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Filling minimality and Lipschitz-volume rigidity of convex bodies among integral current spaces

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Abstract. In this paper we consider metric fillings of boundaries of convex bodies. We show that convex bodies are the unique minimal fillings of their boundary metrics among all integral current spaces. To this end, we also prove that convex bodies enjoy the Lipschitz-volume rigidity property within the category of integral current spaces, which is well known in the smooth category. As further applications of this result, we prove a variant of Lipschitz-volume rigidity for round spheres and answer a question of Perales concerning the intrinsic flat convergence of minimizing sequences for the Plateau problem.

1. Introduction

1.1. Statement of main results. Suppose that Y is a closed orientable smooth manifold equipped with a distance function inducing the same topology. Following Gromov [25] the *filling volume* $\mathrm{FillVol}_\infty(Y)$ is defined to be the infimum over the volumes of complete Riemannian manifolds X bounded by Y. A Riemannian manifold M is called *minimal filling* if $\mathrm{Vol}^n(M) = \mathrm{FillVol}_\infty(\partial M)$, where ∂M is equipped with the subspace metric. Calculating filling volumes and finding minimal fillings is notoriously difficult. Even the filling volume of simple spaces, such as S^1 endowed with the angular metric, is unknown. Whether the minimal filling in this case is the round hemisphere is called Gromov's filling area conjecture and, despite remarkable partial results [6, 39, 49], it remains widely open.

The deep work of Burago and Ivanov [14, 15] shows that Riemannian manifolds which are C^3-close to full dimensional submanifolds of Euclidean or hyperbolic space are minimal fillings. See also [40] for a recent generalization of their work to symmetric spaces of negative curvature. However not every Y admits a smooth minimal filling as defined above. By a result of Wenger [46] the situation changes when, in the definition of the filling volume, smooth Riemannian manifolds are replaced by integral current spaces, see [43]. These are spaces which carry an analytically defined analogue of a fundamental class. In particular, there are well defined notions of boundary and volume for such spaces, and hence it is possible to define the

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filling volume of a closed integral current space as well as when an integral current space is a minimal filling, see, e.g., [38, Section 2.7].

Integral current spaces include, but are not limited to, compact oriented manifolds equipped with a metric that is bi-Lipschitz equivalent to a Riemannian (or Finsler) metric. As a Lipschitz manifold, a convex body $C \subset \mathbb{R}^n$ admits a canonical integral current $[\![C]\!]$ of multiplicity one which satisfies in particular $\partial[\![C]\!] = [\![\partial C]\!]$. The first main result in this paper is that convex bodies are the unique minimal fillings of their boundaries among all integral current spaces.

Theorem 1.1. *Let $C \subset \mathbb{R}^n$ be a convex body and suppose $\iota : \partial C \to X$ is an isometric embedding into an integral current space $X = (X, T)$ such that $\iota_\#[\![\partial C]\!] = \partial T$. Then*

$$\mathbf{M}^{\mathrm{ir}}(T) \geq \mathrm{Vol}^n(C) \tag{1.1}$$

with equality if and only if the map ι extends to an isometry $C \to X$.

Here, $\mathbf{M}^{\mathrm{ir}}(T)$ denotes the inscribed Riemannian mass of T, a variant of the usual mass $\mathbf{M}(T)$ as introduced by Ambrosio and Kirchheim [4]. We refer to Section 2.5 for the precise definitions and mention here that $\mathbf{M}^{\mathrm{ir}}(T)$, resp. $\mathbf{M}^{\mathrm{ir}}(\partial T)$, correspond to the volume $\mathrm{Vol}^n(X)$, resp. boundary volume $\mathrm{Vol}^{n-1}(\partial X)$, when X is a Riemannian manifold. Furthermore, one always has $\mathbf{M}(T) \leq \mathbf{M}^{\mathrm{ir}}(T)$. If we additionally assume that X is infinitesimally Euclidean, in the sense of having property (ET) [36], then the two notions of mass agree. We remark that the mass estimate (1.1) remains true if $\mathbf{M}^{\mathrm{ir}}(T)$ is replaced by $\mathbf{M}(T)$; see Lemma 6.1. This answers a question by Sormani mentioned in her talk [42, 57:15–58:00]. However, the substantial part of Theorem 1.1 is the rigidity, which remains open for the Ambrosio–Kirchheim mass $\mathbf{M}$.

The proof of Theorem 1.1 is based on ideas of Burago and Ivanov [14, 15]. They use the following observation: *Let X, M be closed orientable Riemannian manifolds of the same dimension and $f : X \to M$ a 1-Lipschitz map of degree one. If $\mathrm{Vol}^n(X) \leq \mathrm{Vol}^n(M)$, then f is a metric isometry.* Variants of this statement have been obtained and applied by Besson, Courtois and Gallot [9], Burago and Ivanov [13–15] and Cecchini, Hanke and Schick [16]. Generalizations to the singular settings of Alexandrov and limit RCD spaces were obtained by Li [33] and Li and Wang [35]. See also [34] for an overview on these so-called *Lipschitz-volume rigidity* results. The following variant, which is needed in the proof of Theorem 1.1, does not impose curvature assumptions on the domain space X and is hence of independent interest.

Theorem 1.2. *Let (X, T) be an n-dimensional integral current space, let $C \subset \mathbb{R}^n$ be a convex body and let $f : X \to \mathbb{R}^n$ be a 1-Lipschitz map such that $f_\#(\partial T) = [\![\partial C]\!]$. If $\mathbf{M}(\partial T) \leq \mathrm{Vol}^{n-1}(\partial C)$ and $\mathbf{M}(T) \leq \mathrm{Vol}^n(C)$, then f is an isometry onto C.*

Note that the condition on the degree of f is replaced by $f_\#(\partial T) = [\![\partial C]\!]$. Using Federer's constancy theorem, it is easy to see that this implies $f_\# T = [\![C]\!]$.

As a corollary of Theorem 1.2 we derive the Lipschitz-volume rigidity of the round sphere S^n with respect to the inscribed Riemannian area functional. Here *round sphere* refers to the standard sphere S^n endowed with its intrinsic metric as a Riemannian manifold. Indeed for simple reasons S^n endowed with the subspace metric of $\mathbb{R}^{n+1}$ cannot be Lipschitz-volume rigid; see Section 8 below.

Corollary 1.3. *Let (X, T) be an n-dimensional integral current space with $\partial T = 0$ and let $f : X \to S^n$ be a 1-Lipschitz map such that $f_\# T = [\![S^n]\!]$. If $\mathbf{M}^{\mathrm{ir}}(T) \le \mathrm{Vol}^n(S^n)$, then f is an isometry.*

Besides the proofs of Theorem 1.1 and Corollary 1.3, as another application of Theorem 1.2 we answer the following question of Perales concerning the Euclidean unit ball B^n. This question was promoted in the same talk by Sormani [42, 53:12–55:40].

Question 1.4 (Perales). Assume (M_i) is a sequence of compact orientable Riemannian n-manifolds with

$$\lim_{i\to\infty} \mathrm{Vol}^n(M_i) \le \mathrm{Vol}^n(B^n)$$

which converges in the intrinsic flat sense to a limit space X. Assume further that $f_i : M_i \to \mathbb{R}^N$ are 1-Lipschitz maps such that $f_{i\#}[\![\partial M_i]\!] = [\![S^{n-1}]\!]$ for all $i \in \mathbb{N}$, the f_i take values in a compact set $K \subset \mathbb{R}^N$, and (f_i) converges in the sense of the Sormani Arzelà–Ascoli theorem [41] to a limit map $f : X \to K$. Does it follow that f is an isometry $X \to B^n$?

Informally, Question 1.4 asks about the interplay of the "extrinsic" flat convergence of integral currents in $\mathbb{R}^N$ and the intrinsic flat convergence of the corresponding "intrinsic metrics" on the currents. In general, Question 1.4 has a negative answer, see Example 7.4 below. However, as a consequence of Theorem 1.2 the answer is positive if one imposes a suitable bound on the boundary volumes.

Corollary 1.5. *Let $C \subset \mathbb{R}^n$ be a convex body. Suppose (X_i) is a sequence of integral current spaces with*

$$\liminf_{i\to\infty} \mathbf{M}(\partial T_i) \le \mathrm{Vol}^{n-1}(\partial C) \quad \text{and} \quad \liminf_{i\to\infty} \mathbf{M}(T_i) \le \mathrm{Vol}^n(C)$$

which converges in the intrinsic flat sense to a limit space X. Assume further that $f_i : X_i \to \mathbb{R}^N$ are 1-Lipschitz maps such that $f_{i\#}(\partial T_i)$ flat converges to $[\![\partial C]\!]$, the f_i take values in a compact set $K \subset \mathbb{R}^N$, and (f_i) converges in the sense of Theorem 7.1 to a 1-Lipschitz map $f : X \to K$. Then f is an isometry $X \to C$.

1.2. Strategy of proof. We prove the filling volume rigidity Theorem 1.1 by deducing it from the Lipschitz-volume rigidity Theorem 1.2 following the arguments of Burago and Ivanov [14, 15]. The key idea is to consider a linear isometric embedding $\Phi : \mathbb{R}^n \to \mathscr{L}$ into the injective Banach space $\mathscr{L} := L^\infty(S^{n-1})$. There is an inner product on $\mathscr{L}$ whose induced norm agrees with $\|\cdot\|_\infty$ on $\Phi(\mathbb{R}^n)$ and which does not increase the inscribed Riemannian masses of rectifiable currents in $\mathscr{L}$ (see Lemma 6.2). The inner product induces an orthogonal projection onto $\Phi(\mathbb{R}^n)$ which, together with the injectivity of $\mathscr{L}$, implies the existence of a map $f : X \to \mathbb{R}^n$ with $f_\#(\partial T) = [\![\partial C]\!]$ that does not increase inscribed Riemannian volumes. Although the argument of Burago–Ivanov showing that f is 1-Lipschitz does not generalize directly to integral current spaces, Theorem 1.1 follows from a double application of Theorem 1.2 which circumvents this problem.

To prove our Lipschitz-volume rigidity result, Theorem 1.2, we use a recently established decomposition result for 1-dimensional currents [10] to obtain the case $n = 1$. The general case $n \ge 2$ can be reduced to the 1-dimensional case by considering suitable slicings of the current T. Let $H \subset \mathbb{R}^n$ be an $(n-1)$-dimensional hyperplane and $\varrho : \mathbb{R}^n \to H$ the ortho-

gonal projection onto H. We define $\widehat{\varrho} = \varrho \circ f$. The slices $T_p = \langle T, \widehat{\varrho}, p\rangle$, which are defined for $\mathcal{H}^{n-1}$-almost every $p \in H$, are 1-dimensional integral currents and satisfy $f_\# T_p = [\![C_p]\!]$, where $C_p = C \cap \varrho^{-1}(p)$ is isometric to a closed interval in $\mathbb{R}$. We prove that the integral current space $X_p = (\operatorname{set} T_p, T_p)$ satisfies

$$f_\# T_p = [\![C_p]\!], \quad \mathbf{M}(T_p) \le \mathcal{H}^1(C_p) \quad \text{and} \quad \mathbf{M}(\partial T_p) \le \mathcal{H}^0(\partial C_p).$$

The crucial step of the proof is to show the last inequality concerning the mass of ∂T_p. This follows essentially from Lemmas 3.1 and 3.2, which show that f is mass-preserving in a certain sense. Thus we may deduce from the case $n = 1$ that f is an isometry

$$\operatorname{spt}\langle T, \widehat{\varrho}, p\rangle \to \operatorname{spt}\langle [\![C]\!], \varrho, p\rangle$$

for $\mathcal{H}^{n-1}$-almost every $p \in H$. By Proposition 4.1 below this suffices to conclude that f is an isometry.

To deduce Corollary 1.3, we verify that the assumptions of Theorem 1.2 are satisfied for the coning map $Cf : CX \to CS^n = B^{n+1}$.

1.3. Organization. The paper is organized as follows. After reviewing definitions and well established facts concerning measure theory, volume functionals and metric currents in Section 2, we prove the basic properties of mass preserving Lipschitz maps needed in this paper in Section 3.

The proof of Theorem 1.2 is given in Section 4, starting with the special 1-dimensional case in Section 4.1, the reduction to the 1-dimensional case in Section 4.2, and the conclusion in Section 4.3. Section 5 is devoted to the proof of Corollary 1.3, while Theorem 1.1 is proved in Section 6, with the filling estimate proved in Section 6.1, and the rigidity statement in Section 6.2. In Section 7.1 we discuss intrinsic flat convergence, while Section 7.2 is devoted to the counterexample to the question of Perales and the proof of Corollary 1.5. Lastly, in Section 8, we discuss possible extensions of our results and further counterexamples.

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After the completion of the first version of this article, we were informed by Raquel Perales that she and Giacomo Del Nin have obtained a similar result to Theorem 1.2 independently. In their work [20] they discuss in detail the motivation for Question 1.4, which is to give a direct argument for a gap in the proof of [27, Theorem 1.3]. See also [28] and [1, 2, 26].

2. Preliminaries

2.1. Basic notation and definitions. We write $\mathbb{N} = \{1, 2, \dots\}$ for the set of positive integers. Moreover, we let $\mathbb{R}^n$ denote the set of n-tuples of real numbers with the convention that $\mathbb{R}^0$ consists of exactly one point. A *convex body* is a compact convex subset of $\mathbb{R}^n$ with

non-empty interior. Let $X = (X, d)$ be a metric space. We denote by $B_X(x, r)$ (or simply $B(x, r)$) the closed ball of radius r centered at x, and by B^n the n-dimensional Euclidean unit ball $B_{\mathbb{R}^n}(0, 1)$. Unless otherwise specified, subsets of X are always endowed the subspace metric. We write $\overline{X}$ for the metric completion of X and we tacitly identify X with its canonical isometric copy in $\overline{X}$. A map $f : X \to Y$ between metric spaces X, Y is called *L-Lipschitz*, for some constant $L \geq 0$, if $d(f(x), f(y)) \leq L d(x, y)$ for all x, $y \in X$. The smallest $L \geq 0$ such that f is L-Lipschitz is denoted by $\operatorname{Lip} f$. We use $\mathrm{LIP}(X)$ to denote the set of Lipschitz functions $X \to \mathbb{R}$. A metric space Y is called *injective* if whenever X is a metric space, $A \subset X$ and $f : A \to Y$ is a 1-Lipschitz map, then f can be extended to a 1-Lipschitz map $\bar{f} : X \to Y$. A well-known theorem of McShane (see, e.g., [11, Theorem 1.27]) states that $\mathbb{R}$ is an injective metric space. Hence, each of the Banach spaces $\ell^n_\infty := (\mathbb{R}^n, \|\cdot\|_\infty)$ is injective as well. We say that X and Y are *bi-Lipschitz equivalent* if there exists a bijection $f : X \to Y$ such that f and f^{-1} are both Lipschitz maps. A separable metric space X is called *Lipschitz n-manifold* if every $x \in X$ has a closed neighborhood which is bi-Lipschitz equivalent to B^n.

2.2. Measure theory. As is common in geometric measure theory, we follow the convention that a *measure* on a metric space X is a countably subadditive function

$$\mu : \mathcal{P}(X) \to [0, \infty]$$

such that $\mu(\emptyset) = 0$. Throughout the forthcoming discussion μ will always be a measure on X. We say that $A \subset X$ is *μ-measurable* if

$$\mu(M \cap A) + \mu(M \setminus A) = \mu(M)$$

for all $M \subset X$. The collection of all μ-measurable subsets of X forms a σ-algebra and the restriction of μ to this σ-algebra is countably additive. The measure μ is *Borel* if all Borel subsets of X are μ-measurable. Furthermore, μ is *Borel regular* if μ is Borel and for every $M \subset X$ there is a Borel set $A \subset X$ such that $M \subset A$ and $\mu(M) = \mu(A)$. If X is separable and a Borel subset of its completion $\overline{X}$, then every finite Borel measure μ on X is *tight*, see [37, Theorem 3.2]). The latter means that for every Borel set $A \subset X$ and $\varepsilon > 0$, there is a compact set $K \subset A$ with $\mu(A \setminus K) < \varepsilon$. The *support* of μ is the closed set $\operatorname{spt}\mu$ which consists of those $x \in X$ such that $\mu(B(x, r)) > 0$ for every $r > 0$. If μ is a finite Borel measure, then its support is separable. We say that μ is *concentrated* on $A \subset X$ if $\mu(X \setminus A) = 0$. If X is separable, then μ is concentrated on its support (see [23, Theorem 2.2.16]). Indeed, it is consistent with (but not implied by) ZFC that the latter holds true without assuming X to be separable (compare [23, Section 2.1.6 and Theorem 2.2.16]). In particular, if one assumes this additional set-theoretic axiom, finite Borel measures on *every* complete metric space are tight. In [4, 10] this is a standing assumption because there the authors want to treat also currents in non-separable metric spaces X, and even for separable X some arguments therein rely on embedding X isometrically into the non-separable Banach space ℓ^∞. For the proof of Theorems 1.1 and 1.2 it can be avoided to assume this axiom. But since we will not justify the use of auxiliary results from the articles [4, 10], the reader is invited to also consider it an additional standing assumption throughout this paper.

2.3. Volumes of rectifiable spaces. By Carathéodory's criterion (see, e.g., [23, p. 75]) the Hausdorff n-measure on X, which will be denoted by $\mathcal{H}^n_X$ (or simply $\mathcal{H}^n$), is Borel regular. In this paper, following a common convention, we normalize $\mathcal{H}^n_X$ so that $\mathcal{H}^n_{\mathbb{R}^n}$ equals the

Lebesgue measure $\mathcal{L}^n$. Hausdorff measures can sometimes be calculated in terms of the so-called area formula. The area formula relies on the following metric version of the Rademacher theorem due to Kirchheim [30]: *if $f : A \to X$ is a Lipschitz map from a Borel set $A \subset \mathbb{R}^n$, then, for almost every $p \in A$, there exists a seminorm* $\operatorname{md} f_p$ *on $\mathbb{R}^n$ such that*

$$d(f(q), f(p)) = \operatorname{md} f_p(q - p) + o(|q - p|)$$

as $q \to p$ with $q \in A$. Now the area formula [30] states that the function $x \mapsto \#\{f^{-1}(x)\}$ is $\mathcal{H}^n_X$-measurable and

$$\int_A \operatorname{Jac}^b(\operatorname{md} f_p)\,\mathrm{d}p = \int_X \#\{f^{-1}(x)\}\,\mathrm{d}\mathcal{H}^n(x). \tag{2.1}$$

Here $\operatorname{Jac}^b(\sigma)$ denotes the *Busemann Jacobian* of a seminorm $\sigma : \mathbb{R}^n \to [0, \infty)$ which is defined as $\omega_n / \mathcal{L}^n(B_\sigma)$, where B_σ is the unit ball of σ and $\omega_n := \mathcal{L}^n(B^n)$. More generally, a map $\operatorname{Jac}^\bullet : \Sigma^n \to [0, \infty)$ from the space Σ^n of seminorms on $\mathbb{R}^n$ is a *Jacobian* if

(i) $\operatorname{Jac}^\bullet(|\cdot|) = 1$ for the standard Euclidean norm $|\cdot|$ on $\mathbb{R}^n$,

(ii) $\operatorname{Jac}^\bullet(\sigma') \le \operatorname{Jac}^\bullet(\sigma)$ if $\sigma' \le \sigma$,

(iii) $\operatorname{Jac}^\bullet(\sigma \circ T) = |\det T| \operatorname{Jac}^\bullet(\sigma)$ for any linear map $T : \mathbb{R}^n \to \mathbb{R}^n$.

The properties above are known as *normalization, monotonicity,* and *transformation law*, respectively. See [17, Section 4.1] and the references therein for a more detailed overview. Two Jacobians which play an important role in this paper are Gromov's *mass∗ Jacobian*, defined by

$$\operatorname{Jac}^{m*}(\sigma) := \sup_P \frac{2^n}{\mathcal{L}^n(P)},$$

where the supremum is taken over all parallelepipeds P containing B_σ, and Ivanov's *inscribed Riemannian Jacobian*

$$\operatorname{Jac}^{\mathrm{ir}}(\sigma) := \frac{\omega_n}{\mathcal{L}^n(J(B_\sigma))},$$

where $J(B_\sigma) \subset \mathbb{R}^n$ is the *John ellipsoid* of B_σ, that is, the ellipsoid of maximal $\mathcal{L}^n$-measure contained in B_σ. It follows from John's theorem (compare, e.g., [5]) that if $\sigma_1 \in \Sigma^{n_1}$ and $\sigma_2 \in \Sigma^{n_2}$, then

$$\operatorname{Jac}^{\mathrm{ir}}(\sigma_1 \times \sigma_2) = \operatorname{Jac}^{\mathrm{ir}}(\sigma_1) \cdot \operatorname{Jac}^{\mathrm{ir}}(\sigma_2), \tag{2.2}$$

where $\sigma_1 \times \sigma_2 \in \Sigma^{n_1+n_1}$ is defined by $(\sigma_1 \times \sigma_2)(v_1, v_2) = \sqrt{\sigma_1(v_1)^2 + \sigma_2(v_2)^2}$.

A metric space X is called *n-rectifiable* if there are Borel subsets $\{A_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^n$ and bi-Lipschitz embeddings $\{\varphi^i : A_i \to X\}_{i\in\mathbb{N}}$ such that $\mathcal{H}^n_X$ is concentrated on $\bigcup_{i\in\mathbb{N}} \varphi^i(A_i)$. Without loss of generality, one can assume additionally that $\varphi^i(A_i)$ and $\varphi^j(A_j)$ are disjoint if $i \neq j$. For n-rectifiable spaces, the *density*

$$\Theta_n(B, x) = \lim_{r\to 0} \frac{\mathcal{H}^n(B \cap B(x, r))}{\omega_n r^n}$$

exists and is equal to one for $\mathcal{H}^n$-almost every $x \in B$. Moreover, by the area formula (2.1) the Hausdorff n-measure of a Borel set $B \subset X$ is given by

$$\mathcal{H}^n(B) = \sum_{i\in\mathbb{N}} \int_{(\varphi^i)^{-1}(B)} \operatorname{Jac}^b(\operatorname{md} \varphi^i_p)\,\mathrm{d}p.$$

When X has "non-Euclidean tangents" (i.e., the metric differentials $\operatorname{md}\varphi_p^i$ are not necessarily induced by inner products) different Jacobians yield distinct notions of volume. Indeed, every Jacobian $\operatorname{Jac}^\bullet$ gives rise to a Borel regular measure $\mu_X^\bullet$ on X by setting

$$\mu_X^\bullet(B) := \sum_{i\in\mathbb{N}} \int_{(\varphi^i)^{-1}(B)} \operatorname{Jac}^\bullet(\operatorname{md}\varphi_p^i)\,\mathrm{d}p \tag{2.3}$$

for Borel sets $B \subset X$. It follows from the chain rule for metric differentials and the transformation law (iii) that this does not depend on the choice of the coordinate charts $\{\varphi^i\}$. Furthermore, by the normalization axiom (i) of Jacobians and (2.1) one always has $\mu_{\mathbb{R}^n}^\bullet = \mathcal{H}^n_{\mathbb{R}^n}$. In Section 2.5 we will see an analogous construction for the mass measure of rectifiable currents.

2.4. Metric currents. Using ideas of De Giorgi [19] and extending the classical theory of currents, which goes back to de Rham [21] and Federer and Fleming [24], Ambrosio and Kirchheim introduced *metric currents* in [4]. Variants of their definitions have been proposed and studied by several authors (see [31,32,47,48]). In this paper we follow the original approach of Ambrosio–Kirchheim and review its basic aspects below.

For each $n \geq 0$ let $\mathcal{D}^n(X)$ denote the set of all tuples $(h, \pi_1, \ldots, \pi_n)$, where $h : X \to \mathbb{R}$ is a bounded Lipschitz function and $\pi_i \in \operatorname{LIP}(X)$.

Definition 2.1. Let X be a complete metric space. We say that an $(n+1)$-multilinear map $T : \mathcal{D}^n(X) \to \mathbb{R}$ is an n-current if the following holds:

(1) $T(h, \pi_1^{(j)}, \ldots, \pi_n^{(j)}) \to T(h, \pi_1, \ldots, \pi_n)$ as $j \to \infty$, whenever $\pi_i^{(j)} \to \pi_i$ pointwise and $\operatorname{Lip}\pi_i^{(j)} \leq C$ for some uniform constant C.

(2) $T(h, \pi_1, \ldots, \pi_n) = 0$ whenever, for some $i \in \{1, \ldots, n\}$, π_i is constant on an open neighborhood of $\operatorname{spt} h$.

(3) There is a finite Borel measure μ on X such that

$$|T(h, \pi_1, \ldots, \pi_n)| \leq \prod_{i=1}^n \operatorname{Lip}\pi_i \int_X |h|\,\mathrm{d}\mu \tag{2.4}$$

for all $(h, \pi_1, \ldots, \pi_n) \in \mathcal{D}^n(X)$.

The minimal measure μ satisfying (2.4) is called the *mass* of T and is denoted by $\|T\|$. Any n-current T extends to a functional $T : L^1(X, \|T\|) \times \operatorname{LIP}(X)^n \to \mathbb{R}$ satisfying (2.4). We define $\mathbf{M}(T) := \|T\|(X)$, $\operatorname{spt} T := \operatorname{spt}\|T\|$, and write $\mathbf{M}_n(X)$ for the vector space of all n-currents on X. It is easy to check that $\mathbf{M}_n(X)$ becomes a Banach space when it is endowed with the norm $\mathbf{M}(\cdot)$. There are natural push-forward, restriction and boundary operators on $\mathbf{M}_n(X)$, which we recall next.

Every Lipschitz map $f : X \to Y$ between complete metric spaces X, Y induces a *push-forward* map $f_\# : \mathbf{M}_n(X) \to \mathbf{M}_n(Y)$ on the level of currents. Indeed, for every $T \in \mathbf{M}_n(X)$ we define

$$f_\# T(h, \pi_1, \ldots, \pi_n) = T(h \circ f, \pi_1 \circ f, \ldots, \pi_n \circ f)$$

for all $(h, \pi_1, \ldots, \pi_n) \in \mathcal{D}^n(Y)$. In particular, we note that

$$\|f_\# T\| \leq (\operatorname{Lip} f)^n f_\# \|T\|. \tag{2.5}$$

If $f : X \to Y$ is a Lipschitz map between arbitrary metric spaces, then f extends to a unique Lipschitz map $\bar{f} : \overline{X} \to \overline{Y}$. By abuse of notation, we will usually write $f_\#$ instead of $\bar{f}_\#$.

Given an n-current $T \in \mathbf{M}_n(X)$, $\ell \in \{0, \dots, n\}$, and an $(\ell+1)$-tuple $\omega = (g, \omega_1, \dots, \omega_\ell)$, where $g : X \to \mathbb{R}$ is a bounded Borel function and $\omega_i \in \mathrm{LIP}(X)$, the *restriction* $T \llcorner \omega$ of T by ω is an $(n-\ell)$-current defined by

$$T \llcorner \omega(h, \pi_1, \dots, \pi_{n-\ell}) = T(h\, g, \omega_1, \dots, \omega_\ell, \pi_1, \dots, \pi_{n-\ell})$$

for all $(h, \pi_1, \dots, \pi_{n-\ell}) \in \mathcal{D}^{n-\ell}(X)$. In particular, $T \llcorner A := T \llcorner \mathbb{1}_A$ is a well-defined n-current for any Borel set $A \subset X$.

Finally, if $n \geq 1$ the *boundary* ∂T of $T \in \mathbf{M}_n(X)$ is the n-multilinear map

$$\partial T : \mathcal{D}^{n-1}(X) \to \mathbb{R}$$

defined by

$$\partial T(h, \pi_1, \dots, \pi_{n-1}) = T(1, h, \pi_1, \dots, \pi_{n-1}).$$

We say that T is a normal current if $\partial T \in \mathbf{M}_{n-1}(X)$. The vector space of all normal n-currents on X is denoted by $\mathbf{N}_n(X)$ and we set $\mathbf{N}_0(X) = \mathbf{M}_0(X)$. The spaces $\mathbf{N}_n(X)$ equipped with the norm $\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$ are Banach spaces, with the convention $\mathbf{N}(T) = \mathbf{M}(T)$ if $T \in \mathbf{N}_0(X)$.

2.5. Rectifiable currents and their Finsler mass. Every $\theta \in L^1(\mathbb{R}^n)$ induces an n-current $[\![\theta]\!] \in \mathbf{M}_n(\mathbb{R}^n)$ given by

$$[\![\theta]\!](h, \pi_1 \dots, \pi_n) = \int_{\mathbb{R}^n} \theta h \det[\partial_i \pi_j]_{i,j=1}^n \, \mathrm{d}\mathcal{H}^n$$

for all $(h, \pi_1, \dots, \pi_n) \in \mathcal{D}^n(\mathbb{R}^n)$. We say that $T \in \mathbf{M}_n(X)$ is *rectifiable* (resp. *integer-rectifiable*) if there are compact sets $K_i \subset \mathbb{R}^n$, functions $\Theta_i \in L^1(\mathbb{R}^n)$ (resp. $\Theta_i \in L^1(\mathbb{R}^n; \mathbb{Z})$) with $\operatorname{spt} \Theta_i \subset K_i$ and bi-Lipschitz embeddings $\varphi_i : K_i \to X$ such that

$$T = \sum_{i \in \mathbb{N}} \varphi_{i\#}[\![\Theta_i]\!] \quad \text{and} \quad \mathbf{M}(T) = \sum_{i \in \mathbb{N}} \mathbf{M}(\varphi_{i\#}[\![\Theta_i]\!]). \tag{2.6}$$

We denote by $\mathcal{R}_n(X)$ (resp. $\mathcal{I}_n(X)$) the collection of all rectifiable (resp. integer-rectifiable) currents on X. The mass of a rectifiable current T has the following very concrete interpretation in terms of the Gromov mass* volume μ^{m*},

$$\|T\|(A) = \sum_{i \in \mathbb{N}} \int_{A \cap \varphi_i(K_i)} |\Theta_i \circ \varphi_i^{-1}(x)| \, \mathrm{d}\mu^{m*}(x) \tag{2.7}$$

for every Borel set $A \subset X$ (see, for example, [49, Lemma 2.5 (2)]). More generally, given a Jacobian $\mathrm{Jac}^\bullet$ and the associated volume measure $\mu_X^\bullet$, the *Finsler mass measure* $\|T\|^\bullet$ is defined by

$$\|T\|^\bullet(A) := \sum_{i \in \mathbb{N}} \int_{A \cap \varphi_i(K_i)} |\Theta_i \circ \varphi_i^{-1}(x)| \, \mathrm{d}\mu_X^\bullet \tag{2.8}$$

for every Borel subset $A \subset X$. It can be shown that this definition is independent of the chosen representation (2.6). It moreover satisfies natural estimates, e.g., $\|f_\# T\|^\bullet \leq (\mathrm{Lip}\, f)^n \|T\|^\bullet$ for

every Lipschitz map $f : X \to Y$, and is comparable to the usual mass measure

$$\|T\|^\bullet \le \|T\|^{\mathrm{ir}} \quad \text{and} \quad C^{-1} \cdot \|T\| \le \|T\|^\bullet \le C \cdot \|T\|$$

for a constant $C > 0$ depending only on n, see [49, Lemma 2.5]. In particular, one has

$$\operatorname{spt}\|T\|^\bullet = \operatorname{spt} T.$$

We call $\mathbf{M}^\bullet(T) := \|T\|^\bullet(X)$ the *Finsler mass* associated to the Jacobian $\mathrm{Jac}^\bullet$.

We remark that $T \in \mathcal{R}_n(X)$ if and only if $\|T\|$ is concentrated on an n-rectifiable set and $\|T\| \ll \mathcal{H}^n$. Moreover, if $T \in \mathcal{R}_n(X)$, then $\|T\|$ is concentrated on the n-rectifiable set

$$\operatorname{set} T = \left\{ x \in X : \liminf_{r \downarrow 0} \frac{\|T\|(B(x,r))}{\omega_n r^n} > 0 \right\}, \tag{2.9}$$

and any Borel set $A \subset X$ on which $\|T\|$ is concentrated contains set T up to an $\mathcal{H}^n$-negligible set (see [4, Theorem 4.6]). The set defined in (2.9) is called *characteristic set* of T.

2.6. Slicing. Let $T \in \mathcal{R}_n(X)$ and $\varrho : X \to \mathbb{R}^k$ be a Lipschitz map with $k \in \{1, \dots, n\}$. In [4, Theorems 5.6 and 5.7], Ambrosio and Kirchheim show that there is a natural slicing operator $\mathbb{R}^k \ni p \mapsto \langle T, \varrho, p\rangle \in \mathcal{R}_{n-k}(X)$ which is defined for $\mathcal{H}^k$-almost every $p \in \mathbb{R}^k$. Each slice $\langle T, \varrho, p\rangle$ is concentrated on $\operatorname{spt} T \cap \varrho^{-1}(p)$, and for every $\psi \in C_c(\mathbb{R}^k)$,

$$\int_{\mathbb{R}^k} \langle T, \varrho, p\rangle \, \psi(p) \, \mathrm{d}p = T \llcorner (\psi \circ \varrho, \varrho_1, \dots, \varrho_k), \tag{2.10}$$

where ϱ_i denotes the ith coordinate function of ϱ. Moreover,

$$\int_{\mathbb{R}^k} \|\langle T, \varrho, p\rangle\| \, \mathrm{d}p = \|T \llcorner (1, \varrho)\|, \tag{2.11}$$

where $(1, \varrho)$ is shorthand for $(1, \varrho_1, \dots, \varrho_k)$. In particular, the following slicing inequality holds true:

$$\int_{\mathbb{R}^k} \mathbf{M}\left(\langle T, \varrho, p\rangle\right) \mathrm{d}p \le (\operatorname{Lip} \varrho)^k \, \mathbf{M}(T).$$

These properties uniquely characterize the slices $\langle T, \varrho, p\rangle$. Indeed, if $T^p \in \mathbf{M}_{n-k}(X)$ are concentrated on $L \cap \varrho^{-1}(p)$ for some σ-compact set L, satisfy $\int_{\mathbb{R}^k} \mathbf{M}(T^p) \, \mathrm{d}p < \infty$ and (2.10), then $T^p = \langle T, \varrho, p\rangle$ for $\mathcal{H}^k$-almost every $p \in \mathbb{R}^k$. Hence, for example, one has that the slicing and the push-forward operator commute. More concretely, if $f : X \to Y$ and $\varrho : Y \to \mathbb{R}^k$ are Lipschitz maps, then

$$f_\# \langle T, \widehat{\varrho}, p\rangle = \langle f_\# T, \varrho, p\rangle \tag{2.12}$$

for $\mathcal{H}^k$-almost every $p \in \mathbb{R}^k$, where $\widehat{\varrho} = \varrho \circ f$.

Naively one might suspect that $\operatorname{spt}\langle T, \varrho, p\rangle = \operatorname{spt} T \cap \varrho^{-1}(p)$ up to a set of $\mathcal{H}^{n-k}$-measure zero. However, as the following well-known example shows this cannot be true in general.

Example 2.2. Fix $n \ge 1$ and let $\{x_i : i \in \mathbb{N}\}$ be a dense subset of $\mathbb{R}^{n+1}$. Further, let $(r_i)_{i \in \mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{i \in \mathbb{N}} r_i^n$ is finite. We put

$$T_i = \partial \llbracket B(x_i, r_i) \rrbracket \quad \text{and} \quad T = \sum_{i \in \mathbb{N}} T_i.$$

By construction, T is an integer-rectifiable n-current. Moreover, it is easy to check that

$$\operatorname{spt} T = \mathbb{R}^{n+1},$$

and thus

$$\operatorname{spt} T \cap \varrho^{-1}(p) \cong \mathbb{R}$$

for every orthogonal projection ϱ onto a hyperplane. But $\langle T, \varrho, p\rangle$ is an integer-rectifiable 0-current for $\mathcal{H}^n$-almost every $p \in \mathbb{R}^n$. In particular, the support of $\langle T, \varrho, p\rangle$ consists of finitely many points, and so it cannot be equal to $\operatorname{spt} T \cap \varrho^{-1}(p)$ up to a set of $\mathcal{H}^0$-measure zero.

The following lemma shows that such an equality is true if instead of $\operatorname{spt}\langle T, \varrho, p\rangle$ and $\operatorname{spt} T$ the corresponding characteristic sets are considered. The proof is based on (after isometrically embedding X into ℓ^∞) an application of the coarea inequality [22], and [4, Theorems 9.1, 9.5 and 9.7]. We omit the details.

Lemma 2.3. *If $T \in \mathcal{R}_n(X)$ and $\varrho : X \to \mathbb{R}^k$ with $k \le n$ is a Lipschitz map, then up to a set of $\mathcal{H}^{n-k}$-measure zero*

$$\operatorname{set}\langle T, \varrho, p\rangle = \operatorname{set} T \cap \varrho^{-1}(p) \tag{2.13}$$

for $\mathcal{H}^k$-almost every $p \in \mathbb{R}^k$.

Notice that if $k = n$, then (2.13) is an actual equality, since the empty set is the only $\mathcal{H}^0$-null set.

2.7. Integral current spaces. The space of integral currents $\mathbf{I}_n(X)$ is defined as

$$\mathbf{I}_n(X) = \mathscr{I}_n(X) \cap \mathbf{N}_n(X).$$

Integral currents are the most important class of currents in this article. The seminal boundary-rectifiability theorem [4, Theorem 8.6] states that $\partial T \in \mathbf{I}_{n-1}(X)$ whenever $T \in \mathbf{I}_n(X)$ and $n \ge 1$. We also remark that $\mathbf{I}_n(X)$ is a closed additive subgroup of $\mathbf{N}_n(X)$ for every $n \ge 0$ and, if $T \in \mathbf{I}_n(X)$, then $f_\# T \in \mathbf{I}_n(Y)$ for every Lipschitz map $f : X \to Y$. Moreover, for any Lipschitz map $\varrho : X \to \mathbb{R}^k$ one has $\langle T, \varrho, p\rangle \in \mathbf{I}_{n-k}(X)$ for $\mathcal{H}^k$-almost every $p \in \mathbb{R}^k$. The following definition is due to Sormani and Wenger (see [43, Definition 2.46]).

Definition 2.4 (Integral current space). A pair (X, T) is called n-dimensional integral current space if X is a metric space and $T \in \mathbf{I}_n(\overline{X})$ is such that $\operatorname{set} T = X$. We generally do not emphasize the dependence of the integral current space (X, T) on T and denote it only by X.

To any integral current space (X, T), one can naturally associate a boundary

$$\partial X = (\operatorname{set}\partial T, \partial T),$$

which is also an integral current space. Prime examples of integral current spaces are compact connected orientable Lipschitz manifolds.

Example 2.5. Let M be a compact orientable connected Lipschitz n-manifold. Every such manifold admits a finite atlas of bi-Lipschitz maps

$$\psi_i : U_i \to M,$$

where $U_i \subset \mathbb{R}^{n-1} \times [0,\infty)$ are open and the a.e. defined differentials of the coordinate transitions $\psi_j^{-1} \circ \psi_i$ are orientation preserving. By choosing a subordinate Lipschitz partition of unity and defining it locally in the charts, one can as in the smooth case integrate Lipschitz differential forms $h\, d\pi_1 \wedge \cdots \wedge d\pi_n$. In particular, one obtains an (up to sign) uniquely defined fundamental integer-rectifiable current $[\![M]\!] \in \mathscr{I}_n(M)$. Furthermore, by the Lipschitz version of Stokes' theorem, the manifold boundary of M coincides with the current boundary of $[\![M]\!]$. That is $\partial[\![M]\!] = [\![\partial M]\!]$ and hence $[\![M]\!] \in \mathbf{I}_k(M)$. Finally, since the mass measure of $[\![M]\!]$ equals $= \mu_M^{m*}$, and M is locally bi-Lipschitz equivalent to an open set in $\mathbb{R}^{n-1} \times [0,\infty)$, we deduce that $\operatorname{set}([\![M]\!]) = M$. Hence, $(M, [\![M]\!])$ is an integral current space.

Every convex body $C \subset \mathbb{R}^n$ is a compact connected oriented Lipschitz n-manifold. In particular, $[\![C]\!]$ is an integral n-current, and so the term $[\![\partial C]\!]$ appearing in Theorem 1.2 is a well-defined integral $(n-1)$-current and satisfies $[\![\partial C]\!] = \partial[\![C]\!]$. Other examples of integral current spaces are the metric manifolds considered in [7].

3. Mass preserving 1-Lipschitz maps

In this section we make some simple general observations concerning mass preserving 1-Lipschitz maps. The first of these is the following.

Lemma 3.1. *Let X, Y be complete metric spaces, let $f : X \to Y$ be 1-Lipschitz and let $T \in \mathscr{R}_n(X)$. If $\mathbf{M}(T) \le \mathbf{M}(f_\# T)$, then*

$$\|f_\# T\|(A) = \|T\|(f^{-1}(A)) \tag{3.1}$$

for every Borel set $A \subset Y$. Furthermore,

$$f(\operatorname{spt} T) \subset \operatorname{spt} f_\# T \quad \text{and} \quad \|f_\# T\|(Y \setminus f(\operatorname{set} T)) = 0. \tag{3.2}$$

We remark in passing that the inclusion in (3.2) is strict in general. Indeed, let s_n denote the oriented segment in $\mathbb{R}^2$ connecting $(n, \frac{1}{n})$ to $(n, \frac{1}{n+1})$ and consider the integral current $T = \sum_{n\in\mathbb{N}} [\![s_n]\!]$. Letting $f : \mathbb{R}^2 \to \mathbb{R}$ denote the projection onto the second coordinate, we find that $f(\operatorname{spt} T) = (0,1]$, but $\operatorname{spt} f_\# T = [0,1]$.

Proof of Lemma 3.1. The inequality $\|f_\# T\|(A) \le \|T\|(f^{-1}(A))$ is readily implied by the map f being 1-Lipschitz, the characterization of the mass measure given in [4, Proposition 2.7] and the definition of the push-forward. Applying this inequality to A and $Y \setminus A$, we obtain

$$\begin{aligned}\mathbf{M}(f_\# T) &= \|f_\# T\|(A) + \|f_\# T\|(Y \setminus A)\\ &\le \|T\|(f^{-1}(A)) + \|T\|(X \setminus f^{-1}(A)) = \mathbf{M}(T).\end{aligned}$$

By our assumption $\mathbf{M}(T) \le \mathbf{M}(f_\# T)$, this inequality chain must be rigid and hence (3.1) follows.

To finish the proof, we show (3.2). If $y = f(x)$ with $x \in \operatorname{spt} T$ and U is an open neighborhood of y, then $f^{-1}(U)$ is an open neighborhood of x and hence

$$\|f_\# T\|(U) = \|T\|(f^{-1}(U)) > 0.$$

In particular, it follows that $f(\operatorname{spt} T) \subset \operatorname{spt} f_{\#}T$. Finally,

$$\|f_{\#}T\|(Y \setminus f(\operatorname{set} T)) = \|T\|(X \setminus \operatorname{set} T) = 0$$

which completes the proof. □

Our next lemma deals with currents whose push-forward is equal to a current induced by the fundamental class of a Lipschitz manifold.

Lemma 3.2. *Let X be a complete metric space, let $f : X \to \mathbb{R}^N$ be 1-Lipschitz and let $T \in \mathscr{I}_n(X)$ be such that $f_{\#}T = [\![M]\!]$, where $M \subset \mathbb{R}^N$ is a compact n-dimensional Lipschitz submanifold. If further $\mathbf{M}(T) \le \mathcal{H}^n(M)$, then for every Borel set $A \subset X$, it follows that $f(A \cap \operatorname{set} T)$ is $\mathcal{H}^n$-measurable with*

$$\mathcal{H}^n(f(A \cap \operatorname{set} T)) = \|T\|(A),$$
$$\mathcal{H}^0(f^{-1}(p) \cap \operatorname{set} T) = 1$$

for $\mathcal{H}^n$-almost every $p \in M$.

Naively one might hope that f preserves the mass of all Borel subsets $A \subset X$ even in the more general setting of Lemma 3.1. There are however two obstacles. The more obvious one is that multiplicities might add up. This is excluded here by assuming T to be integer-rectifiable and that the push-forward current has "multiplicity one". The more subtle one is that $\operatorname{spt} T \setminus \operatorname{set} T$ is always a $\|T\|$-nullset but might in general have positive $\mathcal{H}^n$-measure (see Example 2.2). In particular, we cannot exclude the possibility that the image of this set does have positive $\mathcal{H}^n$-measure.

Proof of Lemma 3.2. We may suppose $X = \operatorname{spt} T$. In particular, it follows from Lemma 3.1 that $f(X) \subset M$. As $T \in \mathscr{I}_n(X)$, there are Borel sets $B_i \subset \mathbb{R}^n$, bi-Lipschitz embeddings $\varphi_i : B_i \to X$ and Borel functions $\Theta_i : B_i \to \mathbb{Z} \setminus \{0\}$ such that $\operatorname{set} T$ and $S = \bigcup_{i\in\mathbb{N}} \varphi_i(B_i)$ are equal up to a set of $\mathcal{H}^n$-measure zero,

$$T = \sum_{i\in\mathbb{N}} \varphi_{i\#}[\![\Theta_i]\!] \quad \text{and} \quad \|T\| = \sum_{i\in\mathbb{N}} \|\varphi_{i\#}[\![\Theta_i]\!]\|. \tag{3.3}$$

Since $\mathcal{H}^0$ is the counting measure, one has for every Borel set $A \subset X$ that

$$\begin{aligned} \mathcal{H}^n(f(A \cap S)) &\le \int_{\mathbb{R}^N} \mathcal{H}^0(f^{-1}(p) \cap A \cap S)\, \mathrm{d}\mathcal{H}^n(p) \\ &\le \sum_{i\in\mathbb{N}} \int_M \mathcal{H}^0(f^{-1}(p) \cap A \cap \varphi_i(B_i))\, \mathrm{d}\mathcal{H}^n(p). \end{aligned} \tag{3.4}$$

Moreover, using the area formula, that the metric differentials $\operatorname{md}(f \circ \varphi_i)_q$ are almost everywhere Euclidean and the monotonicity of Jacobians, we get

$$\begin{aligned} \int_M \mathcal{H}^0(f^{-1}(p) \cap A \cap \varphi_i(B_i))\, \mathrm{d}\mathcal{H}^n(p) &\le \int_{\varphi_i^{-1}(A)\cap B_i} \operatorname{Jac}^b(\operatorname{md}(f \circ \varphi_i)_q)\, \mathrm{d}q \\ &= \int_{\varphi_i^{-1}(A)\cap B_i} \operatorname{Jac}^{m*}(\operatorname{md}(f \circ \varphi_i)_q)\, \mathrm{d}q \\ &\le \int_{\varphi_i^{-1}(A)\cap B_i} \operatorname{Jac}^{m*}(\operatorname{md}(\varphi_i)_q)\, \mathrm{d}q \end{aligned} \tag{3.5}$$

for every $i \in \mathbb{N}$. Therefore, using that $|\Theta_i(q)| \geq 1$ for all $q \in B_i$, we arrive at

$$\begin{aligned}\mathcal{H}^n(f(A \cap S)) &\leq \sum_{i\in\mathbb{N}} \int_{\varphi_i^{-1}(A)\cap B_i} |\Theta_i(x)| \cdot \mathrm{Jac}^{m*}(\mathrm{md}(\varphi_i)_q)\,\mathrm{d}q \\ &= \sum_{i\in\mathbb{N}} \|\varphi_{i\#}[\![\Theta_i]\!]\|(A) = \|T\|(A),\end{aligned} \tag{3.6}$$

where in the last equality we used (3.3). Since f is Lipschitz, and set T and S agree up to $\mathcal{H}^n$-nullsets, we have that

$$\mathcal{H}^n(f(A \cap \operatorname{set} T)) = \mathcal{H}^n(f(A \cap S))$$

and

$$\mathcal{H}^0(f^{-1}(p) \cap A \cap \operatorname{set} T) = \mathcal{H}^0(f^{-1}(p) \cap A \cap S)$$

for $\mathcal{H}^n$-almost every $p \in M$. Now, as in the proof of Lemma 3.1,

$$\begin{aligned}\mathcal{H}^n(f(\operatorname{set} T)) &\leq \mathcal{H}^n(f(A \cap \operatorname{set} T)) + \mathcal{H}^n(f((X \setminus A) \cap \operatorname{set} T)) \\ &\leq \|T\|(A) + \|T\|(X \setminus A) = \mathbf{M}(T).\end{aligned} \tag{3.7}$$

Lemma 3.1 tells us that $f(\operatorname{set} T) \subset \operatorname{spt}[\![M]\!] = M$, and so

$$\begin{aligned}\mathcal{H}^n(M \setminus f(\operatorname{set} T)) &= \|[\![M]\!]\|(M \setminus f(\operatorname{set} T)) \\ &= \|T\|(X \setminus f^{-1}(f(\operatorname{set} T))) \leq \|T\|(X \setminus \operatorname{set} T) = 0.\end{aligned}$$

In particular, (3.7) is rigid and hence so are (3.6), (3.5) and (3.4). By our previous observations this gives the claimed equalities. □

4. Proof of Theorem 1.2

4.1. The case $n = 1$. In the following we prove Theorem 1.2 for $n = 1$. The general case $n \geq 2$ is proved in Section 4.3 by reducing it to this case. In the proof we use metric 1-currents induced by curves. For a Lipschitz curve $\gamma : [a, b] \to X$ into a metric space X, the integral 1-current $[\![\gamma]\!] := \gamma_\#[\![[a, b]]\!]$ is given by

$$[\![\gamma]\!](h, \pi_1) = \int_a^b h(\gamma(t))(\pi_1 \circ \gamma)'(t)\,\mathrm{d}t, \quad (h, \pi_1) \in \mathcal{D}^1(X).$$

Note that the boundary of $[\![\gamma]\!]$ is given by $\partial[\![\gamma]\!](h) = h(\gamma(b)) - h(\gamma(a))$ for all $h \in \mathcal{D}^0(X)$. If γ is a loop and $\gamma|_{[a,b)}$ is injective, we say that γ is a *simple* Lipschitz loop. By [10, Theorem 5.3], the integral 1-current T admits a decomposition

$$T = \sum_{i\in I} [\![\gamma_i]\!] + \sum_{j\in J} [\![\eta_j]\!],$$

where I, J are countable index sets, each γ_i is an injective Lipschitz curve, each η_j is a simple Lipschitz loop,

$$\mathbf{M}(T) = \sum_{i\in I} \mathbf{M}([\![\gamma_i]\!]) + \sum_{j\in J} \mathbf{M}([\![\eta_j]\!]) = \sum_{i\in I} \ell(\gamma_i) + \sum_{j\in J} \ell(\eta_j)$$

and

$$\mathbf{M}(\partial T) = \sum_{i\in I} \mathbf{M}(\partial[\![\gamma_i]\!]) + \sum_{j\in J} \mathbf{M}(\partial[\![\eta_j]\!]) = 2|I|. \tag{4.1}$$

By assumption,

$$\mathbf{M}(\partial T) = \mathbf{M}(f_\#(\partial[\![B^1]\!])) = \mathbf{M}([\![1]\!] - [\![-1]\!]) = 2$$

and hence (4.1) implies $|I| = 1$. Henceforth we will denote the unique injective curve γ_j by $\gamma : [a, b] \to X$, and by x_1, x_2 the endpoints of γ. Since $\partial T = \partial[\![\gamma]\!] = [\![x_2]\!] - [\![x_1]\!]$ and $f_\#(\partial T) = [\![1]\!] - [\![-1]\!]$, we conclude that $f(x_2) = 1$ and $f(x_1) = -1$. In particular, since f is 1-Lipschitz,

$$2 \le d(x_1, x_2) \le \ell(\gamma) \le \ell(\gamma) + \sum_{j\in J} \ell(\eta_j) = \mathbf{M}(T) \le 2.$$

This implies that $d(x_1, x_2) = 2$, $T = [\![\gamma]\!]$, and γ is a geodesic connecting x_1 to x_2. Since $X = \operatorname{set} T = \operatorname{spt} T = \gamma([a, b])$, we conclude that X is isometric to B^1. In particular, because X is connected and $1, -1 \in f(X)$, it follows that f must be surjective. Since $f : X \to B^1$ is a surjective 1-Lipschitz map, we conclude that f is an isometry.

4.2. From slice-isometry to isometry. The aim of this subsection is to prove the following proposition which shows that, to obtain Theorem 1.2, it suffices to prove that f is an isometry when restricted to certain slices.

Proposition 4.1. *Let $n \ge 2$, X be an integral n-current space, let $C \subset \mathbb{R}^n$ be a convex body and let $f : X \to \mathbb{R}^n$ be a 1-Lipschitz map such that $f_\# T = [\![C]\!]$ and $\mathbf{M}(T) \le \mathcal{H}^n(C)$. Further, suppose that $k \in \{1, \dots, n-1\}$ and for every orthogonal projection $\varrho : \mathbb{R}^n \to \mathbb{R}^k$ the following holds:* For $\mathcal{H}^k$-almost every $p \in \mathbb{R}^k$ the restriction of f is an isometry

$$\operatorname{spt}\langle T, \varrho \circ f, p\rangle \to \operatorname{spt}\langle [\![C]\!], \varrho, p\rangle.$$

Then f is an isometry $X \to C$.

Here, we use the convention that $\varrho : \mathbb{R}^n \to \mathbb{R}^k$ is called *orthogonal projection* if there are a k-plane $H \subset \mathbb{R}^n$ and an isometry $\phi : \mathbb{R}^k \to H$ such that $\phi \circ \varrho$ is equal to the orthogonal projection $\mathbb{R}^n \to H$.

To prove Theorem 1.2, we will apply Proposition 4.1 with $k = n - 1$ to reduce it to the $n = 1$ case handled in the previous subsection. Another natural option would be to take $k = 1$ reducing the theorem to the $n - 1$ case and performing an induction argument. For the proof of Proposition 4.1 we need the following simple consequence of the Lebesgue density theorem and Fubini's theorem.

Lemma 4.2. *Let $n, k \in \mathbb{N}$ with $k < n$ and let $A_1, A_2 \subset \mathbb{R}^n$ be $\mathcal{H}^n$-measurable subsets such that $\mathcal{H}^n(A_i) > 0$. Then there exist an orthogonal projection $\varrho : \mathbb{R}^n \to \mathbb{R}^k$ and an $\mathcal{H}^k$-measurable $E \subset \varrho(A_1) \cap \varrho(A_2)$ with $\mathcal{H}^k(E) > 0$ such that for every $p \in E$ the respective sections $\varrho^{-1}(p) \cap A_i$ are $\mathcal{H}^{n-k}$-measurable with $\mathcal{H}^{n-k}(\varrho^{-1}(p) \cap A_i) > 0$.*

Proof. Let $p_i \in A_i$ be Lebesgue density points, that is, $\Theta_n(A_i, p_i) = 1$, and we set $v := p_1 - p_2$. For $F := A_1 \cap (A_2 + v)$, we claim that $\mathcal{H}^n(F) > 0$. Indeed, using the fact that

$\mathcal{H}^n(A_2 \cap B(p_2, r)) = \mathcal{H}^n((A_2 + v) \cap B(p_1, r))$ for any $r > 0$, and the inclusion-exclusion principle, we find that

$$\begin{aligned}&\mathcal{H}^n(A_1 \cap B(p_1, r)) + \mathcal{H}^n(A_2 \cap B(p_2, r))\\ &\quad = \mathcal{H}^n((A_1 \cup (A_2 + v)) \cap B(p_1, r)) + \mathcal{H}^n(F \cap B(p_1, r))\end{aligned}$$

and consequently $\Theta_n(A_1, p_1) + \Theta_n(A_2, p_2) = 2 \leq 1 + \Theta_n(F, p_1)$, implying that F has density 1 at p_1.

Now we choose an orthogonal projection $\varrho : \mathbb{R}^n \to \mathbb{R}^k$ with $\varrho(v) = 0$. Then

$$\varrho(F) \subset \varrho(A_1) \cap \varrho(A_2 + v) = \varrho(A_1) \cap \varrho(A_2).$$

Since F is $\mathcal{H}^n$-measurable with $\mathcal{H}^n(F) > 0$, it follows from Fubini's theorem that there is an $\mathcal{H}^k$-measurable set $E \subset \varrho(F)$ with $\mathcal{H}^k(E) > 0$ such that for every $p \in E$ the section $\varrho^{-1}(p) \cap F$ is $\mathcal{H}^{n-k}$-measurable with $\mathcal{H}^{n-k}(\varrho^{-1}(p) \cap F) > 0$. Since $F \subset A_1$, $F \subset A_2 + v$ and $\varrho^{-1}(p) + v = \varrho^{-1}(p)$, we have $\mathcal{H}^{n-k}(\varrho^{-1}(p) \cap A_i) > 0$ for $i = 1, 2$. Finally, by Fubini $\varrho^{-1}(p) \cap A_i$ is $\mathcal{H}^{n-k}$-measurable for almost every $p \in \mathbb{R}^k$ and hence we may also assume that $\varrho^{-1}(p) \cap A_i$ is $\mathcal{H}^{n-k}$-measurable for every $p \in E$. □

Proof of Proposition 4.1. Let $x_1, x_2 \in X$ and $\delta > 0$. Note that the balls $B(x_i, \delta)$ are $\|T\|$-measurable with $\|T\|(B(x_i, \delta)) > 0$. By Lemma 3.2, setting $B_i := B(x_i, \delta) \cap \operatorname{set} T$, the sets $L_i := f(B_i)$ are $\mathcal{H}^n$-measurable with $\mathcal{H}^n(L_i) > 0$.

Due to Lemma 4.2, there are an orthogonal projection $\varrho : \mathbb{R}^n \to \mathbb{R}^k$ and a measurable $E \subset \varrho(L_1) \cap \varrho(L_2)$ with $\mathcal{H}^k(E) > 0$ such that $\varrho^{-1}(p) \cap L_i$ is of positive $\mathcal{H}^{n-k}$-measure for every $p \in E$. By Lemma 2.3 we may further assume that for every $p \in E$,

$$\operatorname{set}\langle T, \widehat{\varrho}, p\rangle = \widehat{\varrho}^{\,-1}(p) \cap \operatorname{set} T \tag{4.2}$$

up to an $\mathcal{H}^{n-k}$ null set and by our assumption that for every $p \in E$ the restriction of f defines an isometry $\operatorname{spt}\langle T, \widehat{\varrho}, p\rangle \to \operatorname{spt}\langle [\![C]\!], \varrho, p\rangle$.

Now let $p \in E$. Then for each i the slice $\varrho^{-1}(p) \cap L_i$ is of positive $\mathcal{H}^{n-k}$-measure. As f is 1-Lipschitz and $f(\widehat{\varrho}^{\,-1}(p) \cap B_i) = \varrho^{-1}(p) \cap L_i$, this implies that also $\widehat{\varrho}^{\,-1}(p) \cap B_i$ is of positive $\mathcal{H}^{n-k}$-measure. Thus we deduce from (4.2) that

$$\mathcal{H}^{n-k}(B_i \cap \operatorname{set}\langle T, \widehat{\varrho}, p\rangle) = \mathcal{H}^{n-k}(\widehat{\varrho}^{\,-1}(p) \cap B_i) > 0.$$

In particular, we may respectively choose points $y_i \in B_i \cap \operatorname{spt}\langle T, \widehat{\varrho}, p\rangle$. Since f is an isometric embedding on $\operatorname{set}\langle T, \widehat{\varrho}, p\rangle$, we have that

$$d(y_1, y_2) = |f(y_1) - f(y_2)|.$$

Since $y_i \in B(x_i, \delta)$ and f is continuous, by letting $\delta \to 0$ we conclude that

$$d(x_1, x_2) = |f(x_1) - f(x_2)|.$$

In particular, since $x_1, x_2 \in X$ were arbitrary, it follows that f defines an isometric embedding. By Lemma 3.1, $f(X)$ is dense in C, and so it follows that $\bar{f} : \overline{X} \to C$ is an isometry. Since $f_\# T = [\![C]\!]$, we find that $X = \operatorname{set} T = \operatorname{spt} T = \overline{X}$ and hence the claim follows. □

4.3. Proof of Theorem 1.2. In the following, we suppose that $n \geq 2$. The case when $n = 1$ is treated in Section 4.1. To prove the theorem, it suffices to show that the assumptions

of Proposition 4.1 are satisfied for $k = n-1$. Let $\varrho : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be an orthogonal projection and set $T_p = \langle T, \widehat{\varrho}, p\rangle$, $X_p = \operatorname{set} T_p$, and $C_p = C \cap \varrho^{-1}(p)$, we claim that the following conditions are satisfied for $\mathcal{H}^{n-1}$-almost every $p \in \mathbb{R}^{n-1}$:

(1) (X_p, T_p) is an integral current space,

(2) $f_\# T_p = [\![C_p]\!]$,

(3) $\mathbf{M}(T_p) \le \mathcal{H}^1(C_p)$,

(4) $\mathbf{M}(\partial T_p) \le \mathcal{H}^0(\partial C_p)$.

Condition (1) follows directly from the properties of the slicing operator discussed in Section 2.6.

By applying the constancy theorem (see [24, Corollary 3.13]) to the integral n-cycle $S = f_\# T - [\![C]\!]$, it follows that $S = 0$ and thus $f_\# T = [\![C]\!]$. Alternatively, this can be seen by applying the deformation theorem (see, e.g., [8, Theorem A.2]). Hence, using (2.12), we find that

$$f_\# T_p = f_\# \langle T, \widehat{\varrho}, p\rangle = \langle f_\# T, \varrho, p\rangle = \langle [\![C]\!], \varrho, p\rangle$$

for $\mathcal{H}^{n-1}$-almost every p. Notice that $[\![C_p]\!] \in \mathbf{M}_1(\mathbb{R}^n)$ are concentrated on $\varrho^{-1}(p)$, satisfy (2.10), and $\int_{\mathbb{R}^{n-1}} \mathbf{M}([\![C_p]\!])\,\mathrm{d}p < \infty$. Hence, as these properties uniquely determine the slices $\langle [\![C]\!], \varrho, p\rangle$, it follows that $\langle [\![C]\!], \varrho, p\rangle = [\![C_p]\!]$ for $\mathcal{H}^{n-1}$-almost every p, and thus (2) follows. We proceed by showing (3). By (2), it follows that $\mathbf{M}(T_p) \ge \mathcal{H}^1(C_p)$. Hence, using Fubini and (2.11), we find that

$$\mathcal{H}^n(C) = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(C_p)\,\mathrm{d}p \le \int_{\mathbb{R}^{n-1}} \mathbf{M}(T_p)\,\mathrm{d}p = \mathbf{M}(T \llcorner (1, \widehat{\varrho})) \le \mathbf{M}(T).$$

By our assumption $\mathbf{M}(T) \le \mathcal{H}^n(C)$ this equality is rigid, and so (3) follows.

Finally, we prove (4). By Lemma 3.1, $f(\operatorname{spt}(\partial T)) \subset \partial C$, and hence by Lemma 3.2 for $\mathcal{H}^{n-1}$-almost every p,

$$\mathcal{H}^0(\widehat{\varrho}^{\,-1}(p) \cap \operatorname{set} \partial T) \le 2.$$

However, for $\mathcal{H}^{n-1}$-almost every p we also have by Lemma 2.3 that

$$\widehat{\varrho}^{\,-1}(p) \cap \operatorname{set} \partial T = \operatorname{set}\langle \partial T, \widehat{\varrho}, p\rangle = \operatorname{set} \partial T_p = \operatorname{spt} \partial T_p.$$

Together with $f_\#(\partial T_p) = [\![\partial C_p]\!]$ this implies (4).

Now, since Theorem 1.2 is valid when $n = 1$, it follows that the restriction of f is an isometry $\operatorname{spt}\langle T, \widehat{\varrho}, p\rangle \to \operatorname{spt}\langle [\![C]\!], \varrho, p\rangle$ for $\mathcal{H}^{n-1}$-almost every $p \in \mathbb{R}^{n-1}$. Therefore, as ϱ was arbitrary, Proposition 4.1 tells us that f is an isometry, as desired. □

5. Proof of Corollary 1.3

The *Euclidean cone* CX over a metric space $X = (X, d)$ is the metric space obtained when endowing $X \times [0, 1]$ with the pseudometric

$$d_C((x, r), (y, s)) := \begin{cases} \sqrt{r^2 + s^2 - 2rs\cos(d(x, y))} & \text{if } d(x, y) < \pi, \\ r + s & \text{otherwise,} \end{cases}$$

which defines a metric on the quotient space $CX = X \times [0, 1]/\!\sim$, where $(x, 0) \sim (y, 0)$ for all $x, y \in X$. Compare also [12, Section 3.6]. Observe in particular that the Euclidean cone over

the round sphere S^n is isometric to the flat disk B^{n+1}. We denote by $H : X \times [0,1] \to CX$ and $e : X \to CX$ the Lipschitz maps defined by $H(x,t) = [(x,t)]$ and $e(x) = [(x,1)]$. It is a consequence of the monotonicity of the cosine function on $[0,\pi]$ that e is 1-Lipschitz. For the same reason, also if $f : X \to Y$ is 1-Lipschitz, then $Cf : CX \to CY$ defined by $[(x,r)] \mapsto [(f(x),r)]$ is 1-Lipschitz as well.

For any $T \in \mathbf{I}_n(X)$, we set

$$CT := H_\#(T \times [\![0,1]\!]),$$

where the product current $T \times [\![0,1]\!] \in \mathbf{I}_{n+1}(X \times [0,1])$ is defined as in [8, Section 3.3]. By construction, $CT \in \mathbf{I}_{n+1}(CX)$, and one has

$$\begin{aligned} \operatorname{set}(CT) &= H(\operatorname{set} T \times [0,1]), \qquad \operatorname{spt} CT = H(\operatorname{spt} T \times [0,1]),\\ \partial(CT) &= C(\partial T) + e_\# T, \qquad (Cf)_\# CT = C(f_\# T). \end{aligned}$$

Moreover, if T is represented as in (2.6) by functions $\Theta_i \in L^1(\mathbb{R}^n, \mathbb{Z})$ and bi-Lipschitz embeddings $\varphi_i : K_i \to X$, then setting $\widetilde{\Theta}_i(x,t) := \Theta(x)$, $\widetilde{K}_i := K_i \times [0,1]$ and $\widetilde{\varphi}_i(x,t) := [(\varphi_i(x),t)]$ we find that

$$CT = \sum_{i\in\mathbb{N}} \widetilde{\varphi}_{i\#}[\![\widetilde{\Theta}_i]\!] \quad \text{and} \quad \mathbf{M}(CT) = \sum_{i\in\mathbb{N}} \mathbf{M}(\widetilde{\varphi}_{i\#}[\![\widetilde{\Theta}_i]\!]). \tag{5.1}$$

The following lemma shows that the $\mathbf{M}^{\mathrm{ir}}$-mass of CT is analogous to the volume of cones in Euclidean space.

Lemma 5.1. *If $T \in \mathbf{I}_n(X)$, then*

$$\mathbf{M}^{\mathrm{ir}}(CT) = \frac{1}{n+1} \cdot \mathbf{M}^{\mathrm{ir}}(T). \tag{5.2}$$

Note that for $\mathbf{M}$, instead of (5.2), only a weaker inequality without the factor $\frac{1}{n+1}$ holds, compare [8, Lemma 3.5]. For this reason we can prove Corollary 1.3 only for $\mathbf{M}^{\mathrm{ir}}$.

Proof. For a Lipschitz map $\varphi : K \to X$ with $K \subset \mathbb{R}^n$ we consider the corresponding map $\widetilde{\varphi} : K \times [0,1] \to CX$ as above. Then for almost every $(x,r) \in K \times [0,1]$ one has for every $(v,s) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ that

$$(\operatorname{md} \widetilde{\varphi}_{(x,r)}(v,s))^2 = \lim_{\varepsilon \downarrow 0} \frac{(r+\varepsilon s)^2 + r^2 - 2r(r+\varepsilon s)\cos(d(\varphi(x+\varepsilon v), \varphi(x)))}{\varepsilon^2}.$$

Using that $1 - \cos(x) = \frac{x^2}{2} + O(x^4)$, we deduce

$$\operatorname{md} \widetilde{\varphi}_{(x,r)}(v,s) = \sqrt{r^2 \cdot (\operatorname{md} \varphi_x(v))^2 + s^2}.$$

Thus by (2.2) one has

$$\operatorname{Jac}^{\mathrm{ir}}(\operatorname{md} \widetilde{\varphi}_{(x,r)}) = \operatorname{Jac}^{\mathrm{ir}}(r \cdot \operatorname{md} \varphi_x) = r^n \cdot \operatorname{Jac}^{\mathrm{ir}}(\operatorname{md} \varphi_x).$$

Using this observation, the charts $\widetilde{\varphi}_i$ as in (5.1), and Fubini, we obtain

$$\mathbf{M}^{\mathrm{ir}}(CT) = \int_0^1 r^n \, \mathrm{d}r \cdot \mathbf{M}^{\mathrm{ir}}(T) = \frac{1}{n+1} \cdot \mathbf{M}^{\mathrm{ir}}(T),$$

as desired. □

Proof of Corollary 1.3. Notice that $CX = (CX, CT)$ is an integral current space and $Cf : CX \to B^{n+1}$ is a 1-Lipschitz map with $(Cf)_\# CT = [\![B^{n+1}]\!]$ (see the discussion before Lemma 5.1). Furthermore, by Lemma 5.1,

$$\mathbf{M}(CT) \le \mathbf{M}^{\mathrm{ir}}(CT) = \frac{1}{n+1} \cdot \mathbf{M}^{\mathrm{ir}}(T) \le \frac{1}{n+1} \cdot \mathrm{Vol}^n(S^n) = \mathrm{Vol}^{n+1}(B^{n+1})$$

and

$$\mathbf{M}(\partial(CT)) = \mathbf{M}(e_\# T) \le \mathbf{M}(T) \le \mathbf{M}^{\mathrm{ir}}(T) \le \mathrm{Vol}^n(S^n).$$

Hence Theorem 1.2 implies that $Cf : CX \to B^{n+1}$ is an isometry.

Now if $x, y \in X$ are such that $d_{CX}([(x,1)],[(y,1)]) = d_{B^{n+1}}(f(x), f(y)) < 2$, then $d(x,y) < \pi$ and

$$\sqrt{2 - 2\cos(d(x,y))} = \sqrt{2 - 2\cos(d_{S^n}(f(x), f(y)))}.$$

Since the cosine function is injective on $[0, \pi]$, this implies that $d(x,y) = d_{S^n}(f(x), f(y))$. To prove the equality also for $x, y \in X$ with $d_{B^{n+1}}(f(x), f(y)) = 2$, we choose $z \in X$ with $x \neq z \neq y$. Since f is bijective and $f(z)$ lies on an S^n-geodesic from $f(x)$ to $f(y)$, the previous case gives

$$\begin{aligned} d(x,y) &\le d(x,z) + d(z,y) \\ &= d_{S^n}(f(x), f(z)) + d_{S^n}(f(z), f(y)) \\ &= d_{S^n}(f(x), f(y)). \end{aligned}$$

Since f is 1-Lipschitz, this implies the claim. □

6. Proof of Theorem 1.1

6.1. The lower bound. We start by proving inequality (1.1). It is readily implied by the following lemma since $\mathbf{M} \le \mathbf{M}^{\mathrm{ir}}$.

Lemma 6.1. *Let $C \subset \mathbb{R}^n$ be a convex body and suppose $\iota : \partial C \to X$ is an isometric embedding into an integral current space X such that $\iota_\# [\![\partial C]\!] = \partial T$. Then*

$$\mathbf{M}(T) \ge \mathrm{Vol}^n(C).$$

The following proof is essentially based on an observation due to Gromov [25, Proposition 2.1.A].

Proof. Since ι is an isometric embedding and $\mathrm{spt}\, \iota_\# [\![\partial C]\!] \subset \iota(\partial C)$, we can conclude that $j_\#(\partial T) = [\![\partial C]\!]$, where $j : \partial X \to \partial C$ denotes the inverse of ι. Let $h : \mathbb{R}^n \to \ell^n_\infty$ be the identity map, which is 1-Lipschitz. Obviously, $h \circ j$ is also 1-Lipschitz, and since ℓ^n_∞ is injective, there exists a 1-Lipschitz extension $f : X \to \ell^n_\infty$ of $h \circ j$. Since $\mathbb{R}^n$ and ℓ^n_∞ are bi-Lipschitz equivalent, it follows directly from the constancy theorem (see [24, Corollary 3.13]) that the n-cycle $S = f_\# T - h_\# [\![C]\!]$ is equal to the zero current and thus $f_\# T = h_\# [\![C]\!]$. In particular, $\mathbf{M}(T) \ge \mathbf{M}(h_\# [\![C]\!])$. But

$$\mathbf{M}(h_\# [\![C]\!]) = \mu^{m*}(h(C)) = \mu^{m*}(C) = \mathbf{M}([\![C]\!]),$$

where in the second equality we have used that $\mathrm{Jac}^{m*}(\|\cdot\|_\infty) = 1$. □

6.2. Rigidity. In this subsection we use the techniques of Burago and Ivanov developed in [14, 15] to deduce the rigidity statement in Theorem 1.1 from the Lipschitz-volume rigidity result Theorem 1.2.

For the proof we need the following auxiliary spaces: We denote by $\mathscr{L} := L^\infty(S^{n-1})$ the Banach space of (equivalence classes of) Borel measurable essentially bounded functions $S^{n-1} \to \mathbb{R}$ endowed with the usual norm $\|\cdot\|_\infty$. Furthermore, we consider the space

$$\mathscr{L}_2 := L^2(S^{n-1})$$

equipped with the inner product

$$\langle f, g\rangle_2 := \frac{n}{\mathrm{Vol}^{n-1}(S^{n-1})} \int_{S^{n-1}} fg \, \mathrm{d}\mathcal{H}^{n-1}$$

and the corresponding norm $\|f\|_2 := \sqrt{\langle f, f\rangle_2}$. The need for this particular normalization constant will become clear below. The properties of these spaces relevant for the proof of Theorem 1.1 are summarized in the following lemma.

Lemma 6.2. *The following hold true:*

(1) $\mathscr{L}$ *is injective.*

(2) $\mathscr{L}_2$ *is a Hilbert space.*

(3) *The canonical embedding* $I : \mathscr{L} \to \mathscr{L}_2$ *is Lipschitz and*

$$\|I_\# T\|^{\mathrm{ir}} \le I_\# \|T\|^{\mathrm{ir}} \tag{6.1}$$

for every $T \in \mathcal{R}_n(\mathscr{L})$.

(4) *There is a linear map* $\Phi : \mathbb{R}^n \to \mathscr{L}$ *such that* Φ *and the composition* $I \circ \Phi : \mathbb{R}^n \to \mathscr{L}_2$ *are both isometric embeddings.*

Proof. To prove (1), it suffices to combine McShane's extension theorem with [14, Lemma 5.1]. Moreover, (2) holds true since $\|\cdot\|_2$ is just a rescaling of the usual L^2-norm.

A straightforward application of Hölder's inequality shows that I is $\sqrt{n}$-Lipschitz. To complete the proof of (3), it remains to show (6.1). In light of (2.8) and (2.3) it suffices to show that

$$\mathrm{Jac}^{\mathrm{ir}}(\mathrm{md}(I \circ \varphi)_x) \le \mathrm{Jac}^{\mathrm{ir}}(\mathrm{md}\, \varphi_x) \quad \text{for } \mathcal{H}^n\text{-almost every } x \in E, \tag{6.2}$$

for every bi-Lipschitz map $\varphi : E \to \mathscr{L}$ from a Borel set $E \subset \mathbb{R}^n$. If $\bar\varphi : \mathbb{R}^n \to \mathscr{L}$ is a Lipschitz extension of φ, then $\mathrm{md}\, \varphi_x = \mathrm{md}\, \bar\varphi_x$ for $\mathcal{H}^n$-almost every $x \in E$. Thus we may assume that φ is defined on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be a point where φ admits a metric differential $\mathrm{md}\, \varphi_x$ and $I \circ \varphi$ admits a Fréchet differential $A_x := (I \circ \varphi)'(x) : \mathbb{R}^n \to \mathscr{L}_2$. Observe that

$$\mathrm{md}(I \circ \varphi)_x(v) = \|A_x(v)\|_2, \quad v \in \mathbb{R}^n.$$

We first claim that $V_x := A_x(\mathbb{R}^n) \subset \mathscr{L}$. Indeed, since

$$\left\| \frac{\varphi(x + hv) - \varphi(x)}{h} \right\|_\infty \le \mathrm{Lip}(\varphi) \quad \text{and} \quad A_x(v) = \lim_{h \to 0} \frac{\varphi(x + hv) - \varphi(x)}{h} \text{ in } \mathscr{L}_2$$

for each $v \in \mathbb{R}^n$ it follows that $A_x(v) \in \mathscr{L}$ and, moreover, that

$$\|A_x(v)\|_\infty \le \liminf_{h\to 0} \left\| \frac{\varphi(x+hv)-\varphi(x)}{h} \right\|_\infty = \operatorname{md}\varphi_x(v), \quad v \in \mathbb{R}^n.$$

We now prove (6.2). The claim is trivially true if A_x is not injective. Thus we may assume that $A_x : \mathbb{R}^n \to V_x$ is a linear isomorphism. In particular, we have that

$$\operatorname{Jac}^{\mathrm{ir}}(\operatorname{md}(I\circ\varphi)_x) = |\det(I\circ A_x)|, \quad \operatorname{Jac}^{\mathrm{ir}}(\operatorname{md}\varphi_x) \ge \operatorname{Jac}^{\mathrm{ir}}(s) = |\det L^{-1}|, \tag{6.3}$$

where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism such that $L(B^n)$ is the John ellipsoid of the norm $s = \|A_x(\cdot)\|_\infty$. Note that

$$(A_x \circ L)(B^n) \subset A_x(B_s) = V_x \cap B_{\mathscr{L}},$$

and thus $A_x \circ L : \mathbb{R}^n \to \mathscr{L}$ is 1-Lipschitz. From [14, Lemma 6.1] it now follows that the composition $I \circ A_x \circ L : \mathbb{R}^n \to \mathscr{L}_2$ is area non-increasing, i.e., $|\det(I \circ A_x \circ L)| \le 1$. Thus

$$|\det(I\circ A_x)| = |\det(I\circ A_x\circ L)|\cdot|\det(L^{-1})| \le |\det(L^{-1})|,$$

which by (6.3) implies (6.2).

To prove (4), we consider the linear map $\Phi : \mathbb{R}^n \to \mathscr{L}$ defined by $\Phi_x(p) = \langle x, p\rangle$, $p \in S^{n-1}$, for each $x \in \mathbb{R}^n$. By using the Cauchy–Schwarz inequality, it is easy to check that Φ is an isometric embedding. It remains to show that $I \circ \Phi$ is an isometric embedding as well. This follows from the proof of [14, Lemma 4.6]. Indeed, for all $x \in \mathbb{R}^n$ of unit norm, one has

$$\|I\circ\Phi(x)\|_2^2 = n \fint_{S^{n-1}} \langle x, p\rangle^2 \,\mathrm{d}\mathcal{H}^{n-1}(p) = \fint_{S^{n-1}} \sum_{i=1}^n \langle e_i, p\rangle^2 \,\mathrm{d}\mathcal{H}^{n-1}(p) = 1.$$

By linearity of $I \circ \Phi$ this completes the proof. □

The proofs of rigidity in [14] and [15] rely on a rigidity version of (6.1) that does not apply in our current setting. Nevertheless, by applying our Lipschitz-volume rigidity theorem twice we are able to avoid this difficulty and complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let X be an integral current space and $\iota : \partial C \to X$ an isometric embedding such that $\iota_\#[\![\partial C]\!] = \partial T$. Inequality (1.1) follows immediately from Lemma 6.1. It remains to show that if $\mathbf{M}^{\mathrm{ir}}(T) = \operatorname{Vol}^n(C)$, then ι can be extended to an isometry $C \to X$.

Clearly, we have $\partial X = \iota(\partial C)$. By Lemma 6.2 (1) the map $\Phi \circ \iota^{-1} : \partial X \to \mathscr{L}$ admits a 1-Lipschitz extension $f : X \to \mathscr{L}$. Let $S \coloneqq f_\# T$ and let Z be the integral current space $(\operatorname{set}(I_\# S), I_\# S)$ endowed with the subspace metric of $\mathscr{L}_2$. By Lemma 6.2 (3) and the monotonicity of $\mathbf{M}^{\mathrm{ir}}$ we have

$$\mathbf{M}(I_\# S) \le \mathbf{M}^{\mathrm{ir}}(I_\# S) \le \mathbf{M}^{\mathrm{ir}}(S) \le \mathbf{M}^{\mathrm{ir}}(T) \le \operatorname{Vol}^n(C). \tag{6.4}$$

Lemma 6.2 (2) implies that there is a 1-Lipschitz projection $P : \mathscr{L}_2 \to (I\circ\Phi)(\mathbb{R}^n)$. Notice that $P_\#[\![\partial Z]\!] = (I\circ\Phi)_\#[\![\partial C]\!]$. Now, since $\mathbf{M}([\![\partial Z]\!]) = \operatorname{Vol}^{n-1}(\partial C)$ and $\mathbf{M}([\![Z]\!]) \le \operatorname{Vol}^n(C)$, Theorem 1.2 implies that the restriction of P to Z defines an isometry $Z \to (I\circ\Phi)(C)$. Since $\mathscr{L}_2$ is a Hilbert space and hence uniquely geodesic, we conclude that $Z = (I\circ\Phi)(C)$.

In particular, (6.4) is rigid, and so $\mathbf{M}^{\mathrm{ir}}(I_\# S) = \mathbf{M}^{\mathrm{ir}}(S) = \mathrm{Vol}^n(C)$. Therefore, by (6.1), we get that $\|I_\# S\|^{\mathrm{ir}} = I_\#\|S\|^{\mathrm{ir}}$ and so $I(\operatorname{spt} S) \subset \operatorname{spt} I_\# S$. Thus, since $\operatorname{spt} I_\# S = (I \circ \Phi)(C)$ and $\partial S = \Phi_\#[\![\partial C]\!]$, it follows that $S = \Phi_\#[\![C]\!]$. But $S = f_\# T$ and so

$$\mathrm{Vol}^n(C) = \mathbf{M}(f_\# T) \le \mathbf{M}(T) \le \mathbf{M}^{\mathrm{ir}}(T) \le \mathrm{Vol}^n(C).$$

By Lemma 3.1, it follows that $f(X) \subset \Phi(C)$. Hence, we can apply Theorem 1.2 once again and we conclude that $f : X \to \Phi(C)$ is an isometry. □

7. Intrinsic flat convergence and the Perales question

7.1. Intrinsic flat convergence. Given $T \in \mathbf{I}_n(X)$, let

$$\mathcal{F}_X(T) = \inf\{\mathbf{M}(U) + \mathbf{M}(V) : T = U + \partial V,\ U \in \mathbf{I}_n(X),\ V \in \mathbf{I}_{n+1}(X)\}$$

denote the *flat norm* of T. If the ambient space X is clear from the context we often write $\mathcal{F}(T)$ instead of $\mathcal{F}_X(T)$. We say that $T_i \in \mathbf{I}_n(X)$ *flat converges* to $T \in \mathbf{I}_n(X)$ if $\mathcal{F}(T - T_i) \to 0$ as $i \to \infty$, and that T_i *converges weakly* to T if $T_i(h, \pi_1, \dots, \pi_n) \to T(h, \pi_1, \dots, \pi_n)$ as $i \to \infty$ for every $(h, \pi_1, \dots, \pi_n) \in \mathcal{D}^n(X)$. It is readily verified that flat convergence implies weak convergence. Conversely, if X admits local coning inequalities and $\sup \mathbf{N}(T_i) < \infty$, then weak convergence also implies flat convergence (see [44]).

In [43], Sormani and Wenger introduced a notion of flat convergence for currents which are not necessarily defined on the same metric space. The *intrinsic flat distance* between two integral current spaces X_1, X_2 of the same dimension is defined as

$$d_{\mathcal{F}}(X_1, X_2) = \inf \mathcal{F}_Z(\phi_{1\#} T_1 - \phi_{2\#} T_2),$$

where the infimum is taken over all complete metric spaces Z and all isometric embeddings ϕ_i of X_i into Z. We say that a sequence X_i of integral current spaces *converges in the intrinsic flat sense* to an integral current space X if $d_{\mathcal{F}}(X_i, X) \to 0$ as $i \to \infty$. The following Arzelà–Ascoli-type theorem is due to Sormani.

Theorem 7.1 (see [41, Theorem 6.1]). *Suppose X_i are integral current spaces converging to the integral current space X in the intrinsic flat sense. Further, suppose $f_i : X_i \to Y$ are L-Lipschitz maps to a compact metric space Y. Then there exist an L-Lipschitz map $f : X \to Y$ and a subsequence, also denoted by f_i, and isometric embeddings $\phi_i : X_i \to Z$ and $\phi : X \to Z$ into a complete separable metric space Z such that*

(a) $\mathcal{F}_Z(\phi_{i\#} T_i - \phi_\# T) \to 0$ *as* $i \to \infty$, *and*

(b) $f_i(x_i) \to f(x)$ *whenever* $x_i \in X_i$ *and* $x \in X$ *satisfy* $\phi_i(x_i) \to \phi(x)$.

The map $f : X \to Y$ will be called a *Sormani limit* of the subsequence f_i. We note that for every $x \in X$ there is always such a sequence $x_i \in X_i$ as in Theorem 7.1 (b) above. In particular, if a subsequence has a Sormani limit, then this limit is unique. This follows directly from the next lemma.

Lemma 7.2. *Let Z be a complete metric space and $T_i \in \mathbf{I}_n(Z)$ a sequence flat converging to $T \in \mathbf{I}_n(Z)$. Then for every $z \in \operatorname{spt} T$ there is a sequence $z_i \in \operatorname{set} T_i$ such that $z_i \to z$ as $i \to \infty$.*

Proof. The following argument is due to Wenger [45, Proposition 2.2]. Let $z \in \operatorname{spt} T$ and $\varepsilon > 0$. Using [4, Proposition 2.7], one can show there exists $(h, \pi_1, \dots, \pi_n) \in \mathcal{D}^n(X)$ such that $T(h, \pi_1, \dots, \pi_n) \neq 0$ and $\operatorname{spt} h \subset B(x, \varepsilon)$. As T_i flat converges to T, and therefore in particular converges weakly to T, for any i that is sufficiently large, one has $T_i(h, \pi_1, \dots, \pi_n) \neq 0$. It now follows from Definition 2.1 (3) that for every such i there is $z_i \in \operatorname{spt} T_i \cap \operatorname{spt} h$. Since $\varepsilon > 0$ was arbitrary and set T_i is dense in $\operatorname{spt} T_i$, a sequence $z_i \in \operatorname{set} T_i$ such that z_i converges to z is now easily constructed. □

It turns out that the convergence as in Theorem 7.1 is compatible with push-forwards of currents.

Lemma 7.3. *Let X_i be a sequence of integral current spaces converging in the intrinsic flat sense to an integral current space X. Suppose further that $f_i : X_i \to \mathbb{R}^N$ are uniformly bounded L-Lipschitz maps and let $f : X \to \mathbb{R}^N$ be the Sormani limit of some subsequence f_i. Then $f_{i\#}T_i$ flat converges to $f_\#T$.*

Proof. Since $f_{i\#}T_i = 0$ and $f_\#T = 0$ whenever $N < n$, we may assume in the following that $n \leq N$. Due to Theorem 7.1, there exist a separable complete metric space Z and isometric embeddings $\phi_i : X_i \to Z$ and $\phi : X \to Z$ such that $\phi_{i\#}T_i$ flat converges to $\phi_\#T$ and f_i converges to f in the sense of Theorem 7.1 (b). To simplify the notation, we write $S_i = \phi_{i\#}T_i$ and $S = \phi_\#T$.

By McShane's extension theorem [11, Theorem 1.27] there exist $(\sqrt{N}L)$-Lipschitz maps $F_i : Z \to \mathbb{R}^N$ and $F : Z \to \mathbb{R}^N$ such that $f_i = F_i \circ \phi_i$ for all $i \in \mathbb{N}$ and $f = F \circ \phi$. In particular, $F_{i\#}S_i = f_{i\#}T_i$ and $F_\#S = f_\#T$. Thus, using the triangle inequality, we get

$$\mathcal{F}(f_{i\#}T_i - f_\#T) \leq \mathcal{F}(F_{i\#}S_i - F_{i\#}S) + \mathcal{F}(F_{i\#}S - F_\#S) \tag{7.1}$$

for all $i \in \mathbb{N}$. Since the maps F_i are uniformly Lipschitz and S_i flat converges to S, it follows from (2.5) that $\mathcal{F}(F_{i\#}S_i - F_{i\#}S) \to 0$ as $i \to \infty$.

Next, we show that the other term on the right-hand side of (7.1) also converges to zero. Notice that $F_i \circ \phi$ converges pointwise to $F \circ \phi$. Indeed, let $x \in X$ and let $x_i \in X_i$ be a sequence such that $\phi_i(x_i)$ converges to $\phi(x)$. The existence of such a sequence is guaranteed by Lemma 7.2. Using that $f_i(x_i) = F_i(\phi_i(x_i))$, we get

$$d(f(x), F_i(\phi(x))) \leq d(f(x), f_i(x_i)) + d(F_i(z_i), F_i(z)), \tag{7.2}$$

where $z_i = \phi_i(x_i)$ and $z = \phi(x)$. As f is the Sormani limit of the f_i, we have that $f_i(x_i)$ converges to $f(x)$. Moreover, since $d(F_i(z_i), F_i(z)) \leq L d(z_i, z)$ and $z_i \to z$ as $i \to \infty$, it follows from (7.2) that $F_i(\phi(x))$ converges to $f(x)$, as desired.

Now, since $F_i \circ \phi$ converges pointwise to $F \circ \phi$, by using Definition 2.1 (1), (3) and Lebesgue's dominated convergence theorem, it is easy to check that $F_{i\#}S$ converges weakly to $F_\#S$. Since the sequence is uniformly $\mathbf{N}$-bounded and $\mathbb{R}^N$ admits coning inequalities for $\mathbf{I}_i(\mathbb{R}^N)$ for $i = 1, \dots, n$, it follows that $F_{i\#}S$ flat converges to $F_\#S$. Hence, because of (7.1), $f_{i\#}T_i$ flat converges to $f_\#T$, as desired. □

7.2. Perales question. The following example shows that the Perales question stated in the introduction has a negative answer in general. The argument uses the following observation,

which follows directly from Lemma 7.2: If $T_i \in \mathbf{I}_n(Z)$ flat converges to $T \in \mathbf{I}_n(Z)$ and set T_i Gromov–Hausdorff converges to Y, then spt T admits an isometric embedding into Y.

Example 7.4. Fix $L \in (0, 2)$ and let M_ε denote the flat 2-dimensional Riemannian manifold with boundary depicted in Figure 1. Notice that M_ε admits a natural decomposition into three pieces, namely $M_\varepsilon = B^2_+ \cup R_\varepsilon \cup B^2_-$, where $B^2_\pm$ are isometric to the half-ball $B^2 \cap \{y \geq 0\}$ and R_ε is contained in a rectangle of length 2 and width ε. By construction, $\mathrm{Vol}^2(M_\varepsilon) \to \mathrm{Vol}^2(B^2)$ as $\varepsilon \to 0$. Let $f_\varepsilon : M_\varepsilon \to B^2$ denote the map which collapses R_ε to the x-axis and is the identity on the half-balls $B^2_\pm$. Clearly, f_ε is 1-Lipschitz. Moreover, using that every cycle in $\mathbf{I}_1(S^1)$ is of the form $m \cdot [\![S^1]\!]$ for some $m \in \mathbb{Z}$, it is easy to check that $f_{\varepsilon\#}[\![\partial M_\varepsilon]\!] = [\![S^1]\!]$ for every $\varepsilon > 0$. Let $U \subset \mathbb{R}^2$ denote a slit unit disk where the slit has length L. The Gromov-Hausdorff limit of (M_ε) is equal to the metric completion of U equipped with the intrinsic metric. In particular, (M_ε) thus does not converge to B^2 in the intrinsic flat sense.

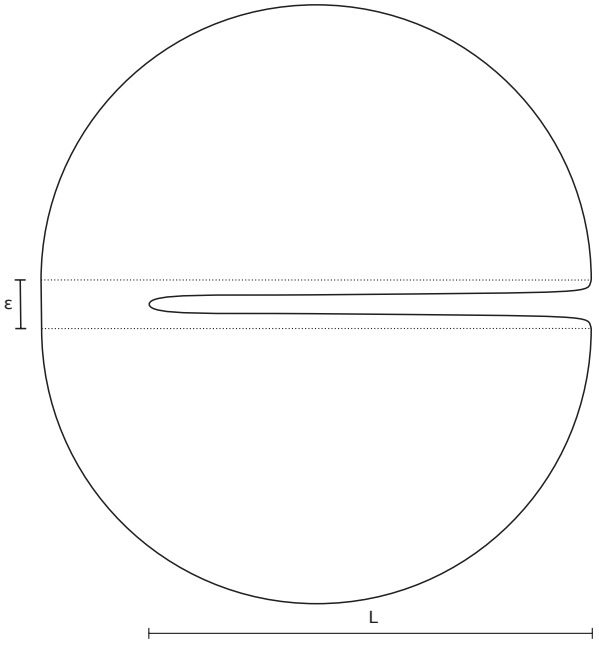

Figure 1. "Flat-football" counterexample to Question 1.4.

Question 1.4 has a positive answer if, in addition, one assumes a suitable bound for the limit of the masses of the boundary currents. We now prove Corollary 1.5, whose statement can be found in the introduction.

Proof of Corollary 1.5. Note that Lemma 7.3 implies that $f_{i\#}T_i$ flat converges to $f_\# T$. Since $\partial(f_{i\#}T_i) = f_{i\#}(\partial T_i)$ flat converges to $[\![\partial C]\!]$, it follows that $\partial(f_\# T) = [\![\partial C]\!]$. Moreover, by the lower semi-continuity of mass (see [4, p. 19]), we have that

$$\mathbf{M}(T) \leq \liminf_{i\to\infty} \mathbf{M}(T_i) \leq \mathrm{Vol}^n(C)$$

and analogously $\mathbf{M}(\partial T) \leq \mathrm{Vol}^{n-1}(\partial C)$. Therefore, by invoking Theorem 1.2, we find that $f : X \to C$ is an isometry. □

Using this corollary, the following result is a direct consequence of the Wenger compactness theorem.

Corollary 7.5. *Let $C \subset \mathbb{R}^n$ be a convex body and (X_i) a sequence of uniformly bounded integral current spaces. Suppose $f_i : X_i \to \mathbb{R}^N$ are 1-Lipschitz maps such that $f_{i\#}(\partial T_i)$ flat converges to $[\![\partial C]\!]$. If*

$$\lim_{i\to\infty} \mathbf{M}(\partial T_i) \leq \mathrm{Vol}^{n-1}(\partial C), \quad \lim_{i\to\infty} \mathbf{M}(T_i) \leq \mathrm{Vol}^n(C),$$

then (X_i) converges in the intrinsic flat sense to C.

Proof. By [45, Theorem 1.2] there exists a subsequence, also denoted by X_i, that converges in the intrinsic flat sense to an integral current space X. Further, notice that since the X_i are uniformly bounded, the f_i take values in a compact set $K \subset \mathbb{R}^N$. Let $f : X \to K$ denote the Sormani limit of a subsequence of (f_i). The existence of such a limit is guaranteed by Theorem 7.1. In particular, f is 1-Lipschitz. Now, Corollary 1.5 tells us that f is an isometry $X \to C$. Since the argument above can be applied to any subsequence of (X_i), it follows that (X_i) converges to C in the intrinsic flat sense, as desired. □

8. Counterexamples and open questions

Theorem 1.2 and Corollary 1.3 show that convex bodies in $\mathbb{R}^n$ and the round sphere S^n have the Lipschitz-volume rigidity property among all integral current spaces. This naturally leads to the question which other metric spaces Y are Lipschitz-volume rigid among integral current spaces. A simple way to come up with non-Lipschitz volume rigid spaces is to consider non-intrinsic metrics. In particular, every compact Lipschitz submanifold $Y \subset \mathbb{R}^N$, which is not a convex subset, does not enjoy the Lipschitz-volume rigidity property when it is endowed with the Euclidean subspace metric. In this case the identity map $Y^{\mathrm{int}} \to Y^{\mathrm{euc}}$ is 1-Lipschitz, volume and boundary volume preserving, but not an isometry.

Note that in situations where Federer's constancy theorem is not valid the boundary push-forward condition is not sufficient (e.g., for non-trivial spaces Y with $\partial Y = 0$). In the following we refer as *Lipschitz-volume rigidity* of an integral current space $Y = (Y, S)$ to the following property: *Suppose X is an integral current space of the same dimension as Y and $f : X \to Y$ is a 1-Lipschitz map such that $f_\# T = S$. If $\mathbf{M}(\partial T) \leq \mathbf{M}(\partial S)$ and $\mathbf{M}(T) \leq \mathbf{M}(S)$, then f is an isometry.*

Question 8.1. Let $Y \subset \mathbb{R}^N$ be a compact orientable connected n-dimensional Lipschitz manifold. Does Y have Lipschitz-volume rigidity among integral current spaces when endowed with its intrinsic metric?

It is not hard to modify the proof of Theorem 1.2 to deduce an affirmative answer when Y is smooth and $n = N$, and hence in particular Y is flat. On the other hand, [18, Example 4.4] suggests that the answer to Question 8.1 is negative for general Lipschitz submanifolds. We suspect that the answer is affirmative when Y is smooth but our proof does not seem amenable for such a generalization in a straightforward way, since it relies on Fubini-type decompositions of $\mathrm{Vol}^n(Y)$.

The situation becomes even more complicated when one allows for non-infinitesimally Euclidean integral current spaces. It turns out that the Lipschitz-volume rigidity of Y can fail even when Y is a convex body in a finite-dimensional normed space. For example, let I^2 be the convex body $[0,1]^2 \subset \mathbb{R}^2$ endowed with the Euclidean metric and let $Y = I^2_\infty$ be the same set endowed with the maximum norm. Then the identity map $f : I^2 \to Y$ is 1-Lipschitz,

$$\mathbf{M}(\llbracket I^2 \rrbracket) = \mu^{m*}(I^2) = 1 = \mu^{m*}(I^2_\infty) = \mathbf{M}(\llbracket Y \rrbracket)$$

and

$$\mathbf{M}(\llbracket \partial I^2 \rrbracket) = \ell(\partial I^2) = 4 = \ell(\partial I^2_\infty) = \mathbf{M}(\llbracket \partial Y \rrbracket),$$

but f is not an isometry.

As discussed in Section 2.3 there is some ambiguity concerning volume as soon as non-Euclidean tangent spaces come into play. The preceding counterexample stems from the observations that $\mathbf{M}$ corresponds to the mass$*$ Jacobian Jac^{m*} in the sense of (2.7) and that $\mathrm{Jac}^{m*}(\sigma)$ is not strictly monotone in σ. Hence, another interesting question would be to investigate whether convex bodies in finite-dimensional normed spaces are Lipschitz-volume rigid among integral current spaces with respect to the Busemann mass $\mathbf{M}^{\mathrm{b}}$ or the Holmes–Thompson mass $\mathbf{M}^{\mathrm{ht}}$.

Concerning Theorem 1.1 we were informed by Roger Züst that Lemma 6.1 and hence the lower bound (1.1) generalizes to convex bodies in finite-dimensional normed spaces. Indeed, for a given Finsler mass $\mathbf{M}^\bullet$ it seems natural to expect that validity of this inequality for all finite-dimensional normed spaces is equivalent to a property that is often called quasi-convexity or semi-ellipticity over $\mathbb{Z}$ in the literature, see [3, 29, 36]. The counterexample above however illustrates that in the setting of normed spaces one can only hope for rigidity when the mass functional is strictly monotone, as is the case for $\mathbf{M}^{\mathrm{b}}$ or $\mathbf{M}^{\mathrm{ht}}$ but not for $\mathbf{M}$ or $\mathbf{M}^{\mathrm{ir}}$.

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