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Consistency of the Flat Flow Solution to the Volume Preserving Mean Curvature Flow

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Abstract

We consider the flat flow solution, obtained via a discrete minimizing movement scheme, to the volume preserving mean curvature flow starting from $C^{1,1}$-regular set. We prove the consistency principle, which states that (any) flat flow solution agrees with the classical solution as long as the latter exists. In particular the flat flow solution is unique and smooth up to the first singular time. We obtain the result by proving the full regularity for the discrete time approximation of the flat flow such that the regularity estimates are stable with respect to the time discretization. Our method can also be applied in the case of the mean curvature flow and thus it provides an alternative proof, not relying on comparison principle, for the consistency between the flat flow solution and the classical solution for $C^{1,1}$-regular initial sets.

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1. Introduction

1.1. Statement of the Main Theorem

In this paper we consider the flat flow solution to the volume preserving mean curvature flow, which is a weak notion of solution obtained via discrete minimizing movement scheme. Our main goal is to prove the full regularity of the flat flow up to the first singular time when the initial set is $C^{1,1}$-regular. As a corollary we obtain the consistency principle between the flat flow and the classical solution.

Let us begin by recalling that a smooth family of sets $(E_t)_{t\in[0,T)} \subset \mathbb{R}^{n+1}$, for some $T > 0$, is a solution to the volume preserving mean curvature flow if it satisfies

$$V_t = -(H_{E_t} - \bar{H}_{E_t}), \tag{1.1}$$

where V_t denotes the normal velocity, H_{E_t} the mean curvature and $\bar{H}_{E_t} := \fint_{\partial E_t} H_{E_t} \, d\mathcal{H}^n$ the integral average of the mean curvature of the evolving boundary ∂E_t. An important feature is that (1.1) can be seen as a L^2-gradient flow of the surface area. Since it also preserves the volume, it can be regarded as the evolutionary counterpart to the isoperimetric problem.

If the initial set E_0 is regular enough, e.g. it satisfies interior and exterior ball conditions, the equation (1.1) has a unique smooth solution for a short interval of time [19]. The classical result by Huisken [28] states that for convex initial sets the classical solution exists for all times and converges exponentially fast to a sphere. Similarly, it follows from [19,44] that if the initial set is close to a local minimum of the isoperimetric problem, the equation (1.1) does not develop singularities and convergences exponentially fast. However, for generic initial sets the equation (1.1) may develop singularities in finite time [40,41]. In fact, unlike the standard mean curvature flow, (1.1) may develop singularities even in the plane and the boundary may also collapse such that the curvature of the evolving boundary stays uniformly bounded up to the singular time. It is therefore natural to find a proper notion of weak solution for (1.1) which is defined for all times even if the flow develops singularities. The crucial difference between (1.1) and the mean curvature flow is that the former is nonlocal and does not satisfy the comparison priciple. Therefore we cannot directly use the notion of viscosity solution to define the level-set solution via the methods introduced by Chen-Giga-Goto [15] and Evans-Spruck [20], although in [33] Kim-Kwon are able to find a viscosity solution for (1.1) for star-shaped sets. Instead, we may use the gradient flow structure to obtain a weak solution called *flat flow* via discrete minimizing movement scheme as first introduced by Almgren-Taylor-Wang [3] and Luckhaus-Stürzenhecker [36] for the mean curvature flow, and then implemented to the volume preserving setting (1.1) by Mugnai-Seis-Spadaro [43]. We give the precise definition in Section 3. The existence of the flat flow solution of (1.1) is proven in [43] and the recent results [16,23,31,32,42] indicate that it has the expected asymptotic behavior. Indeed, it is proven in [31] that in the plane any flat flow solution of (1.1), starting from any set of finite perimeter, converges exponentially fast to a union of equisize disks.

One of the main issues with the flat flow solution is that it has a priori very low regularity. The second issue is that it is not clear if the procedure provides a solution to the equation (1.1) in some weak sense. The first issue is related to the regularity and the second one is the problem of consistency, and it is rather clear that these are closely related to each other. Indeed, the flat flow is obtained as a limit of a discrete minimizing scheme, in the spirit of the Euler implicit method, where the time disretization is led to zero. If the flow remains smooth enough, as the time discretization goes to zero, then one can show that the limiting flat flow provides a solution to the equation (1.1). However, the only case when this seems to be known is the case when the initial set is convex. In this case the construction in [8], which however is slightly different than [43], provides a flow of sets which remains convex and thus gives a solution to (1.1). One may also define a distributional solution to (1.1) (see [43]) and in a recent work Laux [34] proves that this notion of solution, and in fact any gradient-flow calibration, agrees with the classical solution as long as the latter exists (see also [26]).

The issue with regularity and consistency is better understood in the case of the standard mean curvature flow. It is proven in [3] that the flat flow for the mean curvature equation agrees with the classical solution as long as the latter exists. If we are in a situation where the level-set solution is unique, i.e., it does not develop fattening, then due to the result by Chambolle [12] we know that the flat flow coincides with the level-set solution, see also [13,14]. We may then use the result in [21] to conclude that the flat flow is a 'subsolution' to the mean curvature flow in the sense of Brakke and has the partial regularity proven in [9]. Thus we have the consistency and partial regularity for the mean curvature flow when the flow does not develop fattening. In addition, due to the recent result by DePhilippis-Laux [17] together with the classical result in [36], we know that the flat flow is a distributional solution to the mean curvature flow equation when the initial set is mean convex.

As we mentioned above, here we study the regularity of the flat flow solution of (1.1) when the initial set is $C^{1,1}$-regular, which is the same as to say that the set satisfies interior and exterior ball conditions. Throughout the paper we will say that an open set $E \subset \mathbb{R}^{n+1}$ satisfies *uniform ball condition* (which we refer as UBC) with radius $r > 0$ if it satisfies interior and exterior ball condition with radius $r > 0$. If we do not want to emphasize the radius r, we simply say that E satisfies UBC. Our main theorem reads as follows:

Theorem 1.1. *Assume that $E_0 \subset \mathbb{R}^{n+1}$ is an open and bounded set which satisfies UBC with radius r_0. There is time $T_0 > 0$, which depends on r_0 and n, such that any flat flow solution $(E_t)_{t\geq 0}$ of (1.1) starting from E_0 satisfies UBC with radius $r_0/2$ for all $t \leq T_0$. This condition is open in the sense that if $(E_t)_{t\geq 0}$ satisfies UBC with radius r for all $t \leq T$, then there is $\delta > 0$ such that it satisfies UBC with radius $r/2$ for all $t < T + \delta$.*

Moreover, the flat flow $(E_t)_{t\geq 0}$ becomes instantaneously smooth and remains smooth as long as it satisfies UBC. To be more precise, if $(E_t)_{t\geq 0}$ satisfies UBC with radius r for all $t \leq T$, then for every $k \in \mathbb{N}$ it holds that

$$\sup_{t\in(0,T]} \left(t^k \|H_{E_t}\|^2_{H^k(\partial E_t)}\right) \leq C_k, \tag{1.2}$$

where C_k depends on T, n, k, r and $|E_0|$.

In fact, we obtain even stronger result since we prove UBC and the estimate (1.2) directly for the discrete approximative flat flow $(E_t^h)_{t\geq 0}$ such that the estimates hold for all $h \leq h_0$ for constants independent of h. However, we choose to state the regularity result only for the limiting flow since the precise statement, which can be found in Theorem 4.7 and Theorem 5.2, is rather technical. The first part of the theorem is related to the result by Swartz-Yip [47], where the authors prove curvature bounds for the Merriman-Bence-Osher thresholding algorithm for the mean curvature flow.

It is well-known that we have uniqueness among smooth solutions of (1.1). Therefore an important consequence of Theorem 1.1 is the consistency between the notion of flat flow solution and the classical solution of (1.1) when the initial set is $C^{1,1}$-regular.

Corollary 1.2. *Assume that $E_0 \subset \mathbb{R}^{n+1}$ is an open and bounded set which satisfies UBC. Let $(\hat{E}_t)_{t\in[0,T)} \subset \mathbb{R}^{n+1}$ be the classical solution of (1.1) starting from E_0, where $T > 0$ is the maximal time of existence, and let $(E_t)_{t\geq 0} \subset \mathbb{R}^{n+1}$ be a flat flow solution of (1.1) starting from E_0. Then*

$$\hat{E}_t = E_t \qquad \text{for all } t \in [0, T).$$

Let us next briefly comment on the regularity estimate (1.2). The first part of Theorem 1.1 (see Theorem 4.7 in Section 4) provides a bound on UBC for a short time $[0, T_0]$ and the proof of Theorem 4.7 also provides an estimate how the curvature grows in time for the approximative flat flow $(E_t^h)_{t\geq 0}$. However, without higher order regularity bounds we are not able to pass these growth-estimates to the limit as $h \to 0$. Therefore the results of Section 4 only imply the consistency for a short time interval $[0, T_0]$ (see the discussion at the end of Sect. 5). Our main motivation to prove (1.2) is to pass the previously mentioned curvature estimates to the limit as $h \to 0$ by Ascoli-Arzela theorem, and deduce that UBC is, in fact, an open condition and therefore the flat flow agrees with the classical solution over the whole maximal time of existence. Of course, in addition to that, (1.2) quantifies the smoothing effect of the equation in a sharp way.

1.2. An Overview of the Proof

The proof of Theorem 1.1 is divided in three sections and therefore we give here a short overview. We recall that in the minimizing movements scheme, for a fixed time discretization step $h > 0$, we obtain a sequence of sets E_k^h such that $E_0^h = E_0$ is the initial set and E_{k+1}^h is defined inductively as a minimizer of the functional

$$\mathcal{F}_h(E, E_k^h) = P(E) + \frac{1}{h}\int_E d_{E_k^h}\, dx + \frac{1}{\sqrt{h}}\big||E| - m_0\big|,$$

where $d_{E_k^h}$ denotes the signed distance function and $m_0 = |E_0|$. A flat flow is then defined as any cluster point of the discrete flow as $h \to 0$. We first prove in Proposition 3.1 via energy comparison argument, that if E_k^h satisfies UBC with radius r_0 then the subsequent set E_{k+1}^h satisfies the distance estimate

$$|d_{E_k^h}| \leq \frac{C}{r_0} h \qquad \text{on } \partial E_{k+1}^h.$$

The above estimate is crucial as it implies that the speed of the discrete flow is sublinear. It also implies a bound for the mean curvature and the regularity of E_{k+1}^h by applying the Allard's regularity theory [2]. The most crucial part of the proof of the main theorem is then to show that the subsequent set E_{k+1}^h also satisfies UBC with a quantified radius.

We solve this problem by adopting the two-point function method due to Huisken [27] to the discrete setting (see also the works by Andrews [4] and Brendle [10] for an overview of the topic). The idea is to double the variables and to study the maximum and minimum values of the function

$$S_{E_k^h}(x, y) = \frac{(x - y) \cdot \nu(x)}{|x - y|^2}$$

for $x \neq y \in \partial E_k^h$. The point is that the extremal values of $S_{E_k^h}$ are related to the maximal UBC radius of the set E_k^h (see Lemma 4.1). We use the maximum principle to prove the following familiar inequality (see Lemma 4.6):

$$\frac{\|S_{E_{k+1}^h}\|_{L^\infty} - \|S_{E_k^h}\|_{L^\infty}}{h} \leq C\|S_{E_k^h}\|_{L^\infty}^3.$$

By iterating the above estimate, we obtain that the sets E_k^h satisfy UBC for all $k \leq T_0 h^{-1}$, where the constant T_0 is related to the UBC of the initial set. This implies the first part of Theorem 1.1 (see Theorem 4.7). An important technical part in this argument is the discrete version of the formula for $\frac{d}{dt}\nu_{E_t}$ which we derive in Lemma 4.4.

The formula in Lemma 4.4 is, in fact, so simple that we are able to differentiate it multiple times and obtain in Proposition 5.1 a discrete analog for the formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta^k H_{E_t} = \Delta^{k+1} H_{E_t} + \text{lower order terms}, \tag{1.3}$$

where Δ denotes the Laplace-Beltrami operator (see e.g. [38]). The lower order terms are due to the nonlinearity of the equation (1.1) and we need the notation and tools from differential geometry in order to control them. We stress that this is the only part in the paper where we need to introduce higher order covariant derivatives. After we have obtained the discrete version of the formula (1.3) and bounded the lower order error terms, we may adopt the argument from [22] to the discrete setting and obtain the full regularity of the flow. Finally, we point out that the argument can be adopted to the case of the mean curvature flow essentially without any modifications.

2. Notation and Preliminary Results

Throughout this paper, $C_n \in \mathbb{R}_+$ stands for a generic dimensional constant which may change from line to line. We denote the open ball with radius r centered at x by $B_r(x) \subset \mathbb{R}^{n+1}$ and by B_r if it is centered at the origin. We denote by $\mathbf{C}(x, r, R) \subset \mathbb{R}^{n+1}$ the open cylinder

$$\mathbf{C}(x, r, R) := B_r^n(x') \times (-R + x_{n+1}, R + x_{n+1}),$$

where $B_r^n \subset \mathbb{R}^n$ denotes the n-dimensional ball and $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$. For a given set $E \subset \mathbb{R}^{n+1}$ and a radius $r \in \mathbb{R}_+$ we set its r-enlargement $\mathcal{N}_r(E) = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, E) < r\}$. Note that we may alternatively write this as the Minkowski sum $E + B_r$. The notation $\nabla^k F$ stands for k:th order differential of a vector field $F : \mathbb{R}^{n+1} \to \mathbb{R}^m$. For a matrix $\mathcal{A} \in \mathbb{R}^k \otimes \mathbb{R}^k$ we denote by $|\mathcal{A}|$ its Frobenius norm $\sqrt{\operatorname{Tr}(\mathcal{A}^{\mathrm{T}}\mathcal{A})}$ and by $|\mathcal{A}|_{\mathrm{op}}$ its operator norm $\max\{|\mathcal{A}\xi| : \xi \in \mathbb{R}^k, |\xi| = 1\}$.

If a set $S \subset \mathbb{R}^k$ is Lebesgue-measurable, we denote its k-dimensional Lebesgue measure (or volume) by $|S|$. Given a non-empty set $E \subset \mathbb{R}^{n+1}$ we denote the distance function by $\operatorname{dist}_E(x) := \inf_{y\in E} |x - y|$ and the *signed distance function* by $d_E : \mathbb{R}^{n+1} \to \mathbb{R}$, which is defined as

$$d_E(x) := \begin{cases} \operatorname{dist}_E(x), & \text{for } x \in \mathbb{R}^{n+1} \setminus E \\ -\operatorname{dist}_{\mathbb{R}^n\setminus E}(x), & \text{for } x \in E. \end{cases} \tag{2.1}$$

Then clearly it holds that $\operatorname{dist}_{\partial E} = |d_E|$. If for a given point $x \in \mathbb{R}^{n+1}$ there is a unique distance minimizer y_x on ∂E (that is $|x - y_x| = \operatorname{dist}_{\partial E}(x)$), we denote y_x by $\pi_{\partial E}(x)$ and call it the projection of x onto ∂E. For a set of finite perimeter $E \subset \mathbb{R}^{n+1}$ we denote its reduced boundary by $\partial^* E$. Then $P(E; F) = \mathcal{H}^n(\partial^* E \cap F)$ for every Borel set $F \subset \mathbb{R}^{n+1}$ and $P(E) = \mathcal{H}^n(\partial^* E)$.

2.1. Regular Sets and Tangential Differentiation

We will mostly deal with regular and bounded sets $E \subset \mathbb{R}^{n+1}$. As usual, a bounded set $E \subset \mathbb{R}^{n+1}$ is said to be $C^{k,\alpha}$-regular, with $k \geq 1$ and $0 \leq \alpha \leq 1$, if for every $x \in \partial E$ we find a cylinder $\mathbf{C}(x, r, R)$ and a function $f \in C^{k,\alpha}(B_r^n(x'))$ with $|f - x_{n+1}| < R$ such that, up to rotating the coordinates, we may write

$$\operatorname{int}(E) \cap \mathbf{C}(x, r, R) = \{y \in \mathbf{C}(x, r, R) : y_{n+1} < f(y')\}.$$

In particular, ∂E is a compact and embedded $C^{k,\alpha}$-hypersurface. Again, if $\alpha = 0$, we say that E is C^k-regular and if $k = \infty$, we say that E is smooth. If r and R are independent of the choice of x and the $C^{k,\alpha}$-norm of g has a bound, also independent of x, then we say that E is uniformly $C^{k,\alpha}$-regular. We denote the outer unit normal by ν_E, or simply ν if the meaning is clear from the context. Note that $\nu_E \in C^{k-1,\alpha}(\partial E; \partial B_1)$. We always assume that the orientation of ∂E is induced by ν_E. We define the matrix field $P_{\partial E} : \partial E \to \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$ by setting $P_{\partial E} = I - \nu_E \otimes \nu_E$. For a given point $x \in \partial E$ the map $P_{\partial E}(x)$ is the orthogonal projection onto the *geometric tangent plane* $G_x\partial E := \langle \nu_E(x)\rangle^{\perp}$.

For given a vector field $F \in C^l(\mathbb{R}^{n+1}; \mathbb{R}^m)$ with $1 \leq l \leq k$ we define its *tangential differential* along $\Sigma = \partial E$ as a matrix field $\nabla_{\tau_E} F : \partial E \to \mathbb{R}^m \otimes \mathbb{R}^{n+1}$ by setting

$$\nabla_{\tau_E} F = \nabla F P_{\partial E} = \nabla F - (\nabla F \nu_E) \otimes \nu_E. \tag{2.2}$$

When the meaning is clear from the context, we abbreviate E from the notation and write simply $\nabla_\tau F$. In the case $m = n + 1$, the *tangential divergence* of F is defined as $\operatorname{div}_\tau F = \operatorname{Tr}(\nabla_\tau F)$ and the *tangential Jacobian* $J_\tau F$ of F is defined on ∂E as

$$J_\tau F = \sqrt{\det\big((\nabla_\tau F \circ \iota_\tau)^T (\nabla_\tau F \circ \iota_\tau)\big)}, \tag{2.3}$$

where $\iota_\tau(x)$ at $x \in \partial E$ is the inclusion $G_x \partial E \hookrightarrow \mathbb{R}^{n+1}$. In the case $m = 1$, the notation $\nabla_\tau F$ also stands for the *tangential gradient* $P_{\partial E} \nabla F$. Note that $\nabla_\tau F$ is C^{l-1}-regular and independent of how F is extended beyond ∂E. On the other hand, every $G \in C^l(\partial E, \mathbb{R}^m)$, with $1 \leq l \leq k$, admits a C^l-extension $F : \mathbb{R}^n \to \mathbb{R}^m$ so we may extend the concept of tangential differential to concern G simply by setting $\nabla_\tau G = \nabla_\tau F$ and further define the other introduced concepts in a similar manner.

If E is C^k-regular for $k \geq 2$, we may define its *second fundamental form*, with respect to the orientation ν_E, as a matrix field $B_E : \partial E \to \mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$ given by

$$B_E(x) = \sum_i \lambda_i(x) \kappa_i(x) \otimes \kappa_i(x),$$

where the (unit) principal directions $\kappa_1(x), \ldots, \kappa_n(x) \in \langle \nu_E(x) \rangle^\perp$ and the principal curvatures $\lambda_1(x), \ldots, \lambda_n(x)$ at $x \in \partial E$ are given by the orientation ν_E. The corresponding (scalar) mean curvature field H_E is then given pointwise as the sum of the principal curvatures, i.e., $H_E = \operatorname{Tr}(B_E)$. Note that we may simply write

$$B_E = \nabla_\tau \nu_E \quad \text{and} \quad H_E = \operatorname{div}_\tau \nu_E. \tag{2.4}$$

Finally, we define the *tangential Hessian* for given $u \in C^2(\partial E)$ as $\nabla_\tau^2 u = \nabla_\tau(\nabla_\tau u)$ and further the *tangential Laplacian* or the *Laplace-Beltrami* of u as

$$\Delta_\tau u = \operatorname{div}_\tau(\nabla_\tau u) = \operatorname{Tr}(\nabla_\tau^2 u).$$

The tangential Laplacian $\Delta_\tau F$ for $F \in C^2(\partial E; \mathbb{R}^{n+1})$ is defined as $\sum_i \Delta_\tau(F \cdot e_i) e_i$. We will need the following identities on ∂E:

$$\Delta_\tau \mathrm{id} = -H_E \nu_E \quad \text{and} \quad \Delta_\tau \nu_E = -|B_E|^2 \nu_E + \nabla_\tau H_E \ \text{ if } E \text{ is } C^3\text{−regular}. \tag{2.5}$$

The importance of the mean curvature H_E lies in the surface divergence theorem which states that for every $G \in C^1(\partial E; \mathbb{R}^{n+1})$ it holds that

$$\int_{\partial E} \operatorname{div}_\tau G \, d\mathcal{H}^n = \int_{\partial E} H_E (G \cdot \nu_E) \, d\mathcal{H}^n. \tag{2.6}$$

The concept of mean curvature can be generalized to the setting of bounded sets of finite perimeter in the varifold sense. Indeed, for a set of finite perimeter $E \subset \mathbb{R}^{n+1}$, we may define the tangential divergence $\operatorname{div}_\tau F$ of $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ along $\partial^* E$ in the same way as in the regular case by replacing the outer unit normal field with the measure theoretic normal field $\partial^* E \to \partial B_1$ which we also denote by ν_E. Then, if E is a bounded set of finite perimeter and there is $g \in L^1(\partial^* E, \mathcal{H}^n|_{\partial^* E})$ such that

$$\int_{\partial^* E} \operatorname{div}_\tau F \, \mathrm{d}\mathcal{H}^n = \int_{\partial^* E} g(F \cdot \nu_E) \, \mathrm{d}\mathcal{H}^n \tag{2.7}$$

for every $F \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, we say that g is a *generalized mean curvature* of E and denote it by H_E. As mentioned, this is a concept from the context of varifold theory for which we refer to [46] as a standard introduction. Since $\partial^* E$ is $\mathcal{H}^n$-rectifiable set, one may treat the pair $(\partial^* E, \mathcal{H}^n|_{\partial^* E})$ as an rectifiable integral varifold of multiplicity one.

2.2. Riemannian Geometry

We need the notation related to Riemannian geometry and as an introduction to the topic we refer to [35]. Let us assume that $E \subset \mathbb{R}^{n+1}$ is a smooth and bounded set and denote $\Sigma = \partial E$. Since Σ is embedded in $\mathbb{R}^{n+1}$ it has natural metric g induced by the Euclidean metric. Then (Σ, g) is a Riemannian manifold and we denote the inner product on each tangent space $X, Y \in T_x \Sigma$ by $\langle X, Y \rangle$, which we may write in local coordinates as

$$\langle X, Y \rangle = g(X, Y) = g_{ij} X^i Y^j.$$

We extend the inner product in a natural way for tensors. Note that $x \cdot y$ denotes the inner product of two vectors in $\mathbb{R}^{n+1}$. We denote smooth vector fields on Σ by $\mathscr{T}(\Sigma)$ and by a slight abuse of notation we denote smooth k:th order tensor fields on Σ by $\mathscr{T}^k(\Sigma)$. We write X^i for vectors and Z_i for covectors in local coordinates. We denote the Riemannian connection on Σ by $\tilde{\nabla}$ and recall that for a function $u \in C^\infty(\Sigma)$ the covariant derivative $\tilde{\nabla} u$ is a 1-tensor field defined for $X \in \mathscr{T}(\Sigma)$ as

$$\tilde{\nabla} u(X) = \tilde{\nabla}_X u = Xu,$$

i.e., the derivative of u in the direction of X. The covariant derivative of a smooth k-tensor field $F \in \mathscr{T}^k(\Sigma)$, denoted by $\tilde{\nabla} F$, is a $(k+1)$-tensor field and for $Y_1, \dots, Y_k, X \in \mathscr{T}(\Sigma)$ we have the recursive formula

$$\tilde{\nabla} F(Y_1, \dots, Y_k, X) = (\tilde{\nabla}_X F)(Y_1, \dots, Y_k), \tag{2.8}$$

where

$$(\tilde{\nabla}_X F)(Y_1, \dots, Y_k) = XF(Y_1, \dots, Y_k) - \sum_{i=1}^{k} F(Y_1, \dots, \tilde{\nabla}_X Y_i, \dots, Y_k).$$

Here $\tilde{\nabla}_X Y$ is the covariant derivative of Y in the direction of X (see [35]) and since $\tilde{\nabla}$ is the Riemannian connection it holds that $\tilde{\nabla}_X Y = \tilde{\nabla}_Y X + [X, Y]$ for every $X, Y \in \mathscr{T}(\Sigma)$. We denote the k:th order covariant derivative of a function u on Σ by $\tilde{\nabla}^k u \in \mathscr{T}^k(\Sigma)$ and the Laplace-Beltrami operator by Δ. Note that for functions it holds that $\Delta u = \Delta_\tau u$. The notation $\tilde{\nabla}_{i_k} \cdots \tilde{\nabla}_{i_1} u$ means a coefficient of $\tilde{\nabla}^k u$ in local coordinates. We may raise the index of $\tilde{\nabla}_i u$ by using the inverse of the metric tensor g^{ij} as $\tilde{\nabla}^i u = g^{ij}\tilde{\nabla}_j u$. We note that the tangential gradient of $u : \Sigma \to \mathbb{R}$ is equivalent to its covariant derivative in the sense that for every vector field $X \in \mathscr{T}(\Sigma)$ we find a unique vector field $\tilde{X} : \Sigma \to \mathbb{R}^{n+1}$ which satisfies $\tilde{X} \cdot \nu_E = 0$ and

$$\tilde{\nabla}_X u = \nabla_\tau u \cdot \tilde{X}.$$

Similarly it holds that $\tilde{\nabla}^2 u(X, Y) = \nabla_\tau^2 u \tilde{X} \cdot \tilde{Y}$. Finally we recall that the notation ∇^k always stands for the standard Euclidean k:th order differential for an ambient function.

We define the Riemann curvature tensor $R \in \mathscr{T}^4(\Sigma)$ [35,39] via interchange of covariant derivatives of a vector field Y^i and a covector field Z_i as

$$\begin{aligned} \tilde{\nabla}_i\tilde{\nabla}_j Y^s - \tilde{\nabla}_j\tilde{\nabla}_i Y^s &= R_{ijkl}g^{ks}Y^l, \\ \tilde{\nabla}_i\tilde{\nabla}_j Z_k - \tilde{\nabla}_j\tilde{\nabla}_i Z_k &= R_{ijkl}g^{ls}Z_s, \end{aligned} \tag{2.9}$$

where we have used the Einstein summation convention. We may write the Riemann tensor in local coordinates by using the second fundamental form B, which in the Riemannian setting is understood to be 2-form, as

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk}. \tag{2.10}$$

We will also need Simon's identity, which reads as

$$\Delta B_{ij} = \tilde{\nabla}_i\tilde{\nabla}_j H + HB_{il}g^{ls}B_{sj} - |B|^2 B_{ij}. \tag{2.11}$$

Let us next fix our notation for the function spaces. We define the Sobolev space $W^{l,p}(\Sigma)$ in a standard way for $p \in [1, \infty]$, see e.g. [6], denote the Hilbert space $H^l(\Sigma) = W^{l,2}(\Sigma)$ and define the associated norm for $u \in W^{l,p}(\Sigma)$ as

$$\|u\|^p_{W^{l,p}(\Sigma)} = \sum_{k=0}^{l}\int_\Sigma |\tilde{\nabla}^k u|^p \, d\mathcal{H}^n,$$

and, for $p = \infty$,

$$\|u\|_{W^{l,\infty}(\Sigma)} = \sum_{k=0}^{l}\sup_{x\in\Sigma}|\tilde{\nabla}^k u|.$$

The above definition extends naturally for tensor fields. We adopt the convention that $\|u\|_{H^0(\Sigma)} = \|u\|_{L^2(\Sigma)}$ and denote $\|u\|_{C^m(\Sigma)} = \|u\|_{W^{m,\infty}(\Sigma)}$. We remark that we may define the k:th order covariant derivative of a function $u \in C^k(\Sigma)$ and the space $W^{k,p}(\Sigma)$ for $k \geq 2$ as above assuming only that Σ (i.e. the set E for which $\Sigma = \partial E$) is C^k-regular.

Finally we adopt the notation $S \star T$ from [25,38] to denote a tensor formed by contracting some indexes of tensors S and T using the coefficients of the metric tensor g_{ij}. This notation is useful as it implies

$$|S \star T| \leq C|S||T|,$$

where the constant C depends on the 'structure' of $S \star T$.

2.3. Functional and Geometric Inequalities

We will need standard interpolation inequalities on smooth hypersurfaces. Since we will apply them on the moving boundary given by the flow, we need to control the constants in the inequalities. We begin with a simple interpolation on Hölder norms.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^k$ be an open set and let $u \in C^1(\Omega)$, then for every $\alpha \in (0, 1)$*

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq 3\|u\|_{L^\infty(\Omega)}^{1-\alpha}\|u\|_{C^1(\Omega)}^{\alpha}.$$

Proof. The inequality follows from

$$\frac{|u(y)-u(x)|}{|y-x|^\alpha} \leq |u(y)-u(x)|^{1-\alpha}\left(\frac{|u(y)-u(x)|}{|y-x|}\right)^\alpha \leq 2\|u\|_{L^\infty(\Omega)}^{1-\alpha}\|u\|_{C^1(\Omega)}^{\alpha}.$$

□

We continue to introduce functional and geometric inequalities that we need in order to prove the higher order regularity estimates stated at the end of Theorem 1.1. As we already mentioned we do not need any deep results from differential geometry in order to prove the estimate for UBC stated in the beginning of Theorem 1.1. It is only when we deal with higher order derivatives, i.e., higher than two, we need the notation of covariant derivatives. Recall that we always assume that $\Sigma = \partial E$ for a bounded set $E \subset \mathbb{R}^{n+1}$.

Let us first recall the interpolation inequality with Sobolev-norms on embedded surfaces. We use the result from [38, Proposition 6.5] which states that under curvature bound the standard interpolation inequality holds for a uniform constant.

Proposition 2.2. *Assume $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C_0$ and Σ is C^m-regular for $m \geq 2$. Then for integers $0 \leq k \leq l \leq m$ and numbers $p, q, r \in [1, \infty)$, there is $\theta \in [k/l, 1]$ such that for every C^l-regular covariant tensor field T on Σ it holds*

$$\|\tilde{\nabla}^k T\|_{L^p(\Sigma)} \leq C\|T\|_{W^{l,q}(\Sigma)}^{\theta}\|T\|_{L^r(\Sigma)}^{1-\theta}$$

for a constant $C = C(k, l, n, p, q, r, \theta, C_0) \in \mathbb{R}_+$ provided that the following compatibility condition is satisfied

$$\frac{1}{p} = \frac{k}{n} + \theta\left(\frac{1}{q} - \frac{l}{n}\right) + \frac{1}{r}(1-\theta).$$

We denote an index vector by $\alpha \in \mathbb{N}^k$, i.e., $\alpha = (\alpha_1, \ldots, \alpha_k)$ where $\alpha_i \in \mathbb{N}$, and define its norm by

$$|\alpha| = \sum_{i=1}^k \alpha_i.$$

The following inequality is well-known but we prove it for the reader's convenience:

Proposition 2.3. *Assume* $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C$ *and* Σ *is* C^m*-regular for* $m \geq 2$. *Assume* $u_1, \ldots, u_l$ *are* C^m*-regular functions such that* $\|u_i\|_{L^\infty} \leq C$. *Then for an index vector* $\alpha \in \mathbb{N}^l$ *with* $|\alpha| \leq k \leq m$ *and* $p \in (1, \infty)$ *it holds that*

$$\||\tilde{\nabla}^{\alpha_1} u_1| \cdots |\tilde{\nabla}^{\alpha_l} u_l|\|_{L^p(\Sigma)} \leq C_k \sum_{i=1}^{k} \|u_i\|_{W^{k,p}(\Sigma)}.$$

Proof. Without loss of generality we may assume that $|\alpha| = k$. We first use Hölder's inequality to get that

$$\||\tilde{\nabla}^{\alpha_1} u_1| \cdots |\tilde{\nabla}^{\alpha_l} u_l|\|_{L^p(\Sigma)} \leq \|\tilde{\nabla}^{\alpha_1} u_1\|_{L^{\frac{pk}{\alpha_1}}} \cdots \|\tilde{\nabla}^{\alpha_l} u_l\|_{L^{\frac{pk}{\alpha_l}}}.$$

By the interpolation inequality in Proposition 2.2 and by $\|u_i\|_{L^\infty} \leq C$ it holds that

$$\|\tilde{\nabla}^{\alpha_i} u_1\|_{L^{\frac{pk}{\alpha_i}}} \leq C \|u_i\|_{W^{k,p}}^{\frac{\alpha_i}{k}} \|u_i\|_{L^\infty}^{1-\frac{\alpha_i}{k}} \leq C \|u_i\|_{W^{k,p}}^{\frac{\alpha_i}{k}}.$$

Hence we have

$$\||\tilde{\nabla}^{\alpha_1} u_1| \cdots |\tilde{\nabla}^{\alpha_l} u_l|\|_{L^p(\Sigma)} \leq C_k \|u_1\|_{W^{k,p}}^{\frac{\alpha_1}{k}} \cdots \|u_l\|_{W^{k,p}}^{\frac{\alpha_l}{k}}.$$

Since $\alpha_1 + \cdots + \alpha_l = |\alpha| = k$, the claim follows, from Young's inequality. □

If $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is a regular function then its restriction on Σ is also regular. In the next lemma we bound the covariant derivatives of u on Σ with the Euclidean ones. The statement of the lemma is not optimal but it is sharp enough for our purpose. In the proof we will repeatedly use the fact that the k:th order derivative of the composition $f \circ h$ and the product $f \cdot g$ of functions $f, g : \mathbb{R}^m \to \mathbb{R}^k$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$\begin{aligned} \nabla^k (f \circ h) &= \sum_{|\alpha| \leq k-1} \nabla^{1+\alpha_1} h \star \cdots \star \nabla^{1+\alpha_k} h \star \nabla^{1+\alpha_{k+1}} f \\ \nabla^k (f \cdot g) &= \sum_{i+j=k} \nabla^i f \star \nabla^j g. \end{aligned} \tag{2.12}$$

Lemma 2.4. *Assume* Σ *is* C^{k+2}*-regular and* $u \in C^{k+1}(\mathbb{R}^{n+1})$. *Then it holds for all* $x \in \Sigma$ *that*

$$|\tilde{\nabla}^{k+1} u(x)| \leq C_k \sum_{|\alpha| \leq k} \left(1 + |\tilde{\nabla}^{\alpha_1} B_E(x)| \cdots |\tilde{\nabla}^{\alpha_k} B_E(x)|\right) |\nabla^{1+\alpha_{k+1}} u(x)|.$$

Recall that $\tilde{\nabla}^k$ *denotes the k:th order covariant derivative on* Σ *while* ∇^k *is the k:th order Euclidean derivative.*

Proof. The proof follows from basic theory of differential geometry and we merely sketch it. Let us fix $x \in \Sigma$ and choose the coordinates such that $x = 0$ and $\nu_E(0) = e_{n+1}$. Since Σ is C^{k+2}-regular hypersurface we may write it locally as a graph of $f \in C^{k+2}(\mathbb{R}^n)$, i.e., $\Sigma \cap B_r(0) \subset \{(x, f(x)) : x \in \mathbb{R}^n\}$. Note that since $\nu_E(0) = e_{n+1}$ then $\nabla_{\mathbb{R}^n} f(0) = 0$.

We consider the graph coordinates $\Phi^{-1} : B_r^n \to \Phi^{-1}(B_r^n) \subset \Sigma$, $\Phi^{-1}(x) = (x, f(x))$. We denote the points on $\mathbb{R}^n$ by x, the points on Σ by p, $\Phi(p) = \big(x^1(p), \dots, x^n(p)\big)$ and $U = \Phi^{-1}(B_r^n)$. Then the chart $\big(U, (x^i)\big)$ determines coordinate vector fields which we denote by $\frac{\partial}{\partial x^i}\Big|_p$ and recall that they act on smooth functions $v : U \to \mathbb{R}$ at $p = \Phi(x)$ as

$$\frac{\partial}{\partial x^i}\Big|_p v = \tilde{\nabla} v\left(\frac{\partial}{\partial x^i}\right)(p) = \partial_i (v \circ \Phi^{-1})(x),$$

where ∂_i denotes the standard partial derivative in $\mathbb{R}^n$. It holds for the metric tensor and for the Christoffel symbol Γ_{jk}^i (see [35]) for $x \in B_r^n$ that

$$g_{ij}(x) = \delta_{ij} + \partial_i f(x)\partial_j f(x) \quad \text{and} \quad \Gamma_{jk}^i(x) = g^{il}(x)\, \partial_{jk}^2 f(x)\partial_l f(x).$$

Moreover by the recursive formula (2.8) we may write the $(k+1)$:th order covariant derivative of u iteratively (see [35, Lemma 4.8]) as

$$\begin{aligned}\tilde{\nabla}^{k+1} u\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}, \frac{\partial}{\partial x^j}\right) &= \partial_j\left(\tilde{\nabla}^k u\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right)\right) \\ &- \sum_{m=1}^k \tilde{\nabla}^k u\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^l}, \dots, \frac{\partial}{\partial x^{i_k}}\right)\Gamma_{j i_m}^l.\end{aligned} \tag{2.13}$$

Recall that $\tilde{\nabla} u\left(\frac{\partial}{\partial x^i}\right)(p) = \frac{\partial}{\partial x^i}\Big|_p u$.

Using (2.12) we have

$$|\nabla_{\mathbb{R}^n}^{k+1}(u \circ \Phi^{-1})(0)| \leq C_k \sum_{|\alpha| \leq k} \big(1 + |\nabla_{\mathbb{R}^n}^{1+\alpha_1} f(0)| \cdots |\nabla_{\mathbb{R}^n}^{1+\alpha_k} f(0)|\big)|\nabla^{1+\alpha_{k+1}} u(0)|.$$

We use (2.13) and (2.12), and obtain after long but straightforward calculation that

$$|\tilde{\nabla}^{k+1} u(0)| \leq C_k \sum_{|\alpha| \leq k} \big(1 + |\nabla_{\mathbb{R}^n}^{1+\alpha_1} f(0)| \cdots |\nabla_{\mathbb{R}^n}^{1+\alpha_k} f(0)|\big)||\nabla^{1+\alpha_{k+1}} u(0)|.$$

Note that $\nu_E \circ \Phi^{-1} = \frac{(-\nabla_{\mathbb{R}^n} f, 1)}{\sqrt{1 + |\nabla_{\mathbb{R}^n} f|^2}}$. We thus obtain by (2.12) that

$$\begin{aligned}|\nabla_{\mathbb{R}^n}^{l+1} f(0)| &\leq C_l \sum_{|\beta| \leq l} \big(1 + |\nabla^{\beta_1}(\nu_E \circ \Phi^{-1})| \cdots |\nabla^{\beta_l}(\nu_E \circ \Phi^{-1})|\big) \\ &\leq C_l \sum_{|\beta| \leq l-1} \big(1 + |\tilde{\nabla}^{\beta_1} B_E| \cdots |\tilde{\nabla}^{\beta_l} B_E|\big)\end{aligned} \tag{2.14}$$

and the claim follows. □

Next we turn our focus on geometric inequalities on compact hypersufaces. Recall that by classical results e.g. from [6] it holds that $\|u\|_{H^2(\Sigma)} \leq C(\|\Delta u\|_{L^2(\Sigma)} + \|u\|_{L^2(\Sigma)})$ and e.g. in [22] it is proven that $\|u\|_{H^{2k}(\Sigma)} \leq C(\|\Delta u\|_{H^k(\Sigma)} + \|u\|_{L^2(\Sigma)})$. We need these results with a quantitative control on the constant.

Lemma 2.5. *Assume Σ is C^{2k+2}-regular and $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C$. Then for all $u \in C^{2k+1}(\Sigma)$ it holds*

$$\|u\|_{H^{2k}(\Sigma)} \leq C_k(\|\Delta^k u\|_{L^2(\Sigma)} + (1 + \|B_\Sigma\|_{H^{2k-1}(\Sigma)})\|u\|_{L^\infty(\Sigma)})$$

and

$$\|u\|_{H^{2k+1}(\Sigma)} \leq C_k(\|\tilde{\nabla}\Delta^k u\|_{L^2(\Sigma)} + (1 + \|B_\Sigma\|_{H^{2k}(\Sigma)})\|u\|_{L^\infty(\Sigma)}).$$

Proof. We only prove the first inequality since the second follows from the same argument. The proof is similar to [30, Proposition 2.11] but we sketch it for the reader's convenience. Denote $l = 2k$. We begin by noticing that we may interchange the derivatives of the $(l+1)$:th order covariant derivative of u by using (2.9), (2.10), (2.13) and the curvature bound $\|B_\Sigma\|_{L^\infty} \leq C$ (see also [38, Proof of Lemma 7.3])

$$\begin{aligned} &|\tilde{\nabla}_{i_{l+1}} \cdots \tilde{\nabla}_{i_{m+1}} \tilde{\nabla}_{i_m} \cdots \tilde{\nabla}_{i_1} u - \tilde{\nabla}_{i_{l+1}} \cdots \tilde{\nabla}_{i_m} \tilde{\nabla}_{i_{m+1}} \cdots \tilde{\nabla}_{i_1} u| \\ &\quad \leq C_l \sum_{|\alpha| \leq l-1} (1 + |\tilde{\nabla}^{\alpha_1} B_\Sigma| \cdots |\tilde{\nabla}^{\alpha_{l-1}} B_\Sigma|)|\tilde{\nabla}^{\alpha_l} u|. \end{aligned}$$

We leave the details for the reader. This holds pointwise on Σ and we use it without further mentioning. Let us denote $F = \tilde{\nabla}^{2k-2}u$ and denote its components simply by F_β, where $\beta = (i_1, \ldots, i_{2k-2})$. Then it holds by divergence theorem, by interchanging the derivatives and by Proposition 2.3

$$\begin{aligned} &\int_\Sigma |\tilde{\nabla}^{2k} u|^2 \, d\mathcal{H}^n = \int_\Sigma |\tilde{\nabla}^2 F|^2 \, d\mathcal{H}^n \\ &\quad = \int_\Sigma \tilde{\nabla}_i \tilde{\nabla}_j F_\beta \tilde{\nabla}^i \tilde{\nabla}^j F^\beta \, d\mathcal{H}^n = - \int_\Sigma \tilde{\nabla}_j F_\beta \tilde{\nabla}_i \tilde{\nabla}^i \tilde{\nabla}^j F^\beta \, d\mathcal{H}^n \\ &\quad \leq - \int_\Sigma \tilde{\nabla}_j F_\beta \tilde{\nabla}^j \tilde{\nabla}_i \tilde{\nabla}^i F^\beta \, d\mathcal{H}^n \\ &\qquad + C_k \sum_{|\alpha| \leq l-1} \int_\Sigma (1 + |\tilde{\nabla}^{\alpha_1} B_\Sigma|^2 \cdots |\tilde{\nabla}^{\alpha_{l-1}} B_\Sigma|^2)|\tilde{\nabla}^{\alpha_l} u|^2 \, d\mathcal{H}^n \\ &\quad \leq \int_\Sigma \tilde{\nabla}^j \tilde{\nabla}_j F_\beta \tilde{\nabla}_i \tilde{\nabla}^i F^\beta \, d\mathcal{H}^n + C_k(\|u\|^2_{H^{l-1}(\Sigma)} + \|u\|^2_{L^\infty(\Sigma)} \|B_\Sigma\|^2_{H^{l-1}(\Sigma)}) \\ &\quad = \int_\Sigma |\Delta \tilde{\nabla}^{2k-2} u|^2 \, d\mathcal{H}^n + C_k(\|u\|^2_{H^{2k-1}(\Sigma)} + \|u\|^2_{L^\infty(\Sigma)} \|B_\Sigma\|^2_{H^{2k-1}(\Sigma)}). \end{aligned}$$

By interchanging the derivatives and arguing as above we obtain

$$\begin{aligned} \int_\Sigma |\Delta \tilde{\nabla}^{2k-2} u|^2 \, d\mathcal{H}^n &\leq \int_\Sigma |\tilde{\nabla}^{2k-2} \Delta u|^2 \, d\mathcal{H}^n \\ &\quad + C_k(\|u\|^2_{H^{2k-1}(\Sigma)} + \|u\|^2_{L^\infty(\Sigma)} \|B_\Sigma\|^2_{H^{2k-1}(\Sigma)}). \end{aligned}$$

By repeating the argument by replacing u with $\Delta^j u$, for $j = 1, \dots, k-1$, we deduce that

$$\int_\Sigma |\tilde{\nabla}^{2k} u|^2 \, d\mathcal{H}^n \leq \int_\Sigma |\Delta^k u|^2 \, d\mathcal{H}^n + C_k(\|u\|^2_{H^{2k-1}(\Sigma)} + \|u\|^2_{L^\infty(\Sigma)} \|B_\Sigma\|^2_{H^{2k-1}(\Sigma)}).$$

The claim follows from interpolation inequality (Proposition 2.2) as for $\theta \in (0, 1)$ it holds that

$$\|u\|^2_{H^{2k-1}(\Sigma)} \leq \|u\|^{2\theta}_{H^{2k}(\Sigma)} \|u\|^{2(1-\theta)}_{L^\infty(\Sigma)} \leq \varepsilon \|u\|^2_{H^{2k}(\Sigma)} + C_\varepsilon \|u\|_{L^\infty(\Sigma)},$$

where the last inequality follows from Young's inequality. □

Lemma 2.5 together with Simon's identity (2.11), imply

Proposition 2.6. *Assume that* Σ *is* C^{2k+3}*-regular and that* $\|B_\Sigma\|_{L^\infty}, \mathcal{H}^n(\Sigma) \leq C$. *Then it holds that*

$$\|B_\Sigma\|_{H^{2k}(\Sigma)} \leq C_k(1 + \|\Delta^k H_\Sigma\|_{L^2(\Sigma)})$$

and

$$\|B_\Sigma\|_{H^{2k+1}(\Sigma)} \leq C_k(1 + \|\tilde{\nabla}\Delta^k H_\Sigma\|_{L^2(\Sigma)}).$$

2.4. Uniform Ball Condition and Signed Distance Function

In this subsection, we recall some properties related to sets which satisfy UBC as well as properties of signed distance function defined in (2.1). Most of them can be found e.g. in [5,7] while others are more difficult to find. We recall that a set $E \subset \mathbb{R}^{n+1}$ satisfies UBC with given a radius $r \in \mathbb{R}_+$, if it simultaneously satisfies the exterior and interior ball condition with radius r at every boundary point. That is, for every $x \in \partial E$ there are balls $B_r(x_+)$ and $B_r(x_-)$ such that

$$B_r(x_+) \subset \mathbb{R}^{n+1} \setminus E, \quad B_r(x_-) \subset E \quad \text{and} \quad x \in \partial B_r(x_+) \cap \partial B_r(x_-).$$

It is well known, for the experts at least, that UBC for a set implies its boundary being a uniformly $C^{1,1}$-regular hypersurface. We need this property in a quantitative form which states that if $E \subset \mathbb{R}^{n+1}$ satisfies UBC with radius r, then it can be written locally in a cylinder of width $r/2$ as a graph of a $C^{1,1}$-function. Since this result is not easy to find in the literature, we state it and provide a proof here.

Proposition 2.7. *Assume* $E \subset \mathbb{R}^{n+1}$ *satisfies UBC with radius* $r > 0$. *Then for every point* $x \in \partial E$ *we may, by rotating the coordinates if necessary, write the interior of the set locally as a subgraph of a function* $g : B^n_{r/2}(x') \to \mathbb{R}$, *i.e.,*

$$\begin{aligned} \operatorname{int}(E) \cap \boldsymbol{C}(x, r/2, r) &= \{(y', y_{n+1}) \in \boldsymbol{C}(x, r/2, r/2) : y_{n+1} < g(y')\} \ \textit{and} \\ \partial E \cap \boldsymbol{C}(x, r/2, r) &= \{(y', g(y')) : y' \in B^n_{r/2}(x')\}. \end{aligned}$$

The function g is $C^{1,1}$-regular and it holds for all $y' \in B^n_{r/2}(x')$ and $s \in (0, r/2]$

$$|g(y') - g(x')| \le \frac{|y'-x'|^2}{r + \sqrt{r^2 - |y'-x'|^2}},$$

$$|\nabla g(y')| \le \frac{|y'-x'|}{r}\left(1 - \left(\frac{|y'-x'|}{r}\right)^2\right)^{-\frac{1}{2}} \quad \textit{and}$$

$$\sup_{\substack{y_1', y_2' \in B^n_s(x') \\ y_1' \neq y_2'}} \frac{|\nabla g(y_2') - \nabla g(y_1')|}{|y_2' - y_1'|} \le \frac{1}{r}\left(1 - \left(\frac{s}{r}\right)^2\right)^{-\frac{3}{2}}.$$

Moreover, the outer unit normal ν_E on ∂E is $1/r$-Lipschitz continuous in Euclidean metric.

Remark 2.8. We remark, that the converse of Proposition 2.7 also holds true. That is, if $E \subset \mathbb{R}^{n+1}$ is a set such that for every $x \in \partial E$, we may write its boundary locally, by rotating the coordinates, as $\partial E \cap \mathbf{C}(x, r, 2r) \subset \{(y', g(y')) : y' \in B^n_r(x')\}$ with $\|g\|_{C^{1,1}(B^n_r(x'))} \le C/r$, then E satisfies UBC with radius $c\,r$, for a constant $c > 0$ which depends on n and C. This is fairly straightforward to show and we leave it to the reader.

Proof of Proposition 2.7. We remark that UBC with r implies for every $x \in \partial E$ an existence of a unique unit vector $\nu_E(x)$ such that $B_r(x - r\nu_E(x)) \subset E$ and $B_r(x + r\nu_E(x)) \subset \mathbb{R}^{n+1} \setminus E$. Therefore, we have a vector field $\nu_E : \partial E \to \partial B_1$ which later turns out to be the outer unit normal field of E. We first show that ν_E is $1/r$-Lipschitz continuous with respect to Euclidean distance. To this end, fix $x, y \in \partial E$. By the previous observation $B_r(x + r\nu_E(x)) \subset \mathbb{R}^{n+1} \setminus E$ and $B_r(y - r\nu_E(x)) \subset E$ so the balls are disjoint. Similarly, the balls $B_r(x - r\nu_E(x))$ and $B_r(y + r\nu_E(y))$ are disjoint. Hence the distances between the corresponding centerpoints are at least $2r$ and we obtain the inequalities

$$4r^2 \le |x - y + r(\nu_E(x) + \nu_E(y))|^2 \text{ and}$$
$$4r^2 \le |x - y - r(\nu_E(x) + \nu_E(y))|^2.$$

By summing the above inequalities gives us $8r^2 \le 2|x-y|^2 + 4r^2 (1 + \nu_E(x) \cdot \nu_E(y))$ and, again, by subtracting and dividing terms we further obtain

$$1 - \frac{|x-y|^2}{2r^2} \le \nu_E(x) \cdot \nu_E(y) \quad \text{or equivalently} \quad |\nu_E(x) - \nu_E(y)|^2 \le \frac{|x-y|^2}{r^2}. \tag{2.15}$$

In particular, ν_E is $1/r$-Lipschitz.

For given a point $x \in \partial E$, we show the existence of g as claimed. Without loss of generality we may assume $x = 0$ and $\nu_E(0) = e_{n+1}$. Then it holds $B_r(-re_{n+1}) \subset E$ and $B_r(re_{n+1}) \subset \mathbb{R}^{n+1} \setminus E$. Thus, for every $y' \in B^n_{r/2}$ there is a number $t_{y'}$ such that $(y', t_{y'}) \in \partial E$ and

$$|t_{y'}| \le r - \sqrt{r^2 - |y'|^2} = \frac{|y'|^2}{r + \sqrt{r^2 - |y'|^2}}. \tag{2.16}$$

In particular, $|t_{y'}| < |y'|$. Combining (2.15) and (2.16) yields

$$\nu_E(y', t_{y'}) \cdot e_{n+1} \geq \sqrt{1 - \left(\frac{|y'|}{r}\right)^2}. \tag{2.17}$$

Let us show that such a number $t_{y'}$ is unique.

We suppose by contradiction there is $s_{y'} \in (-r, r)\backslash\{t_{y'}\}$ such that $(y', s_{y'}) \in \partial E$. Without loss of generality, we may assume $s_{y'} > t_{y'}$. Since $B_r\big((y', t_{y'}) + r\nu_E(y', t_{y'})\big) \subset \mathbb{R}^{n+1}\backslash E$ and $(y', s_{y'}) \in \partial E$, then the point $(y', s_{y'})$ is not in the ball $B_r\big((y', t_{y'}) + r\nu_E(y', t_{y'})\big)$. Hence, we obtain

$$\begin{aligned} r^2 &\leq |(y', s_{y'}) - \big((y', t_{y'}) + r\nu_E(y', t_{y'})\big)|^2 \\ &= (s_{y'} - t_{y'})^2 - 2r(s_{y'} - t_{y'})\nu_E(y', t_{y'}) \cdot e_{n+1} + r^2. \end{aligned}$$

We first subtract r^2, then divide by $s_{y'} - t_{y'}$ and finally use the estimates (2.16), (2.17) as well as $|y'| < r/2$ to deduce

$$s_{y'} \geq t_{y'} + 2r\nu_E(y', t_{y'}) \cdot e_{n+1} \geq -r + 3\sqrt{r^2 - |y'|^2} > r - \sqrt{r^2 - |y'|^2}.$$

This implies together with $s_{y'} < r$ that $(y', s_{y'}) \in B_r(re_{n+1}) \subset \mathbb{R}^{n+1}\backslash E$ which, in turn, contradicts $(y', s_{y'}) \in \partial E$ and, hence, $t_{y'}$ is a unique value in $(-r, r)$ satisfying $(y', t_{y'}) \in \partial E$.

Thus, the function $g : B^n_{r/2} \to \mathbb{R}$, given by the relation $g(y') = t_{y'}$, satisfies

$$\begin{aligned} \operatorname{int}(E) \cap \mathbf{C}(0, r/2, r/2) &= \{(y', y_{n+1}) \in \mathbf{C}(0, r/2, r/2) : y_{n+1} < g(y')\} \text{ and} \\ \partial E \cap \mathbf{C}(0, r/2, r/2) &= \{(y', g(y')) : y' \in B^n_{r/2}\}. \end{aligned} \tag{2.18}$$

Again, (2.16) gives us the bound on $|g(y')|$ as claimed. The condition (2.17) implies that for every $y' \in B^n_{r/2}$ there are open sets $y' \in V \subset B^n_{r/2}$, $(y', g(y')) \in U \subset \mathbf{C}(0, r/2, r/2)$ and functions $\psi_+, \psi_- \in C^\infty(V)$ such that $\partial B_r((y', g(y')) \pm r\nu_E(y', g(y')) \cap U$ are the graphs of $\psi_\pm$ respectively. Then $\psi_- \leq g \leq \psi_+$ in V and $\psi_-(w) = g(w) = \psi_+(w)$ implying the differentiability of g at y' with $\nabla g(y') = \nabla\psi_\pm(y')$. Moreover, we deduce that $\nu_E(y', g(y'))$ is the outer unit normal of $\{(z', z_{n+1}) \in V \times \mathbb{R} : z_{n+1} > \psi_+(z')\}$ at $(y', g(y'))$ and thus

$$\nu_E(y', g(y')) = \frac{(-\nabla\psi_+(y'), 1)}{\sqrt{1 + |\nabla\psi_+(y')|^2)}} = \frac{(-\nabla g(y'), 1)}{\sqrt{1 + |\nabla g(y')|^2}}. \tag{2.19}$$

Since now g and ν_E are continuous, (2.19) implies that ∇g is continuous too. Thus, E is C^1-regular and ν_E is the actual outer unit normal of E. We combine (2.17) and (2.19) to observe

$$|\nabla g(y')| \leq \frac{|y'|}{r}\left(1 - \left(\frac{|y'|}{r}\right)^2\right)^{-\frac{1}{2}}. \tag{2.20}$$

To conclude the Lipschitz estimate, if $y'_1, y'_2 \in B^n_s$ for given $s \in (0, r/2]$, then the uniform ball condition implies that $(y'_1, g(y'_1)) \notin B_r((y'_2, g(y'_2)) \pm r\nu_E(y'_2, g(y'_2))$

and $(y_2', g(y_2')) \notin B_r((y_1', g(y_1')) \pm r\nu_E(y_1', g(y_1'))$. Hence, using (2.19), we obtain the estimates

$$r^2 \leq \left|(y_2', g(y_2')) \pm r\frac{(-\nabla g(y_2'), 1)}{\sqrt{1+|\nabla g(y_2')|^2}} - (y_1', g(y_1'))\right|^2 \quad \text{and}$$

$$r^2 \leq \left|(y_1', g(y_1')) \pm r\frac{(-\nabla g(y_1'), 1)}{\sqrt{1+|\nabla g(y_1')|^2}} - (y_2', g(y_2'))\right|^2.$$

By summing these inequalities and simplifying, we have

$$\begin{aligned} &\pm(y_2' - y_1') \cdot (\nabla g(y_2') - \nabla g(y_1')) \\ &\leq \frac{\sqrt{1+|\nabla g(y_2')|^2} + \sqrt{1+|\nabla g(y_1')|^2}}{2r} \left(|y_2' - y_1'|^2 + (g(y_2') - g(y_1'))^2\right). \end{aligned}$$

Thus, by recalling, (2.20) we further estimate that

$$\begin{aligned} &|(y_2' - y_1') \cdot (\nabla g(y_2') - \nabla g(y_1'))| \\ &\leq \frac{\sqrt{1+|\nabla g(y_2')|^2} + \sqrt{1+|\nabla g(y_1')|^2}}{2r} \\ &\left(|y_2' - y_1'|^2 + (g(y_2') - g(y_1'))^2\right) \\ &\leq \frac{\sqrt{1+\sup_{B_s^n}|\nabla g|^2}}{r}\left(1+\sup_{B_s^n}|\nabla g|^2\right)|y_2' - y_1'|^2 \\ &\leq \frac{1}{r}\left(1-\left(\frac{s}{r}\right)^2\right)^{-\frac{3}{2}}|y_2' - y_1'|^2. \end{aligned} \tag{2.21}$$

The desired estimate then follows from (2.21) via a standard mollification argument. □

We recall that a signed distance function d_E of a non-empty set $E \subset \mathbb{R}^{n+1}$ is always 1-Lipschitz and it is differentiable at $x \in \mathbb{R}^n \setminus \partial E$ exactly when the projection $\pi_{\partial E}(x)$ exists on ∂E. Again, UBC for E means the differentiability of d_E in a tubular neighborhood. Indeed, one may show that for a non-empty open set $E \subset \mathbb{R}^{n+1}$ and $r \in \mathbb{R}_+$ the conditions

(i) d_E is differentiable in $\mathcal{N}_r(\partial E)$ and
(ii) E satisfies UBC with radius r

are equivalent. In such a case, the projection $\pi_{\partial E}$ onto ∂E is defined in $\mathcal{N}_r(\partial E)$ as a continuous map and the following fundamental identities hold in $\mathcal{N}_r(\partial E)$:

$$\pi_{\partial E} = \mathrm{id} - d_E \nabla d_E \quad \text{and} \quad \nabla d_E = \nu_E \circ \pi_{\partial E}. \tag{2.22}$$

In particular, $d_E \in C^1(\mathcal{N}_r(\partial E))$. Further, it is fairly simple to conclude that for every $t \in (-r, r)$ the sublevel set $E_t = \{x \in \mathbb{R}^{n+1} : d_E(x) < t\}$ has the level set

$\{x \in \mathbb{R}^{n+1} : d_E(x) = t\}$ as the boundary and satisfies UBC with radius $r - |t|$. Moreover, it holds that

$$d_{E_t} = d_E - t \quad \text{and} \quad \pi_{\partial E_t} = \pi_{\partial E} + t\nu_E \circ \pi_{\partial E} \ \text{ in } \mathcal{N}_{r-|t|}(\partial E_t). \tag{2.23}$$

We may then improve the regularity by showing ∇d_E and $\pi_{\partial E}$ are locally Lipschitz continuous in $\mathcal{N}_r(\partial E)$ and obtain quantitative estimates for the Lipschitz constants in smaller tubes.

Lemma 2.9. *Assume $E \subset \mathbb{R}^{n+1}$ satisfies UBC with radius $r > 0$. Then for every $0 < \rho < r$ and $x, y \in \overline{\mathcal{N}_\rho(\partial E)}$ it holds that*

$$|\pi_{\partial E}(x) - \pi_{\partial E}(y)| \leq \frac{r}{r-\rho}|x-y| \ \text{ and } \ |\nabla d_E(x) - \nabla d_E(y)| \leq \frac{1}{r-\rho}|x-y|.$$

Proof. It is enough to prove the first estimate, since the second estimate follows from the first via Proposition 2.7 and the second identity of (2.22). We first show that the estimates hold locally, i.e., for every $x \in \mathcal{N}_r(\partial E)$,

$$\operatorname{Lip}(\pi_{\partial E}, x) \leq \frac{r}{r - |d_E(x)|} \tag{2.24}$$

To this end, we show that, for every $x \in \partial E$ and $y \in B_{r/4}(x)$, it holds that

$$|\pi_{\partial E}(y) - x|^2 \leq \left(1 + \frac{4}{r - |d_E(y)|}|d_E(y)|\right)|y - x|^2. \tag{2.25}$$

We may assume that $x = 0$, $\nu_E(0) = e_{n+1}$ and $y \notin E$. Let $g : B^n_{r/2} \to \mathbb{R}$ be as in Proposition 2.7. Since $|y| < r/4$, then $y \in \mathbf{C}(r/2, r/2, 0)$ implying $|d_E(y)| \leq |y_{n+1} - g_n(y')|$ and, hence, we make a technical observation

$$d_E^2(y) \leq 2d_E(y)(y_{n+1} - g(y')). \tag{2.26}$$

Thus, using Proposition 2.7, (2.22), (2.26) and Young's inequality, we estimate that

$$\begin{aligned}
|\pi_{\partial E}(y)|^2 &= |y|^2 - 2d_E(y)\, y \cdot \nabla d_E(y) + d_E^2(y) \\
&= |y|^2 - 2d_E(y) y_{n+1} + d_E^2(y) - 2d_E(y)\, y \cdot (\nabla d_E(y) - e_{n+1}) \\
&\leq |y|^2 - 2d_E(y) g(y') - 2d_E(y)\, y \cdot (\nu_E(\pi_{\partial E}(y)) - \nu_E(0)) \\
&\leq |y|^2 + 2\frac{|d_E(y)|}{r}|y'|^2 + 2\frac{d_E(y)}{r}|y||\pi_{\partial E}(y)| \\
&\leq |y|^2 + 2\frac{|d_E(y)|}{r}|y|^2 + \frac{|d_E(y)|}{r}|y|^2 + \frac{|d_E(y)|}{r}|\pi_{\partial E}(y)|^2,
\end{aligned}$$

and (2.25) follows. Suppose next $y_1, y_2 \in B_\rho(x)$ for given $x \in \partial E$ and $0 < \rho < r/9$. The sublevel set E_t, for $t = d_E(y_2)$, satisfies UBC with radius $r - \rho$ and $y_2 \in \partial E_t$. Since $|y_1 - y_2| < 2\rho \leq (r - \rho)/4$, then by applying (2.25) for ∂E_t we have

$$|\pi_{\partial E_t}(y_1) - y_2| \leq \left(1 + \frac{8\rho}{r - 2\rho}\right)^{\frac{1}{2}} |y_1 - y_2|. \tag{2.27}$$

On the other hand, first recalling the second identity in (2.23) and then applying Proposition 2.7 gives us

$$|\pi_{\partial E_t}(y_1) - y_2| = |\pi_{\partial E_t}(y_1) - \pi_{\partial E_t}(y_2)| \geq \left(1 - \frac{\rho}{r}\right) |\pi_{\partial E}(y_1) - \pi_{\partial E}(y_2)|,$$

so, by combining the estimate above with (2.27) yields $\mathrm{Lip}(x, \pi_{\partial E}) = 1$. Hence, we deduce that

$$\mathrm{Lip}(x, \pi_{\partial E_t}) = 1 \tag{2.28}$$

for every $t \in (-r, r)$ and $x \in \partial E_t$. By using (2.23) and Proposition 2.7 similarly as previous, we infer (2.24) from (2.28).

Finally, for the first estimate of the claim, we may assume $x, y \in \mathcal{N}_\rho(\partial E)$. Let $J_{yx} := \{tx + (1-t)y : t \in [0, 1]\}$ be the line segment between them. If $J_{yx} \subset \mathcal{N}_\rho(\partial E)$, then the first estimate of the claim follows from (2.24). Otherwise, there are $0 < t_1 \leq t_2 < 1$ such that $tx + (1-t)y \in \mathcal{N}_\rho(\partial E)$ for every $t \in [0, t_1) \cup (t_2, 1]$ and $z_i = t_i x + (1-t_i)y \in \partial \mathcal{N}_\rho(\partial E)$ for $i = 1, 2$. Since $d_E(z_1) = \rho = d_E(z_2)$, then Proposition 2.7 and (2.22) imply

$$|\pi_{\partial E}(z_1) - \pi_{\partial E}(z_2)| \leq \frac{r}{r - \rho} |z_1 - z_2|.$$

On the other hand, due to (2.24) we have

$$|\pi_{\partial E}(x) - \pi_{\partial E}(z_1)| \leq \frac{r}{r-\rho}|x - z_1| \text{ and } |\pi_{\partial E}(z_2) - \pi_{\partial E}(y)| \leq \frac{r}{r-\rho}|z_2 - y|$$

and we conclude the proof. □

If E is $C^{k,\alpha}$-regular, with $k \geq 2$ and $0 \leq \alpha \leq 1$, then $d_E \in C^{k,\alpha}(\mathcal{N}_r(\partial E))$ and $\pi_{\partial E} \in C^{k-1,\alpha}(\mathcal{N}_r(\partial E); \mathbb{R}^{n+1})$. In particular, (2.22) holds everywhere in $\mathcal{N}_r(\partial E)$. Then it holds

$$\nabla^2 d_E = B_E \quad \text{and} \quad \Delta d_E = H_E \quad \text{on} \quad \partial E. \tag{2.29}$$

In particular, we deduce from Lemma 2.9 and (2.29) that

$$\|H_E\|_{L^\infty(\partial E)} \leq \frac{n}{r} \quad \text{and} \quad \sup_{\partial E} |B_E|_{\mathrm{op}} \leq \frac{1}{r}. \tag{2.30}$$

Differentiating $\nabla d_E \cdot \nabla d_E = 1$ yields $\nabla^2 d_E \nabla d_E = 0$ in $\mathcal{N}_r(\partial E)$. Again, by differentiating the first identity in (2.22) we obtain

$$\nabla \pi_{\partial E} = I - \nabla d_E \otimes \nabla d_E - d_E \nabla^2 d_E \quad \text{in} \quad \mathcal{N}_r(\partial E). \tag{2.31}$$

The second identity in (2.22) says that $\nabla d_E = \nabla d_E \circ \pi_{\partial E}$ in $\mathcal{N}_r(\partial E)$. Thus, by differentiating this and by using the properties of the distance function mentioned before we have

$$\nabla^2 d_E = (\nabla^2 d_E)^T = \nabla \pi_{\partial E}(\nabla^2 d_E \circ \pi_{\partial E}) = \big(I - d_E \nabla^2 d_E\big)(B_E \circ \pi_{\partial E}) \text{ in } \mathcal{N}_r(\partial E). \tag{2.32}$$

We write this as

$$\nabla^2 d_E \big(I + d_E (B_E \circ \pi_{\partial E})\big) = B_E \circ \pi_{\partial E}.$$

It follows from (2.30) that the matrix field $I + d_E(B_E \circ \pi_{\partial E})$ is invertible in $\mathcal{N}_r(\partial E)$. Therefore, we have

$$\nabla^2 d_E = (B_E \circ \pi_{\partial E})\big(I + d_E(B_E \circ \pi_{\partial E})\big)^{-1} \ \text{ in } \mathcal{N}_r(\partial E). \tag{2.33}$$

By combining (2.22), (2.31), (2.29) and (2.33), we may decompose $\nabla \pi_{\partial E}$ as

$$\nabla \pi_{\partial E} = I - \nu_E \circ \pi_{\partial E} \otimes \nu_E \circ \pi_{\partial E} - d_E(B_E \circ \pi_{\partial E})\big(I + d_E(B_E \circ \pi_{\partial E})\big)^{-1} \text{ in } \mathcal{N}_r(\partial E). \tag{2.34}$$

By using a fairly standard calibration argument (see e.g. [1, Lemma 4.1]) we conclude that UBC implies so called Λ-*minimizer* condition.

Lemma 2.10. *Assume that $E \subset \mathbb{R}^{n+1}$ is an open and bounded set which satisfies UBC with radius $r > 0$. Then for every set of finite perimeter F it holds that*

$$P(E \cap F) \le P(F) + \frac{n+1}{r}|F \setminus E| \quad \textit{and}$$
$$P(E \cup F) \le P(F) + \frac{n+1}{r}|E \setminus F|.$$

In particular, $P(E) \le \frac{n+1}{r}|E|$.

Proof. The argument is a quantitative version of [1, Lemma 4.1]. We will prove that for every set of finite perimeter F it holds that

$$P(E) \le P(F) + \frac{n+1}{r}|F \Delta E|. \tag{2.35}$$

Then the two inequalities in the statement follow by using (2.35) with $E \cup F$ and $E \cap F$ in place of F and using the fact [37, Lemma 12.22] that

$$P(E \cup F) + P(E \cap F) \le P(E) + P(F).$$

The third inequality follows by using (2.35) with $F = \emptyset$.

By a standard approximation argument for the sets of finite perimeter [37, Thm 13.8] we may assume that F is smooth. In turn, we may approximate also E by a sequence of smooth sets E_k in the C^1-sense such that E_k satisfies UBC with radius r_k such that $r_k \to r$. Therefore, by simplicity we assume that also E is smooth.

For each $k \in \mathbb{N}$ we construct a vector-field $X_k \in C_0^{0,1}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ such that

- (i) $X_k = \nu_E$ on ∂E,
- (ii) $|X_k| \le 1$ in $\mathbb{R}^{n+1}$ and
- (iii) $\|\operatorname{div} X_k\|_{L^\infty(\mathbb{R}^{n+1})} \le (n + 1 + k^{-1})/r$.

To this aim, we first define $\eta_k : \mathbb{R} \to \mathbb{R}$ by setting $\eta_k(t) = \max\{0, 1 - (1 + 1/k)|t|/r\}$ and then set $X_k = (\eta_k \circ d_E)\nabla d_E$. Clearly X_k is a Lipschitz continuous vector field supported in $\overline{\mathcal{N}_{r/(1+k^{-1})}(\partial E)}$ and satisfies the properties (i) and (ii). We further compute that

$$\operatorname{div} X_k = (\eta_k \circ d_E)\Delta d_E + \eta_k' \circ d_E \ \text{ in } \mathcal{N}_{r/(1+k^{-1})}(\partial E) \setminus \partial E.$$

Hence, it follows from Lemma 2.9, $\nabla^2 d_E \nabla d_E = 0$ in $\mathcal{N}_r(\partial E)$ as well as the definition of η_k that $|\operatorname{div} X_k| \leq (n+1+k^{-1})/r$ in $\mathcal{N}_{r/(1+k^{-1})}(\partial E)\backslash\partial E$. Since the sets $\overline{\mathcal{N}_{r/(1+k^{-1})}(\partial E)}$ and $\mathcal{N}_{r/(1+k^{-1})}(\partial E)\backslash\partial E$ agree in the L^1-sense we infer that X_k satisfies (iii). By using the properties (i) – (iii) for X_k as well as the divergence theorem we estimate

$$\begin{aligned} P(E) - P(F) &\leq \int_{\partial E} X_k \cdot \nu_E \, \mathrm{d}\mathcal{H}^n - \int_{\partial F} X_k \cdot \nu_F \, \mathrm{d}\mathcal{H}^n \\ &= \int_E \operatorname{div} X_k \, \mathrm{d}x - \int_F \operatorname{div} X_k \, \mathrm{d}x \leq \int_{E\Delta F} |\operatorname{div} X_k| \, \mathrm{d}x \\ &\leq \frac{n+1+k^{-1}}{r} |F \Delta E|, \end{aligned}$$

and thus, by letting $k \to \infty$, the above yields (2.35). □

Suppose that E' is a connected component of a set E which satisfies UBC with r. If $|E'| < \infty$, then E' is bounded and we may control its diameter in terms of r and $|E'|$. Indeed, by the above approximation we may assume that E is smooth. Then by (2.29) we have $|H_E| \leq n/r$ on ∂E. Thus, combining Lemma 2.10 and Topping's generalization [48] of Simon's diameter control [45] gives us the estimate

$$\operatorname{diam}(E') \leq C_n \int_{\partial E'} |H_{E'}|^{n-1} \, \mathrm{d}\mathcal{H}^n \leq \frac{C_n}{r^n} |E'| \tag{2.36}$$

for a dimensional constant $C_n \in \mathbb{R}_+$. Finally, we need the following interpolation result:

Lemma 2.11. *Assume $E \subset \mathbb{R}^{n+1}$ is an open and bounded set which satisfies UBC with radius $r > 0$. If U is an open set containing ∂E and $u \in C^2(U)$, then*

$$\|\nabla_\tau u\|^2_{L^\infty(\partial E)} \leq 4 \|u\|_{L^\infty(\partial E)} \left(\sup_{\partial E} |\nabla^2 u|_{\mathrm{op}} + \frac{\|\nabla_\tau u\|_{L^\infty(\partial E)}}{r} \right).$$

Proof. By the above approximation argument we may assume that E is smooth.

We first observe that for a bounded function $f \in C^2(\mathbb{R})$ it holds

$$\|f'\|^2_{L^\infty(\mathbb{R})} \leq 4 \|f\|_{L^\infty(\mathbb{R})} \|f''\|_{L^\infty(\mathbb{R})}. \tag{2.37}$$

Indeed, let us fix a $t \in \mathbb{R}_+$. We may assume that $f'(t) > 0$, since otherwise we consider the function $-f$ instead of f. Let I be a maximal open interval containing t such that $f' > 0$ in I so f is strictly increasing there. Then there is a decreasing sequence $(\tilde{t}_i)_i \in (\inf I, t)$ converging to $\inf I$ such that $f'(\tilde{t}_i) \to 0$ as $i \to \infty$. Since f is strictly increasing in I, it is invertible there. Hence, we may compute, for every $i \in \mathbb{N}$,

$$\begin{aligned} |f'(t)|^2 - |f'(\tilde{t}_i)|^2 &= \int_{\tilde{t}_i}^t \frac{\mathrm{d}}{\mathrm{d}s} (f'(s))^2 \mathrm{d}s \\ &= 2 \int_{\tilde{t}_i}^t f''(s) f'(s) \mathrm{d}s = 2 \int_{\tilde{t}_i}^t f''(f^{-1}(f(s))) f'(s) \mathrm{d}s \end{aligned}$$

$$= 2\int_{f(\tilde{t}_i)}^{f(t)} f''(f^{-1}(\tau))\mathrm{d}\tau \leq 4\|f\|_{L^\infty(\mathbb{R})}\|f''\|_{L^\infty(\mathbb{R})},$$

and thus, by letting $i \to \infty$, we obtain $|f'(t)|^2 \leq 4\|f\|_{L^\infty(\mathbb{R})}\|f''\|_{L^\infty(\mathbb{R})}$, and (2.37) follows.

Since ∂E is compact we find $x \in \partial E$ such that $|\nabla_\tau u(x)| = \|\nabla_\tau u\|_{L^\infty(\partial E)}$. We may assume that $|\nabla_\tau u(x)| > 0$. The connected component of ∂E containing x is geodesically complete and, hence, we find a smooth unit speed geodesic curve $\gamma : \mathbb{R} \to \partial E$ satisfying $\gamma(0) = x$ and $\gamma'(0) = \nabla_\tau u(x)/|\nabla_\tau u(x)|$. Then we define a C^2-regular function $f = u \circ \gamma$. Note that $f'(0) = \|\nabla_\tau u\|_{L^\infty(\partial E)}$ and that

$$f'' = \gamma' \cdot (\nabla^2 u \circ \gamma)\gamma' + \gamma'' \cdot (\nabla_\tau u \circ \gamma). \tag{2.38}$$

By differentiating the identity $0 = d_E \circ \gamma$ twice and recalling the identities (2.22) and (2.29) we obtain $0 = \gamma' \cdot (B_E \circ \gamma)\gamma' + \gamma'' \cdot (\nu_E \circ \gamma)$. Since γ is a geodesic curve, then $|\gamma'' \cdot (\nu_E \circ \gamma)| = |\gamma''|$ and hence we infer from the previous that $|\gamma''| \leq |B_E \circ \gamma|_{\mathrm{op}}$. By combing this with (2.38) and using (2.30) gives us

$$|f''| \leq \left(|\nabla^2 u \circ \gamma|_{\mathrm{op}} + |B_E \circ \gamma|_{\mathrm{op}}|\nabla_\tau u \circ \gamma|\right) \leq \left(\sup_{\partial E}|\nabla^2 u|_{\mathrm{op}} + \frac{\|\nabla_\tau u\|_{L^\infty(\partial E)}}{r}\right).$$

Thus, by observing $\|f\|_{L^\infty(\mathbb{R})} \leq \|u\|_{L^\infty(\partial E)}$, the claim follows from (2.37). □

3. Definition of the Flat Flow and the First Regularity Estimates

Let us begin by recalling the definition of the minimizing movements scheme and the flat flow solution of (1.1) from [43]. Assume that $E_0 \subset \mathbb{R}^{n+1}$ is a bounded set of finite perimeter. For given a time step $h \in \mathbb{R}_+$ we construct a parametrized family $(E_t^h)_{t\geq 0}^\infty$ of sets of finite perimeter by an iterative minimizing procedure called minimizing movements, where

$$\begin{aligned} &E_t^h = E_0 \ \text{ for every } 0 \leq t < h \text{ and} \\ &E_t^h = E_{h\lfloor t/h\rfloor}^h \text{ is a minimizer of the functional } \mathcal{F}_h(\,\cdot\,, E_{t-h}^h) \text{ for every } t \geq h. \end{aligned} \tag{3.1}$$

Here for a generic bounded set of finite perimeter $E \subset \mathbb{R}^{n+1}$ the functional $\mathcal{F}_h(\cdot\, E)$, in the class of the bounded set of finite perimeter, is defined as

$$\mathcal{F}_h(F, E) = P(F) + \frac{1}{h}\int_F d_E \,\mathrm{d}x + \frac{1}{\sqrt{h}}\big||F| - m_0\big|, \tag{3.2}$$

for $m_0 = |E_0|$. We call the family $(E_t^h)_{t\geq 0}^\infty$, defined in (3.1), *an approximative flat flow solution* of (1.1) starting from E_0. We note that there is always a minimizer for (3.2) but it might not be unique. By [43] we know that there is a subsequence of approximative flat flows $(E_t^{h_l})_{t\geq 0}$ which converges to a parametrized family $(E_t)_{t\geq 0}$ for a.e. t in the L^1-sense, where for every $t > 0$ the set E_t is a set of finite perimeter with $|E_t| = |E_0|$. Any such limit is called a flat flow solution of (1.1) starting from E_0.

Let us turn our focus back on a generic minimizer of (3.2), where we assume that $|E| = m_0$. We then simply denote any minimizer for $\mathcal{F}_h(\,\cdot\, E)$ by $E^h_{\min}$. One has to be careful in the definition of the functional in (3.2), since the sets of finite perimeter are only defined up to measure zero. We avoid this issue by modifying a set of finite perimeter in a $L^1(\mathbb{R}^{n+1})$ -negligible set and choose as in [37, Rmk 15.3] a representative which topological boundary agrees with the closure of its measure theoretical boundary. Thus, we always use the convention $\partial F = \overline{\partial^* F}$ for the initial set and the minimizers. We also remark that if E is empty, then we use the convention $d_E = \infty$ everywhere to ensure that $E^h_{\min}$ is empty too.

Next, we recall some basic properties regarding the minimizers. First, it is easy to conclude $P(E^h_{\min}) \leq P(E)$. Moreover, $E^h_{\min}$ satisfies the distance property

$$\sup_{E^h_{\min}\Delta E} \ |d_E| \leq \gamma_n \sqrt{h} \tag{3.3}$$

for a dimensional constant $\gamma_n \in \mathbb{R}_+$, see [43, Prop 3.2]. Second, $E^h_{\min}$ has a generalized mean curvature satisfying the Euler-Lagrange equation

$$\frac{d_E}{h} = -H_{E^h_{\min}} + \lambda^h \tag{3.4}$$

in the distributional sense (2.7) on $\partial^* E^h_{\min}$, where the Lagrange multiplier satisfies $|\lambda^h| = 1/\sqrt{h}$ in the case $|E^h_{\min}| \neq m_0$, see [43, Lemma 3.7]. Third, it is easy to see that $E^h_{\min}$ is always a so called (Λ, r) *-minimizer* with suitable $\Lambda, r \in \mathbb{R}_+$ satisfying $\Lambda r \leq 1$ (see [37] for the definition). Thus, by the standard regularity theory [37, Thm 26.5 and Thm 28.1] the reduced boundary $\partial^* E^h_{\min}$ is relatively open in $\partial E^h_{\min}$ and an embedded $C^{1,\alpha}$-regular hypersurface with any $0 < \alpha < 1/2$, and the Hausdorff dimension of the singular part $\partial E^h_{\min} \setminus \partial^* E^h_{\min}$ is at most $n - 7$. Thus, by standard Schauder estimates one may show that $\partial^* E^h_{\min}$ is in fact $C^{2,\alpha}$-regular and (3.4) holds in the classical sense on $\partial^* E^h_{\min}$. Consequently, we may always consider $E^h_{\min}$ as an open set.

We may improve the distance estimate (3.3) as well as regularity properties of $E^h_{\min}$, if we impose more regularity on E. We divide our approach into two steps. The first result states that if E is bounded and satisfies UBC with radius $r_0 > 0$ and h is sufficiently small, then the left hand side of (3.3) is bounded linearly in h, the Lagrange multiplier λ^h is bounded, the generalized mean curvature $H_{E^h_{\min}}$ is bounded in the L^∞-sense and $E^h_{\min}$ has the volume m_0.

Proposition 3.1. *Assume $E \subset \mathbb{R}^{n+1}$ is an open and bounded set of volume m_0 which satisfies UBC with radius r_0. There are positive numbers $h_0 = h_0(n, m_0, r_0)$ and $C_0 = C_0(n, m_0, r_0)$ and a dimensional constant $C_n \in \mathbb{R}_+$ such that if $h \leq h_0$, then*

$$\sup_{E^h_{\min}\Delta E} \ |d_E| \leq \frac{C_n}{r_0} h, \quad \|H_{E^h_{\min}}\|_{L^\infty} + |\lambda^h| \leq C_0 \quad \textit{and} \quad |E^h_{\min}| = m_0.$$

Proof. We prove first part of the claim, i.e., the distance estimate. If $|E^h_{\min} \Delta E| = 0$, then it follows from the openness of $E^h_{\min}$ and E as well as the property $\partial E^h_{\min} = \overline{\partial^* E^h_{\min}}$ and $\partial E = \overline{\partial^* E}$ that $E^h_{\min} \Delta E = \emptyset$ and there is nothing to prove. Thus, we may assume that $|E^h_{\min} \Delta E| > 0$ and further set that

$$d_+ = \sup_{E^h_{\min} \Delta E} d_E \quad \text{and} \quad d_- = \inf_{E^h_{\min} \Delta E} d_E.$$

To conclude the first part of the claim, we show, under the assumption $|E^h_{\min} \Delta E| > 0$, the validity of the implication

$$\sqrt{h} \leq \frac{r_0}{\max\{n+1, 8\gamma_n\}} \quad \Longrightarrow \quad d_- < 0 < d_+ \text{ and } d_+ - d_- \leq \frac{4(n+1)}{r_0} h. \tag{3.5}$$

Thus, (3.5) and our earlier observation gives us the implication

$$\sqrt{h} \leq \frac{r_0}{\max\{n+1, 8\gamma_n\}} \quad \Longrightarrow \quad \sup_{E^h_{\min} \Delta E} |d_E| \leq \frac{4(n+1)}{r_0} h. \tag{3.6}$$

To prove (3.5), we assume by contradiction that $d_- \geq 0$ which implies $E \subset E^h_{\min}$ due to the openness of E and, hence, $|E^h_{\min} \backslash E| = |E^h_{\min} \Delta E| > 0$. Using (2.35) with $r = r_0$, the previous observation, $|E| = m_0$, and the assumption on h yields

$$\begin{aligned}
\mathcal{F}_h(E, E) &\leq P(E^h_{\min}) + \frac{1}{h} \int_E d_E \,\mathrm{d}x + \frac{n+1}{r_0} |E^h_{\min} \setminus E| \\
&< P(E^h_{\min}) + \frac{1}{h} \int_{E^h_{\min}} d_E \,\mathrm{d}x + \frac{n+1}{r_0} |E^h_{\min} \setminus E| \\
&= \mathcal{F}_h(E^h_{\min}, E) + \left(\frac{n+1}{r_0} - \frac{1}{\sqrt{h}} \right) |E^h_{\min} \setminus E| \leq \mathcal{F}_h(E^h_{\min}, E),
\end{aligned}$$

contradicting the minimality of $E^h_{\min}$ and, hence, $d_- < 0$. Similarly we obtain $d_+ > 0$.

On the other hand, $\sqrt{h} \leq r_0/(8\gamma_n)$ implies via (3.3) that $E^h_{\min} \Delta E \subset\subset \mathcal{N}_{r_0/4}(\partial E)$. In particular, $-r_0/2 < d_- < 0 < d_+ < r_0/2$ and for every $t \in (d_-, d_+)$ the sublevel set $E_t = \{x : d_E(x) < t\}$ satisfies UBC with $r_0/2$ and $|E^h_{\min} \backslash E_t|, |E_t \backslash E^h_{\min}| > 0$. By using a suitable continuity argument, we infer from the previous that for every $t < d_+$, sufficiently close to d_+, there is $\tilde{t} \in (d_-, r_+)$ such that $|E^h_{\min} \backslash E_t| = |E_{\tilde{t}} \backslash E^h_{\min}| > 0$ and $\tilde{t} \to d_-$ as $t \to d_+$. For such a pair $(t, \tilde{t})$ we set

$$F = (E_t \cap E^h_{\min}) \cup E_{\tilde{t}}.$$

Clearly, F is a bounded set of finite perimeter and $|F| = |E^h_{\min}|$. Thus, using F as a competitor against $E^h_{\min}$ with respect to $\mathcal{F}_h(\,\cdot\, E)$ we obtain

$$P(E^h_{\min}) \leq P(F) + \frac{1}{h} \int_{E_{\tilde{t}} \backslash E^h_{\min}} d_E \,\mathrm{d}x - \frac{1}{h} \int_{E^h_{\min} \backslash E_t} d_E \,\mathrm{d}x$$

$$\begin{aligned}
&\le P(F) + \frac{\tilde{t}}{h}|E_{\tilde{t}} \setminus E^h_{\min}| - \frac{t}{h}|E^h_{\min} \setminus E_t| \\
&= P(F) + \frac{\tilde{t} - t}{h}|E_{\tilde{t}} \setminus E^h_{\min}|. \qquad (3.7)
\end{aligned}$$

In turn, applying Lemma 2.10 to E_t and $E_{\tilde{t}}$ gives us

$$\begin{aligned}
P(F) &= P((E_t \cap E^h_{\min}) \cup E_{\tilde{t}}) \\
&\le P(E_t \cap E^h_{\min}) + \frac{n+1}{r_0/2}|E_{\tilde{t}} \setminus E^h_{\min}| \\
&\le P(E^h_{\min}) + \frac{n+1}{r_0/2}|E^h_{\min} \setminus E_t| + \frac{n+1}{r_0/2}|E_{\tilde{t}} \setminus E^h_{\min}| \\
&= P(E^h_{\min}) + \frac{4(n+1)}{r_0}|E_{\tilde{t}} \setminus E^h_{\min}|. \qquad (3.8)
\end{aligned}$$

We combine (3.7) and (3.8) and recall $|E_{\tilde{t}} \setminus E^h_{\min}| > 0$ to observe that

$$\frac{t - \tilde{t}}{h} \le \frac{4(n+1)}{r_0}.$$

Thus, by letting $t \to d_+$, we obtain the second estimate in (3.5).

To prove the second part of the claim, we denote by C a generic positive constant which may change its value from the line to line but depends only on n, m_0 and r_0. We fix any connected component E^i of E. By Lemma 2.10 and (2.36) we have $\mathrm{diam}(E^i) \le C$ and $P(E) \le C$. If E^j is a connected component of E distinct to E^i, then UBC with r_0 guarantees $\mathrm{dist}(E^i, E^j) \ge r_0$. Assuming that $\sqrt{h} \le r_0/\max\{n+1, 8\gamma_n\}$ we have, by (3.3), (3.6), openness of $E^h_{\min}$ and $\overline{\partial^* E^h_{\min}} = \partial E^h_{\min}$, that $E^h_{\min} \Delta E \subset\subset \mathcal{N}_{r_0/4}(\partial E)$ and $|d_E/h| \le 4(n+1)/r_0$ on $\partial^* E^h_{\min}$. Again, we infer from the previous observations that for the intersection $\tilde{E}^i = E^h_{\min} \cap (E^i + B_{r_0/4})$ it holds $\partial^* \tilde{E}^i = \partial^* E^h_{\min} \cap (E^i + B_{r_0/4})$, $H_{\tilde{E}^i} = H_{E^h_{\min}}|_{\partial^* \tilde{E}^i}$, $\mathrm{diam}(\tilde{E}^i) \le C + r_0/2 \le C$ and $|\tilde{E}^i| \ge |B_{r_0/2}|$. Using the divergence theorems and the Euler-Lagrange equation (3.4), which holds in the sense of (2.7) on $\partial^* \tilde{E}^i$, we compute that

$$\begin{aligned}
\lambda^h(n+1)|\tilde{E}^i| = \int_{\partial^* E^i} \lambda^h (\mathrm{id} \cdot \nu_{\tilde{E}^i})\, \mathrm{d}\mathcal{H}^n &= \int_{\partial^* \tilde{E}^i} \left(H_{\tilde{E}^i} + \frac{d_E}{h}\right)(\mathrm{id} \cdot \nu_{\tilde{E}^i})\, \mathrm{d}\mathcal{H}^n \\
&= nP(\tilde{E}^i) + \int_{\partial^* \tilde{E}^i} \frac{d_E}{h}\,(\mathrm{id} \cdot \nu_{\tilde{E}^i})\, \mathrm{d}\mathcal{H}^n.
\end{aligned}$$

By translating the coordinates, we may assume $0 \in \tilde{E}^i$ so $|\mathrm{id}| \le \mathrm{diam}(\tilde{E}^i) \le C$ on $\partial^* E^i$. Since we also have $P(\tilde{E}^i) \le P(E^h_{\min}) \le P(E) \le C$, $|\tilde{E}^i| \ge |B_{r_0/2}|$ and $|d_E/h| \le 4(n+1)/r_0$ on $\partial^* \tilde{E}^i$, we infer from the previous computation $|\lambda^h| \le C_0$ for $C_0 = C_0(n, m_0, r_0) \in \mathbb{R}_+$. Therefore, using the Euler-Lagrange equation (3.4) and the first estimate again we have, by possibly increasing C_0, that $\|H_{E^h_{\min}}\|_{L^\infty(\partial^* E^h_{\min})} + |\lambda^h| \le C_0$. Finally, if $|E^h_{\min}| \ne m_0$, then $|\lambda^h| = 1/\sqrt{h}$. Thus, assuming $h \le (2C_0)^{-2}$ excludes this possibility and hence it must hold $|E^h_{\min}| = m_0$. □

Proposition 3.1, allows us to deduce, via Allard's regularity theorem, that the singular set of minimizer is in fact empty. Further, standard Schauder estimates gives us a quantitative, albeit non-sharp, UBC for a minimizer.

Lemma 3.2. *Assume $E \subset \mathbb{R}^{n+1}$ is an open and bounded set of volume m_0 which satisfies UBC with radius r_0. There are positive numbers $h_0 = h_0(n, m_0, r_0)$ and $c_0 = c_0(n, m_0, r_0)$ such that if $h \le h_0$, then $\partial E \backslash \partial E^* = \varnothing$, $E^h_{\min}$ is $C^{3,\alpha}$-regular with any $0 < \alpha < 1$ and $E^h_{\min}$ satisfies UBC with radius $c_0 h^{1/3}$. In particular, (3.4) is satisfied in the classical sense on $\partial E^h_{\min}$. Moreover, if E is C^k-regular, with $k \ge 2$, then $E^h_{\min}$ is C^{k+2}-regular.*

Proof. We divide the proof into two steps. Recall that we may assume $E^h_{\min}$ to be open. In the proof, C denotes a generic positive constant which may change its value from line to line but it depends only on n, m_0 and r_0.

Step 1: By using Allard's regularity theorem we show that the topological boundary $\partial E^h_{\min}$ agrees with the reduced boundary $\partial^* E^h_{\min}$ when h is sufficiently small. To be more precise, we show that there exist positive numbers $\rho = \rho(n, m_0, r_0)$ and $h_1 = h_1(n, m_0, r_0, \rho)$ such that if $h \le h_1$ and $x \in \partial E^h_{\min}$, then, by possibly rotating the coordinates, there is a function $f \in C^{1,1/3}\left(B^n_\rho(x')\right)$ such that

$$\mathbf{C}(x, \rho, 2\rho) \cap E^h_{\min} = \{y \in \mathbf{C}(x, \rho, 2\rho) : y_{n+1} < f(y)\} \tag{3.9}$$

and f satisfies the estimates

$$\|\nabla f\|_{L^\infty(B^n_\rho(x'))} \le 1 \quad \text{and} \quad \|\nabla f\|_{C^{0,\frac{1}{3}}(B^n_\rho(x'))} \le C. \tag{3.10}$$

In particular, (3.9) implies that $\partial^* E = \partial E$ and hence, by our earlier discussion, we conclude that $E^h_{\min}$ is $C^{2,\alpha}$-regular with any $0 < \alpha < 1/2$. We may assume that h_1 is chosen so small that via Proposition 3.1 the boundary $\partial E^h_{\min}$ is contained in $\mathcal{N}_{r_0/2}(\partial E)$. Since $d_E \in C^{1,1}(\mathcal{N}_{r_0/2}(\partial E))$, then recalling the Euler-Lagrange equation (3.4) we may write the generalized mean curvature of $E^h_{\min}$ as a restriction of a $C^{1,1}$-function to $\partial E^h_{\min}$. Therefore, by using standard Schauder estimates, one may show that $E^h_{\min}$ is actually $C^{3,\alpha}$-regular with any $0 < \alpha < 1$. Also, the same method gives us $C^{k+2,\alpha}$-regularity for any $k \ge 2$, if E is already known to be $C^{k,\alpha}$-regular. This is well-known procedure and we leave it to the reader.

The claim of Step 1 follows essentially from [46, Thm 2.5.2], if we prove that for every $x \in \partial E^h_{\min}$ and $\varepsilon \in \mathbb{R}_+$ there are positive numbers $\rho = \rho(n, m_0, r_0, \varepsilon)$ and $\tilde{h} = \tilde{h}(n, m_0, r_0, \rho, \varepsilon)$ such that if $h \le \tilde{h}$, then

$$\frac{\mathcal{H}^n(B_\rho(x) \cap \partial^* E^h_{\min})}{|B^n_\rho|} \le 1 + \varepsilon \quad \text{and} \tag{3.11}$$

$$\rho^{\frac{1}{3}} \left(\int_{B_\rho(x) \cap \partial^* E^h_{\min}} |H_{E^h_{\min}}|^{\frac{3n}{2}} \, \mathrm{d}\mathcal{H}^n \right)^{\frac{2}{3n}} \le \varepsilon. \tag{3.12}$$

We fix $\varepsilon > 0$ and initially assume $h \le h_0$, where h_0 is from Proposition 3.1. It follows from Proposition 3.1 and the fact $\partial E^h_{\min} = \overline{\partial^* E^h_{\min}}$ that

$$(\overline{E^h_{\min}} \cup \overline{E}) \setminus (E^h_{\min} \cap E) \subset \mathcal{N}_{Ch}(\partial E). \tag{3.13}$$

Thus, we may assume that $(\overline{E^h_{\min}} \cup \overline{E}) \setminus (E^h_{\min} \cap E) \subset \mathcal{N}_{r_0/2}(\partial E)$ where the projection $\pi_{\partial E}$ is well-defined. Proposition 3.1 also gives us $|E^h_{\min}| = m_0$. Next, we fix $x \in \partial E^h_{\min}$. Without loss of generality, we may assume $\pi_{\partial E}(x) = 0$ and $\nu_E(0) = e_{n+1}$. Then it follows from Proposition 2.7 that there is $g \in C^{1,1}(B^n_{r_0/2})$ such that $|g(y')| < |y'|^2/r_0$, $|\nabla g(y')| < 2|y'|/r_0$ for every $y' \in B^n_{r_0/2}$ and

$$\mathbf{C}(0, r_0/2, r_0/2) \cap E = \{y \in \mathbf{C}(0, r_0/2, r_0/2) : y_{n+1} < g(y')\}.$$

We have, for every $0 < \rho < r_0/4$, a density bound

$$P(E; \mathbf{C}(0, \rho, r_0/2)) = \int_{B^n_\rho} \sqrt{1 + |\nabla g|^2}\, \mathrm{d}y' \le (1 + C\rho^2)|B^n_\rho|. \tag{3.14}$$

Suppose that $y \in \mathbf{C}(0, \rho, r_0/2) \cap ((\overline{E^h_{\min}} \cup \overline{E}) \setminus (E^h_{\min} \cap E))$ for $0 < \rho < r_0/4$. Recalling (3.13), we may assume that $\pi_{\partial E}(y) \in \mathbf{C}(0, r_0/2, r_0/2)$ and since $|\nabla g| \le C$ in $B^n_{r_0/2}$ we estimate that

$$\begin{aligned} |y_{n+1} - g(y')| &\le |y - \pi_{\partial E}(y)| + |\pi_{\partial E}(y) - (y', g(y'))| \\ &\le |y - \pi_{\partial E}(y)| + C|(\pi_{\partial E}(y))' - y'| \le Ch. \end{aligned}$$

It follows then from Fubini's theorem that

$$\left|\mathbf{C}(0, \rho, r_0/2) \cap \left((\overline{E^h_{\min}} \cup \overline{E}) \setminus (E^h_{\min} \cap E)\right)\right| \le C\rho^n h \quad \text{and} \tag{3.15}$$

$$\mathcal{H}^n\left(\partial\mathbf{C}(0, \rho, r_0/2) \cap \left((\overline{E^h_{\min}} \cup \overline{E}) \setminus (E^h_{\min} \cap E)\right)\right) \le C\rho^{n-1} h \tag{3.16}$$

for $0 < \rho < r_0/4$. We define for such ρ a comparison set F_ρ by setting

$$F_\rho = (E^h_{\min} \setminus \mathbf{C}(0, \rho, r_0/2)) \cup (E \cap \mathbf{C}(0, \rho, r_0/2)),$$

and we make the following technical observations: first, since $E^h_{\min} \cap E$ is open and contained in F_ρ, then $\mathcal{H}^n(\partial^* F_\rho \cap (E^h_{\min} \cap E)) = 0$. Second, $\partial^* F_\rho \subset \overline{E^h_{\min}} \cup \overline{E}$. With help of these, (3.14) and (3.16) we estimate

$$\begin{aligned} P(F_\rho) &= P(F_\rho; \mathbf{C}(0, \rho, r_0/2)) + P(F_\rho; \partial\mathbf{C}(0, \rho, r_0/2)) \\ &\quad + P(F_\rho; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) \\ &= P(E; \mathbf{C}(0, \rho, r_0/2)) + \mathcal{H}^n(\partial^* F_\rho \cap \partial\mathbf{C}(0, \rho, r_0/2)) \\ &\quad + P(E^h_{\min}; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) \\ &\le P(E; \mathbf{C}(0, \rho, r_0/2)) + P(E^h_{\min}; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) \\ &\quad + \mathcal{H}^n\left(\partial\mathbf{C}(0, \rho, r_0/2) \cap \left((\overline{E^h_{\min}} \cup \overline{E}) \setminus (E^h_{\min} \cap E)\right)\right) \end{aligned}$$

$$\leq (1 + C\rho^2)|B^n_\rho| + P(E^h_{\min}; \mathbb{R}^{n+1} \setminus \overline{\mathbf{C}(0, \rho, r_0/2)}) + C\rho^{n-1}h.$$

Thus, the inequality $\mathcal{F}_h(E^h_{\min}, E) \leq \mathcal{F}_h(F_\rho, E)$, (3.13), (3.15), $|E^h_{\min}| = m_0$ and the definition of F_ρ yield

$$\begin{aligned}
&P(E^h_{\min}; \mathbf{C}(0, \rho, r_0/2)) + \frac{1}{h}\int_{\mathbf{C}(0,\rho,r_0/2))\cap(E^h_{\min}\Delta E)} |d_E|\,\mathrm{d}x \\
&\quad\leq (1 + C\rho^2)|B^n_\rho|_n + \frac{1}{\sqrt{h}}||F_\rho| - m_0| + C\rho^{n-1}h \\
&\quad\leq (1 + C\rho^2)|B^n_\rho| + \frac{1}{\sqrt{h}}|\mathbf{C}(0, \rho, r_0/2) \cap (E^h_{\min}\Delta E)| + C\rho^{n-1}h \\
&\quad\leq (1 + C\rho^2)|B^n_\rho| + C(\rho^n\sqrt{h} + \rho^{n-1}h).
\end{aligned}$$

Recall that for the fixed point $x \in \partial E^h_{\min}$ it holds $x = d_E(x)e_{n+1}$ with $|d_E(x)| \leq Ch$. Thus we may assume $B_\rho(x) \subset \mathbf{C}(0, \rho, r_0/2)$ for $0 < \rho < r_0/4$. Hence, the above estimate yields

$$P(E^h_{\min}; B_\rho(x)) \leq (1 + C\rho^2)|B^n_\rho| + C(\rho^n\sqrt{h} + \rho^{n-1}h). \tag{3.17}$$

Moreover, it holds $\|H_{E^h_{\min}}\|_{L^\infty(\partial^* E^h_{\min})} \leq C$ by Proposition 3.1, $P(E^h_{\min}) \leq P(E)$ and $P(E) \leq C$ by Lemma 2.10. Therefore,

$$\rho^{\frac{1}{3}}\left(\int_{B_\rho(x)\cap\partial^* E^h_{\min}} |H_{E^h_{\min}}|^{\frac{3n}{2}}\,\mathrm{d}\mathcal{H}^n\right)^{\frac{2}{3n}} \leq C\rho^{\frac{1}{3}}.$$

We infer from the previous estimate and (3.17) the existence of numbers $\tilde{h}$ and ρ satisfying (3.11) and (3.12).

Step 2: We assume that $h \leq h_1$ and fix $x \in \partial E^h_{\min}$. We may assume that $x = 0$ and $\nu_{E^h_{\min}}(0) = e_{n+1}$. According to Step 1, up to a possible rotation of the coordinates, there is $f \in C^3(B^n_{\rho_1}(x'))$ with $f(0) = \nabla f(0) = 0$ satisfying (3.9) and (3.10). We use Schauder estimate in a quantitative manner to prove there is a positive $h_0 = h_0(n, m_0, r_0) \leq h_1$ such that $h \leq h_0$ implies

$$\|\nabla^2 f\|_{L^\infty(B^n_{\rho/2})} \leq Ch^{-\frac{1}{3}}. \tag{3.18}$$

Once we have proven (3.18) then the claim that $E^h_{\min}$ satisfies UBC with radius $c_0h^{1/3}$ follows in a straightforward manner as we discussed in Remark 2.8.

Thus, we are left to prove (3.18). We may write $H_{E^h_{\min}}$ in local coordinates as the mean curvature of the subgraph $\{(y', y_{n+1} : y' \in B^n_\rho,\ y_{n+1} < f(y')\}$, that is,

$$H_{E^h_{\min}}(y', f(y')) = -\operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right)(y') = -\mathrm{Tr}\left(\mathcal{A}(y')\nabla^2 f(y')\right). \tag{3.19}$$

It follows from (3.10) that $\mathcal{A}$ is uniformly elliptic and bounded in the $C^{0,1/3}$-sense. To be more precise, we have

$$\inf_{y'\in B^n_\rho}\min_{\xi\in\partial B^n_1}\mathcal{A}(y')\xi\cdot\xi\geq 1/C \quad\text{and}\quad \max_{ij}\|[\mathcal{A}]_{ij}\|_{C^{0,\frac{1}{3}}(B^n_\rho)}\leq C.$$

Thus, by using standard Schauder interior estimate [24], (3.10) and (3.19), we obtain

$$\begin{aligned}\|\nabla^2 f\|_{C^{0,\frac{1}{3}}(B^n_{\rho/2})} &\leq C\left(\|u\|_{C^{0,\frac{1}{3}}(B^n_\rho)}+\|f\|_{L^\infty(B^n_\rho)}\right)\\ &\leq C\left(\|u\|_{C^{0,\frac{1}{3}}(B^n_\rho)}+1\right),\end{aligned}\tag{3.20}$$

where $u : B^n_{\rho_1}\to\mathbb{R}^n$ is given by $u(y')=H_{E^h_{\min}}(y',f(y'))$. We may assume h is chosen sufficiently small so that via Proposition 3.1 we have $\|u\|_{L^\infty(B^n_\rho)}\leq C$. Again, (3.10) implies $|\nabla u(y')|\leq C|\nabla_\tau H_{E^h_{\min}}(y',f(y'))|$ for every $y'\in B^n_\rho$. On the other hand, by (tangentially) differentiating the Euler-Lagrange equality (3.4) we obtain $|\nabla_\tau H_{E^h_{\min}}(y',f(y'))|\leq 1/h$ for every $y'\in B^n_\rho$. Hence, $\|\nabla u\|_{L^\infty(B^n_\rho)}\leq C/h$ and since $\|u\|_{L^\infty(B^n_\rho)}\leq C$, assuming $h\leq 1$ yields $\|u\|_{C^1(B^n_\rho)}\leq C/h$. Again, Lemma 2.1 yields $\|u\|_{C^{0,1/3}(B^n_\rho)}\leq Ch^{-1/3}$ and hence, by recalling (3.20), we conclude the existence of $h_0=h_0(n,m_0,r_0)$ satisfying (3.18) for all $h\leq h_0$. □

Remark 3.3. We may replace the exponent $1/3$ with a generic $0<\alpha<1$ in the proof of Lemma 3.2. Then, naturally, h_0 and c_0 also depend on α. UBC with radius r_0 for E and UBC with radius $c_0h^{1/3}$ for $E^h_{\min}$ imply together with the distance estimate of Proposition 3.1 and (2.22) that there is $h_0=h_0(n,m_0,r_0)$ such that if $h\leq h_0$, then $\nabla d_E\cdot\nu_{E^h_{\min}}>0$ on $\partial E^h_{\min}$ and the projection $\pi_{\partial E}$ is injective on $\partial E^h_{\min}$.

4. Uniform Ball Condition for Short-Time

In this section, we adopt the two-point function method to prove that if the initial set E_0 satisfies UBC with radius r_0, then there are positive numbers h_0 and T_0 such that

$$h\leq h_0\implies E^h_t \text{ satisfies UBC with radius } r_0/2 \text{ for } 0\leq t\leq T_0,\tag{4.1}$$

where the approximative flow $(E^h_t)_{t\geq 0}$ starting from E_0 is defined as in (3.1). For more precise statement, see Theorem 4.7 at the end of the section. As we have seen in Lemma 3.2, UBC for an initial set is crucial, as it guarantees that the corresponding minimizer of the energy (3.2) has improved regularity and an initial quantitative bound on UBC although the latter depends on h. In this section, we improve the previous non-sharp estimate on UBC for the minimizer by showing the minimizer satisfies almost the same UBC as the initial set.

The original idea of the two-point function goes back to [27], where it is used to study the regularity of the classical solution to the mean curvature flow. We refer to [10] for a comprehensive overview of the topic and mention also the works [4, 11, 18] which have inspired us. Here we will show that the method can be applied to the approximative flat flow at the level of discrete time scale. We will assume that the approximative flat flow is related to the volume preserving mean curvature flow but the arguments hold with essentially no modifications also in the case of the mean curvature flow.

4.1. Two-Point Function Method

The main idea is to double the variables and, given a set $E \subset \mathbb{R}^{n+1}$ satisfying UBC, to study the function S_E defined for $(x, y) \in \partial E \times \partial E$ with $x \neq y$ as

$$S_E(x, y) := \frac{(x-y) \cdot \nu_E(x)}{|x-y|^2}. \tag{4.2}$$

It is known, but we will include the proof below, that the maximum value of $|S_E|$ is explicitly related to the maximal UBC for E. In other words, doubling the variables allows us to quantify the maximal UBC via the function S_E. It is interesting that the idea of doubling the variables is also used in [29] to study regularity of solutions of nonlinear PDEs.

For the next lemma we note that if a set E satisfies UBC with radius r, then it satisfies UBC with every $0 < \rho < r$. We define r_E to be the supremum of such radii and recalling our previous discussion we may write this as

$$r_E = \sup\{r > 0 : d_E \text{ is differentiable in } \mathcal{N}_r(\partial E)\}. \tag{4.3}$$

Note that $r_E > 0$. We use the abbreviation $\|S_E\|_{L^\infty} := \sup\{|S_E(x, y)| : x, y \in \partial E,\ x \neq y\}$.

Lemma 4.1. *Let $E \subset \mathbb{R}^{n+1}$ be an open and bounded set satisfying UBC. Then it holds that*

$$2\|S_E\|_{L^\infty} = \frac{1}{r_E} \text{ and } \frac{|\nu(x) - \nu(y)|}{|x-y|} \leq 2\|S_E\|_{L^\infty} \text{ for every } x, y \in \partial E \text{ with } x \neq y,$$

where r_E is defined in (4.3). In the case E is C^2-regular, we also have $|H_E|, |B_E| \leq 2n\|S_E\|_{L^\infty}$ on ∂E.

Proof. Let us first show $2\|S_E\|_{L^\infty} \geq 1/r_E$. First of all, we infer from the boundedness of E that $r_E < \infty$. Since E does not satisfy UBC with given a radius $r \in (r_E, \infty)$, there is $z \in \mathcal{N}_r(\partial E)$ such that d_E is not differentiable at z. Hence, there are distinct points $x, y \in \partial E$ such that $|z - x| = |d_E(z)| = |z - y|$. Without loss of generality, we may assume $z = 0$ which implies $|x| = |y| < r$ and $\nu_E(x) = \pm x/|x|$. Thus,

$$\begin{aligned} |S_E(x, y)| &= \left| \frac{\langle x/|x|, x-y \rangle}{|x-y|^2} \right| = \frac{1}{|x|} \left| \frac{|x|^2 - \langle x, y \rangle}{|x-y|^2} \right| \\ &= \frac{1}{2|x|} \left| \frac{|x|^2 - 2\langle x, y \rangle + |y|^2}{|x-y|^2} \right| \end{aligned}$$

$$= \frac{1}{2|d_E(x)|} > \frac{1}{2r}$$

and we conclude the inequality $2\|S_E\|_{L^\infty} \geq 1/r_E$.

To conclude the opposite estimate, we choose $0 < r < r_E$. Let $x, y \in \partial E$ be distinct points. Since E satisfies UBC with r, we have $|d_E(x \pm r\nu_E(x))| = r$ and, hence,

$$r^2 \leq |x \pm r\nu_E(x) - y|^2 = r^2 \pm 2r\langle \nu_E(x), x - y\rangle + |x - y|^2.$$

By subtracting and dividing terms we obtain $\pm 2S_E(x, y) \leq 1/r$. We let $r \to r_E$ to obtain $2\|S_E\|_{L^\infty} \leq 1/r_E$. Thus, $2\|S_E\|_{L^\infty} = 1/r_E$. The rest of the claim is a direct consequence of the previous identity, (2.30) and Proposition 2.7. □

An obvious consequence of Lemma 4.1 is that for every open and bounded set $E \subset \mathbb{R}^{n+1}$ it holds

$$\|S_E\|_{L^\infty} \geq c_0 \tag{4.4}$$

for a positive constant $c_0 = c_0(n, |E|)$.

We will also use the regularized version of S_E, which we define for any $\varepsilon \in \mathbb{R}_+$ as $S_{E,\varepsilon} : \partial E \times \partial E \to \mathbb{R}$,

$$S_{E,\varepsilon}(x, y) := \frac{(x - y) \cdot \nu_E(x)}{|x - y|^2 + \varepsilon}. \tag{4.5}$$

As in the case of S_E, we use the abbreviation $\|S_{E,\epsilon}\|_{L^\infty} = \max\{|S_{E,\varepsilon}(x, y)| : (x, y) \in \partial E \times \partial E\}$. The idea behind considering $S_{E,\varepsilon}$ instead of S_E is that, on the one hand, $S_{E,\epsilon} \to S$ pointwise in $\partial E \times \partial E \setminus \{(x, x) : x \in \partial E\}$ as ϵ tends to zero (in particular, $\|S_{E,\epsilon}\|_{L^\infty} \uparrow \|S_E\|_{L^\infty}$) and, on the other hand, we may differentiate $S_{E,\epsilon}$ on the product $\partial E \times \partial E$ provided that E is sufficiently regular. The following calculations are similar to [4,18] but we give them in order to be self-consistent.

Let us first differentiate $S_{E,\varepsilon}$ in the case E is C^2-regular. In the computations, the notations ∇_τ^x and ∇_τ^y stand for the tangential differentiation along ∂E with respect to x and y -variables respectively. Recalling the basic identities (2.4) as well as observing $B_E\nu_E = 0$ and $\nabla_\tau \mathrm{id} = P_{\partial E}$ on ∂E we compute

$$\begin{aligned} \nabla_\tau^x S_{E,\varepsilon}(x, y) &= \frac{\nabla_\tau^x\big((x - y) \cdot \nu_E(x)\big)}{|x - y|^2 + \varepsilon} - \frac{(x - y) \cdot \nu_E(x)}{(|x - y|^2 + \varepsilon)^2}\nabla_\tau^x |x - y|^2 \\ &= \frac{B_E(x)(x - y) - 2S_{E,\varepsilon}(x, y)\, P_{\partial E}(x)(x - y)}{|x - y|^2 + \varepsilon}. \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \nabla_\tau^y S_{E,\varepsilon}(x, y) &= \frac{\nabla_\tau^y\big((x - y) \cdot \nu_E(x)\big)}{|x - y|^2 + \varepsilon} - \frac{(x - y) \cdot \nu_E(x)}{(|x - y|^2 + \varepsilon)^2}\nabla_\tau^y |x - y|^2 \\ &= \frac{P_{\partial E}(y)\big(-\nu_E(x) + 2S_{E,\varepsilon}(x, y)(x - y)\big)}{|x - y|^2 + \varepsilon} \end{aligned} \tag{4.7}$$

for every $(x, y) \in \partial E \times \partial E$. We immediately obtain the following identities at critical points:

Lemma 4.2. *Let* $E \subset \mathbb{R}^{n+1}$ *be a bounded and* C^2*-regular set. Assume* $(x, y) \in \partial E \times \partial E$ *is a local maximum or a local minimum point of* $S_{E,\varepsilon}$ *defined in* (4.5)*. Then it holds that*

$$B_E(x)(x-y) = 2S_{E,\varepsilon}(x,y)P_{\partial E}(x)(x-y) \ \text{ and} \tag{4.8}$$

$$P_{\partial E}(y)\nu_E(x) = 2S_{E,\varepsilon}(x,y)P_{\partial E}(y)(x-y). \tag{4.9}$$

Moreover, the condition $r_E > \sqrt{\varepsilon}$ *implies*

$$\nu_E(y) = \frac{\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y)}{\big(\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y)\big)\cdot \nu_E(y)}. \tag{4.10}$$

Proof. Since (x, y) is a critical point for the functions $S_{E,\varepsilon}(x, \,\cdot\,)$ and $S_{E,\varepsilon}(\,\cdot\, y)$, then the equality (4.8) follows from (4.6) and the equality (4.9) follows from (4.7). Using $P_{\partial E}(y) = I - \nu_E(y) \otimes \nu_E(y)$ and (4.9) we have

$$\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y) = \left[\big(\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y)\big)\cdot \nu_E(y)\right]\nu_E(y).$$

The equality (4.10) thus follows once we show that

$$\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y) \neq 0. \tag{4.11}$$

We argue by contradiction and assume $\nu_E(x) = 2S_{E,\varepsilon}(x,y)(x-y)$. Then it holds $S_{E,\varepsilon}(x,y) \neq 0$ and the definition of $S_{E,\varepsilon}(x,y)$ implies

$$S_{E,\varepsilon}(x,y) = \frac{(x-y)\cdot\nu_E(x)}{|x-y|^2+\varepsilon} = 2S_{E,\varepsilon}(x,y)\frac{|x-y|^2}{|x-y|^2+\varepsilon}.$$

Therefore, we have $|x-y| = \sqrt{\varepsilon}$. On the other hand, the contradiction assumption, the definition of $S_{E,\varepsilon}$ and Lemma 4.1 together yield that

$$1 = |\nu_E(x)| = 2|S_{E,\varepsilon}(x,y)|\,|x-y| = 2|S_{E,\varepsilon}(x,y)|\sqrt{\varepsilon} \le 2\|S_E\|_{L^\infty}\sqrt{\varepsilon} = \frac{\sqrt{\varepsilon}}{r_E},$$

which is impossible, by the assumption that $r_E > \sqrt{\varepsilon}$. □

If E has higher regularity and ε is sufficiently small, we may naturally extract more information at local extreme points. Indeed, if E is C^3-regular, then by maximum principle at a local maximum (minimum) point $(x, y) \in \partial E \times \partial E$ of $S_{E,\varepsilon}$ it holds that

$$\Delta_\tau^x S_{E,\varepsilon}(x,y) + 2\,\mathrm{div}_\tau^x \nabla_\tau^y S_{E,\varepsilon}(x,y) + \Delta_\tau^y S_{E,\varepsilon}(x,y) \overset{(\geq)}{\leq} 0. \tag{4.12}$$

We calculate the LHS of (4.12) in the next lemma.

Lemma 4.3. *Let* $E \subset \mathbb{R}^{n+1}$ *be a bounded and* C^3*-regular set with* $r_E > \sqrt{\varepsilon}$*. At a local maximum (minimum) point* $(x, y) \in \partial E \times \partial E$ *of* $S_{E,\varepsilon}$ *it holds that*

$$\begin{aligned}&\frac{\nabla_\tau H_E(x)\cdot(x-y)}{|x-y|^2+\varepsilon} + \frac{(\nu_E(x)\cdot\nu_E(y))\,H_E(y) - H_E(x)}{|x-y|^2+\varepsilon}\\ &\overset{(\geq)}{\leq} |B_E(x)|^2 S_{E,\varepsilon}(x,y) - 2H_E(x)S_{E,\varepsilon}^2(x,y) - 2H_E(y)S_{E,\varepsilon}(y,x)S_{E,\varepsilon}(x,y).\end{aligned}$$

Proof. First, we compute the terms on the LHS of (4.12) by taking tangential divergences of (4.6) and (4.7) with respect to x and y -variables. In the computations, we use the identities (2.5) and the fact that the gradients $\nabla_\tau^x S_{E,\varepsilon}(x,y)$ and $\nabla_\tau^y S_{E,\varepsilon}(x,y)$ vanish. Omitting all the details we obtain by straightforward calculation

$$\begin{aligned}\Delta_\tau^x S_{E,\varepsilon}(x,y) &= \operatorname{div}_\tau^x(\nabla_\tau^x S_{E,\varepsilon}(x,y))\\ &= \operatorname{div}_\tau^x\left(\frac{B_E(x)(x-y) - 2S_{E,\varepsilon}(x,y)\,P_{\partial E}(x)(x-y)}{|x-y|^2+\varepsilon}\right)\\ &= \frac{\nabla_\tau H_E(x)\cdot(x-y)}{|x-y|^2+\varepsilon} + \frac{H_E(x)}{|x-y|^2+\varepsilon} - 2S_{E,\varepsilon}(x,y)\frac{n}{|x-y|^2+\varepsilon}\\ &\qquad - |B_E|^2 S_{E,\varepsilon}(x,y) + 2S_{E,\varepsilon}^2(x,y)\,H_E(x),\end{aligned}$$

$$\begin{aligned}\Delta_\tau^y S_{E,\varepsilon}(x,y) &= \operatorname{div}_\tau^y(\nabla_\tau^y S_{E,\varepsilon}(x,y))\\ &= \operatorname{div}_\tau^y\left(-\frac{P_{\partial E}(y)\nu_E(x) + 2S_{E,\varepsilon}(x,y)\,P_{\partial E}(y)(x-y)}{|x-y|^2+\varepsilon}\right)\\ &= \frac{(\nu_E(x)\cdot\nu_E(y))\,H_E(y)}{|x-y|^2+\varepsilon} - 2S_{E,\varepsilon}(x,y)\frac{n}{|x-y|^2+\varepsilon}\\ &\qquad + 2S_{E,\varepsilon}(x,y)S_{E,\varepsilon}(y,x)\,H_E(y)\end{aligned}$$

and

$$\begin{aligned}\operatorname{div}_\tau^x \nabla_\tau^y S_{E,\varepsilon}(x,y) &= \operatorname{div}_\tau^x\left(-\frac{P_{\partial E}(y)\nu_E(x) + 2S_{E,\varepsilon}(x,y)\,P_{\partial E}(y)(x-y)}{|x-y|^2+\varepsilon}\right)\\ &= -\frac{H_E(x)}{|x-y|^2+\varepsilon} + \frac{\big(B_E(x)\nu_E(y)\big)\cdot\nu_E(y)}{|x-y|^2+\varepsilon}\\ &\quad + 2S_{E,\varepsilon}(x,y)\,\frac{n}{|x-y|^2+\varepsilon}\\ &\quad - 2S_{E,\varepsilon}(x,y)\,\frac{\big(P_{\partial E}(x)\nu_E(y)\big)\cdot\nu_E(y)}{|x-y|^2+\varepsilon}.\end{aligned}$$

Collecting the terms and applying the inequality (4.12), we obtain that at a local maximum (minimum) point it holds that

$$\begin{aligned}0 \overset{(\le)}{\ge} &\frac{\nabla_\tau H_E(x)\cdot(x-y)}{|x-y|^2+\varepsilon} + \frac{(\nu_E(x)\cdot\nu_E(y))\,H_E(y) - H_E(x)}{|x-y|^2+\varepsilon}\\ &- |B_E|^2 S_{E,\varepsilon}(x,y) + 2S_{E,\varepsilon}^2(x,y)\,H_E(x) + 2S_{E,\varepsilon}(x,y)S_{E,\varepsilon}(y,x)\,H_E(y)\\ &+ 2\frac{\big(B_E(x)\nu_E(y)\big)\cdot\nu_E(y)}{|x-y|^2+\varepsilon} - 4S_{E,\varepsilon}(x,y)\frac{\big(P_{\partial E}(x)\nu_E(y)\big)\cdot\nu_E(y)}{|x-y|^2+\varepsilon}.\end{aligned}$$

The claim follows once we show that the last line above vanishes, i.e., that

$$\big(B_E(x)\nu_E(y)\big)\cdot\nu_E(y) = 2S_{E,\varepsilon}(x,y)\big(P_{\partial E}(x)\nu_E(y)\big)\cdot\nu_E(y). \tag{4.13}$$

Since $r_E > \sqrt{\varepsilon}$, this follows by first applying the equalities (4.8) and (4.10) in Lemma 4.2 and recalling $B_E(x)\nu_E(x) = 0$:

$$\begin{aligned} B_E(x)\nu_E(y) &= -2S_{E,\varepsilon}(x,y)\frac{B_E(x)(x-y)}{\big(\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y)\big)\cdot\nu_E(y)} \\ &= -4S^2_{E,\varepsilon}(x,y)\frac{P_{\partial E}(x)(x-y)}{\big(\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y)\big)\cdot\nu_E(y)}. \end{aligned}$$

Then we use (4.10) to deduce that

$$P_{\partial E}(x)\nu_E(y) = -2S_{E,\varepsilon}(x,y)\frac{P_{\partial E}(x)(x-y)}{\big(\nu_E(x) - 2S_{E,\varepsilon}(x,y)(x-y)\big)\cdot\nu_E(y)},$$

and (4.13) follows. □

In conclusion, by combining Lemma 4.1 and Lemma 4.3, we obtain that if a bounded C^3-regular set $E \subset \mathbb{R}^{n+1}$ satisfies $r_E > \sqrt{\varepsilon}$, then at a local maximum (minimum) point $(x,y) \in \partial E \times \partial E$ of $S_{E,\varepsilon}$ it holds that

$$\underset{(-)}{+}\left(\frac{\nabla_\tau H_E(x)\cdot(x-y)}{|x-y|^2+\varepsilon} + \frac{(\nu_E(x)\cdot\nu_E(y))\,H_E(y) - H_E(x)}{|x-y|^2+\varepsilon}\right) \le C_n\|S_E\|^3_{L^\infty}. \tag{4.14}$$

4.2. *Short-Time Uniform Ball Estimate*

Let us turn our focus on how to prove (4.1) for an approximative flat flow solution $(E^h_t)_{t\ge 0}$ defined in (3.1) when the initial set E_0 satisfies UBC with given a radius r_0. Assuming we may control the evolution of the quantity $\|S_{E^h_t}\|_{L^\infty}$, then thanks to Lemma 4.1 we also control (from below) the maximal UBC for E^h_t.

We motivate ourselves by considering first the continuous and embedded setting. Assume $(E_t)_t$ is a smooth flow and let ν_t and V_t denote the outer unit normal of E_t and the normal velocity of the flow on ∂E_t respectively. Then one may use the fact that for fixed t there is a smooth *normal parametrization* $(\Phi^t_s)_s$ of the flow such that $\Phi^t_0 = \mathrm{id}$ and $\partial_s\Phi^t_s = [V_s\,\nu_s]\circ\Phi^t_s$. This follows essentially from [5, Thm 8]. It is straightforward to calculate that for such a parametrization

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi^t_{t+s}\bigg|_{s=0} = V_t\,\nu_t \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}s}(\nu_{E_{t+s}}\circ\Phi^t_{t+s})\bigg|_{s=0} = -\nabla_\tau V_t \quad\text{on } \partial E_t. \tag{4.15}$$

In the case of volume preserving mean curvature flow, we have $V_s = -(H_s - \bar{H}_s)$, where H_s is the scalar mean curvature on ∂E_s and $\bar{H}_s$ its integral average over ∂E_s. If x and y are distinct points on ∂E_t, then by using (4.15) and the previous identity, we may compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}S_{E_{t+s}}(\Phi^t_s(x),\Phi^t_s(y))\bigg|_{s=0} &= \frac{\nabla_\tau H_E(x)\cdot(x-y)}{|x-y|^2} + \frac{(\nu_E(x)\cdot\nu_E(y))\,H_E(y) - H_E(x)}{|x-y|^2} \\ &\quad + R_t(x,y), \end{aligned} \tag{4.16}$$

where the remainder term $R_t(x,y)$ has a bound $|R_t(x,y)| \leq C_n \|S_{E_t}\|^3_{L^\infty}$. Suppose that $\|S_{E_t}\|_{L^\infty} = \pm S_{E_t}(x,y)$ and the function $s \mapsto \|S_{E_{t+s}}\|_{L^\infty}$ is differentiable at $s=0$, then we deduce

$$\frac{\mathrm{d}}{\mathrm{d}s}\|S_{E_{t+s}}\|_{L^\infty}\bigg|_{s=0} = \pm\frac{\mathrm{d}}{\mathrm{d}s} S_{E_{t+s}}(\Phi^t_s(x), \Phi^t_s(y))\bigg|_{s=0}.$$

Again, the estimate (4.14) also holds for S_E when the points are distinct. Thus, by possibly increasing C_n, we infer from above and (4.16) that

$$\frac{\|S_{E_{t+s}}\|_{L^\infty} - \|S_{E_t}\|_{L^\infty}}{s} \leq C_n \|S_{E_t}\|^3_{L^\infty}, \tag{4.17}$$

provided that $s \neq 0$ is sufficiently small.

The idea is to mimic the previous argument in the discrete setting for an approximative flat flow $(E^h_t)_{t\geq 0}$. To this end, we need to approximate the two-point functional by its ε-regularized version. We consider the element E^h_t and its consequent set E^h_{t+s}. For sake of brevity, we use the shorthand notations $E_1 = E^h_t$ and $E_2 = E^h_{t+s}$ for the rest of the subsection. First, we want to find a discrete version of the equalities in (4.15). Suppose that an element E_1 satisfies UBC and h is so small that by the discussion of the previous section we have that E_2 is C^1-regular set, $\partial E_2 \subset \mathcal{N}_{r_{E_1}}(\partial E_1)$ and $\nabla d_{E_1} \cdot \nu_{E_2} > 0$ on ∂E_2 are satisfied.

Then it is natural to project the boundary ∂E_2 to ∂E_1 by the projection $\pi_{\partial E_1}$ and, hence, using the identities in (2.22) we have

$$\frac{\mathrm{id} - \pi_{\partial E_1}}{h} = \frac{d_{E_1}}{h}(\nu_{E_2} \circ \pi_{\partial E_1}) \quad \text{on } \partial E_2,$$

which can be seen as a discrete time counterpart of the first identity in (4.15). In the next simple but crucial lemma, we derive a relation between ν_{E_2} and $\nu_{E_1} \circ \pi_{\partial E_1}$ for $x \in \partial E_2$.

Lemma 4.4. *Assume that $E_1 \subset \mathbb{R}^{n+1}$ is an open set satisfying UBC, E_2 is a C^1-regular set such that $\partial E_2 \subset \mathcal{N}_{r_{E_1}}(\partial E)$ and $\nabla d_{E_1} \cdot \nu_{E_2} > 0$ on ∂E_2. Then*

$$\nu_{E_1} \circ \pi_{\partial E_1} = \nabla_{\tau_2} d_{E_1} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}\, \nu_{E_2} \quad \text{on } \partial E_2.$$

Proof. By using the second identity of (2.22) for d_{E_1}, as well as the definition of a tangential gradient, the following holds on ∂E_2:

$$\nu_{E_1} \circ \pi_{\partial E_1} = \nabla d_{E_1} = P_{\partial E_2} \nabla d_{E_1} + (\nabla d_{E_1} \cdot \nu_{E_2})\nu_{E_2} = \nabla_{\tau_2} d_{E_1} + (\nabla d_{E_1} \cdot \nu_{E_2})\nu_{E_2}.$$

Since $|\nu_{E_1} \circ \pi_{\partial E_1}| = 1 = |\nu_{E_2}|$ and $\nabla_{\tau_2} d_{E_1} \cdot \nu_{E_2} = 0$, then the previous decomposition implies $|\nabla d_{E_1} \cdot \nu_{E_2}| = \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}$. Thus, the claim follows from the assumption $\nabla d_{E_1} \cdot \nu_{E_2} > 0$ on ∂E_2. □

The equality in the statement of Lemma 4.4 gives us a discrete analog for the second equality in (4.15) as

$$\nu_{E_2} - \nu_{E_1} \circ \pi_{\partial E_1} = -\nabla_{\tau_2} d_{E_1} + \frac{|\nabla_{\tau_2} d_{E_1}|^2}{1 + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}}\, \nu_{E_2} \text{ on } \partial E_2. \tag{4.18}$$

or equivalently

$$\begin{aligned} \nu_{E_2} - \nu_{E_1} \circ \pi_{\partial E_1} = &- \left(\frac{1}{\sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}} \right) \nabla_{\tau_2} d_{E_1} \\ &+ \frac{|\nabla_{\tau_2} d_{E_1}|^2}{\sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} + 1 - |\nabla_{\tau_2} d_{E_1}|^2}\, \nu_{E_1} \circ \pi_{\partial E_1} \text{ on } \partial E_2, \end{aligned} \tag{4.19}$$

which will be useful later. We need yet one technical lemma related to the projection $\pi_{\partial E_1}$ on the consequent boundary ∂E_2.

Lemma 4.5. *Let* $E_1, E_2 \subset \mathbb{R}^{n+1}$ *be open and bounded sets satisfying UBC. If* $\partial E_2 \subset \mathcal{N}_{r_{E_1}/2}(\partial E_1)$*, then for any* $x, y \in \partial E_2$ *satisfying* $\pi_{\partial E_1}(x) \neq \pi_{\partial E_1}(y)$ *it holds that*

$$\begin{aligned} &\Big| |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 - |x-y|^2 \Big| \\ &\quad \leq C_0 \|d_{E_1}\|_{L^\infty(\partial E_2)} \left(\|S_{E_1}\|_{L^\infty} + \|S_{E_2}\|_{L^\infty} + \|d_{E_1}\|_{L^\infty(\partial E_2)} \|S_{E_2}\|^2_{L^\infty} \right) |x-y|^2, \end{aligned}$$

where $C_0 \geq 1$ *is a universal constant.*

Proof. First, we obtain from (2.22) and the definition of S_{E_1} that

$$\begin{aligned} &|\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 - |x-y|^2 \\ &= -2 d_{E_1}(x) S_{E_1}(\pi_{\partial E_1}(x), \pi_{\partial E_1}(y)) |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ &\quad - 2 d_{E_1}(y) S_{E_1}(\pi_{\partial E_1}(y), \pi_{\partial E_1}(x)) |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ &\quad - \Big| d_{E_1}(x)(\nu_{E_1} \circ \pi_{\partial E_1})(x) - d_{E_1}(y)(\nu_{E_1} \circ \pi_{\partial E_1})(y) \Big|^2. \end{aligned}$$

Thus,

$$\begin{aligned} &\Big| |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 - |x-y|^2 \Big| \\ &\leq 4 \|d_{E_1}\|_{L^\infty(\partial E_2)} \|S_{E_1}\|_{L^\infty} |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ &\quad + 2 |d_{E_1}(x)|^2 |(\nu_{E_1} \circ \pi_{\partial E_1})(x) - (\nu_{E_1} \circ \pi_{\partial E_1})(y)|^2 + 2 |d_{E_1}(x) - d_{E_1}(y)|^2 \\ &\leq 4 \|d_{E_1}\|_{L^\infty(\partial E_2)} \|S_{E_1}\|_{L^\infty} |\pi_{\partial E_1}(x) - \pi_{\partial E_1}(y)|^2 \\ &\quad + 2 \|d_{E_1}\|^2_{L^\infty(\partial E_2)} |(\nu_{E_1} \circ \pi_{\partial E_1})(x) - (\nu_{E_1} \circ \pi_{\partial E_1})(y)|^2 \\ &\quad + 2 |d_{E_1}(x) - d_{E_1}(y)|^2. \end{aligned}$$

The normal ν_{E_1} is $1/r_{E_1}$-Lipschitz continuous by Proposition 2.7 and $\pi_{\partial E_1}$ is 2-Lipschitz continuous in $\mathcal{N}_{r_{E_1}/2}(\partial E_1)$ by Lemma 2.9. On the other hand, recalling

Lemma 4.1 we conclude $\|d_{E_1}\|_{L^\infty(\partial E_2)}\|S_{E_1}\|_{L^\infty} \le 1/4$. Hence, we infer from the previous estimate that

$$\Big||\pi_{\partial E_1}(x)-\pi_{\partial E_1}(y)|^2-|x-y|^2\Big| \le 24\|d_{E_1}\|_{L^\infty(\partial E_2)}\|S_{E_1}\|_{L^\infty}\,|x-y|^2+2|d_{E_1}(x)-d_{E_1}(y)|^2. \tag{4.20}$$

Thus, we are left to estimate the term $|d_{E_1}(x)-d_{E_1}(y)|^2$ on the boundary ∂E_2.

We divide this into two cases. First, suppose that $|x-y| \ge r_{E_2}/2$. Then using Lemma 4.1 we obtain

$$|d_{E_1}(x)-d_{E_1}(y)|^2 \le \frac{4\|d_{E_1}\|^2_{L^\infty(\partial E_2)}}{r_{E_2}^2}|x-y|^2 \le 16\|d_{E_1}\|^2_{L^\infty(\partial E_2)}\|S_{E_2}\|^2_{L^\infty}|x-y|^2. \tag{4.21}$$

Suppose then $|x-y| < r_{E_1}/2$. We define a C^1-extension $\tilde d_{E_1} : \mathcal{N}_{r_{E_2}}(\partial E_2) \to \mathbb{R}$ of the restriction $d_{E_1}|_{\partial E_2}$ by setting $\tilde d_{E_1} = d_{E_1}\circ\pi_{\partial E_2}$. Then $\nabla\tilde d_{E_1} = \nabla\pi_{\partial E_2}\nabla_{\tau_2}d_{E_1}\circ\pi_{\partial E_2}$ and by Lemma 2.9 $|\nabla\pi_{\partial E_2}|_{\mathrm{op}} \le 2$ in $\mathcal{N}_{r_{E_2}/2}(\partial E_2)$ so that $|\nabla\tilde d_{E_1}| \le 2\|\nabla_{\tau_2}\tilde d_{E_1}\|_{L^\infty(\partial E_2)}$. Since the line segment J_{yx} belongs to $\mathcal{N}_{r_{E_2}/2}(\partial E_2)$, we have

$$|d_{E_1}(x)-d_{E_1}(y)|^2 \le 4\|\nabla_{\tau_2}d_{E_1}\|^2_{L^\infty(\partial E_2)}|x-y|^2. \tag{4.22}$$

By Lemma 2.9 we have $|\nabla^2 d_{E_1}|_{\mathrm{op}} \le 2/r_{E_1}$ in $\mathcal{N}_{r_{E_1}}(\partial E_1)$. Therefore, by using Lemma 2.11 and Lemma 4.1 we get an estimate

$$\begin{aligned}
&\|\nabla_{\tau_2}d_{E_1}\|^2_{L^\infty(\partial E_2)}\\
&\le 4\|d_{E_1}\|_{L^\infty(\partial E_2)}\left(\sup_{\partial E_2}|\nabla^2 d_{E_1}|_{\mathrm{op}}+\frac{\|\nabla_\tau d_{E_1}\|_{L^\infty(\partial E_2)}}{r_{E_2}}\right)\\
&\le 16\|d_{E_1}\|_{L^\infty(\partial E_2)}\left(\|S_{E_1}\|_{L^\infty}+\|S_{E_2}\|_{L^\infty}\right).
\end{aligned} \tag{4.23}$$

Thus, we gather the estimate as claimed from (4.20), (4.21), (4.22) and the estimate above. □

We are now ready prove an analogous estimate to (4.17) in the discrete setting.

Lemma 4.6. *Assume that $E_1 \subset \mathbb{R}^{n+1}$ is an open and bounded set, with $|E_1| = m_0$, which satisfies UBC with radius $r_0 \in \mathbb{R}_+$. Let E_2 be any minimizer of the energy $\mathcal{F}_h(\,\cdot\,E_1)$ defined in (3.2). Then there is $h_0 = h_0(n, m_0, r_0)$ such that for $h \le h_0$ E_2 is C^3-regular and*

$$\frac{\|S_{E_2}\|_{L^\infty}-\|S_{E_1}\|_{L^\infty}}{h} \le C_n\|S_{E_1}\|^3_{L^\infty}.$$

If in addition E_1 is C^k-regular, then E_2 is C^{k+2}-regular.

Proof. As previously, $C = C(n, m_0, r_0) > 0$ may change from line to line. We find $h_0 = h_0(n, m_0, r_0) \in \mathbb{R}_+$ such that assuming $h \le h_0$ implies that the conclusions of Proposition 3.1, Lemma 3.2 and Remark 3.3 are valid. Let us quickly summarize what we have achieved so far. First, E_2 is open and bounded, C^3-regular set, or

C^{k+2}-regular set provided that E_1 is C^k-regular, and it satisfies UBC with radius $c_0 h^{1/3}$ for a constant $c_0 = c_0(n, m_0, r_0) > 0$. Hence, by Lemma 4.1 we have apriori estimate

$$\|S_{E_2}\|_{L^\infty} \le C h^{-\frac{1}{3}}. \tag{4.24}$$

Second, ∂E_2 is "close" to ∂E_1. To be more precise, we have $\|d_{E_1}\|_{L^\infty(E_2)} \le C_n h / r_0$ and we may assume that $\partial E_2 \subset \mathcal{N}_{r_0/2}(\partial E_1)$. Moreover, it holds that $\nabla d_{E_1} \cdot \nu_{E_2} > 0$ on ∂E_2 and $\pi_{\partial E_1}$ is injective on ∂E_2. Third, we have the Euler-Lagrange equation (3.4) on ∂E_2 in the classical sense.

Thus, we assume that $h \le h_0$. We might need to shrink h_0 but always in a way that we preserve the dependency $h_0 = h_0(n, m_0, r_0)$. By combining the estimate $\|d_{E_1}\|_{L^\infty(E_2)} \le C_n h / r_0$ from Proposition 3.1 with Lemma 4.1 and (4.24) and by possibly shrinking h_0 we obtain

$$\frac{\|d_{E_1}\|_{L^\infty(E_2)}}{h} \le C_n \|S_{E_1}\|_{L^\infty} \quad \text{and} \quad \|S_{E_2}\|_{L^\infty} \|d_{E_1}\|_{L^\infty(E_2)} \le 1. \tag{4.25}$$

Then, by (3.4), Lemma 4.1 and the first estimate in (4.25), the Lagrange multiplier λ^h can be controlled as

$$|\lambda^h| \le \frac{\|d_{E_1}\|_{L^\infty(E_2)}}{h} + \|H_{E_2}\|_{L^\infty(\partial E_2)} \le C_n(\|S_{E_1}\|_{L^\infty} + \|S_{E_2}\|_{L^\infty}). \tag{4.26}$$

The claim follows once we show

$$\frac{\|S_{E_2}\|_{L^\infty} - \|S_{E_1}\|_{L^\infty}}{h} \le C_n \left(\|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3 \right). \tag{4.27}$$

Indeed, assuming the above holds true we have by Lemma 4.1 and (4.24)

$$\|S_{E_2}\|_{L^\infty} - \|S_{E_1}\|_{L^\infty} \le C_n r_0^{-3} h + C h^{\frac{1}{3}} \|S_{E_2}\|_{L^\infty}$$

and, hence, recalling (4.4) and shrinking h_0, if neccessary, we obtain $\|S_{E_2}\|_{L^\infty} \le 2\|S_{E_1}\|_{L^\infty}$. Thus, reiterating the previous inequality via (4.27) yields the claim.

To prove (4.27), we initially fix any $\varepsilon < r_{E_2}^2$ and choose $(x, y) \in \partial E_2 \times \partial E_2$ such that $|S_{E_2,\varepsilon}(x, y)| = \|S_{E_2,\varepsilon}\|_{L^\infty}$. Since $\|S_{E_2,\varepsilon}\|_{L^\infty} > 0$, then $x \neq y$ and, hence, the injectivity of $\pi_{\partial E_1}$ on ∂E_2 ensures that $\pi_{\partial E_1}(x) \neq \pi_{\partial E_1}(x)$. In order to simplify our notations, we write $\pi = \pi_{\partial E_1}$ and $H_2 = H_{E_2}$ for short. By using the definition in (4.5), the identities (2.22) and (4.18) as well as the Euler-Lagrange equation we may decompose the difference quotient as

$$\begin{aligned}
&\frac{1}{h}\big(S_{E_2,\varepsilon}(x, y) - S_{E_1,\varepsilon}(\pi(x), \pi(y))\big) \\
&\qquad = \frac{(x - y) \cdot \nabla_{\tau_2} H_2(x)}{|x - y|^2 + \varepsilon} \\
&\qquad\quad + \frac{\big(\nu_{E_1}(x) \cdot \nu_{E_2}(y)\big) H_2(y) - H_2(x)}{|x - y|^2 + \varepsilon} \\
&\qquad\quad + \frac{1}{h} \frac{|\nabla_{\tau_2} d_{E_1}(x)|^2}{1 + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}(x)|^2}} S_{E_2,\varepsilon}(x, y)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\lambda^h - \frac{d_{E_1}(y)}{2h}\right)\frac{|\nu_{E_1}(x)-\nu_{E_1}(y)|^2}{|x-y|^2+\varepsilon}\\
&+\frac{d_{E_1}(y)}{2h}\frac{|\nu_{E_1}(\pi(x))-\nu_{E_1}(\pi(y))|^2}{|x-y|^2+\varepsilon}\\
&+\frac{1}{h}\left(\frac{|\pi(x)-\pi(y)|^2-|x-y|^2}{|x-y|^2+\varepsilon}\right)S_{E_1,\varepsilon}(\pi(x),\pi(y)). \qquad (4.28)
\end{aligned}
$$

Next, we estimate the last four terms on the RHS. First, since $\partial E_2 \subset \mathcal{N}_{r_0/2}(\partial E) \subset \mathcal{N}_{r_{E_1}/2}(\partial E_1)$, we have the estimate (4.23) for $\|\nabla_{\tau_2} d_{E_1}\|^2_{L^\infty(\partial E_2)}$ and, hence, recalling the first estimate in (4.25) we have

$$
\begin{aligned}
&\left|\frac{1}{h}\frac{|\nabla_{\tau_2} d_{E_1}(x)|^2}{1+\sqrt{1-|\nabla_{\tau_2} d_{E_1}(x)|^2}}S_{E_2,\varepsilon}(x,y)\right|\\
&\quad\le \frac{16\|d_{E_1}\|_{\partial E_2}}{h}\left(\|S_{E_1}\|_{L^\infty}\|S_{E_2}\|_{L^\infty}+\|S_{E_2}\|^2_{L^\infty}\right) \qquad (4.29)\\
&\quad\le C_n\left(\|S_{E_1}\|^3_{L^\infty}+\|S_{E_2}\|^3_{L^\infty}\right).
\end{aligned}
$$

For the next term, we use Lemma 4.1, the first estimate in (4.25) and (4.26) to obtain

$$
\begin{aligned}
\left|\left(\lambda^h - \frac{d_{E_1}(y)}{2h}\right)\frac{|\nu_{E_1}(x)-\nu_{E_1}(y)|^2}{|x-y|^2+\varepsilon}\right| &\le C_n\left(|\lambda^h| + \|S_{E_1}\|_{L^\infty(\partial E_1)}\right)\|S_{E_2}\|^2_{L^\infty(\partial E_2)}\\
&\le C_n\left(\|S_{E_1}\|^3_{L^\infty(\partial E_1)}+\|S_{E_2}\|^3_{L^\infty(\partial E_2)}\right). \qquad (4.30)
\end{aligned}
$$

By Proposition 2.7 ν_{E_1} is $1/r_0$-Lipschitz and by Lemma 2.9 π is 2-Lipschitz continuous in $\mathcal{N}_{r_0/2}(\partial E_1)$. Thus, by Lemma 4.1 and the first inequality in (4.25), we estimate the second last term as

$$
\begin{aligned}
\left|\frac{d_{E_1}(y)}{2h}\frac{|\nu_{E_1}(\pi(x))-\nu_{E_1}(\pi(y))|^2}{|x-y|^2+\varepsilon}\right| &\le C_n\|S_{E_1}\|_{L^\infty}\frac{1}{r_0^2}\frac{|\pi(x)-\pi(y)|^2}{|x-y|^2} \qquad (4.31)\\
&\le C_n\|S_{E_1}\|^3_{L^\infty}.
\end{aligned}
$$

Finally, by using Lemma 4.5 and the identities in (4.25), we have

$$
\begin{aligned}
&\left|\frac{1}{h}\left(\frac{|\pi(x)-\pi(y)|^2-|x-y|^2}{|x-y|^2+\varepsilon}\right)S_{E_1,\varepsilon}(\pi(x),\pi(y))\right|\\
&\le C_n\frac{\|d_{E_1}\|_{L^\infty(\partial E_2)}}{h}\left(\|S_{E_1}\|_{L^\infty}+\|S_{E_2}\|_{L^\infty}+\|d_{E_1}\|_{L^\infty(\partial E_2)}\|S_{E_2}\|^2_{L^\infty}\right)\|S_{E_1}\|_{L^\infty}\\
&\le C_n\left(\|S_{E_1}\|^3_{L^\infty}+\|S_{E_2}\|^3_{L^\infty}\right). \qquad (4.32)
\end{aligned}
$$

We infer from (4.28), (4.29), (4.30), (4.31) and (4.32) the expression

$$
\frac{S_{E_2,\varepsilon}(x,y)-S_{E_1,\varepsilon}(\pi(x),\pi(y))}{h}
$$

$$= \frac{(x-y)\cdot\nabla_{\tau_2}H_2(x)}{|x-y|^2+\varepsilon} + \frac{\big(\nu_{E_1}(x)\cdot\nu_{E_2}(y)\big)\, H_2(y) - H_2(x)}{|x-y|^2+\varepsilon}$$
$$+ R,$$

where for the remainder term it holds $|R| \le C_n \left(\|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3\right)$. Since (x, y) is a maximum (or minimum) point for $S_{E_2,\varepsilon}$, then we conclude from (4.14)

$$\frac{\|S_{E_2,\varepsilon}\|_{L^\infty} - \|S_{E_1,\varepsilon}\|_{L^\infty}}{h} \le C_n \left(\|S_{E_1}\|_{L^\infty}^3 + \|S_{E_2}\|_{L^\infty}^3\right).$$

Since now $\|S_{E_i,\varepsilon}\|_{L^\infty} \uparrow \|S_{E_i,\varepsilon}\|_{L^\infty}$ for $i = 1, 2$ as ε tends to zero, the above yields (4.27) and we conclude the proof. □

We may now prove the main result of this section which is the UBC estimate for the approximative flat flow.

Theorem 4.7. *Let $E_0 \subset \mathbb{R}^{n+1}$ be an open and bounded set which satisfies UBC with radius $r_0 \in \mathbb{R}_+$ and let m_0 denote its volume. There are $h_0 = h_0(n, m_0, r_0) \in \mathbb{R}_+$ and $T_0 = T_0(n, r_0) \in \mathbb{R}_+$ such that if $h \le h_0$, then any approximative flat flow $(E_t^h)_{t\ge 0}$ of* (1.1) *starting from E_0 satisfies UBC with radius $r_0/2$ for all $t \le T_0$. Moreover, E_t^h is $C^{1+2\lfloor t/h\rfloor}$-regular for every $0 \le t \le T_0$.*

Proof. By a slight abuse of notation, we set h_0 to be as in Lemma 4.6 for the parameters n, m_0 and $r_0/2$. Then we choose

$$T_0 = \frac{r_0^2}{4C_n}\,, \tag{4.33}$$

where the dimensional constant is the same as in Lemma 4.6. We assume that $h \le h_0$ and consider an approximative flat flow $(E_t^h)_{t\ge 0}$ starting from E_0 obtained via the minimizing movements scheme (3.1). We may assume $h \le T_0$, since otherwise the proof is trivial. Since E_0 satisfies UBC with radius r_0, we have by Lemma 4.1 that $\|S_{E_0}\|_{L^\infty} = 1/(2r_0)$. Then we set

$$K = \sup\left\{k \in \mathbb{N} : E_t^h \ \text{ satisfies UBC with } \ \|S_{E^h_{lh}}\|_{L^\infty} \le \frac{1}{r_0} \ \text{ for } \ 0 \le l \le k\right\}.$$

Note that if E^k_{kh} is a bounded set satisfying UBC with $\|S_{E^h_k}\|_{L^\infty} \le 1/r_0$, then thanks to Lemma 4.1 we know that it satisfies UBC with radius $r_0/2$. Thus, it follows from the construction of $(E_t^h)_{t\ge 0}$, the choice of h_0, and Lemma 4.6 that $E^h_{(k+1)h}$ is a bounded C^3-regular set satisfying

$$\|S_{E^h_{(k+1)h}}\|_{L^\infty} \le \|S_{E^h_k}\|_{L^\infty} + C_n h \|S_{E^h_k}\|_{L^\infty}^3 \le \|S_{E^h_k}\|_{L^\infty} + C_n r_0^{-3} h.$$

Since $h_0 \le T_0$, then the choices in (4.33) imply that K is well-defined. By summing the above from $k = 0$ to $k = K$ we obtain

$$\frac{1}{r_0} \le \|S_{E^h_{(K+1)h}}\|_{L^\infty} \le \|S_{E_0}\|_{L^\infty} + \frac{C_n}{r_0^3}(K+1)h = \frac{1}{2r_0} + \frac{C_n}{r_0^3}(K+1)h.$$

This yields $K \ge \lfloor T_0/h\rfloor$ and, hence, it follows from the construction (3.1) that E_t^h satisfies UBC with radius $r_0/2$ for every $0 \le t \le T_0$. The last claim then follows directly from Lemma 4.6. □

5. Higher Regularity

In this section we utilize the short-time UBC from previous section and prove the full regularity of the flat flow solution of (1.1). It is well known that the classical solution for the mean curvature flow is well defined as long as the second fundamental form stays bounded [39]. For the volume preserving flow this is not enough as the flow may develop singularities even if it stays regular [40,41]. However, if the flow in addition satisfies UBC then these singularities do not occur. In this section we show that the approximative flat flow becomes instantaneously smooth and stays smooth as long as it satisfies UBC. We will prove this via energy estimates.

Our starting point is the formula in Lemma 4.4, which for sets E_1 and E_2 as in the lemma, gives the formula which relates their normals as

$$\nu_{E_1} \circ \pi_{\partial E_1} = \nabla_{\tau_2} d_{E_1} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}\, \nu_{E_2} \quad \text{on } \partial E_2.$$

Recall that ∇_{τ_2} denotes the tangential gradient on ∂E_2. Assume now further that E_2 is a minimizer of the functional $\mathcal{F}_h(\,\cdot\, E_1)$ defined in (3.2). We may use the Euler-Lagrange equation (3.4) and have

$$\nu_{E_1} \circ \pi_{\partial E_1} = -h\, \nabla_{\tau_2} H_{E_2} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}\, \nu_{E_2} \quad \text{on } \partial E_2. \tag{5.1}$$

This identity is simple enough for us to differentiate multiple times and this in turn gives us formula which is the discrete analog of the identity (see e.g. [38, Lemma 3.5])

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta^k H_{E_t} = \Delta^{k+1} H_{E_t} + \text{lower order terms.} \tag{5.2}$$

Let us, for the sake of clarification, show how we obtain the discrete version of (5.2) for $k = 0$ from (5.1), which reads as follows

$$\begin{aligned} &\sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2} H_{E_2} - H_{E_1} \circ \pi_{\partial E_1} \\ &\quad = h\, \Delta_{\tau_2} H_{E_2} + h^2\, A_2(\cdot) \nabla_{\tau_2} H_{E_2} \cdot \nabla_{\tau_2} H_{E_2} + a_1(\cdot) d_{E_1} \quad \text{on } \partial E_2, \end{aligned} \tag{5.3}$$

where the function $a_1(\cdot)$ and the matrix field $A_2(\cdot)$ depend smoothly on d_{E_1}, $\nu_{E_1} \circ \pi_{\partial E_1}$ ν_{E_2}, $B_{E_1} \circ \pi_{\partial E_1}$ and B_{E_2}. In particular, since E_1 and E_2 satisfy UBC with radius $r_0/2$, then $a_1(\cdot)$ and $A_2(\cdot)$ are uniformly bounded.

Indeed, by applying the tangential divergence on (5.1) we have

$$\operatorname{div}_{\tau_2} \left(\nu_{E_1} \circ \pi_{\partial E_1} \right) = -h\, \Delta_{\tau_2} H_{E_2} + \sqrt{1 - |\nabla_{\tau_2} d_{E_1}|^2}\, H_{E_2} \quad \text{on } \partial E_2.$$

In order to calculate the LHS, we use (2.22), (2.32) and (2.33) to obtain

$$\begin{aligned} \nabla \big(\nu_{E_1} \circ \pi_{\partial E_1} \big) &= \nabla^2 d_{E_1} = B_{E_1} \circ \pi_{\partial E_1} (I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} \\ &= B_{E_1} \circ \pi_{\partial E_1} - d_{E_1} \left(I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1} \right)^{-1} (B_{E_1} \circ \pi_{\partial E_1})^2 \end{aligned}$$

which holds in the tubular neighborhood $\mathcal{N}_{r_0}(\partial E_1)$, where we also used the fact

$$(B_{E_1} \circ \pi_{\partial E_1})(I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} = (I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1})^{-1} (B_{E_1} \circ \pi_{\partial E_1}).$$

Again from (5.1) we have

$$\nu_{E_2} = \frac{1}{\sqrt{1-|\nabla_{\tau_2} d_{E_1}|^2}} \left(h\,\nabla_{\tau_2} H_{E_2} + \nu_{E_1} \circ \pi_{\partial E_1}\right)$$

on ∂E_2. Using the above identities and the fact $B_{E_1}\nu_{E_1} = 0$ on ∂E_1, we have the following equality on ∂E_2

$$\begin{aligned}
&\operatorname{div}_{\tau_2}\left(\nu_{E_1} \circ \pi_{\partial E_1}\right) = \operatorname{Tr}\big((I - \nu_{E_2} \otimes \nu_{E_2})\nabla^2 d_{E_1}\big)\\
&= H_{E_1} \circ \pi_{\partial E_1} - d_{E_1} \operatorname{Tr}\Big(\left(I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1}\right)^{-1} (B_{E_1} \circ \pi_{\partial E_1})^2\Big)\\
&\quad - \frac{h^2}{1-|\nabla_{\tau_2} d_{E_1}|^2}\Big(\left(I + d_{E_1} B_{E_1} \circ \pi_{\partial E_1}\right)^{-1} (B_{E_1} \circ \pi_{\partial E_1})\Big)\nabla_{\tau_2} H_{E_2} \cdot \nabla_{\tau_2} H_{E_2}.
\end{aligned}$$

The equation (5.3) then follows from the previous calculations and from the identity

$$(\nu_{E_1} \circ \pi_{\partial E_1}) \cdot \nu_{E_2} = \sqrt{1-|\nabla_{\tau_2} d_{E_1}|^2} \quad \text{on } \partial E_2, \tag{5.4}$$

which is a direct consequence of Lemma 4.4.

We may differentiate the equality (5.3) further and obtain a discrete version of (5.2) for every order k. This will produce several nonlinear error terms which have rather complicated structure. However, by introducing sufficiently efficient notation we are able to identify the structure of these error terms and by using UBC and the interpolation inequality from Proposition 2.2 we are able to reproduce the argument from [22] in the discrete setting. The following proposition is the core of the proof for the higher order regularity.

Proposition 5.1. *Assume that $E_1 \subset \mathbb{R}^{n+1}$ is an open and bounded set, with $|E_1| = m_0$, which satisfies UBC with radius r_0 and let E_2 be any minimizer of $\mathcal{F}_h(\,\cdot\,E_1)$ defined in (3.2). There is $h_0 = h_0(n, m_0, r_0)$ such that if $h \le h_0$ and E_1 is C^{2m+3}-regular for $m = 0, 1, 2, \dots$ then*

$$\begin{aligned}
&\Delta_{\tau_2}^m H_{E_2} - (\Delta_{\tau_1}^m H_{E_1}) \circ \pi_{\partial E_1} = h\,\Delta_{\tau_2}^{m+1} H_{E_2} + h\,R_{2m} \quad \text{and}\\
&\nabla_{\tau_2}\Delta_{\tau_2}^m H_{E_2} - (\nabla_{\tau_1}\Delta_{\tau_1}^m H_{E_1}) \circ \pi_{\partial E_1}\\
&\ = h\,\nabla_{\tau_2}\Delta_{\tau_2}^{m+1} H_{E_2} - \partial_{\nu_{E_2}}(\Delta_{\tau_1}^m H_{E_1} \circ \pi_{\partial E_1})\nu_{E_2} + h\,R_{2m+1}
\end{aligned}$$

on ∂E_2 and the error term R_l for $l = 0, 1, 2, \dots$ satisfies the estimate

$$\|R_l\|_{L^2(\partial E_2)}^2 \le C_l \left(1 + \|B_{E_2}\|_{H^{l+1}(\partial E_2)}^2 + \|B_{E_1}\|_{H^l(\partial E_1)}^2\right),$$

where $C_l = C_l(l, n, m_0, r_0)$.

We note that so far we have not used any results from differential geometry. In fact, we need the notation from geometry only to prove Proposition 5.1. Therefore, instead of giving the proof of Proposition 5.1, which is technically challenging, we show first how we may use it to obtain the regularity estimate (1.2) in the statement of Theorem 1.1. Here is the main result of this section.

Theorem 5.2. *Let E_0 be an open and bounded set, with $|E_0| = m_0$, and let $(E_t^h)_{t\geq 0}$ be an approximative flat flow starting from E_0 defined in (3.1). For given $r_0 \in \mathbb{R}_+$ there is $h_0 = h_0(n, m_0, r_0) \in \mathbb{R}_+$ such that if $h \leq h_0$, E_t^h satisfies UBC with radius r_0 in $[0, T]$ and if $(l+2)h \leq T$ for a given $l \in \mathbb{N} \cup \{0\}$, then we have*

$$\sup_{t\in[(l+2)h,T]} \Big((t-lh)^l \|H_{E_t^h}\|^2_{H^l(\partial E_t^h)}\Big) + \int_{(l+2)h}^T (t-lh)^l \|H_{E_t^h}\|^2_{H^{l+1}(\partial E_t^h)}\,\mathrm{d}t \leq C,$$

for a constant $C = C(l, n, m_0, r_0, T)$.

Proof. In the proof, C and C_m denote a positive real number which may change their values but always in a manner that we have the dependencies $C = C(n, m_0, r_0)$ and $C_m = C_m(m, n, m_0, r_0, T)$. We use the abbreviation $E_k = E_{kh}^h$ for $k = 0, 1, 2, \ldots$

First, by Proposition 3.1, Lemma 3.2, Remark 3.3 and Theorem 4.7, we find $h_0 = h_0(n, m_0, r_0) > 0$ such that if $h \leq h_0$ and E_k is C^{2k+1}-regular, bounded set of volume m_0, which satisfies uniform ball condition with radius r_0, then the consequent set E_{k+1} is C^{2k+3}-regular, bounded and of volume m_0, with

$$\|d_{E_k}\|_{L^\infty(\partial E_{k+1})} \leq Ch < r_0/2.$$

Moreover, E_{k+1} satisfies UBC with radius $r_0/2$ and the projection $\pi_{\partial E_k} : \partial E_{k+1} \to \partial E_k$ is injective. We may then prove that, for $k \geq 1$, $\pi_{\partial E_k} : \partial E_{k+1} \to \partial E_k$ is a diffeomorphism with

$$J_{\tau_{k+1}}\pi_{\partial E_k} \geq 1 - Ch > 0 \ \text{ on } \ \partial E_{k+1}, \tag{5.5}$$

where the tangential Jacobian $J_{\tau_{k+1}}\pi_{\partial E_k}$ of $\pi_{\partial E_k}$ on ∂E_{k+1} is defined in (2.3). Indeed, since $\partial E_{k+1} \subset \mathcal{N}_{r_0/2}(\partial E_k)$, then $\pi_{\partial E_k}$ is C^1-regular map on ∂E_{k+1}. Recalling the injectivity of the projection we are remain to prove (5.5). By (2.31) we may write

$$\nabla\pi_{\partial E_{k+1}} = I - \nabla d_{E_k} \otimes \nabla d_{E_k} - d_{E_k}\nabla^2 d_{E_k} \ \text{ on } \ \partial E_{k+1}.$$

Thus, it follows from the definition in (2.3) and $\nabla^2 d_{E_k}\nabla d_{E_k} = 0$ in $\mathcal{N}_{r_0}(\partial E_k)$ that for given a point $x \in \partial E_{k+1}$ there is an orthonormal basis $v_1, \ldots, v_n$ of $G_x\partial E_{k+1}$ such that

$$J_{\tau_{k+1}}\pi_{\partial E_k}(x) = \prod_{i=1}^n \Big|\Big(I - \nabla d_{E_k}(x) \otimes \nabla d_{E_k}(x) - d_{E_k}(x)\nabla^2 d_{E_k}(x)\Big)v_i\Big|$$

$$= \prod_{i=1}^n \Big(1 - (\nabla d_{E_k}(x)\cdot v_i)^2 - 2d_{E_k}(x)\nabla^2 d_{E_k}(x)v_i \cdot v_i + |d_{E_k}(x)|^2|\nabla^2 d_{E_k}(x)v_i|^2\Big)^{\frac{1}{2}}.$$

Since $\partial E_{k+1} \subset \mathcal{N}_{r_0/2}(\partial E_k)$, then Lemma 2.9 yields $\sup_{\partial E_{k+1}} |\nabla^2 d_{E_k}|_{\mathrm{op}} \leq C$. Further, since E_{k+1} satisfies UBC with radius $r_0/2$, then by Lemma 2.11 and by the previous estimates we deduce

$$|\nabla d_{E_k}(x)\cdot v_i|^2 \leq |\nabla_{\tau_2} d_{E_k}(x)|^2$$

$$
\begin{aligned}
&\le 4\|d_{E_k}\|_{L^\infty(\partial E_{k+1})}\left(\sup_{\partial E_{k+1}}|\nabla^2 d_{E_k}|_{\mathrm{op}}+\frac{\|\nabla d_{E_k}\|_{L^\infty(\partial E_{k+1})}}{r_0/2}\right)\\
&\le Ch.
\end{aligned}
$$

Therefore, by combining the previous observations and shrinking h_0, if needed, we obtain (5.5). Again, by possibly shrinking h_0, we may assume that the implications of Proposition 5.1 hold true for the parameters m_0 and $r_0/2$.

Let us from now on assume that the sets E_t^h satisfy UBC with radius r_0 for every $t \in [0, T]$. Let us denote $K = \lfloor T/h \rfloor$. Then the previous discussion holds for every E_k and $k = 0, 1, 2, \ldots, K$. For the sake of presentation, we use abbreviations $\|B_{E_k}\|_{L^2} = \|B_{E_k}\|_{L^2(\partial E_k)}$, $\|B_{E_k}\|_{H^{2m}} = \|B_{E_k}\|_{H^{2m}(\partial E_k)}$ etc.

After the initialization, we prove the claim by induction and to this aim we begin by proving the main regularity estimates. We claim that for every $m = 0, 1, 2, \ldots,$ with $m \le K - 2$, and every $k = m + 1, m + 2, \ldots, K$ it holds that

$$
\|\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} \le (1 + C_m h)\|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2} - h\|\nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} + C_m h \tag{5.6}
$$

and

$$
\|\nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} \le (1+C_m h)\|\nabla_{\tau_k}\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2} - h\|\Delta^{m+1}_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} + C_m h. \tag{5.7}
$$

We first prove (5.6) and fix m. Recall that for $k \ge m+1$ the set E_k is C^{2m+3}-regular. Therefore, by Proposition 5.1, it holds for every $k = m + 1, m + 2, \ldots, K$ that

$$
\Delta^m_{\tau_{k+1}} H_{E_{k+1}} - (\Delta^m_{\tau_k} H_{E_k}) \circ \pi_{\partial E_k} = h\, \Delta^{m+1}_{\tau_{k+1}} H_{E_{k+1}} + h\, R_{2m,k} \ \text{ on } \ \partial E_{k+1},
$$

where the remainder term $R_{2m,k}$ satisfies

$$
\|R_{2m,k}\|^2_{L^2} \le C_m\big(1 + \|B_{E_{k+1}}\|^2_{H^{2m+1}} + \|B_{E_k}\|^2_{H^{2m}}\big).
$$

Again, since E_k and E_{k+1} satisfy UBC with radius $r_0/2$ and $|E_k| = m_0 = |E_{k+1}|$, then $\|B_{E_k}\|_{L^\infty}, \|B_{E_k}\|_{L^\infty} \le C$ by (2.30) and $P(E_k), P(E_{k+1}) \le C$ by Lemma 2.10. Therefore, we may use Proposition 2.6 and Young's inequality to deduce that

$$
\begin{aligned}
&\|B_{E_k}\|^2_{H^{2m}} \le C_m\left(1 + \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\right) \quad \text{and}\\
&\|B_{E_{k+1}}\|^2_{H^{2m+1}} \le C_m\left(1 + \|\nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2}\right).
\end{aligned} \tag{5.8}
$$

We also observe that $\|\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} \le C_m \|B_{E_{k+1}}\|_{H^{2m}}$. Let $\varepsilon \in (0,1)$ be a number which we will choose later. By using the previous observations, (5.5), Young's inequality, and integration by parts we estimate as follows:

$$\begin{aligned}
&\|\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} - \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\\
&\quad\le \int_{\partial E_{k+1}} |\Delta^m_{\tau_{k+1}} H_{E_{k+1}}|^2 - |\Delta^m_{\tau_k} H_{E_k} \circ \pi_{\partial E_k}|^2 \, \mathrm{d}\mathcal{H}^n + Ch \int_{\partial E_{k+1}} |\Delta^m_{\tau_k} H_{E_k} \circ \pi_{\partial E_k}|^2 \, \mathrm{d}\mathcal{H}^n\\
&\quad\le \int_{\partial E_{k+1}} |\Delta^m_{\tau_{k+1}} H_{E_{k+1}}|^2 - |\Delta^m_{\tau_k} H_{E_k} \circ \pi_{\partial E_k}|^2 \, \mathrm{d}\mathcal{H}^n + \frac{Ch}{1-Ch} \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\\
&\quad\le 2\int_{\partial E_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}} (\Delta^m_{\tau_{k+1}} H_{E_{k+1}} - \Delta^m_{\tau_k} H_{E_k} \circ \pi_{\partial E_k}) \, \mathrm{d}\mathcal{H}^n + Ch \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\\
&\quad= 2h\int_{\partial E_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}} (\Delta^{m+1}_{\tau_{k+1}} H_{E_{k+1}} + R_{2m,k}) \, \mathrm{d}\mathcal{H}^n + Ch \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\\
&\quad\le -2h\|\nabla_{\tau_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} + \varepsilon h \|R_{2m,k}\|^2_{L^2} + \frac{h}{\varepsilon} \|\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} + Ch \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\\
&\quad\le -2h\|\nabla_{\tau_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2}\\
&\qquad + C_m h \left(\varepsilon \|B_{E_{k+1}}\|^2_{H^{2m+1}} + \frac{1}{\varepsilon}\|B_{E_{k+1}}\|^2_{H^{2m}}\right) + C_m h \left(1 + \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\right)\\
&\quad\le -2h\|\nabla_{\tau_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} + C_m h \left(\varepsilon \|\nabla_{\tau_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} + \frac{1}{\varepsilon}\|B_{E_{k+1}}\|^2_{H^{2m}}\right)\\
&\qquad + C_m h \left(1 + \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\right).
\end{aligned}$$

By choosing $\varepsilon = (1 + C_m)^{-1}/2$, the previous estimate yields

$$\begin{aligned}
&\|\Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} - \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\\
&\qquad\le -\frac{3h}{2}\|\nabla_{\tau_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}}\|^2_{L^2} + C_m h \|B_{E_{k+1}}\|^2_{H^{2m}} + C_m h \left(1 + \|\Delta^m_{\tau_k} H_{E_k}\|^2_{L^2}\right).
\end{aligned} \tag{5.9}$$

Since $\|B_{E_{k+1}}\|_{L^\infty}, P(E_{k+1}) \le C$, we may use Proposition 2.2 to find $\theta = \theta(m,n) \in (0,1)$ such that

$$\|B_{E_{k+1}}\|^2_{H^{2m}} \le C_m \|B_{E_{k+1}}\|^{2\theta}_{H^{2m+1}} \|B_{E_{k+1}}\|^{2(1-\theta)}_{L^\infty} \le C_m \varepsilon \|B_{E_{k+1}}\|^2_{H^{2m+1}} + C_m \varepsilon^{-\frac{\theta}{1-\theta}}$$

for any $\varepsilon \in (0,1)$, where the last inequality follows from Young's inequality and the curvature bound. Thus, by combing the above with (5.9) and (5.8) the estimate (5.6) follows with a suitable choice of ε.

Let us then prove (5.7). The argument is similar than above and we only point out the main differences. Now Proposition 5.1 gives for every $k = m+1, m+2, \ldots, K$ the formula

$$\begin{aligned}
&\nabla_{\tau_{k+1}} \Delta^m_{\tau_{k+1}} H_{E_{k+1}} - (\nabla_{\tau_k} \Delta^m_{\tau_k} H_{E_k}) \circ \pi_{\partial E_k}\\
&\quad= h\, \nabla_{\tau_{k+1}} \Delta^{m+1}_{\tau_{k+1}} H_{E_{k+1}} - \partial_{\nu_{E_{k+1}}} (\Delta^m_{\tau_k} H_{E_k} \circ \pi_{\partial E_k}) \nu_{E_{k+1}} + h\, R_{2m+1,k} \quad \text{on } \partial E_{k+1},
\end{aligned}$$

where

$$\|R_{2m+1,k}\|^2_{L^2} \le C_m \left(1 + \|B_{E_{k+1}}\|^2_{H^{2m+2}} + \|B_{E_k}\|^2_{H^{2m+1}}\right),$$

and, again, by using Proposition 2.6 and Young's inequality we have estimates

$$\|B_{E_k}\|^2_{H^{2m+1}} \leq C_m(1+\|\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\|^2_{L^2}) \quad \text{and}$$
$$\|B_{E_{k+1}}\|^2_{H^{2m+2}} \leq C_m\left(1+\|\Delta^{m+1}_{\tau_{k+1}}H_{E_{k+1}}\|^2_{L^2}\right).$$

We use the previous observations, the Cauchy-Schwarz inequality, the estimate $\|\nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}}H_{E_{k+1}}\|_{L^2} \leq C_m\|B_{E_{k+1}}\|^2_{H^{2m+1}}$ and argue as in proving (5.6) to deduce that

$$\begin{aligned}
&\|\nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}}H_{E_k}\|^2_{L^2} - \|\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\|^2_{L^2}\\
&\quad\leq \int_{\partial E_{k+1}} |\nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}}H_{E_{k+1}}|^2 - |\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\circ\pi_{\partial E_k}|^2\,\mathrm{d}\mathcal{H}^n + Ch\,\|\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\|^2_{L^2}\\
&\quad\leq 2\int_{\partial E_{k+1}} \nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}}H_{E_{k+1}}\cdot(\nabla_{\tau_{k+1}}\Delta^m_{\tau_{k+1}}H_{E_{k+1}} - \nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\circ\pi_{\partial E_k})\,\mathrm{d}\mathcal{H}^n\\
&\qquad + Ch\,\|\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\|^2_{L^2}\\
&\quad\leq -2h\|\Delta^{m+1}_{\tau_{k+1}}H_{E_{k+1}}\|_{L^2} + \varepsilon h\,\|R_{2m+1,k}\|^2_{L^2(\partial E_{k+1})} + \frac{C_m}{\varepsilon}h\,\|B_{E_{k+1}}\|^2_{H^{2m+1}}\\
&\qquad + Ch\,\|\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\|^2_{L^2}\\
&\quad\leq -2h\|\Delta^{m+1}_{\tau_{k+1}}H_{E_{k+1}}\|_{L^2}\\
&\qquad + C_m h\left(\varepsilon\,\|\Delta^{m+1}_{\tau_{k+1}}H_{E_{k+1}}\|^2_{L^2} + \frac{1}{\varepsilon}\|B_{E_{k+1}}\|^2_{H^{2m+1}}\right) + C_m h\,\left(1+\|\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\|^2_{L^2}\right)\\
&\quad\leq -\frac{3h}{2}\|\Delta^{m+1}_{\tau_{k+1}}H_{E_{k+1}}\|_{L^2} + C_m h\,\|B_{E_{k+1}}\|^2_{H^{2m+1}} + C_m h\,\left(1+\|\nabla_{\tau_k}\Delta^m_{\tau_k}H_{E_k}\|^2_{L^2}\right).
\end{aligned}$$

Again, Proposition 2.2 implies that there is $\theta = \theta(m,n) \in (0,1)$ such that

$$\|B_{E_{k+1}}\|^2_{H^{2m+1}} \leq C_m\|B_{E_{k+1}}\|^{2\theta}_{H^{2m+2}}\|B_{E_{k+1}}\|^{2(1-\theta)}_{L^\infty}$$

and we may proceed as previously to obtain (5.7).

Let us then prove the claim by induction. To be more precise, under the assumption $h \leq h_0$, we claim that, for every $l \in \mathbb{N}\cup\{0\}$ it holds that

$$\max_{l+2\leq k\leq K}\big((k-(l+1))h\big)^l\|H_{E_k}\|^2_{H^l} + \sum_{k=l+2}^{K} h\big((k-(l+1))h\big)^l\|H_{E_k}\|^2_{H^{l+1}}\,\mathrm{d}t \leq C_l \tag{5.10}$$

for $C_l = C_l(l,n,m_0,r_0,T)$, provided that $(l+2)h \leq T$. Since $t-lh \leq 3\lfloor t/h\rfloor h - 3(l+1)h$ for every $t \geq (l+2)h$, then by multiplying (5.10) by 3^l and recalling the definition for the approximative solution in (3.1), we obtain the statement of the theorem.

Let us consider first the case $l = 0$. Since $P(E_k), \|B_{E_k}\|_{L^\infty} \leq C$, then $\|H_{E_k}\|_{L^2} \leq C$ for every $k = 0,1,\ldots,K$. By combining this with (5.6) gives us that for every $k = 1,2,\ldots,K-1$

$$\|H_{E_{k+1}}\|^2_{L^2} - \|H_{E_k}\|^2_{L^2} \leq -h\|\nabla_{\tau_{k+1}}H_{E_{k+1}}\|^2_{L^2} + Ch.$$

We sum over $k = 1, 2, \dots, K-1$ and use $\|H_{E_k}\|_{L^2} \leq C$ as well as $Kh \leq T$ to obtain

$$\|H_{E_K}\|_{L^2}^2 + \sum_{k=1}^{K-1} h \|\nabla_{\tau_{k+1}} H_{E_{k+1}}\|_{L^2}^2 \leq \|H_{E_1}\|_{L^2}^2 + CKh \leq CT.$$

Thus, we conclude that (5.10) holds in the case $l = 0$.

Let us then assume that (5.10) holds for $l-1$, where $l \in \mathbb{N}$. We assume that $(l+2)h \leq T$ and prove (5.10) for l. To this aim, we denote $K' = K - l$ and $E'_k = E_{k+l}$. Again, let τ_k denote the tangential differentiation along $\partial E'_k$. Thus, the induction assumption reads as

$$\max_{1 \leq k \leq K'} (kh)^{l-1} \|H_{E'_k}\|_{H^{l-1}}^2 + \sum_{k=1}^{K'} h (kh)^{l-1} \|H_{E'_k}\|_{H^l(\partial E_t^h)}^2 \leq C_{l-1}. \tag{5.11}$$

We divide the argument into two cases depending whether l is even or odd.

Let us first assume that l is even and thus is of the form $l = 2m$ for $m = 1, 2, \dots$. By binomial expansion it holds $(k+1)^{2m} - k^{2m} \leq 2m(k+1)^{2m-1}$. Therefore, by multiplying (5.6) by $k^{2m}h^{2m}$ we deduce, for every $k = 0, 1, 2, \dots, K'$,

$$\begin{aligned}
&(k+1)^{2m} h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 - k^{2m} h^{2m} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 \\
&\qquad = \big((k+1)^{2m} - k^{2m}\big) h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 \\
&\qquad\quad + k^{2m} h^{2m} \big(\|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 - \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2\big) \\
&\qquad \leq 2m(k+1)^{2m-1} h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 + C_m k^{2m} h^{2m+1} \left(1 + \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2\right) \\
&\qquad\quad - k^{2m} h^{2m+1} \|\nabla_{\tau_{k+1}} \Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2.
\end{aligned}$$

Fix any $j = 2, \dots K'$. Summing the previous estimate from $k = 0$ to $k = j-1$ and using the fact $K'h \leq T$ yields

$$\begin{aligned}
&j^{2m} h^{2m} \|\Delta_{\tau_j}^m H_{E'_j}\|_{L^2}^2 \\
&\leq C_m \sum_{k=0}^{j-1} (k+1)^{2m-1} h^{2m} \|\Delta_{\tau_{k+1}}^m H_{E'_{k+1}}\|_{L^2}^2 \\
&\quad + C_m \sum_{k=0}^{j-1} (kh) k^{2m-1} h^{2m} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 + C_m \sum_{k=0}^{j-1} h\, k^{2m} h^{2m} \\
&\quad - \sum_{k=0}^{j-1} k^{2m} h^{2m+1} \|\nabla_{\tau_{k+1}} \Delta_{\tau}^m H_{E'_{k+1}}\|_{L^2}^2 \\
&\leq C_m (1+T) \sum_{k=1}^{j} k^{2m-1} h^{2m} \|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 \\
&\quad + C_m \int_0^{K'h} s^{2m}\, \mathrm{d}s - \sum_{k=1}^{j} h\, (k-1)^{2m} h^{2m} \|\nabla_{\tau_k} \Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2
\end{aligned}$$

$$\leq C_m(1+T)\sum_{k=1}^{K'} h\, k^{2m-1}h^{2m-1}\|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2$$

$$+ C_m T^{2m+1} - \sum_{k=1}^{j} h\,(k-1)^{2m}h^{2m}\|\nabla_{\tau_k}\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2.$$

Thus, reordering the previous estimate and using the induction assumption (5.11) gives us

$$\begin{aligned}(j-1)^{2m}h^{2m}\|\Delta_{\tau_j}^m H_{E'_j}\|_{L^2}^2 &+ \sum_{k=1}^{j} h\,(k-1)^{2m}h^{2m}\|\nabla_{\tau_k}\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2\\ &\leq C_m(1+T)\sum_{k=1}^{K'} k^{2m-1}h^{2m}\|\Delta_{\tau_k}^m H_{E'_k}\|_{L^2}^2 + C_m T^{2m+1}\\ &\leq C_m(1+T)\sum_{k=1}^{K'} h(kh)^{l-1}\|H_{E'_k}\|_{H^l}^2 + C_m T^{2m+1}\\ &\leq C_m C_{l-1} + C_m T^{2m+1}.\end{aligned}$$

After substituting $E'_k = E_{k+l}$ and reindexing we have for every $j = l+2,\dots,K$

$$\begin{aligned}&\big((j-(l+1))h\big)^l\|\Delta_{\tau_j}^m H_{E_k}\|_{L^2}^2 + \sum_{k=l+1}^{j} h\,\big((k-(l+1))h\big)^l\|\nabla_{\tau_k}\Delta_{\tau_k}^m H_{E_k}\|_{L^2}^2\\ &\leq C_m C_{l-1} + C_m T^{2m+1}.\end{aligned}$$

Since we have $\|B_{E_k}\|_{L^\infty}, P(E_k) \leq C$ for every $k \leq K$, then by combining the estimates of Proposition 2.6 with the previous estimate and using $Kh \leq T$ we obtain (5.10).

The case when l is odd is similar. In this case, we have $l = 2m+1$ for some $m \in \mathbb{N}\cup\{0\}$. Thus, by using (5.7) in the place of (5.6) we may proceed as in the previous case. □

Let us then focus on Proposition 5.1. We will begin by proving two technical lemmas which involve high order derivatives of d_E and $\pi_{\partial E}$. To overcome the technicalities we adopt the notation where A_i denotes a generic tensor field, which depends on the distance function, the normal and the second fundamental form in a smooth way, i.e.,

$$A_i = A_i(d_E, \nu_E\circ\pi_{\partial E}, B_E\circ\pi_{\partial E}) \ \text{ in } \mathcal{N}_{r/2}(\partial E). \tag{5.12}$$

We also adopt here the notation $S \star T$ to denote a tensor formed by contraction on some indexes of tensors S and T. If the set E satisfies UBC, then the quantities d_E, ν_E and $B_E\circ\pi_{\partial E}$ are uniformly bounded in $\mathcal{N}_{r/2}(\partial E)$, we may treat A_i in (5.12) as a bounded coefficient.

It is immediate that it holds for $x \in \partial E$ and $u \in C^2(\partial E)$ that

$$\nabla(u \circ \pi_{\partial E})(x) = \nabla_\tau u(x) \quad \text{and} \quad \Delta_{\mathbb{R}^{n+1}}(u \circ \pi_{\partial E})(x) = \Delta_\tau u(x).$$

Let us then derive related formulas for points $x \in \mathcal{N}_{r/2}(\partial E)$ outside ∂E.

Lemma 5.3. *Assume $E \subset \mathbb{R}^{n+1}$, with $\Sigma = \partial E$, is bounded and C^3-regular set which satisfies UBC with radius r. Then it holds for $u \in C^2(\partial E)$ in $\mathcal{N}_{r/2}(\partial E)$ that*

$$\nabla(u \circ \pi_{\partial E}) = \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E} - d_E \nabla^2 d_E \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E}$$

and

$$\begin{aligned} \nabla^2(u \circ \pi_{\partial E}) =& (P_{\partial E} \circ \pi_{\partial E})(\nabla^2(u \circ \pi_{\partial E}) \circ \pi_{\partial E}) \\ & - \nabla d_E \otimes \nabla^2 d_E \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E} \\ & + d_E\, A_1 \star \nabla^2(u \circ \pi_{\partial E}) \circ \pi_{\partial E} \\ & + d_E\, A_2 \star \nabla(B_E \circ \pi_{\partial E}) \circ \pi_{\partial E} \star \nabla(u \circ \pi_{\partial E}) \circ \pi_{\partial E} \end{aligned}$$

where A_1, A_2 are tensor fields as in (5.12). Moreover, if Σ is in addition C^{k+2}-regular and $u \in C^k(\Sigma)$ for $k \in \mathbb{N}$, then for all $x \in \mathcal{N}_{r/2}(\partial E)$ we may estimate

$$\begin{aligned} &|\nabla^k(u \circ \pi_{\partial E})(x)| \\ &\quad \leq C_k \sum_{|\alpha| \leq k} \big(1 + |\tilde{\nabla}_\Sigma^{\alpha_1} B_E(\pi_{\partial E}(x))| \cdots |\tilde{\nabla}_\Sigma^{\alpha_{k-1}} B_E(\pi_{\partial E}(x))|\big)\, |\tilde{\nabla}_\Sigma^{\alpha_k} u(\pi_{\partial E}(x))|. \end{aligned}$$

Here $\tilde{\nabla}_\Sigma$ denotes the covariant derivative on Σ.

Proof. Let us denote $\hat{u} = u \circ \pi_{\partial E}$ and $\pi = \pi_{\partial E}$ for short. Since π is projection it holds

$$\hat{u}(x) = \hat{u}(\pi(x))$$

for all $x \in \mathcal{N}_{r/2}(\partial E)$. By differentiating this we obtain

$$\nabla \hat{u}(x) = \nabla \pi(x) \nabla \hat{u}(\pi(x)).$$

The first claim then follows from (2.31) and from $\nabla \hat{u} \cdot (\nu_E \circ \pi) = 0$. The second claim follows by differentiating the first and by writing $\nabla^2 d_E(x)$, $\nabla^3 d_E(x)$ and $\nabla \pi$ in a geometric way by using (2.33) and (2.34).

In order to prove the third claim we observe that we may write the second equality simply as

$$\nabla^2 \hat{u}(x) = A_1(x) \star \nabla^2 \hat{u}\big(\pi(x)\big) + A_2(x) \star \nabla(B_E \circ \pi)(\pi_{\partial E}(x)) \star \nabla \hat{u}(\pi(x)).$$

By differentiating this $(k-2)$-times and by using (2.12) and (2.34) we deduce that

$$|\nabla^k \hat{u}(x)| \leq C_k \sum_{|\alpha| \leq k} C\big(1 + |\nabla^{\alpha_1}(B_E \circ \pi)(\pi(x))| \cdots |\nabla^{\alpha_{k-1}}(B_E \circ \pi)(\pi(x))|\big) |\nabla^{\alpha_k} \hat{u}(\pi(x))|.$$

The claim follows once we show that for all $y \in \Sigma$ it holds that

$$|\nabla^l \hat{u}(y)| \leq C_l \sum_{|\beta| \leq l} \big(1 + |\tilde{\nabla}^{\beta_1} B_E(y)| \cdots |\tilde{\nabla}^{\beta_l} B_E(y)|\big) |\bar{\nabla}^{\beta_{l+1}} u(y)|, \tag{5.13}$$

which is the opposite estimate as to that of Lemma 2.4.

We argue as in the proof of Lemma 2.4 and assume $y = 0$, $\nu_E(0) = e_{n+1}$ and write the surface Σ locally as a graph of f, i.e., $\Sigma \cap B_r \subset \{(x', f(x')) : x' \in \mathbb{R}^n\}$ and extended f to $\mathbb{R}^{n+1}$ trivially as $f(x', x_{n+1}) = f(x')$. We may then write the metric tensor and the Christoffel symbols in coordinates as

$$g_{ij}(x') = \delta_{ij} + \partial_i f(x') \partial_j f(x') \quad \text{and} \quad \Gamma^i_{jk}(x') = g^{il}(x') \partial^2_{jk} f(x') \partial_l f(x').$$

Since $\nu_E = \frac{(-\nabla_{\mathbb{R}^n} f, 1)}{\sqrt{1 + |\nabla_{\mathbb{R}^n} f|^2}}$ and $\nabla \hat{u} \cdot (\nu_E \circ \pi) = 0$, we have

$$\partial_{n+1} \hat{u}(y) = \sum_{i=1}^{n} \partial_i f(\pi(y)) \cdot \partial_i \hat{u}(y). \tag{5.14}$$

Let us denote the lth order differential of the function $x' \to \hat{u}(x', 0)$ as $\nabla^l_{\mathbb{R}^n} \hat{u}$. Then by applying first (5.14) and (2.12), and then (2.14) we deduce that

$$\begin{aligned} |\nabla^l \hat{u}(0)| &\leq C_l \sum_{|\beta| \leq l-1} \big(1 + |\nabla^{\beta_1} (\nabla f \circ \pi)(0)| \cdots |\nabla^{\beta_{l-1}} (\nabla f \circ \pi)(0)|\big) |\nabla^{1+\beta_l}_{\mathbb{R}^n} \hat{u}(0)| \\ &\leq C_l \sum_{|\gamma| \leq l-1} \big(1 + |\tilde{\nabla}^{\gamma_1} B_E(0))| \cdots |\tilde{\nabla}^{\gamma_{l-1}} B_E(0)|\big) |\nabla^{1+\gamma_l}_{\mathbb{R}^n} \hat{u}(0)|. \end{aligned}$$

Denote the local chart given by the coordinate parametrization by Φ, i.e., $\Phi^{-1}(x') = (x', f(x'))$ and note that $\hat{u}(\Phi^{-1}(x')) = u(\Phi^{-1}(x'))$. Fix an index vector $\beta = (\beta_1, \ldots, \beta_n, 0)$ with $|\beta| = m$. Then by (2.12) and (2.14) we obtain after straightforward calculations

$$\begin{aligned} |\nabla^\beta (u \circ \Phi^{-1})(0)| &\geq |\nabla^\beta \hat{u}(0)| - C_m \sum_{|\gamma| \leq m-1} \big(1 + |\nabla^{1+\gamma_1} f(0)| \cdots |\nabla^{1+\gamma_{m-1}} f(0)|\big) ||\nabla^{\gamma_m} \hat{u}(0)| \\ &\geq |\nabla^\beta \hat{u}(0)| - C_m \sum_{|\gamma| \leq m-1} \big(1 + |\tilde{\nabla}^{\gamma_1} B_E(0)| \cdots |\tilde{\nabla}^{\gamma_{m-1}} B_E(0)|\big) ||\nabla^{\gamma_m} \hat{u}(0)|. \end{aligned}$$

From here we deduce by an inductive argument that

$$|\nabla^\beta \hat{u}(0)| \leq C_m \sum_{|\gamma| \leq m} \big(1 + |\tilde{\nabla}^{\gamma_1} B_E(0)| \cdots |\tilde{\nabla}^{\gamma_m} B_E(0)|\big) ||\nabla^{\gamma_{m+1}} (u \circ \Phi^{-1})(0)|.$$

Finally using the definition of the covariant derivative and the expression of the Christoffel symbols we obtain arguing as in the proof of Lemma 2.4 that

$$|\nabla^m (u \circ \Phi^{-1})(0)| \leq C_m \sum_{|\gamma| \leq m} \big(1 + |\tilde{\nabla}^{\gamma_1} B_E(0)| \cdots |\tilde{\nabla}^{\gamma_m} B_E(0)|\big) ||\tilde{\nabla}^{\gamma_{m+1}} u(0)|.$$

Hence, we have (5.13) and the third claim follows. □

Let us from now on assume $E_1, E_2 \subset \mathbb{R}^{n+1}$ are as in Proposition 5.1. We write the equality (5.3) by using the Euler-Lagrange equation (3.4) as

$$H_{E_2} - H_{E_1} \circ \pi_{\partial E_1} = h\, \Delta_{\tau_2} H_{E_2} + h\, \rho_0(\cdot) \tag{5.15}$$

on ∂E_2, where the error function is of the form

$$\rho_0(x) = A_1(x) + h\, A_2(x) \star \nabla_{\tau_2} H_{E_2}(x) \star \nabla_{\tau_2} H_{E_2}(x). \tag{5.16}$$

Here and in the rest of the section $A_i(\cdot)$ denotes a tensor field which depends smoothly on d_{E_1}, $\nu_{E_1} \circ \pi_{\partial E_1}$, ν_{E_2}, $B_{E_1} \circ \pi_{\partial E_1}$ and on B_{E_2}. i.e.,

$$A_i(x) = A_i\big(d_{E_1}(x), \nu_{E_1}(\pi_{\partial E_1}(x)), \nu_{E_2}(x), B_{E_1}(\pi_{\partial E_1}(x)), B_{E_2}(x)\big). \tag{5.17}$$

The following lemma is a consequence of Lemma 5.3.

Lemma 5.4. *Assume that the sets $E_1, E_2 \subset \mathbb{R}^{n+1}$ are as in Proposition 5.1. Then it holds for $u \in C^2(\partial E_1)$ on ∂E_2*

$$\begin{aligned}\Delta_{\tau_2}(u \circ \pi_{\partial E_1}) =& \Delta_{\tau_1} u \circ \pi_{\partial E_1} + h\, A_1 \star \nabla^2(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ &+ h^2\, A_2 \star \nabla^2(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2} \\ &+ h\, A_3 \star \nabla(B_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\ &+ h\, A_4 \star \nabla_{\tau_2} B_{E_2} \star \nabla(u \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}.\end{aligned}$$

Proof. Let us denote $\hat{u} = u \circ \pi_{\partial E_1}$ and $\pi = \pi_{\partial E_1}$ for short. Recall that we may write the Laplace-Beltrami on ∂E_2 as

$$\Delta_{\tau_2} \hat{u} = \Delta_{\mathbb{R}^{n+1}} \hat{u} - (\nabla^2 \hat{u}\, \nu_{E_2} \cdot \nu_{E_2}) - H_{E_2} \partial_{\nu_{E_2}} \hat{u}, \tag{5.18}$$

where $\Delta_{\mathbb{R}^{n+1}} \hat{u} = \mathrm{Tr}(\nabla^2 \hat{u})$ denotes the Euclidean Laplacian. Recall that $P_{\partial E_1} = I - \nu_{E_1} \otimes \nu_{E_1}$ stands for the projection on the (geometric) tangent space. We deduce by applying the trace on the second equality in Lemma 5.3, by $\nabla^2 d_{E_1} \nabla d_{E_1} = 0$, and by the Euler-Lagrange equation (3.4) that it holds on ∂E_2

$$\begin{aligned}\Delta_{\mathbb{R}^{n+1}} \hat{u} = \mathrm{Tr}(\nabla^2 \hat{u}) &= \Delta_{\tau_1} u \circ \pi + h\, A_1 \star (\nabla^2 \hat{u} \circ \pi) \\ &+ h\, A_3 \star \nabla(B_{E_1} \circ \pi) \circ \pi \star (\nabla \hat{u} \circ \pi).\end{aligned} \tag{5.19}$$

Similarly, we have

$$\begin{aligned}(\nabla^2 \hat{u}\, \nu_{E_2}) \cdot \nu_{E_2} =& \big((P_{\partial E_1} \circ \pi)(\nabla^2 \hat{u} \circ \pi)\, \nu_{E_2}\big) \cdot \nu_{E_2} \\ &- (\nabla d_{E_1} \cdot \nu_{E_2})\big(\nabla^2 d_{E_1} (\nabla \hat{u} \circ \pi) \cdot \nu_{E_2}\big) \\ &+ h\, \tilde{A}_1 \star (\nabla^2 \hat{u} \circ \pi) + h\, \tilde{A}_3 \star \nabla(B_{E_1} \circ \pi) \circ \pi \star (\nabla \hat{u} \circ \pi).\end{aligned} \tag{5.20}$$

We write

$$\begin{aligned}&\big((P_{\partial E_1} \circ \pi)(\nabla^2 \hat{u} \circ \pi)\, \nu_{E_2}\big) \cdot \nu_{E_2} \\ &\quad = \big((P_{\partial E_1} \circ \pi)(\nabla^2 \hat{u} \circ \pi)\, (\nu_{E_2} - \nu_{E_1} \circ \pi)\big) \cdot (\nu_{E_2} - \nu_{E_1} \circ \pi)\end{aligned}$$

and

$$
\begin{aligned}
&(\nabla d_{E_1} \cdot \nu_{E_2})\big(\nabla^2 d_{E_1}(\nabla \hat{u} \circ \pi) \cdot \nu_{E_2}\big) \\
&\quad = (\nabla d_{E_1} \cdot \nu_{E_2})\big(\nabla^2 d_{E_1}(\nabla \hat{u} \circ \pi) \cdot (\nu_{E_2} - \nu_{E_1} \circ \pi)\big).
\end{aligned}
$$

We then use (4.19) to write $\nu_{E_2} - \nu_{E_1} \circ \pi$ as

$$
\nu_{E_2} - \nu_{E_1} \circ \pi = a_1 \nabla_{\tau_2} d_{E_1} + a_2\, (\nu_{E_1} \circ \pi)
$$

for functions a_1 and a_2 which depend on $|\nabla_{\tau_2} d_{E_1}(x)|^2$. Therefore we may write (5.20) by the Euler-Lagrange equation (3.4) as

$$
\begin{aligned}
(\nabla^2 \hat{u}\, \nu_{E_2}) \cdot \nu_{E_2} = {}& h\, A_1 \star (\nabla^2 \hat{u} \circ \pi) \\
& + h^2\, A_2 \star (\nabla^2 \hat{u} \circ \pi) \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2} \\
& + h\, A_3 \star \nabla(B_{E_1} \circ \pi) \circ \pi \star (\nabla \hat{u} \circ \pi) \\
& + h\, A_4 \star \nabla_{\tau_2} H_{E_2} \star (\nabla \hat{u} \circ \pi).
\end{aligned} \tag{5.21}
$$

We use the first equality in Lemma 5.3, (4.19) and the Euler-Lagrange equation (3.4) to write on ∂E_2

$$
\begin{aligned}
\partial_{\nu_{E_2}} \hat{u} &= (\nabla \hat{u} \circ \pi) \cdot \nu_{E_2} + h\, A_3 \star (\nabla \hat{u} \circ \pi) \\
&= (\nabla \hat{u} \circ \pi) \cdot (\nu_{E_2} - \nu_{E_1} \circ \pi) + h\, A_3 \star (\nabla \hat{u} \circ \pi) \\
&= h\, A_4 \star \nabla_{\tau_2} H_{E_2} \star (\nabla \hat{u} \circ \pi) + h\, A_3 \star (\nabla \hat{u} \circ \pi).
\end{aligned} \tag{5.22}
$$

The claim then follows from (5.18), (5.19), (5.21) and (5.22). □

We may now prove Proposition 5.1.

Proof of Proposition 5.1. We prove only the first equality since the second follows by differentiating the first. We point out that since E_1 is C^{2m+3}-regular, then by Lemma 3.2 the set E_2 is C^{2m+5}-regular. In particular, we have the necessary regularity for the proceeding calculations. To that aim we recall that by (5.15) it holds

$$
H_{E_2} - H_{E_1} \circ \pi_{\partial E_1} = h\, \Delta_{\tau_2} H_{E_2} + h\, \rho_0 \qquad \text{on } \partial E_2, \tag{5.23}
$$

where

$$
\rho_0(x) = A_1(x) + h\, A_2(x) \star \nabla_{\tau_2} H_{E_2}(x) \star \nabla_{\tau_2} H_{E_2}(x).
$$

We differentiate (5.23), use Lemma 5.4 and have on ∂E_2

$$
\Delta_{\tau_2} H_{E_2}(x) - \Delta_{\tau_1} H_{E_1} \circ \pi_{\partial E_1} = h\, \Delta_{\tau_2}^2 H_{E_2} + h\rho_2 + h\, \Delta_{\tau_2} \rho_0,
$$

where

$$
\begin{aligned}
\rho_2 = {}& A_1 \star \nabla^2 (H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\
& + h\, A_2 \star \nabla^2 (H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2} \\
& + A_3 \star \nabla (B_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla (H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \\
& + A_4 \star \nabla_{\tau_2} B_{E_2} \star \nabla (H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}.
\end{aligned}
$$

We continue and deduce by an iterative argument that it holds on ∂E_2

$$\Delta_{\tau_2}^m H_{E_2} - \Delta_{\tau_1} H_{E_1}^m \circ \pi_{\partial E_1} = h\,\Delta_{\tau_2}^{m+1} H_{E_2} + h \sum_{k=0}^{m} \Delta_{\tau_2}^{m-k} \rho_{2k},$$

where ρ_0 is defined in (5.16) and ρ_{2k} for $k \geq 1$ is

$$\begin{aligned}\rho_{2k} &= A_1 \star \nabla^2(\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}\\ &\quad + h\,A_2 \star \nabla^2(\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla_{\tau_2} H_{E_2} \star \nabla_{\tau_2} H_{E_2}\\ &\quad + A_3 \star \nabla(B_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1} \star \nabla(\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}\\ &\quad + A_4 \star \nabla_{\tau_2} B_{E_2} \star \nabla(\Delta_{\tau_1}^{k-1} H_{E_1} \circ \pi_{\partial E_1}) \circ \pi_{\partial E_1}.\end{aligned}$$

We have thus derived a formula for the error terms in the statement of Proposition 5.1, i.e., we have

$$R_{2m}(x) = \sum_{k=0}^{m} \Delta_{\tau_2}^{m-k} \rho_{2k}(x).$$

We need to estimate the norm $\|R_{2m}\|_{L^2(\Sigma_2)}$, where $\Sigma_2 = \partial E_2$. The idea is that the total amount of derivatives acting on the curvature terms in $\Delta_{\tau_2}^{m-k}\rho_{2k}$ is for most of the terms at most $2m$. The only difference is the second row in the definition of ρ_{2k}, which total amount of derivatives is higher but it has an extra h as a coefficient. Therefore we need to treat this term more carefully.

Recall that the tensor fields $A_i(\cdot)$ depend on d_{E_1}, $\nu_{E_1} \circ \pi_{\partial E_1}$, ν_{E_2}, $B_{E_1} \circ \pi_{\partial E_1}$ and on B_{E_2} as stated in (5.17). Denote $\pi = \pi_{\partial E_1}$ for short. We use repeatedly (2.12), Lemma 2.4 and the last inequality in Lemma 5.3 and obtain after long but straightforward calculations the following pointwise estimate for all $x \in \partial E_2$:

$$\begin{aligned}\left|\Delta_{\tau_2}^{m-k}\rho_{2k}(x)\right| \leq C &+ C \sum_{|\alpha| \leq 2m} |\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)|\\ &+ C \sum_{|\alpha| \leq 2m} |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))|\\ &+ Ch \sum_{|\alpha| \leq 2m} (|\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))|) \cdots\\ &\quad (|\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))|) \cdots\\ &\quad \cdots |\tilde{\nabla}^{1+\alpha_{2m+1}} H_{E_2}(x)|\,|\tilde{\nabla}^{1+\alpha_{2m+2}} H_{E_2}(x)|.\end{aligned} \tag{5.24}$$

We use the uniform curvature bounds $\|B_{\Sigma_1}\|_{L^\infty}, \|B_{\Sigma_2}\|_{L^\infty} \leq C$ and Proposition 2.3 to estimate

$$\sum_{|\alpha| \leq 2m} \||\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| \cdots |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)|\|_{L^2(\Sigma_2)} \leq C \|B_{\Sigma_2}\|_{H^{2m}(\Sigma_2)}$$

and

$$\begin{aligned}&\sum_{|\alpha|\le 2m} \||\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))|\cdots|\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))|\|_{L^2(\Sigma_2)}\\ &\qquad \le C \sum_{|\alpha|\le 2m} \||\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(y)|\cdots|\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(y)|\|_{L^2(\Sigma_1)} \le C\|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)}.\end{aligned}$$

We are left with the last term in (5.24). As we already mentioned, this term has different scaling with respect to h. We use the Euler-Lagrange equation (3.4), (4.23) and $\|d_{E_1}\|_{L^\infty(\partial E_2)} \le Ch$ from Proposition 3.1 to deduce that

$$\|\tilde{\nabla} H_{E_2}\|^2_{L^\infty(\Sigma_2)} \le \frac{C}{h}.$$

Therefore we have, by Proposition 2.3, that

$$\begin{aligned}&h \sum_{|\alpha|\le 2m} \|(|\tilde{\nabla}^{\alpha_1} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_1} B_{\Sigma_1}(\pi(x))|)\cdots\\ &\quad (|\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_2}(x)| + |\tilde{\nabla}^{\alpha_{2m}} B_{\Sigma_1}(\pi(x))|)\cdots\\ &\quad \cdots|\tilde{\nabla}^{1+\alpha_{2m+1}} H_{E_2}(x)|\,|\tilde{\nabla}^{1+\alpha_{2m+2}} H_{E_2}(x)|\|_{L^2(\Sigma_2)}\\ &\le Ch\|\tilde{\nabla} H_{E_2}\|^2_{L^\infty(\Sigma_2)}\big(\|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)} + \|B_{\Sigma_2}\|_{H^{2m}(\Sigma_2)}\big)\\ &\quad + Ch\|\tilde{\nabla} H_{E_2}\|_{L^\infty(\Sigma_2)}\|H_{E_2}\|_{H^{2m+1}(\Sigma_2)}\\ &\le C\|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)} + C\|B_{\Sigma_2}\|_{H^{2m}(\Sigma_2)} + C\sqrt{h}\|H_{E_2}\|_{H^{2m+1}(\Sigma_2)}\\ &\le C\|B_{\Sigma_1}\|_{H^{2m}(\Sigma_1)} + C\|B_{\Sigma_2}\|_{H^{2m+1}(\Sigma_2)}\end{aligned}$$

when $h \le 1$, and the claim follows. □

Let us conclude this section by discussing briefly how we obtain Theorem 1.1 and Corollary 1.2 from the results in Sects. 4 and 5. We obtain first from Lemma 4.6 and from Theorem 4.7 that the approximative flow $(E_t^h)_k$ satisfies UBC with radius $r_0/2$ for $t \le T_0$ and we have

$$\frac{\|S_{E^h_{t+h}}\|_{L^\infty} - \|S_{E^h_t}\|_{L^\infty}}{h} \le C_n\|S_{E^h_t}\|^3_{L^\infty}. \tag{5.25}$$

Then we use Theorem 5.2 to deduce that for $t \in [\delta, T_0]$ the sets E_t^h are uniformly C^3-regular when h is small enough. By Ascoli-Arzela theorem we may pass the estimate (5.25) to the limit as $h \to 0$ and conclude that the function $t \mapsto \sup_{s\le t}\|S_{E_s}\|_{L^\infty}$ is locally Lipschitz continuous and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\sup_{s\le t}\|S_{E_s}\|_{L^\infty}\big) \le C_n\big(\sup_{s\le t}\|S_{E_s}\|_{L^\infty}\big)^3 \tag{5.26}$$

for almost every $t \ge 0$ as long as $\sup_{s\le t}\|S_{E_s}\|_{L^\infty}$ remains bounded. The inequality (5.26) implies that UBC is an open condition in time. To be more precise if the flat flow $(E_t^h)_t$, starting from E_0, satisfies $\sup_{t\leq T}\|S_{E_t}\|_{L^\infty}\leq C$, then by (5.26) there is $\delta>0$ such that

$$\sup_{t\leq T+\delta}\|S_{E_t}\|_{L^\infty}\leq 2C.$$

This together with the estimate in Theorem 5.2 implies Theorem 1.1.

The consistency principle follows from the regularity in a rather straightforward way. Indeed, we obtain by the uniform regularity of the approximate flat flow $(E_t^h)_{t\in[0,T]}$ and by the Euler-Lagrange equation (3.4) that the signed distance function satisfies

$$\partial_t d_{E_t}(x)=\Delta_{\mathbb{R}^{n+1}} d_{E_t}(\pi_{\partial E_t}(x))+f(t)$$

for $t\leq T$ and for x in a neighborhood of ∂E_t, where $f(t)$ is a bounded function ot time. From here we may conclude that the flat flow satisfies

$$V_t=-H_{E_t}+f(t).$$

Since the flat flow preserves the volume then necessarily $f(t)=\fint_{\partial E_t} H_{E_t}\,\mathrm{d}\mathcal{H}^n$ and thus it is a solution to (1.1).

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Declarations

Conflicts of interest The authors declare that they have no conflicts of interest.

References

1. ACERBI, E., FUSCO, N., MORINI, M.: Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.* **322**, 515–557, 2013

2. Allard, W.K.: On the first variation of a Varifold. *Annals of Math.* **95**, 417–491, 1972
3. Almgren, F., Taylor, J., Wang, L.: Curvature-driven flows: a variational approach. *SIAM J. Optim.* **31**, 387–438, 1993
4. Andrews, B.: Noncollapsing in mean-convex curvature flow. *Geom. Topol.* **16**, 1413–1418, 2012
5. Ambrosio, L., Dancer, N.: Calculus of variations and partial differential equations: topics on geometrical evolution problems and degree theory. Springer-Verlag, Berlin-Heidelberg (2000)
6. Aubin, T.: Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin (1999)
7. Bellettini, G.: *Lecture notes on mean curvature flow, barriers and singular perturbations*. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie), 12. Edizioni della Normale, Pisa, 2013
8. Bellettini, G., Caselles, V., Chambolle, A., Novaga, M.: The volume preserving crystalline mean curvature flow of convex sets in $\mathbb{R}^N$. *J. Math. Pure Appl.* **92**, 499–527, 2009
9. Brakke, K.A.: *The Motion of a Surface by its Mean Curvature.* Math. Notes 20, Princeton Univ. Press, Princeton, NJ 1978
10. Brendle, S.: *Two-point functions and their applications in geometry.* Bull. Amer. Math. Soc. (N.S.) **51**, 581–596 2014
11. Brendle, S.: A sharp bound for the inscribed radius under mean curvature flow. *Invent. Math.* **202**, 217–237, 2015
12. Chambolle, A.: An algorithm for mean curvature motion. *Interfaces and free Boundaries* **6**, 195–218, 2004
13. Chambolle, A., Morini, M., Ponsiglione, M.: *Minimizing movements and level set approaches to nonlocal variational geometric flows.* Geometric partial differential equations, 93–104, CRM Series, 15, Ed. Norm., Pisa, 2013
14. Chambolle, A., Morini, M., Ponsiglione, M.: Existence and uniqueness for a crystalline mean curvature flow. *Comm. Pure Appl. Math.* **70**, 1084–1114, 2017
15. Chen, Y.G., Giga, Y., Goto, S.: *Uniqueness and existence of viscosity solutions of generalized mean curvature.* Proc. Japan Acad. Ser. A Math. Sci. **65**, 207–210 1989
16. De Gennaro, D., Kubin, A.: *Long time behaviour of the discrete volume preserving mean curvature flow in the flat torus*. Calc. Var. Partial Differ. Equ. **62** (2023), no. 3, Paper No. 103
17. De Philippis, G., Laux, T.: *Implicit time discretization for the mean curvature flow of mean convex sets*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **21**, 911–930, 2020
18. De Philippis, G., Goldman, M.: *A two-point function approach to connectedness of drops in convex potentials*, Comm. Anal. Geom. **30** (2022), no. 4, 815–841.
19. Escher, J., Simonett, G.: The volume preserving mean curvature flow near spheres. *Proc. Am. Math. Soc.* **126**, 2789–2796, 1998
20. Evans, L.C., Spruck, J.: Motion of level sets by mean curvature I. *J. Differential Geom.* **33**, 635–681, 1991
21. Evans, L.C., Spruck, J.: Motion of level sets by mean curvature IV. *Journal of Geometric Analysis* **5**(1), 77–114, 1995
22. Fusco, N., Julin, V., Morini, M.: The Surface Diffusion Flow with Elasticity in Three Dimensions. *Arch. Rational. Mech. Anal.* **237**, 1325–1382, 2020
23. Fusco, N., Julin, V., Morini, M.: *Stationary sets and asymptotic behavior of the mean curvature flow with forcing in the plane*, J. Geom. Anal. **32** (2022), no. 2, Paper No. 53.
24. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer-Verlag, Berlin (2001)
25. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. *J. Diff. Geom.* **17**, 255–306, 1982
26. Hensel, S., Laux, T.: *A new varifold solution concept for mean curvature flow: Convergence of the Allen-Cahn equation and weak-string uniqueness.* Preprint 2021

27. Huisken, G.: A distance comparison principle for evolving curves. *Asian J. Math.* **2**, 127–133, 1998
28. Huisken, G.: The volume preserving mean curvature flow. *J. Rein. Angew. Math* **382**, 35–48, 1987
29. Ishii, H., Lions, P.-L.: Viscosity solutions of fully nonlinear secondorder elliptic partial differential equations. *J. Differential Equations* **83**, 26–78, 1990
30. Julin, V., La Manna, D. A.: *A priori estimates for the motion of charged liquid drop: A dynamic approach via free boundary Euler equations.* Preprint 2021
31. Julin, V., Morini, M., Ponsiglione, M., Spadaro, E.: *The Asymptotics of the Area-Preserving Mean Curvature and the Mullins-Sekerka Flow in Two Dimensions*. Preprint 2021
32. Julin, V., Niinikoski, J.: Quantitative Alexandrov theorem and asymptotic behavior of the volume preserving mean curvature flow. *Anal. PDE* **16**, 679–710, 2023
33. Kim, I., Kwon, D.: Volume preserving mean curvature flow for star-shaped sets. *Comm. Partial Diffenretial Equations* **45**, 414–455, 2020
34. Laux, T.: *Weak-Strong uniqueness for volume-preserving mean curvature flow.* Preprint 2022
35. Lee, J. M.: *Riemannian manifolds. An introduction to curvature.* Graduate Texts in Mathematics, **176**. Springer-Verlag, New York, 1997
36. Luckhaus, S., Stürzenhecker, T.: Implicit time discretization for the mean curvature flow equation. *Calc. Var. PDEs* **3**, 253–271, 1995
37. Maggi, F.: *Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory*. Cambridge Studies in Advanced Mathematics, 135. Cambridge University Press, Cambridge 2012
38. Mantegazza, C.: Smooth geometrc evolutions of hypersurfaces. *Geom. Funct. Anal.* **12**, 138–182, 2002
39. Mantegazza, C.: *Lecture notes on Mean Curvature Flow.* Progress in Mathematics **290**. Birkhäuser/ Springer, Basel 2011
40. Mayer, U.F.: A singular example for the average mean curvature flow. *Experimental Mathematics* **10**, 103–107, 2001
41. Mayer, U.F., Simonett, G.: Self-intersections for the surface diffusion and the volume-preserving mean curvature flow. *Differential and Integral Equations* **13**, 1189–1199, 2000
42. Morini, M., Ponsiglione, M., Spadaro, E.: Long time behaviour of discrete volume preserving mean curvature flows. *J. Reine Angew. Math.* **784**, 27–51, 2022
43. Mugnai, L., Seis, C., Spadaro, E.: *Global solutions to the volume-preserving mean-curvature flow.* Calc. Var. Partial. Diff. Eq. **55**, Art. 18, 23 pp. 2016
44. Niinikoski, J.: Volume preserving mean curvature flows near strictly stable sets in flat torus. *J. Differential Equations* **276**, 149–186, 2021
45. Simon, L.: Existence of surfaces minimizing the Willmore functional. *Comm. Anal. Geom.* **1**, 281–326, 1992
46. Simon, L.: *Introduction to Geometric Measure Theory.* Tsinghua Lectures 2014
47. Swartz, D., Yip, N.K.: Convergence of diffusion generated motion to motion by mean curvature. *Comm. Partial Differential Equations* **42**, 1598–1643, 2017
48. Topping, P.: Relating diameter and mean curvature for submanifolds of Euclidean space. *Comment. Math. Helv.* **83**, 539–546, 2008

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