Dimension comparison and $\mathbb{H}$-regular surfaces in Heisenberg groups

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Abstract

In this thesis we study a specific Carnot group which is the n-th Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$. Carnot groups are simply connected nilpotent Lie groups whose Lie algebra admits a stratification. The Heisenberg group $\mathbb{H}^n$ is one of the easiest examples of non-commutative Carnot groups.

In the first part of the thesis we recall some preliminaries about measure theory and Heisenberg groups which we need. We also prove some useful inequalities which are needed to prove the main theorem.

The main result of the thesis is that in any Heisenberg group $\mathbb{H}^n$ there exists a $\mathbb{H}$-regular hypersurface which has a Euclidean Hausdorff dimension of $2n + \frac{1}{2}$. This generalizes a construction in [KSC04] from $n = 1$ to $n > 1$. To prove the main result we need a dimension comparison theorem in general Heisenberg groups. We will prove such a dimension comparison theorem in the thesis combining ideas from the proofs in [BRSC03](for $\mathbb{H}^1$) and [BTW09](for Carnot groups).

Dimension comparison theorem gives us information about the absolute continuity of the Hausdorff measure when comparing the measure in Euclidean and Heisenberg point of view. As a corollary we obtain Hausdorff dimension comparison which gives us lower and upper bounds for the Heisenberg Hausdorff dimension of a set $A \subset \mathbb{H}^n$. More precisely the results are about comparing Hausdorff measures and dimension when computed in the Euclidean and the Heisenberg distance, respectively.

Tiivistelmä

Tässä tutkielmassa tutkimme Carnot-ryhmiä erikoistapauksessa, jossa Carnot-ryhmä $\mathbb{G}$ on yleinen Heisenbergin ryhmä $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$. Carnot-ryhmät ovat yhdesti yhtenäisiä nilpotentteja Lien ryhmiä, joiden Lien algebra $\mathfrak{h}$ muodostuu k-asteisesta stratifikaatiosta. Heisenbergin ryhmä $\mathbb{H}^n$ on helpoimpia esimerkkejä ei-vaihdannaisesta Carnot-ryhmästä.

Tutkielmassa esitellään ja todistetaan esitietoja mittateoriasta ja Heisenbergin ryhmistä, joita tarvitsemme tutkielmassa. Lisäksi todistetaan hyödyllisiä epäyhtälöitä, joita tarvitaan päätuloksen todistamiseen.

Tutkielman yhtenä päätuloksena todistamme, että missä tahansa Heisenbergin ryhmässä $\mathbb{H}^n$ on olemassa $\mathbb{H}$-säännöllinen hyperpinta, jonka Euklidinen Hausdorffin dimensio on $2n + \frac{1}{2}$. Päätulos yleistää paperissa [KSC04] rakennetun esimerkin tapauksessa $n = 1$ tapaukseen $n > 1$. Tämän tuloksen todistamiseen tarvitsemme dimensionvertailuteoreemaa yleisessä Heisenbergin ryhmässä. Dimensionvertailuteoreema todistetaan tutkielmassa, ja sitä varten todistamme pari propositiota, jotka kertovat Hausdorffin mitan ja pallomitan absoluuttisesta jatkuvuudesta mielivaltaisissa metrisissä avaruuksissa. Pääteorioiden todistusten ideat saamme yhdistämällä tutkimuspapereissa [BRSC03] ja [BTW09] olevia ajatuksia.

Dimensionvertailuteoreema antaa meille tietoa Hausdorffin mitan absoluuttisesta jatkuvuudesta, kun vertaamme sitä Euklidessa ja Heisenbergin tapauksessa. Dimension vertailun avulla saamme seurauksena Hausdorffin dimension vertailuteoreeman, joka on tämän tutkielman toinen päätulos. Sitä käyttämällä voimme verrata joukon Hausdorffin dimensiota Heisenbergin ryhmän ja Euklidisen avaruuden välillä.

Contents

Introduction

The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$ has been an active research topic for at least couple of decades now. The Heisenberg group equipped with the left-invariant Heisenberg distance d_H is an example of a space that is different from Euclidean space. Semmes observed that the space $(\mathbb{H}^n, d_H)$ does not bi-Lipschitzly embed into any Euclidean space. Heisenberg groups provide a rich mathematical theory for example regarding sub-Riemannian geometry or geometric measure theory in non-Euclidean space. In this thesis we focus on the geometric measure theoretic point of view.

We prove two main theorems in thesis. The first one is Corollary 30, which tells us that in any Heisenberg group $\mathbb{H}^n$ we have the next inequalities:

$$\beta_-(\dim_E(A)) \leq \dim_H A \leq \beta_+(\dim_E(A))$$

for every $A \subset \mathbb{H}^n$, where

$$\beta_-(\alpha) = \begin{cases} \alpha,\ 0 \leq \alpha \leq 2n \\ 2\alpha - 2n,\ 2n \leq \alpha \leq 2n+1 \end{cases}$$

and

$$\beta_+(\alpha) = \begin{cases} 2\alpha,\ 0 \leq \alpha \leq 1 \\ \alpha + 1,\ 1 \leq \alpha \leq 2n+1. \end{cases}$$

Here $\dim_E$ and $\dim_H$ denote the Hausdorff dimensions in the Euclidean and Heisenberg metric, respectively.

The second one is Theorem 52 which says that in any Heisenberg group $\mathbb{H}^n$ there exists an $\mathbb{H}$-regular surface $S \subset \mathbb{H}^n$ such that

$$\dim_E S = 2n + \frac{1}{2}.$$

The dimension comparison result in $\mathbb{H}^n$ is known by the paper of Balogh, Tyson and Warhurst in [BTW09]. The proof in this thesis is different from what they used in their paper [BTW09]. In this thesis we obtain the result as a corollary from Theorem 29, which tells us that for $\alpha \geq 0$ we have

$$\mathcal{H}_H^{\min\{2\alpha,\alpha+1\}} \ll \mathcal{H}_E^{\alpha}$$

and

$$\mathcal{H}_E^{\min\{\alpha,n+\alpha/2\}} \ll \mathcal{H}_H^{\alpha}$$

and we get this theorem as a consequence of a result in abstract metric spaces. In [BTW09] they obtain the result as a consequence of Hausdorff measure comparison [BTW09, Proposition 3.1 p.575]. The second result Theorem 52 is a generalisation of [KSC04, Theorem 3.1 and Remark 3.3 p.9], which combined tells that in $\mathbb{H}^1$ there exists an $\mathbb{H}$-regular surface $S \subset \mathbb{H}^1$ such that

$$\dim_E S = \frac{5}{2}.$$

The results are interesting because, for instance :

* If one wants to do geometric measure theory on $(\mathbb{H}^n, d_H)$, one needs an analog of smooth surfaces in Euclidean spaces as building blocks. The $\mathbb{H}$-regular surfaces have been proposed as such building blocks. The result shows that they can be very different from Euclidean C^1 surfaces and look like fractals. In contrast, Euclidean hypersurfaces in $\mathbb{R}^{2n+1}$ have (Euclidean) Hausdorff dimension $2n$.

* Lines in $\mathbb{R}^{2n+1}$ are 1-dimensional in Euclidean metric, but can be 1- or 2-dimensional in Heisenberg metric depending on their position and direction. The dimension comparison theorem provides a range of values that the Heisenberg Hausdorff dimension can assume for a set with given Euclidean Hausdorff dimension. The range is known to be sharp, but we do not prove this here.

Chapter 1 is about preliminaries that we will need throughout this thesis. In the first Section 1.1 we recall some basic definitions of measure theory and metric spaces. We also recall some useful lemmas and properties for Borel regular measures and theorems such as coarea inequality and mass distribution principle. In Section 1.2 we introduce the Heisenberg group $\mathbb{H}^n$ and see some basic theory about it. We will also see how the Euclidean metric and Heisenberg metric are related to each other. Lastly we will see some examples about the Hausdorff dimension of different sets from Euclidean and Heisenberg point of view which leads us towards the dimension comparison theorem in Chapter 2.

Chapter 2 is divided into two sections. In Section 2.1 we first prove a proposition about absolute continuity of Hausdorff measures in arbitrary metric spaces $(X, d_1), (X, d_2)$. We will also get the same kind of a result for spherical measure. In Section 2.2 we prove a comparison theorem in Heisenberg groups for Hausdorff dimension with respect to the Euclidean and Heisenberg metric, respectively. We also get some covering results for Heisenberg metric and Euclidean metric and absolute continuity results for Hausdorff measure and spherical measure when we apply the results from Section 2.1 to the Heisenberg group.

Chapter 3 is about the generalization of the example which Kirchheim and Serra Cassano constructed in $\mathbb{H}^1$. In Section 3.1 we define an auxiliary function $h : \mathbb{R} \to \mathbb{R}$ needed for the construction of the surface. A function with the same properties was already used in [KSC04], but here we follow the simpler construction in [Raj08]. The function has two crucial properties where one is that the Hausdorff dimension of $h^{-1}(t)$ for all points $t \in \mathbb{R}$ is at least $\frac{1}{2}$. In Section 3.2 we generalize the example of Kirchheim and Serra Cassano from $\mathbb{H}^1$ to general $\mathbb{H}^n$. Before the generalization we define what an $\mathbb{H}$-regular hypersurface is and see some easy examples about it. Then lastly with the dimension comparison theorem and the lower bound of the Euclidean Hausdorff dimension of S that we prove, we get the wanted theorem that there is an $\mathbb{H}$-regular surface $S \subset \mathbb{H}^n$ such that the $\dim_E S = 2n + \frac{1}{2}$. This theorem generalizes the example of Kirchheim and Serra Cassano.

Notation

- $\mathcal{P}(X)$ power set of X
- $B(p, r)$ closed ball of radius $r > 0$ centered at the point p
- $U(p, r)$ open ball of radius $r > 0$ centered at the point p
- d_E Euclidean metric
- d_H Heisenberg metric
- $\mathrm{diam}(A) := \sup\{d(x, y) : x, y \in A\}$
- $\mathcal{H}^s_d(A)$ s-dimensional Hausdorff measure of a set A with respect to the metric d
- $\mathcal{S}^s_d(A)$ s-dimensional spherical measure of a set A with respect to the metric d
- $\mathcal{L}^n(A)$ n-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^n$
- $\dim_H(A)$ Hausdorff dimension of a set A in an abstract metric space or in $(\mathbb{H}^n, d_H)$
- $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ ceiling function
- $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ floor function
- $\ll$ absolute continuity of a measure with respect to another measure

Chapter 1

Preliminaries

In this chapter we introduce some basic definitions and results that we need throughout this thesis. This chapter is divided into two parts. In the first part we revise some basic measure theory and some properties for metric spaces. The second part introduces Heisenberg groups and some properties that they have. The main references for this chapter are [Mat95], [CDPT07] and [BRSC03].

1.1 Measure theory and metric spaces

In this thesis we shall call *measure* what is usually called *outer measure*. Let's start with some basic definitions.

Definition 1. A set function $\mu : \mathcal{P}(X) := \{A : A \subset X\} \to [0, \infty] = \{t : 0 \leq t \leq \infty\}$ is called a measure if

1. $\mu(\emptyset) = 0$,
2. $\mu(A) \leq \mu(B)$ whenever $A \subset B$,
3. $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ whenever $A_1, A_2, ... \subset X$.

Definition 2. Let μ and λ be measures on X. Then λ is absolutely continuous with respect to μ denoted by $\lambda \ll \mu$ if for all $E \subset X$:

$$\mu(E) = 0 \text{ implies } \lambda(E) = 0,$$

cf [Mat95, Definition 2.11] for $X = \mathbb{R}^n$.

Definition 3. A mapping $d : X \times X \to [0, \infty[$ is a distance function or a metric in a set X, if it has the next properties

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$, and
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Now the pair (X,d) is called a metric space.

In this thesis we denote by $B(p,r)$ the closed ball with radius $r>0$ centered at point p and by $U(p,r)$ the open ball with radius $r>0$ centered at point p. For any $A\subset X$ we denote $\operatorname{diam}(A) := \sup\{d(x,y) : x,y\in A\}$ and we make an agreement that $\operatorname{diam}(\emptyset)^0 = 0$.

Next we recall the definition of Hausdorff measure and spherical measure on a metric space (X,d).

Definition 4. Let (X,d) be a metric space. When $0\le s<\infty$ and $0<\delta\le\infty$, then for $A\subset X$ we let

$$\mathcal{H}^s_{d,\delta}(A) := \inf\left\{\sum_{n=1}^{\infty}(\operatorname{diam}(E_n))^s : E_n\subset X, \operatorname{diam}(E_n)\le\delta, A\subset\bigcup_{n\in\mathbb{N}}E_n\right\}$$

and

$$\mathcal{S}^s_{d,\delta}(A) := \inf\left\{\sum_{n=1}^{\infty}(\operatorname{diam}(B(x_n,r_n))^s : B(x_n,r_n)\subset X, \operatorname{diam}(B(x_n,r_n))\le\delta, A\subset\bigcup_{n\in\mathbb{N}}B(x_n,r_n)\right\}.$$

We define the s-dimensional Hausdorff measure $\mathcal{H}^s_d$ on (X,d) and the s-dimensional spherical measure $\mathcal{S}^s_d$ on (X,d) by

$$\mathcal{H}^s_d(A) := \lim_{\delta\to 0}\mathcal{H}^s_{d,\delta}(A)$$

and

$$\mathcal{S}^s_d(A) := \lim_{\delta\to 0}\mathcal{S}^s_{d,\delta}(A)$$

respectively. If the metric is clear from the context we will write $\mathcal{H}^s_\delta(A), \mathcal{H}^s(A)$ instead of $\mathcal{H}^s_{d,\delta}(A), \mathcal{H}^s_d(A)$.

Remark 5. We assume throughout Section 1.1 that the metric space (X,d) is separable just to ensure that countable δ-covers always exist, to avoid technicalities.

Also we recall the definition of Hausdorff dimension for a set A.

Definition 6. The Hausdorff dimension of a set $A\subset X$ is

$$\dim_H(A) = \inf\{s\ge 0 : \mathcal{H}^s_d(A) = 0\} = \sup\{s\ge 0 : \mathcal{H}^s_d(A) = \infty\}.$$

Definition 7. Let μ be a measure on a metric space (X,d). The measure μ is Borel regular if it is a Borel measure, i.e. every Borel set is μ-measurable, and if for every $A\subset X$ there is a Borel set $B\subset X$ such that $A\subset B$ and $\mu(A)=\mu(B)$.

For the definition of μ-measurable sets, see [Mat95, 1.3].

Next as in [BRSC03, Prop. 2.2], we list some basic properties of Hausdorff measure and spherical measure which are needed later on.

Proposition 8. *1. Let (X, d) be a metric space. Then $\mathcal{H}_d^s$ and $\mathcal{S}_d^s$ are Borel regular measures and*

$$\mathcal{H}_d^s(A) \leq \mathcal{S}_d^s(A) \leq 2^s \mathcal{H}_d^s(A)$$

for all $A \subset X$, $s \geq 0$.

2. Let (X_i, d_i) $(i = 1, 2)$ be two metric spaces and let $f : (X_1, d_1) \to (X_2, d_2)$ be an L-Lipschitz continuous map, i.e

$$d_2(f(x), f(y)) \leq L d_1(x, y) \qquad (\text{for all } x, y \in X_1).$$

Then

$$\mathcal{H}_{d_2}^s(f(A)) \leq L^s \mathcal{H}_{d_1}^s(A)$$

for all $A \subset X_1$, $s \geq 0$.

3. Let (X, d) be a metric space, $t \in (0, 1)$, and let $d_t(x, y) := (d(x, y))^t$ for $x, y \in X$. Then d_t is a distance on X and

$$\mathcal{H}_d^s(A) = \mathcal{H}_{d_t}^{\frac{s}{t}}(A)$$

for all $A \subset X$, $s \geq 0$.

4. Let (X, d) be a metric space and let $Y \subset X$. Denote by d_Y the metric on Y induced by d. Then

$$\mathcal{H}_d^s(A) = \mathcal{H}_{d_Y}^s(A)$$

for all $A \subset Y$, $s \geq 0$.

5. Properties 2 and 3 hold if Hausdorff measure is replaced by spherical measure.

Proof. Only property 3 will be proved. For the other points we refer the reader to standard textbooks in geometric measure theory such as [Mat95].

First claim is that d_t is a distance:

1. $d_t(x, x) = (d(x, x))^t = 0$ because $d(x, x) = 0$.
2. $d_t(x, y) = (d(x, y))^t > 0$ because $d(x, y) > 0$ if and only if $x \neq y$.
3. $d_t(x, y) = (d(x, y))^t = (d(y, x))^t = d_t(y, x)$ because $d(x, y) = d(y, x)$.
4. To prove the triangle inequality we use the fact that for $a, b \geq 0, t \in (0, 1)$ it holds $(a + b)^t \leq a^t + b^t$. Notice that the function $f(x) = x^t$ is concave on the interval $[0, \infty)$ with $f(0) = 0$. Now for $a, b \geq 0$ we get

   $$(a+b)^t = \frac{a}{a+b}(a+b)^t + \frac{b}{a+b}(a+b)^t \leq \left(\frac{a}{a+b}(a+b)\right)^t + \left(\frac{b}{a+b}(a+b)\right)^t = a^t + b^t.$$

   Now

   $$d_t(x, z) = (d(x, z))^t \leq (d(x, y) + d(y, z))^t \leq (d(x, y))^t + (d(y, z))^t = d_t(x, y) + d_t(y, z).$$

The second claim is that $\mathcal{H}^s_d(A) = \mathcal{H}^{s/t}_{d_t}(A)$ for $A \subset X$. In this proof the notation $\mathrm{diam}E^d_n := \sup\{d(x,y) : x, y \in E_n\}$ tells in which metric we are taking the supremum. Let $\{E_n\}_{n\in\mathbb{N}}$ be a δ-cover for set the A in metric d. Now we get

$$\sum_{n\in\mathbb{N}} (\mathrm{diam}E^d_n)^s = \sum_{n\in\mathbb{N}} (\mathrm{diam}E^d_n)^{\frac{ts}{t}} = \sum_{n\in\mathbb{N}} (\mathrm{diam}E^{d_t}_n)^{\frac{s}{t}}.$$

Now taking infimum of all δ-covers and letting $\delta \to 0$ we get $\mathcal{H}^s_d(A) = \mathcal{H}^{s/t}_{d_t}(A)$. Notice that a δ-cover for d is a δ^t-cover for d_t. □

Next we have a lemma that can be used to give a lower bound on the Hausdorff dimension of a set in metric spaces.

Lemma 9. *(Mass distribution principle): Let (X, d) be a metric space, let $A \subset X$ and let $s > 0$. Suppose there is a measure μ on X with the following property:*

1. *$\mu(B(x, r) \cap A) \le c_\mu r^s$ for all $x \in X$ and $0 < r \le R$, where c_μ,$R > 0$ are fixed constants.*

Then one can conclude that $\mu(A) \le c_\mu \mathcal{H}^s_d(A)$.

In particular, if additionally $\mu(A) > 0$ holds true, then $\mathcal{H}^s_d(A) > 0$.

Proof. Let $\delta < R$. Let $\{E_n\}_{n\in\mathbb{N}}$ be a cover for A where $E_n \subset X$ and $\mathrm{diam}(E_n) \le \delta$. Now $A \subset \bigcup_{n\in\mathbb{N}} E_n \cap A$ and if $x_n \in E_n$, then $E_n \subset B(x_n, \mathrm{diam}(E_n))$. Now we get that

$$\begin{aligned}
\mu(A) &\le \mu(\bigcup_{n\in\mathbb{N}} B(x_n, \mathrm{diam}(E_n)) \cap A) \\
&\le \sum_{n\in\mathbb{N}} \mu(B(x_n, \mathrm{diam}(E_n)) \cap A) \\
&\le \sum_{n\in\mathbb{N}} c_\mu (\mathrm{diam}(E_n))^s = c_\mu \sum_{n\in\mathbb{N}} (\mathrm{diam}(E_n))^s.
\end{aligned}$$

Now taking infimum over covers and let $\delta \to 0$ we get $\mu(A) \le c_\mu \mathcal{H}^s_d(A)$. If $\mu(A) > 0$ then $\mathcal{H}^s_d(A) > 0$ can be instantly concluded from the proof above. □

Theorem 10. *Suppose X is a metric space, suppose μ is a Borel-regular measure on X, and suppose that $X = \cup_{j=1}^\infty V_j$, where $\mu(V_j) < \infty$ and V_j is open for each $j = 1, 2, \ldots$. Then*

$$\mu(A) = \inf_{O\ open, A\subset O} \mu(O) \tag{1.1}$$

for each subset $A \subset X$, and

$$\mu(A) = \sup_{K\ closed, K\subset A} \mu(K) \tag{1.2}$$

for each μ-measurable subset $A \subset X$.

Proof. Proof can be found in [Hol16, Theorems 1.24 and 1.26]. □

Definition 11. Let (X, d) be a metric space and $A \subset X$ with $\operatorname{diam}(A) > 0$. A measure μ on X is said to be an s-Ahlfors regular measure supported on A, if A is closed and there exists a constant $C \geq 1$, such that $\mu(X \setminus A) = 0$ and

$$\frac{1}{C} r^s \leq \mu(B(x, r)) \leq C r^s \text{ for all } x \in A, 0 < r \leq \operatorname{diam}(A),\ r < \infty.$$

If the condition is satisfied for $A = X$, we also say that μ is an s-Ahlfors regular measure on X.

Ahlfors regularity is a useful property for a measure to have like the next lemma shows.

Lemma 12. *Let (X, d) be a metric space, and suppose that μ is a Borel regular measure on X with the property that there are positive constants K and s such that*

$$K^{-1} r^s \leq \mu(B(x, r)) \leq K r^s$$

for all $x \in X$ and $0 < r \leq \operatorname{diam}(X)$, $r < \infty$. Then there is a constant C depending only on K and s so that $C^{-1}\mu(E) \leq \mathcal{H}^s(E) \leq C\mu(E)$ for all sets $E \subset X$.

Proof. "$\mu(E) \lesssim \mathcal{H}^s(E)$" The measure μ satisfies the condition (1) from Lemma 9 which tells us that $\mu(E) \leq K\mathcal{H}^s(E)$.

"$\mathcal{H}^s(E) \leq C\mu(E)$" Let O be an open set such that $E \subset O$. We claim that

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(O) \leq C\mu(O), \tag{1.3}$$

where the constant C depends only on K and s.

Taking infimum over all open sets O such that $E \subset O$ we can conclude that

$$\mathcal{H}^s(E) \leq C \inf_{O \text{ open}, E \subset O} \mu(O) \overset{(1.1)}{=} C\mu(E).$$

To justify line (1.3) notice because O is open then for every $x \in O$ there is $\tilde{r}_x$ such that $B(x, \tilde{r}_x) \subset O$ and $\tilde{r}_x$ can be chosen to be less than a given small enough δ. Now by the $5r$-covering lemma [Sim83, Theorem 3.3] there is a subcollection of these balls $\{B(x, \tilde{r}_x) : x \in O\}$ such that $O \subset \bigcup_{i \in I} B(x_i, 5r_i)$ and $B(x_i, r_i)$ are pairwise disjoint. Since the metric space (X, d) is separable it follows that the index set I is countable. Now

$$\begin{aligned}
\mathcal{H}^s_{10\delta}(O) &\leq \sum_{i \in I} (\operatorname{diam}(B(x_i, 5r_i)))^s \leq \sum_{i \in I} 2^s 5^s r_i^s \\
&\leq K 10^s \sum_{i \in I} \mu(B(x_i, r_i)) = K10^s \mu(\bigcup_{i \in I} B(x_i, r_i)) \leq K 10^s \mu(O).
\end{aligned}$$

Now letting $\delta \to 0$ we get $\mathcal{H}^s(O) \leq K10^s\mu(O)$. □

For sets supporting s-Ahlfors regular measures we get a nice corollary from Lemma 12.

Corollary 13. *Let $A \subset X$ be a closed set in a metric space (X, d) for which there exists an s-Ahlfors regular and Borel regular measure μ supported on A. Then,* $\dim_H(A) = s$.

Proof. By s-Ahlfors regularity we know that $C^{-1}r^s \leq \mu(B(x,r)) \leq Cr^s$ for every $r \in (0, \operatorname{diam}(A)], r < \infty$ and $x \in A$ which already tells us that μ is not a zero measure. Now using Lemma 9 we get that $\mathcal{H}^s(A) > 0$ which means that $\dim_H(A) \geq s$.

For $\dim_H(A) \leq s$ we have two cases.

1. If A is a bounded set then using Lemma 12 we get
$$\mathcal{H}^s(A) \leq C\mu(A) \leq C(\operatorname{diam}(A))^s < \infty$$
which gives us $\dim_H A \leq s$.

2. If A is a unbounded set then we use the fact that in metric spaces unbounded set can be represented as countable union of bounded sets and $\dim_H A = \sup_{n \in \mathbb{N}} \dim_H A_n$ when $A = \bigcup_{n \in \mathbb{N}} A_n$. Then by the conclusion of Lemma 12 for $E = A_n$
$$\mathcal{H}^s(A_n) \leq C\mu(A_n) \leq C(\operatorname{diam}(A_n))^s < \infty.$$
Now we know that $\dim_H A_n \leq s$ for every A_n from which we can conclude that $\sup_{n \in \mathbb{N}} \dim_H A_n \leq s$ and now using the fact that $\dim_H A = \sup_{n \in \mathbb{N}} \dim_H A_n$ gives the claim.

□

Definition 14. A measure μ on a metric space X is said to be doubling if there exists constant $C \geq 1$ such that
$$0 < \mu(B(x, 2r)) \leq C\mu(B(x,r)) < \infty$$
for all $x \in X$ and $r > 0$.

Definition 15. A metric space (X, d) is said to be doubling if there exists $N \in \mathbb{N}$ such that for every $r > 0$ every ball of radius $2r$ can be covered with at most N balls of radius r.

Lemma 16. *If a metric space (X, d) admits a doubling Borel measure then (X, d) is metrically doubling.*

Proof. Let us take r-separated points $\{x_i\}_{i=1}^m$ from $B(x, 2r)$. Now the balls $U(x_i, r/2)$ are disjoint and notice that $U(x_i, r/2) \subset U(x, (5r)/2)$ because if $q \in U(x_i, r/2)$ then $d(q, x) \leq d(q, x_i) + d(x_i, x) < r/2 + 2r = \frac{5}{2}r$. From the same kind of argument one can see that $U(x, \frac{5}{2}r) \subset U(x_i, \frac{9}{2}r)$ holds for every $i \in \{1, ..., m\}$. Let x_1 be a point such that $\mu(B(x_1, r/2)) \leq \mu(B(x_i, r/2))$ for all $i \in \{1, ..., m\}$. Now we get
$$\begin{aligned} m\mu(U(x_1, r/2)) &\leq \mu(\bigcup_{i=1}^m U(x_i, r/2)) \leq \mu(U(x, \frac{5}{2}r)) \\ &\leq \mu(B(x_1, \frac{9}{2}r)) \overset{\text{Doubling measure}}{\leq} C^4 \mu(B(x_1, r/2)), \end{aligned}$$

where C is the doubling constant of μ. That tells us that $m \leq C^4$ which means m is bounded and independent of x and r. By adding finitely many points if necessary, we may assume that $\{x_i\}_{i=1}^m$ is a maximal r-separated set, and still $m \leq C^4$. Claim $\{B(x_i, r)\}_{i=1}^m$ is a cover for $B(x, 2r)$. If $\{B(x_i, r)\}_{i=1}^m$ does not cover $B(x, 2r)$ that means there exists a point $q \in B(x, 2r)$ such that $q \notin B(x_i, r)$ for any $i \in \{1, ..., m\}$. If $q \notin B(x_i, r)$ for any $i \in \{1, ..., m\}$ that tells us $d(q, x_i) > r$ which means $\{x_i\}_{i=1}^m$ is not maximal set of r-separated points, which is a contradiction. $\square$

Lastly recall the coarea inequality. This inequality can be found, for example, from [EH21].

Theorem 17. *If $f : X \to Y$ is a L-Lipschitz map between metric spaces, $A \subset X$, $\infty > s \geq t \geq 0$, then*

$$\int_Y^* \mathcal{H}^{s-t}(f^{-1}(y) \cap A) d\mathcal{H}^t(y) \leq (L)^t C_{s,t} \mathcal{H}^s(A),$$

where $C_{s,t} > 0$ is a universal constant. Moreover if X is boundedly compact i.e., bounded and closed sets in X are compact, A is $\mathcal{H}^s$-measurable, and $\mathcal{H}^s(A) < \infty$, then the function

$$y \to \mathcal{H}^{s-t}(f^{-1}(y) \cap A)$$

is $\mathcal{H}^t$-measurable and then the upper integral $\int_Y^*$ can be replaced with the usual integral.

1.2 Heisenberg groups

In this thesis we consider the n-th Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$ as a homogeneous group endowed with the left invariant, homogeneous Heisenberg distance d_H defined as follows.

Definition 18. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$ where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If $p = (x, y, t)$ and $q = (x', y', t') \in \mathbb{H}^n$ then the group multiplication $* : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ is given by

$$p * q = (x, y, t) * (x', y', t') := (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')),$$

where $x \cdot y$ denotes standard scalar product on $\mathbb{R}^n$.

The inverse of p is $p^{-1} := (-x, -y, -t)$ and $e = 0$ is the identity of $\mathbb{H}^n$. For any $q \in \mathbb{H}^n$ and $r > 0$, we denote $l_q : \mathbb{H}^n \to \mathbb{H}^n$ as the left translation $p \mapsto q * p = l_q(p)$ and $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ as the dilation

$$p \mapsto (rx, ry, r^2 t) = \delta_r(p).$$

Dilations are automorphisms of $\mathbb{H}^n$. Dilation map is bijective, it maps identity to identity and

$$\begin{aligned}
\delta_r(p * q) &= \delta_r(x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')) \\
&= (r(x + x'), r(y + y'), r^2(t + t' + 2(x' \cdot y - x \cdot y'))) \\
&= \delta_r(p) * \delta_r(q)
\end{aligned}$$

which proves that the dilation is automorphism.

$\mathbb{H}^n$ is endowed with a gauge also known as Korányi norm,

$$\|p\|_H := ((|x|^2 + |y|^2)^2 + t^2)^{1/4}$$

where $|x|^2 = \sum_{i=1}^n x_i^2$. This gauge has the properties that are $\|p*q\|_H \leq \|p\|_H + \|q\|_H$, $\|\delta_r(p)\|_H = r\|p\|_H$, $\|p\|_H \geq 0$ and $\|p\|_H = 0 \Leftrightarrow p = 0$. The triangle inequality is proved for instance in [CDPT07, chapter 2.2.1 page 18]. From the properties that the gauge has we see that it induces a metric called the *Heisenberg distance*:

$$d_H(p, q) := \|p^{-1} * q\|_H.$$

Next we prove a few useful properties of d_H. For all $p, q \in \mathbb{H}^n$,

$$d_H(p, q) = \|p^{-1} * q\|_H = d_H(p^{-1} * q, 0).$$

For all $p, q, z \in \mathbb{H}^n$ and for all $r > 0$,

$$d_H(z * p, z * q) = \|(z * p)^{-1} * z * q\|_H = d_H(p, q) \tag{1.4}$$

and

$$d_H(\delta_r(p), \delta_r(q)) = \|(\delta_r(p))^{-1} * \delta_r(q)\|_H = r\|p^{-1} * q\|_H = r d_H(p, q). \tag{1.5}$$

In this thesis we will use the notation d_E for Euclidean distance, $\mathcal{H}_E^s := \mathcal{H}_{d_E}^s$, $\mathcal{H}_H^s := \mathcal{H}_{d_H}^s$, $\dim_E$ and $\dim_H$ for respective Hausdorff dimensions.

The Heisenberg group $\mathbb{H}^n$ is a Lie group and the standard basis of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$ is given by

$$X_i := \partial/\partial x_i + 2y_i\partial/\partial t,\ Y_i := \partial/\partial y_i - 2x_i\partial/\partial t,\ T := \partial/\partial t,$$

for $i = 1, ..., n$. One can read more about Lie theory for example from [Kir08]. The vector fields $X_1, ..., X_n$ and $Y_1, ..., Y_n$ span the horizontal subspace $\mathfrak{h}_1$. Denoting $\mathfrak{h}_2$ as the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed as

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The vector fields $X_1, ..., X_n$ and $Y_1, ..., Y_n$ span a vector bundle, the so-called horizontal bundle $\mathbf{H}\mathbb{H}^n$, where $\mathbf{H}\mathbb{H}_p^n := \text{span}\{X_i(p), ..., X_n(p), Y_1(p), ..., Y_n(p)\}$ for all $p \in \mathbb{H}^n$, which can be canonically identified with a vector subbundle of the tangent vector bundle $\mathbf{T}\mathbb{H}^n \equiv \mathbf{T}\mathbb{R}^{2n+1}$.

Lemma 19. *A set $A \subset \mathbb{H}^n$ is bounded with respect to d_H if and only if it is bounded with respect to d_E.*

Proof. "$\Leftarrow$" Let $b_E := b \geq 1$ be such that $d_E(p, 0) \leq b$ for all $p = (x, y, t) \in A$. This tells us that $|x|^2 + |y|^2 \leq b^2$ and $t^2 \leq b^2$. Now we get

$$\begin{aligned} d_H(p, 0) = ((|x|^2 + |y|^2)^2 + t^2)^{1/4} &\leq ((b^2)^2 + b^2)^{1/4} = (b^4 + b^2)^{1/4} \\ &\leq (b^4 + b^4)^{1/4} = 2^{1/4} b. \end{aligned}$$

"$\Rightarrow$" Let $b_H := b \geq 1$ be such that $d_H(p,0) \leq b$ for all $p = (x,y,t) \in A$. This tells us that $|x|^2 + |y|^2 \leq b^2$ and $t \leq b^2$. Now we get

$$d_E(p,0)^2 = |x|^2 + |y|^2 + t^2 \leq b^2 + b^4 = (1+b^2)b^2 \leq 2b^4.$$

Now, taking square root both sides, we get $d_E(p,0) \leq 2^{1/2}b^2$. □

There is the following relationship between the distances d_E and d_H in $\mathbb{R}^{2n+1}$.

Lemma 20. *Let A be a bounded subset of $(\mathbb{R}^{2n+1}, d_E)$ and let $b \geq 1$ be such that $\sup_{p\in A} d_E(p,0) \leq b$. Then there is a positive constant $c = c(b)$ such that for $p, q \in A$ we have*

$$\frac{1}{c} d_E(p,q) \leq d_H(p,q) \leq c(d_E(p,q))^{\frac{1}{2}}. \tag{1.6}$$

In particular the identity map $\mathrm{Id} : (\mathbb{R}^{2n+1}, d_E) \to (\mathbb{R}^{2n+1}, d_H)$ *is a homeomorphism.*

Proof. Let us start showing inequalities from left to right, denoting $p = (x,y,t)$ and $q = (x', y', t')$.

$$\begin{aligned} d_E(p,q)^2 &= |x-x'|^2 + |y-y'|^2 + |t-t'|^2 \\ &= |x-x'|^2 + |y-y'|^2 + |t-t' - 2(x\cdot y' - x'\cdot y) + 2(x\cdot y' - x'\cdot y)|^2 \\ &\leq |x-x'|^2 + |y-y'|^2 + (|t-t' - 2(x\cdot y' - x'\cdot y)| + |2(x\cdot y' - x'\cdot y)|)^2 \\ &\leq |x-x'|^2 + |y-y'|^2 + 3|t-t' - 2(x\cdot y' - x'\cdot y)|^2 + 12|(x\cdot y' - x'\cdot y)|^2 \quad (1.7) \\ &\leq c_b d_H(q^{-1} * p, 0)^2. \quad (1.8) \end{aligned}$$

Where we used

$$|t-t' - 2(x\cdot y' - x'\cdot y)|^2 \leq 6b^2|t-t' - 2(x\cdot y' - x'\cdot y)|$$

and

$$\begin{aligned} |x\cdot y' - x'\cdot y|^2 = |x\cdot(y'-y) - (x'-x)\cdot y|^2 &\leq (|x\cdot(y'-y)| + |(x'-x)\cdot y|)^2 \\ &\leq 3|y'-y|^2|x|^2 + 3|x'-x|^2|y|^2 \leq 3b^2(|y'-y|^2 + |x'-x|^2) \end{aligned}$$

to go from (1.7) to (1.8). For the other inequality

$$\begin{aligned} d_H(q^{-1} * p, 0)^4 &= (|x-x'|^2 + |y-y'|^2)^2 + |t-t' - 2(x\cdot y' - x'\cdot y)|^2 \\ &\leq (|x-x'|^2 + |y-y'|^2)^2 + (|t-t'| + 2|(x\cdot y' - x'\cdot y)|)^2 \\ &\leq b^2(|x-x'|^2 + |y-y'|^2) + 3|t-t'|^2 + 12|(x\cdot y' - x'\cdot y)|^2 \\ &\leq c_b d_E(p,q)^2. \end{aligned}$$

Now we have proved the wanted inequalities.

Clearly the identity map is bijective and the inverse function of the identity map is the identity map. Let us show that the identity map is continuous at $p \in \mathbb{H}^n$. Let $\epsilon > 0$ and $1 > \delta > 0$. If $d_E(p,q) \leq \delta$ then we get for some $c = c(|p|)$.

$$d_H(f(p), f(q)) = d_H(p,q) \leq c(d_E(p,q))^{1/2} \leq c(\delta)^{1/2} < \epsilon$$

when we choose $\delta < \frac{\epsilon^2}{c^2}$. The continuity of the inverse map can be proved by using the other inequality, which proves that the map Id is a homeomorphism. □

Lemma 20 actually tells us more about the identity maps. For a set A that is bounded with respect to d_E (or d_H), the identity map $\mathrm{Id} : (A, d_E) \to (A, d_H)$ is $\frac{1}{2}$-Hölder continuous and $\mathrm{Id} : (A, d_H) \to (A, d_E)$ is Lipschitz continuous.

From Proposition 8(2) and (1.4)-(1.5) we get a useful corollary for Hausdorff measure of left-translated or dilated sets.

Corollary 21. *Let $A \subset \mathbb{H}^n$, $q \in \mathbb{H}^n$ and $s, r \in (0, \infty)$. Then*

$$\mathcal{H}^s_H(l_q(A)) = \mathcal{H}^s_H(A),$$

$$\mathcal{H}^s_H(\delta_r(A)) = r^s \mathcal{H}^s_H(A).$$

Lemma 22. $\mathcal{H}^{2n+2}_H = c\mathcal{L}^{2n+1}$ *for a suitable positive constant c.*

This follows from the fact that $(2n+2)$-dimensional Hausdorff measure and Lebesgue measure are both left Haar measures on $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$ for which we have the following fact. The fact can be found from [LD21, page 168 fact 6.5.16].

Fact 1. Left-Haar measures and right-Haar measures that are finite and not zero on compact sets with nonempty interior are unique up to a multiplication by a constant.

One can find the proof for the uniqueness of Haar measure for instance from [Coh13, Theorem 9.2.6 p.290] in case of locally compact topological groups.

Proof of Lemma 22. $\mathcal{L}^{2n+1}$ is $(2n+2)$-Ahlfors regular on $(\mathbb{H}^n, d_H)$. It comes from the fact that $\det(D(l_p)) = 1$ and $\det(D(\delta_r)) = r^{2n+2}$. Now the area formula (see e.g., [Sim83, Chapter 2]) tells us that

$$\mathcal{L}^{2n+1}(B_H(p,r)) = \mathcal{L}^{2n+1}(l_p(\delta_r(B_H(0,1)))) = \mathcal{L}^{2n+1}(\delta_r(B_H(0,1))) = r^{2n+2}\mathcal{L}^{2n+1}(B_H(0,1)),$$

where $0 < \mathcal{L}^{2n+1}(B_H(0,1)) < \infty$. Now using Lemma 12 when $\mu = \mathcal{L}^{2n+1}$ we get that $\mathcal{H}^{2n+2}_H$ and $\mathcal{L}^{2n+1}$ are comparable. Now the claim follows from Fact 1 since $\mathcal{H}^{2n+2}_H$ and $\mathcal{L}^{2n+1}$ are both non-trivial locally finite left-Haar measures on $\mathbb{H}^n$. Note that the left invariance of $\mathcal{H}^{2n+2}_H$ follows from Corollary 21. The left invariance of $\mathcal{L}^{2n+1}$ follows from $\det(D(l_p)) = 1$. $\square$

Lastly lets look at some examples which show that the dimension of sets can be different when looking them in d_E or d_H.

Example 23. $\dim_H \mathbb{H}^1 = 4$ but $\dim_E(\mathbb{R}^3) = 3$.

Proof. When $n = 1$ we are studying $\mathbb{H}^1$ and by the proof of Lemma 22 we know that $\mathcal{L}^3$ is Ahlfors 4-regular on $(\mathbb{H}^1, d_H)$. Now by Corollary 13 we get that $\dim_H(\mathbb{H}^1) = 4$. Analogously, $\mathcal{L}^3$ is 3-regular on $(\mathbb{R}^3, d_E)$ and hence $\dim_E(\mathbb{R}^3) = 3$. $\square$

Example 24. $\dim_H(xt - plane) = 3$ but $\dim_E(xt - plane) = 2$.

Proof. Notice that the $xt - plane = \{(x, 0, t) \in \mathbb{H}^1\}$ is a subgroup of $\mathbb{H}^1$. The inverse of $(x, 0, t)$ is $(-x, 0, -t)$, $(0, 0, 0)$ is in the plane and $(x, 0, t) * (x', 0, t') = (x + x', 0, t + t') \in xt - plane$. Now the dilation is $\delta_r((x, 0, t)) = (rx, 0, r^2 t)$. Now we can see that $\det(D(\delta_r)) = r^3$ and $\det(D(l_p)) = 1$ for $\delta_r(x, t) := (rx, r^2 t)$ and $l_{(x,t)}(x', t') := (x+x', t+t')$. From this we can conclude that the $dim_H(xt-plane) = 3$ by an analogous argument as in Example 23, identifying $(x, 0, t)$ with $(x, t) \in \mathbb{R}^2$. On the other hand, one can observe that $\mathcal{H}_E^2$ restricted to the $xt - plane$ is 2-Ahlfors regular and Borel regular measure supported on the $xt - plane$ from which we can conclude that $\dim_E(xt - plane) = 2$. $\square$

Example 25. $\dim_H(L)$ can be 2 or 1 but $\dim_E(L) = 1$ for lines $L \subset \mathbb{H}^1$.

Proof. If L is the $t - axis$, then the Hausdorff dimension with respect to d_H will be 2. If L is, for instance, the $x - axis$, then the dimension will be 1. This can be seen with the same argument as above for those two cases. Also every line in $xy - plane$ through the origin has the same Hausdorff dimension with respect to d_H. This follows from the fact that every line in xy-plane through the origin is just rotation around the t-axis from line $l = \{(x, 0, 0) : x \in \mathbb{R}\}$. Notice that rotation around the t-axis is isometry for d_H. Lets look at map $\mathbf{R}_\theta : (x, y, t) \mapsto (R_\theta(x, y), t)$ where

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Now

$$\begin{aligned}
&d_H(\mathbf{R}_\theta(x, y, t), \mathbf{R}_\theta(x', y', t')) = d_H((R_\theta(x, y), t), (R_\theta(x', y'), t')) \\
&= d_H((x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), t), (x'\cos(\theta) - y'\sin(\theta), x'\sin(\theta) + y'\cos(\theta), t')) \\
&= \ldots \\
&= ((((-(y' - y)\sin(\theta) + (x' - x)\cos(\theta))^2 + ((x' - x)\sin(\theta) + (y' - y)\cos(\theta))^2)^2 \\
&+ (t' - t + 2(xy' - x'y)\cos^2(\theta) + 2(y'x - yx')\sin^2(\theta))^2)^{1/4} \\
&= (((x' - x)^2 + (y' - y)^2)^2 + (t' - t + 2(xy' - x'y))^2)^{1/4} = d_H((x, y, t), (x', y', t'))
\end{aligned}$$

which proves that rotation around the t-axis is an isometry. Regarding $\dim_E(L)$, one can observe that $\mathcal{H}_E^1$ restricted to any line $L \subset \mathbb{R}^3$ is a 1-Ahlfors regular and Borel regular measure supported on L, from which we can conclude that $\dim_E(L) = 1$. $\square$

Chapter 2

Dimension comparison theorem

In this chapter the main goal is to prove a dimension comparison theorem in the general Heisenberg group $\mathbb{H}^n$. Essentially we are interested what the dimension gap can be when comparing the set between Heisenberg and Euclidean space. For example in Example 23 the gap is 1. The main result, Theorem 29, is a generalisation of the $\mathbb{H}^1$ case which is done in the paper [BRSC03]. Note that this problem has been solved in the general case of Carnot groups in [BTW09]. The proof in this chapter follows essentially the approach in [BTW09], but is presented as a consequence of dimension comparison results in abstract metric spaces. One statement is also proven in a more constructive way, following [BRSC03] and generalizing from $n = 1$ to $n > 1$. This chapter is divided into two sections. In the first section we prove theorems that we need in general metric spaces. In section two the goal is to prove Propositions 31 and 36 because the dimension comparison theorem follows from them instantly.

2.1 Dimension comparison in metric spaces

In this section we prove results about the absolute continuity of Hausdorff measure and spherical measure in arbitrary metric spaces $(X, d_1), (X, d_2)$ which will give useful information for different ranges of dimensional exponents. For the proof of the second result we need a covering result that we will prove.

Proposition 26. *Let $k \in (0, 1]$ and $\alpha > 0$, d_1, d_2 be metrics on a set X such that $d_2(p, q) \leq c(d_1(p, q))^k$ holds for all $p, q \in X$. Then*

$$\mathcal{H}^{\alpha}_{d_2} \ll \mathcal{H}^{\alpha \cdot k}_{d_1}.$$

Proof. Suppose that $\mathcal{H}^{\alpha \cdot k}_{d_1}(A) = 0$ for some $A \subset X$. Denote $d_1^k(p, p') := (d_1(p, p'))^k$ if $p, p' \in X$. Now by Proposition 8(3) we also have that $\mathcal{H}^{\alpha \cdot k}_{d_1}(A) = \mathcal{H}^{\alpha}_{d_1^k}(A) = 0$. Now by assumption there is a positive constant c such that for all $p, p' \in A$ we have

$$d_2(p, p') \leq c(d_1(p, p'))^k. \tag{2.1}$$

By the inequality (2.1) and Proposition 8(2) with $X_1 = X_2 = A$, $f = \mathrm{Id}$, d_1^k,d_2 and $L = c$ we get

$$\mathcal{H}^{\alpha}_{d_2}(A) \leq c^{\alpha} \mathcal{H}^{\alpha}_{d_1^k}(A).$$

Now we conclude that

$$\mathcal{H}^{\alpha}_{d_2}(A) \leq c^{\alpha}\mathcal{H}^{\alpha}_{d_1^k}(A) = c^{\alpha}\mathcal{H}^{\alpha\cdot k}_{d_1}(A) = 0.$$

□

Proposition 27. *Let $\kappa \in (0,1)$, and let $(X,d_1),(X,d_2)$ be metric spaces such that for every bounded set $A \subset (X,d_2)$, there exists a constant $C_A < \infty$ such that $d_1 \leq C_A d_2^{\kappa}$. Assume that there exists a Borel measure μ on (X,d_2) that is Ahlfors s_1-regular for d_1 and s_2-regular for d_2. Then, for every bounded set $B \subset (X,d_2)$, there exists $N = N_B < \infty$ depending only on B,κ, and the Ahlfors regularity constant of μ with respect to d_1 and d_2, such that for every $B_{d_1}(p,r) \subset B$ and $0 < r < \min\{1,[2\mathrm{diam}_{d_2}(X)]^{\kappa}\}$ we can find $B_{d_2}(p_1,r^{1/\kappa}),...,B_{d_2}(p_k,r^{1/\kappa})$ satisfying:*

1. *$B_{d_1}(p,r) \subset \bigcup_{i=1}^{k} B_{d_2}(p_i,r^{1/\kappa})$.*
2. *$k \leq N\frac{r^{s_1}}{r^{\frac{1}{\kappa}s_2}}$.*

Proof. Let $\{p_i\}_{i=1}^{k}$ be $r^{1/\kappa}$-separated points with respect to d_2 in $B_{d_1}(p,r) \subset B$. Notice that open balls $U_{d_2}(p_i,r^{1/\kappa}/2)$ are disjoint. If $q \in U_{d_2}(p_i,r^{1/\kappa}/2)$ then $d_1(q,p) \leq d_1(q,p_i)+d_1(p_i,p) \leq C_A d_2(q,p_i)^{\kappa}+r \leq C_A r/2^{\kappa}+r \lesssim r$, where A is the $\frac{1}{2}$-neighbourhood of B with respect to d_2. Now from the inequality we get that $\bigcup_{i=1}^{k} U_{d_2}(p_i,r^{1/\kappa}/2) \subset B_{d_1}(p,\tilde{C}r)$ with $\tilde{C} = (\frac{C_A}{2^{\kappa}}+1)$. Now we get

$$C_2^{-1}k(\frac{r^{1/\kappa}}{2})^{s_2} \leq \mu(\bigcup_{i=1}^{k} U_{d_2}(p_i,r^{1/\kappa}/2)) \leq \mu(B_{d_1}(p,\tilde{C}r)) \leq C_1(\tilde{C}r)^{s_1},$$

from which we can conclude that $k \lesssim \frac{r^{s_1}}{r^{\frac{1}{\kappa}s_2}}$. Here the constants C_i, $i = 1,2$, come from the Ahlfors regularity assumptions. Next we show that $\{B_{d_2}(p_i,r^{1/\kappa})\}_{i=1}^{k}$ covers $B_{d_1}(p,r)$ if we take maximal $r^{1/\kappa}$-separated points $\{p_i\}_{i=1}^{k}$ from $B_{d_1}(p,r)$. Indeed if $B_{d_1}(p,r)$ is not covered by $\{B_{d_2}(p_i,r^{1/\kappa})\}_{i=1}^{k}$ then there exists a point $q \in B_{d_1}(p,r)$ such that $d_2(q,p_i) > r^{1/k}$ which means that $\{p_i\}_{i=1}^{k}$ is not maximal and that is a contradiction. □

Proposition 28. *Let $L \geq 1$ and $M \geq 1$. Let $(X,d_1),(X,d_2)$ be metric spaces with the same class of bounded sets. Assume that for every bounded set $B \subset X$ there is constant $N = N_B < \infty$ such that every ball $B_{d_1}(p,r) \subset B$, $0 < r < 1$ can be covered by $B_{d_2}(p_1,r^L),...,B_{d_2}(p_k,r^L)$ with $k \leq Nr^{-M}$. Then the spherical measure satisfies the following absolute continuity property ($\alpha \geq 0$ arbitrary):*

$$\mathcal{S}^{\frac{\alpha+M}{L}}_{d_2} \ll \mathcal{S}^{\alpha}_{d_1}.$$

By Proposition 8(1), it then also holds that $\mathcal{H}^{\frac{\alpha+M}{L}}_{d_2} \ll \mathcal{H}^{\alpha}_{d_1}$.

Proof. The case $\alpha = 0$ is trivial, so assume that $\alpha > 0$. Suppose that $\mathcal{S}^{\alpha}_{d_1}(A) = 0$ for some $A \subset X$. Let $x_0 \in X$ and $A_n := A \cap B_{d_1}(x_0,n)$ for all $n \in \mathbb{N}$. Given

$n \in \mathbb{N}, 0 < \delta < 1, \epsilon > 0$ arbitrary, there is a covering of A_n with balls $\{B_{d_1}(p_i, r_i)\}_{i \in \mathbb{N}}$ $r_i \leq \delta/2$ which intersect A_n and

$$\sum_{i=1}^{\infty} (\mathrm{diam}_{d_1}(B_{d_1}(p_i, r_i)))^{\alpha} \leq \frac{\epsilon}{2^{\frac{\alpha+M}{L}} N},$$

where $N = N_B$ is the constant associated to $B_n := B_{d_1}(x_0, n+1)$. Since

$$\mathcal{S}_{d_2,\delta}^{\frac{\alpha+M}{L}}(A) \leq \mathcal{S}_{d_2,\delta}^{\frac{\alpha+M}{L}}\left(\bigcup_i \{A \cap B_{d_1}(p_i, r_i) : \mathrm{diam}(B_{d_1}(p_i, r_i)) > 0\}\right),$$

we may suppose without loss of generality that $\mathrm{diam}_{d_1}(B_{d_1}(p_i, r_i)) > 0$ for all i, and by decreasing the radius of the balls if necessary, we may assume that

$$\mathrm{diam}_{d_1}(B_{d_1}(p_i, r_i)) \geq r_i. \tag{2.2}$$

By assumption every $B_{d_1}(p_i, r_i)$ can be covered by $\{B_{d_2}(p_{i,j}, r_i^L)\}_{i \in \mathbb{N}, j \in \{1,\dots,k(i)\}}$ and $k(i) \leq N r_i^{-M}$. Now $\{B_{d_2}(p_{i,j}, r_i^L)\}_{i \in \mathbb{N}, j \in \{1,\dots,k(i)\}}$ also covers A_n. Now for every $m \in \mathbb{N}$

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{k(i)} (\mathrm{diam}_{d_2}(B_{d_2}(p_{i,j}, r_i^L)))^{\frac{\alpha}{L}+\frac{M}{L}} &= \sum_{i=1}^{m}\sum_{j=1}^{k(i)} (\mathrm{diam}_{d_2}(B_{d_2}(p_{i,j}, r_i^L)))^{\alpha/L} \mathrm{diam}_{d_2}(B_{d_2}(p_{i,j}, r_i^L))^{\frac{M}{L}} \\
&\leq \sum_{i=1}^{m} (2r_i^L)^{\alpha/L} \sum_{j=1}^{k(i)} (2r_i^L)^{\frac{M}{L}} \\
&\overset{(2.2)}{\leq} \sum_{i=1}^{m} 2^{\alpha/L} (\mathrm{diam}_{d_1}(B_{d_1}(p_i, r_i)))^{\alpha} k(i) (2r_i^L)^{\frac{M}{L}} \\
&\overset{\text{assumption}}{\leq} \sum_{i=1}^{m} (\mathrm{diam}_{d_1}(B_{d_1}(p_i, r_i)))^{\alpha} 2^{\alpha/L} N r_i^{-M} (2r_i^L)^{\frac{M}{L}} \\
&\leq \frac{\epsilon}{2^{\frac{\alpha+M}{L}} N} 2^{\frac{\alpha+M}{L}} N \leq \epsilon.
\end{aligned}$$

The above also tells us that $\mathcal{S}_{d_2,\delta}^{\frac{\alpha}{L}+\frac{M}{L}}(A_n) \leq \epsilon$. Since $0 < \delta < 1, \epsilon > 0$ are arbitrary we get $\mathcal{S}_{d_2}^{\frac{\alpha}{L}+\frac{M}{L}}(A_n) = 0$. Finally,

$$\mathcal{S}_{d_2}^{\frac{\alpha}{L}+\frac{M}{L}}(A) = \mathcal{S}_{d_2}^{\frac{\alpha}{L}+\frac{M}{L}}(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mathcal{S}_{d_2}^{\frac{\alpha}{L}+\frac{M}{L}}(A_n) = 0.$$

□

2.2 Dimension comparison theorem in Heisenberg group

Our main theorems in this chapter which we want to prove are Theorem 29 and Corollary 30.

Theorem 29. *(Dimension comparison theorem). Let $\alpha \geq 0$. Then the following statements hold in $\mathbb{H}^n$*

1. $$\mathcal{H}_H^{\min\{2\alpha,\alpha+1\}} \ll \mathcal{H}_E^{\alpha}$$

   i.e. $\mathcal{H}_H^{\min\{2\alpha,\alpha+1\}}$ is absolutely continuous with respect to $\mathcal{H}_E^{\alpha}$.

2. $$\mathcal{H}_E^{\min\{\alpha,n+\alpha/2\}} \ll \mathcal{H}_H^{\alpha}$$

   i.e. $\mathcal{H}_E^{\min\{\alpha,n+\alpha/2\}}$ is absolutely continuous with respect to $\mathcal{H}_H^{\alpha}$.

As a corollary, we obtain the Hausdorff dimension comparison.

Corollary 30. *In the Heisenberg group $\mathbb{H}^n$ we have:*

$$\beta_-(\dim_E(A)) \leq \dim_H A \leq \beta_+(\dim_E(A))$$

for every $A \subset \mathbb{H}^n$, where

$$\beta_-(\alpha) = \begin{cases} \alpha, \ 0 \leq \alpha \leq 2n \\ 2\alpha - 2n, \ 2n \leq \alpha \leq 2n+1 \end{cases}$$

and

$$\beta_+(\alpha) = \begin{cases} 2\alpha, \ 0 \leq \alpha \leq 1 \\ \alpha + 1, \ 1 \leq \alpha \leq 2n+1. \end{cases}$$

Proof. Let $\alpha = \dim_E(A)$. Now let $\alpha' > \alpha$. For α' we know that $\mathcal{H}_E^{\alpha'}(A) = 0$. By Theorem 29 we get that $\mathcal{H}_H^{\min\{2\alpha',\alpha'+1\}}(A) = 0$ which tells us that $\dim_H(A) \leq \min\{2\alpha', \alpha'+1\}$ for all $\alpha' > \alpha$. We get the inequality by contradicition: if $\dim_H(A) > \min\{2\alpha', \alpha'+1\}$ then $\mathcal{H}_H^{\min\{2\alpha',\alpha'+1\}}(A) = \infty$ which is not possible. Now letting $\alpha' \to \alpha$ we get that $\dim_H(A) \leq \min\{2\alpha, \alpha+1\}$ which tells us that $\dim_H(A) \leq \beta_+(\alpha)$.

For the other inequality let $\alpha = \dim_H(A)$. By the same argument as above one can show that $\dim_E(A) \leq \min\{\alpha, n + \frac{\alpha}{2}\}$. Now we have two cases to check. If $\beta_-(\dim_E(A)) = \max\{\dim_E(A), 2\dim_E(A) - 2n\} = \dim_E(A)$ then

$$\beta_-(\dim_E(A)) = \max\{\dim_E(A), 2\dim_E(A) - 2n\} = \dim_E(A) \leq \min\{\alpha, n + \frac{\alpha}{2}\} \leq \alpha = \dim_H(A).$$

The other case is that if $\beta_-(\dim_E(A)) = \max\{\dim_E(A), 2\dim_E(A) - 2n\} = 2\dim_E(A) - 2n$ then we get

$$\begin{aligned} \beta_-(\dim_E(A)) &= \max\{\dim_E(A), 2\dim_E(A) - 2n\} = 2\dim_E(A) - 2n \\ &\leq 2(n + \frac{\alpha}{2}) - 2n = \alpha = \dim_H(A). \end{aligned}$$

These two cases show that $\beta_-(\dim_E(A)) \leq \dim_H(A)$ which concludes the proof. $\square$

Let us recall Examples 23 - 25. These examples show the sharpness of the dimension comparison theorem for $\mathbb{H}^1$ and $\alpha = 1$ ($\beta_-(1) = 1, \beta_+(1) = 2$) and $\alpha = 2$ for the upper bound ($\beta_+(2) = 3$), and $\alpha = 3$ ($\beta_-(3) = 4 = \beta_+(3)$).

Now from Proposition 26 we get the first big proposition that we need for the dimension comparison theorem.

Proposition 31. *The Hausdorff measures on $\mathbb{H}^n$ satisfies the following absolute continuity properties*

1. $\mathcal{H}^\alpha_H \ll \mathcal{H}^{\alpha/2}_E$.
2. $\mathcal{H}^\alpha_E \ll \mathcal{H}^\alpha_H$.

Proof. Now by Lemma 20 our assumptions needed for Proposition 26 are true for metrics d_E and d_H when restricted to bounded subsets of $\mathbb{H}^n$ and since $\mathbb{H}^n = \bigcup_{n\in\mathbb{N}} B_E(0,n)$ it suffices to consider bounded sets. Now choosing $k = 1/2$ in the first case and $k = 1$ in the second case then the Proposition 26 gives the claim instantly. $\square$

Analogously as in [BRSC03], we will also prove a second proposition of similar kind because Proposition 31 does not give good results for large values of α. For example, if $\alpha = 4$ then we are in situation $\mathcal{H}^4_E \ll \mathcal{H}^4_H$ but since for every $A \subset \mathbb{H}^1$ we have $\mathcal{H}^4_E(A) = 0$, this does not tell us anything new. That is why we also need the other big proposition because it is going to fix our problem with larger values of α.

The next statement shows us how to almost optimally cover a Euclidean ball with Heisenberg balls.

Proposition 32. *Let A be a bounded subset of $(\mathbb{R}^{2n+1}, d_E)$. Then there exists $N = N(A) \in \mathbb{N}$ such that for any Euclidean ball $B_E(p,r)$ with $p \in A$ and $0 < r < 1$ we can find Heisenberg balls $B_H(p_1,r), ..., B_H(p_k,r)$ satisfying:*

1. $B_E(p,r) \subset \bigcup_{i=1}^k B_H(p_i,r)$.
2. $k \leq \frac{N}{r}$.

Proof. Recall that the Lebesgue measure is a $(2n+2)$-regular measure for d_H and $(2n+1)$-regular for d_E. Now Lemma 19, Lemma 20 and Proposition 27 give us the claim with $\kappa = 1, s_1 = 2n+1, s_2 = 2n+2, d_1 = d_E, d_2 = d_H$. $\square$

The next proposition is basically the same as one above but other way around. It tells us how many Euclidean balls do we need to cover a Heisenberg ball.

Proposition 33. *Given a bounded subset A of $(\mathbb{R}^{2n+1}, d_E)$, there is $N = N(A) \in \mathbb{N}$ such that for any Heisenberg ball $B_H(p,r)$ with $p \in A$ and $0 < r < 1$ we can find $B_E(p_1,r^2), ..., B_E(p_k,r^2)$ satisfying:*

1. $B_H(p,r) \subset \bigcup_{i=1}^k B_E(p_i,r^2)$.
2. $k \leq \frac{N}{r^{2n}}$.

Remark 34. Proposition 33 follows from the Ahlfors regularity of the Lebesgue measure, Lemma 20, Proposition 27 with $\kappa = \frac{1}{2}, s_1 = 2n+2, s_2 = 2n+1, d_1 = d_H, d_2 = d_E$ but I will also present an alternative proof following [BRSC03, Proposition 3.5].

Before we can prove the proposition we want to know how Euclidean balls behave under group translation.

Lemma 35. *Let A be a bounded subset of $(\mathbb{R}^{2n+1}, d_E)$ i.e $A \subset B_E(0,b)$. For $r > 0$, $p = (x,y,t) \in \mathbb{R}^{2n+1}$ and $p' = (x',y',t') \in A$ we have.*

$$l_{p'}(B_E(p,r)) \subset B_E(l_{p'}(p),(2b+1)r).$$

Moreover if $l_{p'}(p) + (x'',y'',t'') \in l_{p'}(B_E(p,r))$ then $|(x'',y'')| \leq r$.

Proof. Let $(x+x''',y+y''',t+t''') \in B_E(p,r)$.

$$\begin{aligned}
&l_{p'}(x+x''',y+y''',t+t''') \\
&\quad = p' * (x+x''',y+y''',t+t''') \\
&\quad = (x'+x+x''',y'+y+y''',t'+t+t'''+2((x+x''')\cdot y' - x'\cdot(y+y'''))) \\
&\quad = l_{p'}(p) + (x''',y''',t'''+2(x'''\cdot y' - x'\cdot y''')).
\end{aligned}$$

Now we can calculate the Euclidean norm of the second term

$$\begin{aligned}
&|(x''',y''',t'''+2(x'''\cdot y' - y'''\cdot x'))|^2 \\
&\quad = |x'''|^2 + |y'''|^2 + |t'''+2(x'''\cdot y' - y'''\cdot x')|^2 \\
&\quad \overset{\text{C−S}}{\leq} |x'''|^2 + |y'''|^2 + (|t'''| + 2|p'||(x''',y''',t''')|)^2 \\
&\quad = |x'''|^2 + |y'''|^2 + |t'''|^2 + 4|p'||(x''',y''',t''')||t'''| + 4|p'|^2|(x''',y''',t''')|^2 \\
&\quad \leq |(x''',y''',t''')|^2 + 4b|(x''',y''',t''')||t'''| + 4b^2|(x''',y''',t''')|^2 \\
&\quad \leq r^2 + 4br^2 + 4b^2r^2 = (2b+1)^2r^2.
\end{aligned}$$

Now taking square root both sides we get $|(x''',y''',t'''+2(x'''\cdot y' - x'\cdot y''')| \leq (2b+1)r$. Hence $l_{p'}(x+x''',y+y''',t+t''') \in B_E(l_{p'}(p),(2b+1)r)$. Notice that if $l_{p'}(p) + (x'',y'',t'') \in l_{p'}(B_E(p,r))$ then $l_{p'}(p) + (x'',y'',t'') = l_{p'}(p) + (x''',y''',...)$ which means that $(x'',y'') = (x''',y''') \in \mathrm{proj}_t(B_E(0,r))$ which tells us that $|(x'',y'')| \leq r$. □

Now we can prove Proposition 33 the alternative way.

Proof of Proposition 33. First we will prove the following for B_H at the origin: There is $K \in \mathbb{N}$ such that given $0 < r < 1$ we can find $B_E(p_1,r^2),...,B_E(p_l,r^2)$ satisfying:

$$B_H(0,r) \subset \bigcup_{i=1}^{l} B_E(p_i,(2n+1)^{1/2}r^2) \text{ and } l \leq \frac{K}{r^{2n}}.$$

First denote O_t as the vertical axis. For $p = (0,0,t) \in O_t$ and $r > 0$ we denote $B_{H,\infty}(p,r)$ the set of points $p' = (x',y',t') \in \mathbb{H}^n$ satisfying $|(x',y')| \leq r$ and $|t-t'| \leq r^2$. Now $B_{H,\infty}(p,r)$ is a flat box with height $2r^2$ centered at $p \in \mathbb{H}^n$ and its orthogonal projection along the vertical axis is a disk of radius r. Now one can see easily that $B_H(0,r) \subset B_{H,\infty}(0,r)$. Let $p \in B_H(0,r)$. Now $\|p\|_H^4 = (|x|^2+|y|^2)^2+|t|^2 \le r^4$ from which we get $|x|^2+|y|^2 \le r^2$ and $|t| \le r^2 \Rightarrow p \in B_{H,\infty}(0,r)$. Now Heisenberg boxes can be covered with Euclidean boxes which can be covered by Euclidean balls. Formally let $\tilde{p} = (\tilde{x},\tilde{y},\tilde{t}) \in \mathbb{H}^n$ and $C(\tilde{p},r^2)$ be a Euclidean box $C(\tilde{p},r^2) := \{(\tilde{x},\tilde{y},\tilde{t}) : |\tilde{x}_j - x_j|, |\tilde{y}_j - y_j|, |\tilde{t}-t| \le r^2, j \in \{1,...,n\}\}$. Now notice that the height of the Heisenberg box and the Euclidean box are same. So we need to calculate how many Euclidean r^2-boxes do we need to cover the Heisenberg box $B_{H,\infty}(0,r)$ in the $2n$-dimensions. So the amount of Euclidean boxes that we need is

$$\lceil (\frac{2r}{2r^2})^{2n} \rceil = \lceil \frac{1}{r^{2n}} \rceil,$$

which in the original paper [BRSC03] is denoted by $([\frac{1}{r}]+1)^{2n}$ for $n=1$. Now each cube $C(x_i,r^2)$ is contained in Euclidean ball of radius $(2n+1)^{1/2}r^2$. Hence, the claim is proved for $K := 2^{2n}$ since $[\frac{1}{r}]+1 < \frac{2}{r}$. Now given $A \subset B_E(0,b)$ with $p \in A$ and $0 < r < 1$, we can find $B_E(p_1,(2n+1)^{1/2}r^2), ..., B_E(p_l,(2n+1)^{1/2}r^2)$ covering for $l_{p^{-1}}(B_H(p,r)) = B_H(0,r)$ so that $lr^{2n} \le K$. Now there exist constant $M = M(b,n)$ and balls $B_E(p_1, r^2/(2b+1)), ..., B_E(p_k, r^2/(2b+1))$ that cover $B_H(0,r)$ such that $kr^{2n} \le KM$. There is such covering because every ball $B_E(p_i,(2n+1)^{1/2}r^2) \subset \bigcup_{j=1}^m B_E(p_{i_j}, r^2/(2b+1))$. Notice that m is bounded by number $\lceil (2b+1)^{2n+1}(2n+1)^{n+1/2} \rceil$. To cover a line length of $(2n+1)^{1/2}2r^2$ with radius of $2r^2/(2b+1)$ balls we need

$$\frac{2r^2(2n+1)^{1/2}}{\frac{2r^2}{2b+1}} = (2n+1)^{1/2}(2b+1)$$

amount of them. To cover all $2n+1$ dimensions we need $\lceil ((2n+1)^{1/2}(2b+1))^{2n+1} \rceil = \lceil (2b+1)^{2n+1}(2n+1)^{n+1/2} \rceil$ amount of balls. Now defining $M := \lceil (2b+1)^{2n+1}(2n+1)^{n+1/2} \rceil$, $N := MK$ and with Lemma 35 we get

$$\begin{aligned} B_H(p,r) = l_p(l_{p^{-1}}(B_H(p,r)) &\subset l_p(\bigcup_{i=1}^{l}\bigcup_{j=1}^{m} B_E(p_{i_j}, r^2/(2b+1))) \\ &= \bigcup_{i=1}^{l}\bigcup_{j=1}^{m} l_p(B_E(p_{i_j}, r^2/(2b+1))) \overset{\text{L.35}}{\subset} \bigcup_{i=1}^{l}\bigcup_{j=1}^{m} B_E(l_p(p_{i_j}), r^2). \end{aligned}$$

Now defining $k := l \cdot m$ proves the proposition with N depending only on n and b. □

Proposition 36. *The spherical measure satisfies the following absolute continuity properties ($\alpha \ge 0$ arbitrary) on $\mathbb{H}^n$:*

1. $\mathcal{S}_H^{\alpha+1} \ll \mathcal{S}_E^{\alpha}$.
2. $\mathcal{S}_E^{n+\frac{\alpha}{2}} \ll \mathcal{S}_H^{\alpha}$.

Proof. The first case comes instantly when applying Proposition 28 to metrics $d_1 = d_E$, $d_2 = d_H$ and choosing $L = 1 = M$. You can apply the Proposition 28 because Proposition 32 holds.

Second case also follows instantly from Proposition 28 when choosing $d_2 = d_E$, $d_1 = d_H$ and $L = 2$ and $M = 2n$. You can apply Proposition 28 because Proposition 33 holds. □

Now we are ready to proof the dimension comparison theorem.

Proof of Theorem 29. 1. If $\min\{2\alpha, \alpha + 1\} = 2\alpha$ then $\mathcal{H}_H^{2\alpha} \ll \mathcal{H}_E^{\alpha}$ by Proposition 31. For the other case we get by Proposition 36 and Proposition 8(1) $\mathcal{H}_H^{\alpha+1} \ll \mathcal{S}_H^{\alpha+1} \ll \mathcal{S}_E^{\alpha} \ll \mathcal{H}_E^{\alpha}$ which proves the case 1.

2. If $\min\{\alpha, n + \alpha/2\} = \alpha$ then $\mathcal{H}_E^{\alpha} \ll \mathcal{H}_H^{\alpha}$ by Proposition 31. For the other case by Proposition 36 and Proposition 8(1) $\mathcal{H}_E^{n+\alpha/2} \ll \mathcal{S}_E^{n+\alpha/2} \ll \mathcal{S}_H^{\alpha} \ll \mathcal{H}_H^{\alpha}$ which proves the case 2.

Thus the dimension comparison theorem is proved. □

Remark 37. In the paper [BRSC03] they have shown the sharpness of the dimension comparison theorem in case of $\mathbb{H}^1$ with different exponents of α and in [BT05] sharpness for the remaining exponents was shown. For references one can see [BRSC03, Theorem 1.2] and [BTW09, Theorem 2.6](in general Carnot groups).

Chapter 3

An $\mathbb{H}$-regular surface in $\mathbb{H}^n$ with large Euclidean dimension

In this chapter we want to construct an intrinsic regular surface in $\mathbb{H}^n$ which has Heisenberg Hausdorff dimension $2n+1$ but the Euclidean Hausdorff dimension is $2n+\frac{1}{2}$ (Theorem 52). The construction for $n = 1$ is due to Kirchheim and Serra Cassano in [KSC04], and in this thesis we generalize it to all $n \in \mathbb{N}$. This generalisation answers a question in [BTW09, p.596] in the specific case where our Carnot group is $\mathbb{G} = \mathbb{H}^n$. The example that we construct here has the smallest gap between $\dim_E$ and $\dim_H$ that is possible for an $\mathbb{H}$-regular surface.

This chapter is divided into two sections. In the first section we construct an auxiliary function that we are going to use in the contruction of the surface. In the second section we are going to construct the surface with the wanted Euclidean Hausdorff dimension.

3.1 Construction of the function

In this section we are going to construct a function which we are going to need for the generalisation of the example. We are not going to follow the construction of the original paper [KSC04]. Instead we present the construction from Tapio Rajala's paper [Raj08] where he also answered a question which in the original paper they could not answer.

Now for the construction we define for every $m \in \mathbb{N}, k \in \{0, 1, 2\}$ and $i \in \{0, 1, ..., m-1\}$ a map $f^m_{mk+i} : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f^m_{mk+i}(x, y) = \left(\frac{x + mk + i}{3m}, (-1)^k \frac{y + i}{m} + \frac{1 + (-1)^{k+1}}{2} \right).$$

Using the above we define set functions $F^m : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2), m \in \mathbb{N}$, by letting

$$F^m(A) = \bigcup_{i=0}^{3m-1} f^m_i(A)$$

Figure 3.1: The blue rectangles are the sets $f_i^m([0,1]^2)$ for $m = 3$ (left) and $m = 4$ (right). In the left picture the blue rectangles form the set $F^3([0,1]^2)$ and in the right picture the blue rectangles form the set $F^4([0,1]^2)$.

for every $A \subset \mathbb{R}^2$. In our case we are only interested in F^3 and F^4. To get some intuition what the functions F^3 and F^4 do, see Figure 3.1 which shows what happens to $[0,1]^2$ when mapped with F^3 or F^4.

The next proposition tells us about the crucial property that the functions have.

Proposition 38. *There is a function $h : \mathbb{R} \to \mathbb{R}$ such that*

1. *for all $t \in \mathbb{R}$ the Euclidean Hausdorff dimension of $h^{-1}(t)$ is at least $\frac{1}{2}$.*
2. *for each $m \geq 1$ we have*

$$\lim_{r \to 0_+} \frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x) - h(y)| : |x - y| \leq r\} = 0.$$

Proof. First we want to construct the graph of our wanted function. For this we will mainly use function F^3 but F^4 will also be used to improve the modulus of continuity. Now define sequence of closed sets $(G_i)_{i=0}^{\infty}$ with

$$G_i = F^{j_1} \circ \cdots \circ F^{j_i}([0,1]^2),$$

where $j_k = 4$ if $\sqrt{k} \in \mathbb{N}$ otherwise $j_k = 3$. Now let $G = \cap_i G_i$. Notice that G is also closed because every G_i is closed.

Let's show that G is a graph of a function over $[0,1]$. Notice that the projection of G that is $\mathrm{proj}(G) = [0,1]$. Because the projection of G is $[0,1]$ it tells us that for every $p \in [0,1]$ the intersection $G \cap \{x = p\} \neq \emptyset$. Notice that for every $p \in [0,1]$ there is a unique y such that $G \cap \{x = p\} = \{(p,y)\}$. There cannot be y_1 and y_2 such that $G \cap \{x = p\} \supseteq \{(p,y_1)\} \cup \{(p,y_2)\}$ when $y_1 \neq y_2$ because with each iterations G_i the functions F^3 or F^4 scale the height of the rectangles by 3^{-1} or 4^{-1}. Because $y_1 \neq y_2$ then the distance is positive between the points which means that there is some index i such that after enough iterations G_i the points y_1 and y_2 cannot be in the same rectangles anymore. For above reasons set G is a graph of a function.

Let g be the function such that G is the graph of g and extend g to whole $\mathbb{R}$ by defining $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x + k) = g(x) + k$$

for every $k \in \mathbb{Z}$ and $x \in [0,1]$.

Proof of 1: Take $t \in [0,1]$ (it suffices to consider this case), identify $\mathbb{R}$ with $\mathbb{R} \times \{t\}$, and construct a measure μ on interval $[0,1]$ using the construction intervals in $G_i \cap \{(x,t) : x \in \mathbb{R}\}$. For every i distribute the measure evenly on the intervals. Precisely this means that the measure of each interval is at most 3^{-i} in the i:th iteration. This is a standard technique in fractal geometry and one can read more about this from [Fal97, Page 9, (5)]. Now take $\epsilon > 0$ and an $n_\epsilon \in \mathbb{N}$ so that $9^{-n\epsilon} \leq (\frac{3}{4})^{\frac{\lfloor \sqrt{n} \rfloor}{2}}$ for every $n \geq n_\epsilon$. To justify the existance of n_ϵ notice that

$$9^{-n\epsilon} \leq (\frac{3}{4})^{\frac{1}{2}\lfloor \sqrt{n} \rfloor} \Leftrightarrow -2n\epsilon \log(3) \leq \frac{1}{2}\lfloor \sqrt{n} \rfloor \log(\frac{3}{4}) \Leftrightarrow \epsilon \geq -\frac{\lfloor \sqrt{n} \rfloor}{n} \frac{\log(3/4)}{4\log(3)}$$

and $\frac{\lfloor \sqrt{n} \rfloor}{n} \to 0$ when $n \to \infty$. Now for every construction interval $X_i \in G_n \cap \{(x,t) : x \in \mathbb{R}\}, n \geq n_\epsilon$, we get

$$\begin{aligned} 0 < \mu(X_i) &\leq 3^{-n} \leq 3^{-n} 9^{n\epsilon} (\frac{3}{4})^{\frac{1}{2}\lfloor \sqrt{n} \rfloor} \\ &= 9^{-\frac{n}{2}} 9^{n\epsilon} 9^{\frac{1}{2}\lfloor \sqrt{n} \rfloor} 12^{-\frac{1}{2}\lfloor \sqrt{n} \rfloor} \leq (9^{-n+\lfloor \sqrt{n} \rfloor} 12^{-\lfloor \sqrt{n} \rfloor})^{\frac{1}{2}-\epsilon} = \mathrm{diam}(X_i)^{\frac{1}{2}-\epsilon}, \end{aligned}$$

where the last equality comes from the fact that intervals of F^3 have length of $\frac{1}{9}$ and intervals of F^4 have length of $\frac{1}{12}$ and $G_i = F^{j_1} \circ \cdots \circ F^{j_i}([0,1]^2)$. Now by Lemma 9 we get $\mathcal{H}^{\frac{1}{2}-\epsilon}(\{h^{-1}(t)\}) > 0$ which means that $\dim_H(h^{-1}(t)) \geq \frac{1}{2} - \epsilon$. Now letting $\epsilon \to 0$ we get $\dim_H(h^{-1}(t)) \geq \frac{1}{2}$. One can see that the requirements for Lemma 9 are fine by first seeing that

$$\tilde{c}\mathrm{diam}(X_i) \leq \mathrm{diam}(X_j) \leq \mathrm{diam}(X_i),$$

where X_i is an interval from iteration i and X_j interval from iteration $i+1$. Now taking $0 < r \ll 1$ and let $n \in \mathbb{N}$ such that $\mathrm{diam}(X_j) \leq r < \mathrm{diam}(X_i)$. Now $\mu(B(x,r))$ intersects at most two intervals from the i-th iteration which gives us that

$$\mu(B(x,r)) \leq 2\mathrm{diam}(X_i)^{\frac{1}{2}-\epsilon} \leq 2\left(\frac{1}{\tilde{c}}\right)^{\frac{1}{2}-\epsilon} \mathrm{diam}(X_j)^{\frac{1}{2}-\epsilon} \leq 2\left(\frac{1}{\tilde{c}}\right)^{\frac{1}{2}-\epsilon} r^{\frac{1}{2}-\epsilon}.$$

Proof of 2: Take $0 < r < 1$ and the largest $n = n(r) \in \mathbb{N}$ such that $r \leq 9^{-n+\lfloor \sqrt{n} \rfloor} 12^{-\lfloor \sqrt{n} \rfloor}$ holds. Now for every $x, y \in \mathbb{R}$ with $|x-y| < r$ we have

$$|h(x) - h(y)| \leq 2 \cdot 3^{-n+\lfloor \sqrt{n} \rfloor} 4^{-\lfloor \sqrt{n} \rfloor}.$$

We get this inequality from the idea of Figure 3.2 and from the observation that there cannot be jump discontinuity because how the function f^m_{mk+i} is constructed. Jump discontinuity here means that there cannot be a gap between two consecutive rectangles.

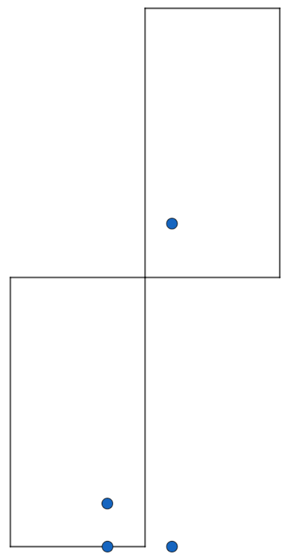

Figure 3.2: Distance between two points is less or equal to width of the rectangle $\Rightarrow$ the distance of function values has to be less or equal to height of two rectangles

Thus for every $m \in \mathbb{N}$ we have

$$\begin{aligned}
&\frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x)-h(y)| : |x-y| \le r\} \\
&\le \frac{((n+1-\lfloor\sqrt{n+1}\rfloor)\log(9)+\lfloor\sqrt{n+1}\rfloor\log(12))^m}{3^{-n-1+\lfloor\sqrt{n+1}\rfloor}12^{-\frac{1}{2}\lfloor\sqrt{n+1}\rfloor}} 2\cdot 3^{-n+\lfloor\sqrt{n}\rfloor}4^{-\lfloor\sqrt{n}\rfloor} \\
&\le (2\log(12)(n+1))^m 6\cdot(\frac{4}{3})^{\frac{1}{2}\lfloor\sqrt{n+1}\rfloor}(\frac{3}{4})^{\frac{1}{2}\lfloor\sqrt{n}\rfloor}(\frac{3}{4})^{\frac{1}{2}\lfloor\sqrt{n}\rfloor} \\
&= (2\log(12)(n+1))^m 6(\frac{3}{4})^{\frac{1}{2}\lfloor\sqrt{n}\rfloor}(\frac{4}{3})^{\frac{1}{2}(\lfloor\sqrt{n+1}\rfloor-\lfloor\sqrt{n}\rfloor)} \\
&\le (2\log(12)(n+1))^m 24(\frac{3}{4})^{\frac{1}{2}\lfloor\sqrt{n}\rfloor}.
\end{aligned}$$

Since the right-hand side tends to 0 as $n \to \infty$, for all $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$, and $0 < r_0 < 1$, such that if $0 < r < r_0$, then $n(r) > N_0$ and

$$\frac{\log{(\frac{1}{r})}^m}{\sqrt{r}} \sup\{|h(x)-h(y)| : |x-y| \le r\} < \epsilon.$$

□

Remark 39. Property 2 ensures that the function h is $\frac{1}{2}$-Hölder continuous on [0,1] since for every $\epsilon > 0$ there exists $0 < r < r(\epsilon)$ such that

$$|h(x)-h(y)| \le \epsilon \log\left(\frac{1}{|x-y|}\right)^{-m} \sqrt{|x-y|}$$

if $|x-y| < r(\epsilon)$. Conversely one can notice if h were an arbitrary $\frac{1}{2}$-Hölder continuous the limit could go to infinity:

$$\frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x)-h(y)| : |x-y| \leq r\} \lesssim \frac{\log((\frac{1}{r}))^m}{\sqrt{r}} r^{\frac{1}{2}}.$$

For Lipschitz functions Property 2 in Proposition 38 is trivially satisfied but the first property does not hold anymore. Indeed, if h were L-Lipschitz,

$$\lim_{r\to 0_+} \frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x)-h(y)| : |x-y| \leq r\} \leq \lim_{r\to 0_+} \frac{\log((\frac{1}{r}))^m}{\sqrt{r}} Lr = 0,$$

but now using Theorem 17 one can see that

$$\int_{\mathbb{R}} \mathcal{H}^0(h^{-1}(t) \cap [0,1]) d\mathcal{H}^1(t) \lesssim \mathcal{H}^1([0,1]) < \infty.$$

Now $\mathcal{H}^0$ is the counting measure and the above tells us that for $\mathcal{H}^1$ almost every $t \in \mathbb{R}$ the points in $h^{-1}(t)$ that intersect with interval $[0,1]$ has to be finite which means that $\dim_H h^{-1}(t) \geq \frac{1}{2}$ cannot happen for almost every t.

3.2 Generalisation of an example by Kirchheim and Serra Cassano

Before we start proving Theorem 52 by adapting the construction from [KSC04] to higher dimensions we need more definitions and some propositions/theorems which we will take for granted.

Definition 40. If Ω is an open subset of $\mathbb{H}^n$ and $f \in C^1(\Omega)$ we define the horizontal gradient of f as

$$\nabla_H f := (X_1 f, ..., X_n f, Y_1 f, ..., Y_n f),$$

where X_i, Y_i are as in Section 1.2.

One can also extend the definition above to work for functions that are not in Euclidean C^1.

Definition 41. We say that function f is differentiable along X_i (Y_i) at P_0 if the map $\lambda \mapsto f(l_{P_0}(\delta_\lambda e_i))$ (respectively: $\lambda \mapsto f(l_{P_0}(\delta_\lambda e_{n+i}))$) is differentiable at $\lambda = 0$, where e_k is the k-th vector of the canonical basis of $\mathbb{R}^{2n+1}$.

Clearly, if $f \in C^1(\Omega)$ then f is differentiable along X_i and Y_i at all points of Ω. Hence, if we set for each f differentiable along X_i and Y_i at P_0 the horizontal gradient to be

$$\nabla_H f = \sum_{n=1}^{n} (X_i f) X_i + (Y_i f) Y_i$$

then this definition extends the one given above.

If $\Omega \subset \mathbb{H}^n$ we shall denote $C^1_{\mathbb{H}}(\Omega)$ the set of continuous real functions in Ω such that $\nabla_H f$ is continuous in Ω.

Definition 42. We say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular hypersurface if for every $p \in S$ there exists an open ball $U(p,r)$ and a function $f \in C^1_{\mathbb{H}}(U(p,r))$ such that

1. $S \cap U(p,r) = \{q \in U(p,r) : f(q) = 0\}$
2. $\nabla_H f(p) \neq 0$.

Example 43. The (y,t)-plane is a $\mathbb{H}$-regular hypersurface in $\mathbb{H}^1$.

Proof. Let function f be $f(x,y,t) = x$. Notice that f is continuous and the horizontal gradient of f is also continuous. One can see this by calculating the horizontal gradient

$$\begin{aligned}\nabla_H f(x,y,t) &= ((\partial_x + 2y\partial_t)f(x,y,t), (\partial_y - 2x\partial_t)f(x,y,t)) \\ &= (\partial_x f(x,y,t) + 2y\partial_t f(x,y,t), \partial_y f(x,y,t) - 2x\partial_t f(x,y,t)) \\ &= (\partial_x x + 2y\partial_t x, \partial_y x - 2x\partial_t x) = (1,0).\end{aligned}$$

The horizontal gradient $\nabla_H f(x,y,t) = (1,0) \neq (0,0)$ which tells us that the horizontal gradient of f is continuous and not zero. Let $p \in (y,t)$-plane then $p = (0, y_0, t_0)$. Now one can see that $(y,t) - plane \cap U((0,y_0,t_0), r) = \{(0,y,t) \in U((0,y_0,t_0), r) : f(0,y,t) = 0\}$ because $f(x,y,t) = 0$ if and only if $x = 0$. □

Remark 44. One can also show with similar argument as above that (x,t)-plane is also $\mathbb{H}$-regular hypersurface. In case of (x,t)-plane the function f would be $f(x,y,t) = y$ and then $\nabla_H f(x,y,t) = (0,1)$. The (x,y)-plane is not a $\mathbb{H}$-regular hypersurface, but we will not prove it here. Instead we try similar argument for (x,y)-plane. Let function f be $f(x,y,t) = t$. Now the function f is continuous and

$$\begin{aligned}\nabla_H f(x,y,t) &= ((\partial_x + 2y\partial_t)f(x,y,t), (\partial_y - 2x\partial_t)f(x,y,t)) \\ &= (\partial_x f(x,y,t) + 2y\partial_t f(x,y,t), \partial_y f(x,y,t) - 2x\partial_t f(x,y,t)) \\ &= (\partial_x t + 2y\partial_t t, \partial_y t - 2x\partial_t t) = (2y, -2x)\end{aligned}$$

which shows that $\nabla_H f(x,y,t)$ is also continuous. Now notice that $\nabla_H f(x,y,t) = 0$ when $p = (0,0,0)$ and the origin is a point from the (x,y)-plane.

Definition 45. The characteristic set of S is defined by

$$C(S) := \{p \in S : T_p S = \mathbf{H}_p \mathbb{H}^n\}.$$

Fact 2. Euclidean C^1 surfaces with no characteristic points are $\mathbb{H}$-regular.

One can see from the computations in Example 43 and Remark 44 that the (x,t)- and (y,t)-planes have no characteristic points but in the (x,y)-plane the origin is a characteristic point.

Definition 46. Let $(x,y,t), p \in \mathbb{H}^n$. We set

$$\pi_p((x,y,t)) = x_1 X_1(p) + ... + x_n X_n(p) + y_1 Y_1(p) + ... + y_n Y_n(p).$$

The map $p \to \pi_p((x,y,t))$ is a smooth section of $\mathbf{H}\mathbb{H}^n$.

Definition 47. We denote by $\langle \cdot, \cdot \rangle_p$ the scalar product on the fiber $\mathbf{H}_p\mathbb{H}^n$ that makes $\{X_1(p), ..., X_n(p), Y_1(p), ..., Y_n(p)\}$ orthonormal. We also write $|\cdot|_p := \sqrt{\langle \cdot, \cdot \rangle_p}$.

The next theorem was proven in [FSSC01, Theorem 6.8].

Theorem 48 (Whitney Extension Theorem). *Let $F \subset \mathbb{H}^n$ be a closed set, assume $f : F \to \mathbb{R}, k : F \to \mathbf{H}\mathbb{H}^n$ are continuous functions. We set*

$$R(p, q) := \frac{f(p) - f(q) - \langle k(q), \pi_q(q^{-1} \cdot p) \rangle_q}{d_H(q, p)},$$

and, if $K \subset F$ is a compact set,

$$\rho_K(\delta) := \sup\{|R(p, q)| : p, q \in K, 0 < d_H(p, q) < \delta\}.$$

If $\rho_K(\delta) \to 0$ as $\delta \to 0$ for every compact set $K \subset F$, then there exists $\tilde{f} : \mathbb{H}^n \to \mathbb{R}, \tilde{f} \in C^1_{\mathbb{H}}(\mathbb{H}^n)$ such that

$$\tilde{f}_{|F} \equiv f, \quad \nabla_{\mathbb{H}} \tilde{f}_{|F} \equiv k.$$

The last theorem that we need before we can prove the main theorem is from [Bal03].

Theorem 49. *Let $Q \subset \mathbb{R}^n$ be the unit cube and let $F : Q \to \mathbb{R}^n$ be a C^1 smooth vector field. Define $A_{g,F} := \{(x, y) \in Q : \nabla g(x, y) = F(x, y)\}$ for $g : Q \to \mathbb{R}$ C^1. Then the following are true.*

1. *For any $\epsilon > 0$ there exists $f_\epsilon : Q \to \mathbb{R}, f_\epsilon \in \bigcap_{0<\alpha<1} C^{1,\alpha}$ such that*

$$\mathcal{L}^n(A_{f_\epsilon, F}) \geq 1 - \epsilon.$$

2. *There exists a constant $K < \infty$ such that*

$$|\nabla f_\epsilon(z) - \nabla f_\epsilon(w)| \leq K(1 + |\log(|z - w|)|)^K |z - w| \quad \text{for all } z, w \in Q. \tag{3.1}$$

Remark 50. Property 2 can be found at the end of the proof of [Bal03, Theorem 4.1 page 80].

Remark 51. Consider a hypersurface $S \subset \mathbb{R}^{2n+1}$ of the form $S = \{(x, y, f(x, y)) : (x, y) \in Q\}$ where $f : Q \to \mathbb{R}$ is a C^1 function on the unit cube Q in $\mathbb{R}^{2n}$. Then $(x, y, f(x, y)) \in C(S)$ if and only if

$$\frac{\partial f}{\partial x_j}(x, y) = 2y_j, \; \frac{\partial f}{\partial y_j}(x, y) = -2x_j, \; j = 1, ..., n.$$

Let us consider the vector field $F : Q \to \mathbb{R}^{2n}$ by $F(x, y) = (2y, -2x)$. The characteristic set $C(S)$ is clearly related to the subset of Q on which the gradient of f coincides with the vector field F.

More precisely, if

$$A_{f,F} := \{(x, y) \in Q : \nabla f(x, y) = F(x, y)\},$$

then

$$C(S) = \{((x, y), f(x, y)) : (x, y) \in A_{f,F}\}.$$

Now we have everything necessary to prove the main theorem.

Theorem 52. *There exists an $\mathbb{H}$-regular surface $S \subset \mathbb{H}^n$ such that*

$$\dim_E S = 2n + \frac{1}{2} \tag{3.2}$$

Proof. Throughout this proof, we will for simplicity denote $d := d_H$ and $\mathcal{H}^s = \mathcal{H}^s_E$. To prove the theorem we need two things. The first is Corollary 30 which we already have proved. The second thing what we need is the next inequality

$$\mathcal{H}^{(4n+1-\epsilon)/2}(S) > 0 \tag{3.3}$$

for all $\epsilon \in (0, 1)$. The theorem follows from these two.

Let us start proving the inequality. By Theorem 49 there is a C^1-function $g : Q = [0, 1]^{2n} \to \mathbb{R}$ and a constant $K \leq \infty$ such that

(3.4)
$$\mathcal{L}^{2n}(A_g) > \frac{1}{2} \quad \text{where} \quad A_g := \{(x, y) \in Q : \nabla g((x, y)) = (2y_1, ..., 2y_n, -2x_1, ..., -2x_n)\},$$

and g satisfies property (2) in Theorem 49.

Observe that $\|g\|_\infty < \infty$ because g is continuous on the compact set Q. For the function g we have the next inequality

(3.5)
$$|g(z) - g(w) - \langle \nabla g(w), z - w\rangle| \leq K'|z - w|^2(1 + |\log(|z - w|)|)^K \quad \text{for all } z, w \in Q,$$

where $K' < \infty$ is a constant depending only on K.

One can show this using the mean value theorem and the Cauchy-Schwarz inequality

$$\begin{aligned}
&|g(z) - g(w) - \langle \nabla g(w), z - w\rangle| \\
&\overset{\text{m.v.t}}{=} |\langle \nabla g(y), z - w\rangle - \langle \nabla g(w), z - w\rangle| \text{ for some } y \text{ on the segment between } z \text{ and } w \\
&= |\langle \nabla g(y) - \nabla g(w), z - w\rangle| \\
&\overset{\text{C-S}}{\leq} |\nabla g(y) - \nabla g(w)||z - w| \\
&\leq K(1 + |\log(|y - w|)|)^K |y - w||z - w| \\
&\leq K'(1 + |\log(|z - w|)|)^K |z - w|^2,
\end{aligned}$$

where $K' = K'(K)$ is a constant depending on $K \geq 1$.

Let us justify the last inequality. Let $K \geq 1$ then there exists $K' = K'(K)$ such that for $0 < r < R$:

$$(1 + |\log(r)|)^K r \leq K'(1 + |\log(R)|)^K R. \tag{$*$}$$

Proof : Define $\phi(r) := (1 + |\log(r)|)^K r$. Now ϕ is monotone increasing on $[1, \infty)$ as a product of monotone increasing functions. There exists $r(K) < 1$ such that ϕ is monotone increasing on $(0, r(K)]$. One can see this by computing the derivative of ϕ:

$$\phi'(r) = (1 + |\log(r)|)^K - K(1 + |\log(r)|)^{K-1} = (1 + |\log(r)|)^{K-1}\left((1 + |\log(r)|) - K\right).$$

The first term $(1+|\log(r)|)^{K-1} > 0$ and the second term $((1+|\log(r)|)-K) \geq 0$ when $|\log r| \geq K-1$. Lastly for $r \in (r(K), 1)$ and $R > r$ we estimate :

$$\begin{aligned}
(1+|\log(r)|)^K r &\leq (1+|\log(r(K))|)^K r \\
&\leq (1+|\log(r(K))|)^K R \\
&\leq (1+|\log(r(K))|)^K (1+|\log(R)|)^K R,
\end{aligned}$$

where the last inequality comes from fact that for arbitrary $R \in (0,\infty)$ we have that $R \leq (1+|\log(R)|)^K R$. Now we have justified the last inequality.

Next choose the function h from Proposition 38 and set $F^* := A_g \times [-\|g\|_\infty, 1+\|g\|_\infty]$. Note that F^* is a closed subset of $\mathbb{H}^n$. Now define the function $f^* : F^* \to \mathbb{R}$ by

$$f^*(x,y,t) = h(t-g(x,y)) \;\text{ if }\; (x,y,t) \in F^*$$

and the section $k^* : F^* \to \mathbf{H}\mathbb{H}^n$ as

$$k^*((x,y,t)) := (0,...,0) \;\text{ if }\; (x,y,t) \in F^*.$$

Reminder : Our goal is to construct the desired surface S. This desired surface S will be obtained as $S = \Omega \cap \{f=0\}$, where Ω is an open neighborhood of F^* and f is defined in $(**)$ on p.38 using the extension of f^* guaranteed by Theorem 48.

Next we want to show that the assumption of Theorem 48 holds true for f^*, k^* and F^*. The continuity of k^*: Let $p=(x,y,t), q=(x',y',t') \in F^*$ and $\epsilon, \delta > 0$ then

$$|k^*(p) - k^*(q)| = 0 < \epsilon \;\text{ always when }\; |p-q| \leq \delta.$$

The continuity of f^* follows from the fact that g is a continuous function and h is also continuous function. The continuity of h follows from Proposition 38. Lastly we need to show that

$$\lim_{\rho\to 0} \rho_{F^*}(\delta) = 0 \tag{3.6}$$

where ρ_{F^*} is the function defined in Theorem 48 for $K = F^*$. Notice that for any $p=(x,y,t), q=(x',y',t') \in F^*$ we have that

$$\begin{aligned}
|g(x,y) - g(x',y') - 2(x\cdot y' - x'\cdot y)| &= |g(x,y) - g(x',y') - 2(y'\cdot(x-x') - x'\cdot(y-y'))| \\
&= |g(x,y) - g(x',y') - (2y'\cdot(x-x') - 2x'\cdot(y-y')))| \\
&= |g(x,y) - g(x',y') - \langle \nabla g(x',y'), (x,y)-(x',y')\rangle| \\
&\leq K' d(p,q)^2 (1+|\log(d(p,q))|)^K.
\end{aligned}$$

We got the last inequality from (3.5) followed by $(*)$ using $|(x,y)-(x',y')| \leq d(p,q)$.

One can also see from $p^{-1} * q = (-x+x', -y+y', -t+t'+2(x\cdot y' - x'\cdot y))$ that

$$|t'-t+2(x\cdot y' - x'\cdot y)| \leq K' d(p,q)^2 (1+|\log(d(p,q))|)^K.$$

With these two inequalities one can show that

$$|(t-g(x,y)) - (t'-g(x',y'))| \leq 2K' d(p,q)^2 (1+|\log(d(p,q))|)^K.$$

Proof:

$$\begin{aligned}
&|(t-g(x,y))-(t'-g(x',y'))| \\
&= |t-t'-2(x\cdot y'-x'\cdot y)+g(x',y')-g(x,y)+2(x\cdot y'-x'\cdot y)| \\
&\leq |t-t'-2(x\cdot y'-x'\cdot y)|+|g(x',y')-g(x,y)+2(x\cdot y'-x'\cdot y)| \\
&= |t'-t+2(x\cdot y'-x'\cdot y)|+|g(x,y)-g(x',y')-2(x\cdot y'-x'\cdot y)| \\
&\leq 2K'd(p,q)^2(1+|\log(d(p,q))|)^K.
\end{aligned}$$

Now using Proposition 38 property 2 we get

$$\begin{aligned}
|f^*(p)-f^*(q)| &= |h(t-g(x,y))-h(t'-g(x',y'))| \\
&\leq \bar{K}\frac{d(p,q)(1+|\log(d(p,q))|)^{K/2}}{|\log(d(p,q)(1+|\log(d(p,q))|)^{K/2})|^{K+1}}.
\end{aligned}$$

One can show this by choosing $r = 2K'd(p,q)^2(1+|\log(d(p,q))|)^K$. Notice that $r \to 0$ if and only if $d(p,q) \to 0$. We get by Proposition 38 for $M = K+1$ that for given $\tilde{K} > 0$ there is $r_0(\tilde{K}) = r_0 > 0$ such that if $r < r_0$, then :

$$\begin{aligned}
&|f^*(p)-f^*(q)| = |h(t-(g(x,y)))-h(t'-g(x',y'))| \\
&\leq \sup\{|h(t-g(x,y))-h(t'-g(x',y'))| : |(t-g(x,y))-(t'-g(x',y'))| \leq r\} \\
&\leq \tilde{K}\frac{\sqrt{r}}{\log((1/r))^{K+1}} \\
&= \bar{K}\frac{d(p,q)(1+|\log(d(p,q))|)^{K/2}}{|\log(d(p,q)(1+|\log(d(p,q))|)^{K/2})|^{K+1}},
\end{aligned}$$

where the constant $\bar{K}$ depends on $\tilde{K}$ and K'. The above tells us that

$$\frac{|f^*(p)-f^*(q)|}{d(p,q)} \leq \tilde{K}\frac{(1+|\log(d(p,q))|)^{K/2}}{|\log(d(p,q)(1+|\log(d(p,q))|)^{K/2})|^{K+1}},$$

if $d(p,q)$ is small enough.

Observing the limits one can see that

$$\lim_{t\to 0_+} (1+|\log(t)|)^{K/2} = \infty$$

and

$$\lim_{t\to 0_+} |\log(t(1+|\log(t)|)^{K/2})|^{K+1} = \infty.$$

The exponent $K+1$ in the denominator is bigger than the exponent in the numerator $K/2$ and we will show that in fact

$$\lim_{t\to 0_+} \frac{(1+|\log(t)|)^{K/2}}{|\log(t(1+|\log(t)|)^{K/2})|^{K+1}} = 0,$$

and in particular, the expression in the limit is bounded as $t \to 0_+$. Let's justify the statement above formally. Claim: $\lim_{t\to 0_+} \psi(t) = 0$ for

$$\begin{aligned}\psi(t) &= \frac{(1+|\log(t)|)^{K/2}}{|\log(t(1+|\log(t)|)^{K/2})|^{K+1}} \\ &= \frac{(1+|\log(t)|)^{K/2}}{\left(-\log(t) - \frac{K}{2}\log(1-\log(t))\right)^{K+1}}\end{aligned}$$

for $t \in (0,1]$ small enough that $t(1+|\log(t)|)^{K/2} < 1$. First we use the fact that

$$\ln(1+x) \le \frac{x}{\sqrt{x+1}} \text{ for } x \ge 0, \text{ applied to } x = -\log(t).$$

The inequality gives us that

$$\log(1-\log(t)) \le \frac{-\log(t)}{\sqrt{1-\log(t)}}. \tag{3.7}$$

Now using the inequality (3.7) we get

(3.8)

$$-\log(t) - \frac{K}{2}\log(1-\log(t)) \ge -\log(t) + \frac{K}{2}\frac{\log(t)}{\sqrt{1-\log(t)}} = -\log(t)\left(1 - \frac{K/2}{\sqrt{1-\log(t)}}\right).$$

There exists a number $t(K) \in (0,1)$ such that for all $t \in (0, t(K))$:

$$1 - \frac{K/2}{\sqrt{1-\log(t)}} \ge \frac{1}{2}. \tag{3.9}$$

Now by (3.8) and (3.9) we get that

$$-\log(t) - \frac{K}{2}\log(1-\log(t)) \ge -\frac{\log(t)}{2}.$$

Combining the information about the inequalities gives us

$$\begin{aligned}\psi(t) &= \left(\frac{(1-\log(t))}{-\log(t) - \frac{K}{2}\log(1-\log(t))}\right)^{\frac{K}{2}} \frac{1}{\left(-\log(t) - \frac{K}{2}\log(1-\log(t))\right)^{\frac{K}{2}+1}} \\ &\le \left(\frac{1-\log(t)}{\frac{-\log(t)}{2}}\right)^{\frac{K}{2}} \frac{1}{\left(\frac{-\log(t)}{2}\right)^{\frac{K}{2}+1}}.\end{aligned}$$

Now the first term $\to 2^{K/2}$ as $t \to 0$ and the second term $\to 0$ as $t \to 0$ which proves the claim.

One can conclude from the above justification that the limit

$$\frac{|f^*(p) - f^*(q)|}{d(p,q)} \to 0 \quad \text{if } p, q \in F^* \text{ and } d(p,q) \to 0,$$

thus (3.6) holds.

Therefore we can apply Theorem 48 and extend $f^* : F^* \to \mathbb{R}$ to a function $\tilde{f}^* : \mathbb{H}^n \to \mathbb{R}$, $\tilde{f}^* \in C^1_{\mathbb{H}}(\mathbb{H}^n)$ such that

$$\nabla_{\mathbb{H}} \tilde{f}^*_{|F^*} \equiv 0. \tag{3.10}$$

Next we can define a $C^1_{\mathbb{H}}(\mathbb{H}^n)$ function $f : \mathbb{H}^n \to \mathbb{R}$ as

$$f(x, y, t) := \tilde{f}^*(x, y, t) - x_1. \tag{$**$}$$

Now by construction and (3.10), because

$$|\nabla_{\mathbb{H}} f|_p = |(-1, 0, ..., 0)|_{\mathbb{R}^{2n}} = 1 \text{ for all } p \in F^*$$

there is an open set $\Omega \supset F^*$ such that

$$|\nabla_{\mathbb{H}} f|_p \neq 0 \text{ for all } p \in \Omega.$$

Defining

$$S := \Omega \cap \{f = 0\},$$

then S is an $\mathbb{H}$-regular surface.

One can observe that

(3.11)

$$\begin{aligned} S \supset A : &= \bigcup_{(x,y) \in A_g} (\{(x, y)\} \times ((h^{-1}(x_1) + g(x, y) \cap [-\|g\|_\infty, 1 + \|g\|_\infty]))) \\ &= \{(x, y, t) : (x, y) \in A_g, t \in \big(h^{-1}(x_1) + g(x, y)\big) \cap [-\|g\|_\infty, 1 + \|g\|_\infty]\}. \end{aligned}$$

One can also see that if $m \in h^{-1}(x_1) \cap [0, 1]$ then $m + g(x, y) \in [-\|g\|_\infty, 1 + \|g\|_\infty]$. Using Theorem 17 when $s = 2n + \frac{1-\epsilon}{2}$, $t = 2n$, $X = A$, $Y = A_g$ and $P(x, y, t) = \text{proj}_t(x, y, t) = (x, y)$ as our function one gets

$$\begin{aligned} &\int_{A_g}^* \mathcal{H}^{(1-\epsilon)/2}(h^{-1}(x_1) \cap [0, 1]) dxdy \\ &\quad \leq \int_{A_g}^* \mathcal{H}^{(1-\epsilon)/2}((h^{-1}(x_1) + g(x, y)) \cap [-\|g\|_\infty, 1 + \|g\|_\infty]) dxdy \\ &\quad = \int_{A_g}^* \mathcal{H}^{(1-\epsilon)/2} \left(P^{-1}(x, y) \cap A\right) dxdy \\ &\quad \leq c_{\epsilon,n} \mathcal{H}^{(4n+1-\epsilon)/2}(A). \end{aligned} \tag{3.12}$$

Note that $\mathcal{H}^{2n}(A_g) > 0$ and the integrand is strictly positive. Since $\epsilon > 0$ we get from Proposition 38 property 1 that

$$\int_{A_g}^* \mathcal{H}^{(1-\epsilon)/2}(h^{-1}(x_1) \cap [0, 1]) dxdy > 0. \tag{3.13}$$

Now from (3.11), (3.12) and (3.13) we get (3.3).

Now we have everything to prove that $\dim_E S = 2n + \frac{1}{2}$. The inequality (3.3) gives us that $\dim_E S \geq 2n + \frac{1}{2}$ when letting $\epsilon \to 0$. To show that $\dim_E S \leq 2n + \frac{1}{2}$ we can use Corollary 30. Let $\alpha := \dim_E S \geq 2n + \frac{1}{2} > 2n$ then $\beta := \dim_H S \geq \beta_-(\alpha) = 2\alpha - 2n$ $\Rightarrow \alpha \leq \frac{\beta}{2} + n = 2n + \frac{1}{2}$ in the last equality we used fact from [FSSC03] which tells us that $\dim_H S = 2n + 1$. Now we have proved that $\dim_E S = 2n + \frac{1}{2}$.

□

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