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Jet spaces over Carnot groups

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Abstract. Jet spaces over $\mathbb{R}^n$ have been shown to have a canonical structure of stratified Lie groups (also known as Carnot groups). We construct jet spaces over stratified Lie groups adapted to horizontal differentiation and show that these jet spaces are themselves stratified Lie groups. Furthermore, we show that these jet spaces support a prolongation theory for contact maps, and in particular, a Bäcklund type theorem holds. A byproduct of these results is an embedding theorem that shows that every stratified Lie group of step $s+1$ can be embedded in a jet space over a stratified Lie group of step s.

1. Introduction

1.1. Overview

Classical jet spaces are vector bundles $\mathrm{J}^m(\mathbb{R}^n;\mathbb{R}^\ell) \to \mathbb{R}^n$ endowed with a horizontal bundle $\mathcal{H}^m \subset T\,\mathrm{J}^m(\mathbb{R}^n;\mathbb{R}^\ell)$, such that jets of functions $\mathbb{R}^n \to \mathbb{R}^\ell$ are exactly those sections that are tangent to $\mathcal{H}^m$. See [14] for further reference.

It is well known that these jet spaces $\mathrm{J}^m(\mathbb{R}^n;\mathbb{R}^\ell)$ carry a structure of stratified Lie group for which $\mathcal{H}^m$ is left-invariant, see [16]. *Stratified Lie groups* are also called *Carnot groups*, and we use the two terms as synonyms.

In Proposition 6.5.1 of [8], it is shown that a stratified Lie group of step 2 can be embedded into a classical jet space $\mathrm{J}^1(\mathbb{R}^n;\mathbb{R}^\ell)$ for a suitable choice of n and ℓ. By "embedding" we mean that there is a strata-preserving injective homomorphism of Lie groups. In contrast, stratified Lie groups of step larger than 2 do not always embed in a classical jet space, see Remark 7.1. Proceeding heuristically leads to considering the substitution of $\mathbb{R}^n$ in $\mathrm{J}^m(\mathbb{R}^n;\mathbb{R}^\ell)$ with a stratified Lie group.

In this paper, we consider jets adapted to horizontal differentiation of smooth functions f from a stratified group G into a vector space W. The stratification of G induces an "intrinsic" notion of degree both for polynomials and for left-invariant differential operators. Following [6], it is natural to reorganize the derivatives of f using this intrinsic degree. Equivalently, one can reorganize the homogeneous polynomials of the Taylor expansion of f. See also [9, 10].

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Such a reorganization allows us to construct vector bundles $\mathtt{J}^m(G;W) \to G$ endowed with a horizontal bundle $\mathcal{H}^m \subset T\,\mathtt{J}^m(G;W)$ that characterizes sections defined as "horizontal" jets of functions $G \to W$. We call such spaces *jet spaces over* G and we prove two fundamental facts of jet spaces, that is, a prolongation and a de-prolongation theorem.

Finally, we show that every stratified Lie group of step $s+1$ can be embedded into some $\mathtt{J}^s(G';W)$ for a stratified Lie group G' of step s.

1.2. Detailed description of results

For a vector space W and a simply connected Lie group G with stratified Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^s V_i$, bracket generated by V_1, our starting point is to consider the space $\mathtt{HD}^k(\mathfrak{g};W)$ of k-multi-linear maps A on V_1 given by k-th order horizontal differentiation of smooth functions $f\colon G \to W$ at the neutral element $e \in G$. More precisely, if $v_1,\dots,v_k \in V_1$ and $\tilde{v}_1,\dots,\tilde{v}_k$ denote the corresponding left invariant vector fields on G, then

$$A(v_1,\dots,v_k) = \tilde{v}_k \dots \tilde{v}_1 f(e). \tag{1.1}$$

We then define $\mathtt{HD}^{\le m}(\mathfrak{g};W) = \bigoplus_{k=0}^m \mathtt{HD}^k(\mathfrak{g};W)$, where we set $\mathtt{HD}^0(\mathfrak{g};W) = W$.

If G is abelian, i.e., $G = \mathbb{R}^n$ for some n, then $\mathtt{HD}^k(\mathbb{R}^n;W)$ is the space of symmetric k-multilinear maps from $\mathbb{R}^n$ to W, and all the constructions and results of this paper are standard. If G is not abelian, we do not have such a clear characterization of $\mathtt{HD}^k(\mathfrak{g};W)$, but we do have an algorithm to construct a basis of it, see Remark 4.8.

The jet space $\mathtt{J}^m(G;W)$ is the simply connected Lie group with Lie algebra

$$\mathfrak{j}^m(\mathfrak{g};W) = \mathfrak{g} \ltimes \mathtt{HD}^{\le m}(\mathfrak{g};W),$$

where the semi-direct product is given by the representation of $\mathfrak{g}$ over $\mathtt{HD}^{\le m}(\mathfrak{g};W)$ that is induced by *right-contractions*. If $v \in V_1$ and $A \in \mathtt{HD}^k(\mathfrak{g};W)$, we define the right contraction of A by v as the $(k-1)$-multi-linear map $v \neg A \in \mathtt{HD}^{k-1}(\mathfrak{g};W)$,

$$v \neg A(v_1,\dots,v_{k-1}) = A(v_1,\dots,v_{k-1},v).$$

In Proposition 3.3, we show that this operation $v \mapsto v\neg$ extends to a Lie algebra anti-morphism $\mathfrak{g} \to \mathtt{End}(\mathtt{HD}^{\le m}(\mathfrak{g};W))$. In the special case $m=0$, we have $\mathfrak{j}^0(\mathfrak{g};W) = \mathfrak{g} \times W$ and $\mathtt{J}^0(G;W) = G \times W$.

The Lie algebra $\mathfrak{j}^m(\mathfrak{g};W)$ turns out to be again stratified of step $\max\{s, m+1\}$, with layers of the form

$$\begin{aligned}
\mathfrak{j}^m(\mathfrak{g};W)_1 &= V_1 \oplus \mathtt{HD}^m(\mathfrak{g};W),\\
\mathfrak{j}^m(\mathfrak{g};W)_2 &= V_2 \oplus \mathtt{HD}^{m-1}(\mathfrak{g};W),\\
\mathfrak{j}^m(\mathfrak{g};W)_3 &= V_3 \oplus \mathtt{HD}^{m-2}(\mathfrak{g};W),\\
&\vdots
\end{aligned}$$

The contact structure $\mathcal{H}^m$ of the jet space $\mathtt{J}^m(G;W)$ is the left invariant distribution defined by the first layer $\mathfrak{j}^m(\mathfrak{g};W)_1$. A smooth function $f\colon G \to W$ defines a section $\mathtt{J}^m f\colon G \to \mathtt{J}^m(G;W)$ by evaluating at points of G the derivatives of f like in (1.1), and we call $\mathtt{J}^m f$ the (*horizontal*) *jet of* f. We show in Proposition 3.8 that a section $\gamma: G \to \mathtt{J}^m(G; W)$ is the jet of a function if and only if $d\gamma$ maps $\mathcal{H}_G$ to $\mathcal{H}^m$, where $\mathcal{H}_G$ is the left-invariant distribution on G defined by V_1.

Once the definition and the structure of $\mathtt{J}^m(G; W)$ are set, we prove three main results. First of all, we demonstrate that contact maps $\mathtt{J}^m(G; W) \to \mathtt{J}^m(G; W)$ are subject to a prolongation theory similar to the classical case and that a Bäcklund type theorem holds.

Note that our "concrete" construction of $\mathtt{J}^m(G; W)$ is diffeomorphic to the manifold $G \times \mathtt{HD}^{\leq m}(\mathfrak{g}; W)$, see Section 3.4. This identification allows us to define a map $\pi_m: \mathtt{J}^{m+1}(G; W) \to \mathtt{J}^m(G; W)$ that is the projection along $\mathtt{HD}^{m+1}(\mathfrak{g}; W)$, see Section 5.1. Note that this map π_m is not a group morphism, however, it is a smooth submersion that maps the horizontal bundle of $\mathtt{J}^{m+1}(G; W)$ to the horizontal bundle of $\mathtt{J}^m(G; W)$.

Theorem A (Prolongation theorem). *Let G be a Carnot group and let W be a vector space. Suppose $m \geq 0$, $\Omega \subset \mathtt{J}^m(G; W)$ is open, and that $F: \Omega \to \mathtt{J}^m(G; W)$ is a contact map. If $\pi_m: \mathtt{J}^{m+1}(G; W) \to \mathtt{J}^m(G; W)$ denotes the projection along $\mathtt{HD}^{m+1}(\mathfrak{g}; W)$, then there is an open set $\hat{\Omega} \subset \mathtt{J}^{m+1}(G; W)$, determined by Theorem 5.2, and a unique contact map $\hat{F}: \hat{\Omega} \to \mathtt{J}^{m+1}(G; W)$ such that*

$$\pi_m \circ \hat{F} = F \circ \pi_m.$$

A more precise statement is Theorem 5.2, where we give a precise definition of the open set $\hat{\Omega}$. The map $\hat{F}$ in Theorem A is called *prolongation of* F, see Section 5.3.

It should be noted that Theorem A loses its significance if the set $\hat{\Omega}$ is not specified, as considering the empty set would render the statement trivially true. In general, the exact size of the set $\hat{\Omega}$ stated in Theorem 5.2 remains uncertain to us. For instance, when F is constant, $\hat{\Omega}$ is empty. However, if F is close enough to the identity map, $\hat{\Omega}$ is non-empty. Additionally, when F is the prolongation of a contact map, $\hat{\Omega}$ corresponds to $\pi_m^{-1}(\Omega)$, as explained in Remark 5.4. As a consequence of the following Theorem B, this is always the case when $m \geq 2$. We do not know if it can be proven that $\pi_m(\hat{\Omega}) = \Omega$ also when $m = 1$ and F is a diffeomorphism.

Theorem B (De-prolongation theorem). *Let G be a Carnot group and let W be a vector space. If $m \geq 2$, then every contact diffeomorphism $\mathtt{J}^m(G; W) \to \mathtt{J}^m(G; W)$ is the prolongation of a contact diffeomorphism $\mathtt{J}^1(G; W) \to \mathtt{J}^1(G; W)$.*

Moreover, suppose that one of the following conditions is satisfied:

(A) $\dim(W) > 1$, *or*

(B) *for every $v \in V_1 \setminus \{0\}$, there is $v' \in V_1$ with $[v, v'] \neq 0$.*

Then every contact diffeomorphism $\mathtt{J}^1(G; W) \to \mathtt{J}^1(G; W)$ is the prolongation of a contact diffeomorphism $\mathtt{J}^0(G; W) \to \mathtt{J}^0(G; W)$.

We can also state Theorem B for contact diffeomorphisms that are defined on domains in $\mathtt{J}^m(G; W)$, in which case the de-prolongation will only be local a priori. A precise result is stated in Theorems 6.1 and 6.2. We show in Remark 6.3 that the second part of Theorem B is sharp.

Finally, we prove that stratified groups embed into jet spaces. It has already been proved by Montgomery in [8], Section 6.5.1, that every manifold of dimension n endowed with a distribution of rank k is *locally smoothly embeddable* into $\mathtt{J}^1(\mathbb{R}^k; \mathbb{R}^{n-k})$. Our result differs from Montgomery's as we seek a *global embedding as stratified Lie groups*, that is, we want to reconstruct a stratified Lie group G as a stratified subgroup of some jet space. This is simply not possible with standard jet spaces, as we explain in Remark 7.1. Indeed, if a stratified group G with Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^{s} V_j$ can be embedded as stratified group in a classical jet space $\mathrm{J}^m(\mathbb{R}^n; W)$, then we must have $[V_i, V_j] = 0$ for all $i, j > 1$.

Notice that, of course, G embeds into $\mathrm{J}^m(G; W)$, by construction. However, we prove that we can reconstruct G inside the jet space of a stratified group with lower step.

Theorem C (Embedding into jet spaces). *Let G be a stratified group of step $s+1$. Then there exists a stratified group G' of step s and a vector space W such that G embeds as a stratified group in $\mathrm{J}^s(G'; W)$.*

In Theorem C, if the stratification of the Lie algebra of G is $\bigoplus_{j=1}^{s+1}$, then G' is the quotient $G/\exp(V_{s+1})$ and W is the vector space V_{s+1}. See Section 7 for details.

1.3. Further developments

We conclude the introduction with a list of further developments that a curious reader might be interested to study.

Our construction of horizontal jets for functions from a stratified Lie group G into a vector space W suggests a similar construction for horizontal jets of contact functions valued in a stratified Lie group W, using the so-called Pansu derivatives.

The fact that the first layer V_1 in $\mathfrak{g}$ is bracket generating plays a central role, because it allows to write all derivatives in terms of horizontal derivatives. However, the fact that $\mathfrak{g}$ is stratified seems to be only a technical aid. This suggests that one could consider polarized Lie groups, that is, Lie groups whose Lie algebra has a chosen subspace that is bracket generating. More specifically, one could carry out a similar study of horizontal jets from a polarized Lie group into a vector space, or horizontal jets of contact maps between polarized Lie groups.

Jet spaces are used most commonly in the geometric study of PDEs. Although our research was mainly driven by embedding problems, we expect that in comparison with standard jets spaces, the jet spaces constructed here should be better adapted to the study of PDEs involving horizontal vector fields. For instance, for the study of symmetries and equivalences (as in [2, 11, 12]) of sub-Laplacians in Carnot groups.

In this paper, we have not considered metric geometries on the spaces involved. However, once a scalar product is chosen on V_1, G becomes a sub-Riemannian Carnot group and there is a canonical choice of sub-Riemannian structure for $\mathrm{J}^m(G; W)$. A first question is whether the embedding we gave in Theorem C is geodesic, that is, distance preserving. One might also try a description of geodesics in $\mathrm{J}^m(G; W)$, as in [4, 5].

In [7, 13, 18], the fact that certain Carnot groups are jet spaces has been used to study Lipschitz extensions and Lipschitz homotopy groups. One could wonder if a similar strategy can be used with $\mathrm{J}^m(G; W)$.

1.4. Structure of the paper

In the preliminary Section 2, we introduce elementary facts, notation and conventions we will use throughout the paper. The definition of jet spaces and their Lie algebras over stratified Lie groups are contained in Section 3. Section 4 is devoted to a second definition of jet spaces using homogeneous polynomials. We prove the prolongation Theorem A in Section 5 and the de-prolongation Theorem B in Section 6. Section 7 is devoted to the proof of the embedding Theorem C. Finally, we present one example in Section 8.

2. Preliminaries

In this section, we introduce stratified Lie groups and we fix basic notation, conventions and a few known results.

2.1. Notation choices

If V and W are vector spaces, we denote by $\mathtt{Lin}^k(V;W)$ the vector space of all k-multilinear maps from V to W. In particular, $\mathtt{End}(V) = \mathtt{Lin}(V;V)$ is the Lie algebra of all linear maps $V \to V$ with Lie brackets

$$[A,B] = AB - BA. \tag{2.1}$$

If $E \to G$ is a vector bundle over a manifold G, we denote by $\Gamma(E)$ the space of smooth sections of E. If $X \in \Gamma(E)$ and $p \in M$, we write $X(p)$ or X_p for the evaluation of X at p. If G, M are manifolds, we denote by $C^\infty(G;M)$ the space of smooth maps $G \to M$. If G is a Lie group, we denote its identity element by e_G or e, and we identify its Lie algebra with the tangent space at the identity element; if $p \in G$, then $L_p\colon G \to G$ is the left translation $L_p(x) = px$. If $\mathfrak{g}$ is a Lie algebra, we denote by $\mathtt{Der}(\mathfrak{g})$ the space of derivations of $\mathfrak{g}$ and by $\mathtt{Aut}(\mathfrak{g})$ the group of Lie algebra automorphisms of $\mathfrak{g}$. If G is a Lie group, we denote by $\mathtt{Aut}(G)$ the group of Lie group automorphisms of G.

If V is a vector space, let $\mathcal{T}(V) = \bigoplus_{j=1}^{\infty} V^{\otimes j}$ be the tensor algebra of V. We define the tensor spaces $\mathcal{T}^m(V) = V^{\otimes m}$ and $\mathcal{T}^{\le m}(V) = \bigoplus_{k=0}^{m} \mathcal{T}^k(V)$. It is clear that we have $\mathcal{T}^m(V)^* = \mathcal{T}^m(V^*)$.

2.2. Vector fields

Let G be a smooth manifold and W a finite-dimensional real vector space. We consider the differential of a smooth map $f\colon G \to W$ as a map $\mathrm{d}f\colon TG \to W$. More precisely, if $v \in T_pG$, then $\mathrm{d}f(v) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(\gamma(t)) \in W$ for any $\gamma\colon \mathbb{R} \to G$ smooth with $\gamma(0) = p$ and $\gamma'(0) = v$.

Vector fields $\tilde{v} \in \Gamma(TG)$ are differential operators $C^\infty(G;W) \to C^\infty(G;W)$: if $f\colon G \to W$ is smooth, then $\tilde{v}f\colon G \to W$ is the map $\tilde{v}f(p) = \mathrm{d}f(\tilde{v}(p))$. The Lie bracket of vector fields $\tilde{v}, \tilde{w} \in \Gamma(TG)$ is defined as the commutator in the algebra of differential operators, i.e.,

$$[\tilde{v}, \tilde{w}] = \tilde{v}\tilde{w} - \tilde{w}\tilde{v}. \tag{2.2}$$

2.3. Vector fields on a Lie group

Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $X \in C^\infty(G;\mathfrak{g})$, i.e., X is a smooth function $G \to \mathfrak{g}$, we denote by $\tilde{X}$ the corresponding vector field on G defined by $\tilde{X}(p) =$

$\mathrm{d}L_p[X(p)]$. The vector field $\tilde{X}$ is characterized by means of the Maurer–Cartan form ω_G as the only vector field $\tilde{X} \in \Gamma(TG)$ such that $\omega_G(\tilde{X}(p)) = X(p)$ for all $p \in G$. The (left) Maurer–Cartan form is the vector-valued 1-form $\omega_G\colon TG \to \mathfrak{g}$ such that $v = DL_p[\omega_G(v)]$ for all $p \in G$ and $v \in T_pG$. We define Lie brackets on $C^\infty(G;\mathfrak{g})$ as follows: if $X, Y\colon G \to \mathfrak{g}$, then we define $[X, Y](p) = \mathrm{d}L_p^{-1}[\tilde{X}, \tilde{Y}](p) \in \mathfrak{g}$. Notice that left-invariant vector fields $\tilde{X}\colon G \to \Gamma(TG)$ correspond to constant functions $G \to \mathfrak{g}$, and that if $X, Y\colon G \to \mathfrak{g}$ are constant, then $[X, Y](p) = [X(p), Y(p)]$ for all p.

If $f, g \in C^\infty(G)$ and $X, Y\colon G \to \mathfrak{g}$ are constant, then

$$[fX, gY] = f \cdot (\tilde{X}g) \cdot Y - g \cdot (\tilde{Y}f) \cdot X + fg[X, Y]. \tag{2.3}$$

Sometimes we will use right-invariant vector fields. For $x \in \mathfrak{g}$, we denote by $x^\dagger$ the right-invariant vector field on G with $x^\dagger(e_G) = x$.

2.4. Derivatives along vectors and linear vector fields

Let V be a vector space. As vector spaces are instances of abelian Lie groups, we are consistent with the notation introduced in Section 2.3 by setting

$$\tilde{v}f(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(p + tv)$$

for any vector $v \in V$.

We will need the following elementary observation. Notice that a linear map $X \in \mathtt{End}(V)$ defines a vector field $\tilde{X}(p) = Xp$. So, if $X, Y \in \mathtt{End}(V)$, then we have two ways of taking Lie brackets: as linear maps in (2.1), $[X, Y]_{\mathtt{End}(V)} = XY - YX \in \mathtt{End}(V)$ and as vector fields in (2.2), $[\tilde{X}, \tilde{Y}]_{\Gamma(TV)} = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X} \in \Gamma(TV)$.

Lemma 2.1. *If $X, Y \in \mathtt{End}(V)$, then $[\tilde{X}, \tilde{Y}]_{\Gamma(TV)} = -\widetilde{[X, Y]}_{\mathtt{End}(V)}$.*

Proof. Let $f\colon V \to \mathbb{R}$ be a smooth function and $p \in V$. Notice that

$$\begin{aligned}
\tilde{X}\tilde{Y}f(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (\tilde{Y}f)(p + sXp) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f\big((p + sXp) + tY(p + sXp)\big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f(p + sXp + tYp + stYXp) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(f(p + sXp + tYp) + f(p + tYp + stYXp)\big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(f(p + sXp + tYp) + f(p + tYp) + f(p + stYXp)\big) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(f(p + sXp + tYp) + f(p + tYp) + sf(p + tYXp)\big),
\end{aligned}$$

where we used the formula $\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{r=0} g(r, r) = \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=0} (g(r, 0) + g(0, r))$. Similarly,

$$\tilde{Y}\tilde{X}f(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(f(p + sXp + tYp) + f(p + tYp) + sf(p + tXYp)\big).$$

Hence,

$$[\tilde{X},\tilde{Y}]f(p) = \tilde{X}\tilde{Y}f(p) - \tilde{Y}\tilde{X}f(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(f(p+tYXp) - f(p+tXYp)\big)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(p+t(YXp-XYp)) = (\widetilde{-[X,Y]p})f(p). \qquad \blacksquare$$

2.5. Vector-valued differential forms

Let G be a smooth manifold and W a vector space. We define *W-valued 1-forms* as

$$\Omega^1(G;W) := \Gamma(\mathtt{Lin}(TG;W)),$$

where $\mathtt{Lin}(TG;W)$ is the vector bundle over G whose fiber at $p \in G$ is the space $\mathtt{Lin}(T_pG;W)$ of linear maps from T_pG to W.

Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$. Via the Maurer–Cartan form, we identify TG with $G \times \mathfrak{g}$, and thus $\mathtt{Lin}(TG;W)$ with $G \times \mathtt{Lin}(\mathfrak{g};W)$. In particular,

$$\Omega^1(G;W) \simeq C^\infty(G;\mathtt{Lin}(\mathfrak{g};W)).$$

In other words, if $\omega \in C^\infty(G;\mathtt{Lin}(\mathfrak{g};W))$, then we define $\tilde{\omega} \in \Omega^1(G;W)$ as $\tilde{\omega}(p)(v) = \omega(p)(\mathrm{d}L_p^{-1}v)$ for $v \in T_pG$.

A *left-invariant W-valued 1-form* is an element $\tilde{\omega}$ of $\Omega^1(G;W)$ such that, for all $p, g \in G$, $\tilde{\omega}(gp) = \tilde{\omega}(p) \circ \mathrm{d}L_g^{-1}$. In particular, left-invariant W-valued 1-forms correspond to constants in $C^\infty(G;\mathtt{Lin}(\mathfrak{g};W))$.

Any W-valued 1-form $\tilde{\omega} \in \Omega^1(G;W)$ defines a subset $\mathrm{Ker}(\tilde{\omega}) \subseteq TG$ given by all vectors $v \in TG$ such that $\tilde{\omega}(v) = 0$. In particular, $\mathrm{Ker}(\tilde{\omega})$ is a subbundle of TG if $\tilde{\omega}$ is left-invariant.

2.6. Anti-semi-direct product

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras, and $\psi\colon \mathfrak{g} \to \mathtt{Der}(\mathfrak{h})$ an anti-morphism of Lie algebras, that is, a linear map such that $\psi([x,y]) = -[\psi(x),\psi(y)]$. For a given ψ, we obtain a Lie algebra $\mathfrak{j} = \mathfrak{g} \ltimes_\psi \mathfrak{h}$ consisting of the vector space $\mathfrak{g} \times \mathfrak{h}$ equipped with the bracket

$$[(x,X),(y,Y)] = ([x,y], -[X,Y] + \psi(y)X - \psi(x)Y).$$

Let G and H be the corresponding connected simply connected Lie groups of $\mathfrak{g}$ and $\mathfrak{h}$. Then there is a anti-morphism of Lie groups $\phi\colon G \to \mathtt{Aut}(H)$ (that is, $\phi(ab) = \phi(b)\phi(a)$) such that $\phi(\exp(x))_* = e^{\psi(x)} \in \mathtt{Aut}(\mathfrak{h})$ for all $x \in \mathfrak{g}$. We obtain a Lie group $\mathtt{J} = G \ltimes_\phi H$ on $G \times H$ with Lie algebra $\mathfrak{j} = \mathfrak{g} \ltimes_\psi \mathfrak{h}$ by setting $(a,A)(b,B) = (ab, B\phi(b)A)$.

Assume that $\mathfrak{h}$ is abelian, that is, a finite-dimensional vector space. Then

$$[(x,X),(y,Y)] = ([x,y], \psi(y)X - \psi(x)Y),$$
$$(a,A)(b,B) = (ab, B + \phi(b)A),$$
$$(\exp(x),A)(\exp(y),B) = (\exp(x)\exp(y), B + e^{\psi(y)}A). \tag{2.4}$$

If $(a, A) \in \mathtt{J}$ and $(x, X) \in \mathfrak{j}$, then the differential of the left translation $L_{(a,A)}$ at $(e_G, 0)$ applied to (x, X) is

$$dL_{(a,A)}|_{(e_G,0)} \begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} dL_a & 0 \\ \psi(\cdot)A & \mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} \tilde{x}(a) \\ \psi(x)A + X \end{pmatrix},$$

where $\tilde{x}(a)$ is the left-invariant vector field $\tilde{x}$ generated by x and evaluated at a, i.e., $\tilde{x}(a) = \mathrm{d}L_a[x]$. Hence, if $(x, X) \in \mathfrak{j}$, the corresponding left-invariant vector field $\widetilde{(x, X)}$ computed at $(a, A) \in \mathtt{J}$ is

$$\widetilde{(x, X)}(a, A) = (\tilde{x}(a), \psi(x)A + X).$$

In order to describe the exponential map $\exp_{\mathtt{J}} \colon \mathfrak{j} \to \mathtt{J}$, we need to find a curve $t \mapsto (a(t), A(t)) \in \mathtt{J}$ such that $a(0) = e_G$, $A(0) = 0$, $a'(t) = \tilde{x}(a(t))$ and $A'(t) = \psi(x)A(t) + X$. By standard considerations, we get

$$\exp_{\mathtt{J}}(x, X) = \Big(\exp(x), \int_0^1 e^{(1-s)\psi(x)} X \,\mathrm{d}s\Big). \tag{2.5}$$

2.7. Stratified Lie groups

A *stratification* for a Lie algebra $\mathfrak{g}$ is a direct sum decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$, where $[V_1, V_j] = V_{1+j}$ for $1 \le j \le s$, and $[V_i, V_j] = 0$ for $i + j > s$. The projection $\mathfrak{g} \to V_i$ will be denoted by Π_i. A *stratified Lie group* is a connected, simply connected nilpotent Lie group whose Lie algebra is stratified. Stratified Lie groups are also known as *Carnot groups* and we use the two terms as synonyms.

The subbundle of TG given pointwise by $\{\tilde{v}(p) \in T_pG : v \in V_1,\ p \in G\}$ is called the *horizontal bundle* and denoted $\mathcal{H}_G$. Furthermore, for every $\lambda > 0$, the *dilation* $\delta_\lambda \colon \mathfrak{g} \to \mathfrak{g}$ is the Lie algebra isomorphism defined by $\Pi_i \circ \delta_\lambda = \lambda^i \Pi_i$. By conjugating the Lie algebra dilation with the exponential map, we get a corresponding dilation $G \to G$, also denoted δ_λ, which by definition, is a group isomorphism.

Lemma 2.2. *Let $U \subset G$ be an open set, let $F \colon U \to G$ be a smooth map such that $dF(\mathcal{H}_G \cap TU) \subset \mathcal{H}_G$, and let $p_0 \in U$ be such that $dF|_{\mathcal{H}_G(p_0)}$ is injective. Then $dF(p_0)$ is a linear isomorphism.*

Proof. We say that two vector fields $\tilde{X}_1$ and $\tilde{X}_2$ are F-related if

$$dF(p)\tilde{X}_1(p) = \tilde{X}_2(F(p))$$

for all p. If $\tilde{X}_1$ and $\tilde{Y}_1$ are F-related to $\tilde{X}_2$ and $\tilde{Y}_2$, respectively, then $[\tilde{X}_1, \tilde{Y}_1]$ is F-related to $[\tilde{X}_2, \tilde{Y}_2]$, see Proposition 1.55 in [17].

Up to shrinking U, we assume that U is a neighborhood of p_0, where $dF|_{\mathcal{H}_G(p)}$ is injective for all $p \in U$. For $p \in U$, define $d_G F(p) \colon \mathfrak{g} \to \mathfrak{g}$ by setting

$$d_G F(p)v = dL^{-1}_{F(p)}(dF(p)(dL_p v)).$$

Since $dF(\mathcal{H}_G \cap TU) \subset \mathcal{H}_G$ and $dF|_{\mathcal{H}_G(p)}$ is injective for all $p \in U$, then we have a smooth function $d_G F|_{V_1} \colon U \to \mathtt{GL}(V_1)$, where $\mathtt{GL}(V_1)$ is the group of invertible linear maps $V_1 \to V_1$. Given $X_2 \colon G \to V_1$ smooth, define $X_1^F \colon U \to V_1$ as

$$X_2^F(p) = (d_G F(p)|_{V_1})^{-1} X_2(F(p)).$$

It is clear that the corresponding vector fields $\tilde{X}_1^F$ and $\tilde{X}_2$ are F-related.

Now, since there is a basis of $T_{F(p_0)}G$ whose elements are evaluations at $F(p_0)$ of iterated Lie brackets of some horizontal vector fields, each element of such a basis is in the image of $dF(p_0)$. Therefore, $dF(p_0)$ is surjective, and thus a linear isomorphism. ■

3. The jet space over a Carnot group

In this section, we will construct the jet space $\mathtt{J}^m(G;W)$ of m-order jets of functions from a stratified Lie group G to a vector space W. To highlight the difference from the standard jets, we sometimes specify that these are *horizontal* jets, that is, they are constructed using horizontal derivatives. The Lie group $\mathtt{J}^m(G;W)$ is derived from its stratified Lie algebra $\mathfrak{j}^m(\mathfrak{g};W) = \bigoplus_k \mathfrak{j}^m(\mathfrak{g};W)_k$. At the end of the section, we characterize by means of jets of functions the left-invariant distribution $\mathcal{H}^m \subset T\mathtt{J}^m(G;W)$ generated by the first layer $\mathfrak{j}^m(\mathfrak{g};W)_1$.

In order to help the reader in the following abstract discussion, we shortly present our algebraic construction of jet spaces when G is abelian, that is, $G = \mathbb{R}^n$ for some n.

If $f:\mathbb{R}^n \to W$ is a smooth function, then we have partial derivatives of order k at $p \in \mathbb{R}^n$ defined as

$$A^k_{f,p}[\partial_{j_1},\dots,\partial_{j_k}] := \partial_{j_k}\cdots\partial_{j_1} f(p).$$

Notice that $A^k_{f,p}$ is a symmetric k-multilinear map and that every symmetric k-multilinear W-valued map A represents the derivatives of order k of a function f at any point p (take the homogeneous polynomial $f(x) = \frac{1}{k!}A[x-p,\dots,x-p]$). Denote by $\mathtt{Sym}^k(\mathbb{R}^n;W)$ the space of all symmetric k-multilinear maps valued in W. When $k=1$, we have that $\mathtt{Sym}^1(\mathbb{R}^n;W) = \mathtt{Lin}(\mathbb{R}^n;W)$, and when $k=0$, we set $\mathtt{Sym}^0(\mathbb{R}^n;W) := W$. We extend these notions to the non-commutative setting in Section 3.1.

We construct the Lie algebra

$$\mathfrak{j}^m(\mathbb{R}^n;W) = \mathbb{R}^n \oplus W \oplus \mathtt{Lin}(\mathbb{R}^n;W) \oplus \mathtt{Sym}^2(\mathbb{R}^n;W) \oplus \cdots \oplus \mathtt{Sym}^m(\mathbb{R}^n;W),$$

with Lie brackets given by the following rule: if $A \in \mathtt{Sym}^k(\mathbb{R}^n;W)$ and $v \in \mathbb{R}^n$, then $[A,v] \in \mathtt{Sym}^{k-1}(\mathbb{R}^n;W)$ is the contraction by v, that is,

$$[A,v][v_1,\dots,v_{k-1}] = A[v_1,\dots,v_{k-1},v].$$

Since we are dealing with symmetric maps, it does not matter which slot of A we contract, whereas in the non-commutative setting the slot choice does matter, see Section 3.2.

With these Lie brackets, $\mathfrak{j}^m(\mathbb{R}^n;W)$ turns out to be a stratified Lie algebra with stratification

$$\begin{aligned}
\mathfrak{j}^m(\mathbb{R}^n;W)_1 &= \mathbb{R}^n \oplus \{0\} \oplus \dots \{0\} \oplus \{0\} \oplus \mathtt{Sym}^m(\mathbb{R}^n;W),\\
\mathfrak{j}^m(\mathbb{R}^n;W)_2 &= \{0\} \oplus \{0\} \oplus \dots \{0\} \oplus \mathtt{Sym}^{m-1}(\mathbb{R}^n;W) \oplus \{0\},\\
&\ \ \vdots\\
\mathfrak{j}^m(\mathbb{R}^n;W)_{m+1} &= \{0\} \oplus W \oplus \{0\} \oplus \dots \{0\} \oplus \{0\}.
\end{aligned}$$

In Section 3.3 we develop the same construction in the non-abelian case.

The jet space $\mathtt{J}^m(\mathbb{R}^m; W)$ is then the simply connected Lie group with Lie algebra $\mathfrak{j}^m(\mathbb{R}^n; W)$. The semi-direct product structure easily facilitates the construction of a model of $\mathtt{J}^m(\mathbb{R}^m; W)$ using $\mathfrak{j}^m(\mathbb{R}^n; W)$ and exponential coordinates of the second kind, see Section 3.4. These are the standard coordinates on jet spaces: we denote by x the coordinates in $\mathbb{R}^n$, u the coordinates in W and u_I coordinates in $\mathtt{Sym}^k(\mathbb{R}^n; W)$, where $I = (I_1, \dots, I_n)$ is a multi-index with $\sum_{j=1}^n I_j = k$. In other words, if $u\colon \mathbb{R}^n \to W$ is a smooth function, u_I should represent the derivative $u_I(p) = \partial_1^{I_1} \cdots \partial_n^{I_n} u(p)$. The horizontal distribution on $\mathtt{J}^m(\mathbb{R}^n; W)$ (also called Cartan distribution) is the intersection of the kernels of the 1-forms

$$\omega_I := \mathrm{d}u_I - \sum_{j=1}^n u_{I+e_j}\, \mathrm{d}x^j.$$

We use a formally similar formula to define the horizontal distribution in Section 3.5.

The above 1-forms can be understood as follows: if $u\colon \mathbb{R}^n \to W$ is a smooth function and $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is a smooth curve, then the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t} u_I(\gamma(t)) = \sum_{j=1}^n u_{I+e_j}(\gamma(t))\, \mathrm{d}x^j[\gamma'(t)].$$

It follows that the curve $\mathbb{R} \to \mathtt{J}(\mathbb{R}^n; W)$ given by

$$t \mapsto (\gamma(t), u(\gamma(t)), \dots, u_I(\gamma(t)), \dots)$$

is tangent to the kernels of the 1-forms ω_I. In fact, this is a characterization of the Cartan distribution, and we prove the same in the non-abelian setting in Section 3.6.

A non-abelian example is described in Section 8.

3.1. Horizontal derivatives

For $f\colon G \to W$ smooth, $k \in \mathbb{N}$ and $p \in G$, we define $A^k_{f,p}$ as the k-multi-linear map from V_1 to W defined by

$$A^k_{f,p}(v_1, \dots, v_k) = \tilde{v}_k \cdots \tilde{v}_1 f(p).$$

If $k = 0$, we set $A^0_{f,p} := f(p)$. Notice that if we define $f_p(x) = f(px)$, then the left invariance of the vector fields $\tilde{v}_i$ implies that $A^k_{f,p} = A^k_{f_p,e}$.

For $k \geq 1$, we denote by $\mathtt{HD}^k(\mathfrak{g}; W)$ the vector subspace of $\mathtt{Lin}^k(V_1; W)$ of all such k-multi-linear maps. For $k = 0$, we set $\mathtt{HD}^0(\mathfrak{g}; W) = W$, coherently with the definition of $A^0_{f,p}$. We then define

$$\mathtt{HD}^{\leq m}(\mathfrak{g}; W) = \bigoplus_{k=0}^m \mathtt{HD}^k(\mathfrak{g}; W).$$

Elements of $\mathtt{HD}^{\leq m}(\mathfrak{g}; W)$ are a priori only sums of derivatives of functions $G \to W$. Corollary 4.6 will show that every element in $\mathtt{HD}^{\leq m}(\mathfrak{g}; W)$ is the derivative of a function, that is, for every $p \in G$, there is $f\colon G \to W$ smooth such that the k-th component A^k of A, is $A^k_{f,p}$.

We call elements of $\mathtt{HD}^{\leq m}(\mathfrak{g}; W)$ (*horizontal*) *jets of order m*. Since we do not use other notions of jets, we will often drop the specification "horizontal".

3.2. Right contractions

If $k \geq 2$ and $A \in \mathtt{Lin}^k(V_1; W)$, then for $v \in V_1$, the *right contraction* of A by v, denoted by $v \neg A$, is the element in $\mathtt{Lin}^{k-1}(V_1; W)$ given by

$$v \neg A(v_1, \ldots, v_{k-1}) = A(v_1, \ldots, v_{k-1}, v).$$

Moreover, if $k = 1$, then $v \neg A = A(v)$, and if $k = 0$, then $v \neg A = 0$.

Lemma 3.1. *If $A \in \mathtt{HD}^k(\mathfrak{g}; W)$ and $v \in V_1$, then $v \neg A \in \mathtt{HD}^{k-1}(\mathfrak{g}; W)$. Furthermore, if $k \geq 1$ and $A = A^k_{f,p}$, then it follows that*

$$v \neg A = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A^{k-1}_{f,p\exp(tv)}. \tag{3.1}$$

Proof. The first assertion of the lemma is a consequence of (3.1) since $t \mapsto A^{k-1}_{f,p\exp(tv)}$ is a smooth curve in the vector space $\mathtt{HD}^{k-1}(\mathfrak{g}; W)$.

Turning to the proof of (3.1), we note first that if $v_1, v_2 \in V_1$, then

$$\begin{aligned} \tilde{v}_2\tilde{v}_1 f_p(y) = \tilde{v}_2(\tilde{v}_1 f_p)(y) &= \frac{\mathrm{d}}{\mathrm{d}s_2}\Big|_{s_2=0} (\tilde{v}_1 f_p)(y\exp(s_2v_2)) \\ &= \frac{\mathrm{d}}{\mathrm{d}s_2}\Big|_{s_2=0} \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} f(py\exp(s_2v_2)\exp(s_1v_1)). \end{aligned}$$

Iterating this procedure, we obtain

$$\tilde{v}_k \cdots \tilde{v}_1 f_p(e) = \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}s_k}\Big|_{s_k=0} f(p\exp(s_kv_k)\cdots\exp(s_1v_1)), \tag{3.2}$$

where the derivations $\frac{\mathrm{d}}{\mathrm{d}s_i}\big|_{s_i=0}$ commute with each other. We now obtain (3.1) by direct calculation as follows. If $k = 1$, then (3.1) is trivial since $A^0_{f,p} = f(p)$. If $k \geq 2$, then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A^{k-1}_{f,p\exp(tv)}(v_1, \ldots, v_{k-1}) \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}s_{k-1}}\Big|_{s_{k-1}=0} f(p\exp(tv)\exp(s_{k-1}v_{k-1})\cdots\exp(s_1v_1)) \\ &\quad= \tilde{v}\tilde{v}_{k-1}\cdots\tilde{v}_1 f(p) = A^k_{f,p}(v_1, \ldots, v_{k-1}, v) = v \neg A(v_1, \ldots, v_{k-1}). \end{aligned}$$

∎

Lemma 3.2. *The linear span of the image of the bilinear map $\neg\colon V_1 \times \mathtt{HD}^k(\mathfrak{g}; W) \to \mathtt{HD}^{k-1}(\mathfrak{g}; W)$ is $\mathtt{HD}^{k-1}(\mathfrak{g}; W)$, for all $k \geq 1$.*

Proof. Let $\mathcal{H} = \operatorname{span}\{v \neg A : v \in V_1,\ A \in \mathtt{HD}^k(\mathfrak{g}; W)\}$. For a given $f \in C^\infty(G; W)$, define $\psi_f\colon G \to \mathtt{HD}^{k-1}(\mathfrak{g}; W)$ by $\psi_f(p) = A^{k-1}_{f,p}$. If γ is a smooth horizontal path in G and s_0 is in the domain of γ, then there is $v \in V_1$ such that $\gamma'(s_0) = \tilde{v}(\gamma(s_0))$. Hence, by (3.1),

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=s_0} \psi_f(\gamma(s)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \psi_f(\gamma(s_0)\exp(tv)) = v \neg A^k_{f,\gamma(s_0)}. \tag{3.3}$$

Since V_1 bracket generates $\mathfrak{g}$, every pair of distinct points $q_1, q_2 \in G$ are connected by a horizontal curve and (3.3) implies that $A^{k-1}_{f,q_1} - A^{k-1}_{f,q_2} \in \mathcal{H}$.

For any $B \in \mathtt{HD}^{k-1}(\mathfrak{g}; W)$ and $p \neq e_G$, there exists f such that $B = A^{k-1}_{f,p}$. Let ϕ be a smooth cut off function with value 1 on a neighborhood of p and 0 on a neighborhood of e_G. Applying the argument above, we have $\psi_{\phi f}(p) - \psi_{\phi f}(e_G) \in \mathcal{H}$, which implies $\psi_{\phi f}(p) = B \in \mathcal{H}$ since $\psi_{\phi f}(e_G) = 0$. ∎

Given $v \in V_1$, we now extend the definition of the right contraction by v to a map $v \neg \colon \mathtt{HD}^{\leq m}(\mathfrak{g}; W) \to \mathtt{HD}^{\leq m}(\mathfrak{g}; W)$, where for each $A \in \mathtt{HD}^{\leq m}(\mathfrak{g}; W)$, we set $(v \neg A)^k = v \neg A^{k+1}$ for $k < m$ and $(v \neg A)^m = 0$.

In the following proposition, we extend the notion of right contraction $v \neg A$ to vectors v that are not in V_1. The spirit of such an extension is based on two observations. On the one hand, vectors in a higher layer V_k are derivations of order k. On the other hand, higher order derivations are compositions of derivations of order 1. A look at the example in Section 8, and Section 8.5 in particular, can help the reader to clarify the picture.

Proposition 3.3. *The map $V_1 \to \mathrm{End}(\mathtt{HD}^{\leq m}(\mathfrak{g}; W))$, $v \mapsto v \neg$, extends uniquely to a Lie algebra anti-morphism $\mathfrak{g} \to \mathrm{End}(\mathtt{HD}^{\leq m}(\mathfrak{g}; W))$, i.e., to a linear map that satisfies*

$$[v, w] \neg = -[v \neg, w \neg]$$

for every $v, w \in \mathfrak{g}$. Furthermore, the map $v \mapsto v \neg$ satisfies the following properties:

(i) *If $v \in V_\ell$ and $A \in \mathtt{HD}^{\leq m}(\mathfrak{g}; W)$, then*

$$(v \neg A)^k = 0 \quad \text{for all } k > m - \ell,$$

and if $A = A^{\leq m}_{f,p}$ for some f and p, then

$$(v \neg A)^k = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A^k_{f,p\exp(tv)} \quad \text{for all } 0 \leq k \leq m - \ell.$$

(ii) *If $v \in V_a$, $w \in V_b$, $A \in \mathtt{HD}^{\leq m}(\mathfrak{g}; W)$, and if $A = A^{\leq m}_{f,p}$ for some f and p, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A^k_{f,p\exp(t[v,w])} = (w \neg (v \neg A) - v \neg (w \neg A))^k \quad \text{for all } 0 \leq k \leq m - a - b.$$

Proof. We will show that there exists an extension of $v \mapsto v \neg$ satisfying (i) and (ii), and then we will show that this extension is a Lie algebra anti-morphism.

We start with the map $v \mapsto v \neg$ defined only for $v \in V_1$. Recall that if $f \colon G \to W$ and $v, w \in \mathfrak{g}$, then

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} (f(\exp(sv)\exp(tw)) - f(\exp(tw)\exp(sv))) = \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0} f(\exp(h[v,w])). \tag{3.4}$$

Moreover, if $v, w \in V_1$, then

$$\begin{aligned} \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} A^k_{f,p\exp(sv)\exp(tw)} &= \left.\frac{\partial}{\partial s}\right|_{s=0} (w \neg A^{k+1}_{f,p\exp(sv)}) \\ &= w \neg \Big(\left.\frac{\partial}{\partial s}\right|_{s=0} A^{k+1}_{f,p\exp(sv)} \Big) = w \neg (v \neg A^{k+2}_{f,p}), \end{aligned} \tag{3.5}$$

where we used Lemma 3.1 and the linearity of $w \neg$. It follows that (ii) holds for $a = b = 1$. Clearly, (i) holds for $\ell = 1$, by Lemma 3.1 and the definition of $\neg$.

Let $1 \le n \le s-1$, and suppose that we have already extended $\neg$ to a bilinear map

$$\neg : \bigoplus_{j=1}^{n} V_j \times \mathtt{HD}^{\le m}(\mathfrak{g}; W) \to \mathtt{HD}^{\le m}(\mathfrak{g}; W)$$

satisfying both (i) for $\ell \le n$, and (ii) for $a, b \le n$. For $u \in V_{n+1}$ and $A \in \mathtt{HD}^{\le m}(\mathfrak{g}; W)$, define

$$(u \neg A)^k := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A^k_{f, p\exp(tu)}$$

for $k \le m-n-1$, and $(u \neg A)^k := 0$ for $k \ge m-n$, where f determines A at some p. If $u = [v, w] \in V_{n+1}$ with $v \in V_1$ and $w \in V_n$, then (ii) implies that $(u \neg A)^k = (w \neg (v \neg A) - v \neg (w \neg A))^k$ for $k \le m-n$; hence, $u \neg A$ does not depend on the choice of f and p. Since $u \mapsto u \neg A$ is linear and V_{n+1} is linearly generated by $[V_1, V_n]$, $u \neg A$ does not depend on the choice of f and p for every $u \in V_{n+1}$.

The newly defined bilinear map

$$\neg : \bigoplus_{j=1}^{n+1} V_j \times \mathtt{HD}^{\le m}(\mathfrak{g}; W) \to \mathtt{HD}^{\le m}(\mathfrak{g}; W)$$

satisfies (i) for $\ell \le n+1$ by construction. If $v \in V_a$ and $w \in V_b$, with $a, b \le n+1$, then we can carry out the same computations as in (3.5) and apply (3.4). Thus, we obtain that this partial extension $\neg$ also satisfies (ii) for $a, b \le n+1$.

By iteration, we have proven that an extension of $\neg$ satisfying (i) and (ii) exists. Finally, (ii) implies that $v \mapsto v \neg$ is a Lie algebra anti-morphism, which must be unique because V_1 Lie generates $\mathfrak{g}$. ■

Remark 3.4. From part (i) of Proposition 3.3, we see that if $v \in V_\ell$ and $A \in \mathtt{HD}^k(\mathfrak{g}; W)$, then $v \neg A \in \mathtt{HD}^{k-\ell}(\mathfrak{g}; W)$ (or 0 if $k - \ell < 0$). In particular, $A \in \mathtt{HD}^k(\mathfrak{g}; W)$, which is a k-linear map on V_1, defines a linear map $V_k \to W$. This observation is an expression of the known fact that derivatives along V_k are "intrinsically" derivatives of order k.

More generally, $A \in \mathtt{HD}^{\le m}(\mathfrak{g}; W)$ defines a linear map $A \colon \mathfrak{g} \to \mathtt{HD}^{\le m-1}(\mathfrak{g}; W)$ by setting $A(v) = v \neg A$. From (i) in Proposition 3.3, we see that if $A = A^{\le m}_{f,p}$ and $k + s \le m$, where s is the step of G, then

$$(v \neg A)^k = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A^k_{f, p\exp(tv)}. \tag{3.6}$$

Remark 3.5. We denote by Ad_p the differential at e_G of the map $q \mapsto pqp^{-1}$. For every $f \in C^\infty(G; W)$, $p \in G$, $x \in \mathfrak{g}$ and $k \in \mathbb{N}$, we have

$$A^k_{x^\dagger f, p} = \left(\mathrm{Ad}_p^{-1}(x) \neg A^{\le k+s}_{f,p}\right)^k. \tag{3.7}$$

Indeed, we have that the curves $t \mapsto p^{-1}\exp(tx)p$ and $t \mapsto \exp(t\,\mathrm{Ad}_p^{-1}(x))$ have equal derivative at $t=0$. Therefore, using the formula (3.2), which is still valid for non-horizontal vector fields, and (3.6), we get that for every $v_1,\dots,v_k \in V_1$,

$$\begin{aligned}
A^k_{x^\dagger f,p}[v_1,\dots,v_k] &= \tilde v_k \cdots \tilde v_1 (x^\dagger f)_p(e) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}s_k}\Big|_{s_k=0} (x^\dagger f)(p\exp(s_k v_k)\cdots\exp(s_1 v_1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}s_k}\Big|_{s_k=0} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\exp(tx)p\exp(s_k v_k)\cdots\exp(s_1 v_1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}s_k}\Big|_{s_k=0} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(pp^{-1}\exp(tx)p\exp(s_k v_k)\cdots\exp(s_1 v_1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}s_1}\Big|_{s_1=0} \cdots \frac{\mathrm{d}}{\mathrm{d}s_k}\Big|_{s_k=0} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(p\exp(t\,\mathrm{Ad}_p^{-1}(x))\exp(s_k v_k)\cdots\exp(s_1 v_1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A^k_{f,p\exp(t\,\mathrm{Ad}_p^{-1}(x))}(v_1,\dots,v_k) \\
&= \left(\mathrm{Ad}_p^{-1}(x) \neg A^{\le k+s}_{f,p}\right)^k (v_1,\dots,v_k),
\end{aligned}$$

that is, we get (3.7).

3.3. The Lie algebra of the jet space

If we consider the vector space $\mathtt{HD}^{\le m}(\mathfrak{g};W)$ as an abelian Lie algebra, then the space of its derivations coincides with the space of linear endomorphisms $\mathtt{End}(\mathtt{HD}^{\le m}(\mathfrak{g};W))$. Hence, by Proposition 3.3, the map $v \mapsto v\neg$ is a Lie algebra anti-morphism from $\mathfrak{g}$ to $\mathtt{Der}(\mathtt{HD}^{\le m}(\mathfrak{g};W))$. Applying the results listed in Section 2.6, we obtain the Lie algebra

$$\mathfrak{j}^m(\mathfrak{g};W) := \mathfrak{g} \ltimes \mathtt{HD}^{\le m}(\mathfrak{g};W),$$

where

$$[(v,A),(w,B)] = ([v,w], w\neg A - v\neg B). \tag{3.8}$$

Lemma 3.6. *$\mathfrak{j}^m(\mathfrak{g};W)$ is a stratified Lie algebra of step $\max\{s,m+1\}$, where s is the step of $\mathfrak{g}$. The k-th layer of $\mathfrak{j}^m(\mathfrak{g};W)$ is*

$$\mathfrak{j}^m(\mathfrak{g};W)_k := V_k \times \mathtt{HD}^{m+1-k}(\mathfrak{g};W),$$

where $V_k = \{0\}$ if $k > s$ and $\mathtt{HD}^{m+1-k}(\mathfrak{g};W) = \{0\}$ if $k > m+1$.

Proof. One easily sees that

$$[\mathfrak{j}^m(\mathfrak{g};W)_1, \mathfrak{j}^m(\mathfrak{g};W)_k] \subset \mathfrak{j}^m(\mathfrak{g};W)_{k+1}.$$

Moreover, in the previous equation, the linear span of the left-hand side is equal to the right-hand side because of Lemma 3.2. ∎

3.4. The jet space

The simply connected Lie group corresponding to the Lie algebra $\mathfrak{j}^m(\mathfrak{g}; W)$ is the semi-direct product $\mathtt{J}^m(G; W) = G \ltimes \mathtt{HD}^{\leq m}(\mathfrak{g}; W)$ with group operation given by (2.4) in Section 2.6 as

$$(\exp(v), A)(\exp(w), B) = (\exp(v)\exp(w), B + e^{(w\,\lrcorner)} A).$$

Using the inverse map $\log\colon G \to \mathfrak{g}$ of the exponential map, we may also express the multiplication directly on G by

$$(a, A)(b, B) = (ab, B + e^{(\log(b)\,\lrcorner)} A).$$

The group $\mathtt{J}^m(G; W)$ will be called the *W-valued (horizontal) jet space over G of order m*.

Since $\mathtt{J}^m(G; W)$ is the product of G and the vector space $\mathtt{HD}^{\leq m}(\mathfrak{g}; W)$, it is a smooth vector bundle over G, where for each $(a, A) \in \mathtt{J}^m(G; W)$, we have the following canonical identifications for the tangent and cotangent spaces:

$$\begin{aligned} T_{(a,A)}\mathtt{J}^m(G; W) &\simeq T_a G \times \mathtt{HD}^{\leq m}(\mathfrak{g}; W), \\ T^*_{(a,A)}\mathtt{J}^m(G; W) &\simeq T^*_a G \times \mathtt{HD}^{\leq m}(\mathfrak{g}; W)^*. \end{aligned}$$

For $(x, X) \in \mathfrak{j}^m(\mathfrak{g}; W)$ and $(a, A) \in \mathtt{J}^m(G; W)$, the differential of the left translation $L_{(a,A)}$ at $(0, 0)$ in the direction (x, X) is given by

$$dL_{(a,A)}|_{(0,0)} \begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} dL_a & 0 \\ \cdot\,\lrcorner A & \mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} \tilde{x}(a) \\ x\,\lrcorner A + X \end{pmatrix},$$

where $\tilde{x}(a)$ is the left-invariant vector field $\tilde{x}$ generated by x and evaluated at a, i.e., $\tilde{x}(a) = \mathrm{d}L_a[x]$. The left invariant vector field generated by (x, X), denoted $\widetilde{(x, X)}$, when evaluated at $(a, A) \in \mathtt{J}^m(G; W)$, has the following form:

$$\widetilde{(x, X)}(a, A) = (\tilde{x}(a), x\,\lrcorner A + X).$$

Since G is simply connected and nilpotent, the exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism and it is common to identify $\mathfrak{g}$ with G via exp. In other words, one can reconstruct G on $\mathfrak{g}$ using the BCH formula. This identification is very useful. However, notice that the exponential map $\mathfrak{g} \times \mathtt{HD}^{\leq m}(\mathfrak{g}; W) = \mathfrak{j}^m(\mathfrak{g}; W) \to \mathtt{J}^m(G; W) = G \times \mathtt{HD}^{\leq m}(\mathfrak{g}; W)$ is not the identity map, but it is described in (2.5).

3.5. The Cartan distribution

For $0 \leq \ell < m$, we define a vector-valued form $\omega^\ell \in \Omega^1(\mathtt{J}^m(G; W); \mathtt{HD}^\ell(\mathfrak{g}; W))$, where for each $(a, A) \in \mathtt{J}^m(G; W)$ and all $(x, X) \in T_{(a,A)}\mathtt{J}^m(G; W)$, the value of $\omega^\ell(a, A)$ is given by

$$\omega^\ell(a, A)(x, X) := X^\ell - \sum_{k=1}^{m-\ell} (dL_a^{-1} x)_k \,\lrcorner A^{\ell+k},$$

where $(dL_a^{-1}x)_k = \Pi_k(dL_a^{-1}x) \in V_k$ is the projection of $dL_a^{-1}x$ to V_k. The sum of these forms gives the 1-form $\omega \in \Omega^1(\mathtt{J}^m(G;W);\mathtt{HD}^{\le m-1}(\mathfrak{g};W))$, with

$$\omega(a,A)(x,X) := \sum_{\ell=1}^{m-1} \omega^\ell(a,A)(x,X) = X^{\le m-1} - (dL_a^{-1}x)\neg A.$$

For every $j \in \{1,\dots,s\}$, define the V_j-valued forms $\theta^j \in \Omega^1(\mathtt{J}^m(G;W);V_j)$ by

$$\theta^j(a,A)(x,X) := \Pi_j(dL_a^{-1}x).$$

Lemma 3.7. *The forms ω, and thus its components ω^ℓ, and the forms θ^j are left invariant under the group operation of $\mathtt{J}^m(G;W)$.*

Proof. We show that the forms ω and θ^j are constant on left-invariant vector fields. To this end, let $x \in \mathfrak{g}$ and $X \in \mathtt{HD}^{\le m}(\mathfrak{g};W)$. For every (a,A), we have

$$\begin{aligned}\omega(a,A)[\widetilde{(x,X)}(a,A)] &= \omega(a,A)[\tilde{x}(a), x\neg A + X] \\ &= (x\neg A + X)^{\le m-1} - x\neg A = X^{\le m-1},\end{aligned}$$

where we used that $x\neg A = (x\neg A)^{\le m-1}$. Therefore, ω is constant on left-invariant vector fields $\widetilde{(x,X)}$, and so ω is left invariant.

Similarly, from $\theta^j(a,A)[(x,X)\tilde{}(a,A)] = \theta^j(a,A)[\tilde{x}(a), x\neg A + X] = x_j$, it again follows that θ^j is constant on left-invariant vector fields. ∎

The *Cartan distribution* or *contact jet structure*[1] on $\mathtt{J}^m(G;W)$ is the distribution

$$\mathcal{H}^m = \bigcap_{\ell=0}^{m-1} \ker\omega^\ell \cap \bigcap_{j=2}^{s} \ker\theta^j.$$

In other words, if $(a,A) \in \mathtt{J}^m(G;W)$, the space $\mathcal{H}^m_{(a,A)}$ consists of points of the form $(\tilde{v}(a), B)$, where $v \in V_1$ and $B^\ell - v\neg A^{\ell+1} = 0$ for all $0 \le \ell \le m-1$.

Notice that $\mathcal{H}^m$ is left-invariant by Lemma 3.7 and that $\mathcal{H}^m_{(e,0)} = \mathfrak{j}^m(\mathfrak{g};W)_1$ is the first layer of the stratified Lie algebra $\mathfrak{j}^m(\mathfrak{g};W)$.

Next, we describe a frame for $\mathcal{H}^m$. If $v_1,\dots,v_r$ is a basis of V_1 and $B_1^m,\dots,B_N^m$ is a basis of $\mathtt{HD}^m(\mathfrak{g};W)$, then a basis of $\mathcal{H}^m_{(a,A)}$ is given by the vectors

$$\begin{aligned} &\mathbb{X}_j(a,A) := (\tilde{v}_j(a), v_j\neg A), && j \in \{1,\dots,r\}, \\ &\mathbb{Y}_j(a,A) := (0, B_j^m), && j \in \{1,\dots,N\}. \end{aligned} \tag{3.9}$$

To compute the Lie brackets of the vector fields at (3.9), we first consider vector fields of the form $\mathbb{X}_v(a,A) = (\tilde{v}(a), v\neg A)$, for $v \in V_1$. From Lemma 2.1 and Proposition 3.3, it follows that

$$[\mathbb{X}_v,\mathbb{X}_w](a,A) = \widetilde{[v,w]}(a), w\neg(v\neg A) - v\neg(w\neg A)) = (\widetilde{[v,w]}(a), [v,w]\neg A),$$

[1] We call the subbundle $\mathcal{H}^m$ a "contact structure", although the term "contact" is also used in the literature to indicate more specific distributions. However, our use of the word "contact" is definitely not uncommon, in particular with reference to jet spaces; see, for instance, [12].

which gives the Lie brackets $[\mathbb{X}_j, \mathbb{X}_k]$. The remaining brackets are

$$[\mathbb{X}_j, \mathbb{Y}_k] = (0, v_j \neg B_k^m) \quad \text{and} \quad [\mathbb{Y}_j, \mathbb{Y}_k] = 0.$$

If $\pi_m\colon \mathtt{J}^{m+1}(G;W) \to \mathtt{J}^m(G;W)$ is the projection along $\mathtt{HD}^{m+1}(\mathfrak{g};W)$, then we have span $d\pi_m(\mathcal{H}^{m+1}) = \mathcal{H}^m$, by Lemma 3.2 (cf. Section 5.1). Moreover, if $\pi_G\colon \mathtt{J}^m(G;W) \to G$ is the projection onto the first factor, then $d\pi_G(\mathcal{H}^m) = \mathcal{H}_G$, where $\mathcal{H}_G$ is the left-invariant distribution on G defined by V_1.

3.6. Characterization of jets of functions

Recall that $\mathtt{J}^m(G;W) \to G$ is a vector bundle. For a smooth function $f\colon G \to W$, the *(horizontal) m-jet of f* is the section of $\mathtt{J}^m(G;W)$ defined by

$$\mathtt{J}^m f(p) = (p, A_{f,p}^{\leq m}).$$

Recall also that $\mathcal{H}_G$ is the left-invariant distribution on G defined by V_1.

Proposition 3.8. *Let $\gamma\colon G \to \mathtt{J}^m(G;W)$ be a smooth section. The following statements are equivalent*:

(i) *There exists $f\colon G \to W$ smooth such that $\gamma = \mathtt{J}^m f$,*

(ii) $\gamma^*\omega^\ell|_{\mathcal{H}_G} = 0$ *for every* $0 \leq \ell \leq m-1$,

(iii) $d\gamma(\mathcal{H}_G) \subset \mathcal{H}^m$.

Proof. To begin, note that since γ is assumed to be a section of $\mathtt{J}^m(G;W)$, (ii) and (iii) are equivalent. So, we only need to prove (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii). Assume $\gamma = \mathtt{J}^m f$ and let $p \in G$ and $v \in V_1$. Computing the differential $d\gamma(\tilde{v}(p))$, we get

$$\begin{aligned} d\gamma(\tilde{v}(p)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \gamma(p\exp(tv)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (p\exp(tv), A_{f,p\exp(tv)}^{\leq m}) = (\tilde{v}(p), v \neg A_{f,p}^{\leq m+1}), \end{aligned}$$

where we applied (3.1) in the last identity. It is clear that $d\gamma(\tilde{v}(p))$ is a linear combination of the vectors in (3.9).

(ii) $\Rightarrow$ (i). We write $\gamma(p) = (p, \sum_{j=0}^m \gamma^j(p))$, where $\gamma^j(p) \in \mathtt{HD}^j(\mathfrak{g};W)$. Since γ is smooth, the function $f = \gamma^0$ is a smooth function $G \to W$. We need to show that $\gamma = \mathtt{J}^m f$. To this end, we show by induction that the claim $\gamma^k = A_f^k$ is true for $k = 0, \dots, m$, starting with the fact that for $k = 0$, the claim is true by definition, i.e., $\gamma^0 = f$.

Let $k < m$ and assume $\gamma^k = A_f^k$; we shall show $\gamma^{k+1} = A_f^{k+1}$. Since (ii) holds, for all $p \in G$ and $v \in V_1$, we have

$$0 = \omega^k(\gamma(p))[d\gamma(p)[\tilde{v}(p)]] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \gamma^k(p\exp(tv)) - v \neg \gamma^{k+1}(p).$$

Let $p \in G$ and $v_1, \dots, v_k, v \in V_1$. Then, using the inductive hypothesis and (3.1), we get

$$\begin{aligned}\gamma^{k+1}(p)(v_1, \dots, v_k, v) &= (v \lrcorner \gamma^{k+1}(p))(v_1, \dots, v_k) \\ &= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \gamma^k(p\exp(tv))\Big)(v_1, \dots, v_k) = \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A^k_{f,p\exp(tv)}\Big)(v_1, \dots, v_k) \\ &= (v \lrcorner A^{k+1}_{f,p})(v_1, \dots, v_k) = A^{k+1}_{f,p}(v_1, \dots, v_k, v).\end{aligned}$$

We conclude that $\gamma^{k+1} = A^{k+1}_f$ and the proof is complete. ∎

4. Jets via Taylor expansions

In this section, we build a link between our notion of horizontal jet of a function and the notion of weighted Taylor expansion developed in [6]. We will gain both a second perspective on horizontal jets and a few existence results that are needed to develop a theory of horizontal jets.

Throughout the section, G is a stratified Lie group and W a finite-dimensional real vector space.

4.1. Homogeneity on stratified Lie groups

In this subsection, we introduce polynomials and homogeneous differential operators. We have chosen a synthetic definition of polynomials on stratified Lie groups. However, our definition is equivalent to the one given by Folland and Stein in [6]. Other approaches to polynomials on Lie groups have been considered; see [1] for a discussion of their equivalences. We provide a sketch of some well-known results and concepts with the particular goal of proving the second assertion in Proposition 4.3, which links homogeneous operators to the spaces $\mathtt{HD}^m(\mathfrak{g}; W)$, which were introduced in the previous section. From the combination of Propositions 4.1 and 4.3, we obtain an identification of $\mathtt{HD}^m(\mathfrak{g}; W)$ with the space of polynomials $\mathcal{P}^m_p(G; W)$, for each $p \in G$.

Dilations $\delta_\lambda : G \to G$ and left-translations $L_p : G \to G$ define linear operators δ^*_λ, L^*_p: $C^\infty(G; W) \to C^\infty(G; W)$ via pre-composition, that is,

$$\delta^*_\lambda f = f \circ \delta_\lambda \quad \text{and} \quad L^*_p f = f \circ L_p.$$

We define *dilations centered at* $p \in G$ by

$$\delta_{p,\lambda} = L_p \circ \delta_\lambda \circ L_p^{-1}. \tag{4.1}$$

A function $f \in C^\infty(G; W)$ is a *homogeneous polynomial of weighted degree* $m \in \mathbb{N}$ *at* $p \in G$ if $\delta^*_{p,\lambda} f = \lambda^m f$ for all $\lambda > 0$. We denote the vector space of all homogeneous polynomials of weighted degree $m \in \mathbb{N}$ at $p \in G$ with values in W by $\mathcal{P}^m_p(G; W)$. One can easily check that for every $p \in G$ and $m \in \mathbb{N}$, we have

$$L^*_g(\mathcal{P}^m_p(G; W)) = \mathcal{P}^m_{g^{-1}p}(G; W). \tag{4.2}$$

We finally define the space $\mathcal{P}^{\le m}(G; W) = \bigoplus_{k=0}^m \mathcal{P}^k_e(G; W)$ of *polynomials of weighted degree* $m \in \mathbb{N}$ *on* G. Notice that $\mathcal{P}^k_p(G; W) = \mathcal{P}^k_p(G; \mathbb{R}) \otimes W$.

A linear operator $E: C^\infty(G) \to C^\infty(G)$ is *homogeneous of degree* $k \in \mathbb{Z}$ if

$$\delta_\lambda^* \circ E \circ (\delta_\lambda^*)^{-1} = \lambda^k E$$

for all $\lambda > 0$.

One can easily check that if E_1, E_2 are homogeneous operators of degree k_1 and k_2, respectively, then the composition $E_1 E_2$ is homogeneous of degree $k_1 + k_2$. Moreover, if $f \in \mathcal{P}_e^m(G)$, then the multiplier operator $m_f(u) = fu$ is homogeneous of degree m. If $v \in V_k$, then the differential operator $\tilde{v}$ is homogeneous of order $-k$.

Let $\mathcal{U}(G)$ be the universal enveloping algebra of G, that is, the algebra of all left-invariant differential operators on $C^\infty(G)$. Elements of $\mathcal{U}(G)$ are linear combinations of operators of the type $\tilde{X}_m \cdots \tilde{X}_1$, where each $\tilde{X}_j$ is a left-invariant vector field on G. For $m \in \mathbb{N}$, we denote by $\mathcal{U}^m(G)$ the space of elements in $\mathcal{U}(G)$ that are homogeneous of degree $-m$. One can show that $\mathcal{U}(G) = \bigoplus_{m=1}^\infty \mathcal{U}^m(G)$.

In [6], Folland and Stein gave an apparently different definition of homogeneous polynomials on graded groups using exponential coordinates. However, looking at the Taylor expansion of smooth functions at e, one can easily see that their definition is equivalent to ours. For reasons of convenience, we postpone the proof of this fact to Section 4.3, precisely to Lemma 4.7, where we discuss these objects in coordinates.

We take from [6] two properties of homogeneous polynomials. First, Proposition 1.25 in [6] gives us that $\mathcal{P}_p^m(G; W) \subset \bigoplus_{k=0}^m \mathcal{P}_q^k(G; W)$ for every $p, q \in G$. In turn, this implies that $\mathcal{P}^{\le m}(G; W) = \bigoplus_{k=0}^m \mathcal{P}_p^k(G; W)$ for all $p \in G$. Second, Proposition 1.30 in [6] gives us that $\mathcal{U}^m(G)$ is the dual space of $\mathcal{P}_e^m(G)$, as we summarize in the following proposition.

Proposition 4.1 (Proposition 1.30 in [6]). *For every $p \in G$, the pairing*

$$\langle \cdot | \cdot \rangle_p : \mathcal{U}^m(G) \times \mathcal{P}_p^m(G) \to \mathbb{R}, \quad \langle D | f \rangle_p := Df(p),$$

defines a linear isomorphism ψ_p from $\mathcal{P}_p^m(G)$ to the dual $\mathcal{U}^m(G)^*$. Moreover, ψ_p extends to an isomorphism from $\mathcal{P}_p^m(G; W)$ to $\mathcal{U}^m(G)^* \otimes W$, for every finite-dimensional vector space W.

Corollary 4.2. *If $f \in C^\infty(G; W)$, $p \in G$ and $m \in \mathbb{N}$, then there exists a unique $P_{f,p}^m \in \mathcal{P}_p^m(G; W)$ such that*

$$Df(p) = DP_{f,p}^m(p)$$

for all $D \in \mathcal{U}^m(G)$.

Corollary 4.2 follows directly from Proposition 4.1, because $D \mapsto Df(p)$ defines an element of $(\mathcal{U}^m(G))^*$. The polynomial $\sum_{k=0}^m P_{f,p}^k \in \mathcal{P}^{\le m}(G; W)$ is the *homogeneous Taylor expansion of f at p of order m*.

For the definition of $\mathcal{T}^m(V_1)$, see Section 2.1.

Proposition 4.3. *For every $m \in \mathbb{N}$, the function*

$$\tau : \mathcal{T}^m(V_1) \to \mathcal{U}^m(G), \quad \tau(v_1 \otimes \cdots \otimes v_m) = \tilde{v}_m \cdots \tilde{v}_1,$$

is surjective, and its transpose maps $\mathcal{U}^m(G)^*$ onto $\mathtt{HD}^m(\mathfrak{g}; \mathbb{R}) \subset \mathcal{T}^m(V_1^*)$, that is,

$$\tau^* : \mathcal{U}^m(G)^* \to \mathtt{HD}^m(\mathfrak{g}; \mathbb{R})$$

is a linear isomorphism. Moreover, τ^* extends to a linear isomorphism from $\mathcal{U}^m(G)^* \otimes W$ to $\mathtt{HD}^m(\mathfrak{g}; W) \subset \mathcal{T}^m(V_1^*) \otimes W$.

Proof. The fact that τ is surjective is standard and we do not prove it here.

Since $\tau\colon \mathcal{T}^m(V_1) \to \mathcal{U}^m(G)$ is a surjective linear map, its transpose τ^* is a linear embedding of $\mathcal{U}^m(G)^*$ into $\mathcal{T}^m(V_1^*)$. The proof that τ^* maps $\mathcal{U}^m(G)^*$ onto $\mathtt{HD}^m(\mathfrak{g}; \mathbb{R})$ is divided into two parts.

First, we show that $\tau^*(\mathcal{U}^m(G)^*) \subset \mathtt{HD}^m(\mathfrak{g}; \mathbb{R})$. Let $\alpha \in \mathcal{U}^m(G)^*$. By Proposition 4.1, there is $f \in \mathcal{P}_e^m(G)$ such that $\alpha(D) = Df(e)$ for all $D \in \mathcal{U}^m(G)$. Then

$$\begin{aligned}\tau^*(\alpha)(v_1 \otimes \cdots \otimes v_m) &= \alpha(\tau(v_1 \otimes \cdots \otimes v_m)) \\ &= \tilde{v}_m \cdots \tilde{v}_1 f(e) = A^m_{f,e}(v_1 \otimes \cdots \otimes v_m).\end{aligned}$$

Second, we show that $\tau^*(\mathcal{U}^m(G)^*) \supset \mathtt{HD}^m(\mathfrak{g}; \mathbb{R})$. If $A \in \mathtt{HD}^m(\mathfrak{g}; \mathbb{R})$, then there is $f \in C^\infty(G)$ such that $A = A^m_{f,e}$. Then, the computation above shows that $\tau^*(\alpha) = A$, where $\alpha = \psi_p(P^m_{f,e})$. ∎

4.2. The polynomial jet space

For $P \in \mathcal{P}^{\leq m}(G; W)$ and $v \in \mathfrak{g}$, define

$$v \neg P = v^\dagger P \in \mathcal{P}^{\leq m}(G; W),$$

where $v^\dagger$ denotes the right-invariant vector field on G with $v^\dagger(e_G) = v$. There is no risk of confusion with the right contraction defined in Section 3.2; in the expression "$v \neg X$", the nature of X determines the nature of $\neg$.

Remark 4.4. One can easily show that if $P \in \mathcal{P}_e^m(G; W)$ and $v \in V_k$, then $v \neg P$ is an element of $\mathcal{P}_e^{m-k}(G; W)$. Notice that, in the latter claim, e cannot be substituted with any point $p \in G$. The reason is that, for $p \neq e$, the spaces $\mathcal{P}_p^m(G; W)$ depend on the choice of left over right translations in the definition of $\delta_{p,\lambda}$, see (4.1).

As a direct consequence of the standard equality $[v^\dagger, w^\dagger] = -[v, w]^\dagger$ for v, w in a Lie algebra $\mathfrak{g}$, it follows that the map $v \mapsto v\neg \in \mathtt{End}(\mathcal{P}^{\leq m}(G; W))$ is a Lie algebra anti-morphism of $\mathfrak{g}$, i.e.,

$$v \neg w \neg - w \neg v \neg = -[v, w] \neg,$$

for every $v, w \in \mathfrak{g}$.

Therefore, we can apply the construction presented in Section 2.6 with $\psi(v) = v\neg$. Define $\mathfrak{j}^m_{\mathcal{P}}(\mathfrak{g}; W)$ as $\mathfrak{g} \ltimes_\psi \mathcal{P}^{\leq m}(G; W)$ with Lie brackets

$$[(v, P), (w, Q)] = ([v, w], w \neg P - v \neg Q) = ([v, w], w^\dagger P - v^\dagger Q). \tag{4.3}$$

In parallel to Lemma 3.6, the Lie algebra $\mathfrak{j}^m_{\mathcal{P}}(\mathfrak{g}; W)$ is stratified with layers

$$\mathfrak{j}^m_{\mathcal{P}}(\mathfrak{g}; W)_k = V_k \oplus \mathcal{P}_e^{m+1-k}(G; W),$$

where $V_k = \{0\}$ if $k > s$, and $\mathcal{P}_e^{m+1-k}(G; W) = \{0\}$ if $k > m + 1$. Define

$$\mathtt{J}^m_{\mathcal{P}}(G; W) = G \ltimes \mathcal{P}^{\leq m}(G; W)$$

as the Lie group with operation

$$(\exp(v), P)(\exp(w), Q) = (\exp(v)\exp(w), Q + e^{v^\dagger} P).$$

Theorem 4.5. (i) *For every $p \in G$, the map $\sigma_p : \mathcal{P}^m_p(G;W) \to \mathtt{HD}^m(\mathfrak{g};W)$ given by $\sigma_p(f) = A^m_{f,p}$, is a linear isomorphism.*[2]

(ii) *If $p \in G$, $f \in \mathcal{P}^{\le m}(G;W)$ and $v \in \mathfrak{g}$, then*

$$\sigma_p(v \neg f) = \mathrm{Ad}_p^{-1}(v) \neg \sigma_p(f) \in \mathtt{HD}^{\le m}(G;W). \tag{4.4}$$

(iii) *The map $\sigma : \mathfrak{j}^m_{\mathcal{P}}(\mathfrak{g};W) \to \mathfrak{j}^m(\mathfrak{g};W)$ given by $\sigma(v, f) = (v, A^{\le m}_{f,e})$ is an isomorphism of stratified Lie algebras.*

(iv) *The map $\Phi : \mathtt{J}^m_{\mathcal{P}}(G;W) \to \mathtt{J}^m(G;W)$ given by $\Phi(p, f) = (p, A^{\le m}_{f,e})$ is the isomorphism of Lie groups with $\Phi_* = \sigma$.*

Proof. The composition of the isomorphisms ψ_p form Proposition 4.1 and τ^* from Proposition 4.3 gives the linear isomorphism

$$\sigma_p : \mathcal{P}^m_p(G;W) \xrightarrow{\psi_p} \mathcal{U}^m(G;W)^* \xrightarrow{\tau^*} \mathtt{HD}^m(\mathfrak{g};W), \quad \sigma_p(f) = A^m_{f,p},$$

for each $p \in G$.

The identity (4.4) follows from Remark 3.5. Notice, in particular, that $\sigma_e(x \neg f) = x \neg \sigma_e(f)$. Thus, σ is an isomorphism of stratified Lie algebras. Finally, the last statement is proved by checking that

$$\Phi(\exp_{\mathtt{J}_{\mathcal{P}}}(x, P)) = \exp_{\mathtt{J}}(\sigma(x, P))$$

for every $x \in \mathfrak{g}$ and $P \in \mathcal{P}^{\le m}(G)$, which is a computation with (2.5). ■

An immediate consequence of Theorem 4.5 is the following statement.

Corollary 4.6. *For every $p \in G$ and $A \in \mathtt{HD}^{\le m}(\mathfrak{g};W)$, there is $f \in C^\infty(G;W)$ such that $A^{\le m}_{f,p} = A$ and $A^k_{f,p} = 0$ for all $k > m$. In particular, if $f \in \mathcal{P}^m_p(G;W)$, then $A^k_{f,p} = 0$ for $k \ne m$.*

4.3. Bases

Let $\mathcal{B} = (b_1, \dots, b_n)$ be a basis of $\mathfrak{g}$ adapted to the stratification $\bigoplus_j V_j$, i.e., there is a non-decreasing sequence of integers $\{\mathbf{w}_i\}_{i=1}^n$ such that $b_i \in V_{\mathbf{w}_i}$ for all i. If $I \in \mathbb{N}^n$ is a multi-index, define $\mathbf{w}(I) = \sum_{j=1}^n \mathbf{w}_j I_j$. We denote by x_j the exponential coordinates given by $\mathcal{B}$, i.e., smooth functions $G \to \mathbb{R}$ such that $\exp(\sum_j x_j(p) b_j) = p$ for all $p \in G$. The *homogeneous degree* of a monomial $x^I = \prod_{j=1}^n x_j^{I_j}$ is $\deg(x^I) = \mathbf{w}(I)$.

Lemma 4.7. *Using the above notation, $\{x^I\}_{\mathbf{w}(I)=m}$ is a basis of $\mathcal{P}^m_e(G;\mathbb{R})$.*

[2] Notice that the space $\mathtt{HD}^{\le m}(\mathfrak{g};\mathbb{R})$ does not depend on p, but the way it is identified with $\mathcal{P}^{\le m}_p(G)$ does.

Proof. It is clear that $\{x^I\}_{\mathbf{w}(I)=m}$ is a linearly independent subset of $\mathcal{P}^m_e(G;\mathbb{R})$. To prove the claim, we set $|I| = \sum_{j=1}^n I_j$, so that $|I| \le \mathbf{w}(I) \le s|I|$ because $1 \le w_i \le s$ for all i. Given $f \in \mathcal{P}^m_e(G;\mathbb{R})$, we write the Taylor expansion of f at 0 in exponential coordinates up to order m, that is, $f(x) = \sum_{|I|\le m} f_I x^I + \rho(x)$, where $\rho \in C^\infty(G)$ is such that $|\rho(x)| \le C(|x|^{m+1})$ for $|x| \le 1$ and some $C \ge 0$. Here $|\cdot|$ denotes the Euclidean norm. We rearrange this sum as

$$f(x) = \sum_{|I|\le m} f_I x^I + \rho(x) = \sum_{k=0}^{m} \sum_{\mathbf{w}(I)=k} f_I x^I + \sum_{k=m+1}^{sm} \sum_{\substack{\mathbf{w}(I)=k \\ |I|\le m}} f_I x^I + \rho(x).$$

Therefore, for x fixed and $\lambda > 0$, we have

$$\frac{f(\delta_\lambda x)}{\lambda^m} - \sum_{\mathbf{w}(I)=m} f_I x^I = \sum_{k=0}^{m-1} \lambda^{k-m} \sum_{\mathbf{w}(I)=k} f_I x^I + \sum_{k=m+1}^{sm} \lambda^{k-m} \sum_{\substack{\mathbf{w}(I)=k \\ |I|\le m}} f_I x^I + \frac{\rho(\delta_\lambda x)}{\lambda^m}.$$

Notice that the left-hand side of the latter identity is constant in λ, while on the right-hand side, we have

$$\lim_{\lambda\to 0^+} \Big| \sum_{k=0}^{m-1} \lambda^{k-m} \sum_{\mathbf{w}(I)=k} f_I x^I \Big| = \infty \quad \text{if } \sum_{\mathbf{w}(I)=k} f_I x^I \neq 0 \text{ for some } k,$$

$$\lim_{\lambda\to 0^+} \sum_{k=m+1}^{sm} \lambda^{k-m} \sum_{\substack{\mathbf{w}(I)=k \\ |I|\le m}} f_I x^I = 0, \qquad \lim_{\lambda\to 0^+} \frac{\rho(\delta_\lambda x)}{\lambda^m} = 0,$$

where the last limit follows from the fact $|\delta_\lambda x| \le \lambda |x|$. We therefore obtain that $f(x) = \sum_{\mathbf{w}(I)=m} f_I x^I$. ∎

For a multi-index $I \in \mathbb{N}^n$, define $\tilde{b}^I = \tilde{b}_1^{I_1} \cdots \tilde{b}_n^{I_n} \in \mathcal{U}(G)$. Notice that we obtain $(\delta_\lambda)_* \tilde{b}^I = \lambda^{-\mathbf{w}(I)} \tilde{b}^I$. By the Poincaré–Birkhoff–Witt theorem, see Theorem I.2.7 in [3], $\{\tilde{b}^I\}_{I\in\mathbb{N}^n}$ is a basis of $\mathcal{U}(G)$. Since $\tilde{b}^I \in \mathcal{U}^m(G)$ if and only if $\mathbf{w}(I) = m$, we have that $\{\tilde{b}^I : \mathbf{w}(I) = m\}$ is a basis of $\mathcal{U}^m(G)$.

We have two ways to build a basis of $\mathtt{HD}^m(\mathfrak{g}; W)$: we can take $\{\sigma_e(x^I)\}_{\mathbf{w}(I)=m}$, or the basis dual to $\{\tilde{b}^I\}_{\mathbf{w}(I)=m}$. These are not the same, because it is *not* true that $\langle \tilde{b}^I | x^J \rangle = 0$ if and only if $I = J$.

We choose to describe the second basis, that is, the basis dual to $\{\tilde{b}^I\}_{\mathbf{w}(I)=m}$, for both $\mathtt{HD}^m(\mathfrak{g};\mathbb{R})$ and $\mathcal{P}^m_e(G;\mathbb{R})$. In other words, we will get $\{A_I\}_{\mathbf{w}(I)=m} \subset \mathtt{HD}^m(\mathfrak{g};\mathbb{R})$ such that, for every $f \in C^\infty(G)$ and $p \in G$,

$$A^m_{f,p} = \sum_{\mathbf{w}(I)=m} (\tilde{b}^I f)(p) \cdot A_I.$$

And, for each $p \in G$, we will get $\{P_{p,I}\}_{\mathbf{w}(I)=m} \subset \mathcal{P}^m_p(G;\mathbb{R})$ such that

$$P^m_{f,p} = \sum_{\mathbf{w}(I)=m} (\tilde{b}^I f)(p) \cdot P_{p,I}. \tag{4.5}$$

We can obtain the polynomial $P_{p,I}$ by imposing $\tilde{b}^I P^J(p) = \delta^I_J$ and then compute $A_I = A^m_{P_{e,I},e}$. The identity (4.5) gives us the homogeneous Taylor expansion of $f \in C^\infty(G)$ at p as

$$f \sim \sum_{m\geq 0} \sum_{\mathbf{w}(I)=m} (\tilde{b}^I f)(p) \cdot P_{p,I}.$$

However, we can compute A_I without taking derivatives and exploiting the algebra structure of $\mathcal{U}(G)$. Indeed, compute the functions $\tau_I : \mathcal{T}^m(V_1) \to \mathbb{R}$ given by

$$\tau(\xi) = \sum_{\mathbf{w}(I)=m} \tau_I(\xi)\tilde{b}^I \tag{4.6}$$

for $\xi \in \mathcal{T}^m(V_1)$. If $\{P_{p,I}\}_{\mathbf{w}(I)=m}$ is the basis of $\mathcal{P}^m_p(G)$ dual to $\{\tilde{b}^I\}_{\mathbf{w}(I)=m}$, then

$$\langle \tau^* P_{p,I} | \xi \rangle = \langle P_{p,I} | \tau(\xi) \rangle = \tau_I(\xi).$$

Therefore, A_I is given by

$$\langle A_I | \xi \rangle = \tau_I(\xi) \tag{4.7}$$

for all $\xi \in \mathcal{T}^m(V_1)$. Notice that A_I does not depend on p, while $P_{p,I}$ does. Moreover, we can compute A_I without computing $P_{p,I}$. Clearly, the isomorphism σ_p is given by $\sigma_p(P_{p,I}) = A_I$.

Remark 4.8. We now summarize an algorithm to compute a basis for $\mathtt{HD}^m(\mathfrak{g}; W)$:

(1) Choose a basis $\mathcal{B} = (b_1, \dots, b_n)$ of $\mathfrak{g}$ adapted to the stratification $\bigoplus_{j=1}^s V_j$; we denote by Ξ^m the basis of $\mathcal{T}^m(V_1)$ induced by $\mathcal{B}$.

(2) Compute the list of multi-indices $\mathcal{I}^m = \{I \in \mathbb{N}^n : \mathbf{w}(I) = m\}$.

(3) For each $\xi \in \Xi^m$, write $\tau(\xi)$ in the basis $\{\tilde{b}^I\}_{I\in\mathcal{I}^m}$ as in (4.6), so that we obtain $\{\tau_I(\xi)\}_{I\in\mathcal{I}^m, \xi\in\Xi^m}$.

(4) For each $I \in \mathcal{I}^m$, compute $A_I = \sum_{\xi\in\Xi^m} \tau_I(\xi)\xi^*$, where ξ^* is the element of $\mathcal{T}^m(V_1^*)$ dual to ξ with respect to the basis Ξ^m.

(5) The resulting $\{A_I\}_{I\in\mathcal{I}^m}$ is a basis of $\mathtt{HD}^m(\mathfrak{g}; \mathbb{R})$. Since $\mathtt{HD}^m(\mathfrak{g}; W)$ is the tensor product $\mathtt{HD}^m(\mathfrak{g}; \mathbb{R}) \otimes W$, a basis for $\mathtt{HD}^m(\mathfrak{g}; W)$ is $\{A_I \otimes c\}_{I\in\mathcal{I}^m, c\in\mathcal{C}}$, for a basis $\mathcal{C}$ of W.

See Section 8 for an application.

5. Prolongation of contact maps

5.1. Projection to lower order jet bundles

In this section, we will simultaneously consider $\mathtt{J}^m(G; W)$ and $\mathtt{J}^{m+1}(G; W)$, and therefore, we will denote objects on $\mathtt{J}^{m+1}(G; W)$ with a hat symbol "ˆ". For instance, we use p to represent a point in $\mathtt{J}^m(G; W)$ and $\hat{p}$ to represent a point in $\mathtt{J}^{m+1}(G; W)$. Similarly, ω^ℓ refers to a contact form on $\mathtt{J}^m(G; W)$, while $\hat{\omega}^\ell$ represents a contact form on $\mathtt{J}^{m+1}(G; W)$, and so on.

We denote by $\pi_m \colon \mathtt{J}^{m+1}(G;W) \to \mathtt{J}^m(G;W)$ the projection along $\mathtt{HD}^{m+1}(\mathfrak{g};W)$. This map is not a morphism of Lie groups because $\mathtt{HD}^{m+1}(\mathfrak{g};W)$ is not an ideal of $\mathfrak{j}^{m+1}(\mathfrak{g};W)$. On the other hand, the projection does map the horizontal distribution to the horizontal distribution, although it is not *onto*.

In particular, if $\hat{p} \in \pi_m^{-1}(p)$ then $d\pi_m|_{\hat{p}}(\mathcal{H}^{m+1}_{\hat{p}}) \neq \mathcal{H}^m_p$ (as one can see from (5.2) below), however $\mathcal{H}^m_p$ is the closure of the union of $d\pi_m|_{\hat{p}}(\mathcal{H}^{m+1}_{\hat{p}})$ for all $\hat{p} \in \pi_m^{-1}(p)$. In fact, in the previous statement, as the next lemma shows, it is enough that $\hat{p}$ mearly ranges over any relatively open subset of $\pi_m^{-1}(p)$.

Lemma 5.1. *If $p \in \mathtt{J}^m(G;W)$ and $U \subset \mathtt{J}^{m+1}(G;W)$ is open and satisfies*

$$\pi_m^{-1}(p) \cap U \neq \emptyset,$$

then

$$\mathcal{H}^m_p = \operatorname{span}\Big(\bigcup\{d\pi_m|_{\hat{p}}(\mathcal{H}^{m+1}_{\hat{p}}) : \hat{p} \in \pi_m^{-1}(p) \cap U\}\Big). \tag{5.1}$$

Proof. "$\supset$" Let $\hat{p} = (a, A^{\leq m+1}) \in \pi_m^{-1}(p) \cap U$. Since elements of $\mathcal{H}^{m+1}_{\hat{p}}$ have the form $(\tilde{v}(a), v \neg A^{\leq m+1} + B^{m+1})$, where $v \in V_1$ and $B^{m+1} \in \mathtt{HD}^{m+1}(\mathfrak{g};W)$, it follows that

$$d\pi_m|_{\hat{p}}((\tilde{v}(a), v \neg A^{\leq m+1} + B^{m+1})) = (\tilde{v}(a), v \neg A^{\leq m+1}) \in \mathcal{H}^m_p. \tag{5.2}$$

Since $\hat{p} \in \pi_m^{-1}(p) \cap U$ is arbitrary, we can conclude that the right-hand side of (5.1) is a subset of $\mathcal{H}^m_p$.

"$\subset$" By Lemma 3.2, if $\{v_1, \dots, v_r\} \subset V_1$ is a basis for V_1 and $\{B_1, \dots, B_N\} \subset \mathtt{HD}^{m+1}(\mathfrak{g};W)$ is basis for $\mathtt{HD}^{m+1}(\mathfrak{g};W)$, then $\{v_i \neg B_j : i = 1, \dots, r, j = 1, \dots, N\}$ contains a basis for $\mathtt{HD}^m(\mathfrak{g};W)$. Given $\hat{p} = (a, A^{\leq m+1}) \in \pi_m^{-1}(p) \cap U$, define $\hat{q}_j \in \mathtt{J}^{m+1}(G;W)$ by $\hat{q}_j = (a, A^{\leq m+1} + B_j)$. It follows that $\hat{q}_j \in \pi_m^{-1}(p)$ and by scaling if necessary, we can assume that each B_j is close enough to 0 so that $\hat{q}_j \in U$.

The left invariant field generated by $(v_i, 0)$ evaluated at $\hat{q}_j$ is

$$(\tilde{v}_i(a), v_i \neg (A^{\leq m+1} + B_j)),$$

and by definition belongs to $\mathcal{H}^{m+1}_{\hat{q}_i}$. Furthermore,

$$d\pi_m|_{\hat{q}_j}((\tilde{v}_j(a), v_j \neg (A^{\leq m+1} + B_j))) = (\tilde{v}_j(a), v_j \neg A^{\leq m+1} + v_j \neg B_j) \in H,$$

where H denotes the right-hand side of (5.1). Since $(\tilde{v}_j(a), v_j \neg A^{\leq m+1}) \in H$ for all j, it follows that $(0, v_j \neg B_j) \in H$ for all j, and we conclude that $\mathtt{HD}^m(\mathfrak{g};W) \subset H$. ∎

5.2. Contact maps

Given a smooth map $F \colon \Omega \to \mathtt{J}^m(G;W)$ from an open set $\Omega \subset \mathtt{J}^m(G;W)$, we write it componentwise as

$$F(a, A) = (F_G(a, A), F^0(a, A), \dots, F^m(a, A)),$$

where $F_G\colon \Omega \to G$ and $F^k\colon \Omega \to \mathtt{HD}^k(\mathfrak{g}; W)$ are smooth maps. Furthermore, F is said to be a *contact map* if $dF(\mathcal{H}^m) \subset \mathcal{H}^m$, which in the case $m \geq 1$, is characterized by the following conditions:

$$
\begin{aligned}
&\omega^\ell(F(p))(dF(p)\mathbb{X}_j(p)) = 0, \quad 0 \leq \ell \leq m-1, \text{ for all } j \in \{1,\dots,r\},\\
&\omega^\ell(F(p))(dF(p)\mathbb{Y}_j(p)) = 0, \quad 0 \leq \ell \leq m-1, \text{ for all } j \in \{1,\dots,N\},\\
&\theta^k(F(p))(dF(p)\mathbb{X}_j(p)) = 0, \quad 2 \leq k \leq s, \text{ for all } j \in \{1,\dots,r\},\\
&\theta^k(F(p))(dF(p)\mathbb{Y}_j(p)) = 0, \quad 2 \leq k \leq s, \text{ for all } j \in \{1,\dots,N\},
\end{aligned}
\tag{5.3}
$$

for all $p \in \Omega$. See (3.9) for the definition of $\mathbb{X}_j(p)$ and $\mathbb{Y}_j(p)$. In the case $m = 0$, the conditions reduce to those given by the forms θ^k and simply mean that F is contact map of some open set $\Omega \subseteq G \times W$, where $\mathcal{H}^0_{(e_G,0)} = \mathrm{j}^0(\mathfrak{g}; W) = V_1 \times W$. Note that, in the case $G = \mathbb{R}^n$, the definition is consistent with the usual contact system as defined in [12], Chapter 4.

5.3. Prolongation

Suppose that $\Omega \subset \mathtt{J}^m(G; W)$ is open, and that $F\colon \Omega \to \mathtt{J}^m(G; W)$ is a contact map. If $\hat{\Omega} \subset \mathtt{J}^{m+1}(G; W)$ is open and satisfies $\pi_m(\hat{\Omega}) \subseteq \Omega$, then a smooth map $\hat{F}\colon \hat{\Omega} \to \mathtt{J}^{m+1}(G; W)$ satisfying

$$\pi_m \circ \hat{F} = F \circ \pi_m \tag{5.4}$$

is called a *prolongation of* F if $\hat{F}$ is a contact map. Since $\hat{\omega}^\ell = \pi_m^* \omega^\ell$ for $\ell = 1, \dots, m-1$, and $\hat{\theta}^k = \pi_m^* \theta^k$ for $k = 2, \dots, s$, every map $\hat{F}$ that satisfies condition (5.4), immediately satisfies all the contact conditions (5.3) on $\mathtt{J}^{m+1}(G; W)$ except those corresponding to $\hat{\omega}^m$. In particular, $\hat{F}_G = F_G \circ \pi_m$, $\hat{F}^i = F^i \circ \pi_m$ for $i = 0, \dots, m$, and $\hat{F}^{m+1}$ is determined by the contact conditions corresponding to $\hat{\omega}^m$. More precisely, the conditions

$$\hat{\omega}^m(\hat{F}(\hat{p}))(d\hat{F}(\hat{p})\hat{\mathbb{Y}}_j(\hat{p})) = 0 \quad \text{for all } j \in \{1, \dots, N\} \tag{5.5}$$

are trivial, since from (5.4), we have, for every $j \in \{1, \dots, N\}$,

$$d\pi_m(\hat{F}(\hat{p}))\, d\hat{F}(\hat{p})\, \hat{\mathbb{Y}}_j(\hat{p}) = dF(p)\, d\pi_m(\hat{p})\hat{\mathbb{Y}}_j(\hat{p}) = dF(\pi_m(\hat{p}))0 = 0,$$

which implies that

$$d\hat{F}(\hat{p})\hat{\mathbb{Y}}_j(\hat{p}) \in \ker d\pi_m(\hat{F}(\hat{p}))$$

or, equivalently,

$$d\hat{F}(\hat{p})\, \hat{\mathbb{Y}}_j(\hat{p}) \in \operatorname{span}\{\hat{\mathbb{Y}}_k(F(\hat{p})) : k = 1, \dots, N\}, \tag{5.6}$$

giving (5.5).

It follows that the determining conditions for $\hat{F}^{m+1}$ are the equations

$$\hat{\omega}^m(\hat{F}(\hat{p}))(d\hat{F}(\hat{p})\hat{\mathbb{X}}_j(\hat{p})) = 0 \quad \text{for all } j \in \{1, \dots, r\}. \tag{5.7}$$

The existence of prolongations is governed by the following result.

Theorem 5.2 (Prolongation theorem). *Suppose $m \geq 0$, $\Omega \subset \mathtt{J}^m(G;W)$ is open, and that $F:\Omega \to \mathtt{J}^m(G;W)$ is a contact map. Let $v_1,\dots,v_r$ be a basis of V_1 and, for $j \in \{1,\dots,r\}$, define $\tilde{N}_j:\pi_m^{-1}(\Omega) \to TG$ as*

$$\tilde{N}_j(\hat{p}) = dF_G(\pi_m(\hat{p}))\, d\pi_m(\hat{p})\, \hat{\mathbb{X}}_j(\hat{p}).$$

Define $\hat{\Omega} \subset \pi_m^{-1}(\Omega) \subset \mathtt{J}^{m+1}(G;W)$ as the open set, where $\tilde{N}_1,\dots,\tilde{N}_r$ are pointwise linearly independent. Then there is a unique contact map $\hat{F}:\hat{\Omega} \to \mathtt{J}^{m+1}(G;W)$ such that

$$\pi_m \circ \hat{F} = F \circ \pi_m.$$

Moreover, if F is a diffeomorphism, then $\hat{F}$ is also a diffeomorphism and $\hat{F}^{-1}$ is the prolongation of F^{-1}.

The proof will be given after the proof of Lemma 5.5.

Remark 5.3. Uniqueness of the prolongation $\hat{F}$ is proved only on $\hat{\Omega}$ (cf. (5.16)). For instance, a constant map F is contact and admits infinite prolongations, but $\hat{\Omega} = \emptyset$.

Remark 5.4. We do not know the size of $\hat{\Omega}$ in general. *If F is itself a prolongation of a contact map on $\mathtt{J}^{m-1}(G;W)$, then $\hat{\Omega} = \pi_m^{-1}(\Omega)$.* Notice that, by the results in the following Section 6, F is a prolongation as soon as $m \geq 2$.

Let us prove the above statement about $\hat{\Omega}$. First, notice that if $\hat{p} = (a, A^{\leq m+1})$ and $p = \pi_m(\hat{p})$, then

$$d\pi_m(\hat{p})\hat{\mathbb{X}}_j(\hat{p}) = (\tilde{v}_j(a), v_j \neg A^{\leq m} + v_j \neg A^{m+1}) = \mathbb{X}_j(p) + v_j \neg A^{m+1},$$

where $v_j \neg A^{m+1} \in \mathtt{HD}^m(\mathfrak{g};W)$.

Second, if $\pi_G:\mathtt{J}^m(G;W) \to G$ is the projection onto G, then it follows that the restriction $d\pi_G(q)|_{\mathcal{H}^m}:\mathcal{H}^m(q) \to \mathcal{H}_G(\pi_G(q))$ is surjective with kernel $\mathtt{HD}^m(\mathfrak{g};W)$, for every $q \in \mathtt{J}^m(G;W)$.

Third, if F is a prolongation, then it satisfies (5.6) (without hats). In particular, the span of the vectors $dF(p)\mathbb{X}_j(p)$ is transversal to $\mathtt{HD}^m(\mathfrak{g};W)$. Hence, $d\pi_G(F(p))$ is an isomorphism between the span of the vectors $dF(p)\mathbb{X}_j(p)$ and $\mathcal{H}_G(\pi_G(p))$.

Thus,

$$\tilde{N}_j(\hat{p}) = dF_G(\pi_m(\hat{p}))\, d\pi_m(\hat{p})\hat{\mathbb{X}}_j(\hat{p}) = d\pi_G(F(p))\, dF(p)\, \mathbb{X}_j(p)$$

are linearly independent.

For the proof of Theorem 5.2, we need the following technical lemma.

Lemma 5.5. *Assuming the hypothesis of Theorem 5.2, for every $f:G \to W$ smooth and $a \in G$ such that $\mathtt{J}^{m+1}f(a) \in \hat{\Omega}$, there exist a smooth function $h:G \to W$ and a neighborhood U of a such that*

$$F \circ \mathtt{J}^m f|_U = \mathtt{J}^m h \circ F_G \circ \mathtt{J}^m f|_U. \tag{5.8}$$

Moreover, $F_G \circ \mathtt{J}^m f(U)$ is an open subset of G and $F_G \circ \mathtt{J}^m f$ is a diffeomorphism on U.

The following diagram illustrates the functions involved in the lemma:

$$
\begin{tikzcd}
\mathrm{J}^m(G;W) \arrow[r, "F"] & \mathrm{J}^m(G;W) \arrow[d, shift right, "\pi_G"'] \\
G \arrow[u, "\mathrm{J}^m f"] \arrow[r, "F_G \circ \mathrm{J}^m f"'] & G \arrow[u, dashed, shift right, "\mathrm{J}^m h"']
\end{tikzcd}
\tag{5.9}
$$

Proof. First, notice that if $\hat{p} = (a, A^{\leq m+1})$ and $p = \pi_m(\hat{p}) = (a, A^{\leq m})$, then

$$d\pi_m(\hat{p})\hat{\mathbb{X}}_j(\hat{p}) = (\tilde{v}_j(a), v_j \neg A^{\leq m} + v_j \neg A^{m+1}) = \mathbb{X}_j(p) + (0, v_j \neg A^{m+1}) \in \mathcal{H}^m_p,$$

where $v_j \neg A^{m+1} \in \mathrm{HD}^m(\mathfrak{g}; W)$. Since F and the projection π_G to G are contact, $F_G = \pi_G \circ F$ is also contact. Thus, $\tilde{N}_j$ takes values in $\mathcal{H}_G$. Define

$$N_j(\hat{p}) = dL^{-1}_{F_G(p)} \tilde{N}_j(\hat{p}) \in V_1. \tag{5.10}$$

Second, since F is contact, it follows that for all $0 \leq \ell \leq m-1$ and all $\hat{p} \in \hat{\Omega} \subset \mathrm{J}^{m+1}(G;W)$, we have that

$$
\begin{aligned}
0 &= \omega^\ell(F(p))(dF(p)d\pi_m(\hat{p})\,\hat{\mathbb{X}}_j(\hat{p})) \\
&= dF^\ell(p)\,d\pi_m(\hat{p})\,\hat{\mathbb{X}}_j(\hat{p}) - (dL^{-1}_{F_G(p)}\,dF_G(p)\,d\pi_m(\hat{p})\,\hat{\mathbb{X}}_j(\hat{p}))_1 \neg F^{\ell+1}(p) \\
&= dF^\ell(p)\,d\pi_m(\hat{p})\,\hat{\mathbb{X}}_j(\hat{p}) - dL^{-1}_{F_G(p)}\,\tilde{N}_j(\hat{p}) \neg F^{\ell+1}(p) \\
&= dF^\ell(p)\,d\pi_m(\hat{p})\,\hat{\mathbb{X}}_j(\hat{p}) - N_j(\hat{p}) \neg F^{\ell+1}(p),
\end{aligned}
$$

that is,

$$dF^\ell(p)(d\pi_m(\hat{p})\,\hat{\mathbb{X}}_j(\hat{p})) = N_j(\hat{p}) \neg F^{\ell+1}(p). \tag{5.11}$$

Next, let $f\colon G \to W$ be a smooth function. Notice that

$$
\begin{aligned}
d(F_G \circ \mathrm{J}^m f)(a)\tilde{v}_j(a) &= dF_G(\mathrm{J}^m f(a))\Big(\tilde{v}_j(a), \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A^{\leq m}_{f, a\exp(tv_j)}\Big) \\
&= dF_G(\mathrm{J}^m f(a))(\tilde{v}_j(a), v_j \neg A^{\leq m+1}_{f,a}) \\
&= \tilde{N}_j(\mathrm{J}^{m+1} f(a)).
\end{aligned}
\tag{5.12}
$$

Since the vectors $\tilde{N}_j(\mathrm{J}^{m+1} f(a))$ are linearly independent, Lemma 2.2 ensures that the map $F_G \circ \mathrm{J}^m f\colon G \to G$ is a diffeomorphism from a neighborhood U_1 of a to a neighborhood U of $F_G \circ \mathrm{J}^m f(a)$ in G. Let $\phi\colon U \to U_1$ be its inverse. The computations above, now read as

$$\tilde{v}_j|_{U_1} = d\phi \circ \tilde{N}_j \circ \mathrm{J}^{m+1} f|_{U_1}. \tag{5.13}$$

Define $h\colon U \to W$ by $h = F^0 \circ \mathrm{J}^m f \circ \phi$. We shall prove that for all $0 \leq k \leq m$,

$$(\mathrm{J}^m h)^k = F^k \circ \mathrm{J}^m f \circ \phi, \tag{5.14}$$

which then implies (5.8).

We prove (5.14) by induction on k. For $k = 0$, the identity (5.14) is the definition of h. So we assume (5.14) holds for $k = \ell \le m - 1$ and consider it for $k = \ell + 1 \le m$. Let $b \in U$, $a = \phi(b)$ and $j \in \{1, \dots, r\}$. Then

$$
\begin{aligned}
(5.15)\quad N_j(a, A^{\le m+1}_{f,a}) \neg (\mathtt{J}^m h)^{\ell+1}(b) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathtt{J}^m h)^{\ell}(b \exp(t N_j(a, A^{\le m+1}_{f,a})))\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (F^{\ell} \circ \mathtt{J}^m f)(\phi(b \exp(t N_j(a, A^{\le m+1}_{f,a}))))\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (F^{\ell} \circ \mathtt{J}^m f)(\phi(b) \exp(t v_j))\\
&= dF^{\ell}|_{(a, A^{\le m}_{f,a})}((\tilde{v}_j(a), v_j \neg A^{\le m+1}_{f,a}))\\
&= N_j(a, A^{\le m+1}_{f,a}) \neg F^{\ell+1}(a, A^{\le m}_{f,a}),
\end{aligned}
$$

where the first computation follows from (3.1), the second is the inductive hypothesis, the third follows from (5.13), the fourth is simply the computation of the differential, and the last one follows from (5.11). Since the vectors N_j form a basis of V_1, we conclude that $(\mathtt{J}^m h)^{\ell+1}(b) = F^{\ell+1}(a, A^{\le m}_{f,a})$, that is, (5.14) holds for $k = \ell + 1$. ∎

Proof of Theorem 5.2. The determining conditions at (5.7) become

$$
(5.16)\qquad d\hat{F}^m(\hat{p})\hat{\mathbb{X}}_j(\hat{p}) = N_j(\hat{p}) \neg \hat{F}^{m+1}(\hat{p}) \quad \text{for all } j,
$$

where N_j are defined in (5.10). Given that $\{N_j(\hat{p}) : j = 1, \dots, r\}$ is a basis of V_1, there exists a unique $\hat{F}^{m+1}(\hat{p}) \in \mathtt{Lin}^{m+1}(V_1; W)$ that satisfies (5.16).

We need to show that $\hat{F}^{m+1}(\hat{p}) \in \mathtt{HD}^{m+1}(\mathfrak{g}; W)$. Write $\hat{p} = (a, A^{\le m+1}) \in \hat{\Omega}$, let $f : G \to W$ be a smooth function with $A^{\le m+1} = A^{\le m+1}_{f,a}$, which exists by Corollary 4.6, and let h be a smooth function as in Lemma 5.5. We can perform again the computations in (5.15) with $\ell = m$ until the second to last step and obtain

$$
N_j(a, A^{\le m+1}_{f,a}) \neg (\mathtt{J}^{m+1} h)^{m+1}(F_G(a, A^{\le m})) = dF^m|_{(a, A^{\le m}_{f,a})}[(\tilde{v}_j(a), v_j \neg A^{\le m+1}_{f,a})],
$$

which is (5.16) with $\hat{F}^{m+1}(\hat{p}) = (\mathtt{J}^m h)^{m+1}(F_G(\hat{p})) \in \mathtt{HD}^{m+1}(\mathfrak{g}; W)$.

Finally, suppose that F is a diffeomorphism. Then $dF(\mathcal{H}^m_p) = \mathcal{H}^m_{F(p)}$ for all $p \in \Omega$, and thus F^{-1} is also a contact map. Let $\hat{\Omega}_F$ denote the domain of $\hat{F}$ and let $\hat{\Omega}_{F^{-1}}$ denote the domain of the prolongation of F^{-1} as defined above.

We claim that $\hat{F}(\hat{\Omega}_F) \subset \hat{\Omega}_{F^{-1}}$. Indeed, let $\hat{p} \in \hat{\Omega}_F$ and set $\hat{q} = \hat{F}(\hat{p})$. By the previous discussion, we know that $\hat{q} = \mathtt{J}^{m+1} h(a)$, where h is given by Lemma 5.5. Now notice that, by (5.12), the linear independence of the vectors $\tilde{N}_j(\hat{p})$ is equivalent to the non-singularity of the differential $d(\pi_G \circ F \circ \mathtt{J}^m f)(a)$. Since the diagram (5.9) commutes, the differential $d(\pi_G \circ F^{-1} \circ \mathtt{J}^m h)(b)$, where $b = \pi_G(\hat{q})$, is also non-singular. By (5.12) again, this time with F^{-1} replacing F, we conclude that the vectors $\tilde{N}_j(\hat{q})$ defined for F^{-1} are also linearly independent and thus $\hat{q} \in \hat{\Omega}_{F^{-1}}$.

Let $H : \hat{\Omega}_{F^{-1}} \to \mathtt{J}^{m+1}(G; W)$ be the prolongation of F^{-1}. Since $H \circ \hat{F}$ is a prolongation of $F^{-1} \circ F = \mathrm{Id}_{\Omega}$, and since the prolongation of a map is unique, $H \circ \hat{F} = \mathrm{Id}_{\hat{\Omega}_F}$, i.e., $H = \hat{F}^{-1}$ on $\hat{F}(\hat{\Omega}_F) \subset \hat{\Omega}_{F^{-1}}$ and thus $\hat{F}$ is a diffeomorphism. By symmetry, we get also $H(\hat{\Omega}_{F^{-1}}) \subset \hat{\Omega}_F$ and $H^{-1} = \hat{F}$. In particular, $\hat{F}(\hat{\Omega}_F) = \hat{\Omega}_{F^{-1}}$. ∎

6. De-prolongation of contact maps: Bäcklund's theorem

6.1. The de-prolongation theorems

This section is devoted to the de-prolongation Theorem B, which extends the well-known Lie–Bäcklund theorem (or Bäcklund's theorem) to horizontal jet spaces. For the classical statement of the theorem, see Theorem 3.1 in [16]. See also Theorem 4.32 in [12], the notes at the end of Chapter 5 in [11], or Section 7.2 of [15] for a discussion on the standard result.

The Bäcklund theorem classically states that, for $m \geq 1$, contact transformations of $\mathtt{J}^m(\mathbb{R}^n;\mathbb{R}^k)$ are prolongations of contact transformations of $\mathtt{J}^1(\mathbb{R}^n;\mathbb{R}^k)$. We extend this result in Theorem 6.1. Further, if $k > 1$, then contact transformations of $\mathtt{J}^1(\mathbb{R}^n;\mathbb{R}^k)$ are prolongations of contact transformations of $\mathtt{J}^0(\mathbb{R}^n;\mathbb{R}^k)$, i.e., prolongations of diffeomorphisms $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$, while, if $k = 1$, there are contact transformations of $\mathtt{J}^1(\mathbb{R}^n;\mathbb{R})$ that are not prolongations. Our generalization is Theorem 6.1.

Let G be a stratified Lie group with stratified Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^s V_j$, and let W be a vector space. The first part of Theorem B follows by iterating the following result.

Theorem 6.1. *Assume $m \geq 1$. Denote by $\pi_m\colon \mathtt{J}^{m+1}(G;W) \to \mathtt{J}^m(G;W)$ the projection along $\mathtt{HD}^{m+1}(\mathfrak{g};W)$. Let $\Omega \subset \mathtt{J}^{m+1}(G;W)$ be an open set, define $\Omega' = \pi_m(\Omega)$ and assume that $\pi_m^{-1}(q) \cap \Omega$ is connected for every $q \in \Omega'$. If $F\colon \Omega \to \mathtt{J}^{m+1}(G;W)$ is a contact diffeomorphism, then there is a contact diffeomorphism $F'\colon \Omega' \to \mathtt{J}^m(G;W)$ such that $\pi_m \circ F = F' \circ \pi_m$, i.e., the following diagram commutes*:

$$\begin{tikzcd} \Omega \arrow[r, "F"] \arrow[d, "\pi_m"'] & \mathtt{J}^{m+1}(G;W) \arrow[d, "\pi_m"] \\ \Omega' \arrow[r, "F'"'] & \mathtt{J}^m(G;W). \end{tikzcd}$$

The following theorem states that, with further assumptions, contact diffeomorphisms of $\mathtt{J}^1(G;W)$ are prolongations of contact diffeomorphisms of $\mathtt{J}^0(G;W)$. This is a more precise restatement of the second part of Theorem B.

Theorem 6.2. *Suppose that one of the following conditions is satisfied*:

(A) $\dim(W) > 1$, *or*

(B) $\dim(W) = 1$ *and for every $v \in V_1 \setminus \{0\}$, there is $v' \in V_1$ with $[v, v'] \neq 0$.*

Denote by $\pi_0\colon \mathtt{J}^1(G;W) \to \mathtt{J}^0(G;W)$ the projection along $\mathtt{HD}^1(\mathfrak{g};W)$. Let $\Omega \subset \mathtt{J}^1(G;W)$ be an open set, define $\Omega' = \pi_0(\Omega)$ and assume that $\pi_0^{-1}(q) \cap \Omega$ is connected for every $q \in \Omega'$. If $F\colon \Omega \to \mathtt{J}^1(G;W)$ is a contact diffeomorphism, then there is a contact diffeomorphism $F'\colon \Omega' \to \mathtt{J}^0(G;W)$ such that $\pi_0 \circ F = F' \circ \pi_0$, i.e., the following diagram commutes:

$$\begin{tikzcd} \Omega \arrow[r, "F"] \arrow[d, "\pi_0"'] & \mathtt{J}^1(G;W) \arrow[d, "\pi_0"] \\ \Omega' \arrow[r, "F'"'] & \mathtt{J}^0(G;W). \end{tikzcd} \tag{6.1}$$

Remark 6.3. Theorem 6.2 is sharp, that is, *if both conditions* (A) *and* (B) *are violated, then there exists a contact diffeomorphism* $F\colon \mathrm{J}^1(G;W) \to \mathrm{J}^1(G;W)$ *that is not the lift of a contact diffeomorphism* $\mathrm{J}^0(G;W) \to \mathrm{J}^0(G;W)$.

The construction of such a contact map F is the following. If both conditions (A) and (B) are violated, then $W = \mathbb{R}$ and there is $\hat{v} \in V_1$ such that $[\hat{v}, V_1] = \{0\}$. It follows that $\mathfrak{g}$ is the direct product of a stratified Lie algebra $\mathfrak{g}'$ and $\mathbb{R}$ (take V_1' such that $V_1 = V_1' \oplus \mathbb{R}\hat{v}$ and $\mathfrak{g}'$ the Lie span of V_1'). Then

$$\mathfrak{j}^1(\mathfrak{g};W) \simeq (\mathfrak{g}' \times \mathbb{R}) \oplus \mathbb{R} \oplus ((\mathfrak{g}')^* \times \mathbb{R}^*),$$

with Lie bracket

$$[((v,x),z,(\alpha,y)),((\bar{v},\bar{x}),\bar{z},(\bar{\alpha},\bar{y}))] = (([v,\bar{v}],0),\alpha(\bar{v}) - \bar{\alpha}(v) + (\bar{y}x - y\bar{x}),(0,0)).$$

Define

$$\phi((v,x),z,(\alpha,y)) = ((v,-y),z,(\alpha,x)).$$

Then ϕ is a Lie algebra automorphism of $\mathfrak{j}^1(\mathfrak{g};W)$. Indeed,

$$\begin{aligned}
&[\phi((v,x),z,(\alpha,y)),\phi((\bar{v},\bar{x}),\bar{z},(\bar{\alpha},\bar{y}))] \\
&\qquad = [((v,-y),z,(\alpha,x)),((\bar{v},-\bar{y}),\bar{z},(\bar{\alpha},\bar{x}))] \\
&\qquad = (([v,\bar{v}],0),\alpha(\bar{v}) - \bar{\alpha}(v) + (-y\bar{x} + \bar{y}x),(0,0)) \\
&\qquad = [((v,x),z,(\alpha,y)),((\bar{v},\bar{x}),\bar{z},(\bar{\alpha},\bar{y}))].
\end{aligned}$$

The map $F\colon \mathrm{J}^1(G;W) \to \mathrm{J}^1(G;W)$ is the unique Lie group automorphism with $F_* = \phi$. Since $\phi(\mathfrak{j}^1(\mathfrak{g};W)_1) = \mathfrak{j}^1(\mathfrak{g};W)_1$, F is a contact diffeomorphism. By construction, there is no F' that makes the diagram (6.1) commute, not even locally.

6.2. Characteristic vector fields and proof of Theorem 6.1

De-prolongation results are based on the existence of a particular type of characteristic vector field.

For an open set $\Omega \subset \mathrm{J}^m(G;W)$, let $T\mathrm{J}^m_\Omega = \bigcup_{p\in\Omega} T_p\mathrm{J}^m(G;W)$ and $\mathcal{H}^m_\Omega = \bigcup_{p\in\Omega}\mathcal{H}^m_p$. Starting with $L^1_\Omega = \Gamma(\mathcal{H}^m_\Omega)$, we define a filtration of $\Gamma(T\mathrm{J}^m_\Omega)$ inductively by setting $L^{i+1}_\Omega = L^i_\Omega + [L^1_\Omega, L^i_\Omega]$ for $i = 1,\dots,s-1$. If $F\colon \Omega \to \mathrm{J}^m(G;W)$ is a smooth contact diffeomorphism, then $F_*L^i_\Omega = L^i_{f(\Omega)}$, since $F_*L^1_\Omega = L^1_{F(\Omega)}$. A Cauchy characteristic of order i over Ω is a vector field $X \in \Gamma(L^i_\Omega)$ such that $[X,\Gamma(L^i_\Omega)] \subset \Gamma(L^i_\Omega)$. The set $\mathcal{C}^i_\Omega$ of all Cauchy characteristics of order i over Ω is a Lie algebra thanks to the Jacobi identity. If $F\colon \Omega \to \mathrm{J}^m(G;W)$ is a smooth contact diffeomorphism, then $F_*\mathcal{C}^i(\Omega) \subseteq \mathcal{C}^i(F(\Omega))$, since

$$F_*[X,\Gamma(L^i_\Omega)] = [F_*X, F_*\Gamma(L^i_\Omega)] = [F_*X,\Gamma(L^i_{F(\Omega)})] \subset F_*\Gamma(L^i_\Omega) = \Gamma(L^i_{F(\Omega)}).$$

The particular notion of characteristic we use here is a vector field $\tilde{X} \in \Gamma(L^1_\Omega)$ such that $[\tilde{X},[\tilde{X},\tilde{Y}]] \subset \Gamma(L^2_\Omega)$ for all $\tilde{Y} \in \Gamma(L^1_\Omega)$. The characteristic property of these vector fields is preserved by contact transformation. Indeed, if $F\colon \Omega \to \mathrm{J}^m(G;W)$ is a smooth contact diffeomorphism, then

$$\begin{aligned}
[F_*\tilde{X},[F_*\tilde{X},\Gamma(L^1_{F(\Omega)})]] &= [F_*\tilde{X},[F_*\tilde{X},F_*\Gamma(L^1_\Omega)]] \\
&= F_*[\tilde{X},[\tilde{X},\Gamma(L^1_\Omega)]] \subset F_*\Gamma(L^2_\Omega) = \Gamma(L^2_{F(\Omega)}).
\end{aligned}$$

This observation together with the characterization of characteristic vector fields, that is proved in Lemma 6.4 below, allow us to prove the de-prolongation result in Theorem 6.1.

Once again, we will heavily use the conventions described in Section 2.3 to represent vector fields on Lie groups. In particular, notice the difference between "$[X_p, Y_p]$" and "$[X,Y]_p$".

Lemma 6.4. *Assume $m \geq 2$. Denote by Π_3 the projection $\mathtt{j}^m(\mathfrak{g};W) \to \mathtt{j}^m(\mathfrak{g};W)_3$ given by the stratification of $\mathtt{j}^m(\mathfrak{g};W)$. Let $X\colon \mathtt{J}^m(G;W) \to \mathtt{j}^m(\mathfrak{g};W)_1$ be a horizontal vector field and write $X = v^X + A^X$, where $v^X\colon \mathtt{J}^m(G;W) \to V_1$ and $A^X\colon \mathtt{J}^m(G;W) \to \mathtt{HD}^m(\mathfrak{g};W)$. Then the following are equivalent:*

(i) $v^X = 0$,

(ii) *X is characteristic, that is, $\Pi_3([X,[X,Y]]) = 0$ for every horizontal vector field $Y\colon \mathtt{J}^m(G;W) \to \mathtt{j}^m(\mathfrak{g};W)_1$.*

Proof. (i) $\Rightarrow$ (ii) If $Y\colon \mathtt{J}(G;W) \to \mathtt{j}^m(G;W)_1$ is a horizontal vector field and we write $Y = v^Y + A^Y$, then for every $p \in \mathtt{J}(G;W)$,

$$\Pi_3([X,[X,Y]]_p) = \Pi_3([X_p,[X_p,Y_p]]) = [A^X_p,[A^X_p,v^Y_p]] + [A^X_p,[A^X_p,A^Y_p]] = 0,$$

where the first identity is justified by writing X and Y in a basis of $\mathtt{j}^m(G;W)_1$ and then applying (2.3).

(ii) $\Rightarrow$ (i) Let $B \in \mathtt{HD}^m(\mathfrak{g};W)$ and let $Y = A^Y \equiv B$ be a constant vector field. Then, for every $p \in G$,

$$0 = \Pi_3([X,[X,Y]]_p) = \Pi_3([X_p,[X_p,Y_p]]) = [v^X,[v^X,B]] = v^X \neg (v^X \neg B).$$

Since B is arbitrary in $\mathtt{HD}^m(\mathfrak{g};W)$ and $m \geq 2$, we conclude that $v^X = 0$. ■

Proof of Theorem 6.1. Let $\mathcal{V} \subset T\mathtt{J}^{m+1}(G;W)$ be the left-invariant vector bundle defined by $\mathtt{HD}^{m+1}(\mathfrak{g};W)$.

We claim that $dF_p(\mathcal{V}) = \mathcal{V}_{F(p)}$ for every $p \in \Omega$. We let $A_1,\dots,A_\ell$ be a basis of $\mathtt{HD}^{m+1}(\mathfrak{g};W)$ and consider the corresponding left-invariant vector fields $\tilde{A}_1,\dots,\tilde{A}_\ell$, which form a frame for $\mathcal{V}$. Fix $k \in \{1,\dots,\ell\}$. By Lemma 6.4, $\tilde{A}_k$ is a characteristic vector field. Since F is contact, $F_*\tilde{A}_k$ is also a characteristic vector field. Lemma 6.4 implies that $F_*\tilde{A}_k|_{F(p)} \in \mathcal{V}_{F(p)}$ for every $p \in \Omega$. We have that $dF_p(\tilde{A}_1|_p),\dots,dF_p(\tilde{A}_\ell|_p)$ are linearly independent and belong to $\mathcal{V}_{F(p)}$, and thus they form a basis of this vector space. The claim is thus proven.

Finally, notice that the fibers $\pi_m^{-1}(q)$, for $q \in \mathtt{J}^m(G;W)$, are the integral manifolds of $\mathcal{V}$. We conclude that there exists a smooth map F' so that the above diagram commutes. The fact that F' is contact map follows from Lemma 5.1. ■

6.3. Proof of Theorem 6.2 with condition (A)

Lemma 6.5. *Let $X,Y\colon \mathtt{J}^m(G;W) \to \mathtt{j}^m(G;W)_1$ be horizontal vector fields and let $p \in \mathtt{J}^m(G;W)$. The commutator $[X,Y]_p$ is horizontal if and only if $[X_p,Y_p] = 0$.*

Proof. Write X and Y in a basis of $\mathtt{j}^m(G;W)_1$ and then apply (2.3). ■

Corollary 6.6. *If $\mathcal{V} \subset \mathcal{H}^m$ is a horizontal involutive subbundle, then, for every $p \in \mathtt{J}^m(G; W)$, the space $dL_p^{-1}(\mathcal{V}_p)$ is an abelian subalgebra of $\mathfrak{j}^m(G; W)_1$.*

Proposition 6.7. *Assume condition (A), that is,* $\dim(W) > 1$. *If R is an abelian subalgebra of $\mathfrak{j}^1(G; W)$ contained in $\mathfrak{j}^1(G; W)_1 = V_1 \times \mathtt{HD}^1(\mathfrak{g}; W)$ such that* $\dim(R) = \dim(V_1) \cdot \dim(W)$*, then $R = \mathtt{HD}^1(\mathfrak{g}; W)$.*

Proof. We let $m = \dim(V_1)$ and $n = \dim(W) > 1$, so that $\dim(R) = mn$. Notice that $\mathtt{HD}^1(\mathfrak{g}; W) = \mathtt{Lin}(V_1; W)$ and thus $\dim(\mathtt{HD}^1(\mathfrak{g}; W)) = mn$.

Let $\pi: V_1 \times \mathtt{HD}^1(\mathfrak{g}; W) \to V_1$ be the projection to the first factor and set $a = \dim(\pi(R))$ and $b = \dim(R \cap \mathtt{HD}^1(\mathfrak{g}; W))$. Notice that $\dim(R) = a + b$ since the restriction of π to R is a linear map with kernel of dimension b and image of dimension a. Moreover,

$$
\begin{aligned}
b = \dim(R \cap \mathtt{HD}^1(\mathfrak{g}; W)) &= mn + mn - \dim(R + \mathtt{HD}^1(\mathfrak{g}; W)) \\
&\geq mn + mn - (mn + m) = mn - m.
\end{aligned}
$$

Let $v_1, \dots, v_a \in V_1$ be a basis of $\pi(R)$ and define the map $\phi: \mathtt{HD}^1(\mathfrak{g}; W) \to W^a$ as $\phi(\alpha) = (\alpha(v_1), \dots, \alpha(v_a))$. Then ϕ is surjective and thus $\dim(\ker \phi) = mn - an$.

Since R is an abelian Lie algebra, if $(0, \beta) \in R \cap \mathtt{HD}^1(\mathfrak{g}; W)$ and $(v_j, \alpha) \in R$ for some j, then

$$
0 = [(0, \beta), (v_j, \alpha)] = \beta(v_j).
$$

Therefore, $\phi(\beta) = 0$, and so $b \leq \dim(\ker \phi) = mn - an$. All in all, we obtain

$$
mn = \dim(R) = a + b \leq a + mn - an,
$$

that is,

$$
0 \leq a(1 - n).
$$

Since $n > 1$, a must be zero. ■

Theorem 6.2 with condition (A) is a special case of the previous proposition.

Proof of Theorem 6.2 with condition (A). Assume condition (A), i.e., $\dim(W) > 1$. Let $\mathcal{V} \subset T\mathtt{J}^1(G; W)$ be the left-invariant vector bundle defined by $\mathtt{HD}^1(\mathfrak{g}; W)$. We claim that $dF_p(\mathcal{V}) = \mathcal{V}_{F(p)}$ for every $p \in \Omega$. Indeed, $\mathcal{V}$ is a horizontal involutive subbundle of rank $\dim(V_1) \cdot \dim(W)$, and so $dF(\mathcal{V})$ is also a horizontal involutive subbundle of the same rank since F is a contact diffeomorphism. From Corollary 6.6, we obtain that for every q in the image of F, $dF_q(\mathcal{V})$ is the left translation of an abelian subalgebra R_q in $\mathfrak{j}(\mathfrak{g}; W)_1$. By Proposition 6.7, we get $R_q = \mathtt{HD}^1(\mathfrak{g}; W)$, i.e., $dF_q(\mathcal{V}) = \mathcal{V}_q$, as claimed.

From this point, the proof concludes as for Theorem 6.1 in Section 6.2. ■

6.4. Proof of Theorem 6.2 with condition (B)

Proof of Theorem 6.2 with condition (B). Assume condition (B), i.e., $\dim(W) = 1$ and for every $v \in V_1 \setminus \{0\}$, there is $v' \in V_1$ with $[v, v'] \neq 0$. It follows that $\mathtt{HD}(\mathfrak{g}; W) = V_1^*$ and $\mathfrak{j}^1(\mathfrak{g}; W)_1 = V_1 \oplus V_1^*$, while $\mathfrak{j}^1(\mathfrak{g}; W)_2 = V_2 \oplus W$. For every $X \in \mathfrak{j}(\mathfrak{g}; W)_1$, define

$$
R_X = \{Y \in \mathfrak{j}(\mathfrak{g}; W)_1 : [X, Y] = 0\} \quad \text{and} \quad \delta(X) = \dim R_X.
$$

If $X = (0, \alpha)$, where $\alpha \in V_1^*$ is nonzero and $Y = (w, \beta) \in \mathfrak{j}(\mathfrak{g}; W)_1$, then $[X, Y] = \alpha(w)$ is zero if and only if $w \in \ker(\alpha)$. Therefore,

$$\delta(0, \alpha) = 2\dim(V_1) - 1 \quad \text{whenever } \alpha \neq 0. \tag{6.2}$$

If $X = (v, \alpha)$ with $v \neq 0$ and $Y = (w, \beta)$, then $[X, Y] = 0$ if and only if $[v, w] = 0$ and $\alpha(w) - \beta(v) = 0$. Since ad_v is nontrivial on V_1 by assumption, the projection of R_X to V_1 has dimension at most $\dim(V_1) - 1$. Moreover, each fiber in R_X of this projection has dimension $\dim(V_1) - 1$, because $\alpha(w) - \beta(v) = 0$ is a nontrivial linear equation in β, when w is fixed. We conclude that

$$\delta(v, \alpha) \leq 2\dim(V_1) - 2 \quad \text{whenever } v \neq 0. \tag{6.3}$$

Fix $p \in \Omega$. We claim that, for every $X \in \mathfrak{j}(\mathfrak{g}; W)_1$,

$$\delta(F_*\tilde{X}|_{F(p)}) = \delta(X). \tag{6.4}$$

If $Y \in \mathfrak{j}(\mathfrak{g}; W)_1$ is such that $[X, Y] = 0$, then $F_*([\tilde{X}, \tilde{Y}]) = [F_*\tilde{X}, F_*\tilde{Y}] = 0$ because F is a diffeomorphism. Since F is contact, both $F_*\tilde{X}$ and $F_*\tilde{Y}$ are horizontal vector fields. Thus, we get from Lemma 6.5 that $[(F_*\tilde{X})_{F(p)}, (F_*\tilde{Y})_{F(p)}] = 0$. Therefore, $F_*\tilde{Y}_{F(p)} \in R_{F_*\tilde{X}_{F(p)}}$. Since F^{-1} is also a contact diffeomorphism we conclude (6.4).

From (6.4) and the previous computation of $\delta(X)$ in (6.2) and (6.3), we obtain that if $\alpha \in V_1^*$ then $F_*\tilde{\alpha}_{F(p)} \in V_1^*$ for every $p \in \Omega$.

From this point, the proof concludes as for Theorem 6.1 in Section 6.2. ∎

7. Embedding of Carnot groups into jet spaces

In this section we prove Theorem C and compute its application to groups of step 2 and 3.

7.1. Proof of Theorem C

For this proof, we will extensively use the identification of the group with its Lie algebra via the exponential map. More explicitly, if $\mathfrak{g}$ is the Lie algebra of (a nilpotent, simply connected Lie group) G, then we define, for $x, y \in \mathfrak{g}$,

$$xy = \log(\exp(x)\exp(y)),$$

which, via the BCH formula, has an explicit polynomial expression. If f is a smooth function and $v \in \mathfrak{g}$, then the corresponding left and right invariant vector fields take the form

$$\tilde{v} f(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(p(tv)) \quad \text{and} \quad v^\dagger f(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f((tv)p).$$

Let G be the stratified group from Theorem C with stratified Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^{s+1} V_j$. For $x \in \mathfrak{g}$, we write $x_j = \Pi_j(x) \in V_j$ and $x' = \sum_{j \leq s} x_j \in \mathfrak{g}'$. The BCH formula gives a map $\eta\colon \mathfrak{g} \times \mathfrak{g} \to V_{s+1}$, which is a polynomial, such that

$$\Pi_{s+1}(xy) = x_{s+1} + y_{s+1} + \eta(x, y).$$

The function η has the following properties:

(1) $\eta(x, y)$ depends only on x' and y',

(2) $\eta(\delta_\lambda x, \delta_\lambda y) = \lambda^{s+1}\eta(x, y)$,

(3) since

$$\Pi_{s+1}((xy)z) = x_{s+1} + y_{s+1} + z_{s+1} + \eta(x,y) + \eta(xy,z)$$

and

$$\Pi_{s+1}(x(yz)) = x_{s+1} + y_{s+1} + z_{s+1} + \eta(y,z) + \eta(x,yz),$$

we have

$$\eta(xy,z) + \eta(x,y) = \eta(x,yz) + \eta(y,z). \tag{7.1}$$

Define $\mathfrak{g}' = \bigoplus_{j=1}^{s} V_j$ with Lie brackets $[x,y]' = [x,y] - \Pi_{s+1}([x,y])$, i.e., $\mathfrak{g}' \simeq \mathfrak{g}/V_{s+1}$. We identify again the group G' with $\mathfrak{g}'$ endowed with the group operation given by the BCH formula. One can easily check that

$$(xy)' = (x'y')' = (x'y')_{\mathfrak{g}'},$$

where the last term is the group operation given by the BCH formula on $\mathfrak{g}'$.

By Theorem 4.5, the polynomial jet space $\mathrm{J}^s_{\mathcal{P}}(G';W)$ is isomorphic to the jet space $\mathrm{J}^s(G';W)$ defined in Section 3.4. We will define an injective morphism of stratified Lie algebras $\phi\colon \mathfrak{g} \to \mathrm{j}^s_{\mathcal{P}}(G';V_{s+1})$. We consider linear maps ϕ of the following form: For every $k \in \{1,\dots,s+1\}$, there is a linear map $\phi_k\colon V_k \to \mathcal{P}^{s+1-k}_{e_{G'}}(G';V_{s+1})$ such that for all $v \in V_k$,

$$\phi(v) = v + \phi_k(v) \in V_k \oplus \mathcal{P}^{s+1-k}_{e_{G'}}(G';V_{s+1}) = \mathrm{j}^s_{\mathcal{P}}(\mathfrak{g}';V_{s+1})_k. \tag{7.2}$$

Notice that $\phi_{s+1}\colon V_{s+1} \to V_{s+1}$. Any such map ϕ is a morphism of stratified Lie algebras if and only if it is a Lie algebra morphism, because it already preserves the stratification. Furthermore, ϕ is a Lie algebra morphism if and only if for every $v \in V_i$ and $w \in V_j$, we have $[\phi(v),\phi(w)] = \phi([v,w])$, that is, by (4.3),

$$w \neg \phi_i(v) - v \neg \phi_j(w) = \phi_{i+j}([v,w]) \qquad \text{if } i+j < s+1, \tag{7.3}$$

$$w \neg \phi_i(v) - v \neg \phi_j(w) = [v,w] + \phi_{s+1}([v,w]) \quad \text{if } i+j = s+1. \tag{7.4}$$

For $v \in V_k$, $1 \le k \le s+1$, we define

$$\phi_k(v) := (v_x^\dagger \eta)(e,y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta(tv,y), \tag{7.5}$$

by which we mean that $\phi_k(v)$ is the polynomial in y resulting from deriving η in x along the right invariant vector field $v^\dagger$ and evaluating at $x = e$. Similarly, we will also write $v_y^\dagger$ to denote the derivation in y. Notice that since η depends only on y', $\phi_k(v)$ is actually a function on G'.

We claim that $(v_x^\dagger \eta)(e,y) \in \mathcal{P}^{s+1-k}_{e_{G'}}(G';V_{s+1})$ when $v \in V_k$. Indeed, for every $\lambda > 0$,

$$\begin{aligned}
(v_x^\dagger \eta)(e,\delta_\lambda y) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta((tv),\delta_\lambda y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta(\delta_\lambda(\lambda^{-k}tv),\delta_\lambda y) \\
&= \lambda^{s+1} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta((\lambda^{-k}tv),y) = \lambda^{s+1-k} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta((tv),y) \\
&= \lambda^{s+1-k} (v_x^\dagger \eta)(e,y).
\end{aligned}$$

Therefore, $(v_x^\dagger \eta)(e,y) \in \mathcal{P}^{s+1-k}_{e_{G'}}(G';V_{s+1})$, and ϕ is therefore well defined.

Next, we show that ϕ does in fact satisfy (7.3) and (7.4). Let $v \in V_i$ and $w \in V_j$ for $1 \le i, j \le s+1$. Then, using (7.1),

$$\begin{aligned} w \neg \phi_i(v)(y) &= w_y^\dagger(v_x^\dagger \eta)(e, y) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (v_x^\dagger \eta)(e, (tw)y) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} \eta(rv, (tw)y) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} (\eta((rv)(tw), y) + \eta(rv, tw) - \eta(tw, y)) \\ &= (w_x^\dagger v_x^\dagger \eta)(e, y) + \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} \eta(rv, tw). \end{aligned}$$

Notice that

$$\eta(rv, tw) - \eta(tw, rv) = \Pi_{s+1}\big(\log(\exp(rv)\exp(tw)) - \log(\exp(tw)\exp(rv))\big),$$

and that the BCH formula implies that

$$\log(\exp(rv)\exp(tw)) = rv + tw + \frac{rt}{2}[v, w] + P(rv, tw),$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}r}\big|_{r=0} P(rv, tw) = 0$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} (\eta(rv, tw) - \eta(tw, rv)) = \Pi_{s+1}([v, w]).$$

We conclude that

$$\begin{aligned} (w \neg \phi_i(v) - v \neg \phi_j(w))(y) &= (w_x^\dagger v_x^\dagger \eta)(e, y) - (v_x^\dagger w_x^\dagger \eta)(e, y) + \Pi_{s+1}([v, w]) \\ &= (-[v^\dagger, w^\dagger]_x \eta)(e, y) + \Pi_{s+1}([v, w]) \\ &= ([v, w]_x^\dagger \eta)(e, y) + \Pi_{s+1}([v, w]), \end{aligned}$$

where we used the standard relation $-[v^\dagger, w^\dagger] = [v, w]^\dagger$ between right and left invariant vector fields.

We have thus shown that the map ϕ defined as in (7.2) with (7.5) is a morphism of stratified Lie algebras $\phi\colon \mathfrak{g} \to \mathfrak{j}^s(\mathfrak{g}'; V_{s+1})$. The only missing property we need to conclude is that ϕ is injective. Since $\phi|_{V_k}$ is injective for every $k < s+1$ and since ϕ preserves the stratification, we only need to check that $\phi|_{V_{s+1}}$ is injective. But $\phi|_{V_{s+1}}$ is the identity, because if $v \in V_{s+1}$, then $(v_x^\dagger \eta)(e, y) = 0$, as η does not depend on x_{s+1}. ■

7.2. Example: step 2

As an example, we will show how stratified groups of step 2 embed into the standard jet spaces over abelian groups. Let $\mathfrak{g} = V_1 \oplus V_2$ be a stratified Lie algebra of step 2. Then

$$\mathfrak{j}^1(V_1; V_2) = V_1 \oplus V_2 \oplus \mathtt{Lin}(V_1; V_2),$$

with Lie brackets

$$[(v_1, v_2, A), (w_1, w_2, B)] = (0, A(w_1) - B(v_1), 0).$$

Notice that

$$\log(\exp(v)\exp(w)) = v + w + \frac{1}{2}[v,w] = (v_1 + w_1) + \Big(v_2 + w_2 + \frac{1}{2}[v_1, w_1]\Big),$$

so that $\eta(v,w) = \frac{1}{2}[v_1, w_1]$. The embedding constructed in Theorem C is

$$\phi(v_1, v_2) = \Big(v_1, v_2, \frac{1}{2}[v_1, \cdot]\Big).$$

7.3. Example: step 3

Let $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3$ be a stratified Lie algebra of step 3. Then $\mathfrak{g}' = V_1 \oplus V_2$ and

$$\mathrm{j}^2(V_1 \oplus V_2; V_3) = (V_1 \oplus V_2) \oplus V_3 \oplus \mathtt{Lin}(\mathfrak{g}'; V_3) \oplus \mathtt{HD}^2(\mathfrak{g}'; V_3),$$

with Lie brackets

$$\begin{aligned}&[(v_1, v_2; v_3; A^1; A^2), (w_1, w_2; w_3; B^1; B^2)] \\ &\quad = \big(0, [v_1, w_1]; A^1(w_1) - B^1(v_1) + A^2(w_2) - B^2(v_2); A^2(w_1) - B^2(v_1); 0\big).\end{aligned}$$

Notice that

$$\begin{aligned}\log(\exp(v)\exp(w)) &= v + w + \frac{1}{2}[v,w] + \frac{1}{12}([v,[v,w]] + [w,[w,v]]) \\ &= (v_1 + w_1) + \Big(v_2 + w_2 + \frac{1}{2}[v_1, w_1]\Big) \\ &\quad + \Big(v_3 + w_3 + \frac{1}{2}([v_1, w_2] + [v_2, w_1])\Big) \\ &\quad + \frac{1}{12}([v_1,[v_1,w_1]] + [w_1,[w_1,v_1]]),\end{aligned}$$

so that

$$\eta(v,w) = \frac{1}{2}([v_1, w_2] + [v_2, w_1]) + \frac{1}{12}([v_1,[v_1,w_1]] + [w_1,[w_1,v_1]]).$$

Therefore,

$$\begin{aligned}\phi_1(v_1)(w) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \eta(tv_1, w) = \frac{1}{2}[v_1, w_2] + \frac{1}{12}[w_1,[w_1,v_1]], \\ \phi_2(v_2)(w) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \eta(tv_2, w) = \frac{1}{2}[v_2, w_1], \\ \phi_3(v_3)(w) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \eta(tv_3, w) = 0 \quad \text{(as expected)}.\end{aligned}$$

We obtain

$$\phi(v_1, v_2, v_3) = \Big(v_1, v_2; v_3; \frac{1}{2}[v_2, \Pi_1(\cdot)]; \frac{1}{2}[v_1, \Pi_2(\cdot)] + \frac{1}{12}[\Pi_1(\cdot), [\Pi_1(\cdot), v_1]]\Big),$$

as a map into $\mathrm{J}^2_{\mathcal{P}}(G'; V_3)$.

If we want to write the components of ϕ as multilinear maps (instead of polynomials), we can use Theorem 4.5. So, if $P: G' \to V_3$ is the map $P(x) = [\Pi_1(x), [\Pi_1(x), v_1]]$, then we need to compute $A^2_{P,e}(x_1, y_1) = \tilde{y}_1 \tilde{x}_1 P(e)$, which we can compute using (3.2). We obtain

$$A^2_{P,e}(x_1, y_1) = [x_1, [y_1, v_1]] + [y_1, [x_1, v_1]]$$

Similarly, for $Q(x) = [v_1, \Pi_2(x)]$, we get $A^2_{Q,e}(x_1, y_1) = \frac{1}{2}[v_1, [y_1, x_1]]$.

Therefore, as a map into $\mathtt{J}^2(G'; V_3)$, we have

$$\phi(v_1, v_2, v_3) = \Big(v_1, v_2; v_3; \frac{1}{2}[v_2, \cdot]; \frac{1}{2}[v_1, [y, x]] + \frac{1}{12}[x, [y, v_1]] + [y, [x, v_1]]\Big),$$

where x and y are the place holders to indicate the ordered entries of the bilinear map.

Remark 7.1. Notice that if $G = \mathbb{R}^n$ is abelian, then $\mathtt{J}^m(\mathbb{R}^n; W)$ has the property

$$[\mathfrak{j}^m(\mathbb{R}^n; W)_i, \mathfrak{j}^m(\mathbb{R}^n; W)_j] = 0 \quad \text{for all } i, j > 1. \tag{7.6}$$

Every stratified subgroup of $\mathtt{J}^m(\mathbb{R}^n; W)$ must also satisfy (7.6). Therefore, not all stratified Lie group of step larger than 2 can be embedded in a standard jet space.

8. Example: the first Heisenberg group

8.1. The first Heisenberg group

The Lie algebra $\mathfrak{h}$ of the *first Heisenberg group* $\mathbb{H}$ is the stratified three-dimensional Lie algebra with basis X, Y, Z and with the only non-trivial bracket relation $Z = [X, Y]$. The stratification $\mathfrak{h} = V_1 \oplus V_2$ is given by $V_1 = \operatorname{span}\{X, Y\}$ and $V_2 = \mathbb{R}Z$. In exponential coordinates (x, y, z), we have

$$\tilde{X}(x, y, z) = \partial_x - \frac{y}{2}\partial_z, \quad \tilde{Y}(x, y, z) = \partial_y + \frac{x}{2}\partial_z, \quad \tilde{Z}(x, y, z) = \partial_z.$$

We will apply the algorithm described in Remark 4.8 to obtain a basis for $\mathtt{HD}^m(\mathfrak{h}; \mathbb{R})$ for $m \in \{1, 2, 3\}$. Moreover, we compute the corresponding basis of $\mathcal{P}^m_e(G; \mathbb{R})$. The corresponding basis for $\mathcal{P}^m_p(G; \mathbb{R})$ can be obtained via (4.2).

8.2. A basis for $\mathtt{HD}^1(\mathfrak{h}; \mathbb{R})$

We have already chosen the basis $\mathcal{B} = \{X, Y\}$ of V_1, which has a dual basis $\{X^*, Y^*\}$ for V_1^*. The set of multi-indices is $\mathcal{I}^1 = \{(1,0,0), (0,1,0)\}$. A basis of $\mathcal{U}^1(\mathbb{H})$ is $\{\tilde{X}, \tilde{Y}\}$. The map $\tau: \mathcal{T}^1(V_1) \to \mathcal{U}^1(\mathbb{H})$ is given by $\tau(X) = \tilde{X}$ and $\tau(Y) = \tilde{Y}$, so, by (4.7),

$$A_{(1,0,0)} = X^* \quad \text{and} \quad A_{(0,1,0)} = Y^*$$

form a basis of $\mathtt{HD}^1(\mathfrak{h}; \mathbb{R})$. The basis of $\mathcal{P}^1_e(\mathbb{H}; \mathbb{R})$ dual to $\{\tilde{X}, \tilde{Y}\}$ is $\{x, y\}$.

8.3. A basis for $\mathtt{HD}^2(\mathfrak{h};\mathbb{R})$

In this case, we have $\Xi^2 = \{X\otimes X, X\otimes Y, Y\otimes X, Y\otimes Y\}$. The set of multi-indices is $\mathcal{I}^2 = \{(2,0,0),(1,1,0),(0,2,0),(0,0,1)\}$. A basis of $\mathcal{U}^2(\mathbb{H})$ is $\{\tilde X^2, \tilde X\tilde Y, \tilde Y^2, \tilde Z\}$. We can compute the map $\tau\colon \mathcal{T}^2(V_1)\to\mathcal{U}^2(\mathbb{H})$ as

$$\begin{aligned}\tau(X\otimes X) &= \tilde X\tilde X, \quad \tau(X\otimes Y) = \tilde Y\tilde X = \tilde X\tilde Y - \tilde Z,\\ \tau(Y\otimes X) &= \tilde X\tilde Y, \quad \tau(Y\otimes Y) = \tilde Y\tilde Y.\end{aligned}$$

Notice the specular order of X and Y on the two sides of the equality. A basis of $\mathtt{HD}^2(\mathfrak{h};\mathbb{R})$ is

$$\begin{aligned}A_{(2,0,0)} &= X^*\otimes X^*, \quad A_{(1,1,0)} = X^*\otimes Y^* + Y^*\otimes X^*,\\ A_{(0,2,0)} &= Y^*\otimes Y^*, \quad A_{(0,0,1)} = -X^*\otimes Y^*.\end{aligned}$$

Notice that $\mathtt{HD}^2(\mathfrak{h};\mathbb{R}) = \mathcal{T}^2(V_1^*)$.

To compute a basis of $\mathcal{P}^2_e(\mathbb{H};\mathbb{R})$ dual to $\{\tilde X^2, \tilde X\tilde Y, \tilde Y^2, \tilde Z\}$ we need to first compute the action of each element of the latter basis to $\{x^2, xy, y^2, z\}$, which is a basis of $\mathcal{P}^2_e(\mathbb{H};\mathbb{R})$. We present this action in the following table:

	$\tilde X^2$	$\tilde X\tilde Y$	$\tilde Y^2$	$\tilde Z$
x^2	2	0	0	0
xy	0	1	0	0
y^2	0	0	2	0
z	0	1/2	0	1

Therefore, the basis of $\mathcal{P}^2_e(\mathbb{H};\mathbb{R})$ dual to $\{\tilde X^2, \tilde X\tilde Y, \tilde Y^2, \tilde Z\}$ is $\{x^2/2, xy, y^2/2, z - xy/2\}$.

8.4. A basis for $\mathtt{HD}^3(\mathfrak{h};\mathbb{R})$

In this case, we have

$$\begin{aligned}\Xi^3 = \{&X\otimes X\otimes X, X\otimes X\otimes Y, X\otimes Y\otimes X, X\otimes Y\otimes Y,\\ &Y\otimes X\otimes X, Y\otimes X\otimes Y, Y\otimes Y\otimes X, Y\otimes Y\otimes Y\}.\end{aligned}$$

The set of multi-indices is

$$\mathcal{I}^3 = \{(3,0,0),(2,1,0),(1,2,0),(0,3,0),(1,0,1),(0,1,1)\},$$

and a basis of $\mathcal{U}^3(\mathbb{H})$ is

$$\{\tilde X^3, \tilde X^2\tilde Y, \tilde X\tilde Y^2, \tilde Y^3, \tilde X\tilde Z, \tilde Y\tilde Z\}.$$

We are now giving the formulas without description:

$$\begin{aligned}\tau(X\otimes X\otimes X) &= \tilde X\tilde X\tilde X = \tilde X^3,\\ \tau(X\otimes X\otimes Y) &= \tilde Y\tilde X\tilde X = \tilde X^2\tilde Y - 2\tilde X\tilde Z,\end{aligned}$$

$$\begin{aligned}
\tau(X\otimes Y\otimes X) &= \tilde{X}\tilde{Y}\tilde{X} = \tilde{X}^2\tilde{Y} - \tilde{X}\tilde{Z},\\
\tau(X\otimes Y\otimes Y) &= \tilde{Y}\tilde{Y}\tilde{X} = \tilde{X}\tilde{Y}^2 - 2\tilde{Y}\tilde{Z},\\
\tau(Y\otimes X\otimes X) &= \tilde{X}\tilde{X}\tilde{Y} = \tilde{X}^2\tilde{Y},\\
\tau(Y\otimes X\otimes Y) &= \tilde{Y}\tilde{X}\tilde{Y} = \tilde{X}\tilde{Y}^2 - \tilde{Y}\tilde{Z},\\
\tau(Y\otimes Y\otimes X) &= \tilde{X}\tilde{Y}\tilde{Y} = \tilde{X}\tilde{Y}^2,\\
\tau(Y\otimes Y\otimes Y) &= \tilde{Y}\tilde{Y}\tilde{Y} = \tilde{Y}^3;
\end{aligned}$$

$$\begin{aligned}
A_{(3,0,0)} &= X^*\otimes X^*\otimes X^*,\\
A_{(2,1,0)} &= X^*\otimes X^*\otimes Y^* + X^*\otimes Y^*\otimes X^* + Y^*\otimes X^*\otimes X^*,\\
A_{(1,2,0)} &= X^*\otimes Y^*\otimes Y^* + Y^*\otimes X^*\otimes Y^* + Y^*\otimes Y^*\otimes X^*,\\
A_{(0,3,0)} &= Y^*\otimes Y^*\otimes Y^*,\\
A_{(1,0,1)} &= -2X^*\otimes X^*\otimes Y^* - X^*\otimes Y^*\otimes X^*,\\
A_{(0,1,1)} &= -2X^*\otimes Y^*\otimes Y^* - Y^*\otimes X^*\otimes Y^*.
\end{aligned}$$

Finally, we present the actions in the following table:

	$\tilde{X}^3$	$\tilde{X}^2\tilde{Y}$	$\tilde{X}\tilde{Y}^2$	$\tilde{Y}^3$	$\tilde{X}\tilde{Z}$	$\tilde{Y}\tilde{Z}$
x^3	6	0	0	0	0	0
x^2y	0	2	0	0	0	0
xy^2	0	0	2	0	0	0
y^3	0	0	0	6	0	0
xz	0	1	0	0	1	0
yz	0	0	1	0	0	1

From this table, we get the basis of $\mathcal{P}^3_e(\mathbb{H};\mathbb{R})$ dual to $\{\tilde{X}^3, \tilde{X}^2\tilde{Y}, \tilde{X}\tilde{Y}^2, \tilde{Y}^3, \tilde{X}\tilde{Z}, \tilde{Y}\tilde{Z}\}$:

$$\left\{\frac{x^3}{6}, \frac{x^2y}{2}, \frac{xy^2}{2}, \frac{y^3}{6}, xz - \frac{x^2y}{2}, yz - \frac{xy^2}{2}\right\}.$$

8.5. The Lie algebra $\mathfrak{j}^2(\mathfrak{h};\mathbb{R})$

We will describe in Tables 1 and 2 the Lie algebra structure of

$$\mathfrak{j}^2(\mathfrak{h};\mathbb{R}) = \mathfrak{h}\oplus\mathbb{R}\oplus \mathtt{HD}^1(\mathfrak{h};\mathbb{R})\oplus \mathtt{HD}^2(\mathfrak{h};\mathbb{R}).$$

The following computation for one of the Lie brackets might be instructive to the reader:

$$\begin{aligned}
[A_{(0,0,-1)}, A_{(0,0,1)}] &= [Z, X^*\otimes Y^*] = -Z\neg(X^*\otimes Y^*) = [X\neg, Y\neg](X^*\otimes Y^*)\\
&= (X\neg Y\neg - Y\neg X\neg)(X^*\otimes Y^*) = (X\neg X^* - Y\neg 0) = T.
\end{aligned}$$

The second identity is an application of the definition (3.8), while the third one is an application of Proposition 3.3. The symbol T denotes the standard basis element of $\mathbb{R}$, that is, 1.

1	2	3	4	5	6	7	8	9	10
$A_{(-1,0,0)}$ X	$A_{(0,-1,0)}$ Y	$A_{(2,0,0)}$ $X^* \otimes X^*$	$A_{(1,1,0)}$ $X^* \otimes Y^* + Y^* \otimes X^*$	$A_{(0,2,0)}$ $Y^* \otimes Y^*$	$A_{(0,0,1)}$ $-X^* \otimes Y^*$	$A_{(0,0,-1)}$ Z	$A_{(1,0,0)}$ X^*	$A_{(0,1,0)}$ Y^*	$A_{(0,0,0)}$ T

Table 1. Reference table of explicit symbols used in Table 2

$*$	$A_{(-1,0,0)}$	$A_{(0,-1,0)}$	$A_{(2,0,0)}$	$A_{(1,1,0)}$	$A_{(0,2,0)}$	$A_{(0,0,1)}$	$A_{(0,0,-1)}$	$A_{(1,0,0)}$	$A_{(0,1,0)}$	$A_{(0,0,0)}$
$A_{(-1,0,0)}$	0	$A_{(0,0,-1)}$	$-A_{(1,0,0)}$	$-A_{(0,1,0)}$	0	0	0	$-A_{(0,0,0)}$	0	0
$A_{(0,-1,0)}$	$-A_{(0,0,-1)}$	0	0	$-A_{(1,0,0)}$	$-A_{(0,1,0)}$	$A_{(1,0,0)}$	0	0	$-A_{(0,0,0)}$	0
$A_{(2,0,0)}$	$A_{(1,0,0)}$	0	0	0	0	0	0	0	0	0
$A_{(1,1,0)}$	$A_{(0,1,0)}$	$A_{(1,0,0)}$	0	0	0	0	0	0	0	0
$A_{(0,2,0)}$	0	$A_{(0,1,0)}$	0	0	0	0	0	0	0	0
$A_{(0,0,1)}$	0	$-A_{(1,0,0)}$	0	0	0	0	$-A_{(0,0,0)}$	0	0	0
$A_{(0,0,-1)}$	0	0	0	0	0	$A_{(0,0,0)}$	0	0	0	0
$A_{(1,0,0)}$	$A_{(0,0,0)}$	0	0	0	0	0	0	0	0	0
$A_{(0,1,0)}$	0	$A_{(0,0,0)}$	0	0	0	0	0	0	0	0
$A_{(0,0,0)}$	0	0	0	0	0	0	0	0	0	0

Table 2. Lie bracket relations in $\mathfrak{j}^2(\mathfrak{h};\mathbb{R})$.

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