}} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \kappa \int_X g\rho \, d\mathcal{H}^2 \tag{5}$$

for every Borel function $g \colon X \to [0, \infty]$. If $p \geq 2$, then (5) holds with constant $\kappa = 4/\pi$.

We expect that the sharp constant $\kappa = 4/\pi$ holds also for $p < 2$. Endowing the Euclidean plane with the supremum norm $\|(x_1, x_2)\| \coloneqq \sup \{|x_1|, |x_2|\}$ and considering $u(x_1, x_2) = x_1$ shows the sharpness of Theorem 1.6 when $p \geq 2$.

As hinted at earlier, (3) implies (2) for Lipschitz functions in spaces satisfying strong geometric assumptions, for example, $\mathcal{H}^n$ being doubling and supporting a $(1,1)$-Poincaré inequality, cf. [Che99] (or [IPS22, ES21]). Without the further geometric assumptions, the pointwise Lipschitz constant may be much larger than the minimal upper gradient, so (3) does not always imply (2). Indeed, we have the following consequence of Theorem 5.3.

Theorem 1.7. *For every $n \geq 2$, there exist a metric n-manifold $X \subset \mathbb{R}^{n+1}$ and a Cantor set $C \subset X$ such that for $u(x_1, x_2, \ldots, x_n) = x_1$, the following holds:*

$$0 < \mathcal{H}^n(C) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(C \cap u^{-1}(t)) \, dt \quad \text{and} \quad \rho = \chi_{X \setminus C} \quad \text{is an upper gradient of } u|_X.$$

In particular, (2) fails for $g = \chi_C$ with any constant.

Theorem 1.7 illustrates that the inequality (2) does not hold in the Lipschitz class. For this reason, there is no obvious generalization of Theorem 1.6 in dimension two.

Question 1.8. Does Question 1.3 have a positive answer on metric n-manifolds for monotone functions $u \colon X \to \mathbb{R}$ with (locally) integrable upper gradients?

1.1. **Two-dimensionality.** A noticeable assumption in Theorem 1.6 is the setting of metric surfaces. Our methods do not lend themselves to obvious generalization even to metric n-manifold setting for $n \geq 3$. To illustrate the point, we argue as follows. When Theorem 1.4 is applied on a metric n-manifold and $u(x) = d(x, x_0)$, we conclude that almost every level set of u satisfies $\mathcal{H}^{n-1}(u^{-1}(t)) < \infty$. When t is small enough, $u^{-1}(t)$ is contained in a neighbourhood Ω of x_0 homeomorphic to $\mathbb{R}^n$, and Alexander's duality guarantees the existence of a continuum $C \subset u^{-1}(t)$ separating Ω into two or more connected components. In the particular case of $n = 2$, $\mathcal{H}^1(C) < \infty$ guarantees that C is a rectifiable path which allows us to control the oscillation of a Sobolev function f on C in terms of $\int_C \rho \, d\mathcal{H}^1$, where ρ is any upper gradient of f. More precisely, we have

$$|f(x) - f(y)| \leq \int\limits_C \rho \, d\mathcal{H}^1 \quad \text{for every } x, y \in C. \tag{6}$$

Without further geometric assumption on X, (6) does not seem to have a straight-forward generalization for $n > 2$ since, e.g., $\mathcal{H}^{n-1}(C) < \infty$ for continua C does not imply the existence of a Lipschitz parametrization from $[0, 1]^{n-1}$. The key point is that (6) allows one to deduce that Sobolev analysis on metric surfaces is rich without any need for further assumptions on X.

To handle the full range $1 \leq p \leq \infty$, we provide two proofs of Theorem 1.6. The inequality involving the non-sharp constant holds in the full range but the sharp constant for $p \geq 2$ in Theorem 1.6 is based on recent advances in uniformization theory of metric surfaces: the existence of (weakly) quasiconformal homeomorphisms onto metric surfaces [NR22], see Section 4.1 for further discussion.

1.2. **Monotone functions.** Typical examples of monotone functions include solutions to the p-Laplace equation or other elliptic PDE's of divergence form. The calculus of variations approach to p-harmonic functions can be developed in metric (measure) spaces, cf. [BB11]. In fact, an important first step in the uniformization theorem by the third named author in [Raj17] was to construct a specific 2-harmonic function on a general metric surface. The constructed function satisfies the following slightly weaker property.

Definition 1.9. A function $u \colon X \to \mathbb{R}$ is *weakly monotone* if for every open V compactly contained in X,

$$\sup_V u \leq \sup_{\partial V} u < \infty \quad \text{and} \quad \inf_V u \geq \inf_{\partial V} u > -\infty.$$

Deducing continuity of weakly monotone functions is of interest. This is one of our main results.

Theorem 1.10. *Suppose that X is a metric surface and $u \colon X \to \mathbb{R}$ is weakly monotone. If u has a locally p-integrable upper gradient, for $p \geq 2$, then u is continuous. In particular, u is monotone.*

The key idea in the proof of Theorem 1.10 is a *non-sharp version* of the coarea inequality for weakly monotone functions with locally integrable upper gradients. Our approach is based on a related result [RR19], and in fact, Theorem 1.10 generalizes related continuity results from [RR19] for all ranges $p \geq 2$ and for a large class of problems. Moreover, together with Theorem 1.6, we obtain a *sharp* duality of modulus lower bound in [RR19, Theorem 1.3]; we note that the sharp duality lower bound was first proved in [EBPC22, Corollary 1.2] with different methods and in greater generality. When $p < 2$, the continuity conclusion does not hold even in the plane, cf. Example 3.10.

Given an inequality of the form (5) for monotone functions, a consequence of [Nta20, Theorem 1.5] follows for metric surfaces.

Corollary 1.11. *Let U be a metric surface homeomorphic to $\mathbb{R}^2$, and $p \geq 1$. If a monotone $u\colon U \to \mathbb{R}$ has a locally p-integrable upper gradient, then for almost every $t \in u(U)$ the following properties hold:*

(a) The level set $u^{-1}(t)$ is a locally rectifiable properly embedded topological 1-manifold.
(b) Each component of $u^{-1}(t)$ is homeomorphic to $\mathbb{R}$.

We say that a set $K \subset U$ is a *properly embedded topological 1-manifold* if K is closed and every $y \in K$ is contained in $I \subset K$, relatively open in K, with I homeomorphic to $\mathbb{R}$. Corollary 1.11 plays a key role when we establish the sharp version of (5) for $p \geq 2$.

Notation. We shall write $u^{-1}(t)$ to mean $\{x\colon u(x) = t\}$. The α-dimensional Hausdorff measure is denoted by $\mathcal{H}^\alpha$. The upper integral of any function on a measure space (X, μ) is denoted by $\int^* f \, d\mu$. If f is μ-measurable then it agrees with the usual integral. We use $\#A$, χ_A and $\overline{A}$ to denote, resp., the cardinality, the characteristic function, and closure of a set A. The closed ball $\{y\colon d(y,x) \leq r\}$ is denoted by $\overline{B}(x,r)$, which might not coincide with the closure of the open ball $B(x,r) = \{y\colon d(y,x) < r\}$.

2. Preliminaries

Let X be a metric space. For all $Q \geq 0$, the *Q-dimensional Hausdorff measure*, or the *Hausdorff Q-measure*, of a set $E \subset X$ is defined by

$$\mathcal{H}^Q_X(E) = \frac{\omega_Q}{2^Q} \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^Q \colon E \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i < \delta \right\},$$

where the dimensional constant ω_Q is chosen so that $\mathcal{H}^n_{\mathbb{R}^n}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers. In particular, $\omega_1 = 2$ and $\omega_2 = \pi$. We typically omit the subscript X from the definition.

Given a set $K \subset X$, a function $f\colon K \to \mathbb{R}$ is *Lipschitz* if

$$\mathrm{LIP}(f) := \sup_{x,y \in K, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty.$$

The supremum on the left is the *Lipschitz constant* of f. We say that f is L-Lipschitz if $\mathrm{LIP}(f) \leq L$.

For a given Lipschitz $f\colon K \to \mathbb{R}$ and $x \in K$, we define the *pointwise Lipschitz constant* of f as

$$\mathrm{lip}\,(f)(x) = \inf_{r>0} \sup_{0<s\leq r} \sup_{y \in B(x,s) \cap K} \frac{|f(y) - f(x)|}{s}.$$

In fact, this definition of $\operatorname{lip}(f)$ coincides with the one in (4).

Let X be a metric surface. For each $1 \leq p < \infty$, we say $\rho\colon X \to [-\infty, \infty]$ belongs to $L^p(X)$ if ρ is measurable and

$$\|\rho\|_{L^p(X)} \coloneqq \left(\int_X |\rho|^p \, d\mathcal{H}^2 \right)^{1/p} < \infty.$$

For $p = \infty$, $\|\rho\|_{L^\infty(X)}$ is the smallest $C \in [0, \infty]$ for which $|\rho| \leq C$, $\mathcal{H}^2$-almost everywhere, and denote $\rho \in L^\infty(X)$ if ρ is measurable and $\|\rho\|_{L^\infty(X)} < \infty$. In case $\rho \in L^p(X)$, we say that ρ is p-integrable. Local p-integrability refers to being p-integrable on each compact subset of the space; recall that X is locally compact so the space X can be covered by open sets whose closures are compact, justifying the nomenclature.

2.1. **The upper integral.** Let $E \subset X$ be a set with $\mathcal{H}^Q(E) < \infty$ for $Q = 2$ (resp. $Q = 1$). For any function $\rho\colon E \to [0, \infty]$ we define the *upper integral* of ρ (with respect to $\mathcal{H}^Q$) to be

$$\int_E^* \rho \, d\mathcal{H}^Q := \inf \left\{ \int_E \rho' \, d\mathcal{H}^Q \colon \rho' \text{ is } \mathcal{H}^Q\text{-measurable and } \rho \leq \rho',\ \mathcal{H}^Q\text{-almost everywhere} \right\}.$$

We use some elementary properties of the upper integral. If $0 \leq \rho_1(x) \leq \rho_2(x)$, $\mathcal{H}^Q$-almost everywhere in E, then

$$\int_E^* \rho_1 \, d\mathcal{H}^Q \leq \int_E^* \rho_2 \, d\mathcal{H}^Q.$$

The monotone convergence theorem holds for upper integrals. Namely, if $0 \leq \rho_1(x) \leq \rho_2(x) \leq \dots$ is an increasing sequence of (not necessarily measurable) functions, and for $\mathcal{H}^Q$-almost every $x \in E$, $\rho(x) = \lim_{n\to\infty} \rho_n(x)$, then

$$\int_E^* \rho \, d\mathcal{H}^Q = \lim_{n\to\infty} \int_E^* \rho_n \, d\mathcal{H}^Q.$$

Lastly, for an arbitrary $\rho\colon E \to [0, \infty]$, $\int_E^* \rho \, d\mathcal{H}^Q = 0$, if and only if $\rho = 0$, $\mathcal{H}^Q$-almost everywhere in E.

2.2. **Rectifiable curves and path integrals.** A *path* in a metric space X is a continuous map $\gamma\colon [a, b] \to X$. The *length* $\ell(\gamma)$ of γ is the smallest value $L \in [0, \infty]$ for which

$$\sum_{i=1}^k d(\gamma(t_i), \gamma(t_{i-1})) \leq L,$$

for every choice of $k \in \mathbb{N}$ and $a = t_0 \leq t_1 \leq \dots \leq t_k = b$. We say that γ is *rectifiable* if $\ell(\gamma) < \infty$.

Suppose $\gamma\colon [a,b] \to X$ is a path and $\rho\colon X \to [0,\infty]$ is Borel. Then the *path integral* of ρ over γ is

$$\int_\gamma \rho \, ds := \int_X \#(\gamma^{-1}(x))\rho(x)\, d\mathcal{H}^1.$$

Here $\#(\gamma^{-1}(x)) = \infty$ if $\gamma^{-1}(x)$ is not finite and otherwise $\#(\gamma^{-1}(x))$ is the cardinality of the set $\gamma^{-1}(x)$. If ρ is not Borel, we define

$$\int_\gamma \rho \, ds := \int_X^{*} \#(\gamma^{-1}(x))\rho(x)\, d\mathcal{H}^1.$$

Remark 2.1. Note that whenever $E \subset [a,b]$ is Borel and $\gamma\colon [a,b] \to X$ is a path, then $\gamma(E)$ is analytic [Fed69, 2.2.10]. This implies that $\gamma(E)$ is $\mathcal{H}^1$-measurable [Fed69, 2.2.13]. This allows us to prove that $x \mapsto \#(\gamma^{-1}(x))$ is $\mathcal{H}^1$-measurable. The key observation is to fix a sequence of countable Borel partitions $(\mathcal{K}_n)$ of $[a,b]$ such that the supremum of the diameters of the elements of $\mathcal{K}_n$ converges to zero as $n \to \infty$ and each $\mathcal{K}_{n+1}$ refines $\mathcal{K}_n$, i.e., each $E \in \mathcal{K}_n$ is a countable union of some elements of $\mathcal{K}_{n+1}$. Now, measurability follows from

$$\#(\gamma^{-1}(x)) = \lim_{n\to\infty} \sum_{E \in \mathcal{K}_n} \chi_{\gamma(E)}(x) \quad \text{for every } x \in X.$$

If $\gamma\colon [a,b] \to X$ is rectifiable, then there exists a unique path $\gamma_s\colon [0,\ell(\gamma)] \to X$ such that $\gamma = \gamma_s \circ h$, where $h\colon [a,b] \to [0,\ell(\gamma)]$ is continuous, nondecreasing and onto, and $\ell(\gamma_s|_{[0,s]}) = s$ for all $0 \le s \le \ell(\gamma)$. The path γ_s is called *the arclength parametrization* of γ. Recall that the arclength parametrization is 1-Lipschitz, cf. [HKST15, Section 5].

Let $\gamma\colon [a,b] \to X$ be a rectifiable path in a metric space, and let $\gamma_s\colon [0,\ell(\gamma)] \to X$ be the arclength parametrization of it. For a Borel function $\rho\colon X \to [0,+\infty]$, the path integral of ρ over γ can be computed as follows:

$$\int_\gamma \rho \, ds = \int_0^{\ell(\gamma)} \rho(\gamma_s(t))\, dt.$$

The equality follows from the area formula for paths, proved for example in [Fed69, Theorem 2.10.13.].

A path $\gamma\colon [a,b] \to X$ is *absolutely continuous* if $\ell(\gamma) < \infty$ and if γ maps sets of Lebesgue measure zero to sets of $\mathcal{H}^1$-measure zero. For absolutely continuous curves, there is a third way to compute the path integral of Borel functions. For this purpose, we denote

$$|\gamma'|(t) := \lim_{h\to 0} \frac{d(\gamma(t+h),\gamma(t))}{|h|}$$

whenever the limit exists. When the limit exists, we refer to $|\gamma'|(t)$ as the metric speed of γ at t. It turns out that for any rectifiable path, the limit exists almost everywhere in $[a,b]$ [Dud07]. Recall that in the Euclidean setting, the metric speed coincides with the modulus of the usual derivative.

With the additional assumption of absolute continuity, we obtain the following.

Lemma 2.2 ([Dud07]). *Suppose that $\gamma\colon [a,b] \to X$ is absolutely continuous and $\rho\colon X \to [0,+\infty]$ is Borel. Then*

$$\int_\gamma \rho \, ds = \int_a^b \rho(\gamma(t))|\gamma'|(t)\, dt.$$

2.3. **Modulus of path families.** We continue considering a metric surface X. We typically denote a collection of paths by Γ and refer to Γ as a *path family*. A Borel function $\rho\colon X \to [0,\infty]$ is *admissible* for a path family Γ if

$$\int_\gamma \rho \, ds \geq 1 \quad \text{for every } \gamma \in \Gamma.$$

Then, for each $1 \leq p < \infty$, we denote

$$\operatorname{mod}_p \Gamma = \inf \left\{ \int_X \rho^p \, d\mathcal{H}^2_X \colon \rho \text{ is admissible for } \Gamma \right\}.$$

In case $p = \infty$, we set

$$\operatorname{mod}_p \Gamma = \inf \left\{ \|\rho\|_{L^\infty(X)} \colon \rho \text{ is admissible for } \Gamma \right\}.$$

The set function $\Gamma \mapsto \operatorname{mod}_p \Gamma$ is an outer measure.

We say that Γ is *p-negligible* if $\operatorname{mod}_p \Gamma = 0$. The following characterization of negligible paths is an effective tool. Notice that this characterization does not require the notion of modulus and could be given as a definition of modulus zero without defining modulus, see, e.g. [HKST15, Lemma 5.2.8] for a proof.

Lemma 2.3. *Let $1 \leq p \leq \infty$. A path family Γ is* p-negligible *if and only if there exists an $L^p(X)$-integrable Borel function $h\colon X \to [0,\infty]$ such that $\int_\gamma h \, ds = \infty$ for every $\gamma \in \Gamma$.*

2.4. **Sobolev analysis.** Let X be a metric surface and Y a metric space. Let $u\colon X \to Y$ be a map and $\rho\colon X \to [0,\infty]$ a Borel function. If $\gamma\colon [a,b] \to X$ is rectifiable, we say that the triple (u, ρ, γ) satisfies the *upper gradient inequality* if

$$d(u(\gamma(a)), u(\gamma(b))) \leq \int_\gamma \rho \, ds.$$

If the triple (u, ρ, γ) satisfies the upper gradient inequality for every path outside a p-negligible family, we say that ρ is a *p-weak upper gradient* of u. If the exceptional set of paths is empty, we say ρ is an *upper gradient* of u.

If u has a p-integrable p-weak upper gradient, then there exists a p-weak upper gradient ρ such that $\rho \leq \rho'$ almost everywhere for every other p-integrable p-weak upper gradient ρ' of u. For $1 \leq p < \infty$, this is proved in [HKST15, Theorem 6.3.20] and for $p = \infty$ a similar argument works, cf. [Mal13]. Any p-minimal p-weak upper gradient of u is denoted by ρ_u; we typically omit the p from the notation since p is clear from the context.

Whenever $1 \leq p \leq \infty$, we write $u \in D^{1,p}(X;Y)$ whenever u has a p-integrable p-weak upper gradient. In case $Y = \mathbb{R}$, we also use $u \in D^{1,p}(X)$. We note that Lemma 2.3

implies that u has a p-integrable upper gradient whenever u has a p-integrable p-weak upper gradient, see e.g. [HKST15, Lemma 6.2.2]. For a thorough exposition of the topic of Sobolev analysis on metric measure spaces, see [HKST15].

We recall the following fact.

Lemma 2.4. *Let X be a metric surface and Y a metric space. Let $u\colon X \to Y$ be a map with a p-integrable p-weak upper gradient ρ. Let Γ_0 denote the collection of all rectifiable paths $\gamma\colon [a,b] \to X$ for which one of the following occurs: $u\circ\gamma$ is not rectifiable, $\ell(u\circ\gamma) > \int_\gamma \rho\, ds$, or $\int_\gamma \rho\, ds = \infty$. Then Γ_0 is p-negligible.*

Proof. For $1 \leq p < \infty$, the claim follows from [HKST15, Propositions 6.3.2 and 6.3.3]. The same argument also works for $p = \infty$. □

2.5. **Some topology of the plane and continua.** In this section, we consider a metric surface U homeomorphic to $\mathbb{R}^2$. For many of the topological results of this section, this is a crucial assumption.

We first recall some topological results about separation of sets and continua. Recall that $A \subset U$ is a *continuum* if A is compact and connected. We say a set $A \subset U$ *separates* x and y if they belong to different connected components of $U \setminus A$.

Lemma 2.5 ([Wil79, Chapter 2, Lemma 5.20])**.** *If $K \subset U$ is compact and $x, y \in U \setminus K$ are separated by K, then there exists a continuum $C \subset K$ such that x and y are separated by C. In particular, if a compact set separates two points, then a connected component of the compact set separates them.*

A continuous image of a compact subinterval of $\mathbb{R}$ is called a *Peano continuum*. The next result claims that continua with finite 1-dimensional Hausdorff measure are Peano continua. An example of a continuum that is not a Peano continuum is the Warsaw circle.

Lemma 2.6 ([RR19, Proposition 5.1])**.** *Let $K \subset U$ be a continuum. If $\mathcal{H}^1(K) < \infty$, then there exists a 1-Lipschitz surjection $\gamma\colon [0, 2\mathcal{H}^1(K)] \to K$ such that $\#(\gamma^{-1}(x)) \leq 2$ for $\mathcal{H}^1$-almost every $x \in K$.*

Given two sets $K \subset V \subset U$, we say that $K \subset V$ is *relatively closed (resp. open) in V* if there exists a closed (resp. open) set $K' \subset U$ for which $K = K' \cap V$. When the set V is clear from the context, we say that K is relatively closed (resp. open).

For Peano continua, there is a stronger conclusion than in Lemma 2.5.

Lemma 2.7 ([Wil79, Chapter 4, Theorem 6.7])**.** *If a Peano continuum $K \subset U$ separates $x, y \in U \setminus K$, then there exists a subcontinuum $K' \subset K$ that is homeomorphic to the unit circle $\mathbb{S}^1$ and separates x and y.*

We say that a Peano continuum K is *simple* if it is homeomorphic either to a point, to a compact interval, or to the unit circle $\mathbb{S}^1$. A homeomorphic image of a compact interval is called *an arc*. Observe that any subcontinuum of a simple Peano continuum is again simple.

Lemma 2.8. *Let $\mathcal{F}$ be a collection of pairwise disjoint Peano continua in U and $\mathcal{F}' \subset \mathcal{F}$ the subcollection containing the ones that are not simple. Then $\mathcal{F}'$ is countable.*

Proof. A point x_0 in a Peano continuum K is a *junction point*, if there exists three (compact) arcs E_1, E_2, E_3 in K that meet at x_0 but are otherwise disjoint. A Peano continuum is not simple if and only if it contains a junction point [Nta20]. Thus every element in $\mathcal{F}'$ has a junction point. A theorem by Moore [Moo28, Theorem 1] states that there cannot be an uncountable collection of pairwise disjoint Peano continua in U if each of them contains a junction point. In particular, $\mathcal{F}'$ must be countable. □

Suppose that $f\colon [0,1] \to K \subset U$ is a homeomorphism and $\mathcal{H}^1(K) < \infty$. Then $P(t) = \mathcal{H}^1(f([0,t]))$ is strictly increasing, continuous, and bounded. Then $\gamma(t) = f \circ P^{-1}(t)$ is a Lipschitz path. With this fact, the lemma below readily follows.

Lemma 2.9. *Let K be a simple Peano continuum with $\mathcal{H}^1(K) < \infty$. Then there exists a surjective Lipschitz $\gamma\colon [a,b] \to K$, injective outside its end points.*

For the purposes of future sections, we establish the following.

Lemma 2.10. *Let $E \subset U$ be a closed and connected set. If $\mathcal{H}^1(E) < \infty$, then there exists a sequence of continua $(E_n)_{n=1}^{\infty}$ with $E_n \subset E_{n+1}$ and $E = \bigcup_{n=1}^{\infty} E_n$.*

Proof. We may assume that E is not a continuum, since otherwise the claim is trivial.

We claim that E is locally connected. That is, given any point $x_0 \in E$ and a relatively open set $x_0 \in W \subset E$, there is a connected and relatively open $x_0 \in U \subset W$. We argue by contradiction. Then, by [Why63, Theorem (12.1), p. 18], there is $x_0 \in E$, a relatively open and bounded $W \subset E$ with $x_0 \in W$, a sequence of continua $(F_i)_{i=1}^{\infty}$, pairwise disjoint, with $F_i \subset \overline{W} \cap E$, converging to a continuum F of positive diameter. Therefore, there exists $c > 0$ such that for all large i, $\operatorname{diam} F_i \geq c$. Since for every $N \in \mathbb{N}$,

$$\mathcal{H}^1(W) \geq \mathcal{H}^1\left(\bigcup_{i=1}^{N} F_i\right) = \sum_{i=1}^{N} \mathcal{H}^1(F_i) \geq \sum_{i=1}^{N} \operatorname{diam} F_i,$$

we obtain a contradiction by passing to the limit $N \to \infty$. Hence E is locally connected. Having verified that E is locally connected, connected and locally compact, [Why63, Theorem (5.3), p. 38] proves that E is locally path connected, i.e. given any point $x_0 \in E$ and relatively open set $x_0 \in W \subset E$, there is a path connected and relatively open $x_0 \in U \subset W$.

We show that the claim follows from the local path connectedness. To this end, fix $y_1 \in E$. Fix a sequence $(V_n)_{n=1}^{\infty}$ for which $y_1 \in V_1$, $\overline{V}_n \subset V_{n+1} \subset U$, $E \cap \overline{V}_n$ is compact, V_n is open, and $U = \bigcup_{n=1}^{\infty} V_n$. By the local path connectivity and connectedness of E, every $y \in E$ is contained in some path $\gamma\colon [0,1] \to E$ with $\gamma(0) = y_1$ and $\gamma(1) = y$. For large enough m, we have that $V_m \supset |\gamma|$. This implies that if K_n denotes the path connected component of $E \cap V_n$ containing y_1, we have $E = \bigcup_{n=1}^{\infty} K_n$. By setting $E_n := \overline{K}_n$, we obtain a sequence of continua in E for which $E = \bigcup_{n=1}^{\infty} E_n$.

□

2.6. **Level sets of Lipschitz functions.** In this section, we consider an open subset U of a metric surface X with $\mathcal{H}^2(U) < \infty$. We begin with the following formulation of *Eilenberg's inequality*, proved later as Lemma 5.2:

Lemma 2.11 ([EH21])**.** *Let $f\colon U \to \mathbb{R}$ be L-Lipschitz. Then, for every Borel $g\colon U \to [0,\infty]$,*

$$\int_{\mathbb{R}}^{*} \int_{f^{-1}(s)} g \, d\mathcal{H}^1 \, ds \leq L\frac{4}{\pi} \int_{U} g \, d\mathcal{H}^2.$$

Remark 2.12. When the integral on the right-hand side of the inequality in Lemma 2.11 is finite, the *upper integral* $\int^*$ can be replaced by the usual integral since in that case

$$s \mapsto \int_{f^{-1}(s)} g \, d\mathcal{H}^1$$

is Borel measurable.

Lemma 2.13. *Let $U \subset X$ be homeomorphic to $\mathbb{R}^2$ and $\mathcal{H}^2(U) < \infty$. Fix a Lipschitz function $f\colon U \to \mathbb{R}$ and denote $I = f(U)$. Then for almost every $s \in I$:*

(1) $\mathcal{H}^1(f^{-1}(s)) < \infty$ and each continuum $E \subset f^{-1}(s)$ is a simple Peano continuum;

Furthermore, suppose that $N \subset U$ satisfies $\mathcal{H}^2(N) = 0$ and Γ_0 is a p-negligible path family in U, for some $1 \leq p \leq \infty$. Then, for almost every $s \in I$,

(2) for every continuum $E \subset f^{-1}(s)$ of positive diameter, there exists a surjective Lipschitz path $\gamma\colon [0,1] \to E$ that is injective outside its end points;
(3) for every absolutely continuous $\gamma\colon [a,b] \to f^{-1}(s)$, $\int_\gamma \chi_N \, ds = 0$. Moreover, if there exists $M \in \mathbb{N}$ such that $\#(\gamma^{-1}(x)) \leq M$ for $\mathcal{H}^1$-almost every $x \in X$, then $\gamma \notin \Gamma_0$.

Proof. Applying Lemma 2.11 to $g = \chi_U$ implies that $\mathcal{H}^1(f^{-1}(s)) < \infty$ for almost every s. By Lemma 2.6, each compact set $K \subset f^{-1}(s) \cap U$ is a Peano continuum. Lemma 2.8 implies that except for a countable set of values s, all continua $E \subset f^{-1}(s)$ are simple. This proves (1). Claim (2) follows from Lemma 2.9.

Next, we establish (3). Since Γ_0 has p-modulus zero, there exists an $L^p(U)$-integrable Borel function $h\colon U \to [0,\infty]$ such that $\infty = \int_\gamma h \, ds$ for every $\gamma \in \Gamma_0$.

Notice that h is also in $L^1(U)$. So, by Eilenberg's inequality, $\int_{f^{-1}(t)} h \, d\mathcal{H}^1 < \infty$ for almost every t. Hence, if $\gamma\colon [a,b] \to f^{-1}(t)$ satisfies $\#(\gamma^{-1}(x)) \leq M$ for $\mathcal{H}^1$-almost every $x \in U$, then

$$\int_\gamma h \, ds \leq M \int_{f^{-1}(t)} h \, d\mathcal{H}^1 < \infty.$$

Thus $\gamma \notin \Gamma_0$.

To prove the remaining claim, first fix a Borel set $\tilde{N} \supset N$ such that $\mathcal{H}^2(\tilde{N}) = 0$. Observe that, by the Eilenberg inequality,

$$0 = \int_{f^{-1}(t)} \chi_{\widetilde{N}} \, d\mathcal{H}^1 = \mathcal{H}^1(f^{-1}(t) \cap \widetilde{N}) \quad \text{for almost every } t.$$

In particular, $\mathcal{H}^1(f^{-1}(t) \cap N) = 0$ for almost every t. For every such t, for every path $\gamma \colon [a, b] \to f^{-1}(t)$, $\int_\gamma \chi_N \, ds = 0$. □

3. Coarea inequality and continuity for monotone functions

In this section we establish the coarea inequality (Equation (5)), with the non-sharp constant κ, for weakly monotone functions (Definition 1.9) with p-integrable upper gradients, and prove their continuity (Theorem 1.10) when $p \geq 2$.

Throughout this section, we fix a metric surface X and a metric surface $U \subset X$ homeomorphic to $\mathbb{R}^2$ and satisfying $\mathcal{H}^2(U) < \infty$. We also fix a weakly monotone function $u \colon U \to \mathbb{R}$. We moreover assume that u has a p-weak upper gradient $\rho \in L^p(U)$. We do not impose restrictions on $1 \leq p \leq \infty$ unless stated otherwise. Note that $L^p(U) \subset L^1(U)$ since $\mathcal{H}^2(U) < \infty$.

3.1. Initial bounds on oscillations.

Proposition 3.1. *Fix a compact $E \subset U$ and let $f(z) := d(z, E)$. Fix $\epsilon_E > 0$ such that $f^{-1}([0, \epsilon_E])$ is compactly contained in U. Then, denoting $E_r := f^{-1}([0, r])$, we have*

$$2\mathcal{H}^1(u(E_r)) \leq \int_{f^{-1}(r)} \rho \, d\mathcal{H}^1 \quad \text{for almost every } r \in (0, \epsilon_E).$$

Moreover, if $0 < r < \epsilon_E$ and E' is a continuum with $E' \subset E_r$, then

$$2 \operatorname{diam} u(E') \leq \int_{f^{-1}(s)} \rho \, d\mathcal{H}^1 \quad \text{for almost every } s \in (r, \epsilon_E). \tag{7}$$

Corollary 3.2. *For every $x_0 \in U$ and every $r > 0$ for which $\overline{B}(x_0, 2r)$ is compact in U, the function $f(z) = d(z, x_0)$ satisfies*

$$2r\mathcal{H}^1(u(\overline{B}(x_0, r))) \leq \int_r^{2r} \int_{f^{-1}(s)} \rho \, d\mathcal{H}^1 \, ds \leq \frac{4}{\pi} \int_{B(x_0, 2r)} \rho \, d\mathcal{H}^2.$$

Proof. We apply Proposition 3.1 for $E = \{x_0\}$. Then $E_s = \overline{B}(x_0, s)$ for each $0 < s < 2r$. Moreover, if $r < s < 2r$, then $\mathcal{H}^1(u(E_r)) \leq \mathcal{H}^1(u(E_s))$. Since $s \mapsto \mathcal{H}^1(u(E_s))$ is nondecreasing, it is measurable. Subsequently,

$$2r\mathcal{H}^1(u(E_r)) = \int_r^{2r} 2\mathcal{H}^1(u(E_r)) \, ds \leq \int_r^{2r} 2\mathcal{H}^1(u(E_s)) \, ds \leq \int_r^{2r} \int_{f^{-1}(s)} \rho \, d\mathcal{H}^1 \, ds.$$

The proof is completed once we apply the Eilenberg inequality. □

We apply the following lemma during the proof of Proposition 3.1.

Lemma 3.3. *Fix $N \subset U$ with $\mathcal{H}^2(N) = 0$, and let Γ_0 be a p-negligible collection of paths. Let $E \subset U$ be compact and recall the notation (and choice of $\epsilon_E > 0$) from Proposition 3.1. Then for almost every $s \in (r, \epsilon_E)$, there exists a finite collection of Jordan domains $(W_i)_{i=1}^N$, compactly contained in U, that satisfy the following:*

(1) $E_r \subset \bigcup_{i=1}^N W_i$;
(2) $\partial W_i \subset f^{-1}(s)$ for every i and $(\overline{W}_i)_{i=1}^N$ are pairwise disjoint;
(3) for every continuum $C \subset \bigcup_{i=1}^N \partial W_i$, there exists a surjective Lipschitz parametrization $\gamma\colon [0,1] \to C$, injective except possibly for $\gamma(0) = \gamma(1)$, such that $\gamma \notin \Gamma_0$ and $\int_\gamma \chi_N \, ds = 0$.

Proof of Lemma 3.3. Fix $x_1 \in U \setminus f^{-1}([0, \epsilon_E])$ and $0 < r < r' < s < \epsilon_E$ for now. Observe that there is a finite number of components $(V_i)_{i=1}^N$ of $f^{-1}([0, r'))$ covering E_r. Recalling Lemma 2.5, for every V_i, there exists a connected component $F_i \subset f^{-1}(s)$ separating V_i from x_1. For almost every $s \in (r', \epsilon_E)$, each connected component of $f^{-1}(s)$ is a simple Peano continuum by Lemma 2.13 (1). On the other hand, Lemma 2.7 guarantees that F_i contains a continuum homeomorphic to $\mathbb{S}^1$. Given that F_i is simple, this happens only if F_i itself is that subset.

Let $K_i \subset U$ denote the unique set homeomorphic to $[0,1]^2$ satisfying $\partial K_i = F_i$. We denote the interior of K_i by W_i. By construction, $x_1 \in U \setminus K_i$. Observe that if $F_i \cap F_j \neq \emptyset$, then necessarily $K_i = K_j$. Indeed, if $F_i \cap F_j \neq \emptyset$, then $F_i = F_j$ must hold since $F_i \cup F_j$ is connected and the connected components of $f^{-1}(s)$ are simple. Next, observe that an arbitrary compactly contained Jordan domain $V \subset U$ is the unique component of $U \setminus \partial V$ homeomorphic to the plane. Hence $K_i = K_j$. With this observation, by removing possible duplicate indices, we may assume that $(K_i)_{i=1}^M$ are pairwise disjoint. We have now verified requirements (1) and (2). Towards (3), we simply need to apply (2) and (3) from Lemma 2.13 (and Lemma 2.9 to obtain a parametrization that is injective outside its end points). $\square$

Proof of Proposition 3.1. Let Γ_0 be the p-negligible family identified in Lemma 2.4 and $r \in (0, \epsilon_E)$. We apply Lemmas 2.13 and 3.3 for $N = \emptyset$ and Γ_0, and obtain a negligible set of $s \in (r, \epsilon_E)$ outside which the conclusions of the Lemmas hold. We may also assume that the upper gradient inequality holds for all absolutely continuous paths $\gamma\colon [a,b] \to f^{-1}(s)$ with ($\mathcal{H}^1$-essentially) bounded multiplicity and that $\int_{f^{-1}(s)} \rho \, d\mathcal{H}^1 < \infty$.

Let W_i be the Jordan domains given by Lemma 3.3. Observe that $u(W_i)$ is contained in an interval I_i of length $\operatorname{diam} u(W_i)$, so

$$\mathcal{H}^1(u(W_i)) \leq \mathcal{H}^1(I_i) = \operatorname{diam} u(W_i). \tag{8}$$

By the weak monotonicity of u (Definition 1.9), we have

$$\operatorname{diam} u(W_i) \leq \sup_{x,y \in \partial W_i} |u(y) - u(x)|. \tag{9}$$

Observe that ∂W_i admits a Lipschitz parametrization $\gamma\colon [0,1] \to \partial W_i$ injective outside the end points. As $\gamma \notin \Gamma_0$, the composition $u \circ \gamma$ is absolutely continuous. Therefore, there exists a pair of points x_0 and y_0 on ∂W_i such that

$$\sup_{x,y\in\partial W_i} |u(y)-u(x)| = |u(y_0)-u(x_0)|.$$

Since ∂W_i can be separated into two subarcs γ_1, γ_2, overlapping only at x_0 and y_0, we can choose the one that yields

$$\sup_{x,y\in\partial W_i} |u(y)-u(x)| = |u(y_0)-u(x_0)| \leq \int_{\gamma_j} \rho\, ds \leq \frac{1}{2}\int_{\partial W_i} \rho\, ds\,, \tag{10}$$

where we applied the upper gradient inequality. Combining (8), (9), and (10) and using subadditivity of the Hausdorff measure we obtain

$$2\mathcal{H}^1(u(E_r)) \leq 2\sum_{i=1}^N \mathcal{H}^1(u(W_i)) \leq \sum_{i=1}^N \int_{\partial W_i} \rho\, ds \leq \int_{f^{-1}(s)} \rho\, d\mathcal{H}^1 < \infty.$$

The last inequality uses the fact that the boundaries of the Jordan domains W_i are pairwise disjoint.

We first prove the second part of the proposition. To this end, suppose that E is a continuum in E_r. Then E_r is contained in a single W_i. Therefore

$$2\operatorname{diam} u(E_r) \leq 2\operatorname{diam} u(W_i) \leq \int_{\partial W_i} \rho\, d\mathcal{H}^1 \leq \int_{f^{-1}(s)} \rho\, d\mathcal{H}^1.$$

This proves the second part of the claim.

Towards proving the first claim of the proposition, let $N_r \subset (r, \epsilon_E)$ be the negligible set so that for all $s \in (r, \epsilon_E) \setminus N_r$,

$$2\mathcal{H}^1(u(E_r)) \leq \int_{f^{-1}(s)} \rho\, d\mathcal{H}^1. \tag{11}$$

Fix a countable dense subset $(r_i)_{i=1}^\infty$ of $(0, \epsilon_E)$, and let M be the union of all N_{r_i} and the collection of discontinuity points of $(0, \epsilon_E) \ni r' \mapsto 2\mathcal{H}^1(u(E_{r'}))$. Observe that the discontinuity points form a countable set since the function of interest is nondecreasing and locally bounded. Clearly M is negligible.

Now fix $s \in (0, \epsilon_E) \setminus M$. Then (11) yields

$$2\mathcal{H}^1(u(E_s)) = \sup_{r_i<s} 2\mathcal{H}^1(u(E_{r_i})) \leq \int_{f^{-1}(s)} \rho\, d\mathcal{H}^1. \tag{12}$$

Finally, this proves the first claim of the proposition (with s playing the role of r). □

The following oscillation result is key in establishing initial topological properties of u.

Corollary 3.4. *Fix $x_0 \in U$ and $r > 0$ for which $\overline{B}(x_0, 2r)$ is compact in U. Let V be the connected component of $B(x_0, r)$ containing x_0. Then*

$$2r \sup_{x,y\in V} |u(y)-u(x)| \leq \frac{4}{\pi} \int_{B(x_0,2r)} \rho\, d\mathcal{H}^2. \tag{13}$$

Proof. We apply Proposition 3.1 for $E = \{x_0\}$ and $f(z) = d(z, E)$. Then (7) shows that

$$2 \sup_{x,y \in V} |u(y) - u(x)| \leq \int_{f^{-1}(s)} \rho \, d\mathcal{H}^1 \quad \text{for almost every } s \in (r, 2r).$$

Now, (13) follows by integrating over the interval $(r, 2r)$ and applying the Eilenberg inequality. □

We later show continuity for weakly monotone functions with p-integrable upper gradients, $p \geq 2$. The following lemma is a weaker continuity result.

Lemma 3.5. *Let G denote the collection of all $x \in U$ with*

$$\limsup_{r \to 0^+} r^{-2} \int_{B(x,2r)} \rho \, d\mathcal{H}^2 < \infty.$$

Then u is continuous at every $x \in G$. In particular, u is continuous $\mathcal{H}^2$-almost everywhere in U.

To emphasize, here continuity is with respect to the original domain and not just the continuity of $u|_G$.

Proof. The complement of G has negligible $\mathcal{H}^2$-measure due to the integrability of ρ; we apply [Fed69, Theorem 2.10.19 (3)] to the measure $\mu = \rho\mathcal{H}^2$ to deduce that the 2-dimensional upper density of μ is finite at $\mathcal{H}^2$-almost every $x \in X$. Consider an arbitrary $x_0 \in G$. Since the collection of the sets V from Corollary 3.4 form a neighbourhood basis of x_0, continuity of u at x_0 follows from (13) and the defining property of G. □

3.2. **First proof of the coarea inequality.** In this section we prove Theorem 1.6 for weakly monotone functions and weak upper gradients, with the non-sharp constant κ.

Theorem 3.6. *Let X be a metric surface and $p \geq 1$. If $u\colon X \to \mathbb{R}$ is a weakly monotone function with a locally p-integrable p-weak upper gradient ρ, then for $\kappa = (4/\pi) \cdot 200$,*

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \kappa \int_X g\rho \, d\mathcal{H}^2 \quad \textit{for every Borel } g\colon X \to [0, \infty]. \tag{14}$$

Proof. It suffices to show that

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t) \cap U} g \, d\mathcal{H}^1 \, dt \leq \kappa \int_U g\rho \, d\mathcal{H}^2 \quad \text{for every Borel } g\colon U \to [0, \infty] \tag{15}$$

for any subset $U \subset X$ homeomorphic to $\mathbb{R}^2$ with $\mathcal{H}^2(U) < \infty$ and $g \in L^p(U)$. Indeed, we can cover X with countably many such subsets U_j, apply (15) to the restrictions of g to $U_j \setminus \cup_{\ell=1}^{j-1} U_\ell$ on U_j, and sum up the results to get (14).

Moreover, it suffices to prove (15) for $g = \chi_E$, the characteristic function of an open set $E \subset U$ compactly contained in U, due to standard approximation via simple functions. Fix $\epsilon_0 > 0$ such that $\overline{B}(x, 20\epsilon_0)$ is compactly contained in U for every $x \in E$.

Step 1: Consider the collection G of all $x \in E$ for which for all $0 < \epsilon < \epsilon_0$ there exists some $0 < r < \epsilon$ with

$$\int\limits_{B(x,10r)} \rho \, d\mathcal{H}^2 \leq 200 \int\limits_{B(x,r)} \rho \, d\mathcal{H}^2.$$

Fix an arbitrary $0 < \epsilon < \epsilon_0$. By the 5r-covering theorem [Fed69, 2.8.4], there exists a countable (possibly finite) collection of pairwise disjoint balls $\{B(x_j, r_j)\}_j$, for which $x_j \in G$, $10r_j < \min\{\epsilon, d(X \setminus E, x_j)\}$. We write $B_j = B(x_j, r_j)$ for short. The collection $\{B(x_j, 5r_j)\}_j$ covers G, and

$$\int\limits_{10B_j} \rho \, d\mathcal{H}^2 \leq 200 \int\limits_{B_j} \rho \, d\mathcal{H}^2 \quad \text{for each } j.$$

For each j, we find a Borel set $C_j \supset u(5B_j)$ with $\mathcal{H}^1(C_j) = \mathcal{H}^1(u(5B_j))$. Then

$$g_\epsilon(t) = \sum_j 10r_j \chi_{C_j}(t)$$

is Borel measurable. Then, by Corollary 3.2, applied with $r = 5r_j$,

$$\int\limits_{\mathbb{R}} g_\epsilon(t) \, dt \leq \sum_j \frac{4}{\pi} \int\limits_{10B_j} \rho \, d\mathcal{H}^2.$$

The defining property of the B_j and the inclusion $\bigcup_j B_j \subset E$ yield

$$\sum_j \int\limits_{10B_j} \rho \, d\mathcal{H}^2 \leq 200 \int\limits_E \rho \, d\mathcal{H}^2.$$

Thus,

$$\int\limits_{\mathbb{R}} g_\epsilon(t) \, dt \leq \frac{4}{\pi} 200 \int\limits_E \rho \, d\mathcal{H}^2.$$

Suppose that $x \in \overline{u^{-1}(t)} \cap G$ for some given $t \in \mathbb{R}$. Then x is contained in some $5B_j$. By openness of $5B_j$ one shows that $t \in u(5B_j)$ for every such j. Hence the definition of the Hausdorff content $\mathcal{H}^1_\epsilon$ yields

$$\mathcal{H}^1_\epsilon(\overline{u^{-1}(t)} \cap G) \leq \sum_{j:\ t \in u(5B_j)} 10r_j \leq \sum_j 10r_j \chi_{u(5B_j)}(t) \leq g_\epsilon(t).$$

Since ϵ was arbitrary, by applying monotone convergence theorem, we conclude

$$\int\limits_{\mathbb{R}}^{*} \mathcal{H}^1(\overline{u^{-1}(t)} \cap G) \, dt = \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}}^{*} \mathcal{H}^1_\epsilon(\overline{u^{-1}(t)} \cap G) \, dt \leq \frac{4}{\pi} 200 \int\limits_E \rho \, d\mathcal{H}^2.$$

Step 2: Consider $F = E \setminus G$. We claim that

$$\int_{\mathbb{R}}^{*} \mathcal{H}^1(\overline{u^{-1}(t)} \cap F)\, dt = 0. \tag{16}$$

This will complete the proof of inequality (15) for $g = \chi_E$. Indeed, having verified (16), by Step 1 and subadditivity property of the upper integral we deduce

$$\int_{\mathbb{R}}^{*} \mathcal{H}^1(\overline{u^{-1}(t)})\, dt = \int_{\mathbb{R}}^{*} \mathcal{H}^1(\overline{u^{-1}(t)} \cap G)\, dt \leq \frac{4}{\pi} 200 \int_E \rho\, d\mathcal{H}^2.$$

So it remains to establish (16). By definition of G, for every $x \in F$, there exists $k_x \in \mathbb{N}$, such that for any $j > k_x$,

$$\int_{B(x,10^{-j})} \rho\, d\mathcal{H}^2 \leq 200^{-(j-k_x)} \int_{B(x,10^{-k_x})} \rho\, d\mathcal{H}^2. \tag{17}$$

By monotone convergence, it is sufficient to establish (16) with F replaced with

$$F_k = \left\{x \in F \colon k_x \leq k, d(x, X \setminus E) > 10^{-k}\right\} \quad \text{for arbitrary } k \in \mathbb{N}.$$

We fix $k \in \mathbb{N}$ and $j - 1 > k$ for now. By the definition of the Hausdorff measure, there exists a countable collection of balls B_m that cover F_k, with radii r_m that satisfy $r_m \leq 10^{-j}$ and $2r_m \geq \operatorname{diam} B_m \geq r_m$, such that each B_m intersects F_k, and

$$\frac{4}{\pi} \sum_m (\operatorname{diam} B_m)^2 - (1/j) \leq 4\mathcal{H}^2(F_k). \tag{18}$$

In fact, we may require the balls to be centered at the set F_k.[1]

For each $m \in \mathbb{N}$, let $j_m \in \mathbb{Z}$ be the largest integer for which $2r_m \leq 10^{-j_m}$. Observe from the inequalities $10^{-j_m} < 20r_m \leq 20 \cdot 10^{-j}$ that $j_m \geq j - 1$. Using these observations, we deduce from Corollary 3.2 and (17) that

$$2r_m \mathcal{H}^1(u(B_m)) \leq \frac{4}{\pi} \int_{2B_m} \rho\, d\mathcal{H}^2 \leq \frac{4}{\pi} 200^{-(j_m-k)} \int_U \rho\, d\mathcal{H}^2. \tag{19}$$

As before, we consider $g_j(x) = \sum_m 2r_m \chi_{C_m}$ for Borel sets $C_m \supset u(B_m)$ with $\mathcal{H}^1(C_m) = \mathcal{H}^1(u(B_m))$. By arguing as in Step (1), the definition of $\mathcal{H}^1_{1/j}$ yields that

$$\mathcal{H}^1_{1/j}(\overline{u^{-1}(t)} \cap F_k) \leq g_j(t) \quad \text{for all } t \in \mathbb{R}.$$

By (upper) integrating over $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}}^{*} \mathcal{H}^1_{1/j}(\overline{u^{-1}(t)} \cap F_k)\, dt \leq \sum_m 2r_m \mathcal{H}^1(u(B_m)).$$

[1]After an initial choice of a covering according to the definition of $\mathcal{H}^2$, replace each set by a closed ball centered on F_k and radius equal to the diameter of the set. (Hence, the factor 4 on the right.)

We apply now (19) and the inequalities $1 < 20 \cdot 10^{j_m} \cdot r_m$ and $j_m \geq j-1$, and obtain

$$\begin{aligned}\int_{\mathbb{R}}^{*} \mathcal{H}^1_{1/j}(\overline{u^{-1}(t)} \cap F_k)\,dt &\leq \sum_m 2r_m \mathcal{H}^1(u(B_m)) \leq \frac{4}{\pi}\sum_m 200^{-(j_m-k)}\int_U \rho\,d\mathcal{H}^2 \\ &\leq \frac{200^k \cdot 4}{\pi}\left(\int_U \rho\,d\mathcal{H}^2\right)\sum_m 200^{-j_m}(20\cdot 10^{j_m} r_m)^2 \\ &\leq \frac{200^k \cdot 4}{\pi}\left(\int_U \rho\,d\mathcal{H}^2\right) 20^2 \sum_m 2^{-(j-1)} r_m^2 \\ &\leq \frac{200^k \cdot 4}{\pi}\left(\int_U \rho\,d\mathcal{H}^2\right) 20^2 2^{-(j-1)} \sum_m r_m^2.\end{aligned}$$

Now, we apply (18) and pass to the limit as $j \to \infty$, and conclude

$$\int_{\mathbb{R}}^{*} \mathcal{H}^1(\overline{u^{-1}(t)} \cap F_k)\,dt = 0.$$

Passing to the limit $k \to \infty$ yields (16) and the proof is complete. $\square$

3.3. **Continuity of weakly monotone functions.** In this section we prove Theorem 1.10: weakly monotone functions with p-integrable upper gradients are continuous when $p \geq 2$. In other words, for this range of p, *weakly monotone* functions are *monotone* functions. We prove this result by a refined study of the topology of the level sets of such functions.

The standing assumptions in this section are that U is homeomorphic to $\mathbb{R}^2$ with $\mathcal{H}^2(U) < \infty$, and $u\colon U \to \mathbb{R}$ is weakly monotone (Definition 1.9). We moreover assume that u has a p-integrable upper gradient ρ, $1 \leq p < \infty$.

We start with the following topological lemma, cf. [Nta20, Corollary 2.8], which says that connected components of the closures of the level sets of weakly monotone functions "leave every compact set".

Proposition 3.7. *Let $E \subset \overline{u^{-1}(t)} \cap U$ be a connected component. Then $\emptyset \neq E \setminus K$ for every compact $K \subset U$.*

Remark 3.8. Since the results are applied to cases where U is a subset of a metric surface X, we maintain the notation $\overline{u^{-1}(t)} \cap U$ to emphasize that we are taking the closure relative to the subspace topology of U.

Proof. Aiming for a contradiction, suppose to the contrary that $E \setminus K = \emptyset$ for some compact $K \subset U$. Then E itself is compact. Consider then a Jordan domain $W \supset E$ compactly contained in U. We denote $A := \overline{W} \cap \overline{u^{-1}(t)}$, and observe that E is also a connected component of A.

Since A is a compact subset of U, with U being homeomorphic to $\mathbb{R}^2$, for each open set $V \supset E$, there exists a Jordan domain V' compactly contained in V, with $E \subset V'$ and $A \cap \partial V' = \emptyset$; see [Why58, Corollary 3.11, p.35]. Below, we apply this result for $V = W$.

We apply Lemmas 2.5 and 2.13 to $f(z) = d(z, \partial V')$ as follows: Let

$$\epsilon_0 = \min\left\{d(U \setminus \overline{W}, \partial V'), d(A, \partial V')\right\},$$

observing that $f^{-1}([0,s])$ is compact in $\overline{W} \setminus A$ for every $0 < s < \epsilon_0$. Then, for every $0 < s < \epsilon_0$, there exists a connected component $F_s \subset f^{-1}(s) \cap V'$ separating E from $\partial V'$; recall Lemma 2.5. By applying Lemma 2.13, for almost every such s, we may assume that F_s is simple (thus, homeomorphic to $\mathbb{S}^1$, cf. Lemma 2.7) and admits a Lipschitz parametrization along which u is absolutely continuous. We fix one such s and let V_s be the Jordan domain bounded by F_s that contains E. Since $E \subset V_s \cap \overline{u^{-1}(t)}$, Definition 1.9 implies

$$t \in \left[\inf_{\partial V_s} u, \sup_{\partial V_s} u\right].$$

Given that $u|_{\partial V_s}$ is continuous, there exists $x_0 \in \partial V_s$ such that $u(x_0) = t$. But then, $x_0 \in u^{-1}(t) \cap \partial V_s$, contradicting $A \cap \partial V_s = \emptyset$. □

The next result can be compared to Lemma 2.13. However, here u is not necessarily Lipschitz, and instead of the Eilenberg inequality we use the coarea inequality of Theorem 3.6. Recall that a simple Peano continuum is a set homeomorphic either to a point, or a compact interval, or to $\mathbb{S}^1$.

Proposition 3.9. *Let $u\colon U \to \mathbb{R}$ be a weakly monotone function with a p-integrable p-weak upper gradient, for some $2 \leq p \leq \infty$. Denote $I := u(U)$. Then, for almost every $t \in I$,*

(1) $u^{-1}(t) = \overline{u^{-1}(t)} \cap U$ and every continuum $E \subset \overline{u^{-1}(t)} \cap U$ is a simple Peano continuum, and,
(2) $\mathcal{H}^1(\overline{u^{-1}(t)} \cap U) < \infty$.

Suppose, moreover, that Γ_0 is a path family with p-modulus zero and $N_0 \subset U$ is an $\mathcal{H}^2$-negligible set. Then, for almost every $t \in I$,

(3) for every continuum $E \subset \overline{u^{-1}(t)} \cap U$ of positive diameter, there exists a surjective Lipschitz path $\gamma\colon [0,1] \to E$ that is injective outside its end points; and
(4) for every absolutely continuous $\gamma\colon [0,1] \to \overline{u^{-1}(t)} \cap U$, $\int_\gamma \chi_{N_0}\, ds = 0$. Moreover, if there exists $M \in \mathbb{N}$ such that $\#(\gamma^{-1}(x)) \leq M$ for $\mathcal{H}^1$-almost every $x \in \overline{u^{-1}(t)} \cap U$, then $\gamma \notin \Gamma_0$.

Proof. We fix a minimal p-weak upper gradient ρ of u. Let Γ_1 denote the negligible collection of rectifiable paths for which the triple (u, ρ, γ) fails the upper gradient inequality or along which ρ fails to be path integrable.

Recall the negligible family Γ_0 and the set N_0 with $\mathcal{H}^2(N_0) = 0$ from our assumptions, and recall also that u is continuous outside some N_1 with $\mathcal{H}^2(N_1) = 0$, as stated in

Lemma 3.5. Fix an arbitrary Borel set $B \supset N_0 \cup N_1$ with $\mathcal{H}^2(B) = 0$. We consider a Borel function $G\colon U \to [0,\infty]$ in $L^p(U)$ satisfying

$$\int_\gamma G\,ds = \infty, \quad \text{for all} \quad \gamma \in \Gamma_0 \cup \Gamma_1.$$

Let $\widehat{\rho} \coloneqq \rho + (1+G)\epsilon + \infty \cdot \chi_B$ for an arbitrary $\epsilon > 0$. Observe that $\widehat{\rho}$ is Borel, $\widehat{\rho} \in L^p(U)$, and that the upper gradient inequality holds for the triples $(u, \widehat{\rho}, \gamma)$ for *every* rectifiable path.

Let Γ_2 denote the collection of all paths $\gamma\colon [a,b] \to U$ satisfying

$$\int_\gamma \widehat{\rho}\,ds = \infty.$$

In particular, if $\gamma \notin \Gamma_2$ is absolutely continuous, then $u \circ \gamma$ is absolutely continuous. Note also that if γ contains a subpath in $\Gamma_0 \cup \Gamma_1$, then $\gamma \in \Gamma_2$. Moreover, as $\widehat{\rho}$ is $L^p(U)$-integrable, Γ_2 is p-negligible.

In case $p = \infty$, $\widehat{\rho} \in L^\infty(U)$ and we may apply Theorem 3.6 to deduce that for almost all $t \in u(U)$,

$$\int_{\overline{u^{-1}(t)} \cap U} \widehat{\rho}\,d\mathcal{H}^1 \leq \|\widehat{\rho}\|_\infty \mathcal{H}^1(\overline{u^{-1}(t)} \cap U) < \infty.$$

For $2 \leq p < \infty$, we claim that for almost every $t \in u(U)$,

$$\int_{\overline{u^{-1}(t)} \cap U} (\widehat{\rho})^{p-1}\,d\mathcal{H}^1 < \infty. \tag{20}$$

Indeed, we apply the coarea inequality Theorem 3.6 to the Borel function $g = (\widehat{\rho})^{p-1}$ and the 1-weak upper gradient $\widehat{\rho}$ of u. Then Hölder's inequality and (20) imply

$$\int_{\overline{u^{-1}(t)} \cap U} \widehat{\rho}\,d\mathcal{H}^1 < \infty \tag{21}$$

for almost all $t \in u(U)$. So the conclusion (21) holds for every $2 \leq p \leq \infty$.

We are now in a position to establish Claims (1) to (4).

We establish Claim (4) first. To this end, consider an absolutely continuous $\gamma\colon [a,b] \to \overline{u^{-1}(t)} \cap U$ for an arbitrary t satisfying the conclusion (21). Then, as $\widehat{\rho} \geq \infty \cdot \chi_B$, we conclude $\mathcal{H}^1(B \cap \overline{u^{-1}(t)}) = 0$. Therefore $\int_\gamma \chi_B\,ds = 0$ by definition of the path integral. Next, in addition, we assume that $\#(\gamma^{-1}(x)) \leq M$ for $\mathcal{H}^1$-almost every $x \in \overline{u^{-1}(t)} \cap U$. Then the definition of the path integral implies

$$\int_\gamma \widehat{\rho}\,ds \leq M \int_{\overline{u^{-1}(t)} \cap U} \widehat{\rho}\,d\mathcal{H}^1.$$

Thus, if t satisfies the conclusion (21), then $\gamma \notin \Gamma_2$. Then conclusion (4) follows for any γ as above. In particular, Claim (4) holds.

Next, since $\widehat{\rho} \geq \epsilon$, we conclude $\mathcal{H}^1(\overline{u^{-1}(t)} \cap U) < \infty$ for every t satisfying the conclusion (21). In particular, Claim (2) holds.

Let $\mathcal{G}$ denote the collection of $t \in u(U)$ satisfying conclusion (21). We assume for now that $u^{-1}(t) = \overline{u^{-1}(t)} \cap U$ for every $t \in \mathcal{G}$. We prove the latter in the next paragraph. We then show how to establish (1). Let $\mathcal{F} \subset \mathcal{G}$ denote the collection of t for which $u^{-1}(t)$ contains a non-simple continuum. Then $\mathcal{F}$ is countable by Lemma 2.8. Claim (1) follows from this observation.

Next, we claim that whenever $t \in \mathcal{G}$, then $u^{-1}(t) = \overline{u^{-1}(t)} \cap U$. To this end, we first observe that every connected component of $\overline{u^{-1}(t)} \cap U$ has positive $\mathcal{H}^1$-measure by Proposition 3.7. We consider an arbitrary connected component E of $\overline{u^{-1}(t)} \cap U$. Then Lemma 2.10 implies the existence of continua $(E_n)_{n=1}^{\infty}$ such that $E_n \subset E_{n+1} \subset E$, $E = \bigcup_{n=1}^{\infty} E_n$ and $\mathcal{H}^1(E_1) > 0$. Then, by Lemma 2.6, there exists a surjective Lipschitz $\gamma_n \colon [0,1] \to E_n$ satisfying $1 \leq \#(\gamma_n^{-1}(x)) \leq 2$ for $\mathcal{H}^1$-almost every $x \in E_n$. As in the proof of Claim (4), we conclude $\gamma_n \notin \Gamma_2$. This yields that $u \circ \gamma_n$ is absolutely continuous. As γ_n has zero length in B, by considering a constant speed reparametrization of γ_n, we may therefore assume that $\gamma_n^{-1}(B)$ has negligible Lebesgue measure. Also, as u is continuous at every $\gamma_n(s)$ for every $s \in [0,1] \setminus \gamma_n^{-1}(B)$, the absolute continuity of $u \circ \gamma_n$ implies $u \circ \gamma_n(s) = t$ for every $s \in [0,1]$. In particular, $E_n \subset u^{-1}(t)$ for every $n \in \mathbb{N}$. The conclusion $u^{-1}(t) = \overline{u^{-1}(t)} \cap U$ follows by the arbitrariness of $n \in \mathbb{N}$ and the component E.

To finish, we show Claim (3). Now outside a countable family of t satisfying conclusion (21), every continuum $E \subset \overline{u^{-1}(t)} \cap U$ is simple. When $\operatorname{diam} E > 0$, such sets can be parametrized by a Lipschitz path that is injective outside its end points, as we recall from Lemma 2.4. Thus Claim (3) follows. Since Claims (1) to (4) were proved, the proof is complete. $\square$

We are now ready to prove Theorem 1.10.

Proof of Theorem 1.10. It suffices to prove continuity for $u \colon U \to \mathbb{R}$ satisfying the standing assumptions of this section. Let $x_0 \in U$, and consider the numbers $s_1 = \liminf_{y \to x_0} u(y)$ and $s_2 = \limsup_{y \to x_0} u(y)$. Then x_0 is a point of discontinuity for u if and only if $s_1 < s_2$.

We assume that u is discontinuous at x_0 and derive a contradiction. To this end, from Lemma 3.3 we obtain $\epsilon_0 > 0$ and a nested collection of Jordan domains U_r compactly contained in U, for almost every $0 < r < \epsilon_0$, for which $u|_{\partial U_r}$ is continuous, $d(x_0, y) = r$ for every $y \in \partial U_r$ and $\bigcap_r U_r = \{x_0\}$. Note that the continuity of $u|_{\partial U_r}$ follows from the existence of a Lipschitz parametrization of ∂U_r injective outside its end points, such that u is absolutely continuous along the parametrization.

Continuity of $u|_{\partial U_r}$ implies that $u(\partial U_r)$ is connected. Also, Definition 1.9 implies that $u(\partial U_r) \supset (s_1, s_2)$. Since r is arbitrary, we conclude $x_0 \in \overline{u^{-1}(t)}$ for every $s_1 < t < s_2$. Proposition 3.7 yields the existence of a connected component $E_t \subset \overline{u^{-1}(t)} \cap U$ containing x_0, with $\operatorname{diam} E_t > 0$. On the other hand, Proposition 3.9 implies $u|_{E_t} = t$ for almost every such t. This is a contradiction since (s_1, s_2) has positive measure. $\square$

Example 3.10. Assumption $p \geq 2$ in Theorem 1.10 cannot be relaxed even in the standard $\mathbb{R}^2$. Indeed, $u\colon \mathbb{D} \to \mathbb{R}$ defined by $u(x) = x_1 + x_1/|x|$ for $x \neq 0$ and $u(0) = 0$ is weakly monotone and discontinuous at the origin. Moreover, $u \in D^{1,p}(\mathbb{D})$ for all $p < 2$. See [IKO01, Sect. 3] for further details.

4. Coarea inequality via weakly quasiconformal mappings

In this section we give a second proof of the coarea inequality for monotone functions using suitable parametrizations of the metric surfaces from Euclidean domains. The motivation arises from the fact that in the Euclidean setting, better than an inequality, we actually have an *equality*, known as the the coarea formula (Theorem 1.2).

4.1. **Weakly quasiconformal maps.**

Definition 4.1 (Quasiconformal maps)**.** Given metric spaces Ω and X, endowed with locally finite $\mathcal{H}^2$-measures, a homeomorphism $f\colon \Omega \to X$ is *quasiconformal* if there exists a $K \geq 1$ such that

$$K^{-1} \operatorname{mod} \Gamma \leq \operatorname{mod} f\Gamma \leq K \operatorname{mod} \Gamma,$$

for every family Γ of continuous paths. Here $f\Gamma$ denotes the collection of all $f \circ \gamma$ for which $\gamma \in \Gamma$. We shall say K-*quasiconformal* to emphasize the role of K.

The third named author established in [Raj17] necessary and sufficient conditions for a domain $U \subset X$ homeomorphic to $\mathbb{R}^2$ in a metric surface X to admit a quasiconformal parametrization from $\mathbb{R}^2$ or the disk $\mathbb{D}$. That is, there to exist a quasiconformal homeomorphism $\varphi\colon \Omega \to U$ for $\Omega = \mathbb{D}$ or $\Omega = \mathbb{R}^2$. As a sufficient condition, we note the following,

$$\sup_{x \in U} \limsup_{r \to 0^+} \frac{\mathcal{H}^2(\overline{B}(x,r))}{\pi r^2} < \infty, \text{ see [RRR21]}.$$

Romney observed in [Rom19] that whenever such a parametrization exists, there exists a $\pi/2$-quasiconformal homeomorphism $\varphi\colon U \to \Omega \subset \mathbb{R}^2$. More generally, we say that $U \subset X$ is a *quasiconformal surface* if every point $x_0 \in U$ is contained in an open set $U' \subset X$ which admits a quasiconformal parametrization. It is now understood that every quasiconformal surface is a $\pi/2$-quasiconformal image of a Riemannian surface [Iko21]. In fact, for every quasiconformal surface X, there exists a Riemannian surface Y and a quasiconformal homeomorphism $f\colon X \to Y$ satisfying

$$\frac{2}{\pi} \operatorname{mod} \Gamma \leq \operatorname{mod} f\Gamma \leq \frac{4}{\pi} \operatorname{mod} \Gamma \tag{22}$$

for every path family Γ. Both inequalities are best possible. In general, there are geometric obstructions for a metric surface to be a quasiconformal surface. A typical example involves considering a length space X obtained from the plane $\mathbb{R}^2$ by collapsing the closed disk $\overline{\mathbb{D}}$ to a point and endowing the quotient space with the induced length distance. No neighbourhood of the collapsed disk on X can be quasiconformal homeomorphic to a subset of the plane. More subtle examples were recently considered in [IR22, Iko22, NR22]. Fortunately, every metric surface is a *weakly quasiconformal* image of a smooth Riemannian surface. Before formulating the precise statement, we need a definition.

Definition 4.2 (Weakly quasiconformal parametrization.)**.** Given metric spaces V and U, endowed with locally finite $\mathcal{H}^2$-measures, a map $f\colon V \to U$ is *weakly K-quasiconformal* if it satisfies the following:

(a) f is a uniform limit of homeomorphisms $g\colon V \to U$,
(b) there exists a $K \geq 1$ such that

$$\operatorname{mod}\Gamma \leq K \operatorname{mod} f\Gamma, \tag{23}$$

for every path family Γ.

Recently, Ntalampekos and Romney established the following result which was conjectured by the third named author and Wenger, cf. [IR22, Question 1.1].

Theorem 4.3 (Theorem 1.3, [NR22])**.** *If U is a metric surface, then there exists a Riemannian surface V and a weakly $(4/\pi)$-quasiconformal $f\colon V \to U$.*

Theorem 4.3 was proved earlier by Meier and Wenger [MW21] and Ntalampekos and Romney [NRar], under the assumption that U is locally geodesic. See also [LW17]. Observe that the mapping f in Theorem 4.3 satisfies only the upper bound in the stronger result (22). Moreover, the example above shows that the one-sided inequality (23) cannot be upgraded to quasiconformality.

Properties of weakly quasiconformal mappings have been extensively studied in the metric surface setting in [NRar] and in greater generality in [Wil12, ILP21]. In particular, we have the following.

Lemma 4.4 (Theorem 7.1, [NRar])**.** *Let $f\colon V \to U$ be a continuous map between metric surfaces. Then f satisfies $\operatorname{mod}\Gamma \leq K \operatorname{mod}(f\Gamma)$, for all path families, if and only if f has a locally 2-integrable 2-weak upper gradient ρ for which*

$$\int_{f^{-1}(E)} \rho^2 \, d\mathcal{H}^2 \leq K\mathcal{H}^2(E) \quad \textit{for every Borel set } E \subset U. \tag{24}$$

Note, in particular, that if (24) holds for some locally 2-integrable 2-weak upper gradient, it also holds for the minimal 2-weak upper gradient ρ_f. Consider next $\nu(E) := \int_{f^{-1}(E)} \rho_f^2 \, d\mathcal{H}^2$ for every Borel set $E \subset U$. Inequality (24) is equivalent to requiring that ν is locally finite and that

$$\nu(E) \leq K\mathcal{H}^2(E) \quad \text{for all Borel } E \subset U.$$

This implies that

$$\int_U g \, d\nu \leq K \int_U g \, d\mathcal{H}^2 \quad \text{for any Borel function } g\colon U \to [0, \infty].$$

In other words,

$$\int_V g(f(x))\rho_f^2(x) \, d\mathcal{H}^2(x) \leq K \int_X g \, d\mathcal{H}^2 \quad \text{for any Borel function } g\colon U \to [0, \infty]. \tag{25}$$

4.2. **Pullback of monotone functions by weakly quasiconformal maps.** In this section, $f\colon V \to U$ is weakly K-quasiconformal where $V \subset \mathbb{R}^2$ is open and simply connected and U is a metric surface satisfying $\mathcal{H}^2(U) < \infty$.

Later, $u\colon U \to \mathbb{R}$ will be a (weakly) monotone function in the sense of Definition 1.9. Throughout, we stick to the notation $v \coloneqq u \circ f$. The aim of this section is to prove that v inherits important regularity and structural properties of u.

The modulus inequality (23) allows us to pullback Dirichlet functions using f. More precisely, we have the following.

Lemma 4.5. *Let V and U be metric surfaces and $f\colon V \to U$ weakly quasiconformal. If $u\colon U \to [-\infty, \infty]$ has a 2-integrable 2-weak upper gradient ρ, then $\rho'(x) = \rho(f(x))\rho_f(x)$ is a 2-integrable 2-weak upper gradient of $v \coloneqq u \circ f$. In particular,*

$$\rho_v(x) \leq \rho_u(f(x))\rho_f(x) \quad \textit{for } \mathcal{H}^2\textit{-a.e. } x \in V.$$

Proof. Let Γ_0 denote the family of paths $\gamma\colon [a,b] \to U$ for which either

(1) $u \circ \gamma$ fails to be absolutely continuous,
(2) $\int_\gamma \rho\, ds = \infty$, or
(3) there exists an interval $I \subset [a,b]$ such that $\ell(u \circ \gamma|_I) > \int_I (\rho(\gamma(t)))|\gamma'|(t)\, dt$.

Then Γ_0 has negligible 2-modulus since ρ is a 2-integrable 2-weak upper gradient of u. In particular, for every absolutely continuous $\Gamma_0 \not\ni \gamma\colon [a,b] \to U$,

$$|(u \circ \gamma)'|(t) \leq \rho(\gamma(t))|\gamma'|(t) \quad \text{for almost every } t \in [a,b];$$

see, e.g., [HKST15, Proposition 6.3.3].

Let Γ_1 denote the family of paths $\gamma\colon [a,b] \to V$ for which

(1) $f \circ \gamma \in \Gamma_0$,
(2) $f \circ \gamma$ fails to be absolutely continuous,
(3) $\int_\gamma \rho_f\, ds = \infty$, or
(4) there exists an interval $I \subset [a,b]$ with $\ell(f \circ \gamma|_I) > \int_I (\rho_f(\gamma(t)))|\gamma'|(t)\, dt$.

Then Γ_1 has negligible 2-modulus since f is weakly quasiconformal. Now, for each absolutely continuous $\Gamma_1 \not\ni \gamma\colon [a,b] \to V$, we have

$$|(v \circ \gamma)'|(t) \leq \rho(f(\gamma(t)))|(f \circ \gamma)'|(t) \leq \rho(f(\gamma(t)))\rho_f(\gamma(t))|\gamma'|(t) \quad \text{for a.e. } t \in [a,b],$$

where [HKST15, Proposition 6.3.3] was used again. Hence

$$\ell(v \circ \gamma) \leq \int_\gamma (\rho \circ f)\rho_f\, dt = \int_\gamma \rho'\, dt.$$

So, $\rho'(x) = \rho(f(x))\rho_f(x)$ is a 2-integrable 2-weak upper gradient — the integrability follows from (25). $\square$

Lemma 4.6. *Suppose that V and U are metric surfaces and $f\colon V \to U$ is a uniform limit of homeomorphisms $f_n\colon V \to U$. If $u\colon U \to \mathbb{R}$ is a monotone function, then $v = u \circ f$ is a monotone function.*

Proof. If $f_n\colon V \to U$ are homeomorphisms converging to f uniformly, then it is easy to check that $v_n := u \circ f_n$ converge to $v := u \circ f$ uniformly on compact sets $K \subset V$. Now, if Ω is an open and compactly contained subset of V, then from the facts that $v_n|_{\overline{\Omega}} \to v|_{\overline{\Omega}}$ uniformly and that each v_n is monotone, we conclude

$$\sup_{x\in\partial\Omega} v(x) = \lim_{n\to\infty} \sup_{x\in\partial\Omega} v_n(x) = \lim_{n\to\infty} \sup_{x\in\Omega} v_n(x) = \sup_{x\in\Omega} v(x).$$

Similar argument holds for inf in place of sup, so, the monotonicity of v follows. □

We recall from Theorem 1.10 that u having a 2-integrable upper gradient and being weakly monotone implies continuity and thus monotonicity. Now combining this with Lemmas 4.5 and 4.6 gives the main result of this subsection.

Theorem 4.7. *Let Y and X be metric surfaces and $f\colon Y \to X$ weakly quasiconformal. If a weakly monotone $u\colon X \to \mathbb{R}$ has a locally 2-integrable 2-weak upper gradient, then $v = u \circ f$ is also weakly monotone and has a locally 2-integrable 2-weak upper gradient.*

4.3. **Second proof of the coarea inequality.** We now prove a version of Theorem 3.6 with the sharp constant when $p \geq 2$.

Theorem 4.8. *Let U be a metric surface and $p \geq 2$. Suppose there exists weakly K-quasiconformal map $f\colon \mathbb{D} \to U$. If a weakly monotone function $u\colon U \to \mathbb{R}$ has a locally p-integrable p-weak upper gradient ρ, then*

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq K \int_U g\rho \, d\mathcal{H}^2 \quad \textit{for every Borel } g\colon U \to [0,\infty].$$

Theorem 1.6 follows by combining Theorem 4.8 when $p \geq 2$ and Theorem 3.6 when $1 \leq p < 2$, respectively.

Proof of Theorem 4.8. Recall that U is homeomorphic to $\mathbb{D}$ by the existence of f. By exhausting U with compactly contained disks, we may assume that $\mathcal{H}^2(U) < \infty$ and that ρ is p-integrable on U. We first recall from Theorem 1.10 that u is monotone. Fix a minimal 2-weak upper gradient ρ_f of f. Write $v := u \circ f$ and recall from Lemma 4.5 that $\rho'(x) = \rho(f(x))\rho_f(x)$ is a 2-integrable 2-weak upper gradient of v. Moreover, Lemma 4.6 shows that v is monotone. Let $g\colon U \to [0,\infty]$ be any Borel function. We claim that

$$\int_{u^{-1}(t)} g \, d\mathcal{H}^1 \leq \int_{v^{-1}(t)} (g \circ f)\rho_f \, d\mathcal{H}^1 \quad \text{for almost all } t \in u(U). \tag{26}$$

To this end, let Γ_0 denote the collection of all absolutely continuous $\theta\colon [a,b] \to \mathbb{D}$ for which there exists an interval $[c,d] \subset [a,b]$ so that $f \circ \theta|_{[c,d]}$ is not absolutely continuous or the triple $(f, \rho_f, \theta|_{[c,d]})$ fails the upper gradient inequality. The family Γ_0 is 2-negligible.

We apply Corollary 1.11 and Proposition 3.9 twice in the following manner. We first apply it for u and conclude that for almost all t, $u^{-1}(t)$ is a properly embedded topological 1-manifold in U and $\mathcal{H}^1(u^{-1}(t)) < \infty$. We next apply it for v and the path family Γ_0. Then, for almost all t, $v^{-1}(t)$ is a properly embedded topological 1-manifold in $\mathbb{D}$, $\mathcal{H}^1(v^{-1}(t)) < \infty$ and every injective absolutely continuous $\theta\colon [a,b] \to v^{-1}(t)$ is in the complement of Γ_0.

Combining the facts from the previous two paragraphs yields the following. For almost all $t \in u(U)$ and every connected component E of $v^{-1}(t)$, there exists an increasing sequence $(E_n)_{n=1}^{\infty}$ of continua exhausting E (Lemma 2.10). Moreover, E and E_n are homeomorphic to $\mathbb{R}$ and $[0,1]$, respectively, and there exists a homeomorphic Lipschitz parametrization $\theta_n \colon [0,1] \to E_n$, $\theta_n \notin \Gamma_0$. Therefore

$$\int_{f(E_n)} g \, d\mathcal{H}^1 \leq \int_{f\circ\theta_n} g \, ds \leq \int_{\theta_n} (g \circ f)\rho_f \, ds = \int_{E_n} (g \circ f)\rho_f \, d\mathcal{H}^1.$$

Here the first inequality and the equality follow from the area formula for paths. The second inequality is a consequence of $\theta_n \notin \Gamma_0$. The sets $f(E_n)$ exhaust $f(E)$, so, we pass to the limit $n \to \infty$ and conclude

$$\int_{f(E)} g \, d\mathcal{H}^1 \leq \int_{E} (g \circ f)\rho_f \, d\mathcal{H}^1.$$

As $v^{-1}(t)$ is a properly embedded topological 1-manifold, it has countably many components. Thus we may take the sum over all the components of $v^{-1}(t)$ and apply subadditivity to conclude (26).

Integrating (26) over $\mathbb{R}$ and applying the Euclidean coarea formula Theorem 1.2 for v yields

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \int_{\mathbb{D}} (g \circ f)\rho_f |\nabla v| \, d\mathcal{H}^2 \quad \text{for every Borel } g \colon U \to [0,\infty]. \tag{27}$$

The application is valid since the weak $(1,2)$-Poincaré inequality yields $v \in L^2(\mathbb{D})$, so, in particular, $v \in W^{1,1}(\mathbb{D})$.

Since $|\nabla v|$ is a 2-minimal 2-weak upper gradient of u, $|\nabla v| \leq \rho' = (\rho \circ f)\rho_f$, $\mathcal{H}^2$-almost everywhere in $\mathbb{D}$. Thus (27) implies

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \int_{\mathbb{D}} ((g\rho) \circ f)\rho_f^2 \, d\mathcal{H}^2 \quad \text{for every Borel } g \colon U \to [0,\infty]. \tag{28}$$

We apply (25) to (28) and conclude

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq K \int_{U} g\rho \, d\mathcal{H}^2 \quad \text{for every Borel } g \colon U \to [0,\infty].$$

□

Combining Theorem 4.8 with the existence of weakly $4/\pi$-quasiconformal parametrizations gives the coarea inequality with the best possible constant $4/\pi$.

Corollary 4.9. *Let X be a metric surface and $p \geq 2$. If $u\colon X \to \mathbb{R}$ is a weakly monotone function with a locally p-integrable p-weak upper gradient ρ, then*

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int_X g\rho \, d\mathcal{H}^2 \quad \textit{for every Borel } g\colon X \to [0,\infty].$$

Proof. Arguing as in the beginning of the proof of Theorem 3.6, we may replace X with surface U as in the proof of Theorem 4.8. The claim now follows from Theorem 4.3 and Theorem 4.8. □

Theorem 1.6 follows from Theorem 3.6 and Corollary 4.9.

5. Lipschitz counterexamples to the coarea inequality

A natural question is whether the coarea inequality holds for all Sobolev functions that may not be monotone. We start this section with the following remark.

Remark 5.1. In any complete metric n-manifold X supporting a weak $(1,1)$-Poincaré inequality and doubling $\mathcal{H}^n$, the minimal p-weak upper gradient of each Lipschitz $u\colon X \to \mathbb{R}$ is equal to the pointwise Lipschitz constant $\operatorname{lip}(u)$ [Che99], whenever $p > 1$. Under these assumptions, the coarea inequality for Lipschitz functions follows from (a localized) Eilenberg inequality. Namely, whenever $u\colon X \to \mathbb{R}$ is Lipschitz, Lemma 5.2 below implies

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(s)} g \, d\mathcal{H}^{n-1} \, ds \leq \frac{2\omega_{n-1}}{\omega_n} \int_X g\rho_u \, d\mathcal{H}^n \quad \text{for every Borel } g\colon X \to [0,\infty]. \tag{29}$$

When $p > 1$, the conclusion (29) applying [Che99] holds even if such geometric assumptions are valid only locally on X, cf. [IPS22, Theorem 1.1]. The equality $\operatorname{lip}(u) = \rho_u$ is now known to hold also in the $p = 1$ case, cf. [ES21, Theorem 1.10.].

Lemma 5.2. *Let X be a metric space with a locally finite $\mathcal{H}^n$-measure. Let $u\colon X \to \mathbb{R}$ be Lipschitz, $(E_i)_{i=1}^{\infty}$ a Borel decomposition of X, and $h = \sum_{i=1}^{\infty} \chi_{E_i} \operatorname{lip}(u|_{E_i})$. Then*

$$\int_{\mathbb{R}}^{*} \int_{u^{-1}(s)} g \, d\mathcal{H}^{n-1} \, ds \leq \frac{2\omega_{n-1}}{\omega_n} \int_X gh \, d\mathcal{H}^n \quad \textit{for every Borel } g\colon X \to [0,\infty].$$

Proof. We first note that it is sufficient to prove the claim when $g = \chi_A$ for an arbitrary Borel set $A \subset X$ since the general claim follows via approximation by simple functions.

Lemma 3.10 and Theorem 3.15 of [EH21] establish that, for each Borel set $E \subset X$,

$$\int_{\mathbb{R}}^{*} \mathcal{H}^{n-1}(E \cap u^{-1}(s)) \, ds \leq \Phi^{1,n-1}(E,u), \tag{30}$$

where

$$\Phi^{1,n-1}(E,u) = \frac{\omega_{n-1}}{2^{n-1}} \lim_{\delta \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} u(A_i))^{n-1} \operatorname{diam}(A_i) \colon \operatorname{diam} A_i < \delta, E \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

It is clear directly from the definition that $\Phi^{1,n-1}(E,u) = \Phi^{1,n-1}(E,u|_E)$ for every set E. That $\Phi^{1,n-1}(\cdot,u)$ is a Borel regular outer measure follows from the Carathéodory's construction and from the fact that $(\operatorname{diam} u(A))^{n-1} \operatorname{diam} A = (\operatorname{diam} u(\overline{A}))^{n-1} \operatorname{diam} \overline{A}$ for every set $A \subset X$, where $\overline{A}$ denotes the closure of A.

Moreover, Lemma 3.9 of [EH21] shows

$$\Phi^{1,n-1}(E,u|_E) \leq \frac{2\omega_{n-1}}{\omega_n} \mathrm{LIP}(u|_E) \mathcal{H}^n(E), \quad \text{for every } E \subset X. \tag{31}$$

As a preliminary result, we claim that

$$\Phi^{1,n-1}(E,u) \leq \frac{2\omega_{n-1}}{\omega_n} \int_E \operatorname{lip}(u|_E) \, d\mathcal{H}^n, \quad \text{for any Borel } E \subset X. \tag{32}$$

Since $\mathcal{H}^n$ is locally finite, it suffices to establish (32) for Borel sets $E \subset X$ with $\mathcal{H}^n(E) < \infty$. Notice that if $A \subset B$, then $\operatorname{lip}(u|_A)(x) \leq \operatorname{lip}(u|_B)(x)$ for every $x \in A$.

Next we adapt an argument from [Sch16, Lemma 3.157]: for each $\epsilon > 0$ and $\eta > 0$, by Lusin–Egorov, there exist triples $(K_i, \lambda_i, r_i)_{i=1}^{\infty}$ such that

(1) K_i are pairwise disjoint compact sets and $\mathcal{H}^n(E \setminus \bigcup_i K_i) = 0$;
(2) $\lambda_i \geq 0$;
(3) $\operatorname{diam} K_i < r_i < \eta$; and
(4) for each $(x,r) \in K_i \times (0, r_i]$,

$$\lambda_i - \epsilon \leq \sup_{s \leq r} \sup_{y \in E \cap B(x,s)} \frac{|u(y) - u(x)|}{s} \leq \lambda_i \,.$$

The third and fourth points imply that $u|_{K_i}$ is λ_i-Lipschitz in K_i. Then, by the countable subadditivity of $\Phi^{1,n-1}(\cdot,u)$, by (1), and (31), we obtain

$$\begin{aligned} \Phi^{1,n-1}(E,u) &\leq \sum_{i=1}^{\infty} \frac{2\omega_{n-1}}{\omega_n} \int_{K_i} \left(\epsilon + \sup_{s \leq r_i} \sup_{y \in E \cap B(x,s)} \frac{|u(y) - u(x)|}{s} \right) d\mathcal{H}^n \\ &\leq \frac{2\omega_{n-1}}{\omega_n} \epsilon \mathcal{H}^n(E) + \frac{2\omega_{n-1}}{\omega_n} \int_E \sup_{s \leq \eta} \sup_{y \in E \cap B(x,s)} \frac{|u(y) - u(x)|}{s} \, d\mathcal{H}^n. \end{aligned}$$

Since the lower bound and the last upper bound are independent of the particular decomposition, we may pass to the limits $\epsilon \to 0^+$ and $\eta \to 0^+$, apply dominated convergence, and conclude (32).

Fix an arbitrary Borel set $A \subset X$. We apply (32) for each Borel set $A \cap E_i$, for each $i \in \mathbb{N}$, where $(E_i)_{i=1}^{\infty}$ is the Borel decomposition from the assumptions. Given that

$\operatorname{lip}(u|_{A\cap E_i})(x) \le \operatorname{lip}(u|_{E_i})(x)$ for every $x \in A \cap E_i$, we conclude that

$$\Phi^{1,n-1}(A \cap E_i, u) \le \frac{2\omega_{n-1}}{\omega_n} \int_{A\cap E_i} h \, d\mathcal{H}^n \quad \text{for every } i \in \mathbb{N}. \tag{33}$$

Now (30) and (33) yield that

$$\int_{\mathbb{R}}^{*} \mathcal{H}^{n-1}(u^{-1}(t) \cap A)\, dt \le \frac{2\omega_{n-1}}{\omega_n} \int_A h \, d\mathcal{H}^n \quad \text{for every Borel } A \subset X.$$

This inequality completes the proof. □

We next show that the monotonicity condition cannot be disposed of in our main result, Theorem 1.6, even in the Lipschitz category. Thus, in order to have the coarea inequality hold for all Lipschitz or Sobolev functions one needs further geometric conditions on the space, such as the ones in Remark 5.1.

Theorem 5.3. *There exists an n-rectifiable metric n-manifold $X \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(X) < \infty$ and a Cantor set $C \subset X$ such that, whenever $u\colon X \to \mathbb{R}$ is Lipschitz and $1 \le p \le \infty$, the p-minimal p-weak upper gradient $\rho_{u,p}$ of u satisfies*

$$\rho_{u,p}(x) = 0, \quad \textit{for } \mathcal{H}^n\textit{-a.e. } x \in C.$$

Moreover, the orthogonal projection $u(x_1, \dots, x_n) = x_1$ satisfies

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(u^{-1}(t) \cap C)\, dt > 0. \tag{34}$$

In particular, any universal coarea inequality of the form (2) fails for the pair $(u, \rho_{u,p})$.

The constructed surface X has the following property: When applying Lemma 5.2 with the Borel decomposition $E_1 = C$, $E_2 = X \setminus C$, every Lipschitz $u\colon X \to \mathbb{R}$ and every Borel set $E \subset X$ satisfy, for every $p \ge 1$,

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(u^{-1}(t) \cap E)\, dt = \int_E \chi_{E_1} \operatorname{lip}(u|_{E_1})\, d\mathcal{H}^n + \int_E \chi_{E_2} \rho_{u,p}\, d\mathcal{H}^n\,. \tag{35}$$

Theorem 5.3 implies Theorem 1.7 directly.

We will prove Theorem 5.3 after recalling a method for construction of some special metric n-manifolds that is of independent interest. The construction is fairly standard, see, e.g. [Fed69, 4.2.25, pages 420-423], [HZ16], [Raj17, Proposition 17.1] for closely related constructions. We explain in the case $n = 2$ as the construction straight-forwardly generalizes to $n \ge 3$.

Recall the well-known example of a four-corner Cantor set with positive area. The set C is the intersection of a nested sequence of unions of squares, beginning with the unit square $Q_1^1 = [0,1]^2$. To fix some notation, at stage k, we have the squares Q_j^k, for $j = 1, \dots, 4^{k-1}$. Let q_j^k be the centers of the squares Q_j^k. Each Q_j^k, and by a slight abuse of terminology, resp. each q_j^k, has four children that we denote by $Q_{j(1)}^{k+1}, \dots, Q_{j(4)}^{k+1}$, resp. $q_{j(1)}^{k+1}, \dots, q_{j(4)}^{k+1}$.

Toward the construction of our metric surface X, begin with the (open) square $(0,1)^2 \times \{1\} \subset \mathbb{R}^3$ as stage $k=1$. Remove a small disk from $(0,1)^2 \times \{1\}$ centered at q_1^1 and let S_1^1 be its boundary. Let $S_1^2, \ldots, S_4^2$ be four small(er) circles lying in $\mathbb{R}^2 \times \{1/2\}$ centered, resp., at $q_1^2, \ldots, q_4^2$. There is a smooth surface whose boundary is precisely the (five) circles just described. We can visualize the surface as a tube starting at S_1^1 and then branching into four sub-tubes that then get glued to $S_1^2, \ldots, S_4^2$.

Notice that by choosing $S_1^2, \ldots, S_4^2$ small enough, we can make the area of the tube to be as close to the area of the disc filling S_1^1 as we wish. While almost preserving the area by making the tube thinner, we may also spiral the tubular part to make the length of paths joining S_1^1 to the other boundary components as large as we wish. Except for the boundaries, the surface is constructed in the interior $(0,1)^2 \times (1/2, 1)$.

We continue the construction by joining each circle S_j^2 to four smaller circles lying in $\mathbb{R}^2 \times \{1/3\}$ and centered at the corresponding $q_{j(1)}^3, \cdots, q_{j(4)}^3$. The joining tubes are constructed within $Q_j^2 \times (1/3, 1/2)$, so the construction coming from S_j^2 is completely disjoint from the one for $S_{j'}^2$, for any distinct pair j, j'.

Obviously, there is no obstacle in continuing this construction ad infinitum. The metric surface X is the Hausdorff limit of the manifolds constructed in these steps. Notice that it contains the Cantor set C (under the identification of $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$).

We list the key properties of X relevant for us:

(1) the space X is homeomorphic to $(0,1)^2$.

To see this, notice that each finite stage of the construction retracts to the previous stage. So, a homeomorphism from $X \setminus C$ to $(0,1)^2 \setminus C$ can be constructed as a limit of a sequence of homeomorphisms, each extending the domain of definition of the previous one without altering the map there. It is then easy to see that the final homeomorphism extends uniquely to C as well.

Since we had freedom in altering the areas and lengths, by modifying the connecting tubes at each stage, we further guarantee that

(2) there exists a constant $A > 0$ so that
$$\mathcal{H}^2(X \cap B(x,r)) \leq Ar^2 \quad \text{for all } x \in X \text{ and all } r > 0,$$

and

(3) the minimum lengths of rectifiable curves in X that join a point at step k to a point at step $k+\ell$ goes to infinity as $\ell \to \infty$.

We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. Again we work in $n=2$ as the proof for higher dimensions is basically the same. Let X and C be as above. First note that since C is 2-rectifiable and $X \setminus C$ is smooth, the space X is 2-rectifiable. By property (3) above, there are no rectifiable curves that intersect C other than the constant curves. This means that the minimal upper gradient of any function is zero (a.e.) on C. Therefore, if ρ is the minimal upper gradient of some $u\colon X \to \mathbb{R}$, then

$$\int_C \rho \, d\mathcal{H}^2 = 0 \,. \tag{36}$$

Now, as C is a four-corner Cantor set with $\mathcal{H}^2(C) > 0$, and $u(x_1, x_2, x_3) = x_1$ is the orthogonal projection, by Fubini's theorem

$$\int_{\mathbb{R}} \mathcal{H}^1(u^{-1}(t) \cap C) \, dt > 0 \,.$$

By (36), thus, the coarea inequality fails for the Lipschitz u and any minimal upper gradient of it.

The equality (35) follows for Borel sets $E \subset X \setminus C$ from the smoothness of $X \setminus C$. On the other hand, as $C \subset \mathbb{R}^2 \times \{0\}$, we may consider Borel subsets $E \subset C$ as a subset of the plane. The planar coarea formula then yields

$$\int_{\mathbb{R}} \mathcal{H}^1(u^{-1}(t) \cap E) \, dt = \int_E \operatorname{lip}(u|_C) \, d\mathcal{H}^2 \quad \text{for every Borel } E \subset C;$$

the equality follows by applying the planar coarea formula for a Lipschitz extension of $u|_C$. Then (35) follows from the smoothness of $X \setminus C$. □

Remark 5.4. The last paragraph of the proof of Theorem 5.3 could also be argued using the coarea formula established by Ambrosio and Kirchheim, cf. [AK00, Theorem 9.4].

Acknowledgement. We thank Sylvester Eriksson–Bique for pointing out an initial inaccuracy in the construction of Section 5. We thank the referee for very helpful comments.

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