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Control theory / *Théorie du contrôle*

Local controllability does imply global controllability

La contrôlabilité locale implique la contrôlabilité globale

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Abstract. We say that a control system is locally controllable if the attainable set from any state x contains an open neighborhood of x, while it is controllable if the attainable set from any state is the entire state manifold. We show in this note that a control system satisfying local controllability is controllable. Our self-contained proof is alternative to the combination of two previous results by Kevin Grasse.

Résumé. Nous disons qu'un système de contrôle est localement controllable si les ensembles atteignables à partir de tout état x sont un voisinage de x, tandis que le système est contrôlable si les ensembles atteignables à partir de tout état coïncident avec la variété entière. Nous montrons qu'un système qui est localement controllable est contrôlable. Notre preuve est une alternative à la combinaison de deux résultats précédents par Kevin Grasse.

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1. Introduction

Let M be a connected finite-dimensional smooth manifold. Here we study the controllability properties of a system of the form

$$\dot{x} = F(x, u(t)), \qquad x \in M, \tag{C}$$

where $F : M \times U \to TM$, with $U \subset \mathbb{R}^m$ for some $m \in \mathbb{N}$ and F is $\mathscr{C}^1$. We consider as set of admissible controls either $\mathscr{U}_{pc} = \cup_{T \geq 0}\{u : [0,T] \to U \mid u \text{ piecewise constant}\}$, or the family $\mathscr{U}_\infty = \cup_{T\geq 0} L^\infty([0,T],U)$. Given an admissible control $u \in \mathscr{U}$, where $\mathscr{U}$ is one of these two classes, we denote by $\phi(\cdot, x, u)$ the unique absolutely continuous maximal solution of (C) with initial condition x at time 0. The *attainable set* from a point x in M for system (C) is

$$\mathscr{A}_x = \{\phi(T, x, u) \mid T \geq 0,\ u \in \mathscr{U},\ \phi(\cdot, x, u) \text{ is defined on } [0,T]\}. \tag{1}$$

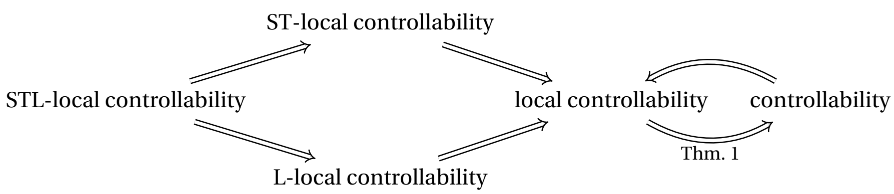
When $y \in \mathscr{A}_x$ we say that y is *attainable from* or *reachable from* x. We say that system (C) is *locally controllable* if the attainable set from any initial state x in M is a neighborhood of x, i.e.,

$$x \in \operatorname{Int} \mathscr{A}_x, \qquad \forall\, x \in M, \tag{2}$$

while it is said to be *controllable* if $\mathscr{A}_x = M$ for each x in M. Observe that sometimes in the literature the expression *local controllability* is used with a different meaning (see, for example, [4, Definition 3.2]).

The notion of local controllability has been studied extensively in the literature, especially in the stronger forms of *small-time local controllability* (ST-local controllability) (for which the attainable set $\mathscr{A}_x$ is replaced by the set of points attainable from x within an arbitrarily small positive time) and *localized local controllability* (L-local controllability) (for which one considers the set of points attainable from x by admissible trajectories that stay in an arbitrarily small neighborhood of x). A combination of the two constraints yields the notion of *small-time localized local controllability* (STL-local controllability). Table 1 contains a scheme of the implications that can be directly deduced from the above definitions, and we refer to Section 2 for a detailed description of these different types of local controllability and the relations between them.

Table 1. Relations between different types of local controllability. As discussed in Section 2, the missing arrows cannot be added to the scheme. The only arrow that needs to be justified here is the one representing the fact that local controllability implies controllability. This is the object of Theorem 1.



Folklore has it that controllability can be deduced from suitable versions of local controllability: for example, in [3, Section 12.3] it is stated (without proof) that STL-local controllability implies controllability. Another example are linear systems for which controllability is known to be equivalent to ST-local controllability. The question whether ST-local controllability implies controllability for a more general control system was formulated for instance in [1, Section 3].

The purpose of this paper is to prove that the weakest version of local controllability is sufficient to deduce controllability, giving in particular a positive answer to the question just mentioned.

Theorem 1. *If system* (C) *is locally controllable, then* (C) *is controllable.*

We mention that if L-local controllability is known, then controllability can be shown with a simpler proof than what is proposed here for Theorem 1 (see Appendix A) but still does not follow immediately from the definitions, since reachability is not a symmetric property. Indeed, the fact that one can reach an open neighborhood of a given initial state does not imply that any point in the neighborhood can be steered back to the initial state.

Let us mention that it is hard to find testable conditions for local controllability to hold. Indeed, in the literature it is more common to find conditions for ST-local controllability since those can be deduced from Lie algebraic arguments (see, e.g., [9] and references therein). It should be noticed that such conditions do not usually provide local controllability at *every* point, since they tipically require that the point at which local controllability is studied is an equilibrium of one of the admissible vector fields. Finally, we note that the interest in ST-local controllability is motivated, for example, by its relation with the continuity of the optimal time function, as explained in [14].

Key steps of the proof of Theorem 1

System (C) is said to be *approximately controllable* if all attainable sets are dense, i.e.,

$$\mathrm{cl}\mathscr{A}_x = M, \qquad \forall\ x \in M,$$

where cl denotes the closure with respect to the topology of M. The first step in the proof of of Theorem 1 is to prove that local controllability implies approximate controllability.

Lemma 2. *If* (C) *is locally controllable, then it is approximately controllable.*

The proof of the lemma, presented in Section 3, relies on the regularity of the flow of (C) for a fixed control, and on the connectedness of M. The second key property is the following lemma.

Lemma 3. *Assume that* (C) *is locally controllable. Then, for every x and y in M,*

$$y \in \mathscr{A}_x \implies x \in \mathscr{A}_y.$$

Lemma 3 is proved by showing that trajectories of (C) can be retraced back by finding a control driving their endpoint to their starting point. More precisely, assume that y is in $\mathscr{A}_x$ and consider a control u such that $y = \phi(T, x, u)$. For t in a left neighborhood of T, the point $\phi(t, x, u)$ can be reached from y due to local controllability. By repeating this argument and concatenating the controls, one can find smaller and smaller $t \geq 0$ such that $\phi(t, x, u)$ can be reached from y. In order to reach $x = \phi(0, x, u)$ (i.e., to prove the lemma) one has to show that the sequence of times t found following such a procedure eventually attains zero, unlike the situation depicted in Figure 1.

These arguments for the proof of Theorem 1 hold for other classes of controls, provided that the control system remains well-posed in the space of absolutely continuous functions and that the set of controls contains the piecewise constant functions. (For conditions of this type, see, e.g., [2, Chapters 2 and 3].)

The fact that local controllability implies global controllability was already treated in [11] for a compact state manifold M. For the noncompact case, in [5] the author shows that local controllability implies global one for piecewise constant controls. A related result in [7] shows that local controllability via bounded measurable controls implies the same via piecewise constant controls, therefore extending the previous result to control systems with bounded measurable controls. Our self-contained proof is alternative to the combination of these two results.

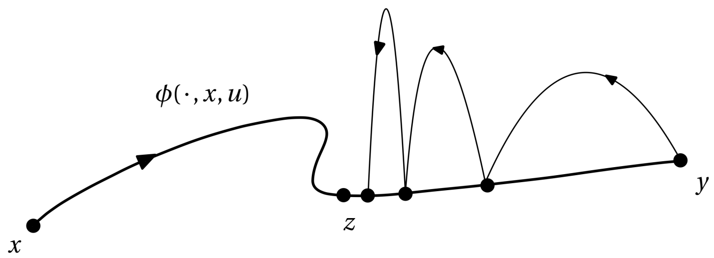


Figure 1. When retracing back the trajectory $\phi(\cdot, p, u)$, the attainable sets might get smaller and smaller and collapse to a point z before reaching x, since a priori their size is not lower semi-continuous. Lemma 3 shows that this situation cannot happen, proving a key step for the proof of Theorem 1.

2. On the local forms of controllability

Let us define the set of points attainable from a point x in M at a time $t > 0$ with trajectories of (C) remaining inside of a domain Ω by

$$\mathscr{A}^t_{x,\Omega} = \{\phi(t, x, u) \mid u \in \mathscr{U},\ \phi(\cdot, x, u) \text{ defined on } [0, t] \text{ with values in } \Omega\},$$

and let

$$\mathscr{A}^{\leq T}_{x,\Omega} = \bigcup_{0<t\leq T} \mathscr{A}^t_{x,\Omega}, \qquad \mathscr{A}_{x,\Omega} = \bigcup_{0<t<+\infty} \mathscr{A}^t_{x,\Omega}.$$

Moreover, let us denote $\mathscr{A}^{\leq T}_x = \mathscr{A}^{\leq T}_{x,M}$ and notice that $\mathscr{A}_x = \mathscr{A}^{\leq +\infty}_{x,M}$. System (C) is said to be small-time locally controllable if

$$x \in \operatorname{Int} \mathscr{A}^{\leq T}_x \qquad \forall\ T > 0,\ \forall\ x \in M; \qquad \text{(ST-locally controllable)}$$

moreover, system (C) is said to be localized locally controllable if

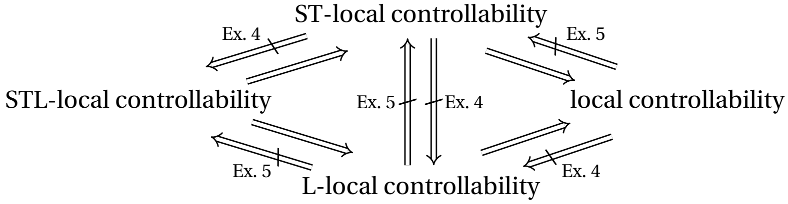
$$x \in \operatorname{Int} \mathscr{A}_{x,\Omega} \qquad \forall\ x \in M,\ \forall\ \Omega \text{ neigh. of } x. \qquad \text{(L-locally controllable)}$$

Finally, system (C) is said to be small-time localized locally controllable if

$$x \in \operatorname{Int} \mathscr{A}^T_{x,\Omega} \qquad \forall\ T > 0,\ \forall\ x \in M,\ \forall\ \Omega \text{ neigh. of } x. \qquad \text{(STL-locally controllable)}$$

One recognizes immediately that the implications contained in Table 1 can be directly deduced from the above definitions, with the exception of Theorem 1. In this section we show that the missing arrows cannot be added to the scheme, provinding the examples summarised in Table 2.

Table 2. A diagram summarizing the fact that the weaker forms of local controllability do not imply the stronger.



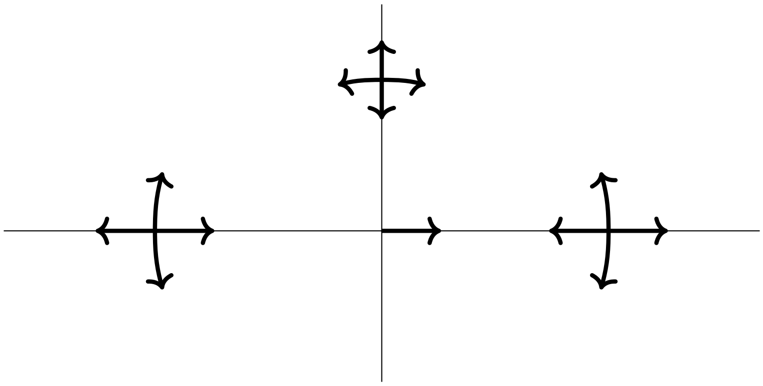


Figure 2. An illustration of the admissible vector fields of the control system in Example 5, which is L-locally controllable everywhere but is not ST-locally controllable at the origin.

Example 4 (ST-local controllability $\not\Rightarrow$ L-local controllability). Recall that a linear control system $\dot{x} = Ax + Bu$ is controllable if and only if it is ST-locally controllable which, in turns, is equivalent to the Kalman condition (see, e.g, [13]). On the other hand, a control system is L-locally controllable only if the range $\mathrm{Im}(B)$ of B is the entire state space. Indeed, if $A\bar{x}$ points outside $\mathrm{Im}(B)$ at some $\bar{x} \in \mathrm{Im}(B)$, considering a linear system of coordinates associated with a basis containing $A\bar{x}$ and a set of generators for $\mathrm{Im}(B)$, we have that the component along $A\bar{x}$ of every admissible trajectory staying in a sufficiently small neighborhood of $\bar{x}$ is increasing. Hence, a necessary condition for L-local controllability is that Ax is in $\mathrm{Im}(B)$ for every $x \in \mathrm{Im}(B)$. If such a condition is satisfied, every admissible trajectory starting from $\mathrm{Im}(B)$ cannot exit it. Therefore, L-local controllability can only hold when $\mathrm{Im}(B)$ is maximal. (See [1] for more results on controllability and local controllability of control-affine systems with unbounded controls and [12] for a detailed study on the minimal time for controllability in the case of finite-dimensional linear autonomous control systems with state constraints and unbounded controls.)

Hence any controllable linear system such that $\mathrm{Im}(B)$ is not maximal is ST-locally controllable without being L-locally controllable. This also proves that a ST-locally controllable system is not necessarily STL-locally controllable, and that a locally controllable system is not necessary L-locally controllable, as indicated in Table 2.

Example 5 (L-local controllability $\not\Rightarrow$ ST-local controllability). We now present an example of a control system in $\mathbb{R}^2$ that is L-locally controllable, but which fails to be ST-locally controllable at the origin. The example shows, in particular, that a L-locally controllable system is not necessarily STL-locally controllable and that a locally controllable system is not necessarily ST-locally controllable, as indicated in Table 2. Let us define, for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$X_0(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_1(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \text{and} \quad X_2(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and let us consider the control system

$$x' = u_0 X_0(x) + u_1 X_1(x) + u_2 X_2(x), \qquad u_0 \in [0,1],\ u_1, u_2 \in [-1,1].$$

An illustration of this system can be found in Figure 2. Outside of the origin this system is STL-locally controllable, since the vector fields X_1 and X_2 are transversal and u_1, u_2 can take both positive and negative values. One can check that the maximal angular velocity is independent of the radius, and that the time needed to complete a semicircle is greater than or equal to π.

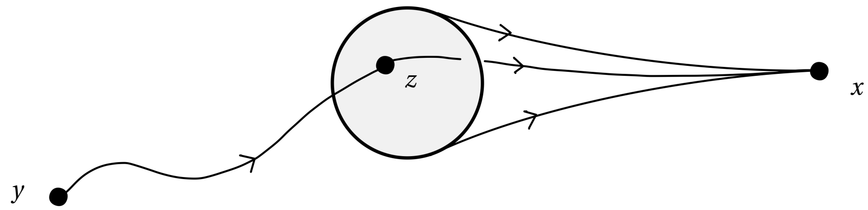


Figure 3. If (C) is approximately controllable and the controllable set to a certain state x has nonempty interior, then x can be reached from any other state in M. This is observed in Remark 6.

3. Proofs

3.1. *Preliminaries*

Let us observe that, fixed a control $u \in \mathscr{U}$, the non-autonomous differential equation (C) is well-posed in the space of absolutely continuous functions. Precisely, for a given initial condition $x \in M$, there exist $T > 0$ and a neighborhood W of x such that $\phi(t, y, u)$ is defined for $(t, y) \in [0, T] \times W$ and absolutely continuous with respect to t. Moreover, for any $t \in [0, T]$ the flow $\phi(t, \cdot, u)$ restricted to W is a local $\mathscr{C}^1$-diffeomorphism (see, e.g., [8, Theorem 6.2]) or [13, Theorem 1]).

Let us denote by $\mathscr{F} = \{F(\cdot, u) \mid u \in U\}$ the family of vector fields of M parametrized by F. For a fixed vector field f in $\mathscr{F}$, we denote by $e^{tf}(y)$ the value at time t of the trajectory of $\dot{x} = f(x)$ starting from y, implicitly assuming that such a trajectory is indeed defined between 0 and t.

Given a point x in M, the *controllable set to* x is the set of points which can be steered to x, i.e.,

$$\mathscr{A}_x^- = \{y \in M \mid x \in \mathscr{A}_y\}.$$

Observe that $\mathscr{A}_x^-$ is the attainable set from x for the control system defined by $-F$, whose solutions are the trajectories of (C) followed in the opposite time direction.

Remark 6. Assume that (C) is approximately controllable. If a point x in M satisfies $\mathrm{Int}\mathscr{A}_x^- \neq \emptyset$, then x can be reached from any other point. Indeed, for any y in M, since $\mathscr{A}_y$ is dense in M it intersects the interior of A_x^-. Thus, there exists $z \in A_y \cap A_x^-$. Since $z \in \mathscr{A}_x$ and $z \in \mathscr{A}_y$, system (C) can be steered from y to x (see Figure 3).

3.2. *Approximate controllability of locally controllable systems*

We are ready to prove Lemma 2.

Proof of Lemma 2. Let $x \in M$. We want to show that $\mathrm{cl}(\mathscr{A}_x)$ is open. By connectedness of M, this implies that $\mathrm{cl}(\mathscr{A}_x) = M$, thus proving the lemma.

Let $y \in \mathrm{cl}(\mathscr{A}_x)$. We claim that for every control $u \in \mathscr{U}$ and $t > 0$ such that $\phi(t, y, u)$ is defined, we have that

$$\phi(t, y, u) \in \mathrm{cl}(\mathscr{A}_x). \tag{3}$$

This concludes the proof of the lemma. Indeed, from (3) it follows that $\mathscr{A}_y \subset \mathrm{cl}(\mathscr{A}_x)$; since $\mathscr{A}_y$ contains y in its interior due to local controllability, this proves that $\mathrm{cl}(\mathscr{A}_x)$ is open.

In order to prove (3), fix any neighborhood V of $\phi(t, y, u)$: we show that V has nonempty intersection with $\mathscr{A}_x$. Consider a neighborhood W of y such that the map $\varphi = \phi(t, \cdot, u)|_W$ is a $\mathscr{C}^1$-diffeomorphism. In particular, $\varphi(W)$ is a neighborhood of $\phi(t, y, u)$, and the set $W' = \varphi^{-1}(V \cap \varphi(W))$ is a neighborhood of y. Since y is in the closure of $\mathscr{A}_x$, there exists $y_1 \in W' \cap \mathscr{A}_x$. Consider an admissible control steering (C) from x to y_1: by concatenating such a control with

u one finds that $\phi(t, y_1, u)$ is in $\mathscr{A}_x$. This implies that $\phi(t, y_1, u)$ belongs to $V \cap \mathscr{A}_x$, proving that $V \cap \mathscr{A}_x$ is nonempty, as required. □

3.3. *Symmetry of attainable sets of locally controllable systems*

Proof of Lemma 3. Let x and y in M be such that $y \in \mathscr{A}_x$. We argue by contradiction supposing that $x \notin \mathscr{A}_y$. We claim that this implies the existence of a point z in M (actually $z \in \mathscr{A}_x$) such that

$$z \notin \mathscr{A}_y \quad \text{and} \quad \operatorname{Int}\mathscr{A}_z^- \neq \emptyset. \tag{4}$$

This yields a contradiction, since the assertions in (4) cannot hold both at the same time due to Remark 6. (Notice that Remark 6 applies to system (C) because of Lemma 2.) The rest of the proof is dedicated to proving the existence of a point z satisfying (4).

Consider $u \in \mathscr{U}$ and $T > 0$ such that $\phi(T, x, u) = y$. Define the absolutely continuous curve $\gamma : [0, T] \to M$ by $\gamma(t) = \phi(t, x, u)$. Let

$$\tau = \inf\{t \in [0, T] \mid \gamma(t) \in \mathscr{A}_y\}.$$

We claim that $\gamma([0, T]) \cap \mathscr{A}_y = \gamma((\tau, T])$ (see Figure 1, in which z plays the role of $\gamma(\tau)$). Indeed, $\gamma^{-1}(\gamma([0, T]) \cap \mathscr{A}_y)$ is open since $\mathscr{A}_y$ is open, and its complement is nonempty since it contains zero (we are assuming that $x \notin \mathscr{A}_y$). Moreover,if a certain $s \in [0, T]$ satisfies $\gamma(s) \in \mathscr{A}_y$, then, for all t in $[s, T]$, one has $\gamma(t) \in \mathscr{A}_y$ since it suffices to concatenate the control steering (C) from y to $\gamma(s)$ with $u|_{[s,t]}$ in order to attain $\gamma(t)$. Up to renaming $\gamma(\tau)$ as x, we can assume that $\tau = 0$. Namely, without loss of generality, we can assume

$$x \notin \mathscr{A}_y \quad \text{and} \quad \phi(t, x, u) \in \mathscr{A}_y, \quad \text{for all } t \in (0, T].$$

Let V be a neighborhood of x contained in $\mathscr{A}_x$. We now construct a parametrization $C_n : I_n \to M$ ($I_n \subset \mathbb{R}^n$ open) of a n-dimensional $\mathscr{C}^1$-embedded submanifold of M satisfying $C_n(I_n) \subset \mathscr{A}_x^-$, and therefore x (or more exactly $\gamma(\tau)$) satisfies (4). This will be done by a recursive argument, by constructing a finite sequence of embeddings $C_k : I_k \to M$ ($I_k \subset \mathbb{R}^k$ open) of class $\mathscr{C}^1$, $k = 1, \ldots, n$, with

$$C_k(s) \in V \quad \text{and} \quad x \in \mathscr{A}_{C_k(s)}, \quad \text{for all } s \in I_k. \tag{5}$$

Let us begin with $k = 1$. Let $v \in U$ (constant) such that $F(x, v) \neq 0$ and denote $f_1 = F(\cdot, v)$. Let I_1 be an open interval of the form $(0, \delta_1)$ such that the map $C_1 : I_1 \to M$ of class $\mathscr{C}^2$ defined by $C_1(t) = e^{-tf_1}(x)$ parameterizes an embedded curve. Since the constant control defined by v belongs to $\mathscr{U}$, then one can reach x from any point in $C_1(I_1)$. Moreover, one has that $C_1(I_1)$ is contained in V, up to choosing δ_1 sufficiently small. Thus, C_1 satisfies (5) for $k = 1$.

Now, suppose having constructed a k-dimensional parameterization C_k of class $\mathscr{C}^1$ satisfying (5), with $1 \leq k \leq n-1$. Fix a point $x_k \in C_k(I_k)$, and consider a control u_k in $\mathscr{U}$ and a time $T_k \geq 0$ such that $x_k = \phi(T_k, x, u_k)$. (We are using here that $x_k \in V \subset \mathscr{A}_x$.) Let W_k be a neighborhood of x such that $\varphi_k = \phi(T_k, \cdot, u_k)|_{W_k}$ is a $\mathscr{C}^1$-diffeomorphism. Then $S_k := \varphi_k^{-1} \circ C_k$ parameterizes an embedded submanifold of dimension k containing x. Moreover,

$$x \in \mathscr{A}_{S_k(s)}, \qquad \forall\, s \in I_k, \tag{6}$$

since $x \in \mathscr{A}_{C_k(s)}$, and $C_k(s) \in \mathscr{A}_{S_k(s)}$ using u_k as control. In particular, we have that $S_k(s) \notin \mathscr{A}_y$ for all $s \in I_k$. As a consequence, since $\phi(t, x, u) \in \mathscr{A}_y$ for all $t \in (0, T]$, we have that

$$S_k(I_k) \cap \{\phi(t, x, u) \mid t \in (0, T]\} = \emptyset. \tag{7}$$

This implies the existence of $t_k \in [0, T]$ and of $\sigma \in I_k$ with $S_k(\sigma) \in V$ such that $F(S_k(\sigma), u(t_k))$ is transverse to $T_{S_k(\sigma)} S_k(I_k)$. Indeed, if one had $F(S_k(s), u(t)) \in T_{S_k(s)} S_k(I_k)$ for all $s \in I_k$ and $t \in [0, T]$, then, by uniqueness of solutions of Cauchy problems with C^1 vector fields, $\phi(t, x, u)$ would stay in $S_k(I_k)$, at least for t sufficiently small. However, this contradicts (7). Moreover, σ

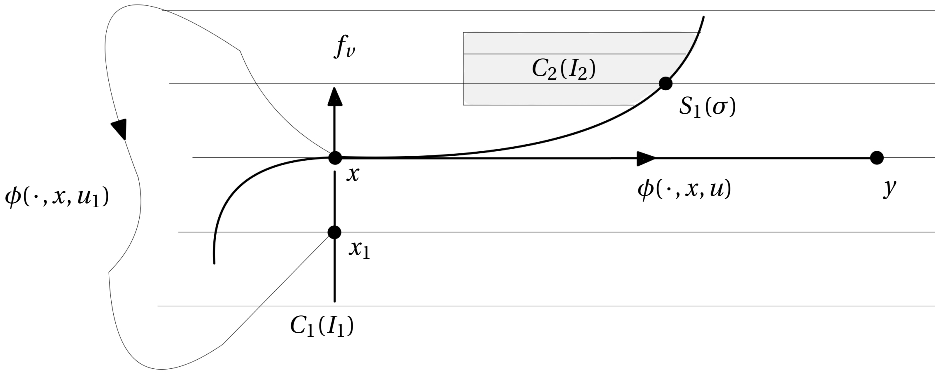


Figure 4. A graphic representation of the iterations in the proof of Lemma 3. One can steer any point in $C_2(I_2)$ to x by first attaining $S_1(I_1)$, then attaining $C_1(I_1)$ via the control u_1 from which one can reach the initial state x.

can be chosen so that $S_k(\sigma)$ is arbitrarily close to x, and in particular so that it belongs to V. Let f_k be the vector field $f_k = F(\cdot, u(t_k))$.

By transversality of $f_k(S_k(\sigma))$ and $T_{S_k(\sigma)}S_k(I_k)$, there exist $\delta_{k+1} > 0$ and an open neighborhood $I'_k \subset I_k$ containing σ such that the map $C_{k+1} : I'_k \times (-\delta_{k+1}, \delta_{k+1}) \to M$ defined by

$$C_{k+1}(s,t) = e^{tf_k} \circ S_k(s), \qquad \forall\ (s,t) \in I'_k \times (-\delta_{k+1}, \delta_{k+1}),$$

is a $\mathscr{C}^1$ parametrization of an embedded submanifold of dimension $k+1$. Moreover, since $C_{k+1}(\sigma,0) = S_k(\sigma) \in V$, the set $I'_k \times (-\delta_{k+1}, \delta_{k+1})$ can be chosen so that $C_{k+1}(I'_k \times (-\delta_{k+1}, \delta_{k+1})) \subset V$. We are now left to observe that $x \in \mathscr{A}_{C_{k+1}(s)}$ for all $s \in I_{k+1} := I'_k \times (-\delta_{k+1}, 0)$. In fact, starting from $C_{k+1}(s)$ one can reach $S_k(I_k)$ using the (constant) control $u(t_k)$ corresponding to f_k, and x can be reached from $S_k(I_k)$ by (6). This concludes the iteration, since $C_{k+1}|_{I_{k+1}}$ satisfies (5). □

3.4. *Conclusion*

Once Lemma 3 is proven, Theorem 1 follows from the following standard argument.

Proof of Theorem 1. Assume that (C) is locally controllable. Define the relation $\sim$ on M by saying that $x \sim y$ if and only if $x \in \mathscr{A}_y$. Thanks to Lemma 3, $\sim$ is an equivalence relation. Due to the local controllability, the equivalence classes are open. Each class is also closed, since its complement is the union of the other classes, and such an union is open. Due to the connectedness of M, there is only one class and system (C) is controllable. □

Appendix A. A simpler proof when L-local controllability holds.

In this appendix we give a simpler proof of the controllability of L-locally controllable systems. We first observe the following.

Proposition 7. *If system* (C) *is L-locally controllable, then for any point $x \in M$ the set $\mathscr{A}_x^-$ has nonempty interior.*

A proof of this proposition can be found below. However, observe that Proposition 7 can be directly deduced from [6, Theorem 5.3], since the property of localized local controllability implies, in the terminology of [6], that (C) has the nontangency property.

Proof. The argument mimics the proof of Krener's theorem [10]. Fix x in M. We claim that there exists $f_1 \in \mathscr{F}$ such that $f_1(x) \neq 0$. Indeed, if that were not the case, any solution $\phi(\cdot, x, u)$ with $u \in \mathscr{U}$ would be constant. Let

$$N_1 = \{e^{-tf_1}(x) \mid t \in (0,\delta)\}$$

for $\delta > 0$. If M is one-dimensional, then we have concluded. Otherwise, we claim that there exist $y_1 \in N_1$ and $f_2 \in \mathscr{F}$ such that $f_1(y_1)$ and $f_2(y_1)$ are linearly independent. Indeed, let V_1 be a neighborhood of $e^{-\frac{\delta}{2}f_1}(x)$ not containing x nor $e^{-\delta f_1}(x)$ and assume that every $f \in \mathscr{F}$ is tangent to $N_1 \cap V_1$. Then the trajectories of (C) starting from $N_1 \cap V_1$ and staying in V_1 cannot quit $N_1 \cap V_1$. This contradicts the localized local controllability property.

Thus, define the embedded two-dimensional submanifold of class $\mathscr{C}^1$

$$N_2 = \{e^{-t_2 f_2} \circ e^{-t_1 f_1}(x) \mid (t_1, t_2) \in I_2 \times (0, \delta_2)\},$$

for a suitable nonempty open subinterval I_2 of $(0,\delta)$ and a suitable $\delta_2 > 0$. If the dimension of M is equal to 2 the proof is concluded, otherwise, reasoning as above, there exist $y_2 =\in N_2$ and $f_3 \in \mathscr{F}$ such that $f_3(y_2)$ is transverse to N_2, i.e.,

$$\dim(\operatorname{span}\{f_3(y_2)\} + T_{y_2}N_2) = \dim(N_2) + 1 = 3.$$

Hence, the differential of the map $(t_1, t_2, t_3) \mapsto e^{-t_3 f_3} \circ e^{-t_2 f_2} \circ e^{-t_1 f_1}(x)$ has full rank in a neighborhood of $(\bar{t}_1, \bar{t}_2, 0)$, where $\bar{t}_1, \bar{t}_2$ are such that $y_2 = e^{-\bar{t}_2 f_2} \circ e^{-t \bar{f}_1}(x)$. We can then iterate the construction up to reaching the dimension of M. □

Now due to Lemma 2, L-locally controllable systems are approximately controllable. Moreover, Proposition 7 ensures that any point x in M satisfies $\operatorname{Int}\mathscr{A}_x^- \neq \emptyset$. Thus, by Remark 6, we deduce that any x in M can be reached from any other point.

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