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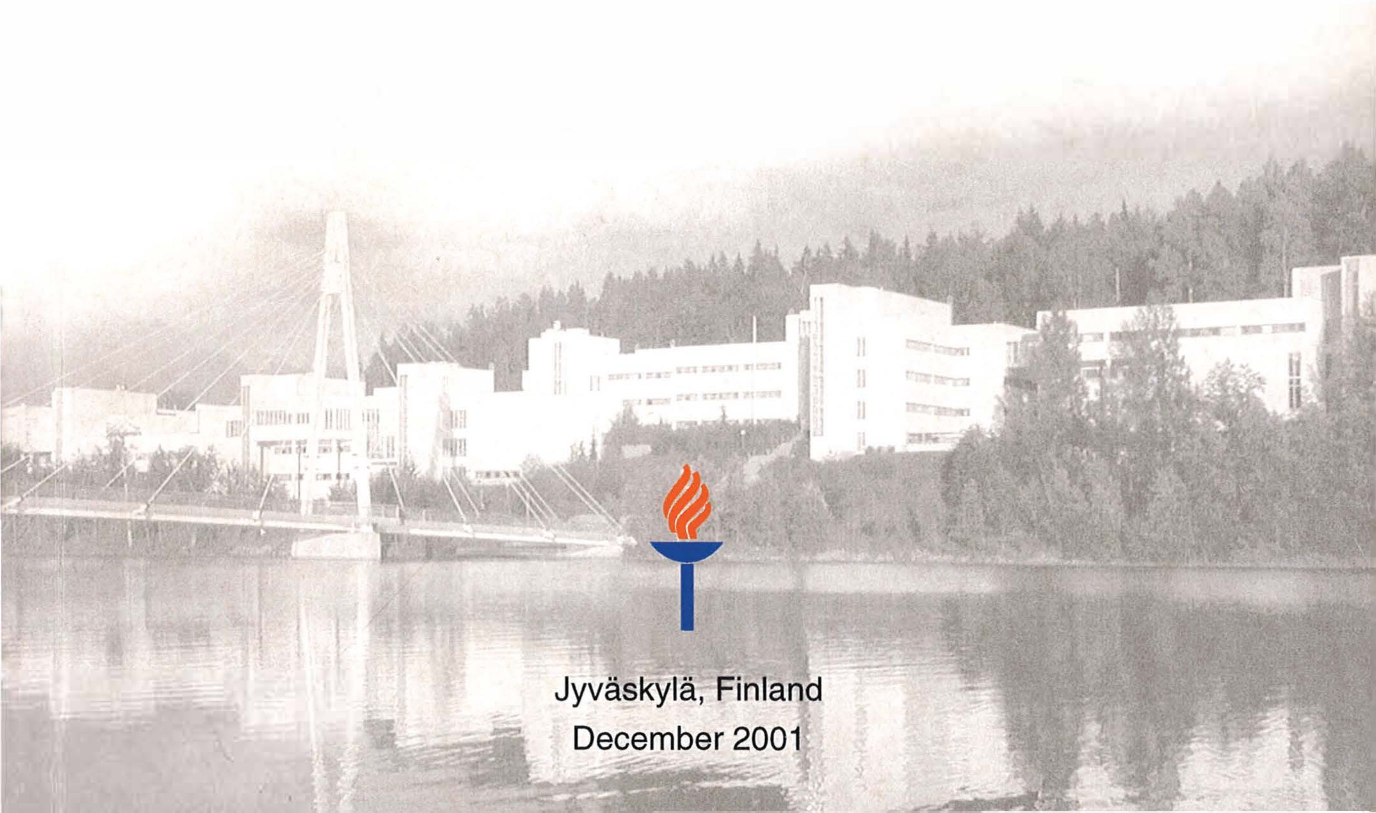
QUANTUM-MECHANICAL MODELS OF BLACK HOLES

BY

PASI REPO

Academic Dissertation
for the Degree of
Doctor of Philosophy



Jyväskylä, Finland
December 2001

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Dedication

◁ to my wife ▷
Aija

◁ to my daughters ▷
Elisa and Anni

Contents

Foreword

Since this thesis contains material which is rather well-known to specialists in the fields of canonical quantum gravity and black hole physics, it may be considered to be somewhat out of the ordinary. Therefore I feel that the motives of my representation should be explained.

During my post-graduate studies my supervisor Jarmo Mäkelä often reminded me not only that I should try to explain physics in as simple terms as possible but he also encouraged me to write my notes in such a manner that they could be used as lecture notes when teaching students here at Jyväskylä. As a result, this academic dissertation for the PhD degree functions as an introduction to black hole physics and its contents should be understandable to a student who has some knowledge about Einstein's general relativity and quantum mechanics. However, I believe now that, at least in most cases, I have attempted to explain the details of black hole physics to myself. For these reasons, I have reviewed the results of ADM formalism, classical black holes, and the preliminaries of quantum black holes.

In addition to the introductory parts this thesis contains some new results found by myself. They have been published in the papers listed in List of Publications. The work leading to the new results in this thesis has been carried out during the years 1997-2000 at the Department of Physics in the University of Jyväskylä.

I should like to express my gratitude to my supervisor and collaborator Dr. Jarmo Mäkelä for his excellent and most inspiring guidance in the vast field of black holes. I am also grateful to my supervisor Dr. Markku Lehto not only for introducing me to the world of physics and to Einstein's theory of general relativity but also because of his invaluable encouragement, comments and advice throughout the work leading to this thesis. I thank Markus Luomajoki and Johanna Piilonen for excellent and productive cooperation. I also want to thank Dr. Jorma Louko for invaluable critique during the preparation of my papers and Dr. Matias Aunola for numerous consultations. I also want to express my gratitude to the Department of Physics for the great working conditions it has provided for my graduate studies.

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Finally, I want to thank my beloved wife Aija and daughters Elisa and Anni for your love and support.

Jyväskylä, December 2001

Pasi Repo

List of Publications

I J. Mäkelä and P. Repo, Quantum-mechanical model of the Reissner-Nordström black hole, Physical Review D **57**, 4899 (1998). https://doi.org/10.1103/PhysRevD.57.4899

II J. Mäkelä and P. Repo, How to interpret black hole entropy?, JYFL-preprint 18/1998, gr-qc/9812075.

https://doi.org/10.48550/arXiv.gr-qc/9812075

III J. Mäkelä and P. Repo, Microscopic black-hole pairs in highly excited states, Classical and Quantum Gravity **18**, 373 (2001).

https://doi.org/10.1088/0264-9381/18/3/302

IV J. Mäkelä, P. Repo, M. Luomajoki and J. Piilonen, Quantum-mechanical model of the Kerr-Newman black hole, Physical Review D **64**, 024018 (2001).

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The author of this thesis has written the paper II, and the most part of the paper III. The author has participated in the writing of the publications I and IV. The mathematical analysis in papers I-III has been performed by the author. The mathematical analysis in paper IV has been performed by the author in collaboration with Johanna Piilonen and Markus Luomajoki.

Prologue

This prologue acts as a preliminary discourse in this thesis. The purpose is to arouse the curiosity of the reader towards the versatile properties of the quantum-mechanical black holes, rather than introduce the specific framework of the research. Let us therefore discuss some general aspects of the main subjects involved in this thesis.

The era of modern science began in 1687 when Sir Isaac Newton published his treatise entitled Philosophiae Naturalis Principia Mathematica. Newton was the first to explain gravitation — the phenomenon where massive bodies fall down when they are released near the surface of the Earth. *Newton's theory of gravitation* is accurate enough to provide the paths of missiles and rockets, and its region of validity can be tested with the orbits of the planets in our solar system. In 1915 Albert Einstein introduced his revolutionary theory of gravitation [1], the *general theory of relativity*. It changed many of the old Newtonian concepts of gravitation and, moreover, it extended the region of validity of the theory of gravitation. However, Einstein's theory has some limitations of its own. To this day our knowledge of gravitation has been based on general relativity. Because of the limitations of general relativity our understanding about gravitation is not complete.

One of the most important problems of modern physics is to find a complete description of gravitation. After all, *gravitation is one of the fundamental interactions* of Nature. One possible route to the complete theory of gravitation is hopefully afforded by certain physical objects where the effects of gravitation become very strong — so strong that even light cannot escape from those objects. These objects are called *black holes.* Since black holes are predicted by general theory of relativity, we use this theory as our starting point. To really understand what exactly are the main problems with the present understanding about gravitation, and why and how black holes are employed when searching for a complete theory of gravitation let us make a brief journey through the world of physics.

Physics attempts to explain Nature. According to our present understanding, all phenomena in Nature can be explained by *four fundamental interactions.* These fundamental interactions are known as the electromagnetic, weak, strong and gravitational interaction. The first three fundamental interactions, for instance, keep protons and neutrons together inside the atomic nucleus, give matter its properties and cause the electric phenomena we observe around us. They also cause the chemical reactions, and are responsible for the radioactive phenomena. The last interaction, gravity, governs every large-scale phenomena such as the mutual motion of celestial bodies, like the Earth and the Sun — but it also affects the smallest particles in the universe. Moreover, we are taught that there are only *three* *underlying theories of physics* which govern all phenomena and every entity in Nature. These theories are *quantum mechanics, thermodynamics*, and *general relativity*. Quantum mechanics describes the behaviour of particles when they interact, thermodynamics tells us how a large collection of particles behaves when the constituents of the collection interact with each other, whereas general relativity is a theory of gravitation telling us how space and time behave and how and why particles fall. In future, when we have a complete understanding of gravitation, we might have only two underlying theories left in physics: some sort of generalized quantum mechanics and thermodynamics. There are also indications that all underlying theories of physics should be able to unite into a theory of everything.

Einstein's general theory of relativity is his second theory of space and time. The first theory is known as the *special theory of relativity*, and it states that not three-dimensional space and one-dimensional time but four-dimensional spacetime remains the same for all inertial observers [2]. This means that space and time are relative, but *spacetime is absolute*. Spacetime is the arena where all phenomena of Nature take place, and it consists of time and of three-dimensional space such that time and space are not separate entities but they form a sort of a union. This very important conclusion was drawn when Einstein required the velocity of light to be the same for all inertial observers. Since the special theory of relativity does not include gravity at all, Einstein had to develop the general theory of relativity.

The general theory of relativity generalizes the special theory of relativity in the sense that in general relativity *spacetime is curved* whereas in the special theory it is flat. The fundamental idea of the general theory is that *matter makes spacetime curved*, and the gravitational interaction between massive bodies is described by the geometry of curved spacetime. More precisely, *gravitation is a manifestation of curvature of spacetime* in the sense that bodies in free fall tend to follow the shortest possible routes, called geodesics, in curved spacetime. This property of general relativity explains why all bodies have the same acceleration in the same gravitational field [3]: *spacetime geodesics are the same for all bodies*. For this reason an observer in free fall does not notice any gravitational field in his immediate vicinity, but every test particle that the observer releases, remains, relative to the observer, in rest or in uniform motion. This means that the observer's coordinate system is locally flat, i.e., locally inertial. Considerations of freely falling observer made Einstein to formulate the *general principle of equivalence*. One version of it states that all physical laws in freely falling coordinate systems are locally the same. This principle has been confirmed several times, and the first known experiments were performed by Galileo Galilei.

The crucial moment in the discovery of the general relativity was in 1907, when Einstein realized the importance of the general principle of equivalence. From that moment it took about eight years for Einstein to find the correct mathematical equations of motion, the so-called *Einstein's field equations*. Einstein's field equations tell us that matter interacts with spacetime making it curved, and on the other hand, they tell us how the spacetime curvature determines the routes of bodies. Einstein's field equations consist of ten coupled second-order nonlinear partial differential equations where the field variable is given by the so-called *metric tensor* $g_{\mu\nu}$. The metric tensor carries all information about the geometry of spacetime which is modelled by a four-dimensional *(pseudo-) Riemannian manifold.* A four-dimensional manifold is a space whose points can be identified by four real numbers. These real numbers give the coordinates of each point.

The essence of general relativity is stated in the *general coordinate invariance* which says that Einstein's field equations must be independent of a choice of spacetime coordinates. In other words, the interaction between spacetime and matter must not depend on the coordinate system one assigns to the four-dimensional spacetime manifold. This property of the theory is manifested in the field equations: they are tensor equations. Tensors are, by definition, objects that remain invariant under general coordinate transformations.

Black-hole spacetimes are solutions to Einstein's field equations. A black hole is, in a certain sense, like any ordinary object which is characterized by the object's mass M, angular momentum J and electric charge Q. The properties of the black hole are uniquely determined by the values of these quantities. However, the black hole is a spacetime region which is invisible to an observer outside the hole. For this reason black holes are called black.

The first black hole solution was found by Karl Schwarzschild in 1916 [4]. It is called the Schwarzschild black hole solution, and it is parametrized by just one parameter, the black hole mass M. The Schwarzschild black hole solution is a spherically symmetric and static black hole. The present number of different types of black holes is four. These are known as the Schwarzschild, the Reissner-Nordström [5, 6], the Kerr [7] and the Kerr-Newman [8] black holes.

A black hole may emerge in Nature in many ways, but usually one considers astrophysical, or effective, black holes and their birth. It has been shown that when a massive star has exhausted all its nuclear fuel it begins to collapse under its own weight. If the mass of the star is large enough, the collapse cannot be halted and it continues until the star has been compressed into a single "point" of a three-dimensional space. This "point" is called a singularity [1]. The final state of a spherically collapsing star is called the *Schwarzschild black hole.* Another important gravitational collapse where black holes could have arisen has perhaps taken place when the universe was very dense. These black holes are very small and they are called *primordial black holes.* Whatever caused the birth of a black hole, the hole is always uniquely determined by M, J and Q, and their properties to an outside observer seem to be identical if their parameters are identical.

According to general relativity *gravitation bends light rays* since massive objects make spacetime curved and light rays also follow the shortest routes in spacetime. When a star has collapsed and the mass bending light rays is situated in the singularity, the spacetime curvature at the singularity is infinite. We know, that in the Schwarzschild spacetime, the spacetime curvature depends only on the distance from the singularity. When one measures the curvature of Schwarzschild spacetime very far from the singularity, it turns out to be approximately zero and the spacetime geometry is nearly flat far from the point of the infinite curvature. Furthermore, one observes

[1] Einstein's general theory of relativity is not valid in singularity. Therefore already the theory itself predicts its own problems.

that there is a certain distance, called the *Schwarzschild radius*

$$R_S = \frac{2MG}{c^2} \quad ,$$

where $G \approx 6.67 \times 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ is Newton's gravitational constant, $c \approx 3,00 \times 10^8$ m/s is the speed of light and M is the mass of the collapsed star, which determines a boundary of a very peculiar spacetime region. This boundary is a two-dimensional spherical surface in spacetime, and it is called the *event horizon.* In other words, at the distance of the Schwarzschild radius from the singularity every Schwarzschild black hole has an event horizon. An event horizon forms an invisible limit of no-escape. The curvature of the spacetime is so large that even light rays that are emitted at the horizon cannot be seen by an observer outside the Schwarzschild radius. Moreover, if we only barely cross the event horizon and are inside the horizon, the curvature of the spacetime prevents us from turning back and returning to the other side of the horizon. The pull of the gravitational field of the collapsed star inside the horizon would suck us towards the singularity. Eventually, we would end our journey in the singularity. Therefore it is not possible to interact with the black hole (except with the gravitational field created by the hole) before one has passed the event horizon, and it is not possible to return from the hole. In other words, black holes shut themselves out of their surroundings. Because of these reasons the spacetime region where the distance from the singularity is less than or equal to the Schwarzschild radius is called a black hole, and such regions can be considered as holes in spacetime.

Thermodynamics is a branch of physics that investigates thermodynamic properties such as the temperature and the entropy of a system consisting of large collection of particles. The essence of thermodynamics is contained in the *four laws of thermodynamics.* The zeroth law of thermodynamics states that the temperature of a system in thermal equilibrium is constant. Thermal equilibrium, in turn, means that the macroscopic properties of the system remain constant. The first law of thermodynamics states that the energy is conserved in macroscopic systems. The second law of thermodynamics states that the entropy of a system cannot decrease in any process. The entropy of a system is a very important quantity. In broad terms, entropy is a measure of disorder in the system: the entropy of the system increases when the particles of the system become disordered. We may say that the entropy in our house increases when our children are free to play and tangle their toys all over the house. Moreover, the entropy of the system is a macroscopic concept, like temperature, but it has a connection with the microscopic properties of the particles that constitute the system. The microscopic properties of the particles, in turn, determine the macroscopic properties of the system. The third law of thermodynamics states that temperature the $T = 0$ K cannot be reached. At this temperature even the remnant oscillations and rotations of the system would die out.

Quantum mechanics was, like general relativity, developed in the beginning of the last century. It is a realization of a large number of experimental tests and theoretical investigations — in particular, of an intense study of atoms — performed at the turn of the 19th and 20th centuries. Another important input was given by the so-called *black body radiation.* A black body is an ideal surface that emits and absorbs perfectly all electromagnetic radiation. When the black body is heated it is seen to radiate with a continuous distribution of all wavelengths of light. Such a radiation is called black body radiation. The shape of the radiation distribution depends only on the temperature of the surface of the black body. Therefore we can determine the temperature of the radiator from the shape of the distribution. However, experimental distributions did not agree with the theoretical predictions. To solve this inconsistency Max Planck made in 1900 the famous invention, called *quantum hypothesis*, which states that for a given wavelength of the radiation energy could have only certain discrete values. Because of the quantum hypothesis the radiation could be emitted from the black body only with certain values of energy, and the theoretical radiation distribution which took into account the quantum hypothesis, coincided with the experimental results. Five years later Einstein was able to find a confirmation to this hypothesis. Today no one cannot elude the fact that in Nature certain quantities are discretized, or, quantized.

The solution to the black body radiation problem presented also evidence that sometimes electromagnetic waves behave like particles. In 1924 Louis de Broglie proposed that Nature loves symmetry and therefore matter behaves in some situations like waves and in others like particles. This is called the *wave-particle dualism.* If we think of a surface wave on water, it is not possible to localize precisely the wave on the water. Analogously, it is impossible to localize with great precision the particle represented by a wave. If one attempts to determine particle's position x then the information about the momentum p of the particle is lost. Of course this contradicts with classical physics, but it has been proved by numerous experiments, and its core is given by *Heisenberg's uncertainty principle*

$$\Delta x \Delta p \geq \frac{h}{2\pi} \quad , \tag{2}$$

where $h \approx 6.63 \times 10^{-34}$ Js is the Planck constant. Heisenberg's uncertainty principle is one of the characteristics of Nature and it cannot be avoided. This very profound and unconventional principle is contained in quantum mechanics. Since in quantum physics particles have no precise position nor momentum (velocity), which tell the state of the motion of particles in classical physics, the classical description of a state of motion is replaced by a concept of quantum state Ψ which is usually called the *wave function* of the system. Mathematically these states are modelled by the so-called state vectors $|\Psi\rangle$ which live in a vector space called the Hilbert space of the system.

Due to Heisenberg's uncertainty principle the real numbers that correspond to classical measurable quantities, like position x and momentum p, must be replaced by *operators*, like $\hat{x}$ and $\hat{p}$, when going over from classical to quantum physics. Operators operate in the *Hilbert space* of the system, and their job is to transform the state vectors to each other. Operators do not follow the ordinary calculational rules of real numbers. For instance, the so-called *commutator*

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{3}$$

of the operators $\hat{A}$ and $\hat{B}$ is not necessarily equal to the null operator.

A special class of equations in quantum mechanics is represented by the *eigenvalue equation*

$$\hat{A}|\Psi\rangle = A|\Psi\rangle \quad , \tag{4}$$

where A is the *eigenvalue* and $|\Psi\rangle$ is the *eigenvector* of the operator $\hat{A}$. A very important postulate of quantum mechanics states that every eigenvalue of an operator representing physical quantity (observable) is a possible result of a measurement of a physical quantity. Now we know how to represent mathematically the states, physical quantities and the possible results of measurements of the system in quantum theory.

The problem now is how to find the correct expressions for the operators corresponding to classical quantities. There is a standard procedure, called *canonical quantization*, to go over from classical to quantum physics. To quantize a classical theory canonically, the classical theory must first be cast into a *Hamiltonian form*, then replace the so-called *phase space variables* [9] by their operator counterparts, and finally write down the so-called canonical commutation relations between the operators. In particular, when one attempts to quantize general relativity canonically, one applies the rules of canonical quantization to the general theory of relativity. Because of that, one has to find Einstein's field equations in the Hamiltonian formulation of general relativity. In 1962 Arnowitt, Deser and Misner discovered the so-called *ADM formulation* which is one of the Hamiltonian formulations of general relativity. The very heart of the ADM formalism lies in the general coordinate invariance of general relativity. When one requires the ADM formulation to satisfy the principle of general coordinate invariance, *four constraint equations* per spacetime point appear. These constraint equations are known as the *Hamiltonian constraint*

$$\mathcal{H} = 0 \quad , \tag{5}$$

and the *diffeomorphism constraints*

$$\mathcal{H}^a = 0 \quad , a = 1, 2, 3 \quad . \tag{6}$$

The Hamiltonian constraint is a manifestation of the invariance of the formulation under time reparametrizations, and the three diffeomorphism constraints imply that the ADM formulation is invariant under three-dimensional spatial coordinate transformations. The constraint equations are functions of the phase space variables of general relativity.

The constraints are very important also in the quantum theory of general relativity since the dynamics of general relativity is already contained in the four constraint equations. However, most of the conceptual and mathematical problems of canonical quantum gravity are caused by the very same constraints. The mathematical problems are technical of nature, but the conceptual problems are more profound. The role played by time, for instance, is not clear even at the time of writing of this thesis [10, 11, 12]. In spite of the problems there are two possible ways to continue the quantization of general relativity. One may either replace the classical constraints by their operator counterparts by just replacing the phase space variables by their operator counterparts in a standard manner, or one may attempt to solve the constraints at the classical level, identify the physical degrees of freedom, and then quantize the theory in the resulting physical phase space. The former of these approaches is known as the *Dirac constraint quantization*, and the latter is called the *reduced phase space quantization.* The Dirac constraint quantization leads to a very important equation called the *Wheeler-DeWitt equation*

$$\hat{\mathcal{H}}\Psi = 0 \ . \tag{7}$$

The Wheeler-DeWitt equation is an operator counterpart of the Hamiltonian constraint, and its solutions has led to an approach called the *loop quantum gravity* [13]. At the moment loop quantum gravity is one of the leading candidates for the quantum theory of gravitation. However, the major part of the work done by the author has been involved with the reduced phase space quantization [14, 15, 16].

What makes the reduced phase space quantization so interesting is that it is perhaps the most straightforward approach available and it gives interesting results and insights into the viable quantum theory of gravity. It offers an application of quantum gravity where only the *physical degrees of freedom of the gravitational field itself are quantized.* In some cases the reduced phase space quantization provides a *finite-dimensional quantum theory* of gravitation. For instance, black hole spacetimes have classically at most six independent physical phase space coordinates. In other words, the reduced phase space quantization applies extremely well when one wishes to investigate the quantum-mechanical properties of black holes. Of course this kind of a reasoning compels one to ask:"Why to study quantum black holes, as black holes are spacetime regions that shut themselves out of their surroundings?" Let us answer this question.

Gravitational field itself is extremely difficult to quantize, and when matter fields are present quantization becomes almost impossible. Even the quantization of matter fields in fixed background spacetime leads to inconsistencies if spacetime is curved. However, the rules of quantum mechanics have been successfully applied to matter fields in black hole spacetimes. In 1974 Stephen Hawking quantized a certain matter field (such as an electromagnetic field) in the Schwarzschild black hole spacetime [17], and he was able to show that *black holes emit black body radiation* called the *Hawking radiation* with a characteristic temperature known as the *Hawking temperature.* This was surprising at that time: Black holes were by definition black. This might sound very peculiar, but it is not. Hawking thought that the radiation emitted from the black hole is not coming from inside the hole but from the immediate vicinity of the event horizon of the hole. He suggested that the strong gravitational field of the hole would cause a quantum-mechanical effet called *pair creation* [2]. In pair creation a particle and its antiparticle can be created, and after they have been created the antiparticle (or particle) goes into the hole and the particle (or antiparticle) seems to come out of the hole. In normal situations a particle and an antiparticle would annihilate each other very soon after they were created.

Hawking's celebrated calculation ensured that *black holes are thermodynamical as well as quantum objects* having certain well-defined temperature and entropy. The importance of this calculation is realized when we think

[2]Pair creation is allowed by Heisenberg's uncertainty principle: energy may decrease or increase for a short period of time without breaking any laws of Nature. We know that $E_0 = mc^2$, in turn, allows energy to be converted into mass of particles.

that it reveals a deep interconnection between quantum mechanics, thermodynamics and gravitation. In that sense we may say that black holes may play a key role when searching for an understanding of the structure of spacetime. In fact, black holes and atoms are very closely related in their importance to the development of physics. A hundred years ago atoms were investigated very intensively and the understanding of the structure of atoms led to quantum mechanics. Now, the intensive study of black holes may lead us to quantum gravity.

Before Hawking's famous result, Jacob Bekenstein (being then a young post-graduate student) was working open-mindedly with certain *laws of black hole physics.* Because of the parallelisms between these laws and the laws of thermodynamics Bekenstein believed that a black hole has an entropy[18, 19]. However, assigning an entropy to a black hole requires that the hole must radiate with a certain temperature. The discovery of the Hawking radiation ensured Bekenstein's anticipations of black holes having an entropy. Therefore the entropy of a black hole is called the *Bekenstein-Hawking entropy.*

Of course there are many unresolved problems in the field of quantum black holes. One of the major problems is that what are the microstates of a black hole constituting the macrostate that corresponds to the Hawking temperature and the Bekenstein-Hawking entropy. It is believed that once we know the correct solution to this problem, also the correct quantum theory of gravity is near. As a black hole is a quantum object, we may study its measurable properties by quantizing the gravitational field created by the hole. Since there are, at least classically, three measurable quantities of a black hole, namely the mass, the angular momentum and the electric charge, the reduced phase space quantization approach provides very efficient quantum-mechanical models of black holes. A closely related problem is to find the correct quantization rule for the area of the event horizon, or area spectrum. The area spectrum can be determined once we know the quantization rules for the mass, the angular momentum and the electric charge of the hole. The area spectrum, among other things, determines the radiation distribution, i.e, the temperature and the entropy of the hole. These are the main subjects of this thesis.

The object of this dissertation is to offer a self-contained introductory review to black hole physics and give a detailed survey of the reduced phase space quantization of the gravitational fields created by black holes. The dissertation consist of two parts. Part I deals with classical and Part II with quantum black holes. Chapter 1 deals with the ADM and the Hamiltonian formulation of general relativity. Chapter 2 provides an elementary introduction to classical black holes and Chapter 3 deals with the reduction of phase space variables corresponding to black hole spacetimes. Chapter 4 presents the results of the Hawking radiation and the black hole entropy. In Chapter 5 we quantize the black hole spacetimes and obtain a certain quantum mechanical equation for black holes. Chapter 6 deals with the question of the exisistence of microscopic black-hole pairs within a very restricted model. In the end we have gathered five appendices. Appendix A involves the so-called *WKB approximation* and the rest of the appendices contain most of the calculations which yield the results obtained by the author.

In Chapter 3 we consider the classical *Hamiltonian formulation* of Reissner-

Nordström and Kerr-Newman black hole spacetimes. Basically, our study is based on the results obtained by Jorma Louko and Stephen Winters-Hilt in Ref. [20], and on an important theorem proved by Tullio Regge and Claudio Teitelboim [21]. We have managed to reduce the phase space of Reissner-Nordström spacetime in detail [14], but the reduction of the phase space of Kerr-Newman spacetime could not be performed in this dissertation. However, there are good reasons to believe that such a reduction exists yielding a certain classical Hamiltonian of Kerr-Newman black holes [16].

In Chapter 5 we proceed to the quantization of the classical Hamiltonian dynamics of Reissner-Nordström and Kerr-Newman black holes. Quantization envolves a straightforward replacement of the classical Hamiltonian by the corresponding Hamiltonian operator which yields a sort of *"Schrödinger equation" of black holes* [16]. That equation is the main result of this dissertation. Our equation has many interesting consequences. For instance, it predicts that the mass, the electric charge, and the angular momentum spectra of black holes are discrete. In particular, the spectrum of the quantity $M^2 - Q^2 - (J/M)^2$ is strictly positive. In the context of Hawking radiation, this is a very interesting result, and it is in agreement with the third law of black hole physics. Therefore it is a strong argument in favor of the physical validity of our model. Also, when we study the eigenvalues of the horizon area of black holes we get a result which is closely related, although not quite identical to the famous proposal made by Jacob Bekenstein [22].

Chapter 6 deviates slightly from the main subject in this dissertation, and it involves *microscopic black-hole pairs.* The main question is whether microscopic black-hole pairs form bound systems in which black holes revolve around each other and what kind of transitions can they then perform. The main result of Chapter 6 is an expression for the transition rates for spontaneous emission of gravitons when the system makes a transitions from one quantum state to another. By using this result we obtain selection rules for possible transitions in system formed by a black-hole pair, and transition rates for allowed transitions. The main observation is that, within the approximations made, the lifetimes of the stationary state of the system are large enough for the energy spectrum of the black-hole pair to be discrete. The energies released in transitions between energy eigenstates are of the order of 10^{22} eV $= 10^6$ J, which is about the same as the energy needed when an automobile is accelerated from rest to a speed of 150 km/h. [23]

Part I

Classical Theory

Chapter 1

ADM Formulation of General Relativity

1.1 Introduction to Geometrodynamics

In this chapter we formulate *Einstein's general relativity* in the Hamiltonian framework. There are many different Hamiltonian formulations of general relativity at our disposal depending on the generalized coordinates used in Hamilton's principle of the gravitational action. One possible Hamiltonian formulation known as the Ashtekar formulation uses as the variables of the configuration space of the system the components of a complex-valued $SU(2)$ connection A^i_a [24].[1] In this chapter, however, we concentrate solely on the so-called *ADM formulation* of general relativity, which was found by R. Arnowitt, S. Deser and C. W. Misner in 1962 [25]. In the ADM formulation the configuration coordinates of the system are taken to be the components of the metric tensor q_{ab} induced on the spacelike hypersurfaces Σ_t of constant time t of the spacetime manifold $\mathcal{M}$ [26]. The ADM formulation provides the most straightforward approach to the study of Hamiltonian dynamics of black holes.

To understand the essential ideas in the Hamiltonian formulation of general relativity, it is very useful to keep in mind certain well-known concepts of the Hamiltonian formulation of classical mechanics. That is because the physical ideas and the mathematical language of these concepts can be straightforwardly utilized in our approach to general relativity. In classical mechanics the *system* consists of all the particles moving in a three-dimensional Euclidean space, and by the concept of *history* of the system during some *time* interval we mean the complete knowledge about the positions of the particles at each instant during that time interval. In the Hamiltonian approach to general relativity, in turn, the notion of time is given by any appropriate function $t = t(x^a)$ $(a = 1, 2, 3)$ corresponding to exactly one spacelike hypersurface Σ_t of constant t in the manifold $\mathcal{M}$, and the history of the system during the time interval $[t_i, t_f]$ corresponds to the four-geometry of the subset of the manifold which is bounded by the two hypersurfaces Σ_{t_i} and Σ_{t_f}. A two-dimensional analogue of this is given in

[1] An ambitious program called *quantum geometry*, or loop quantum gravity, [13] which is a very intense area of research at the moment, rests on the Ashtekar formulation of the Hamiltonian framework.

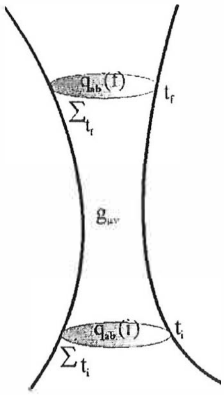

Figure 1.1: Spacelike hypersurfaces $t = constant$ begin their life at the past $t = t_\text{i}$ hypersurface, then go through the tube, and finally end their life at the future $t = t_\text{f}$ hypersurface.

Fig. 1.1.

The number of degrees of freedom of systems considered in classical mechanics is usually small, whereas in general relativity the number of degrees of freedom of any system is infinite. Even when matter fields are not present, and the system consists of spacetime itself only, the number of degrees of freedom of the system is uncountable. For instance, in the covariant formulation of general relativity the components of the metric tensor $g_{\mu\nu}(x^\mu)$ on the four-dimensional (pseudo-)Riemannian manifold in every spacetime point x^μ can be chosen to be the coordinates of the configuration space of the system. The number of spacetime points on the manifold is infinite, and hence the number of degrees of freedom of the system is $10 \cdot \infty$.

Keeping the basic concepts of classical mechanics in mind, we now enter the discussion about the properties of the spacetime manifold. Because of the time evolution of three-geometries in general relativity one often calls it as *geometrodynamics*. In geometrodynamics we investigate the rate of change of the three-geometry of space at one instant of time and obtain knowledge about the four-geometry of spacetime at that time from the knowledge about the time evolution of the three-geometry.

1.1.1 3+1 Decomposition of Spacetime

To employ the components of the three-metric q_{ab} induced on the spacelike hypersurface of constant t as the variables of the configuration space of our system, the ADM formulation, as any Hamiltonian formulation of general relativity, demands a decomposition of the spacetime manifold $\mathcal{M}$ into space and time. However, spacetime itself is dynamic, and therefore no preferred background time coordinate exists. Moreover, the chosen time coordinate t which labels the spacelike hypersurfaces cannot be related to a physical notion of time before the spacetime metric is known. Unfortunately, the metric is the unknown variable of the formulation, but, in general, we may choose the time coordinate in almost any way we like. Any such spacetime splitting into space and time defines an appropriate notion of time coordinate, since we move forward in time by moving from one slice of spacetime to the next.

The slicing of spacetime into space and time requires certain global properties from the spacetime manifold $\mathcal{M}$. We shall always assume that the spacetime manifold is *globally hyperbolic*. Global hyperbolicity of spacetime means, loosely speaking, that it behaves causally well i.e. one cannot alter its past. More precisely, the spacetime manifold $\mathcal{M}$ can be foliated by a one-parameter family of *Cauchy surfaces* $\{\Sigma_t\}$, where every hypersurface Σ_t may be considered as an instant of "time", and hence a global time function $t = t(x^\mu)$ can be chosen such that each hypersurface Σ_t of constant t is a Cauchy surface. A Cauchy surface, in turn is, by definition, a spacelike hypersurface which every non-spacelike curve intersects exactly once. Furthermore, it can then be shown that the spacetime manifold must be topologically equivalent to $\Sigma \times \mathbb{R}$, where Σ is a three-dimensional submanifold diffeomorphic to Σ_t for all t. For a detailed discussion of the global structure of spacetime the reader is encouraged to peruse Ref. [27].

A well-known way to illustrate the 3+1 decomposition of spacetime is to think of the two-dimensional surface of a bread which is cut in very thin slices. Then a certain one-geometry is induced into the crust of the bread. This one-geometry, in turn, depends on the surface of the bread, but it also depends on how the slicing of the bread has been performed. If we know the order of the slices of the bread we can reconstruct the original bread from the slices. In the case of spacetime the chosen time coordinate tells us how to slice the manifold, and if we know how the manifold is sliced, we know how the four-geometry of spacetime can be reconstructed from the three-geometries of the hypersurfaces. In other words, the ADM formulation of general relativity rests basically on the idea that once we know the time evolution of q_{ab} at every point x^a on the spacelike hypersurface, we have the knowledge about the spacetime geometry. Hence, by the history of our system we mean the complete knowledge about the values of the metric q_{ab} at every space point x^a on every spacelike hypersurface Σ_t.

1.1.2 Superspace

In the sixties J. A. Wheeler introduced the concept of *superspace* to mean an infinite-dimensional space of three-geometries, where each point represents a 3-geometry. More precisely, the superspace which is the configuration space of general relativity, is the space of equivalence classes of metric tensors q_{ab} at every hypersurface point on every Σ_t. The two metrics q_{ab} and q'_{ab} are defined to be equivalent if they can be obtained from each other by a diffeomorphism. In other words, the two metrics are considered to be equivalent if they can be transformed into each other by a three-dimensional coordinate transformation on a spacelike hypersurface. The superspace has an interesting interpretation in the ADM formulation of general relativity: Hypersurfaces $\{\Sigma_t\}$ "move" in the superspace in the same way as the particles in classical mechanics move in a three-dimensional space.

1.2 Lagrangian Formulation of General Relativity

There has been a lengthy period of interest in the study of the dynamics in general relativity under *general boundary conditions* of spacetimes [28]. However, we shall concentrate only on *asymptotically flat* spacetimes, which contribute the well-known boundary terms [21, 29]: the *ADM energy*, the *ADM momentum* and the *ADM angular momentum* of spacetime. These boundary terms play a very significant role in the ADM formulation, and also throughout our work. Since we are aiming to investigate the physical properties of *isolated primordial black holes* (see Ch. 2), asymptotically flat spacetimes are precisely those that offer us ideal isolated black-hole systems in general relativity. Despite the boundary contribution, we first formulate the Lagrangian formulation of general relativity without the boundary terms, and we shall not introduce them until in Sec. 1.5.

1.2.1 Einstein-Hilbert Action

We start with the spacetime covariant form of the Lagrangian formulation of general relativity, where the action can be taken to be the *Einstein-Hilbert action*

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G} \int\limits_{\mathcal{M}} d^4x \sqrt{-g}\, R \ , \tag{1.1}$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$ and R is the Riemann curvature scalar of the four-dimensional spacetime manifold $\mathcal{M}$.

It is well known that Einstein's field equations in vacuum can be obtained by varying the action (1.1) with respect to the components of the metric $g_{\mu\nu}$. When we require that the first-order variation δS_{EH} vanishes for arbitrary variations $\delta g_{\mu\nu}$, as $\delta g_{\mu\nu}$ and its derivatives $\delta g_{\mu\nu\,,\alpha}$ vanish at the boundaries of the spacetime manifold $\mathcal{M}$, the correct field equations emerge. We define the variation $\delta g_{\mu\nu}$ of the metric tensor to mean an "infinitesimal" change of the metric tensor at every spacetime point x^μ. It is not necessary and, in fact, in the case of asymptotically flat spacetimes it is utterly wrong, to assume that the variations of the spacetime geometry vanish at the boundaries. However, as mentioned, we first develop the Lagrangian formulation of general relativity without having any concern about the spacetime boundaries, and thus we shall maintain requiring that the variations of the metric and its derivatives vanish at the boundaries. We may also think that our spacetime is a compact manifold $\mathcal{M}$. Either way, the Lagrangian density of pure gravity can be taken to be

$$\mathcal{L}_{\mathrm{EH}}(g_{\mu\nu}, g_{\mu\nu,\alpha}, g_{\mu\nu,\alpha,\beta}) = \frac{\sqrt{-g}}{16\pi G} R \ . \tag{1.2}$$

1.2.2 Gaussian Normal Coordinates

Let us now write the Riemann curvature scalar *density* $\sqrt{-g}R$ in terms of the Christoffel symbols as

$$\sqrt{-g}R = \sqrt{-g}g^{\mu\nu}\left(-\Gamma^{\alpha}_{\mu\alpha\,,\nu} + \Gamma^{\alpha}_{\mu\nu\,,\alpha} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\alpha} - \Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\sigma\nu}\right) \ , \tag{1.3}$$

and further, in the more convenient form

$$\sqrt{-g}R = -g^{\mu\nu}\sqrt{-g}\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\right) + \left[g^{\mu\nu}\sqrt{-g}\left(\delta^{\alpha}_{\mu}\Gamma^{\beta}_{\nu\beta} - \Gamma^{\alpha}_{\mu\nu}\right)\right]_{,\alpha} . \tag{1.4}$$

When these two terms on the right hand side of (1.4) are substituted into the action (1.1), one can convert the volume integral of the second term on the right hand side of the Riemann curvature scalar density (1.4) into a boundary term. The boundary term arises when the Gauss' theorem is applied to curved spacetime in four dimensions. As the variations of the components of the metric tensor and the variations of their derivatives are assumed to vanish at the boundaries of spacetime, we may omit the boundary term. In the effective Lagrangian formulation of pure gravity, without any boundary terms present, we may consider the first term on the right hand side of (1.4), only.

At this point we slice spacetime into space and time, but there is not any preferred time that tells us how to slice the four-geometry into three-geometries. However, there is a gauge freedom which allows us to choose any appropriate gauge we wish to work in. When we choose a specific gauge or, equivalently, a specific coordinate system that makes our dynamical system simple, then, unfortunately, we loose some very important information about the corresponding formulation of the theory. Remarkably, the loss of information can be recovered afterwards, and thus we should not be too much concerned about it at the moment.

Consider the first term in (1.4). We may use the so called *Gaussian normal coordinates* (see, for instance, [30]), where a of freely falling observer is at rest with respect to the spatial coordinates, and the time coordinate gives the proper time of the observer. The line element of spacetime in Gaussian normal coordinates can be written as

$$ds^2 = -dt^2 + q_{ab}dx^a dx^b , \tag{1.5}$$

where $a, b = 1, 2, 3$, and $q_{aa} > 0$ for all a. We note that there has not been made any restrictions on the spacetime geometry itself, since it is possible to show that this kind of a choice can be done for every spacelike hypersurface. It is easy to see that we now have $g_{00} = -1$ and $g_{0a} = g_{a0} = 0$ for every a, and therefore the Christoffel symbol

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\alpha}\right) \tag{1.6}$$

always disappears if two of its three indices become zero. Hence, we may divide the first term in (1.4) into a spatial part and into a part which includes the terms related to our preferred time $t = x^0$:

$$\begin{aligned} -g^{\mu\nu}\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\right) &= q^{mn}\Gamma^{s}_{0s}\Gamma^{0}_{mn} - 2q^{mn}\Gamma^{0}_{an}\Gamma^{a}_{m0} + \Gamma^{s}_{a0}\Gamma^{a}_{0s} \\ &\quad + q^{mn}\left(\Gamma^{s}_{as}\Gamma^{a}_{mn} - \Gamma^{s}_{an}\Gamma^{a}_{ms}\right) , \end{aligned} \tag{1.7}$$

where the last term in the parenthesis is exactly the same term as the term we began with, except that it lives on the spacelike hypersurfaces of constant t only. Therefore, we may use the convenient form of (1.4), and write the last term on the spacelike hypersurfaces as

$$\sqrt{q}q^{mn}\left(\Gamma^{s}_{as}\Gamma^{a}_{mn} - \Gamma^{s}_{an}\Gamma^{a}_{ms}\right) = \left[q^{mn}\sqrt{q}\left(\delta^{a}_{m}\Gamma^{b}_{nb} - \Gamma^{a}_{mn}\right)\right]_{,a} + \sqrt{q}\mathcal{R} . \tag{1.8}$$

where $\sqrt{q}$ denotes the determinant of the metric q_{ab} induced on the spacelike hypersurfaces, and $\mathcal{R}$ is the Riemann curvature scalar on the spacelike hypersurface Σ_t of constant t.

With the help of the hypersurface metric q_{ab} and its time derivative $\dot{q}_{ab}$, we can express the first part of (1.7) in the form

$$q^{mn}\Gamma^s_{0s}\Gamma^0_{mn} - 2q^{mn}\Gamma^0_{an}\Gamma^a_{m0} + \Gamma^s_{a0}\Gamma^a_{0s} = \frac{1}{4}\left(q^{ab}q^{cd} - q^{ac}q^{bd}\right)\dot{q}_{ab}\dot{q}_{cd}\,. \tag{1.9}$$

When the non-boundary term in (1.4) is written in terms of the three-dimensional Riemann curvature scalar, the hypersurface metric q_{ab}, and its time derivatives $\dot{q}_{ab}$ in Gaussian normal coordinates, and then substituted into the action (1.1), we get the Einstein-Hilbert action in a form that is no longer spacetime covariant:

$$\begin{aligned} S_{\text{EH}} &= \frac{1}{16\pi G}\int dt \int_{\Sigma_t} d^3x\sqrt{q}\left[\frac{1}{4}\left(q^{ac}q^{bd} - q^{ab}q^{cd}\right)\dot{q}_{ab}\dot{q}_{cd} + \mathcal{R}\right] \\ &\quad - \frac{1}{16\pi G}\int dt \int_{\Sigma_t} d^3x \left[\sqrt{q}q^{mn}\left(\Gamma^a_{mn} - \delta^a_m\Gamma^b_{nb}\right)\right]_{,a}\,. \end{aligned} \tag{1.10}$$

The last term in (1.10) can be converted into a vanishing spatial boundary term with the help of the Gauss' theorem. Hence, the effective Lagrangian density of pure gravity in Gaussian normal coordinates is

$$\mathcal{L}_{\text{eff}} = \frac{\sqrt{q}}{16\pi G}\left[\frac{1}{4}\left(q^{ac}q^{bd} - q^{ab}q^{cd}\right)\dot{q}_{ab}\dot{q}_{cd} + \mathcal{R}\right]\,. \tag{1.11}$$

The fundamental idea in the ADM formulation is to consider the spatial metric q_{ab} as the dynamical object of general relativity. Now, the effective Lagrangian density (1.11) in Gaussian normal coordinates is composed of the spatial metric and its first-order time derivatives only. Therefore we might anticipate that these concepts constitute appropriate information about the system when the initial hypersurface Σ_t is given. The hypersurface metric describes the intrinsic geometry of the hypersurfaces and the time derivative of the spatial metric tells us how they evolve when they are moved forward in time t. In fact, the time derivative of the three-metric on the hypersurface written in the Gaussian normal coordinates can be shown to reveal how the hypersurface of constant t "bends" in spacetime. In other words, the time derivative of the spatial metric tells us how the hypersurface is embedded in spacetime. In general, the information about the extrinsic properties of the hypersurface is contained in the *extrinsic curvature tensor*. Let us next consider the concept of the extrinsic curvature tensor in detail.

1.2.3 Extrinsic Curvature Tensor

Until further notice, we will forget that the effective Lagrangian (1.11) is written in a specific gauge as we study the exterior properties of a spacelike hypersurface Σ_t. The intrinsic properties of the hypersurface are contained in the spatial metric q_{ab} but, in general, we have no knowledge about the rate of change of the metric as it moves from one instant of time to another. The "bending" or rate of change of the spatial metric of the hypersurface Σ_t in

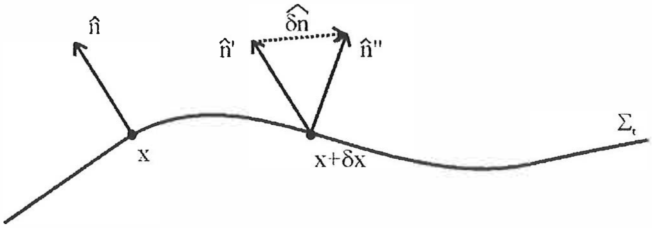

Figure 1.2: A spacetime diagram showing the idea of the extrinsic curvature of a spacelike hypersurface Σ_t. The difference of the unit normal vector $\widehat{\delta n}$ between the shifted vector $\hat{n}'$ and the parallel transported vector $\hat{n}''$ corresponds to the bending of the embedded hypersurface in the spacetime.

spacetime can be found by comparing the unit normal $\hat{n}$ of the hypersurface when this hyperspace is shifted an infinitesimal amount from x^a to $x^a + \delta x^a$, to the parallelly transported [31] unit normal $\hat{n}$ at the point $x^a + \delta x^a$. A schematic picture of the procedure is given in Figure 1.2. The coordinate-independent change of the unit vector normal to the hypersurface is given by the difference vector $\widehat{\delta n}$, whereas this difference can be obtained from the components of *extrinsic curvature tensor* $K^a{}_b$ of the spacelike hypersurface. We define the exterior curvature to be positive when the bases of the two unit normals, $\hat{n}$ and $\hat{n}''$, are closer than their tips. The sign convention of this definition is in agreement with Israel [32] and Wald [30], but disagrees with Arnowitt, Deser and Misner [25], and Misner, Thorne and Wheeler [29]. The general definition of the components of the extrinsic curvature tensor K^a_b $(a, b = 1, 2, 3)$ is

$$K^a{}_b b^\mu{}_{(a)} = \frac{\delta n^\mu}{\delta x^b} \,, \tag{1.12}$$

where $b^\mu_{(a)}$ are the components of the tangent vector of the coordinate curve associated with x^a, and the operator $\frac{\delta}{\delta x^a}$ must be understood as the covariant differentiation of the components n^μ of the unit normal $\hat{n}$ with respect to x^a.

The spatial metric q_{ab} is given by the inner product between the tangent vectors,

$$q_{ab} = b^\mu_{(a)} b^\nu_{(b)} g_{\mu\nu} \,, \tag{1.13}$$

and if we define

$$K_{ab} = q_{as} K^s{}_b \tag{1.14}$$

we can write the definition (1.12) in the viable form

$$K_{ab} = \frac{\delta n^\mu}{\delta x^b} b^\nu{}_{(a)} g_{\mu\nu} \,. \tag{1.15}$$

After considering the components of the shifted unit normal n'^μ and the components of the parallelly transported unit normal n''^μ with respect to the components of the original unit normal n^μ, one can easily show that the covariant differential δn^μ can be written in terms of the covariant derivative of $\hat{n}$ and the covariant differential δx^b:

$$\delta n^\mu := n'^\mu - n''^\mu = n^\mu{}_{;b} \delta x^b \,. \tag{1.16}$$

Therefore, at the hypersurface Σ_t of constant t, (1.15) becomes

$$K_{ab} = n_{a;b} \ , \tag{1.17}$$

which is a very useful expression for the extrinsic curvature tensor.

Now, let us return to our specific gauge and find an expression for the extrinsic curvature tensor in Gaussian normal coordinates. In order to discover the extrinsic curvature tensor in our gauge, we have to choose the covariant components of the unit normal n_μ on the spacelike hypersurface $t = constant$. One possible choice of components in our coordinate system is

$$n_\mu = -|g^{00}|^{-1/2}\delta_{\mu 0} = -\delta_{\mu 0} \ ; \tag{1.18}$$

since such a unit normal $\hat{n}$ of the spacelike hypersurface satisfies the orthogonality condition and it also is a timelike vector:

$$g_{\mu\nu}n^\mu b^\nu_{(a)} = n_\nu b^\nu_{(a)} = -|g^{00}|^{-1/2}\delta_{\nu 0}b^\nu_{(a)} = -|g^{00}|^{-1/2}b^0_{(a)} = 0 \ , \tag{1.19}$$

$$g_{\mu\nu}n^\mu n^\nu = g^{\mu\nu}n_\mu n_\nu = \frac{g^{\mu\nu}}{|g^{00}|}\delta_{\mu 0}\delta_{\nu 0} = \frac{g^{00}}{|g^{00}|} = -1 \ . \tag{1.20}$$

The last equality in (1.19) follows from the relation $b^0_{(a)} = 0$ on the hypersurface Σ_t. Hence, the extrinsic curvature tensor can written in the form

$$K_{ab} = |g^{00}|^{-1/2}\Gamma^0_{ab} \ , \tag{1.21}$$

which is symmetric in ab. From the previous equation (1.21) one can easily see that in Gaussian normal coordinates the extrinsic curvature tensor has a simple expression

$$K_{ab} = \frac{1}{2}\dot{q}_{ab} \ . \tag{1.22}$$

Indeed, the time derivative of the spatial metric tells us how the spacelike hypersurfaces of constant t are embedded in spacetime, and because of that we may write the effective Lagrangian density (1.11) in expressed Gaussian normal coordinates in terms of the spatial metric and the extrinsic curvature tensor:

$$\mathcal{L}_{\text{eff}} = \frac{\sqrt{q}}{16\pi G}\left(K_{ab}K^{ab} - K^2 + \mathcal{R}\right) \ , \tag{1.23}$$

where K is the trace of K_{ab}. The action corresponding to (1.23) can be written as

$$S_{\text{eff}} = \frac{1}{16\pi G}\int dt \int_{\Sigma_t} d^3x\sqrt{q}\left(K_{ab}K^{ab} - K^2 + \mathcal{R}\right) \ , \tag{1.24}$$

which may be taken as the starting point of the ADM formulation of general relativity.

1.3 Hamiltonian Formulation of General Relativity

1.3.1 Canonical Momenta

As it has been mentioned several times before, the basic idea in the ADM formalism is to use the components q_{ab} of the spatial metric as the generalized coordinates of our system. Therefore, when we proceed from the

Lagrangian formalism to the Hamiltonian formalism of general relativity, we follow the standard method which is closely analogous to the corresponding procedure in classical mechanics, and we define the canonical *momentum* π^{ab} conjugate to q_{ab} as

$$\pi^{ab} = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{q}_{ab}} , \tag{1.25}$$

where $\mathcal{L}_{\text{eff}}$ is given by Eq. (1.11). Using Eqs. (1.11) and (1.22) we find that π^{ab} can be written in terms of the extrinsic curvature tensor and the canonical momentum conjugate to q_{ab} is

$$\pi^{ab} = \frac{\sqrt{q}}{16\pi G}\left(K^{ab} - q^{ab}K\right) , \tag{1.26}$$

which can be inverted,

$$K_{ab} = \frac{16\pi G}{\sqrt{q}}\left(\pi_{ab} - \frac{1}{2}q_{ab}\pi\right) , \tag{1.27}$$

where π inside the parenthesis denotes the trace of the canonical momentum π^{ab}, and the indices of the momenta are pushed up and down by the spatial metric q_{ab}. When we use (1.22) once more, we notice that

$$\dot{q}_{ab} = (16\pi G)\, G_{abcd}\pi^{cd} , \tag{1.28}$$

where

$$G_{abcd} = \frac{1}{\sqrt{q}}\left(q_{ab}q_{cd} - q_{ac}q_{bd} - q_{ad}q_{bc}\right) \tag{1.29}$$

is the so-called the *Wheeler-DeWitt metric.*

1.3.2 Hamiltonian

Let us define the *Hamiltonian* $H[q_{ab}, \pi^{ab}]$ on every Σ_t. The Hamiltonian H is of the form

$$H = \int_{\Sigma_t} d^3x \mathcal{H} , \tag{1.30}$$

where $\mathcal{H} = \mathcal{H}(q_{ab}, q_{ab,c}, q_{ab,c,d}, \pi^{ab})$ is called the *Hamiltonian density*:

$$\mathcal{H} := \pi^{ab}\dot{q}_{ab} - \mathcal{L}_{\text{eff}} . \tag{1.31}$$

By using Eq. (1.28) and the definition of the extrinsic curvature tensor we find that the Hamiltonian density can be written as

$$\mathcal{H} = \frac{\sqrt{q}}{16\pi G}\left(K_{ab}K^{ab} - K^2 - \mathcal{R}\right) = \frac{1}{2}(16\pi G)\, G_{abcd}\pi^{ab}\pi^{cd} - \frac{\sqrt{q}}{16\pi G}\mathcal{R} , \tag{1.32}$$

where the latter expression is, to some extent, analogous to the Hamiltonian of classical mechanics: the first term can be considered as the analog of the kinetic energy and the second term can be viewed as the analog of the potential energy. If one goes in the analogy even deeper than this, one finds that the concept of the particle in classical mechanics moving in space is in geometrodynamics replaced by the concept of the hypersurface Σ_t "moving" in space of the three-geometries.

1.3.3 Hamiltonian Equations of Motion

Now we are ready to write down the dynamical equations of our system in Gaussian normal coordinates. These equations are analogous to the Hamiltonian equations of motion for classical particles, and they can be obtained by varying the effective action

$$S_{\text{eff}} = \int dt \int_{\Sigma} d^3x \left(\pi^{ab}\dot{q}_{ab} - \frac{1}{2}(16\pi G)\, G_{abcd}\pi^{ab}\pi^{cd} + \frac{\sqrt{q}}{16\pi G}\mathcal{R} \right) \quad (1.33)$$

with respect to the variables q_{ab} and π^{ab}. The dynamical equations, in principle, are [30]

$$\dot{q}_{ab} = \frac{\delta H}{\delta \pi^{ab}} = \frac{\partial \mathcal{H}}{\partial \pi^{ab}}, \quad (1.34)$$

$$\dot{\pi}^{ab} = -\frac{\delta H}{\delta q_{ab}} = \partial_c \left[\frac{\partial \mathcal{H}}{\partial(\partial_c q_{ab})} \right] - \partial_c\partial_d \left[\frac{\partial \mathcal{H}}{\partial(\partial_c\partial_d q_{ab})} \right] + \frac{\partial \mathcal{H}}{\partial \dot{q}_{ab}} \quad (1.35)$$

for every $a, b = 1, 2, 3$ and for every point on the spacelike hypersurface Σ_t of constant t. The equations (1.34) and (1.35) are equivalent to the equations which may be obtained by varying the Einstein-Hilbert action (1.1) with respect to the spacelike components q_{ab} of the metric tensor $g_{\mu\nu}$ [33]. Hence, the problem is that our dynamical equations are equivalent to Einstein's field equations $G_{ab} = 0$ only, and the equations $G_{00} = 0$ and $G_{a0} = 0$ were lost as we made a choice to work in a specific gauge. The missing equations are recovered in a very natural manner, however.

1.4 Constraints in Pure Gravity

So far, we have constructed the Lagrangian and the Hamiltonian formulations of general relativity in a specific gauge, and because of this choice we have encountered a serious problem: Four of Einstein's field equations have disappeared. For the same reason we now have to face another major problem which is that the foundation of Einstein's theory of gravitation, the *diffeomorphism invariance* of general relativity, is destroyed. The power of the ADM formulation of general relativity lies in the fact that it manages to solve both of these problems at the same time. As the diffeomorphism invariance is recovered, the puzzle of the missing equations is solved. Hence, the two problems are reduced to one problem of recovering the diffeomorphism invariance. Our next task is to improve the Hamiltonian formulation to include the diffeomorphism invariance. When the spacetime covariant formulation is modified into a formulation where time is separated from the spatial coordinates, the diffeomorphism invariance means that the action (1.33) remains invariant under reparametrizations of time t and under spacelike coordinate transformations on the spacelike hypersurface Σ_t of constant t.

1.4.1 The Spirit of the ADM Formulation

To relax the choice of the gauge, and to get an expression for the spacetime line element in a generic form, we have to perform in (1.5) a time

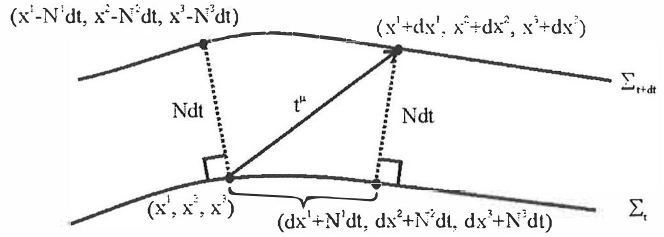

Figure 1.3: When the spatial point x^a on the hypersurface Σ_t is transformed to the spatial point x^a+dx^a on Σ_{t+dt} by means of the lapse function and the shift vector, then the dotted line joining the points x^a on Σ_t and $x^a - N^a dt$ on Σ_{t+dt} is perpendicular to the hypersurface $t = constant$.

reparametrization and a spatial coordinate transformation:

$$dt \longrightarrow dt' = Ndt \ , \tag{1.36}$$

$$dx^a \longrightarrow dx'^a = dx^a + N^a dt \ , \tag{1.37}$$

where $N = N(x^a, t)$, known as the *lapse function*, is any smooth function of time t and hypersurface coordinates x^a, and $N^a = N^a(x^a, t)$, whose components are also smooth functions of time and space coordinates, is called the *shift vector*. The physical interpretation of the lapse function N is that it gives the lapse of proper time between two consecutive spacelike hypersurfaces Σ_t and Σ_{t+dt}. The shift vector, in turn, dictates the three-dimensional spatial coordinate transformation on the hypersurface.

One of the most elegant properties of the ADM formulation is that one is able to perform spatial coordinate transformations by means of the shift vector and to connect consecutive spacelike hypersurfaces Σ by means of the lapse function. Geometrically, the lapse function N makes the vector t^μ joining the two points (x^1, x^2, x^3) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ on the hypersurfaces Σ_t and Σ_{t+dt}, respectively, to be perpendicular to the hypersurface Σ_t. On the other hand, the shift vector and the lapse function together shift the point $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ on the hypersurface Σ_{t+dt} relative to the point (x^1, x^2, x^3) on the hypersurface Σ_t such that the vector t^μ ceases to be orthogonal to the hypersurface $t = constant$. This corresponds to the situation where the point (x^1, x^2, x^3) on the hypersurface Σ_t is first liable to transformation $(dx^1 + N^1 dt, dx^2 + N^2 dt, dx^3 + N^3 dt)$ in accordance with (1.37), and then the transformed point is transported into the point $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ on the hypersurface Σ_{t+dt} by using the lapse function. This is illustrated in Fig. 1.3. In that case we can write the spacetime line element in the form

$$ds^2 = -N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \ , \tag{1.38}$$

or, equivalently,

$$ds^2 = -\left(N^2 - N^a N_a\right) dt^2 + 2q_{ab} N^a dx^b + q_{ab} dx^a dx^b \ , \tag{1.39}$$

where $N_a = q_{ab} N^b$. The metric (1.39) is known as the *ADM metric*.

As we noticed, the geometrical interpretation of the lapse function and the shift vector are intuitively very clear but, besides of that, the physical ideas behind the lapse function and the shift vector give a striking insight into the Hamiltonian formulation of general relativity. First of all, the ADM metric now consists of the lapse function and the shift vector, and therefore the configuration space of the Hamiltonian formulation is increased by four new variables. On the other hand, the corresponding gravitational action must remain invariant under time reparametrizations and spatial coordinate transformations in order to be diffeomorphically invariant. In other words, the action must be independent of N and N^a, and it is independent if and only if its variations with respect to N and N^a vanish. Before we can find out the consequences of introducing diffeomorphism invariance into the formulation, we first have to find an expression for the gravitational action (1.33) in the so-called *ADM form*, which is written in terms of N and N^a.

From the time reparametrization (1.36) it follows that the effective action (1.33) becomes

$$\begin{aligned} S_{\text{eff}} &= \int dt \int_{\Sigma_t} d^3x \left(\pi^{ab}\dot{q}_{ab} - N\left[\frac{1}{2}(16\pi G)\, G_{abcd}\pi^{ab}\pi^{cd} - \frac{\sqrt{q}}{16\pi G}\mathcal{R}\right]\right) \\ &= \int dt \int_{\Sigma_t} d^3x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H}\right) , \end{aligned} \tag{1.40}$$

where we have replaced $\dot{q}_{ab} \longrightarrow \frac{1}{N}\dot{q}_{ab}$ and $\mathcal{H}$ remains invariant since the Hamiltonian density $\mathcal{H}$ does not include any notion of dt.

The action (1.40) is also subject to the spatial coordinate transformation (1.37) determined by the shift vector N^a. Now the change of the action is a bit more subtle than in the time reparametrization. To begin to treat with the action (1.40), we recall an important result from elementary Riemannian geometry: Under infinitesimal coordinate transformations $x^\mu \longrightarrow x^\mu + \xi^\mu$ the spacetime metric $g_{\mu\nu}$ transforms as

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) - \xi_{\mu;\nu}(x) - \xi_{\nu;\mu}(x) . \tag{1.41}$$

On the spacelike hypersurfaces the corresponding metric is q_{ab}, and the infinitesimal displacement vector ξ^μ is replaced by the vector $N^a dt$. Because of that, we find that under spatial coordinate transformations, i.e., under gauge transformations, q_{ab} transforms as

$$q'_{ab} = q_{ab} - N_{a|b}dt - N_{b|a}dt , \tag{1.42}$$

at each point on the spacelike hypersurface. The symbol $N_{a|b}$ denotes covariant differentation of the shift vector with respect to the coordinate x^b on the hypersurface. Furthermore, the time derivative of the spatial metric $\dot{q}_{ab}$ transforms as

$$\dot{q}'_{ab} = \dot{q}_{ab} - N_{a|b} - N_{b|a} . \tag{1.43}$$

Hence, we finally find that under time reparametrizations and under spatial coordinate transformations S_{eff} transforms (when the boundary terms are neglected) to

$$S_{\text{eff}} = \int dt \int_{\Sigma} d^3x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H} + 2N_a\mathcal{H}^a\right) , \tag{1.44}$$

where

$$\mathcal{H}^a := -2\pi^{ab}{}_{|b} \,. \tag{1.45}$$

1.4.2 Constraints

Since the time reparametrizations and spatial coordinate transformations are represented by the lapse function N and the shift vector N^a, the independence of the action (1.44) of N and N^a or, equivalently, the vanishing of the variations of S_{eff} with respect to N and N^a, implies that

$$\mathcal{H} = 0 \,, \tag{1.46}$$

$$\mathcal{H}^a = 0 \,. \tag{1.47}$$

The equation $\mathcal{H} = 0$ is known as the *Hamiltonian constraint* and the three equations $\mathcal{H}^a = 0$ are known as the *diffeomorphism constraints* of general relativity. The use of the terminology "constraint" is justified by the fact that the variables N and N^a should not be viewed as dynamical variables of the formulation, since the action (1.44) does not include any time derivatives or canonically conjugate momenta of the variables, and therefore they have a role similar to the one played by *Lagrange's undetermined multipliers* in classical mechanics.

The interpretation of Eq. (1.46) is that the Hamiltonian formulation remains invariant under time reparametrizations and Eq. (1.47) tells us that the Hamiltonian formulation remains invariant in spacelike coordinate transformations, i.e., in spatial diffeomorphisms. The four constraints together imply that the Hamiltonian formulation does not depend on the 3+1 decomposition of spacetime. Hence we have managed to recover the full diffeomorphism invariance of general relativity. Moreover, the variation of the action (1.44) with respect to the lapse N is equivalent to the variation of the Einstein-Hilbert action (1.1) with respect to g_{00}, since in the ADM metric, $g_{00} = -N^2 + N^a N_a$, and hence the Hamiltonian constraint $\mathcal{H} = 0$ is equivalent to Einstein's field equation $G_{00} = 0$. Furthermore, the variation of the action (1.44) with respect to the shift N_a is equivalent to the variation of the Einstein-Hilbert action (1.1) with respect to g_{a0}, since in the ADM metric, $g_{a0} = N_a$. Because of that the diffeomorphism constraints $\mathcal{H}^a = 0$ are equivalent to Einstein's field equations $G_{a0} = 0$. Hence, by recovering the diffeomorphism invariance of the formulation, we have recovered the missing field equations. The constraints also imply that the real number of degrees of freedom per spacetime point is not 6, the number of the independent components of the spatial metric q_{ab}, but $6 - 4 = 2$.

In a general coordinate system, when the lapse N and the shift N^a are employed, we can write the Hamiltonian of pure gravity as

$$H = \int_{\Sigma_t} d^3x \, (N\mathcal{H} + N_a \mathcal{H}^a) \,, \tag{1.48}$$

and the dynamical equations of the system are still given by (1.34) and (1.35).

When matter fields are present in spacetime, the Hamiltonian (1.48) has to be supplemented by two new terms $\mathcal{H}_{\text{matter}}$ and $\mathcal{H}^a_{\text{matter}}$, which are, respectively, known as the Hamiltonian and the momentum density for matter. By taking into account the Hamiltonian and the momentum density for matter, we get the Hamiltonian of our system in the presence of matter fields

$$H = \int_{\Sigma_t} d^3x \left[N \left(\mathcal{H} + \mathcal{H}_{\text{matter}} \right) + N_a \left(\mathcal{H}^a + \mathcal{H}^a_{\text{matter}} \right) \right] . \tag{1.49}$$

Now, the variations with respect to the lapse N and the shift N_a yield the Hamiltonian and the diffeomorphism constraints in the presence of matter fields:

$$\mathcal{H} + \mathcal{H}_{\text{matter}} = 0 , \tag{1.50}$$

$$\mathcal{H}^a + \mathcal{H}^a_{\text{matter}} = 0 , \tag{1.51}$$

These constraint ensure that the Hamiltonian formulation of general relativity remains unchanged under time reparametrizations and spatial coordinate transformations even in the presence of matter fields.

1.4.3 ADM Action

When the lapse N and the shift N^a differ from unity, the expressions for the extrinsic curvature tensor (1.22) and for the effective action (1.24), written in a general coordinate system in terms of the extrinsic curvature tensor, change considerably. It can be shown, by using (1.21), that the components of the extrinsic curvature tensor can be extracted, when the ADM metric is used, from

$$K_{ab} = \frac{1}{2N} \left(\dot{q}_{ab} - N_{a|b} - N_{b|a} \right) . \tag{1.52}$$

Since K_{ab} and $\mathcal{R}$ are tensors on a spacelike hypersurface, the action (1.24) is invariant under spacelike coordinate transformations defined by the shift N^a, and therefore is independent of the shift N^a, whereas the action integral changes when time reparametrization, defined by the lapse N, is performed. Hence the action (1.24) can be written as

$$S_{\text{ADM}} = \frac{1}{16\pi G} \int dt \int_{\Sigma_t} d^3x \sqrt{q} N \left(K_{ab} K^{ab} - K^2 + \mathcal{R} \right) , \tag{1.53}$$

which is called the *ADM action* of pure gravity. The ADM action (1.53), when the boundary terms are ignored, is the most generic starting point for the Hamiltonian formulation of general relativity. In this thesis we have adopted, in my opinion, a more pedagogical approach to the Hamiltonian formulation as we introduced the diffeomorphism invariance of the formulation by hand, instead of studying the properties of the Hamiltonian formulation by taking the ADM action, written in terms of the ADM metric, as our starting point.

1.5 Boundary Terms in Asymptotically Flat Spacetimes

We are ready to investigate the appropriate boundary terms that must be included in the ADM action (1.53) when our spacetime is assumed to be *asymptotically flat.* It has been noticed over 30 years ago that the Hamiltonian formulation of general relativity depends substantially on whether spatial hypersurfaces Σ are open or closed [34]. If the spacetime manifold composed of the spacelike hypersurfaces is closed, there are no boundary terms to be included in the ADM action at all, and the Hamiltonian (1.49) of the system vanishes when Einstein's field equations are satisfied. This suggests that the total energy of a closed universe is zero. However, when the spatial hypersurfaces are open such that the boundary of a spacelike hypersurface Σ_t of constant t is a two-dimensional surface S such that $\Sigma_t \cup S$ is a compact submanifold with a boundary, then one must supplement the ADM action by additional boundary terms evaluated from certain surface integrals over the surface S transformed to spatial infinity in order to attain correct dynamical equations of motion.

When matter fields are present, the gravitational part of the Hamiltonian constraint does not bring any difficulties at all as one varies the matter part of the Hamiltonian. Moreover, matter fields vanish at spatial infinity faster than the spatial metric, and because of that the surface integrals at infinity containing the matter fields vanish nicely. These reasons cause us to consider in this section pure gravity only[2].

In this section we shall first discuss, in few words, asymptotically flat spacetimes, since the concept of 'infinitely far' definitely demands some clarification. After that, we shall seek for the boundary terms of asymptotically flat spacetimes.

1.5.1 Asymptotically Flat Spacetimes

Roughly speaking, by an asymptotically flat spacetime we mean a certain class of spacetimes which deviate from the flat Minkowski spacetime geometry the less the farther off one moves from the matter source that causes the spacetime to curve. Although the intuitive idea of asymptotic flatness is relatively clear, we must anyway define the asymptotic flatness of spacetime in a more precise manner. A proper definition of asymptotically flat spaces was put forward by R. Penrose [35], and since then it has been investigated and elucidated by many authors (see the treatment in Refs. [27, 30]). Since the studies concerning this particular property of spacetime are in itself a vast subject, we are satisfied here with a rudimentary treatment. We define spacetime to be asymptotically flat if there exists a system of coordinates x^0, x^1, x^2, x^3 such that the metric components in these coordinates, at large distances, behave along the spatial directions as follows:

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(r^{-1}) \text{ as } r \longrightarrow \infty \ , \tag{1.54}$$

where $\eta_{\mu\nu}$ is the metric tensor of flat spacetime, and $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. In other words, we assume that the spatial coordinates become Cartesian in the asymptotic spacelike infinity. In these coordinates we may write the spacetime metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{1.55}$$

which is familiar from the linear field approximation of Einstein's field equations, and therefore we may ignore the higher-order terms in $h_{\mu\nu}$ in all the

[2]The diffeomorphism constraints of matter do not include the spatial metric nor its conjugate momenta.

derived quantities. Definition (1.54) perhaps raises more questions than it answers, but it is adequate to us, since ultimately we will be interested in black-hole spacetimes only. Fortunately, in stationary[3] black-hole spacetimes it is possible to find such a coordinate system that satisfies the condition (1.54).

When we study the properties of spacetime infinitely far away from the mass distribution, we may use a special method developed by R. Penrose known as the *conformal compactification* of spacetime [35]. The basic idea in the process of compactifying the *whole spacetime geometry* is that we bring the spacetime points from inifinity to a finite distance, and thereby we may study the causal structure of infinity. In this process we use coordinate transformations which map the infinite interval $(-\infty, \infty)$ onto the finite interval $(-\pi, \pi)$; thus we obtain a new manifold $\tilde{\mathcal{M}}$ which possesses a spacetime boundary representing the infinity of spacetime. As an enlightening example, let us consider the flat Minkowski spacetime. We perform *conformal transformation* of coordinates from the usual spherical coordinates (t, r, θ, ϕ) of flat spacetime to new spherical coordinates $(\psi, \xi, \theta, \phi)$ such that

$$t + r = \tan\left[\frac{1}{2}(\psi + \xi)\right] , \tag{1.56}$$

$$t - r = \tan\left[\frac{1}{2}(\psi - \xi)\right] , \tag{1.57}$$

$$\theta = \theta , \tag{1.58}$$

$$\phi = \phi , \tag{1.59}$$

where $-\pi < \psi - \xi < \psi + \xi < \pi$. Now, the points at infinity in Minkowski spacetime correspond to the points when $\psi + \xi$ and $\psi - \xi$ take the values $\pm\pi$. Thus the inifinite spacetime has been shrunk into a diagram which is finite in spatial and temporal directions. A two-dimensional figure illustrating the boudaries of Minkowski spacetime give rise to the *conformal diagram* known as the *Carter-Penrose diagram* of Minkowski spacetime. In Fig. 1.4 the coordinates θ and ϕ have been suppressed.

It can be shown that all timelike curves begin at the point i^- called the *past timelike infinity* and end at the point i^+ called the *future timelike infinity*, and the point i^0 is known as the *spatial infinity*, where $r \longrightarrow \infty$ for finite t.[4] All lightlike curves begin at the boundary $\mathfrak{I}^-$ and end at $\mathfrak{I}^+$. The boundaries $\mathfrak{I}^-$ and $\mathfrak{I}^+$ are called the *past null infinity* and the *future null infinity*, respectively. The spacetime region $J^+(\mathfrak{I}^-)$ consists of events which can be causally affected by events in $\mathfrak{I}^-$ and is called the *causal future* of $\mathfrak{I}^-$. Respectively, the *causal past* of $\mathfrak{I}^+$ is denoted by $J^-(\mathfrak{I}^+)$. In the example of Minkowski spacetime, asymptotically flat region of spacetime is bounded by the *boundaries* $\dot{J}^+(\mathfrak{I}^-)$ and $\dot{J}^-(\mathfrak{I}^+)$ and it consists of events in $J^-(\mathfrak{I}^+)$.

In fact, almost for any spacetime it is possible to draw the Carter-Penrose diagram. In particular, analogous conformal coordinate transformations can be performed in black-hole spacetimes (see for example [27]), and because of that it is possible to draw their Carter-Penrose diagrams as well. We will return to the conformal diagrams of black holes in Chapter 2.

[3]For the definition of a stationary black hole, see Sec. 2.5.

[4]Actually, the points i^-, i^+ and all the other points inside the diagram represents a two-sphere S^2.

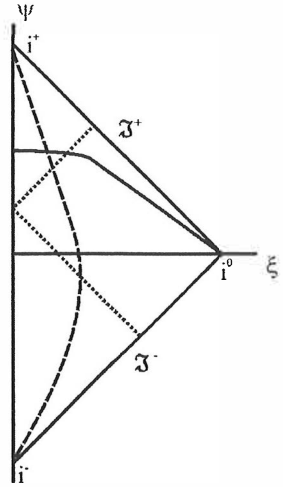

Figure 1.4: Carter-Penrose diagram of Minkowski spacetime when both of the two timelike infinities are included. The figure shows spacelike (continous line), timelike (dashed line) and null (dotted line) geodesics.

Let us consider a region of an asymptotically flat spacetime manifold $\mathcal{M}$ bounded by two spacelike hypersurfaces Σ_t and $\Sigma_{t'}$. It is no longer natural to assume that the variations of the dynamical variables and their canonical momenta vanish or are held fixed at the spatial boundaries. The most natural spatial boundary condition is that the variations preserve asymptotic flatness. By satisfying this condition the boundary terms are required to be added to the gravitational action (1.44). On the road to the Hamiltonian formulation we have ignored all the boundary terms that we have confronted. Therefore we shall calculate the boundary terms arising from the variations of the action (1.44), and, in order to have well-defined equations of motion, we shall add these terms into the action to cancel out the undesired boundary terms.

The role of the surface integrals in the Hamiltonian formulation of general relativity has been extensively investigated and elucidated by T. Regge and C. Teitelboim in Ref. [21]. In fact, Regge and Teitelboim showed in [21] that the boundary terms are closely related to the invariance of the gravitational action under time and spatial coordinate translations and rotations at asymptotic infinity. These translations and rotations are very well known to be associated with the conserved quantities of total energy, linear momentum and angular momentum of an isolated system in classical mechanics. Even in general relativity it is possible to find the corresponding quantities.

1.5.2 ADM Energy

First, we consider the case where the vector t^μ joining two consecutive spacelike hypersurfaces Σ_t and Σ_{t+dt} represents asymptotically a time translation, that is, we choose the lapse N to become some function $N_+(t)$ and the shift N^a is chosen to vanish as the radial coordinate r approaches the spatial infinity. In that case only the term

$$\frac{\sqrt{q}}{16\pi G}N\mathcal{R}$$

in the action (1.44) produces a non-zero boundary term when the action is varied with respect to the dynamical variables and their conjugate momenta. The boundary term corresponding to this term has been obtained, for example, in [30] by varying the Hamiltonian (1.48) with respect to the dynamical variables, and it is shown, in asymptotic Cartesian coordinates, to yield a surface integral

$$E^{\mathrm{ADM}} := \lim_{r\to\infty} \frac{1}{16\pi G} \oint_S dS^n \left(q_{mn,m} - q_{mm,n}\right) , \tag{1.60}$$

where the notation '$\lim_{r\to\infty}$' must be understood as a process where the two-dimensional spatial boundary S of the hypersurface $t = constant$ is carried forward to spatial infinity. This surface integral is known as the *ADM energy* of spacetime and it is nonvanishing, and for this reason it must be accounted in the gravitational action (1.44). However, it is possible to show that exactly the same surface integral (1.60) is attainable when we consider the linear field approximation of the first term on the right hand side of (1.8) multiplied by the lapse N. In fact, we omitted this term in the very beginning of our investigation. We may approximate the Christoffel symbol Γ^a_{mn} as

$$\Gamma^a_{mn} = \frac{1}{2}\delta^{as}\left(h_{am,n} + h_{na,m} - h_{mn,a}\right) + \mathcal{O}(h^2) , \tag{1.61}$$

and therefore

$$\begin{aligned}
&\tfrac{1}{16\pi G}\int dt \int_\Sigma d^3x N \left[q^{mn}\sqrt{q}\left(\delta^a_m \Gamma^b_{nb} - \Gamma^a_{mn}\right)\right]_{,a} \\
&= \tfrac{1}{16\pi G}\int dt \int_\Sigma d^3x \left(h_{mm,n} - h_{mn,m}\right)_{,n} + \mathcal{O}(h^2) \\
&= \tfrac{1}{16\pi G}\int dt \oint_S dS^n \left(h_{mm,n} - h_{mn,m}\right) + \mathcal{O}(h^2) \quad .
\end{aligned} \tag{1.62}$$

The only difference between the surface integrals (1.60) and (1.62) is that the latter is written in terms of the small perturbation h_{ab} at spatial infinity. When the condition (1.54) is taken into acount and Eq. (1.55) is used in the former integral, the two surface integrals become identical. Hence, if we require spacetime to be asymptotically flat then the variation of the action (1.44) with respect to its dynamical varibles q_{ab} brings along the ADM energy (1.62) which must be cancelled out. Because of that, the gravitational action must be supplemented by the so-called *ADM boundary term*

$$S^{\mathrm{ADM}}_{\partial\Sigma} = \int dt N_+(t) E^{\mathrm{ADM}} , \tag{1.63}$$

where the lapse N becomes some function $N_+(t)$ as r approaches to infinity. Hence, the gravitational action (1.44) with the ADM boundary term related to the time translations is

$$S = \int dt \int_\Sigma d^3x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a\right) - \int dt N_+(t) E^{\mathrm{ADM}} . \tag{1.64}$$

The appearance of E_{ADM} is very intriguing and we will study this later in this chapter. Here we shall only point out that when Einstein's field equations are satisfied, the Hamiltonian of the system simply becomes

$$H = N_+(t)E^{\mathrm{ADM}} \ . \tag{1.65}$$

If our universe is compact, there will be no boundary contributions and thus

$$H = 0 \ . \tag{1.66}$$

This suggests very interestingly that the *total energy* of our universe should be zero.

1.5.3 Linear Momentum

In addition to the ADM energy, which is related to time evolution at asymptotic infinity, we have, for the non-vanishing shift N^a at spatial infinity, boundary terms associated with asymptotic spatial coordinate translations. Because of that, we want the vector t^μ to become a pure spatial coordinate translation, that is, we choose the lapse N vanish and the shift N^a_+ becomes some function $N^a_+(t)$ as r goes to infinity. In that case only the variation of the action (1.44) with respect to the momenta conjugate to q_{ab} brings along a term

$$2\int dt \int_\Sigma d^3x \left(N^a \delta\pi_a{}^b\right)_{|b} = -2\lim_{r\to\infty}\int dt N^a_+(t) \oint_S \pi_a{}^b dS_b =: S^{\mathrm{trans}}_{\partial\Sigma} \ , \tag{1.67}$$

which must be cancelled at infinity. The surface integral

$$P_a^{\mathrm{ADM}} := -2\lim_{r\to\infty}\oint_S \pi_a{}^b dS_b \ , \tag{1.68}$$

is known as the *ADM momentum* of spacetime. Now the gravitational action (1.64) must be supplemented by this boundary term also:

$$\begin{aligned} S \ = \ & \int dt \int_\Sigma d^3x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a\right) \\ & - \int dt\, N_+(t)\, E^{\mathrm{ADM}} \\ & + \int dt\, N^a_+(t)\, P_a^{\mathrm{ADM}} \ . \end{aligned} \tag{1.69}$$

1.5.4 Angular Momentum

So far we have considered time translation and spatial coordinate translations at asymptotic infinity. There is one flat-space symmetry still to be investigated, namely the rotational symmetry at spatial infinity. At this time, we choose again not to have time evolution at the asymptotic infinity, i.e., the lapse N vanishes as r approaches infinity, but since we want to have an infinitesimal asymptotic rotation of coordinates around the asymptotic Cartesian coordinate system determined by an infinitesimal angle $\delta\varphi^b$ as r approaches to infinity, we choose

$$N^a dt \overset{r\to\infty}{\longrightarrow} \varepsilon_{abc}\delta\varphi^b x^c \ . \tag{1.70}$$

In that case the variation of the action (1.44) with respect to the momenta conjugate to q_{ab} brings along the same term as in spatial coordinate translations, but the corresponding surface integral differs from (1.68) considerably:

$$2\int dt \int_\Sigma d^3x \left(N_a \delta\pi^{ab}\right)_{|b} = -2\varepsilon_{abc} \lim_{r\to\infty} \int dt\, \omega^b(t) \oint_S x^c \pi^{an} dS_n =: S^{\text{rot}}_{\partial\Sigma} \; , \tag{1.71}$$

where $\omega^b := d\varphi^b/dt$ is the angular velocity of the rotating coordinate system around the Cartesian coordinate system at asymptotic infinity. The surface integral

$$L_b^{\text{ADM}} := -2\varepsilon_{abc} \lim_{r\to\infty} \oint_S dS_n x^c \pi^{an} \; , \tag{1.72}$$

is known as the *ADM angular momentum* of spacetime. Finally, when even the last boundary term is taken into acount, the gravitational action with the appropriate boundary terms in pure gravity can be written as

$$\begin{aligned} S \;=\; & \int dt \int_\Sigma d^3x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a\right) \\ & - \int dt\, N_+(t) \lim_{r\to\infty} \frac{1}{16\pi G} \oint_S dS^b \left(q_{ab,a} - q_{aa,b}\right) \\ & + \int dt\, N_+^a(t)\, 2 \lim_{r\to\infty} \oint_S \pi_a{}^b dS_b \\ & + \int dt\, \omega^b(t)\, 2\varepsilon_{abc} \lim_{r\to\infty} \oint_S dS_n x^c \pi^{an} \; , \end{aligned} \tag{1.73}$$

or equivalently,

$$\begin{aligned} S \;=\; & \int dt \int_\Sigma d^3x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a\right) \\ & - \int dt \left(N_+(t)\, E^{\text{ADM}} + N_+^a(t)\, P_a^{\text{ADM}} + \omega^b L_b^{\text{ADM}}\right) . \end{aligned} \tag{1.74}$$

From the ADM energy (1.60) and from the components of the ADM momentum (1.68) it is possible to form a conserved Lorentz-invariant four-vector known as the ADM energy-momentum of spacetime at spatial infinity:

$$P^\mu := (E^{\text{ADM}}, P^{\text{ADM}\; a}) \; . \tag{1.75}$$

The conservation of the ADM energy and the Lorentz invariance of the ADM four-momentum was investigated by Regge and Teitelboim in [21].

The physical interpretation of the ADM energy-momentum is that the ADM energy gives the total energy of spacetime evaluated at the asymptotic infinity as mentioned before, and the ADM momentum yield the total linear momentum of spacetime evaluated at the asymptotic infinity. It is important to realize that these values do not depend on the properties of the internal structure of spacetime, but the total energy, the linear momentum, and even the angular momentum of spacetime can be read off from the spacetime boundaries. Since the ADM energy is constant under time translations on every Σ, the total energy associated with any spacelike hypersurface Σ is given by the ADM energy E^{ADM}, and, for example, in the Schwarzschild spacetime, which is asymptotically flat, one can show that the ADM energy is given by $E^{\mathrm{ADM}} = M$, where M can be understood as the total mass of the Schwarzschild spacetime. The physical properties of the ADM energy-momentum of a isolated system in general relativity are analogous to the corresponding properties of a particle in special relativity: The total energy of a particle is given by the time component $p^0 = E$ of the four-momentum $p^\mu(x^\mu)$ of the particle, and the mass of the particle is given by $m = \sqrt{\ \ p_\mu p^\mu}$. Now, if the particle is at rest with respect to the coordinate system defined by x^μ, then $E = m$. Thus, the knowledge of the rest frame and the mass m determines the total energy and even the four-momentum of the particle. Despite the fact that local energy density is not a well-defined concept in general relativity, one can define the concept of total energy of spacetime in the case of an asymptotically flat universe. This is very important when we study the quantum mechanics of isolated black holes.

Chapter 2

Black Holes

2.1 Introduction

A *black hole* is a spacetime region in which the gravitational field is so strong that even light cannot escape from it to infinity. In 1916 K. Schwarzschild published the first known exact solution to Einstein's field equations, and it soon turned out that it included a spacetime region which is called the black hole. Today no one who accepts general relativity as a whole cannot elude the theoretical prediction that black holes must exist in our universe. However, this has not always been the case: Before neutron stars were discovered by the end of the sixties [36], black holes were believed not to exist in reality. In spite of the misbelieves, S. Chandrasekhar showed in 1931 that when an ordinary star has exhausted all its nuclear fuel and it has collapsed to a white dwarf, it must continue to collapse due to the gravitational attraction of its own mass to a neutron star if the mass of the white dwarf exceeds 1.2 solar masses [37]. This mass limit is known as the *Chandrasekhar limit.* The Chandrasekhar limit is caused by the outward pressure of the electron gas of the white dwarf. Since electrons are spin-1/2 particles, the electron gas obeys *Pauli's exclusion principle* and therefore it necessarily produces a certain outward pressure that prevents the white dwarf from collapsing if its mass is smaller than the Chandrasekhar limit. Another major input was put forward by R. Oppenheimer and H. Snyder in 1939 [38], as they showed by using a simple relativistic model that a sufficiently massive star continues to collapse infinitely. If the mass of a white dwarf exceeds 1.44 solar masses the white dwarf collapses to a neutron star, and if the mass of the neutron star is more than 3.2 solar masses the pressure of the neutron gas cannot prevent it from collapsing infinitely. When a massive star collapses infinitely, its radius approaches, in a finite proper time, its *gravitational*, or *Schwarzschild, radius*

$$R_{\mathrm{S}} = \frac{2GM}{c^2}\,, \tag{2.1}$$

where $G \approx 6.67 \times 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ is Newton's gravitational constant, $c = 299792458$ m/s is the speed of light, and M is the mass of the star. After the radius of the collapsing star has passed its Schwarzschild radius, even the light emitting from the collapsing matter cannot escape to infinity any longer, but the light rays are trapped inside the surface $r = R_S$. The surface $r = R_S$ is called the *trapped surface* [39]. When the radius of the collapsing star has become smaller than its Schwarzschild radius a far-away observer cannot see the collapsing star anymore. Therefore, if a collapsed matter distribution has a radius smaller than its Schwarzschild radius, we call it a black hole. According to Oppenheimer's and Snyder's model, even after the trapped surface has formed, matter still continues to collapse until it has become a single point in space. This point at all times is called the black hole *singularity*. (See Fig. 2.1.)

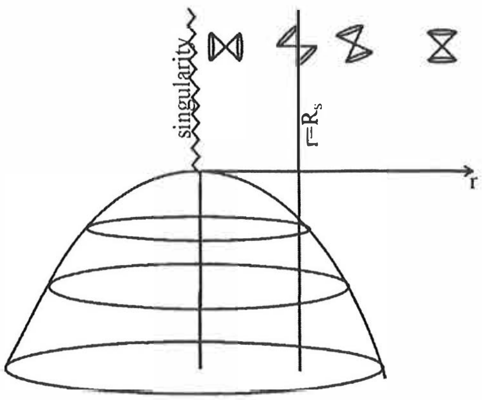

Figure 2.1: This figure represents a spherically symmetric gravitational collapse, when the azimuthal dimension is suppressed. The light cones "tip over" at the surface $r = R_S$. They illustrate the fact that the future of any particle is at the singularity $r = 0$. Note that the singularity produced in spherically symmetric gravitational collapse is spacelike.

There are at least three different known processes of the gravitational collapse. The first process is the gravitational collapse of a star which was just briefly discussed. The second process is the gravitational collapse of the center of a cluster of stars where the whole core of the cluster collapses into a very massive black hole [40]. The third process is a controversial subject at the moment, but it is widely believed that the so-called *primordial black holes* could have been produced in the very dense early universe by a direct gravitational collapse of matter [41]. By the direct gravitational collapse we mean a collapse that is caused by the dense inhomogeneous matter distribution. It has been speculated that although our universe is on the large scale very homogeneous, on the microscopic level the early universe might have been initially inhomogeneous, and therefore these inhomogeneous regions could have collapsed directly to black holes rather than expanded with the universe. The mass scale of the primordial black holes could be anything between the Planck-size black holes with mass $\sim 1\ \mu$g and the black holes in the core of a large galaxy with $\sim 10^{10}$ solar masses. The importance of the Planck-size primordial black holes to physics is highlighted in their quantum effects. In fact, we concentrate on the quantum-mechanical models of a single primordial black hole or black-hole pairs in Chapter 5 and Chapter 6.

All the black hole spacetimes that have been produced by the gravitational collapse of matter are not time-symmetric because of the disappearance of the mass distribution. Black holes that are time-symmetric are called the *eternal black holes*. These black holes may be considered unphysical, but usually it is convenient to investigate the properties of the eternal black holes.

In this chapter we consider some generally important results originating from the 1970's and even from the era at the beginning of Einstein's general relativity. We define the concept of black hole and introduce the different types of black holes and their mechanical laws. Also the *uniqueness theorems* of black holes are presented.

2.1.1 Black Hole, Black Hole Singularity and Naked Singularity

Usually black holes are defined to be spacetime regions were the gravitational attraction becomes so strong that even light cannot escape from it.[1] Therefore black holes appear us as black. However, black holes can be defined without any notion to the terms 'gravitational' and 'light' [27]: In an asymptotically flat spacetime $\mathcal{M}$ a black hole is a closed set $B \subset \mathcal{M}$ such that

$$B = \mathcal{M} - J^{-}(\mathfrak{I}^{+}) \ . \tag{2.2}$$

In other words, a black hole is a spacetime region, which does not belong to $J^{-}(\mathfrak{I}^{+})$. Consider, as an example, the flat Minkowski spacetime. There the whole spacetime manifold is included in $J^{-}(\mathfrak{I}^{+})$ (see Fig. 1.4), and because of that one may say that Minkowski spacetime does not contain black holes. This observation gives us a reason to state that spacetime contains a black hole if and only if $\mathcal{M}$ is not included in $J^{-}(\mathfrak{I}^{+})$. The usual definition 'of no escape' follows from the definition (2.2), since all lightlike curves are asymptotically parallel to the boundary $H^{+} = \dot{J}^{-}(\mathfrak{I}^{+}) \cap \mathcal{M}$ of a black hole. This is illustrated in Fig. 2.6, where the boundary H^{+} is a two-dimensional lightlike surface called the black hole *event horizon*. Hence, in asymptotically flat spacetime, it is impossible, even for light rays to escape from B to the future null infinity $\mathfrak{I}^{+}$, and if light cannot come out of the black hole then all particles and even information are captured inside the hole. It is not possible to interact with the black hole before one has passed the surface of the hole, and it is not possible to return from the hole once the surface of the black hole has been crossed over. In other words, black holes shut themselves out of their surroundings. Therefore black holes are considered as holes in spacetime. We shall investigate the geodesics of freely falling matter in black hole spacetimes in Sec. 2.4.1, and the meaning of the event horizon is discussed further in Sec. 2.2.

General relativity possesses spacetime singularities whenever the metric tensor is not defined. Some of these singularities can be removed by a coordinate transformation, and because of that these coordinate singularities are unphysical. General relativity predicts the existence of true physical singularities also. Such singularities cannot be removed by any coordinate transformation. Physical singularities can be divided into cosmological, naked and black hole singularites. In this chapter we are interested in the singularites inside the black hole only[2]. The black hole singularities are further

[1] The implication of the large gravitational potential was first noticed by J. Michell in 1784.

[2] According to the unproven Cosmic Censorship Conjecture, stated by R. Penrose [42],

characterized by an infinite curvature scalar R. In other words, the black hole singularity can be characterized by undefined metric and by infinitely large spacetime curvature. Another way of understanding the singularities comes out very naturally, when one investigates the incompleteness of timelike or a null geodesics [27]: If there is an incomplete timelike or null geodesic in spacetime, it can be shown that – under certain assumptions – spacetime must have a singularity. By an incomplete geodesic we mean a geodesic that is inextendible in at least one direction and has a finite affine length [43]. Roughly speaking, an incomplete geodesic breaks at some spacetime point. In particular, an incomplete geodesic must begin or end at the spacetime singularity.

In terms of geodesics it is possible to show that under some highly technical assumptions concerning matter and the global properties of spacetime, a closed trapped surface exists and spacetime must contain a singularity. There are several analytical and quite general statements about the exsistence of the singularities. These statements are the so-called *singularity theorems* and they were formulated by R. Penrose and S. W. Hawking [44, 45, 46, 27]. The singularity theorems show their power in proving that even non-spherically symmetric gravitational collapse produce a closed trapped surface. While the passage to the singularity theorems would be very interesting, it would also be a very lengthy journey. Therefore we shall not go into a detailed discussion. A brief intorduction to the singularity theorems is given in [29], and some of the details can be found, for example, in Ref. [30].

In general, the gravitational collapse of a star is very difficult to understand, and indeed, there is no generic model that would describe non-spherically symmetric gravitational collapse. In fact, there is no known mechanism that would cause other than spherically symmetric black holes.

2.2 Schwarzschild Black Hole

The first black hole solution to Einstein's field equations was found by K. Schwarzschild in 1916 [4]. It is called the Schwarzschild solution, and it represents a spherically symmetric vacuum solution. In the Schwarzschild coordinates t, r, θ, ϕ the Schwarzschild line element has the form

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad (2.3)$$

where the parameter M can be identified as the total mass of the gravitational source. This line element has many interesting properties. First of all, as shown by Birkhoff [47], the spacetime geometry is determined by the total mass M, even when the mass distribution performs oscillations or vibrations such that they preserve spherical symmetry. In other words, the Schwarzschild solution remains the only spherically symmetric vacuum solution to Einstein's field equations whether the spherically symmetric gravitational source were static or not. Secondly, the Schwarzschild

in any reasonable spacetime no singularity is ever observationally seen by a far-away observer. The so-called *naked singularities* are not clothed by the event horizon. This unproven conjecture is widely believed to be true, and it constitutes one of the cornerstones of general relativity.

line element (2.3) is time independent and depends only on r, and therefore the reference frame formed by the Schwarzschild coordinates is *static*.

When $r \longrightarrow R_S$, we notice from the Schwarzschild line element that $g_{00} \longrightarrow 0$ and $g_{11} \longrightarrow \pm\infty$. In other words, the point $r = R_S$ is a singularity. However, it turns out that this singularity is an unphysical coordinate singularity, causing no effects on any freely falling observers. For ordinary stars the region $r < R_S$ is filled with matter, and as such it is not a relevant region of spacetime, whereas for black holes that region is admissible. Moreover, we already know that a gravitational collapse produces a black hole, and that a massive, spherically symmetric mass distribution collapses into a spherically symmetric black hole. The exterior region of spherically symmetric black holes, in turn, are described by the Schwarzschild solution. Therefore spherically symmetric non-charged black holes are called Schwarzschild black holes, and for this reason it is interesting to study the behaviour of the Schwarzschild line element also inside the region $r < R_S$ – the black hole itself. When r approaches to zero, we find from the Schwarzschild line element that g_{00} approaches to infinity and the curvature scalar R becomes infinite. Hence, the point $r = 0$ is the true black hole singularity.

It is also important to note that the roles of the time coordinate t and the spatial coordinate r change at the point $r = R_S$: t becomes a spatial coordinate and r records temporal coordinate. This follows from (2.3) where the coefficients of dr^2 and dt^2 reverse their signs when crossing the Schwarzschild radius. This, in turn, implies that every signal of information or particle is compelled to fall into the singularity after it has crossed the trapped surface $r = R_S$. Nothing can escape from the black hole region. Moreover, light cones 'tip over' at the Schwarzschild radius $r = R_S$, such that inside the black hole they always open toward the line $r = 0$, where the singularity lies. The trapped surface $r = R_S$ bounding the hole is called the *event horizon* H of the Schwarzschild black hole. The event horizon acts as a one-way membrane. Anything may go inside the black hole, but nothing can come out of it.

The event horizon is a two-dimensional surface[3]. This can be seen by showing that the three-volume of the surface is zero, whereas the area of the region where $t = 0$ and $r = R_S$ is

$$A = 4\pi R_S^2 \ . \tag{2.4}$$

When $r \longrightarrow \infty$ the Schwarzschild line element (2.3) becomes the flat Minkowski spacetime line element in the spherical coordinates t, r, θ, ϕ. This implies that far away from the mass distribution M the Schwarzschild spacetime is asymptotically flat (according to our definition (1.54)). Later in this chapter we shall discuss the Carter-Penrose diagram of the Schwarzschild spacetime.

2.2.1 Kruskal Diagram of Kruskal Spacetime

The Schwarzschild coordinates are not well-defined at the event horizon. Therefore one can neither continuously describe the geodesics of the infalling matter in terms of t, r, θ and ϕ nor analyze the coordinate singularity

[3]Hawking has shown that the topology of an event horizon of any black hole is S^2 [27].

$r = R_S$, and even the light cones are ill-behaved on the event horizon in the Schwarzschild coordinates. A static coordinate system that removes the coordinate singularity is the Kruskal coordinate system [48, 49]. To illustrate the Schwarzschild spacetime in Kruskal coordinates, let us develop the necessary coordinate transformation step by step.

In terms of the so called "Regge-Wheeler tortoise coordinate"

$$r_* := r + R_S \ln \left| \frac{r}{R_S} - 1 \right| \tag{2.5}$$

we define the outgoing and ingoing Eddington-Finkelstein coordinates $\mathcal{U}, r$ and $\mathcal{V}, r$ such that

$$\mathcal{U} = t - r_* \,, \tag{2.6}$$

$$\mathcal{V} = t + r_* \,. \tag{2.7}$$

Hence we may implicitly express the Schwarzschild coordinate r as a function of $\mathcal{U}$ and $\mathcal{V}$ such that

$$r_* = r + 2GM \ln \left| \frac{r}{2GM} - 1 \right| = \frac{1}{2} (\mathcal{V} - \mathcal{U}) \,, \tag{2.8}$$

and therefore the Schwarzschild line element takes the form

$$ds^2 = \frac{-2GM e^{-\frac{r}{2GM}}}{r} e^{\frac{\mathcal{V}-\mathcal{U}}{4GM}} d\mathcal{U} d\mathcal{V} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \,. \tag{2.9}$$

By further defining the *Kruskal coordinates* u and v such that

$$u = \frac{1}{2} \left(e^{\mathcal{V}/4GM} + e^{-\mathcal{U}/4GM} \right) \,, \tag{2.10}$$

$$v = \frac{1}{2} \left(e^{\mathcal{V}/4GM} - e^{-\mathcal{U}/4GM} \right) \,, \tag{2.11}$$

the Schwarzschild line element becomes

$$ds^2 = \frac{32(GM)^3 e^{-\frac{r}{2GM}}}{r} (dv^2 - du^2) - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \,. \tag{2.12}$$

There is no touch of singularity at $r = 2GM$ in the line element (2.12), and thus the Schwarzschild solution can be extended to all values $r > 0$, whereas the true singularity $r = 0$ still exists. In the Kruskal coordinates the singularity $r = 0$ is represented by the surface $u^2 - v^2 = -1$, and at the event horizon $u^2 - v^2 = 0$ holds. Moreover, u is a spacelike and v is a timelike coordinate.

The relationship between the Schwarzschild and the Kruskal coordinates is

$$\left(\frac{r}{2GM} - 1\right) e^{r/2GM} = u^2 - v^2 \,, \tag{2.13}$$

$$\frac{t}{2GM} = 2 \tanh^{-1} \left(\frac{v}{u} \right) \,. \tag{2.14}$$

With these well-known properties of the Kruskal coordinates it is possible to draw a picture representing the Schwarzschild spacetime in the Kruskal coordinates, when $d\theta = d\phi = 0$. The representation in Fig. 2.2 is called the *Kruskal diagram* [48] of the Schwarzschild spacetime.

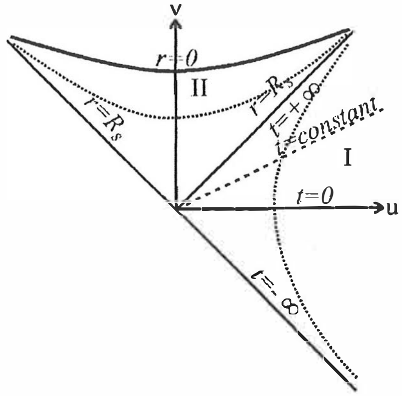

Figure 2.2: This figure represents the Kruskal diagram of the Schwarzschild spacetime. Radial light rays are given by the straight lines $u = \pm v +$ *constant*, the event horizon $r = R_S$ is given by the curves $u = \pm v$. The (dotted) curves of constant r are hyperbolas and the curves $t =$ *constant* are straight (dashed) lines through the origin where $v/u =$ *constant*. At the event horizon $t \longrightarrow \pm\infty$.

In the Kruskal diagram of the Schwarzschild spacetime the light rays are straight lines with the slopes equal to ± 1. Because of that it easy to see that light cannot escape from the spacetime region II, whereas light can go from region I to region II. For this reason the region II represents a black hole and the region I represents its exterior region. The Kruskal diagram 2.2 shows clearly that the spacetime has a boundary at $v = -u$. However, there is nothing that forces the spacetime to have this peculiar property. The boundary can be removed from the Schwarzschild spacetime by completing it into the so-called *Kruskal spacetime* [48] (see Fig. 2.3), known as the *maximal Schwarzschild spacetime*. The completed spacetime allows a spacetime region, where both u and v take negative values. By a maximal spacetime we mean a spacetime manifold where every geodesic either is of infinite length, i.e., it has neither end nor beginning, or it begins or ends on a singularity.

Let us now turn our attention to the Kruskal diagram of the Kruskal spacetime. There we find some striking things: (i) The Kruskal spacetime has two singularities which are the only spacetime boundaries. (ii) Light rays can go from the region III to the black hole region II but not vice versa. Therefore the region III is also an exterior region to the black hole II, and it is identical to the exterior region I. (iii) The regions I and III have no information about each other. (iv) Light rays from the regions I and III can never reach the region IV, whereas light can cross the horizon $r = 2GM$ from the region IV to the regions I and II. Hence, the region IV is a time reversal of the region II, and thus it is called the *white hole*. The white hole singularity $r = 0$ is called the *past singularity* while the black hole singularity refers to the *future singularity*.

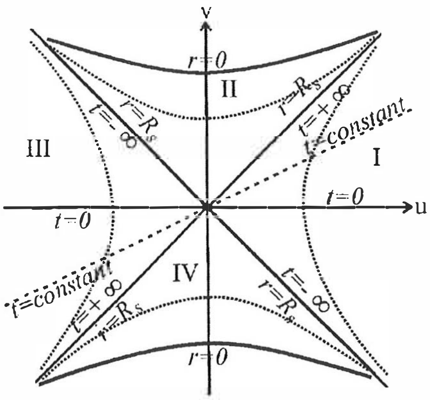

Figure 2.3: This figure represents the maximal Schwarzschild spacetime in the Kruskal coordinates. There are four spacetime regions in Kruskal spacetime. I and III are the asymptotically flat exterior regions. II and IV comprehend the black hole and the time reverse of the black hole i.e. a *white hole.* Only the regions I and II are relevant to the spherically symmetric gravitational collapse.

2.2.2 Einstein-Rosen Bridge

The Kruskal spacetime can be sliced into three-dimensional spacelike hypersurfaces. If the slicing is performed such that the hypersurfaces of constant time do not meet the singularities then the Schwarzschild radius r attains some positive minimum value. In other words, these hypersurfaces with a minimum positive radius coordinate are "tunnels" of the Kruskal spacetime. These tunnels are known as the *Einstein-Rosen bridges* [50] or the *wormholes* or the *Schwarzschild throats.* The three-geometry of the wormholes greatly differs from flatness, and they can be illustrated as in Fig. 2.4.

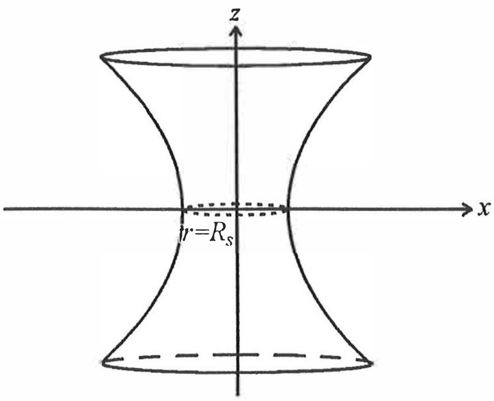

Figure 2.4: The Einstein-Rosen bridge in the case when $t = 0$ and $\theta = \pi/2$. The two-dimensional surface is embedded in $\mathbb{R}^3$.

As one can notice from the Kruskal diagram Fig. 2.3, the Einstein-Rosen bridge connects the two asymptotically flat exterior regions I and IV, and the wormhole serves as a spacelike path from one point to another[4]. However, no signal may exploit the Schwarzschild throat as a passage between two universes: Although the exterior region of the Schwarzschild black hole is static, the interior region is not. The time translation $t \longrightarrow t + dt$ leaves the Schwarzschild geometry unchanged in regions I and III, but in regions II and IV such "time translations" are in fact spatial transformations. When one performs the correct time translations for the hypersurfaces of constant time, they move in the $+v$ direction in the Kruskal diagram 2.3 and enter the region II. Hence, the geometry of the spacelike hypersurface changes and the four-geometry of the Kruskal spacetime changes inside regions II and III. Because of that the wormhole evolves in time. In the beginning of the evolution the wormhole throat is created, then the throat expands, recontracts and finally pinches off even before light signals pass across the bridge (Fig. 2.5).

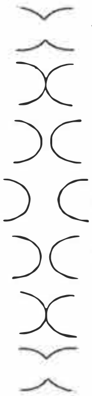

Figure 2.5: This figure illustrates the time evolution of the throat of the Einstein-Rosen bridge. The wormhole throat is first created, then the throat expands, reaches its maximum value $r = R_S$ and then it begins to contract finally pinching off.

2.2.3 Carter-Penrose Diagram of Kruskal Spacetime

We can construct the Carter-Penrose diagram of the Kruskal spacetime by performing a conformal coordinate transformation from the Kruskal coordinates u, v to new coordinates ψ, ξ such that

$$v + u = \tan\frac{1}{2}(\psi + \xi) , \tag{2.15}$$

$$v - u = \tan\frac{1}{2}(\psi - \xi) , \tag{2.16}$$

where $-\pi < \pi - \xi < \psi + \xi < \pi$. It is easy to see that in the new coordinates

$$v^2 - u^2 = \tan\frac{1}{2}(\psi + \xi)\tan\frac{1}{2}(\psi - \xi) , \tag{2.17}$$

[4]The points that are connected by the wormhole may lie in the same universe or in two separate universes.

and so the singularities $v^2 - u^2 = 1$ are given by the line $\psi = \pm\pi/2$. Moreover, the points where $\xi = \pm\pi$ correspond to the case where u approaches $\pm\infty$, and the lines where $v + u \longrightarrow \pm\infty$ or $v - u \dashrightarrow \pm\infty$ transform, respectively, to lines $\psi = \pm\pi - \xi$ and $\psi = \pm\pi + \xi$. Hence, the infinite exterior regions can be mapped to a finite distance. The Carter-Penrose diagram of the Kruskal spacetime is given in Fig. 2.6.

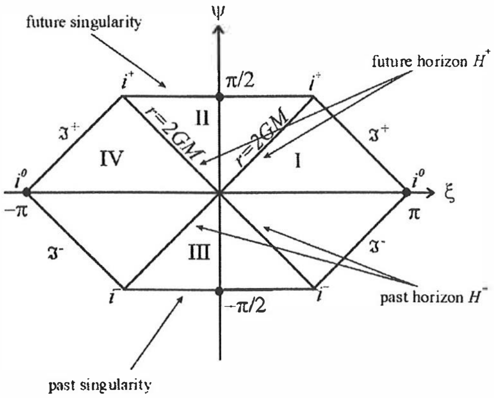

Figure 2.6: The Carter-Penrose diagram of the Kruskal spacetime, showing the infinities and the singularities. The coordinates θ and ϕ have been suppressed.

2.3 Reissner-Nordström Black Hole

The sourceless Einstein-Maxwell equations are

$$G_{\mu\nu} = 2\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2.18}$$

$$F^{\mu\nu}{}_{;\mu} = 0 . \tag{2.19}$$

$F^{\mu\nu}$ is the *electromagnetic field tensor* defined as

$$F^{\mu\nu} = A^{\nu,\mu} - A^{\mu,\nu} , \tag{2.20}$$

where A^{μ} is the *electromagnetic four-vector potential* such that A^0 is the electrostatic scalar potential Φ and the components A^i $(i = 1, 2, 3)$ form the ordinary *electromagnetic vector potential* $\vec{A}$.

The Reissner-Nordström [5, 6] solution

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{GQ}{r^2}\right)dt^2 - \left(1 - \frac{2GM}{r} + \frac{GQ}{r^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.21}$$

is the only exact electrovacuum, static and asymptotically flat solution to Einstein's field equations (2.18)[5]. It represents a charged and spherically symmetric black hole geometry. The Reissner-Nordström solution with

[5]The uniqueness of the Reissner-Nordström solution is discussed in Sec. 2.5.

$Q = 0$ gives as a special case the vacuum solution (2.3) of Schwarzschild spacetime. Maxwell's equations (2.19), in turn, infer that the electrostatic potential in a spherically symmetric static case is

$$A_0 = -\frac{Q}{r}\,, \tag{2.22}$$

where the parameter Q is clearly the electric charge of the spherically symmetric charge distribution. Hence, the parameters M and Q can be interpreted as the mass and the charge of a spherically symmetric charged mass distribution.

2.3.1 Singularities and Horizons

Although the line element (2.21) usually is regarded as a description of the spacetime outside a charged spherically symmetric mass distribution, it is very interesting to study the metric for all $r \in [0, \infty)$. When we perform this we find that the Reissner-Nordström spacetime has a curvature singularity only at the point $r = 0$ and the coordinate singularities lie at the points where

$$\Delta := r^2 - 2GMr + GQ^2 = 0\,. \tag{2.23}$$

This equation is satisfied when

$$r = r_\pm := GM \pm \sqrt{(GM)^2 - GQ^2}\,. \tag{2.24}$$

By proceeding analogously to the Schwarzschild case (see Sec. 2.2.1 and, for more details, see Ref. [27].), one can show that in the Reissner-Nordström spacetime the points where $r = r_\pm$, are coordinate singularities, and that the surfaces $r = r_\pm$ are lightlike. Moreover, the roles of the time coordinate t and the radial coordinate r change at the surfaces $r = r_\pm$. In other words, t is a spacelike and r is a timelike coordinate over the interval $r_- < r < r_+$, and furthermore, the future light cones "tip over" at the lightlike surface $r = r_+$ and again at the lightlike surface $r = r_-$. Because of that even light cannot escape to infinity ($r \longrightarrow \infty$) from the region $r < r_+$ nor can it escape back to the region $r > r-$, once it has crossed the surface $r = r_-$. Therefore the region $r < r_+$ is called the *Reissner-Nordström black hole*, and the region $r > r_+$ is the exterior region of the Reissner-Nordström black hole. The surface $r = r_+$ is called the *exterior horizon* and the surface $r = r_-$ is called the *interior horizon*. (See Fig. 2.7.) The exterior horizon is the event horizon of the Reissner-Nordström black hole.

The number of the coordinate singularities, or equivalently, the number of the horizons (2.24) depends on the numerical values of the parameters M and Q. Therefore we consider three cases: $GM^2 < Q^2$, $GM^2 = Q^2$ and $GM^2 > Q^2$. The case $GM^2 < Q^2$ may be considered unphysical, because there are no horizons, and the singularity $r = 0$ appears naked. (See the footnote at the bottom of the page 37.)

When $GM^2 > Q^2$ the metric (2.21) has two coordinate singularities, since Δ vanishes at both points $r = r_+$ and $r = r_-$. It is important to note that the spacetime is not static in the intermediate region $r_- < r < r_+$, but spacelike slices possess there some time evolution. This dynamics turns out to be crucial for our quantum-mechanical model of the Reissner-Nordström black hole. In Chapter 3 we shall discuss the dynamics of black hole spacetimes.

When $GM^2 = Q^2$ there is only one horizon at the point $r = r_\pm = GM$. This black hole solution is called the *extreme* Reissner-Nordström black hole and the line element (2.21) becomes:

$$ds^2 = \left(1 - \frac{GM}{r}\right)^2 dt^2 - \left(1 - \frac{GM}{r}\right)^{-2} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \quad (2.25)$$

We find that spacetime is static inside as well as outside the horizon $r = GM$, whereas in the case $GM^2 \neq Q^2$ the spacetime has a dynamical region. Moreover, the topology of an extreme Reissner-Nordström black hole in a Euclidean spacetime is $S^2 \times \mathbb{R} \times S^1$, whereas for a nonextreme black hole it is $\mathbb{R}^2 \times S^2$.

2.3.2 Carter-Penrose Diagram of Reissner-Nordström Spacetime

Causal properties of the Reissner-Nordström spacetime can be elucidated by the means of Carter-Penrose diagrams. The process to obtain the diagrams in the three different cases $GM^2 < Q^2$, $GM^2 = Q^2$ and $GM^2 > Q^2$ is analogous to the Schwarzschild case and the details are shown, for example, in Ref. [27]. The crucial point in the process is that the conformal coordinate transformation alters in the three cases. Let us first define for all r:

$$r_* = \begin{cases} r + \frac{r_+^2}{r_+ - r_-} \ln|r - r_+| - \frac{r_-^2}{r_+ - r_-} \ln|r - r_-| \ , \ GM^2 > Q^2 \\ r + GM \ln(r - GM)^2 - \frac{2}{r - GM} \ , \ GM^2 = Q^2 \\ r + GM \ln\left(r^2 - 2GMr + GQ^2\right) + \frac{2}{GQ^2 - G^2M^2} \\ \quad \times \arctan\left(\frac{r - GM}{GQ^2 - G^2M^2}\right) \ , \ GM^2 < Q^2 \end{cases} \quad (2.26)$$

The Eddington-Finkelstein coordinates in Reissner-Nordström spacetime are $\mathcal{U}, r$ or $\mathcal{V}, r$, where

$$\mathcal{U} = t - r_* \ , \quad (2.27)$$
$$\mathcal{V} = t + r_* \ , \quad (2.28)$$

and then the corresponding Kruskal coordinates are

$$u = \frac{1}{2}\left[\exp\left(\frac{r_+ - r_-}{4r_+^2}\mathcal{V}\right) + \exp\left(-\frac{r_+ - r_-}{4r_+^2}\mathcal{U}\right)\right] , \quad (2.29)$$
$$v = \frac{1}{2}\left[\exp\left(\frac{r_+ - r_-}{4r_+^2}\mathcal{V}\right) - \exp\left(-\frac{r_+ - r_-}{4r_+^2}\mathcal{U}\right)\right] . \quad (2.30)$$

Finally, we define the coordinates ψ and ξ such that

$$v + u = \tan\frac{1}{2}(\psi + \xi) \ , \quad (2.31)$$
$$v - u = \tan\frac{1}{2}(\psi - \xi) \ , \quad (2.32)$$

where $-\pi < \psi - \xi < \psi + \xi < \pi$. The Penrose diagram of the maximally extended Reissner-Nordström spacetime, for $GM^2 > Q^2$, $GM^2 = Q^2$ and

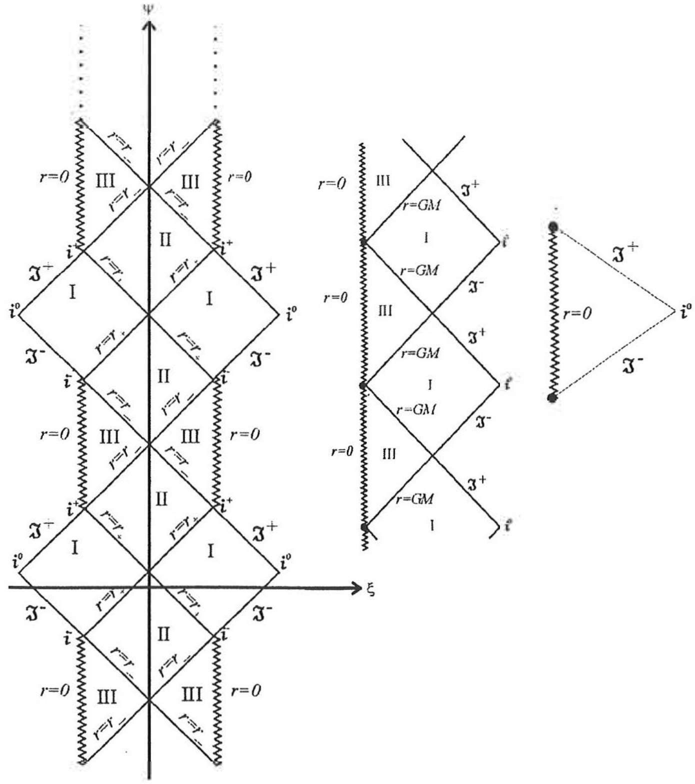

Figure 2.7: The Carter-Penrose diagrams of the maximal Reissner-Nordström spacetime ($GM^2 > Q^2$, $GM^2 = Q^2$ and $GM^2 < Q^2$), showing the infinities and the singularities. The coordinates θ and ϕ have been suppressed. Regions I are asymptotically flat. The only boundaries constitute the singularities $r = 0$, and they are in regions III. Note that the singularities are now *timelike* in contrast to the Schwarzschild spacetime.

$GM^2 < Q^2$ are shown in Fig. 2.7. Now that the singularities $r = 0$ are timelike it is possible to avoid ending up in a singularity and to travel to other universes through the "wormholes" in Reissner-Nordström spacetime. However, it seems that one is not able to get back to our "old" universe. Perhaps one should not be concerned about this, because there is no known process in which Reissner-Nordström black holes could be produced.

2.4 Kerr-Newman Black Hole

The most realistic black hole is, in the Boyer-Lindquist coordinates [51], described by the Kerr solution [7] ($G = 1$)

$$\begin{aligned} ds^2 &= \left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &\quad -\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2\ , \quad (2.33) \end{aligned}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$. It describes an asymptotically flat, axially symmetric geometry for rotating black hole, and it has two parameters M and $a := J/M$ which are, respectively, the mass and the angular momentum J per mass of the system. The Kerr solution is the special case of an electrically charged Kerr solution known as the Kerr-Newman solution [8]. These charged and rotating black holes have three parameters M, a and Q such that the Kerr-Newman metric and the components of the electrostatic potential are

$$\begin{aligned} ds^2 &= \left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &\quad - \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 , \quad (2.34) \\ A_0 &= -\frac{Qr}{\Sigma} , \quad A_3 = \frac{Qa\sin^2\theta r}{\Sigma} , \quad (2.35) \end{aligned}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 + Q^2 - 2Mr$. The Kerr-Newman solutions are the only exact asymptotically flat stationary and axially symmetric vacuum solutions to Einstein's field equations. When $Q = 0$ in Eq. (2.34), we have the neutral Kerr metric (2.33), and when $a = 0$, we have the Reissner-Nordström solution (2.21). When $a = Q = 0$, we recover the Schwarzschild solution (2.3). Hence, stationary black hole spacetime solutions are included in the Kerr-Newman solution.

2.4.1 Geodesic Equation in Kerr-Newman Spacetime

So far we have considered worldlines of various freely falling observers only qualitatively. Now, we are able to investigate the physically relevant, i.e., lightlike and timelike geodesics in every black hole spacetime at one blow as we know that all stationary black holes are included in the Kerr-Newman solution. The geodesic equation for a free particle in any spacetime, when the coordinates x^μ satisfy the constraint

$$\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} =: \delta = 1 \text{ or } 0 , \qquad (2.36)$$

where $\delta = 1$ corresponds to timelike and $\delta = 0$ corresponds to lightlike geodesics, and with the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, can be derived from the Lagrangian for the particle in spacetime:

$$2L = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} , \qquad (2.37)$$

where τ is an affine parameter of the particle along the geodesics. Especially, for the Kerr-Newman spacetime at the equatorial plane $\theta = \pi/2$, the Lagrangian (2.37) has the form

$$L_{KN} = \frac{1}{2}\frac{\Delta - a^2}{r^2}\dot{t}^2 + \frac{a(r^2+a^2-\Delta)}{r^2}\dot{t}\dot{\phi} - \frac{1}{2}\frac{(r^2+a^2)^2 - \frac{1}{2}\Delta a^2}{r^2}\dot{\phi}^2 - \frac{1}{2}\frac{r^2}{\Delta}\dot{r}^2 , \qquad (2.38)$$

where, for example, $\dot{t} = dt/d\tau$. By defining the momenta canonically conjugate to the configuration coordinates in the usual manner, we find that the corresponding momenta conjugate to t, r and ϕ are

$$p_t = \frac{\partial L}{\partial \dot{t}} = \frac{\Delta - a^2}{r^2}\dot{t} + \frac{a(r^2 + a^2 - \Delta)}{r^2}\dot{\phi} , \quad (2.39)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = -\frac{r^2}{\Delta}\dot{r} , \quad (2.40)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = -\frac{(r^2 + a^2)^2 - \Delta a^2}{r^2}\dot{\phi} + \frac{a(r^2 + a^2 - \Delta)}{r^2}\dot{t} . \quad (2.41)$$

It is easy to see that

$$\dot{p}_t = \frac{\partial L}{\partial t} = 0 ,$$
$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0 , \quad (2.42)$$

and therefore p_t and p_ϕ are constants of motion. Thus, let us denote these constants as

$$p_t = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\dot{t} + \left(\frac{2aM}{r} - \frac{Q^2 a}{r^2}\right)\dot{\phi} =: \frac{R^*}{\sqrt{1 + R^{*2}}} , \quad (2.43)$$

$$p_\phi = \left(\frac{2aM}{r} - \frac{Q^2 a}{r^2}\right)\dot{t} - \left[(r^2 + a^2) + \frac{2a^2 M}{r} - \frac{Q^2 a^2}{r}\right]\dot{\phi} =: \ell \quad (2.44)$$

where we have introduced new real-valued parameters R^* and ℓ. We also have

$$p_r = -\frac{r^2}{\Delta}\dot{r} , \quad (2.45)$$

and therefore the Hamiltonian

$$H = p_t\dot{t} + p_r\dot{r} + p_\phi\dot{\phi} - L \quad (2.46)$$

of the free particle in the Kerr-Newman spacetime at the equatorial plane can be shown to coincide with the Lagrangian L:

$$H = L . \quad (2.47)$$

When the constraint (2.36) is satisfied, the Hamiltonian (2.46) is also a constant:

$$H = \frac{1}{2} \text{ or } 0 . \quad (2.48)$$

When the parameters R^* and ℓ are substituted into the Hamiltonian (2.46), we get

$$2H = \frac{R^*}{\sqrt{1 + R^{*2}}}\dot{t} - \frac{r^2}{\Delta}\dot{r}^2 + \ell\dot{\phi} , \quad (2.49)$$

and, moreover, when $\dot{\phi}$ and $\dot{t}$ are expressed in terms of R^* and ℓ as

$$\dot{t} = \frac{1}{\Delta}\left(\frac{r^4 + 2r^2 a^2 - a^2(\Delta - a^2)}{r^2}\frac{R^*}{\sqrt{1 + R^{*2}}} - \frac{a(r^2 + a^2 - \Delta)}{r^2}\ell\right) , \quad (2.50)$$

$$\dot{\phi} = \frac{1}{\Delta}\left(\frac{a(r^2 + a^2 - \Delta)}{r^2}\frac{R^*}{\sqrt{1 + R^{*2}}} - \frac{\Delta - a^2}{r^2}\ell\right) , \quad (2.51)$$

then (2.49) becomes

$$
\begin{aligned}
2H = \delta &= \frac{1}{\Delta}\Bigg[-r^2\dot{r}^2 + \left(r^2 + 2a^2 - \frac{a^2(\Delta - a^2)}{r^2}\right)\frac{R^{*2}}{1+R^{*2}} \\
&- 2a\ell\frac{R^*}{\sqrt{1+R^{*2}}} - 2\frac{a(a^2-\Delta)}{r^2}\ell\frac{R^*}{\sqrt{1+R^{*2}}} \\
&+ \frac{\Delta - a^2}{r^2}\ell^2\Bigg] .
\end{aligned}
\tag{2.52}
$$

When we set $\ell = a\frac{R^*}{\sqrt{1+R^{*2}}}$, Eq. (2.52) gives the radial equation for geodesics of particles in the Kerr-Newman spacetime that plays the same role as the radial geodesics in the Schwarzschild geometry and in the Reissner-Nordström spacetime:

$$
\dot{r}^2 = \frac{R^{*2}}{1+R^{*2}} - \frac{\delta\Delta}{r^2}. \tag{2.53}
$$

If we further restrict ourselves to lightlike particles only, Eq. (2.53) and Eqs. (2.50) and (2.51) reduce to

$$
\begin{aligned}
\dot{r} &= \pm\frac{R^*}{\sqrt{1+R^{*2}}} , &(2.54)\\
\dot{t} &= \frac{r^2+a^2}{\Delta}\frac{R^*}{\sqrt{1+R^{*2}}} , &(2.55)\\
\dot{\phi} &= \frac{a}{\Delta}\frac{R^*}{\sqrt{1+R^{*2}}} . &(2.56)
\end{aligned}
$$

The radial coordinate behaves nicely with respect to the affine parameter, but the time and the azimuthal coordinate do not cross the horizon continuously. To see this, let us obtain the equations for these coordinates:

$$
\begin{aligned}
\frac{dt}{dr} &= \pm\frac{r^2+a^2}{\Delta} , &(2.57)\\
\frac{d\phi}{dr} &= \pm\frac{a}{\Delta} . &(2.58)
\end{aligned}
$$

The solution to Eq. (2.57) is given by

$$
\pm t = r + \frac{r_+^2 + a^2}{r_+ - r_-}\ln|r - r_+| - \frac{r_-^2 + a^2}{r_+ - r_-}\ln|r - r_-| , \tag{2.59}
$$

where $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$, and the solution for Eq. (2.58) is given by

$$
\pm\phi = \frac{a}{r_+ - r_-}\ln|r - r_+| - \frac{a}{r_+ - r_-}\ln|r - r_-| . \tag{2.60}
$$

As one can see, these solutions clearly display the singular behaviour of t *and* ϕ, as the particle approaches the points r_+ and r_-. This character of the t-coordinate was already qualitatively discussed in the Schwarzschild and the Reissner-Nordström spacetimes, where the points r_+ and r_- are defined to correspond to the horizons of the Schwarzschild and the Reissner-Nordström black holes (see Fig. 2.8), respectively.

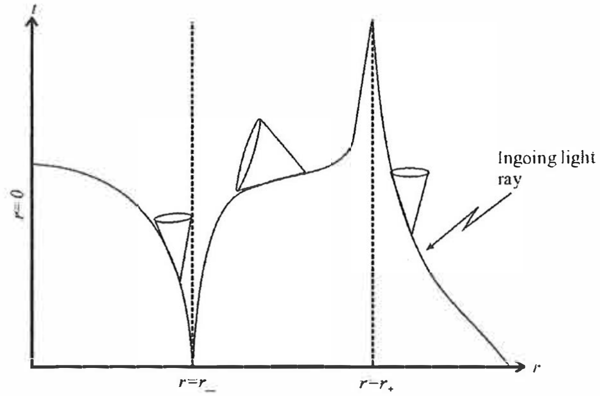

Figure 2.8: A path of a radial light ray is shown in the ill-behaved Boyer-Lindquist coordinates, when the angular part is supressed. This figure especially demonstrates the behaviour of t-coordinate. A similar figure could be drawn for ϕ-coordinate also.

2.4.2 Coordinate Singularities and Horizons

The coordinate singularities of the Kerr-Newman spacetime in the Boyer-Lindquist coordinates lie at the points $r = r_-$ and $r = r_+$ (see Fig. 2.8), and they satisfy the equation $\Delta = 0$. Therefore, the Kerr-Newman spacetime has, like the Reissner-Nordström spacetime, two, one or zero unphysical coordinate singularities depending on whether the expression $M^2 - a^2 - Q^2$ is positive, zero or negative. These singularities can be removed by an appropriate coordinate transformation [27]. However, we shall not perform the transformation. The surfaces $r = r_-$ and $r = r_+$ in the Kerr-Newman spacetime have a similar role the surfaces $r_\pm = M^2 \pm \sqrt{M^2 - Q^2}$ have in the Reissner-Nordström spacetime.

The Kerr-Newman black hole is situated inside the region $r < r_+$, the surface $r = r_+$ being known as the *exterior horizon* or the event horizon of the Kerr-Newman black hole. The surface $r = r_-$ is called the *interior horizon* of the Kerr-Newman black hole. These conclusions can be achieved by the same reasoning we used in the Reissner-Nordström spacetime and by looking at Fig. 2.8.

The true singularity is at the point where $\Sigma = 0$. Hence, we find that the Kerr-Newman black hole singularity lies at the point where $r = 0$ and $\theta = \pi/2$. In fact, it is possible to interpret the singularity $r = 0$ and $\theta = \pi/2$ as a *ring singularity*, with the topology $S^1 \times \mathbb{R}$. The interpretation of a ring singularity arises naturally form the Kerr-Newman metric (2.34) written in the Kerr-Schild coordinates [52]. However, we shall not perform this transformation (the reader may look for the details in Ref. [29]).

The area of the two-dimensional event horizon of the Kerr-Newman black hole is

$$A = \int\limits_{r=r_+} \sqrt{g_{22}g_{33}}dx^2dx^3 = 4\pi\left(r_+^2 + a^2\right) = 16\pi M_{\rm ir}^2\ , \qquad (2.61)$$

where

$$M_{\mathrm{ir}}^2 := \frac{1}{4}\left(r_+^2 + a^2\right) \tag{2.62}$$

is the so-called *irreducible mass* of the black hole. The expression (2.61) for the area of the the event horizon of the Kerr-Newman black hole reduces to the area of the Schwarzschild black hole event horizon when $Q = a = 0$, and to the area of the Reissner-Nordström black hole event horizon when $a = 0$. The irreducible mass describes the minimum value of the black hole mass when the values of M, Q and a are changed in a process where Q and a are to be removed such that the surface area A is increased in the process until all charge and angular momentum have been removed.

It is again important to note that the Kerr-Newman spacetime is not static in the intermediate region $r_- < r < r_+$ but spacelike slices have there certain time evolution. This dynamics is essential to our quantum-mechanical model of the Kerr-Newman black hole, which will be discussed in Chapter 5.

2.4.3 The Ergosphere

The Kerr-Newman spacetime has a very interesting region – called the *ergosphere* – that the spherically symmetric spacetimes do not have. In spherically symmetric spacetimes $g_{00} = 0$ and $g_{11} \longrightarrow \pm\infty$ at the event horizon, whereas in the Kerr-Newman spacetime $g_{00} = 0$ when

$$g_{00} = \frac{\Delta - a^2 \sin^2\theta}{\Sigma} = 0 \Longleftrightarrow r = r_S = M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}\,, \tag{2.63}$$

which is clearly different from $r = r_\pm$. If $0 < \theta < \pi$ then $r_S > r_+$ and when $\theta = 0$ or $\theta = \pi$ then $r_S = r_+$. The region $r_+ < r < r_S$ is called the *ergosphere*. The ergosphere is a region exterior to a rotating black hole, and moreover, we note that $g_{00} > 0$, and therefore t is a spacelike coordinate inside the ergosphere. Hence, every timelike observer unavoidably has angular momentum inside the ergosphere. However, it is possible to construct frames of reference with zero angular momentum. Observers that move with these coordinates are called the zero angular momentum observers (ZAMO), and they can be constructed far from and inside the rotating black holes. One such a coordinate system has been constructed in Ref. [16].

In 1969 R. Penrose proposed that energy could be extracted from a rotating black hole [42]. The ergosphere has a crucial role in this process. It is possible to consider a purely classical wave analogue of Penrose's energy-extraction process, and in that case the phenomenon is known as *super-radiance* [53]. The super-radiance phenomenon has its own analogue in the field of the black hole quantum mechanics. We shall discuss quantum black-holes in Chapter 4.

2.4.4 Carter-Penrose Diagram of Kerr-Newman Spacetime

The global properties of the Reissner-Nordsrtröm black hole and the Kerr and the Kerr-Newman black holes are very similar to each other except that the ring singularity of the Kerr-Newman black hole brings along new allowed regions when we consider the maximal extension of the Kerr-Newman spacetime. In fact, it is possible to allow for the radial coordinate r to have negative values through the interior of the ring singularity. This has been done, for instance, in [27]. Figure 2.9 shows the conformal structure of the maximal Kerr-Newman geometry, with negative values of r. As one can see, the "other" side of the singularity is an asymptotically flat region. However, the maximal geometry of Kerr-Newman spacetime possesses one severe drawback: It allows closed timelike curves, which break the global hyperbolicity of spacetime.

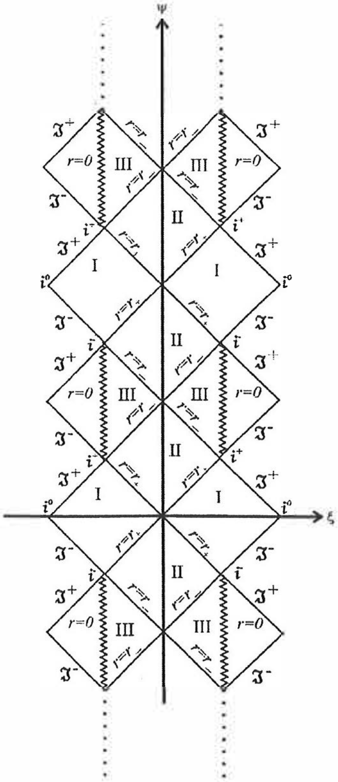

Figure 2.9: The Carter-Penrose diagram of the maximal Kerr-Newman spacetime. The regions I represent the asymptotically flat regions where $r > r_+$, and the regions III include the ring singularity ($-\infty < r < r_-$). The regions II ($r_- < r < r_+$) possess dynamics.

2.5 Stationary Black Holes and Black Hole Mechanics

If the Cosmic Censorship Conjecture holds, then regardless of the exact process of the complete gravitational collapse, a black hole in classical vacuum produced by the gravitational collapse is expected eventually to reach an *equilibrium* with its surroundings [54] and therefore the spacetime geometry around the black hole finally becomes *stationary*. Stationary black holes are either static with zero angular momentum, or they are *axially symmetric*, or they are both [27]. A stationary spacetime is, at least in some frame of reference, time independent but it fails to be time symmetric. In other words, in a stationary spacetime the components of the metric tensor do not depend on the time coordinate and the line element is not symmetric under time reversal $t \dashrightarrow -t$. In contrast, a spacetime is said to be *static* if it is stationary and time symmetric. The axial symmetry of a black hole, loosely speaking, means that the black hole spacetime is invariant with respect to the axis of rotation. For example, the Schwarzschild spacetime is static and the Kerr-Newman spacetime is stationary and axially symmetric. In particular, for a stationary black hole axial symmetry is a necessary property.

In the middle of the 1970's J. A. Wheeler conjectured that a stationary "black hole has no hair" [55]. This metaphorical conjecture is equivalent to the following exact statement: When external gravitational and electromagnetic fields of a collapsing mass distribution finally reach an equilibrium, the resulting black hole geometry is uniquely described by three parameters: mass M, angular momentum J and electric charge Q. Wheeler's "no hair" theorem has been proved in a series of even stronger results of the so-called *uniqueness theorems* of black holes [56]. The remarkable feature of the "no hair" theorem is that in the process of the gravitational collapse external matter fields "forget" all the other properties of matter except the mass, the electric charge and the angular momentum. When an outside observer tries to distinguish two black holes from each other, the only properties the observer has at his disposal are masses, electric charges and angular momenta of the two black holes. Hence, any two black holes with the same mass, angular momentum and electric charge are from the point of view of an external observer exactly identical. This observation does not depend on the process of the gravitational collapse at all. On the other hand, when matter falls into a black hole, only the mass, angular momentum and electric charge of the black hole change. The proofs of the uniqueness theorems are out of the scope of this thesis. However, the uniqueness theorems play a central role in our quantum-mechanical description of black holes, and therefore we shall briefly state their essential physical content.

One of the uniqueness theorems is known as *Birkhoff's theorem* [47]. This theorem says that any spherically symmetric vacuum solution is static and it agrees with the Schwarzschild spacetime. Birkhoff's theorem can be generalized to cover the Reissner-Nordström spacetime by considering the static solutions to the Einstein-Maxwell equations (2.18) and (2.19), and by showing that they reduce to the Reissner-Nordström solution [32]. Especially, Birkhoff's theorem guarantees that the Schwarzschild geometry is the exact exterior geometry that agrees with the interior solution for any spherically symmetric star. Unfortunately, being static does not imply spherical symmetry. However, according to Israel's theorem [32] an asymptotically flat, static and vacuum spacetime geometry exterior to a mass distribution is necessarily Schwarzschild if the spacetime geometry is non-singular everywhere outside and on the boundary of the Schwarzschild black hole.

Rotating black holes are not static but stationary and axisymmetric. According to the Carter-Robinson theorem [57] an asymptotically flat, stationary and *axisymmetric* vacuum black hole spacetime that is non-singular on and outside an event horizon, is the Kerr spacetime characterized by two parameters M and J. This theorem can be stated and proven for the electrically charged Kerr black hole also [56] thus a stationary vacuum final state of any black hole formed in a gravitational collapse must always be a Kerr-Newman black hole which is described by mass M, electric charge Q and angular momentum J.

An analogous lesson about the moral of the uniqueness theorems we have already learnt in thermodynamics of ordinary matter: After a thermodynamical system of ordinary matter reaches an equilibrium with its surroundings, the system is completely described by few macroscopical quantities i.e. by its total energy, temperature etc. In thermodynamics one has managed to formulate the four laws of thermodynamics. Similarly a stationary black hole spacetime is in equilibrium with its surroundings, and because of that they are described by only a few parameters. Now, it would not be surprising if there were laws similar to the laws of thermodynamics for stationary black holes as well. Indeed, in the beginning of the 70's Hawking [58] found the so-called area law of black holes which is similar to the second law of thermodynamics. A bit later Bardeen, Carter and Hawking [59] developed systematically three other laws of black hole physics. These four laws are also called the laws of black hole mechanics. Their importance to the evolution of black hole physics at the quantum-mechanical level cannot be overemphasized. We shall return to the quantum aspects of black holes later in Chapter 4. In the following subsections we formulate the black hole laws in manner analogous to the laws of thermodynamics and discuss their effects.

2.5.1 Zeroth Law

We first introduce the concept of *surface gravity* which plays a major role in the zeroth law of black hole mechanics. The surface gravity

$$\kappa := \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2} = \frac{\left(M^2 - Q^2 - a^2\right)^{1/2}}{2M\left[M + (M^2 - Q^2 - a^2)^{1/2}\right] - Q^2} \tag{2.64}$$

describes the limiting value of the force that must that be exerted at infinity to hold a unit test mass in place. For rotating holes, a test mass cannot be hold in place at the horizon with respect to infinity, but κ is anyway called the surface gravity.

The zeroth law of black hole physics:

κ is constant over the future event horizon of a stationary black hole. (2.65)

This law resembles the zeroth law of thermodynamics, which states that the temperature of a whole system in thermal equilibrium is constant. Therefore the zeroth law hints that the temperature of a stationary black hole should be proportional to κ.

2.5.2 First Law

The first law of black hole physics:

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_{\mathrm{H}}\delta J + \Phi_{\mathrm{H}}\delta Q \ , \tag{2.66}$$

where M is the black hole mass, A is the area of the future event horizon, $J = aM$ is the angular momentum, $\Omega_H := d\phi/dt = a/(r_+^2+a^2)$ is the angular velocity of the event horizon and $\Phi_H = Qr_+/(r_+^2+a^2)$ is the electric potential on the horizon. This is closely analogous to the first law of thermodynamics which states that $\delta E = T\delta S - P\delta V + \mu\delta N$. The remarkable thing is that the term δA in black hole physics has a similar role as δS has in thermodynamics. Moreover, κ, again, plays the role of temperature of a black hole. These analogues are further complemented by the second and third law.

2.5.3 Second Law

The second law of black hole physics – the so-called *area law*:

> *If the Cosmic Censorship hypothesis holds*[6] *then*
>
> $$\delta A \geq 0 \tag{2.67}$$
>
> *in any process.*

This law is closely analogous to the second law of thermodynamics: $\delta S \geq 0$. Again, the area of the event horizon is related to the entropy.

The area theorem has some interesting consequences. First, the horizon area A can be expressed in terms of the irreducible mass (2.62) as in Eq. (2.61). Therefore a special case of the area law is

$$\delta M_{\mathrm{ir}} \geq 0 \ . \tag{2.68}$$

Eq. (2.68) restricts, for example, the amount of energy that can be extracted from a black hole by means of the Penrose process. Secondly, the amount of energy radiated in black hole collisions is limited by the area theorem, and it can be shown that about 29% of the original mass is allowed to be radiated away. The third interesting consequence is that black holes cannot bifurcate. It is easy to show that bifurcation of black holes leads to an inconsistency between the area law and the energy conservation.

2.5.4 Third Law

The third law of black hole physics:

$$\kappa = 0 \textit{ cannot be reached.} \tag{2.69}$$

This is analogous to the third law of thermodynamics which states that $T = 0$ cannot be reached.

As the surface gravity κ is zero for extreme black holes, it follows from the third law of black hole thermodynamics that nonextreme black hole cannot become extreme via any physical process. This is very interesting from our point of view [14, 16].

Is the analogue between thermodynamics and black hole physics merely formal without any physical relevance? Does a black hole have a temperature and an entropy? These questions can be properly answered when quantum

[6] Actually, we should also assume that the stress-energy tensor $T_{\mu\nu}$ satisfies a certain reasonable energy condition.

mechanics is taken into account in Chapter 4. However, parallelisms between the black hole area law and the second law of thermodynamics made J. D. Bekenstein in 1973 to suggest that black holes have entropy S_{BH} which is proportional to its event horizon area A [19]:

$$S_{\mathrm{BH}} \propto A \tag{2.70}$$

or in SI-units

$$S_{\mathrm{BH}} = \gamma \frac{k_{\mathrm{B}} c^3}{\hbar G} A \ , \tag{2.71}$$

where γ a real number of the order of unity. As one would expect, at that time this was universally considered nothing but a speculation.

Chapter 3

Hamiltonian Dynamics of Primordial Black Hole Spacetimes

3.1 Introduction

From ordinary quantum mechanics we know that before we can go from a classical theory to the corresponding canonical quantum theory the classical theory must be cast into the Hamiltonian form. When general relativity is cast into the Hamiltonian form, there are two possible routes to canonical quantization: One may either replace the classical constraints by their operator counterparts, or try to solve the constraints first in the classical level, identify the physical degrees of freedom, and then quantize the system in the resulting physical phase space. The former of these methods is known as the *Dirac constraint quantization*, whereas the latter is known as the *reduced phase space quantization*. Chapters 5 and 6 are devoted to reduced phase space quantization of stationary black hole spacetimes. In this chapter we investigate the classical Hamiltonian dynamics of such spacetimes.

At first sight, it might seem unreasonable to investigate dynamics of stationary spacetimes but even stationary black hole spacetimes have *dynamics*. More precisely, even stationary black hole spacetimes have a region which does not admit a timelike Killing vector field [60]. This means that in a certain spacetime region the black hole spacetime geometry evolves in time no matter how we choose the time coordinate. It is this time evolution of black hole spacetime geometry on which we focus our attention.

To see what this means consider, as an example, the simplest possible black hole, the Schwarzschild black hole. In the curvature coordinates T and R it has the spacetime metric

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dT^2 + \frac{dR^2}{1 - \frac{2M}{R}} + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{3.1}$$

One observes that when $R < 2M$, the coordinate R becomes timelike, and because spacetime geometry inside the event horizon depends on R, it evolves in time. In that region R describes the radius of the wormhole throat of the black hole. In a more precise manner the fact that spacetime inside the event horizon really has dynamics in its geometry can be seen if one considers the conformal diagram of Kruskal spacetime: When $R < 2M$ one cannot move in any timelike direction without changing R, and therefore the geometry of the spacelike hypersurfaces of spacetime. In Reissner-Nordström and Kerr-Newman black hole spacetimes the dynamical region lies in the intermediate region between the outer and the inner horizons of the hole.

In this chapter we shall, in detail, perform the so-called *Hamiltonian reduction* of the phase space of spherically symmetric electrovacuum spacetimes. We shall also state and give some justification to the corresponding—although still unproven—results for the axially symmetric electrovacuum spacetimes containing a black hole. The Hamiltonian formulation of spherically symmetric electrovacuum spaetimes is based on the results found by Louko and Winters-Hilt [20], and by Mäkelä and Repo [14] . The results concerning the Hamiltonian formulation of axially symmetric spacetimes can be found in the paper written by Mäkelä et al. [16] . The Hamiltonian reduction of the spherically symmetric spacetime is performed in Ref. [61].

3.2 Hamiltonian Action of Einstein-Maxwell Theory

The Einstein-Maxwell theory is a theory of an electromagnetic field interacting with a gravitational field. In this section we shall develop the Hamiltonian formulation of such a theory in all details, paying particular attention to the boundary terms appearing in asymptotically flat spacetimes as a consequence of the requirement of internal consistency of the theory.

The action of the Einstein-Maxwell theory can be written, in general, as

$$S = \frac{1}{16\pi} \int\limits_{\mathcal{M}} d^4x \sqrt{-g}\,(R - F_{\mu\nu}F^{\mu\nu}) + (\text{boundary terms}) \,. \tag{3.2}$$

In this equation the integration is performed over the whole four-dimensional spacetime $\mathcal{M}$. Here g is the determinant of the spacetime metric $g_{\mu\nu}$, and

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.3}$$

is the electromagnetic field tensor. A_μ is the electromagnetic four-vector potential. R is the four-dimensional scalar curvature.

As it is well known, we can write the action (3.2) as

$$S = S_\Sigma^{\text{grav}} + S_\Sigma^{\text{em}} + S_{\partial\Sigma}^{\text{grav}} + S_{\partial\Sigma}^{\text{em}} \,, \tag{3.4}$$

where S_Σ^{grav} is given by Eq. (1.24) and

$$S_\Sigma^{\text{em}} := \frac{1}{16\pi} \int dt \int_\Sigma d^3x \sqrt{q} N F_{\mu\nu}F^{\mu\nu} \,; \tag{3.5}$$

$S_{\partial\Sigma}^{\text{grav}}$ and $S_{\partial\Sigma}^{\text{em}}$ are the boundary terms associated with spacelike asymptotic infinities of asymptotically flat spacetimes. In Eqs. (1.24) and (3.5) the spatial integration is performed over the whole spacelike hypersurface Σ for spacetime where the time t is constant.

The properties of the actions S_Σ^{grav} and $S_{\partial\Sigma}^{\text{grav}}$ are well known. Consider now the action S_Σ^{em} of Eq. (3.5). To begin with, consider first the case where the spacetime metric can be written as in Eq. (1.5). In other words, we have chosen Gaussian normal coordinates, where the lapse N is unity, and the shift N^a vanishes identically. In these coordinates we can write

$$S_\Sigma^{\rm em} = \int dt \int_\Sigma d^3x \mathcal{L}^{\rm em} \ , \tag{3.6}$$

where

$$\mathcal{L}^{\rm em} := \frac{1}{16\pi}\sqrt{q}\left\{2q^{ab}\left[\dot{A}_a\dot{A}_b - 2\dot{A}_a(\partial_b A_0) + (\partial_a A_0)(\partial_b A_0)\right] - {}^{(3)}F_{ab}{}^{(3)}F^{ab}\right\} \tag{3.7}$$

is the electromagnetic Lagrangian in curved spacetime. The dot means a time derivative, and we have defined

$$\begin{aligned} {}^{(3)}F_{ab} &:= \partial_a A_b - \partial_b A_a \ , && (3.8)\\ {}^{(3)}F^{ab} &:= q^{am}q^{bn}\,{}^{(3)}F_{mn} \ . && (3.9)\end{aligned}$$

The canonical momentum conjugate to A_a is

$$\pi^a := \frac{\partial \mathcal{L}^{\rm em}}{\partial \dot{A}_a} = \frac{\sqrt{q}}{4\pi}q^{as}\left(\dot{A}_s - \partial_s A_0\right) = \frac{\sqrt{q}}{4\pi}q^{as}F_{0s} \ . \tag{3.10}$$

This relation can be inverted, and we have

$$\dot{A}_b = \frac{4\pi}{\sqrt{q}}\pi_b + \partial_b A_0 \ , \tag{3.11}$$

where we have defined

$$\pi_b := q_{ab}\pi^a \ . \tag{3.12}$$

In terms of π^a we can write the electromagnetic Lagrangian as

$$\mathcal{L}^{\rm em} = \pi^a\dot{A}_a - \left[\frac{2\pi}{\sqrt{q}}q_{ab}\pi^a\pi^b + \pi^a(\partial_a A_0) + \frac{\sqrt{q}}{16\pi}{}^{(3)}F_{ab}{}^{(3)}F^{ab}\right] \ . \tag{3.13}$$

Hence, we get

$$S_\Sigma^{\rm em} = \int dt \int_\Sigma d^3x \left[\pi^a\dot{A}_a - \mathcal{H}^{\rm em} + A_0(\partial_a\pi^a)\right] \ , \tag{3.14}$$

where

$$\mathcal{H}^{\rm em} := \frac{2\pi}{\sqrt{q}}q_{ab}\pi^a\pi^b + \frac{\sqrt{q}}{16\pi}{}^{(3)}F_{ab}{}^{(3)}F^{ab} \ . \tag{3.15}$$

In Eq. (3.14) we have dropped the term $\frac{1}{2}\int dt \int_\Sigma d^3x\partial_a(A_0\pi^a)$ which can be transformed into a boundary term.

We now include, in a manner similar to the one used in Sec. 1.4, the lapse and the shift in our formulation. To include the lapse we replace dt by $dt' = Ndt$ and, because A_0 transforms into

$$A_0' = \frac{\partial x^\mu}{\partial x'^0}A_\mu \ , \tag{3.16}$$

we find that for a general lapse but for a vanishing shift the electromagnetic action is

$$S_\Sigma^{\rm em} = \int dt \int_\Sigma d^3x \left[\pi^a\dot{A}_a - N\mathcal{H}^{\rm em} + A_0(\partial_a\pi^a)\right] \ . \tag{3.17}$$

As we replace dx^a by $dx'^a = dx^a + N^a dt$ we include a non-vanishing shift vector, from which it follows that A_0 is replaced by

$$A'_0 = A_0 - N^s A_s \ . \tag{3.18}$$

Moreover, at the hypersurface where $x^0 = t + dt$, A_a is replaced by

$$A'_a = \frac{\partial x^s}{\partial x'^a} A_s(t+dt, x^b - N^b dt) = A_a + \dot{A}_a dt - (\partial_s A_a) N^s dt - (\partial_a N^s) A_s dt \ . \tag{3.19}$$

Hence, we find that $\dot{A}_a$ must be replaced by

$$\dot{A}'_a = \dot{A}_a - (\partial_s A_a) N^s - (\partial_a N^s) A_s \ . \tag{3.20}$$

Substituting Eqs. (3.18) and (3.20) into Eq. (3.17) we obtain an expression for the electromagnetic action in the presence of a non-vanishing shift:

$$S_\Sigma^{\text{em}} = \int dt \int_\Sigma d^3x \left[\pi^a \dot{A}_a - N\mathcal{H}^{\text{em}} - N^s \mathcal{H}_s^{\text{em}} + A_0(\partial_a \pi^a) \right] \ , \tag{3.21}$$

where we have defined

$$\mathcal{H}_s^{\text{em}} := \pi^a \, {}^{(3)}F_{sa} \ , \tag{3.22}$$

and we have ignored the term $\int dt \int_\Sigma d^3x \partial_a (A_s N^s \pi^a)$.

We are now ready to write down the whole Einstein-Maxwell action without boundary terms. The gravitational part S_Σ^{grav} is a mere ADM action (1.44). Putting the actions (3.21) and (1.44) together we get the Einstein-Maxwell action

$$S_\Sigma = \int dt \int_\Sigma d^3x \left(\pi^{ab} \dot{q}_{ab} + \pi^a \dot{A}_a - N\mathcal{H} - N^a \mathcal{H}_a - A_0 \mathcal{G} \right) \ , \tag{3.23}$$

where

$$\mathcal{H} := \mathcal{H}^{\text{grav}} + \mathcal{H}^{\text{em}} \tag{3.24}$$

is the Hamiltonian constraint,

$$\mathcal{H}_a := \mathcal{H}_a^{\text{grav}} + \mathcal{H}_a^{\text{em}} \tag{3.25}$$

is the diffeomorphism constraint, and

$$\mathcal{G} := -\partial_a \pi^a \tag{3.26}$$

is the Gaussian constraint.

We shall consider asymptotically flat spacetimes. In those kind of spacetimes we must include the boundary terms (1.60), (1.68) and (1.72) related to pure gravity. We still have to include boundary terms related to electromagnetism. First of all, we observe that variation of the action with respect to the momentum π^a conjugate to A_a brings along a term

$$\int dt \int_\Sigma d^3x \partial_a \left(A_0 \delta \pi^a \right) \ ,$$

which must be cancelled at infinity. Hence, we need an electromagnetic boundary term,

$$S_{\partial\Sigma}^{\text{em}} := -\int dt A_0^+(t) Q(t) \ , \tag{3.27}$$

where

$$A_0^+(t) := \lim_{r\to\infty} A_0(t, x^a) \tag{3.28}$$

is the electric potential at infinity, and

$$Q := -\lim_{r\to\infty} \oint \pi_a dS^a \tag{3.29}$$

is the electric charge of spacetime.

We are prepared to write down the whole Einstein- Maxwell action, with appropriate boundary terms. We get

$$\begin{aligned} S &= \int dt \int_\Sigma d^3x \left(\pi^{ab}\dot{q}_{ab} + \pi^a \dot{A}_a - N\mathcal{H} - N^a\mathcal{H}_a - A_0\mathcal{G}\right) \\ &\quad - \int dt \Big[N_+(t)E^{\rm ADM}(t) + N_+^a(t)P_a^{\rm ADM}(t) + \omega^b L_b^{\rm ADM}(t) \\ &\quad + A_0^+(t)Q(t)\Big] , \end{aligned} \tag{3.30}$$

where $E^{\rm ADM}(t)$, $P_a^{\rm ADM}(t)$ and $L_b^{\rm ADM}(t)$ are given by Eqs. (1.60), (1.68) and (1.72), respectively. Because of that, the total Hamiltonian of the Einstein-Maxwell theory is

$$\begin{aligned} H &= \int_\Sigma d^3x \,(N\mathcal{H} + N^s\mathcal{H}_s + A_0\mathcal{G}) + N_+(t)E^{\rm ADM}(t) \\ &\quad + N_+^a(t)P_a^{\rm ADM}(t) + \omega^b L_b^{\rm ADM}(t) + A_0^+(t)Q(t) . \end{aligned} \tag{3.31}$$

Hence, one is left with the last four terms only when the classical constrainsts

$$\mathcal{H} = 0 , \tag{3.32}$$
$$\mathcal{H}_s = 0 , \tag{3.33}$$
$$\mathcal{G} = 0 \tag{3.34}$$

are satisfied.

3.3 Spherically Symmetric Hamiltonian Action of Einstein-Maxwell Theory

In this section we present a classical Hamiltonian formulation of spherically symmetric electrovacuum spacetimes with boundary conditions of Ref. [14]. The classical solutions to Einstein's field equations representing these kinds of spacetimes are uniquely characterized by the mass M and the electric charge Q. The spacetime geometry is described by the Reissner-Nordström line element which takes, in the curvature coordinates T, R, the form

$$ds^2 = \left(1 - \frac{2GM}{R} + \frac{Q}{R^2}\right) dT^2 - \left(1 - \frac{2GM}{R} + \frac{Q}{R^2}\right)^{-1} dR^2 - R^2 d\Omega^2 . \tag{3.35}$$

The spacelike hypersurfaces of constant time t in the spacetime foliation extend from left to right asymptotic infinities in the conformal diagram of the maximally extended Reissner-Nordström spacetime (see Fig. 3.1). The max-

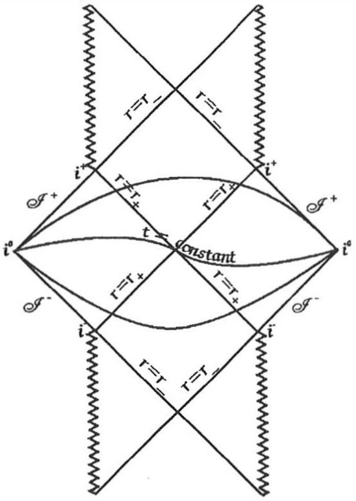

Figure 3.1: The conformal diagram of Reissner-Nordström spacetime. Our spacelike hypersurfaces $t = constant$ begin their evolution at the past $r = r_-$ hypersurface, then go through the bifurcation point [62], and finally end their evolution at the future $r = r_-$ hypersurface.

imally extended Reissner-Nordström spacetime has a periodic geometrical structure. We choose one such a period and let the spacelike hypersurface go through the interior regions of the hole in arbitrary ways. However, if we look at the conformal diagram in Fig. 3.1, we find that it is not possible to push these hypesurfaces beyond the inner horizons, where $r = r_-$. Otherwise the hypersurface would necessarily fail to be spacelike. Hence our study of the Hamiltonian dynamics of Reissner-Nordström spacetimes must be restricted to include, in addition to the left and right exterior regions of the hole, only such an interior region of the hole that lies between two successive $r = r_-$ hypersurfaces in the conformal diagram. Our spacelike hypersurfaces $t = constant$ begin their evolution at the past $r = r_-$ hypersurface, then go through the bifurcation point, and finally end their evolution at the future $r = r_-$ hypersurface. The Hamiltonian formulation of Reissner-Nordström spacetimes was performed in Ref. [20]. However, the considerations of those authors were thermodynamically motivated and so they investigated the exterior regions of the hole, whereas our interest lies in the interior regions of the hole. Therefore our boundary conditions differ greatly from those given by authors Louko and Winters-Hilt, but as to most technical details the discussion goes exactly like in Ref. [20].

3.3.1 Lagrangian Formulation

Our starting point is the general spherically symmetric ADM metric

$$ds^2 = -N^2\, dt^2 + \Lambda^2 (dr + N^r\, dt)^2 + R^2\, d\Omega^2 \ , \tag{3.36}$$

where $d\Omega^2$ is the metric on the unit two-sphere, and the lapse N, the shift N^r, as well as the dynamical variables Λ and R of the spacetime geometry are assumed to be functions of the time coordinate t and the radial coordinate r only. The electromagnetic four-potential is taken to be spherically symmetric such that its nonzero components are

$$A_t := \phi \,, \tag{3.37}$$

$$A_r := \Gamma \,, \tag{3.38}$$

where ϕ and Γ are assumed to be functions of t and r only. The radial coordinate r takes all values from negative to positive infinity. The region, where $r = -\infty$ corresponds to the left-hand side, and the region, where $r = \infty$ corresponds to the right-hand side asymptotically infinite regions in the conformal diagram of Reissner-Nordström spacetime. We shall further assume both the spatial and the spacetime metric to be nondegenerate, and we take both functions Λ and R to be positive.

The spatial part of the line element (3.36) leads via Eq. (1.8) to the three-dimensional curvature scalar

$$\mathcal{R} = -4\Lambda^{-2}R^{-1}R'' + 4\Lambda^{-3}R^{-1}\Lambda'R' - 2\Lambda^{-2}R^{-2}R'^2 + 2R^{-2} \,. \tag{3.39}$$

The nonzero components of the extrinsic curvature tensor K_{ab} with nonvanishing shift N^r are given by Eq. (1.52), and they are:

$$K_{11} = -N^{-1}\Lambda\left[\dot{\Lambda} - (\Lambda N^r)'\right] \,, \tag{3.40}$$

$$K_{22} = -N^{-1}R\left[\dot{R} - R'N^r\right] \,, \tag{3.41}$$

$$K_{33} = \sin^2\theta K_{22} \,. \tag{3.42}$$

Inserting the curvature scalar (3.39) and the nonzero components of the extrinsic curvature tensor into the ADM action (1.24) of pure gravity and integrating over the two-sphere we obtain (in natural units $G = 1 = c$) the gravitational action

$$\begin{aligned} &S_\Sigma^{\mathrm{grav}}[\Lambda, R; N, N^r] \\ &= \int dt \int_{-\infty}^{\infty} dr \Big[- N^{-1}\Big(R(-\dot{\Lambda} + (\Lambda N^r)')(-\dot{R} + R'N^r) \\ &+\frac{1}{2}\Lambda(-\dot{R} + R'N^r)^2\Big) + N\Big(\Lambda^{-2}RR'\Lambda' - \Lambda^{-1}RR'' - \frac{1}{2}\Lambda^{-1}R'^2 \\ &+\frac{1}{2}\Lambda\Big)\Big] \,. \end{aligned} \tag{3.43}$$

Inserting the components of the electromagnetic potential (3.37) and (3.38) in the electromagnetic Lagrangian (3.13) we obtain the electromagnetic action

$$S_\Sigma^{\mathrm{em}}[\Lambda, R, \Gamma; N, N^r, \Phi] = N^{-1}\Lambda^{-1}R^2(\dot{\Gamma} - \Phi')^2 \,. \tag{3.44}$$

Up to boundary terms, we thus get the action

$$\begin{aligned} &S_\Sigma[\Lambda, R, \Gamma; N, N^r, \Phi] \\ &= \int dt \int_{-\infty}^{\infty} dr \Big[- N^{-1}R(-\dot{\Lambda} + (\Lambda N^r)')(-\dot{R} + R'N^r) \end{aligned}$$

$$+\frac{1}{2}N^{-1}\Lambda(-\dot{R}+R'N^r)^2+N^{-1}\Lambda^{-1}R^2(\dot{\Gamma}-\Phi')^2$$
$$+N\left(\Lambda^{-2}RR'\Lambda'-\Lambda^{-1}RR''-\frac{1}{2}\Lambda^{-1}R'^2+\frac{1}{2}\Lambda\right)\Bigg] \ . \tag{3.45}$$

The Einstein-Maxwell equations can be derived by varying the action (3.45). The appropriate boundary terms shall be inserted after passing on to the Hamiltonian formulation.

3.3.2 Hamiltonian Formulation

The canonical momenta conjugate to metric variables Λ, R and Γ are

$$P_\Lambda = \frac{\delta S_\Sigma}{\delta\dot{\Lambda}} = -N^{-1}R(\dot{R}-R'N^r) \ , \tag{3.46}$$
$$P_R = \frac{\delta S_\Sigma}{\delta\dot{R}} = -N^{-1}\left(\Lambda(\dot{R}-R'N^r)+R(\dot{\Lambda}-(\Lambda N^r)')\right) \ , \tag{3.47}$$
$$P_\Gamma = \frac{\delta S_\Sigma}{\delta\dot{\Gamma}} = N^{-1}\Lambda^{-1}R^2(\dot{\Gamma}-\Phi') \ . \tag{3.48}$$

The action (3.45) can be cast into the Hamiltonian action by the Legendre dual transformation [63]

$$S_\Sigma[\Lambda,R,\Gamma,P_\Lambda,P_R,P_\Gamma;N,N^r,\tilde{\Phi}]$$
$$= \int dt \int_{-\infty}^{\infty} dr\left(P_\Lambda\dot{\Lambda}+P_R\dot{R}+P_\Gamma\dot{\Gamma}-N\mathcal{H}-N^r\mathcal{H}_r-\tilde{\Phi}\mathcal{G}\right) \tag{3.49}$$

where the Hamiltonian constraint $\mathcal{H}=0$, the diffeomorphism constraint $\mathcal{H}_r = 0$, and the Gauss constraint $\mathcal{G}=0$ are obtained by varying the action (3.49) with respect to the lapse N, the shift N^r and the quantity $\tilde{\Phi}$, so that the constraints equations are given by

$$\mathcal{H} = -R^{-1}P_RP_\Lambda+\frac{1}{2}R^{-2}\Lambda(P_\Lambda^2+P_\Gamma^2)+\Lambda^{-1}RR''-\Lambda^{-2}RR'\Lambda'$$
$$+\frac{1}{2}\Lambda^{-1}R'^2-\frac{1}{2}\Lambda=0 \ , \tag{3.50}$$
$$\mathcal{H}_r = P_RR'-\Lambda P_\Lambda'-\Gamma P_\Gamma'=0 \ , \tag{3.51}$$
$$\mathcal{G} = -P_\Gamma'=0 \ . \tag{3.52}$$

It is convenient to define the quantity $\tilde{\Phi}$ in terms of the electric potential Φ such that

$$\tilde{\Phi} := \Phi - N^r\Gamma \ , \tag{3.53}$$

which is not a dynamical variable but the Lagrange multiplier associated with the Gauss constraint (3.52). Note that the diffeomorphism constraint (3.51) generates three-dimensional spatial diffeomorphisms in both the gravitational and electromagnetic parts of the formulation.

The Hamiltonian equations of motion can be obtained by varying the action (3.49) with respect to the variables $\Lambda, R, \Gamma, P_\Lambda, P_R$ and P_Γ. The dynamical equations in the presence of the electromagnetic field are

$$\dot{\Lambda} = N(R^{-2}\Lambda P_\Lambda - R^{-1}P_R)+(N^r\Lambda)' \ , \tag{3.54}$$
$$\dot{R} = -NR^{-1}P_\Lambda + N^rR' \ , \tag{3.55}$$

$$
\begin{aligned}
\dot{\Gamma} &= N\Lambda R^{-2}P_{\Gamma} + (N^r\Gamma)' + \tilde{\Phi}' \ , && (3.56)\\
\dot{P}_{\Lambda} &= N\left[-R^{-2}(P_{\Lambda}^2 + P_{\Gamma}^2) - (\Lambda^{-1}R')^2 + 1 + 3\ell^{-2}R^2\right] \\
&\quad -\Lambda^{-2}N'RR' + N^r P_{\Lambda}' \ , && (3.57)\\
\dot{P}_R &= N\left[\Lambda R^{-3}(P_{\Lambda}^2 + P_{\Gamma}^2) - R^{-2}P_{\Lambda}P_R - (\Lambda^{-1}R')' + 3\ell^{-2}\Lambda R\right] \\
&\quad -(\Lambda^{-1}N'R)' + (N^r P_R)' \ , && (3.58)\\
\dot{P}_{\Gamma} &= N^r P_{\Gamma}' \ . && (3.59)
\end{aligned}
$$

3.3.3 Boundary Conditions

Regge and Teitelboim [21] pointed out that in order to evaluate the appropriate boundary terms at the asymptotic infinities, it is essential to adopt asymptotic falloff conditions for the canonical variables and the Lagrange multipliers at infinity. The form of the falloff conditions depends on the chosen spacetime and on the spacetime foliation. We shall adopt a class of falloff conditions that ensure every classical solution to the Einstein-Maxwell theory to be asymptotically flat at asymptotic infinities. Therefore we shall follow and partly modify the treatments of Kuchař [61], and of Louko and Winters-Hilt [20].

Primordial electrovacuum black hole spacetimes are asymptotically flat and we may choose a global coordinate system x^a which is asymptotically Cartesian on the spatial hypersurfaces $t = \mathit{constant}$. Therefore such a coordinate system and the spherical coordinate system r, θ, ϕ are related by the ordinary flat-space spherical coordinate transformation. In particular, at spatial infinity the line element of asymptotically flat Reissner-Nordström spacetime behaves like

$$
ds^2 \sim \left(1 - \frac{2M}{r}\right) dt^2 - \left(\delta_{ab} + 2M\frac{x^a x^b}{r^3}\right) dx^a dx^b \ . \qquad (3.60)
$$

At the asymptotic infinity the spatial metric q_{ab} and the conjugate momenta π^{ab} are required to fall off such that

$$
\begin{aligned}
q_{ab} - \delta_{ab} &\sim r^{-1} \ , && (3.61)\\
\pi^{ab} &\sim r^{-2} \ . && (3.62)
\end{aligned}
$$

These general falloff conditions yield the falloff conditions of the canonical variables. However, primordial black holes have left- and right-hand side spatial asymptotic infinities, where $r \longrightarrow \pm\infty$. The conformal diagram of Reissner-Nordström spacetime is symmetric with respect to the spacelike coordinate r. Hence, the general behaviour of the spatial hypersurfaces at both the asymptotic infinities is similar and the falloff conditions of the canonical data at the right- and left-hand side of the hypersurfaces $t = \mathit{constant}$ may be easily adopted at one blow. At $r \longrightarrow \pm\infty$, we assume that the canonical variables and the Lagrangian multipliers of the theory behave such that

$$
\begin{aligned}
\Lambda(t, r) &= 1 + M_{\pm}(t)|r|^{-1} + O^{\infty}(|r|^{-1-\epsilon}) \ , && (3.63)\\
R(t, r) &= |r| + O^{\infty}(|r|^{-\epsilon}) \ , && (3.64)\\
P_{\Lambda}(t, r) &= O(|r|^{-\epsilon}) \ , && (3.65)
\end{aligned}
$$

$$\begin{aligned} P_R(t,r) &= O^\infty(|r|^{-1-\epsilon}) \ , & (3.66)\\ N(t,r) &= N_\pm(t) + O^\infty(|r|^{-\epsilon}) \ , & (3.67)\\ N^r(t,r) &= O^\infty(|r|^{-\epsilon}) \ , & (3.68)\\ \Gamma(t,r) &= O^\infty(|r|^{-1-\epsilon}) \ , & (3.69)\\ P_\Gamma(t,r) &= Q_\pm(t) + O^\infty(|r|^{-\epsilon}) \ , & (3.70)\\ \tilde{\Phi}(t,r) &= \tilde{\Phi}_\pm(t) + O^\infty(|r|^{-\epsilon}) \ , & (3.71) \end{aligned}$$

where $M_\pm(t) > 0$ and $N_\pm(t) \geq 0$. In these equations, $0 < \epsilon \leq 1$, and $O(|r|^{-\epsilon})$ denotes a term that falls off at infinity as $|r|^{-\epsilon}$ and whose derivatives with respect to $|r|$ fall accordingly as $|r|^{-\epsilon-k}$, where $k = 1, 2, 3, \ldots$. These falloff conditions are consistent with the constraint equations and they are preserved by the dynamical equations of the theory. Thus the canonical action S_Σ is well-defined. Of particular interest is the falloff condition (3.68), which states that the shift vanishes at infinities. This means that our asymptotic coordinate systems are at rest with respect to the hole.

3.3.4 Boundary Terms

As we have previously seen, one must include in asymptotically flat spacetimes certain boundary terms in order to make the variational principle consistent. Our particular interest are the boundary terms in spherically symmetric electrovacuum spacetimes. In this section we shall calculate these boundary terms by using the falloff conditions of the canonical data.

There are no boundary contributions from the ADM linear and angular momentum boundary terms (1.68) and (1.72) since our asymptotic coordinate systems at both infinities were chosen to be at rest with respect to the hole. The only nonzero boundary contributions correspond to the time translations and to the electromagnetic gauge transformations. These boundary terms, in the case of one spatial infinity, are given by Eqs. (1.63) and (3.27). Now we have two asymptotic infinities and because of that there are four boundary terms

$$S^{\text{grav}}_{\partial\Sigma} = -\int dt\,(N_+(t)E_+(t) + N_-(t)E_-(t)) \ , \tag{3.72}$$

$$S^{\text{em}}_{\partial\Sigma} = -\int dt\,\left(A_0^+(t)Q_+(t) - A_0^-(t)Q_-(t)\right) \ , \tag{3.73}$$

where E_+ and E_- are the ADM energies (1.60) at the spatial infinities. Note that when evaluating E_-, the limit $r \longrightarrow \infty$ must be replaced by the limit $r \longrightarrow -\infty$, and the integration is performed over the two-sphere S^2_- at the left-hand side spatial infinity. One can see from the falloff conditions (3.63) and (3.64) that the boundary term $S^{\text{grav}}_{\partial\Sigma}$ takes the form

$$S^{\text{grav}}_{\partial\Sigma} = -\int dt\,(N_+(t)M_+(t) + N_-(t)M_-(t)) \ , \tag{3.74}$$

where $M_+(t)$ and $M_-(t)$ are the ADM energies of spherically symmetric electrovacuum spacetimes. At the asymptotic infinity we see that $A_0^+ = \Phi_+ \sim \tilde{\Phi}_+$ and $A_0^- = \Phi_- \sim \tilde{\Phi}_-$, and therefore the electromagnetic boundary term $S^{\text{em}}_{\partial\Sigma}$ becomes

$$S^{\text{em}}_{\partial\Sigma} = -\int dt\,\left(\tilde{\Phi}_+(t)Q_+(t) - \tilde{\Phi}_-(t)Q_-(t)\right) \ . \tag{3.75}$$

The boundary term $S_{\partial\Sigma}$ is a sum of the terms $S^{\rm grav}_{\partial\Sigma}$ and $S^{\rm em}_{\partial\Sigma}$. Hence the total action is

$$
\begin{aligned}
& S_\Sigma[\Lambda, R, \Gamma, P_\Lambda, P_R, P_\Gamma; N, N^r, \tilde{\Phi}] \\
&= \int dt \int_{-\infty}^{\infty} dr \left(P_\Lambda \dot{\Lambda} + P_R \dot{R} + P_\Gamma \dot{\Gamma} - N\mathcal{H} - N^r \mathcal{H}_r - \tilde{\Phi}\mathcal{G} \right) \\
&- \int dt \Big(N_+(t) M_+(t) + N_-(t) M_-(t) \\
&+ \tilde{\Phi}_+(t) Q_+(t) - \tilde{\Phi}_-(t) Q_-(t) \Big) \ . \qquad (3.76)
\end{aligned}
$$

It should be noted that one is not allowed to vary the action with respect to the functions N and $\tilde{\Phi}$ at infinities but these functions should be kept as prescribed functions of the time t. That is because varying the action with respect to $N_\pm$ would imply vanishing ADM mass -and hence flatness- of spacetime; varying action with respect to $\tilde{\Phi}_\pm$, in turn, would imply vanishing electric charge.

3.3.5 Reconstructing Mass, Electric Charge and Time

Our boundary conditions (3.63)–(3.71) for the spherically symmetric electrovacuum spacetimes ensure that every classical solution is asymptotically flat. As it was mentioned in Sec. 2.3 the Reissner-Nordström spacetime is the unique spherically symmetric asymptotically flat solution to the Einstein-Maxwell theory. The Reissner-Nordström solution is completely characterized by the parameters M and Q which are, respectively, the mass and the charge of Reissner-Nordström spacetime. It was found in Ref. [20] that these parameters can be read off from any small piece of the exterior region of Reissner-Nordström spacetime if the values of the phase space coordinates $\Lambda, R, \Gamma, P_\Gamma, P_R$ and P_Γ are known on a spacelike hypersurface embedded in such a region of the spacetime. We wish to recover from the canonical data the mass and the charge parameters of spacetime when the spatial hypersurfaces extend from the left- to the right-hand side asymptotic infinities. In Ref. [20] it was found how the information about the positions of the spacelike hypersurfaces have been embedded into the Reissner-Nordström spacetime can be read off from the canonical data at any point on the hypersurface. Even the information about electromagnetic gauge fixing can be read off from the phase space coordinates of the theory. In other words, we wish to show that it is possible to tell from the canonical data the mass and the charge of the black hole, the information about the choice of the electromagnetic gauge, and how the hypersurfaces $t = constant$ have been embedded in spacetime.

Let us first consider the charge parameter. The Gauss constraint (3.52) and the dynamical equation (3.59) imply that P_Γ is independent of r and t. Because of that, it is remained unchanged under time reparametrizations and spacelike diffeomorphisms. Equation (3.48), in turn, implies that P_Γ remains unchanged under the electromagnetic gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \xi$, where ξ is an arbitrary function of T and R. Moreover, it was shown in Ref. [20] that

$$Q = P_\Gamma \ . \qquad (3.77)$$

Let us then consider the mass parameter. When the hypersurface foliation

$$T = T(t,r) \quad , \qquad R = R(t,r) \tag{3.78}$$

is implemented in the formulation the mass function takes the expression [20]

$$M = \frac{1}{2}\frac{P_\Lambda^2}{R} + \frac{1}{2}\frac{P_\Gamma^2}{R} + \frac{1}{2}R - \frac{1}{2}\frac{R(R')^2}{\Lambda^2} \tag{3.79}$$

in the exterior region of the hole.

Let us consider the embedding. In Ref. [20] it was found that

$$-T' = R^{-1}F^{-1}\Lambda P_\Lambda P_\Gamma \ , \tag{3.80}$$

where

$$F := \left(\frac{R'}{\Lambda}\right)^2 - \left(\frac{P_\Lambda}{R}\right)^2 \ . \tag{3.81}$$

From Eq. (3.80) one can solve T as a function of r provided that one knows T for one value of r. Keeping t as a constant one can read off from that solution the position of the $t = \textit{constant}$ hypersurface of spacetime in the Reissner-Nordström manifold.

Let us finally consider the electromagnetic gauge. The usual electromagnetic potential

$$A_T = \frac{Q}{R} \tag{3.82}$$

is a solution to Maxwell's equations that involves a specific fixing of the electromagnetic gauge. The general spherically symmetric solution can be written as

$$A_T = \frac{Q}{R} + \frac{\partial}{\partial T}\xi \ , \tag{3.83}$$

$$A_R = \frac{\partial}{\partial R}\xi \ . \tag{3.84}$$

$$\tag{3.85}$$

Hence, the general solution expressed in the coordinates t and r is

$$A_t = \frac{Q}{R}\dot{T} + \dot{\xi} \ , \tag{3.86}$$

$$A_r = \frac{Q}{R}T' + \xi' \ . \tag{3.87}$$

Comparing Eqs. (3.38), (3.77), (3.80) and (3.87) one finds that

$$\xi' = \Gamma + R^{-2}F^{-1}\Lambda P_\Lambda P_\Gamma \ . \tag{3.88}$$

If one knows the gauge function ξ for one value of r, one can solve ξ for any r from Eq. (3.88). Hence, the information about the choice of the electromagnetic gauge is carried by the phase space coordinates of the theory.

At first sight, there might seem to be a difficulty with the horizons $R = r_\pm$, because it follows from Eqs. (3.77), (3.79) and (3.81) that

$$F = 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \ . \tag{3.89}$$

Hence, the quantities T' and ξ' appear to have a singularity when $R = r_\pm$. This problem has been considered in the case $Q = 0$ by Kuchař [61]. His conclusion was that one can nevertheless propagate at a well-determine rate through horizon one's knowledge of T by using Eq. (3.80). The very same arguments can be applied also when $Q \neq 0$. Hence, the event horizon does not pose a problem for our knowledge of T and ξ as functions of r.

3.3.6 Canonical Transformation

The equations (3.77), (3.79), (3.80) and (3.81) were obtained by using Einstein's equations from the known form of the Reissner-Nordström solution. Those equations therefore have a clear geometrical meaning only when the equations of motion are satisfied. Let us now forget for the moment the fact that M and Q are the mass and the charge parameters of the Reissner-Nordström black hole and consider M and Q as functions of the phase space variables given by Eqs. (3.79) and (3.77), even when the equations of motions are not satisfied. It was shown in Ref. [20] that the function Q of Eq. (3.77) has vanishing Poisson brackets with the function M of Eq. (3.79) and the function $-T'$ of Eq. (3.80). Moreover, it was shown that ξ' is canonically conjugate to Q as well as that $-T'$ is canonically conjugate to M, whereas ξ' Poisson commutes with M and $-T'$. Therefore, M and Q may be considered as the new configuration coordinates, and the quantities $-T'$ and $-\xi'$ may be taken to be the corresponding canonical momenta. As in the Schwarzschild case [61], the momentum P_R was modified also in Ref. [20] in order to have a canonical chart such that the canonical transformation [64] from the old to the new phase space becomes

$$M := \frac{1}{2}\frac{P_\Lambda^2}{R} + \frac{1}{2}\frac{P_\Gamma^2}{R} + \frac{1}{2}R - \frac{1}{2}\frac{R(R')^2}{\Lambda^2} \quad , \tag{3.90}$$

$$P_M := R^{-1}F^{-1}\Lambda P_\Lambda \quad , \tag{3.91}$$

$$\mathsf{R} := R \quad , \tag{3.92}$$

$$\begin{aligned} P_{\mathsf{R}} := {} & P_R - \frac{1}{2}R^{-1}\Lambda P_\Lambda - \frac{1}{2}R^{-1}F^{-1}\Lambda P_\Lambda \\ & - R^{-1}\Lambda^{-2}F^{-1}\Big((\Lambda P_\Lambda)'(RR') - (\Lambda P_\Lambda)(RR')'\Big) \\ & + R^{-1}F^{-1}\Lambda P_\Lambda P_\Gamma^2 R^{-2} \quad , \end{aligned} \tag{3.93}$$

$$Q := P_\Gamma \quad , \tag{3.94}$$

$$P_Q := -\Gamma - R^{-2}F^{-1}\Lambda P_\Lambda P_\Gamma \quad . \tag{3.95}$$

This transformation was, indeed, shown in Ref. [20] to be canonical and differentiable in the exterior region of the hole. But with our boundary conditions the transformations really become singular when $F = 0$. In particular, F goes to zero at the horizons and, as a result, the canonical momenta become infinite on the horizons. In other words, the new canonical chart must be used carefully. Outside the horizons the canonical transformation is well-defined and differentiable. The inverse transformation is also well-defined and differentiable.

The new falloff conditions for the new phase space variables at both asymptotic infinities are

$$M(t,r) = M_+(t) + O^\infty(|r|^{-1}) \quad , \tag{3.96}$$

$$
\begin{aligned}
\mathsf{R}(t,r) &= |r| + \ell^2\rho(t)|r|^{-2} + O^\infty(|r|^{-3}) , && (3.97)\\
Q(t,r) &= Q_+(t) + O(|r|^{-1}) , && (3.98)\\
P_M(t,r) &= O^\infty(|r|^{-6}) , && (3.99)\\
P_{\mathsf{R}}(t,r) &= O^\infty(|r|^{-4}) , && (3.100)\\
P_Q(t,r) &= O^\infty(|r|^{-2}) , && (3.101)
\end{aligned}
$$

3.3.7 Hamiltonian Action and Constraints in the New Variables

Using Eqs. (3.52), (3.77) and (3.79), and the expressions of Ref. [20] for the Hamiltonian and the diffeomorphism constraints $\mathcal{H}$ and $\mathcal{H}_r$ in the absence of the cosmological constant, one finds that the spatial derivatives of the mass and the charge functions M and Q can be written in terms of the constraints:

$$
\begin{aligned}
M' &= -\Lambda^{-1}R'\mathcal{H} - \Lambda^{-1}R^{-1}P_\Lambda\mathcal{H}_r + \\
&\quad (\Lambda^{-1}R^{-1}\Gamma P_\Gamma - R^{-1}P_\Gamma)G , && (3.102)\\
Q' &= -G . && (3.103)
\end{aligned}
$$

Hence, the constraint equations (3.50), (3.51) and (3.52) imply that

$$
\begin{aligned}
M' &= 0 , && (3.104)\\
Q' &= 0 . && (3.105)
\end{aligned}
$$

In other words, the mass and charge functions M and Q are constants with respect to r, and therefore we write

$$
\begin{aligned}
M(t,r) &= m(t) , && (3.106)\\
Q(t,r) &= q(t) . && (3.107)
\end{aligned}
$$

Moreover, it was noted in Ref. [20] that the expression for $\mathcal{H}_r$ in the new variables must be

$$
\mathcal{H}_r = P_M M' + P_Q Q' + P_{\mathsf{R}} R' . \tag{3.108}
$$

Eqs. (3.51), (3.104) and (3.105) therefore imply that

$$
P_{\mathsf{R}} = 0 . \tag{3.109}
$$

Hence, the set of new constraints $M' = 0$, $Q' = 0$ and $P_{\mathsf{R}} = 0$ are equivalent, expect on the horizons, to the old set of constraints.

The total action written in terms of the variables M, Q, R and their canonical momenta takes the form

$$
\begin{aligned}
&S[M, \mathsf{R}, Q, P_M, P_{\mathsf{R}}, P_Q; N, N^r, \tilde{\Phi}]\\
&\quad = \int dt \int_{-\infty}^{+\infty} dr \, (P_M \dot{M} + P_Q \dot{Q} + P_{\mathsf{R}} \dot{\mathsf{R}} - N\mathcal{H} - N^r\mathcal{H}_r - \tilde{\Phi}G)\\
&\quad - \int dt \, (N_+ M_+ + N_- M_- + \tilde{\Phi}_+ Q_+ - \tilde{\Phi}_- Q_-) , && (3.110)
\end{aligned}
$$

where the Hamiltonian, the diffeomorphism and the Gaussian constraints are functions of the new variables. It is also possible to write an action in terms of the new canonical varibles by redefining the Lagrange multipliers, but this procedure is not very illuminating. The results of such a procedure can be found in Ref. [20].

3.3.8 Hamiltonian Reduction

When the constraint equations are satisfied, the Hamiltonian action (3.110) takes the form

$$S[m,q,p_m,p_q;N_-,N_+,\tilde{\Phi}_-,\tilde{\Phi}_+] = \int dt[p_m\dot{m}+p_q\dot{q}-(N_++N_-)m-(\tilde{\Phi}_+-\tilde{\Phi}_-)q] \ , \tag{3.111}$$

where we have defined:

$$p_m(t) := \int_{-\infty}^{+\infty} dr\, P_M(t,r) \ , \tag{3.112}$$

$$p_q(t) := \int_{-\infty}^{+\infty} dr\, P_Q(t,r) \ . \tag{3.113}$$

The infinite-dimensional phase space is thus reduced to a phase space which is spanned by just four canonical coordinates. These canonical coordinates are the variables m and q – which can be identified as the mass M and the charge Q of the hole when Einstein's equations are satisfied – and the corresponding canonical momenta p_m and p_q.

The momenta p_m and p_q have an interesting interpretation: because we have defined $P_M := -T'$ and $P_Q := -\xi'$, we find that p_m is simply the difference in the Minkowski time T at the left and right infinities on the spacelike hypersurface $t = constant$. The momentum p_q, in turn, reveals the difference between the choices of the gauge function ξ at the asymptotic infinities of spacetime.

One can read off from Eq. (3.111) the true reduced Hamiltonian of the Reissner-Nordström hole in terms of the variables m and q:

$$H = (N_+ + N_-)m + (\tilde{\Phi}_+ - \tilde{\Phi}_-)q \ . \tag{3.114}$$

The Hamiltonian equations of motion are therefore

$$\dot{m} = \frac{\partial H}{\partial p_m} = 0 \ , \tag{3.115}$$

$$\dot{q} = \frac{\partial H}{\partial p_q} = 0 \ , \tag{3.116}$$

$$\dot{p}_m = -\frac{\partial H}{\partial m} = -(N_+ + N_-) \ , \tag{3.117}$$

$$\dot{p}_q = -\frac{\partial H}{\partial q} = -(\tilde{\Phi}_+ - \tilde{\Phi}_-) \ . \tag{3.118}$$

As one can see from Eqs. (3.115) and (3.116), the mass m and the charge q are constants of motion of the system. This statement is in accordance with the result that m and q are equal to the mass M and the charge Q parameters of the Reissner-Nordström black hole solution.

Now, one could, of course, use the variables m and q and their canonical momenta as the phase space coordinates of the system, and construct a Hamiltonian quantum theory of the Reissner-Nordström hole based on the use of these coordinates. There is, however, a very grave disadvantage with these coordinates: They describe the *static* aspects of the black hole spacetime only. Indeed, we saw in Eqs. (3.115) and (3.116) that the variables m and q are constants of motion of the system. However, there is *dynamics* in the Reissner-Nordström spacetime in the sense that in the region where $r_- < R < r_+$, there is no timelike Killing vector field orthogonal to a spacelike hypersurface. Our next task is to find such phase space coordinates which describe the dynamics of spacetime in a natural manner.

3.3.9 Hamiltonian Dynamics with Charge as an External Parameter

The aim of all physical theories, at least in principle, is to be able to predict the possible outcomes of measurements. When we talk about measurements, however, we need a reference to an *observer* performing these measurements: the possible outcomes of measurements are the possible outcomes of measurements as such as they can be measured, in principle, by a certain observer. The properties of the observer, in turn, motivate the structure of the theory. An interesting point of view to the interpretation of quantum mechanics was suggested by Rovelli [65]. His idea was, in rough terms, that one is not justified to talk about any *absolute* quantum state of a physical system. Instead, one should talk about a quantum state relative to some observer. This idea has given some inspiration to the point of view adopted in this thesis.

In this section we choose the observer in a most simple manner: our observer is at the right hand side asymptotic infinity in the conformal diagram, at rest with respect to the Reissner-Nordström hole. Our aim is to construct a quantum theory of the Reissner-Nordström spacetime from the point of view of such an observer. To this end, we choose the lapse functions $N_\pm(t)$ at asymptotic infinities so that

$$N_+(t) \equiv 1 \ , \quad (3.119)$$

$$N_-(t) \equiv 0 \ . \quad (3.120)$$

In other words, we have chosen the time coordinate at the right infinity to be the proper time of our observer, and we have "frozen" the time evolution at the left infinity. This can be considered justified on grounds that our observer can make observations at just one infinity.

The next task is to fix the functions $\tilde{\Phi}_\pm(t)$. As one can see from Eqs. (3.53) and (3.71), the functions $\tilde{\Phi}_\pm(t)$ are just the electric potentials at asymptotic infinities. It is customary to choose the zero point of the electric potential in such a way that at asymptotic infinities the electric potential vanishes. This choice is compatible with the Reissner-Nordström solution to Einstein-Maxwell equations. Hence, we choose:

$$\tilde{\Phi}_+(t) \equiv \tilde{\Phi}_-(t) \equiv 0 \ . \quad (3.121)$$

With these choices of the lapse functions and the electric potentials the reduced Hamiltonian (3.114) takes the form

$$H = m \ . \quad (3.122)$$

Because of that, the numerical value of our Hamiltonian is just the mass M of the hole. This mass includes, from the point of view of our observer, all the energy of the system, gravitational as well as electromagnetic.

When choosing the phase space coordinates we again refer to the properties of our observer. Our observer sees the exterior regions of the black hole as static, and he is an inertial observer. These properties prompt us to choose the phase space coordinates in such a manner that when the classical equations of motion are satisfied, all the dynamics is, in a certain sense, confined inside the apparent horizon $R = r_+$ of the hole. Moreover, as we shall see in a moment, the choice of the phase space coordinates describing the dynamics of spacetime is related to the choice of slicing of spacetime into space and time. We choose a slicing where the proper time of an observer in a radial free fall through the bifurcation two-sphere coincides with the proper time of our far-away observer at rest. On grounds of the Principle of Equivalence one may view these kind of slicings to be in a preferred position in relating the physical properties of the black hole interior to the physics observed by our far-away observer.

The new phase space coordinates describing the dynamics of the black hole spacetime can be obtained from the phase space coordinates m, q, p_m and p_q by means of an appropriate canonical transformation. To make things simple, we shall in this section consider the charge q as a mere external parameter of the system, having a fixed value Q. In the next section, the charge will be considered as a dynamical variable. Hence, the dimension of the phase space of our system is, in this section, just two.

As in Ref. [66], we perform a canonical transformation from the phase space variables (m, p_m) to the new phase pace variables (a, p_a) such that the relationship between the old and the new phase space variables is (q is now a constant, which we denote by Q):

$$|p_m| = \sqrt{2ma - a^2 - Q^2} + m\sin^{-1}\left(\frac{m-a}{\sqrt{m^2 - Q^2}}\right) + \frac{1}{2}\pi m \; , \tag{3.123}$$

$$p_a = \mathrm{sgn}(p_m)\sqrt{2ma - a^2 - Q^2} \; , \tag{3.124}$$

and we have imposed by hand a restriction:

$$-\pi m \leq p_m \leq \pi m \; . \tag{3.125}$$

With the restriction (3.125) the transformations (3.123) and (3.124) are well-defined and one-to-one. It follows from Eq. (3.124) that

$$m = \frac{p_a^2}{2a} + \frac{1}{2}a + \frac{Q^2}{2a} \; . \tag{3.126}$$

If one substitutes this expression for m to Eq. (3.123), one gets p_m in terms of a and p_a. One finds that the fundamental Poisson brackets between m and p_m remain invariant, and hence the transformation is canonical.

Eqs. (3.122) and (3.126) imply that the classical Hamiltonian takes, in terms of the variables a and p_a, the form

$$H = \frac{p_a^2}{2a} + \frac{1}{2}a + \frac{Q^2}{2a} \; . \tag{3.127}$$

The geometrical interpretation of the variable a is extremely easy to find. We first write the Hamiltonian equation of motion for a:

$$\dot{a} = \frac{\partial H}{\partial p_a} = \frac{p_a}{a} \; , \tag{3.128}$$

and it follows from Eq. (3.126) and the fact that $m = M$ when the classical equations of motion are satisfied that the equation of motion for a is

$$\dot{a}^2 = \frac{2M}{a} - 1 - \frac{Q^2}{a^2} . \tag{3.129}$$

One can see from the Reissner-Nordström metric (3.35) that the equation of motion for an observer in a radial free fall through the bifurcation two-sphere is

$$\dot{R}^2 = \frac{2M}{R} - 1 - \frac{Q^2}{R^2} , \tag{3.130}$$

where the overdot means proper time derivative. As one can see, Eqs. (3.129) and (3.130) are identical. Hence, we can interpret a as the radius of the wormhole throat of the Reissner-Nordström black hole, from the point of view of an observer in a radial free fall through the bifurcation two-sphere. Moreover, we see from Eq. (3.129) that a is confined to be, classically, within the region $[r_-, r_+]$. In other words, our variable a "lives" only within the inner and outer horizons of the Reissner-Nordström black hole, and this is precisely the region in which it is impossible to find a time coordinate in such a way that spacetime with respect to that time coordinate would be static. Hence, both of the requirements we posed for our phase space coordinates are satisfied: dynamics is confined inside the apparent horizon and the time coordinate on the wormhole throat is the proper time of an observer in a radial free fall.

With the interpretation explained above, the restriction (3.125) becomes understandable. One can see from Eq. (3.117) that when the lapse functions $N_\pm$ at asymptotic infinities are chosen as in Eq. (3.119), the canonical momentum p_m conjugate to m is $-t + constant$, where t is the time coordinate of our asymptotic observer. Now, the transformation defined by Eqs. (3.123) and (3.124) involves an identification of the time coordinate t with the proper time of a freely falling observer on the throat. However, as it was noted at the beginning of Sec. 3.3, it is impossible to push the spacelike hypersurfaces $t = constant$ beyond the $R = r_-$ hypersurfaces in the conformal diagram. The proper time a freely falling observer needs to fall from the past $R = r_-$ hypersurface to the future $R = r_-$ hypersurface through the bifurcation two-sphere is, as it can be seen from Eq. (3.130),

$$\Delta t = 2\int_{r_-}^{r_+} \frac{R'\,dR'}{\sqrt{2MR' - R'^2 - Q^2}} = 2\pi M , \tag{3.131}$$

and hence the restriction (3.125) is needed. As one can see from Eq. (3.123), $|p_m| = 0$, when $a = r_+$, and $|p_m| = \pi M$, when $a = r_-$. We have chosen p_m to be posititive, when the hypersurface $t = constant$ lies between the past $R = r_-$ hypersurface and the bifurcation point, and negative when that hypersurface lies between the bifurcation point and the future $R = r_-$ hypersurface.

As to the classical Hamiltonian theory, the only thing one still needs to check is, whether there exist such foliations of the Reissner-Nordström spacetime where the Minkowski time t at asymptotic infinity and the proper time of a freely falling observer at the throat through the bifurcation two-sphere really are the one and the same time coordinate. It is easy to see that time coordinates determining this sort of foliations really exist. As a concrete example, consider a generalization of the so-called Novikov coordinates in the Schwarzschild geometry [29]. More precisely, one takes a collection of freely falling test particles whose initial three-velocity with respect to the curvature coordinates is zero when $T = 0$. If one relates the radial coordinate of spacetime to the positions of these test particles when $T = 0$, and takes the time coordinate to be at every spacetime point the proper time of the test particle falling through that point, one finds that the time coordinate of our distant observer at rest, the Minkowski time, and the proper time of a freely falling observer in the throat are the one and the same time coordinate. It should be noted, however, that all foliations in which the proper time in the throat and the asymptotic Minkowski time are identified, are incomplete, since such foliations, in addition to failing to cover regions outside the past and future $R = r_-$ hypersurfaces, also fail to cover the whole exterior regions of the hole. More precisely, these foliations are valid only when $-\pi M \leq t \leq \pi M$.

3.3.10 Hamiltonian Dynamics with Charge as a Dynamical Variable

To complete the classical Hamiltonian theory we need to consider the charge Q as a dynamical varible. The guiding principle in our search for appropriate canonical variables describing the dynamics of the electromagnetic field is that since our distant observer at rest observes the electromagnetic field outside the event horizon as static, all the dynamics of the electromagnetic field must be confined, classically, inside the horizon.

To find appropriate canonical variables, recall that when the classical equations of motion are satisfied, the only non-vanishing component of the electromagnetic potential A_μ is, in curvature coordinates, the component $A_T = Q/R$. Now, this component is static with respect to time T everywhere outside the horizon. However, since R becomes a timelike coordinate when $r_- < R < r_+$, we find that A_T necessarily has dynamics between the inner and outer horizons of the Reissner-Nordström black hole. In terms of A_T we can write the Reissner-Nordström metric (3.35) as

$$ds^2 = -\left(1 - \frac{2M}{R} + A_T^2\right) dT^2 + \frac{dR^2}{1 - \frac{2M}{R} + A_T^2} + R^2\, d\Omega^2 \quad . \tag{3.132}$$

In what follows, we shall "forget" the explicit dependence of A_T on R and Q, and instead treat A_T as an independent dynamical variable of our theory. However, in all our investigations we shall assume that A_T is independent of T or, more precisely,

$$\frac{\partial A_T}{\partial T} \equiv 0 \quad . \tag{3.133}$$

From this restriction it follows that we can treat A_T as a function of an appropriate time coordinate only. Hence, our phase space, which will be spanned by the throat variables (a, p_a) and the electromagnetic variables (A_T, p_{A_T}), where p_{A_T} is the canonical momentum conjugate to A_T, will be four-dimensional, which is in harmony with the results of Ref. [20]. To complete the classical theory we must just find a canonical transformation from the old phase space variables (m, p_m, q, p_q) to the new phase space variables

(a, p_a, A_T, p_{A_T}), and write the classical Hamiltonian, from the point of view of our distant obsever at rest, in terms of the variables a, p_a, A_T and p_{A_T}.

To find a clue to the expression of the classical Hamiltonian in terms of the gravitational and electromagnetic variables, let us write the Hamiltonian constraint on the timelike geodesic going through the bifurcation two-sphere of the Reissner-Nordtsröm spacetime, using the foliation of Sec. 3.3.9. In that foliation the spacetime metric can be written as

$$ds^2 = -dt^2 + \left(\frac{2M}{a} - 1 - A_T^2\right) dT^2 + a^2\, d\Omega^2 \quad . \tag{3.134}$$

As one can see, T is now a spatial coordinate of spacetime. Hence, we can identify the expression $(\frac{2M}{a} - 1 - A_T^2)^{1/2}$ with the variable $\Lambda(r,t)$ of Sec. 3.3.1. Moreover, we can identify A_T with Γ. The variable $\phi(r,t)$ is assumed to vanish. With these identifications, and treating M as a constant, we find that the Hamiltonian contraint of Ref. [20] written in terms of Λ, R and Γ and their time derivatives, can be written in terms of a and A_T and their time derivatives as

$$\begin{aligned} \mathcal{H} \;=\; & \left(\frac{2M}{a} - 1 - A_T^2\right)^{-1/2} \left[\frac{1}{2}(1 + A_T^2)\dot{a}^2 + aA_T\dot{A}_T\dot{a} + \frac{1}{2}a^2\dot{A}_T^2 \right. \\ & \left. -\frac{M}{a} + \frac{1}{2}(1 + A_T^2)\right] = 0 \quad . \end{aligned} \tag{3.135}$$

From this equation one can solve M:

$$M = \frac{1}{2}a(1 + A_T^2)\dot{a}^2 + a^2 A_T\dot{A}_T\dot{a} + \frac{1}{2}a(1 + A_T^2) \quad . \tag{3.136}$$

It is easy to see that if one substitutes

$$A_T = \frac{Q}{a} \quad , \tag{3.137}$$

and keeps Q as a constant, one gets

$$M = \frac{1}{2}a\dot{a}^2 + \frac{1}{2}a + \frac{Q^2}{2a} \quad . \tag{3.138}$$

Hence, if one interprets the right hand side of Eq. (3.138) as the classical Hamiltonian of the system, one gets, with the substitution (3.137), the same Hamiltonian as in Eq. (3.127).

At this point we define a new variable

$$b := aA_T \quad . \tag{3.139}$$

As a result, Eq. (3.136) becomes simpler:

$$M = \frac{1}{2}a\dot{a}^2 + \frac{1}{2}a\dot{b}^2 + \frac{1}{2}a + \frac{b^2}{2a} \quad . \tag{3.140}$$

Because of that, we are prompted to write the classical Hamiltonian of the Reissner-Nordström black hole, from the point of view of our distant observer at rest, as

$$H = \frac{p_a^2}{2a} + \frac{p_b^2}{2a} + \frac{1}{2}a + \frac{b^2}{2a} \quad , \tag{3.141}$$

where p_a is the canonical momentum conjugate to the throat radius a, and

$$p_b := a\dot{b} \tag{3.142}$$

is the canonical momentum conjugate to the variable b.

We obtained the Hamiltonian (3.141) by means of a guesswork based on the study of the Hamiltonian constraint on the timelike geodesic going through the bifurcation two-sphere, in a specific foliation of the Reissner-Nordström spacetime. The real problem is to find out, whether there exists a well defined one-to-one canonical transformation from the phase space coordinates m, p_m, q and p_q, introduced in Secs. 3.3.7 and 3.3.8, to the phase space coordinates a, p_a, b and p_b such that the Hamiltonian takes the form (3.141) if we choose the lapse functions and the electric potentials at asymptotic infinities as in Sec. 3.3.9.

We shall perform such a transformation in two steps. We first define a canonical momentum p_w conjugate to a yet unknown variable w as

$$p_w := q \ . \tag{3.143}$$

With this choice the classical Hamiltonian takes the form

$$H = \frac{p_a^2}{2a} + \frac{p_w^2}{2a} + \frac{1}{2}a \ . \tag{3.144}$$

The variables m and p_m are expressed in terms of a and p_a as in Eqs. (3.123) and (3.126), but we have replaced Q with p_w.

The next task is to find w. One expects w to be related in one way or another to the momentum p_q conjugate to q. Since p_q defines the electromagnetic gauge, we first write the Hamiltonian in a general gauge,

$$H = \frac{p_a^2}{2a} + \frac{p_w^2}{2a} + \frac{1}{2}a + (\tilde{\phi}_+ - \tilde{\phi}_-)p_w \ , \tag{3.145}$$

which follows from Eq. (3.114). Using Eq. (3.118) we find that the Hamiltonian equation of motion for w is

$$\dot{w} = \frac{\partial H}{\partial p_w} = \frac{p_w}{a} - \dot{p}_q \ . \tag{3.146}$$

An expression of p_q in terms of a, p_a, w and p_w can be gained by integrating the both sides of Eq. (3.146) along the classical trajectory in the phase space:

$$p_q = \int \frac{p_w}{a\dot{a}}\, da - w \ , \tag{3.147}$$

where we shall substitute

$$\dot{a} = -sgn(p_w)\sqrt{\frac{2m}{a} - 1 - \frac{p_w^2}{a}} \ . \tag{3.148}$$

This substitution involves choosing $\dot{p}_q = 0$. When the electric potential is assumed to vanish at asymptotic infinities, this choice can be made. With an appropriate choice of the integration constant we get

$$p_q = sgn(p_m)p_w\left[\sin^{-1}\left(\frac{p_a^2 + p_w^2 - a^2}{\sqrt{(p_a^2 + p_w^2 + a^2)^2 - 4a^2p_w^2}}\right) + \frac{\pi}{2}\right] - w \ , \tag{3.149}$$

where we have made the substitution

$$m = \frac{p_a^2}{2a} + \frac{p_w^2}{2a} + \frac{1}{2}a \quad . \tag{3.150}$$

Eqs. (3.123), (3.124), (3.143) and (3.150) constitute a transformation from the phase space coordinates m, p_m, q and p_q to the phase space coordinates a, p_a, w and p_w. It is easy to see that this transformation is well-defined and canonical. Moreover, the transformation is one-to-one provided we impose the restriction

$$\left| \frac{p_q + w}{p_w} \right| \leq \pi \quad . \tag{3.151}$$

This restriction is related to the fact that we are considering spacetime between two successive $R = r_-$ hypersurfaces. Since $\dot{p}_q$ vanishes when the electric potentials are assumed to vanish at asymptotic infinities, we find that, classically, $w = -q\pi + p_q$ at the past $R = r_-$ hypersurface, $w = p_q$ at the bifurcation point, and $w = q\pi + p_q$ at the future $R = r_-$ hypersurface. In other words, the domain of w is bounded by the fact that the $t = constant$ hypersurfaces cannot be pushed beyond the $R = r_-$ hypersurfaces.

It only remains to find a canonical transformation from the variables w and p_w to the variables b and p_b. We define

$$b \ := \ p_w \sin\left(\frac{w}{p_w}\right) \quad , \tag{3.152}$$

$$p_b \ := \ p_w \cos\left(\frac{w}{p_w}\right) \quad . \tag{3.153}$$

This transformation is well-defined and canonical as well as, with the restriction (3.151), one-to-one. Because of that, we are justified to write the classical Hamiltonian as in Eq. (3.141). Moreover, since it follows from Eqs. (3.152) and (3.153) that

$$p_w^2 = p_b^2 + b^2 \quad , \tag{3.154}$$

we can identify the quantity $p_b^2 + b^2$ as the square of the electric charge of the black hole.

3.4 Axially Symmetric Hamiltonian Action of Einstein-Maxwell Theory

In this section we shall consider a classical Hamiltonian formulation of axially symmetric electrovacuum spacetimes with boundary conditions of Ref. [16]. The complete Hamiltonian formulation of Kerr-Newman spacetimes has not yet been performed, but the study of Hamiltonian dynamics is based on an important theorem proved by Regge and Teitelboim [21]. This theorem states, essentially, that the physical Hamiltonian of an asymptotically flat spacetime with matter fields can be gained if we first solve the classical constraints, and then substitute the solutions to the constraints, in terms of the physical phase space coordinates of the theory, to the boundary terms at asymptotic spacelike infinity. It is unclear whether the assumptions of Regge's and Teitelboim's theorem are valid for Kerr-Newman spacetimes, but we accept this as an unproved hypothesis and see where it takes us.

3.4.1 Boundary Terms in Kerr-Newman Spacetime

As we saw in Chapter 1, one must include, in asymptotically flat spacetimes, certain boundary terms in order to make the variational principle consistent. Now, of particular interest are the boundary terms in Kerr-Newman spacetime, the most general black hole spacetime.

We begin with the Kerr-Newman line element (2.34) in Boyer-Lindquist coordinates. To calculate the boundary terms we must approximate the line element (2.34) at asymptotic infinity, where $r \to \infty$, when only leading order terms are taken into account:

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4J \sin^2\theta}{r} dt d\phi + r^2 \sin^2\theta d\phi^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\theta^2 , \tag{3.155}$$

where $J := Ma$ is the angular momentum of the hole. In Cartesian coordinates this expression takes the form

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4J}{r^3}(xdy - ydx)dt + \left(1 + \frac{2M}{r}\right)(dx^2 + dy^2 + dz^2) ; \tag{3.156}$$

here r is not the same r as in Eq. (3.155). In Eq. (3.155) r is one of the Boyer-Lindquist coordinates, whereas in Eq. (3.156) r is defined to be equal to $(x^2 + y^2 + z^2)^{1/2}$.

We proceed to evaluate the boundary terms. When evaluating the boundary terms the first task is to fix the coordinate system far away from the black hole. In other words, we must fix the lapse N and the shift N^a. In this paper we choose a far-away coordinate system which revolves, with respect to the Cartesian coordinates x, y and z, with an extremely small angular velocity ω around the z-axis. (We must assume ω to be extremely small since otherwise the velocities of the far-away observers would exceed the velocity of light. More precisely, we choose ω to be so small that even for observers who are so far away from the hole that the boundary terms are, to a very good approximation, those calculated at infinity, the velocities are well below the velocity of light.) Because, in flat space, the velocity of an observer at the point $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ revolving with angular velocity $\vec{\omega}$ is

$$\vec{v} = \vec{\omega} \times \vec{r} , \tag{3.157}$$

and because, in Cartesian coordinates, N^a represents the a-component of velocity, we find that $N^a dt$ is given by Eq. (1.70) and therefore

$$N^a = \varepsilon^a{}_{bc} \omega^b x^c , \tag{3.158}$$

where $\varepsilon^a{}_{bc}$ is the Levi-Civita symbol such that $\varepsilon_{123} = 1$.

What sort of boundary terms do show up with this kind of a choice of the shift? We recall from Section 1.5.3 that if the shift N^a is chosen as in Eq. (3.158), we must therefore bring along the boundary term (1.71).

Now, when calculating the boundary term $S^{\text{rot}}_{\partial\Sigma}$ of Eq. (1.71) we should, of course, first perform a coordinate transformation where the spacetime metric (3.156) is replaced by the corresponding expression in revolving coordinates, and then proceed to calculate the boundary term by using this expression. However, when the far-away coordinate system revolves very slowly, we are interested in terms linear in ω only. Taking into account the transformation in the expression of the metric would produce terms quadratic in ω, which can be neglected. Hence, we are allowed to calculate the boundary term (1.71) by using the metric (3.156). Since $\vec{\omega} = \omega\hat{k}$, we have $\omega^1 = 0 = \omega^2$ and $\omega^3 = \omega$, and so the boundary term can be written in the form

$$S^{\rm rot}_{\partial\Sigma} = -2\int dt\, \omega \oint \left(x\pi^2{}_s dS^s - y\pi^1{}_s dS^s\right) . \tag{3.159}$$

By comparing the line element of Eq. (3.156) with the ADM line element of the form

$$ds^2 = -\left(N^2 - N_a N^a\right) dt^2 + 2N_a dx^a dt + q_{ab} dx^a dx^b , \tag{3.160}$$

where N is the lapse function, N^a is a component of the shift vector (a = 1,2,3) and q_{ab} is a spacelike component of the metric tensor associated with the hypersurface, we find that the only nonzero spacelike components of the metric tensor are

$$q_{11} = 1 + \frac{2M}{r} = q_{22} = q_{33} . \tag{3.161}$$

For the components of the shift vector N^a we have

$$N_1 = \frac{2Jy}{r^3},\ N_2 = -\frac{2Jx}{r^3},\ N_3 = 0 , \tag{3.162}$$

where

$$N_a := q_{ab} N^b . \tag{3.163}$$

Comparing Eqs. (3.156) and (3.160) we obtain the lapse function N

$$N = \sqrt{1 - \frac{2M}{r} + \frac{4J^2}{r^6}\left(1 + \frac{2M}{r}\right)^{-1} (x^2 + y^2)} . \tag{3.164}$$

To evaluate the covariant derivatives of N_a we must calculate the Christoffel symbols using the nonzero components of q_{ab} expressed in Eq. (3.161). The nonzero Christoffel symbols needed in the calculations are

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = -\frac{Mx}{r^3}\left(1 + \frac{2M}{r}\right)^{-1} , \tag{3.165}$$

$$\Gamma^2_{11} = \Gamma^2_{33} = \frac{My}{r^3}\left(1 + \frac{2M}{r}\right)^{-1} , \tag{3.166}$$

$$\Gamma^1_{22} = \Gamma^1_{33} = \frac{Mx}{r^3}\left(1 + \frac{2M}{r}\right)^{-1} , \tag{3.167}$$

$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^3_{23} = -\frac{My}{r^3}\left(1 + \frac{2M}{r}\right)^{-1} , \tag{3.168}$$

$$\Gamma^1_{13} = \Gamma^2_{23} = -\frac{Mz}{r^3}\left(1 + \frac{2M}{r}\right)^{-1} . \tag{3.169}$$

By using Eq. (1.52) for the exterior curvature tensor K_{ab}, we get for the nonzero components

$$K_{11} = \frac{2Jxy}{Nr^5}\left(\frac{2M}{r + 2M} - 3\right) , \tag{3.170}$$

$$K_{22} = -\frac{2Jxy}{Nr^5}\left(\frac{2M}{r+2M}-3\right) , \tag{3.171}$$

$$K_{12}=K_{21} = -\frac{J}{Nr^5}\left(x^2-y^2\right)\left(\frac{2M}{r+2M}-3\right) , \tag{3.172}$$

$$K_{13}=K_{31} = \frac{Jyz}{Nr^5}\left(\frac{2M}{r+2M}-3\right) , \tag{3.173}$$

$$K_{23}=K_{32} = -\frac{Jxz}{Nr^5}\left(\frac{2M}{r+2M}-3\right) . \tag{3.174}$$

To evaluate π^{ab}, the canonical momentum conjugate to q_{ab}, by using the expression (1.26) we must first calculate K and $\sqrt{q}$, and we obtain that

$$K := q^{11}K_{11} + q^{22}K_{22} + q^{33}K_{33} = 0 , \tag{3.175}$$

and

$$\sqrt{q} = \left(1+\frac{2M}{r}\right)^{\frac{3}{2}} . \tag{3.176}$$

When the results from Eqs. (3.175) and (3.176) are substituted to Eq. (1.26) and the indices are pushed down by the spatial metric q_{ab}, we get the nonzero components

$$\pi_{11} = \frac{(1+\frac{2M}{r})^{\frac{3}{2}}}{16\pi}\frac{2Jxy}{Nr^5}\left(\frac{2M}{r+2M}-3\right) , \tag{3.177}$$

$$\pi_{22} = -\frac{(1+\frac{2M}{r})^{\frac{3}{2}}}{16\pi}\frac{2Jxy}{Nr^5}\left(\frac{2M}{r+2M}-3\right) , \tag{3.178}$$

$$\pi_{12}=\pi_{21} = -\frac{(1+\frac{2M}{r})^{\frac{3}{2}}}{16\pi}\frac{J}{Nr^5}\left(x^2-y^2\right)\left(\frac{2M}{r+2M}-3\right) , \tag{3.179}$$

$$\pi_{13}=\pi_{31} = \frac{(1+\frac{2M}{r})^{\frac{3}{2}}}{16\pi}\frac{Jyz}{Nr^5}\left(\frac{2M}{r+2M}-3\right) , \tag{3.180}$$

$$\pi_{23}=\pi_{32} = -\frac{(1+\frac{2M}{r})^{\frac{3}{2}}}{16\pi}\frac{Jxz}{Nr^5}\left(\frac{2M}{r+2M}-3\right) . \tag{3.181}$$

The boundary term $S^{\text{rot}}_{\partial\Sigma}$ can be expressed by using the components calculated above and the components of the unit normal on the surface:

$$\begin{aligned} S^{\text{rot}}_{\partial\Sigma} &= -2\int dt\,\omega \oint \Big[xq^{22}\left(\pi_{21}n^1+\pi_{22}n^2+\pi_{23}n^3\right) - \\ &\quad -yq^{11}\left(\pi_{11}n^1+\pi_{12}n^2+\pi_{13}n^3\right)\Big]dS \\ &= \frac{1}{8\pi}\int dt\,\omega J \oint \left(1+\frac{2M}{r}\right)^{\frac{1}{2}}\left(\frac{2M}{r+2M}-3\right)\frac{1}{Nr^5} \\ &\quad \times\Big[\left(x^2-y^2\right)\left(xn^1-yn^2\right)+2xy\left(yn^1-xn^2\right) \\ &\quad +z(x^2+y^2)n^3\Big]dS . \end{aligned} \tag{3.182}$$

This integral is easy to evaluate in the spherical coordinates. We first consider a 2-dimensional spherical surface with radius r. The relations between the spherical coordinates r, θ and ϕ and the Cartesian coordinates x, y and z are

$$x = r\cos\phi\sin\theta,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta\ . \qquad (3.183)$$

The components of the unit normal n^a, $(a = 1,2,3)$ on the spherical surface are

$$n^1 = n_x = \cos\phi\sin\theta,\ n^2 = n_y = \sin\theta\sin\phi,\ n^3 = n_z = \cos\theta \qquad (3.184)$$

and the area element of the sphere is

$$dS = r^2\sin\theta d\theta d\phi\ . \qquad (3.185)$$

In these coordinates the boundary term takes the form

$$\begin{aligned} S^{\rm rot}_{\partial\Sigma} &= \frac{1}{8\pi}\int dt\ \omega J\left(1+\frac{2M}{r}\right)^{\frac{1}{2}}\left(\frac{2M}{r+2M}-3\right) \\ &\times\int\limits_{\theta=0}^{\pi}\int\limits_{\phi=0}^{2\pi}\frac{1}{N}\Big[\left(\cos^2\phi-\sin^2\phi\right)^2\sin^5\theta + \\ &\quad 4\cos^2\phi\sin^2\phi\sin^5\theta+\cos^2\theta\sin^3\theta\Big]d\phi d\theta\ . \end{aligned} \qquad (3.186)$$

As r approaches infinity N goes to unity and so the denominator can be approximated as unity as well. The integration gives then

$$S^{\rm rot}_{\partial\Sigma} \approx \frac{1}{8\pi}\left(1+\frac{2M}{r}\right)^{\frac{1}{2}}\left(\frac{2M}{r+2M}-3\right)\frac{8\pi}{3}\int dt\ \omega J\ , \qquad (3.187)$$

and so the boundary term at spatial infinity, where $r\to\infty$, is

$$S^{\rm rot}_{\partial\Sigma} = -\int dt\,\omega J\ . \qquad (3.188)$$

We must still calculate the ADM boundary term (1.60) as well as the electromagnetic boundary term (3.27). The ADM boundary term of the Kerr-Newman spacetime is, for arbitrary lapse N_+ at infinity,

$$S^{\rm ADM}_{\partial\Sigma} = -\int dt N_+ M\ . \qquad (3.189)$$

To calculate the electromagnetic boundary term we first recall that for Kerr-Newman solution the only nonzero components of A_μ in Boyer-Lindquist coordinates are

$$A_t = -\frac{Qr}{\Sigma}\ , \qquad (3.190)$$

$$A_\phi = \frac{Qar}{\Sigma}\sin^2\theta\ . \qquad (3.191)$$

Using Eqs. (3.10), (3.18) and (3.20) one finds that for general lapse and shift one can write π^a, the canonical momentum conjugate to A_a, as

$$\pi^a = \frac{1}{N}\frac{\sqrt{q}}{4\pi}q^{as}\left(F_{0s}-N^{b(3)}F_{bs}\right)\ . \qquad (3.192)$$

This expression, together with Eqs. (3.190) and (3.191), implies that in Boyer-Lindquist coordinates the only surviving component of π^a is π^r which, in the leading order, can be written very far away from the hole as

$$\pi^r = -\frac{Q}{4\pi r^2} + \mathcal{O}\left(r^{-3}\right) . \tag{3.193}$$

Hence, the electromagnetic boundary term (3.27) is

$$S^{em}_{\partial\Sigma} = -\int dt A_0^+ Q \ , \tag{3.194}$$

as expected. The slow rotation of the asymptotic coordinate system will change the ADM and the electromagnetic boundary terms a bit but the resulting corrections will be of the order of $\mathcal{O}(\omega^2)$ and can therefore be neglected.

3.4.2 Hamiltonian Dynamics of Kerr-Newman Spacetimes

We shall proceed to the study of the Hamiltonian dynamics of maximally extended Kerr-Newman spacetimes. To begin with, consider a foliation of such spacetimes into space and time. Obviously, we want the spacelike hypersurfaces where the time $t = constant$ to cover as great a portion of spacetime as possible. Maximally extended Kerr-Newman spacetimes have a periodic geometrical structure, and we pick up one such period. We choose the spacelike hypersurfaces $t = constant$ such that they begin from the left-hand-side asymptotic infinity, then go through the interior regions of the hole in arbitrary ways, and finally end at the right hand side asymptotic infinity in the conformal diagram. However, such spacelike hypersurfaces cannot be pushed beyond the interior horizons, where the Boyer-Lindquist coordinate is

$$r = r_- := M - \sqrt{M^2 - Q^2 - a^2} \ , \tag{3.195}$$

since otherwise our hypersurfaces would fail to be spacelike. Hence our study of the Hamiltonian dynamics of Kerr-Newman spacetimes must be restricted to include, in addition to the left and the right exterior regions of the Kerr-Newman black hole, only such an interior region of the hole that lies between two successive $r = r_-$ hypersurfaces in the conformal diagram. Our spacelike hypersurface $t = constant$ therefore begins its evolution at the past $r = r_-$ hypersurface, then goes through the bifurcation point where

$$r = r_+ := M + \sqrt{M^2 - Q^2 - a^2} \ , \tag{3.196}$$

and finally ends its evolution at the future $r = r_-$ hypersurface (see Fig. 3.2). Bearing this restriction in mind, we shall go into the Hamiltonian dynamics of Kerr-Newman spacetimes.

The first task is to write the action with appropriate boundary terms. Again, the problem is now that we have *two* asymptotic infinities, and at both of these infinities we have certain boundary terms. When this fact is taken into account, we find that the action takes the form

$$\begin{aligned} S \ &= \int dt \int_\Sigma d^3x \left(p^{ab}\dot{q}_{ab} + p^a \dot{A}_a - N\mathcal{H} - N^s\mathcal{H}_s - A_0\mathcal{G}\right) \\ &\quad - \int dt \left[(N_+ + N_-)m + (A_0^+ - A_0^-)q + (\omega_+ - \omega_-)\iota\right] \ . \end{aligned} \tag{3.197}$$

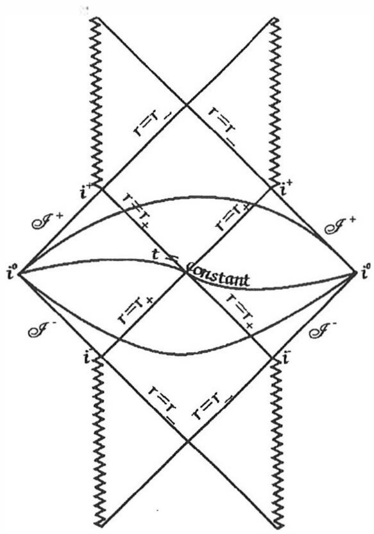

Figure 3.2: The conformal diagram of Kerr-Newman spacetime. Our spacelike hypersurfaces $t = constant$ begin their evolution at the past $r = r_-$ hypersurface, then go through the bifurcation point, and finally end their evolution at the future $r = r_-$ hypersurface.

In this equation, quantities equipped with plus and minus, respectively, are quantities written at the right and the left asymptotic infinities. In particular, ω_+ and ω_- are the angular velocities of far-away coordinate systems around z-axis. Hence, the total Hamiltonian of Kerr-Newman spacetime is

$$\begin{aligned} H_{tot} &= \int_\Sigma d^3x(N\mathcal{H} + N^s\mathcal{H}_s + A_0\mathcal{G}) \\ &\quad +(N_+ + N_-)m + (A_0^+ - A_0^-)q + (\omega_+ - \omega_-)\iota\,. \qquad (3.198) \end{aligned}$$

Now, the problem with our Hamiltonian is that it contains an enormous number of independent degrees of freedom. Indeed, our Hamiltonian may be considered as a function of the hypersurface metric q_{ab} at each point x on the spacelike hypersurface Σ, together with the corresponding canonical momenta π^{ab}. However, the ultimate object of interest in this paper is canonical quantization of the stationary black hole sector of Einstein-Maxwell theory. Stationary black holes, in turn, are characterized by just three independent, classical degrees of freedom, and hence an enormous number of degrees of freedom must be truncated.

For non-rotating black hole spacetimes, the truncation process has already been performed in this chapter. A similar truncation could also be performed for rotating black holes: One begins with asymptotically flat Einstein-Maxwell theory with appropriate symmetries, solves the classical constraints, and is finally left with just six physical, canonical degrees of freedom which may be taken to be the mass m, the electric charge q, and the angular momentum ι of the Kerr-Newman black hole, together with the corresponding canonical momenta p_m, p_q and p_ι.

An important feature of the process explained above, in which the phase space becomes reduced in such a way that only the physical degrees of freedom are left, is that the resulting Hamiltonian, the so-called reduced Hamiltonian, involves the boundary terms only. In particular, the reduced Hamiltonian of Kerr-Newman spacetimes is

$$H^{\text{red}} = (N_+ + N_-)m + (A_0^+ - A_0^-)q + (\omega_+ - \omega_-)\iota \,. \tag{3.199}$$

As a matter of fact, the reduced Hamiltonian may be used as the real, physical Hamiltonian of the system. This was proved by Regge and Teitelboim [21]. They found that if one solves the classical constraints and then substitutes the solutions to the reduced Hamiltonian, then the correct equations of motion for the canonical variables are obtained. More precisely, they showed the following: One assumes that the variables q_{ab} and π^{ab} can be separated by a one-to-one, time independent, functionally differentiable canonical transformation in two sets $(\varphi^\alpha, \pi_\alpha)$ and (ψ^A, π_A) in such a way that

a) the reduced Hamiltonian depends only on φ^α and π_α

b) when π_α are prescribed as functions p_α of spacetime coordinates x which satisfy
$$\dot{p}_\alpha = 0, \tag{3.200}$$
then the constraints $\mathcal{H} = 0$ and $\mathcal{H}_s = 0$ can be solved to express φ^α as functionals
$$\varphi^\alpha = f^\alpha[\psi^A; \pi_A] \tag{3.201}$$
of the remaining canonical variables.

The functional derivatives of f^α with respect to ψ^A and π_A are assumed to exist. If the above conditions are true then Hamilton's equation for the reduced Hamiltonian

$$H^{\text{red}}[\psi^A; \pi_A] = (\text{boundary terms})\bigg|_{\varphi^\alpha = f^\alpha, \pi_\alpha = p_\alpha}, \tag{3.202}$$

together with Eqs. (3.200) and (3.201) are equivalent to Einstein's equations in the particular frame defined by $\pi_\alpha = p_\alpha$.

The proof of this result is easy: The Poisson brackets are invariant under canonical transformation and the Hamiltonian is unchanged in value if the canonical transformation is independent of time. Hence

$$\dot{\psi}^A(x) = \frac{\delta H}{\delta \pi_A(x)}\bigg|_{\varphi^\alpha = f^\alpha, \pi_\alpha = p_\alpha}. \tag{3.203}$$

On the other hand,

$$\begin{aligned} H[\varphi^\alpha; \pi_\alpha, \psi^A; \pi_A]\bigg|_{\varphi^\alpha = f^\alpha, \pi_\alpha = p_\alpha} &= (\text{boundary terms})\bigg|_{\varphi^\alpha = f^\alpha, \pi_\alpha = p_\alpha} \\ &= H^{\text{red}}[\psi^A; \pi_A] \,. \end{aligned} \tag{3.204}$$

Differentiating Eq. (3.204) with respect to π_A gives

$$\int_\Sigma d^3y \frac{\delta H}{\delta\varphi^\alpha(y)}\bigg|_{\varphi^\alpha=f^\alpha,\pi_\alpha=p_\alpha} \frac{\delta f^\alpha(y)}{\delta\pi_A(x)} + \frac{\delta H}{\delta\pi_A(x)}\bigg|_{\varphi^\alpha=f^\alpha,\pi_\alpha=p_\alpha} = \frac{\delta H^{\rm red}}{\delta\pi_A(x)} . \quad (3.205)$$

However, by Eq. (3.200)

$$\dot{\pi}_\alpha(x) = \frac{\delta H}{\delta\varphi^\alpha(x)}\bigg|_{\varphi^\alpha=f^\alpha,\pi_\alpha=p_\alpha} = 0 , \quad (3.206)$$

and therefore

$$\frac{\delta H}{\delta\pi_A(x)}\bigg|_{\varphi^\alpha=f^\alpha,\pi_\alpha=p_\alpha} = \frac{\delta H^{\rm red}}{\delta\pi_A(x)} . \quad (3.207)$$

In other words, $H^{\rm red}$ generates the correct equation of motion for ψ^A. In a completely analogous way one shows that the correct equation of motion is also generated for π_A. Although we have here considered pure gravity only, it is clear that our analysis could be easily generalized to include electromagnetic fields as well.

The real problem is: Are the assumptions of the previous theorem valid for Kerr-Newman spacetimes? In other words, is it possible to divide the phase space of Einstein-Maxwell theory with appropriate symmetries in two parts in a manner explained above? For spherically symmetric, asymptotically flat Einstein-Maxwell theory this has been done in this chapter. For theories having the Kerr-Newman solution as their unique solution to the classical constraints this has not been done. However, there is not an obvious reason why this could not be done, and we state the following hypothesis:

Hypothesis 1 *For an appropriately symmetric, asymptotically flat Einstein-Maxwell theory having the Kerr-Newman solution as its unique solution to the Hamiltonian, diffeomorphism and Gaussian constraints, there exists a one-to-one, time independent, differentiable canonical transformation which divides the phase space* $(q_{ab}, p^{ab}, A_a, p^a)$ *into two sets* (M, Q, J, P_M, P_Q, P_J) *and* (ψ^A, P_A) *in such a way that*

a) the reduced Hamiltonian depends only on M*,* Q*,* J*,* P_M*,* P_Q *and* P_J

b) when M*,* Q *and* J *are prescribed as functions* m*,* q*, and* ι *which satisfy*

$$\dot{m} = \dot{q} = \dot{\iota} = 0 , \quad (3.208)$$

then the constraints can be solved to express the P_M*,* P_Q *and* P_J *as functionally differentiable functionals of* ψ^A *and* P_A*.*

We have been unable to find an exact proof of this hypothesis for Kerr-Newmann black hole spacetimes and, indeed, this is the weak point of our model. However, there are no obvious reasons why it would not be true. In what follows, we shall consider our hypothesis as true and see where it takes us.

The first consequence of our hypothesis is that $H^{\rm red}$ of Eq. (3.199) may be used as the real, physical Hamiltonian of our theory, with m, q and ι as the coordinates of the configuration space. For that reason we shall drop "red" from our Hamiltonian, and denote it simply by H.

Our Hamiltonian implies the following canonical equations of motion:

$$\dot{m} = \frac{\partial H}{\partial p_m} = 0 , \tag{3.209}$$

$$\dot{q} = \frac{\partial H}{\partial p_q} = 0 , \tag{3.210}$$

$$\dot{\iota} = \frac{\partial H}{\partial p_\iota} = 0 , \tag{3.211}$$

$$\dot{p}_m = -\frac{\partial H}{\partial m} = -(N_+ + N_-) , \tag{3.212}$$

$$\dot{p}_q = -\frac{\partial H}{\partial q} = -(A_0^+ - A_0^-) , \tag{3.213}$$

$$\dot{p}_\iota = -\frac{\partial H}{\partial \iota} = -(\omega_+ - \omega_-) , \tag{3.214}$$

where p_m, p_q and p_ι, respectively, are the canonical momenta conjugate to m, q and ι. As expected, m, q and ι are constants in time. The time derivative of p_m depends on the choice of the lapse function at both asymptotic infinities of our spacelike hypersurface, $\dot{p}_q$ depends on the difference between electric potentials at infinities, and $\dot{p}_\iota$ depends on the difference between the angular velocities of far-away coordinate systems.

The quantities $N_\pm$, $A_0^\pm$ and $\omega_\pm$ determine the gauge for our theory. For physical reasons, it is sensible to work in a specific gauge, where

$$N_+ \equiv 1 , \tag{3.215}$$

$$N_- \equiv 0 , \tag{3.216}$$

$$\omega_\pm \equiv 0 , \tag{3.217}$$

$$A_0^\pm \equiv 0 . \tag{3.218}$$

In this gauge the Hamiltonian takes a particularly simple form in terms of the canonical coordinates:

$$H = m . \tag{3.219}$$

The physical sense of this kind of a gauge fixing lies in the fact that we consider Kerr-Newman spacetimes from the point of view of a certain specific observer: Our observer is at rest at the right-hand-side asymptotic infinity, and his time coordinate is the asymptotic Minkowski time, the proper time of such an observer. We have "frozen" the time evolution at the left infinity, which is sensible because our observer can make observations from just one infinity. For such an observer, the classical Hamiltonian of the Kerr-Newman spacetime is just M, the ADM mass of the Kerr-Newman black hole.

Now, the problem with the phase space coordinates m, q, ι, p_m, p_q and p_ι is that they describe the *static* aspects of Kerr-Newman spacetimes only. However, there is *dynamics* in Kerr-Newman spacetimes in the sense that between the event horizons there is a region in which it is impossible to find a timelike Killing vector field. Our next task is to find canonical variables describing the dynamical properties of Kerr-Newman black holes in a natural manner.

When choosing the phase space coordinates, we refer to the properties of our observer: Our observer lies at rest very far away from the hole and he is an inertial observer. For such an observer, the Kerr-Newman spacetime appears as stationary, and all the relevant dynamics of the Kerr-Newman spacetime is, in a certain sense, confined inside the event horizon of the hole. These properties prompt us to choose the phase space coordinates in such a manner that when the classical equations of motion are satisfied, all the dynamics is, in a certain sense, confined inside the event horizon $r = r_+$ of the hole. Moreover, as we shall see in a moment, the choice of the phase space coordinates describing the dynamics of spacetime is related to the choice of slicing of spacetime into space and time. We choose a slicing where the proper time of an observer in a free fall through the bifurcation surface coincides with the proper time of our far-away observer at rest. On grounds of the principle of equivalence one may view these types of slicings to be in a preferred position in relating the physical properties of the black hole interior to the physics observed by our far-away observer.

3.4.3 Hamiltonian Dynamics with Charge and Angular Momentum as External Parameters

To make things simple, consider q and ι first as mere external parameters of the theory, having fixed values Q and J, respectively. In that case our phase space is just two-dimensional being spanned by the phase space coordinates m and p_m. In this two-dimensional phase space we perform the following transformation from the old phase space coordinates m and p_m to the new phase space coordinates R and p_R:

$$\begin{aligned} |p_m| &= \sqrt{2mR - R^2 - Q^2 - a^2} + m\sin^{-1}\left(\frac{m-R}{\sqrt{m^2 - Q^2 - a^2}}\right) \\ &\quad + \frac{1}{2}\pi m \,, \qquad (3.220) \\ p_R &= \operatorname{sgn}(p_m)\sqrt{2mR - R^2 - Q^2 - a^2}\;; \qquad (3.221) \end{aligned}$$

and, moreover, we have imposed by hand the restriction

$$-\pi m \le p_m \le \pi m \,. \qquad (3.222)$$

With the restriction (3.222) the transformation given by Eqs. (3.220) and (3.221) is well-defined and one-to-one. It follows from Eq. (3.221) that

$$m = \frac{1}{2R}(p_R^2 + R^2 + Q^2 + a^2) \,. \qquad (3.223)$$

If one substitutes this expression for m into Eq. (3.220), one gets p_m in terms of R and p_R. One finds that the fundamental Poisson brackets between m and p_m are preserved invariant, and hence the transformation given by Eqs. (3.220) and (3.221) is canonical.

Equations (3.219) and (3.223) imply that the classical Hamiltonian takes, in terms of the variables R and p_R, the form

$$H = \frac{1}{2R}\left(p_R^2 + R^2 + Q^2 + a^2\right) \,. \qquad (3.224)$$

The geometrical interpretation of the variable R is extremely interesting. We first write the Hamiltonian equation of motion for R:

$$\dot{R} = \frac{\partial H}{\partial p_R} = \frac{p_R}{R}\,; \qquad (3.225)$$

and it follows from Eq. (3.224) that when the classical equations of motion for m and p_m are satisfied, then the equation of motion for R is

$$\dot{R}^2 = \frac{2m}{R} - 1 - \frac{Q^2 + a^2}{R^2} \ . \tag{3.226}$$

Now, one can see from the Kerr-Newman metric (2.34) that for an observer falling freely through the bifurcation surface at the equatorial plane $\theta = \pi/2$ such that $\dot{\theta} = \dot{\phi} = 0$, the proper time elapsed when r is changed from r to $r + dr$ is $d\tau$ such that

$$-d\tau^2 = \frac{r^2}{r^2 + a^2 + Q^2 - 2Mr} dr^2 \ , \tag{3.227}$$

and therefore the equation of motion of our observer is

$$\dot{r}^2 = \frac{2M}{r} - 1 - \frac{Q^2 + a^2}{r^2} \ , \tag{3.228}$$

where the dot means proper time derivative. As one can see, Eqs. (3.226) and (3.228) are identical. Hence, we may interpret R as the radius of the wormhole throat of the Kerr-Newman black hole, from the point of view of an observer in a free fall at the equatorial plane such that $\dot{\phi} = 0$ through the bifurcation two-sphere. Moreover, one can see from Eq. (3.226) that R is confined to be, classically, within the region $[r_-, r_+]$. In other words, our variable R "lives" only within the inner and outer horizons of the Kerr-Newman black hole, and this is precisely the region in which it is impossible to find a time coordinate such that spacetime with respect to that time coordinate would be static. Hence, both of the requirements we posed for our phase space coordinates are satisfied: Dynamics is confined inside the apparent horizon and the time coordinate on the wormhole throat is the proper time of a freely falling observer.

With the interpretation explained above, the restriction (3.222) becomes understandable. One can see from Eq. (3.117) that when the lapse functions $N_\pm$ at asymptotic infinities are chosen as in Eqs. (3.215) and (3.216), the canonical momentum p_m conjugate to m is $-t + constant$, where t is the time coordinate of our asymptotic observer. Now, the transformation given by Eqs. (3.220) and (3.221) involves an identification of the time coordinate t with the proper time of a freely falling observer on the throat. However, as it was noted at the beginning of this section, it is impossible to push the spacelike hypersurfaces $t = constant$ beoynd the $r = r_-$ hypersurfaces in the conformal diagram. The proper time a freely falling observer needs to fall from the past $r = r_-$ hypersurface to the future $r = r_-$ hypersurface through the bifurcation surface is, as it can be seen from Eq. (3.228),

$$\Delta t = 2 \int_{r_-}^{r_+} \frac{r' dr'}{\sqrt{2Mr' - r'^2 - Q^2 - a^2}} = 2\pi M \ , \tag{3.229}$$

and hence the restriction (3.222) is needed. As one can see from Eq. (3.220) that $|p_m| = 0$ when $R = r_+$ and $|p_m| = \pi m$ when $R = r_-$. We have chosen p_m to be positive when the hypersurface $t = constant$ lies between the past $r = r_-$ hypersurface and the bifurcation surface, and negative when that hypersurface lies between the bifurcation point and the future $r = r_-$ hypersurface.

Concerning the classical Hamiltonian theory with J and Q as mere external parameters the only thing one still needs to check is whether there exist such foliations of the Kerr-Newman spacetime where the Minkowski time at asymptotic infinity and the proper time of a freely falling observer at the throat through the bifurcation surface really are the one and the same time coordinate. In a certain sense, one may view these observers and their time coordinates as physically equivalent. In Refs. [66] and [14] similar identifications are performed and they are based on the Novikov coordinate system (see, for instance, Ref. [29]), where the time coordinate of a given point is given by the proper time τ of a freely falling observer in the Schwarzchild or Reisner-Nordström spacetime through that point, and the radial coordinate R^* in the Novikov coordinate system is related to the point r where the freely falling observer has begun his journey.

Since the R-coordinate in the classical Hamiltonian (3.224) can be geometrically interpreted as the radius of a wormhole throat at the equatorial plane $\theta = \pi/2$ in the Kerr-Newman black hole, we begin the construction of the slicing with desired properties by considering the Kerr-Newman line element (2.34) written in Boyer-Lindquist coordinates at the equatorial plane. The construction of the particular spacetime foliation proceeds exactly like the investigations performed in Sec. 2.4.1. The only major difference now is that the geodesics of the timelike observer are required to be non-rotating in the intermediate and in the exterior regions of the Kerr-Newman black hole. In other words, we choose $\ell = 0$ and $\delta = 1$, and because of these particular choices we get from Eq. (2.52) that

$$2H_{\mathrm{KN}} = 1 = -\frac{r^2}{\Delta}\dot{r}^2 + \frac{R^*}{\sqrt{1+R^{*2}}}\frac{1}{\Delta}\left(\frac{r^4 + 2r^2a^2 - a^2(\Delta - a^2)}{r^2}\frac{R^*}{\sqrt{1+R^{*2}}}\right). \tag{3.230}$$

As we set $\dot{r} = 0$, Eq. (3.230) yields us a quartic equation for r:

$$\Delta r^2 = \frac{R^{*2}}{1+R^{*2}}\left[r^4 + 2r^2a^2 - a^2(\Delta - a^2)\right] . \tag{3.231}$$

From this equation one can calculate the r-coordinate r_{max} of the point from which an observer in a free fall begins his journey, in terms of R^* which henceforth will be used as a radial coordinate of Kerr-Newman spacetime. Eq. (3.230) implies an implicit expression $r(\tau, R^*)$ for the old radial coordinate r in terms of the new time coordinate τ and the new radial coordinate R^*:

$$\tau = \pm\sqrt{1+R^{*2}}\int_{r_{max}(R^*)}^{r(\tau,R^*)} \frac{r'^2}{\sqrt{R^{*2}(r'^2+a^2)(2Mr'-Q^2) - r'^2\Delta}}\,dr' . \tag{3.232}$$

In this equation the signs + and −, respectively, correspond to the past and the future of the line where the time coordinate t is zero in the conformal diagram. To obtain an explicit expression $r(\tau, R^*)$ for r one should first solve the quartic equation (3.231), and then perform the integration in Eq. (3.232). Solving Eq. (3.231), however, would yield a tremendously complicated expression for r_{max}, and we shall not write it down here. However, it is easy to see that there are always at least two positive roots $r = r_{max} = r_{max}(R^*)$. This can be seen by plotting the both sides of Eq. (3.231) and varying R^*.

Moreover, one finds that if one puts $r = r_{max} = r_+$ then Eq. (3.231) implies $R^* = 0$, and vice versa: if one sets $R^* = 0$, then Eq. (3.231) is solved by $r = r_+$. Hence, we have found that for every $R^* \geq 0$ there is an observer in free fall such that this observer is at rest at the time $t = \tau = 0$ with respect to the old radial coordinate r. When $R^* = 0$ our observer begins his journey at the bifurcation surface and his world line is a straight vertical line in the conformal diagram.

Can we extend this coordinate transformation to the right-hand-side asymptotic infinity? Yes we can, since we may choose the coordinate R^* such that the solution $r = r_{max} > r_+$ is the largest of the roots of Eq. (3.231). When this choice is made, one can show starting from Eq. (3.231), that for large r_{max}

$$R^{*2} \sim -\left(1 + \frac{r_{max}^2 + a^2}{Q^2 - 2Mr_{max}}\right) + \mathcal{O}(r_{max}^{-3}) \,. \tag{3.233}$$

Hence R^* goes to infinity as r_{max} goes to infinity and vice versa. Moreover, the time coordinate τ of an observer at the asymptotic infinity coincides with the proper time τ of a freely falling observer at the wormhole throat.

Another issue to be investigated still is that do the observers rotate or not with respect to the Boyer-Lindquist coordinates? We wrote our Hamiltonian from the point of view of an asymptotic non-rotating observer, and we assumed a foliation in which the time coordinate at the throat is a proper time of a non-rotating observer in a free fall. To show that in our foliation both of the observers are non-rotating we must show that $\dot{\phi} \to 0$ as $r \to r_+$ and $r \to \infty$. The latter case is straightforward, since in the expression

$$\dot{\phi} = \frac{1}{\Delta}\frac{a(r^2 + a^2 - \Delta)}{r^2}\frac{R^*}{\sqrt{1 + R^{*2}}} \,, \tag{3.234}$$

given by Eq. (2.51) when $\ell = 0$, the factor $R^*/\sqrt{1 + R^{*2}}$ approaches unity and the factor in front of it approaches zero. The first case, where $r \to r_+$, is a bit tricky, since we do not know the explicit relation between r and R^* at the bifurcation point. We have solved the tricky part by expanding Eq. (3.231) in terms of r near the bifurcation point. If we take only the zeroth and the first order terms, we find that the point $r = r_{max}$ where $\dot{r} = 0$ is related to R^* by an expression

$$r_{max} \approx \frac{r_+^5\left(2 - \frac{R^{*2}}{1+R^{*2}}\right) - 2Mr_+^4 + a^2 r_+^3 \frac{R^{*2}}{1+R^{*2}} + 4Ma^2 r_+^2 \frac{R^{*2}}{1+R^{*2}} - 3a^2q^2 r_+ \frac{R^{*2}}{1+R^{*2}}}{2r_+^4\left(1 - \frac{R^{*2}}{1+R^{*2}}\right) - 2Mr_+^3 + 2Ma^2 r_+ \frac{R^{*2}}{1+R^{*2}} - 2a^2q^2 \frac{R^{*2}}{1+R^{*2}}} \,, \tag{3.235}$$

which gives that $r_{max} = r_+$ as $R^* \longrightarrow 0$, as it should. Now, when Eq. (3.235) is substituted into Eq. (3.234), and letting $R^* \to 0$, one gets the result

$$\dot{\phi} \longrightarrow 0 \,. \tag{3.236}$$

In other words, we have managed to construct a foliation of Kerr-Newman spacetime with desired properties at the equatorial plane: At the asymptotic infinity the time coordinate is the proper time of a freely falling, non-rotating observer at rest, and at the wormhole throat of a similar non-rotating observer in a radial free fall through the bifurcation surface.

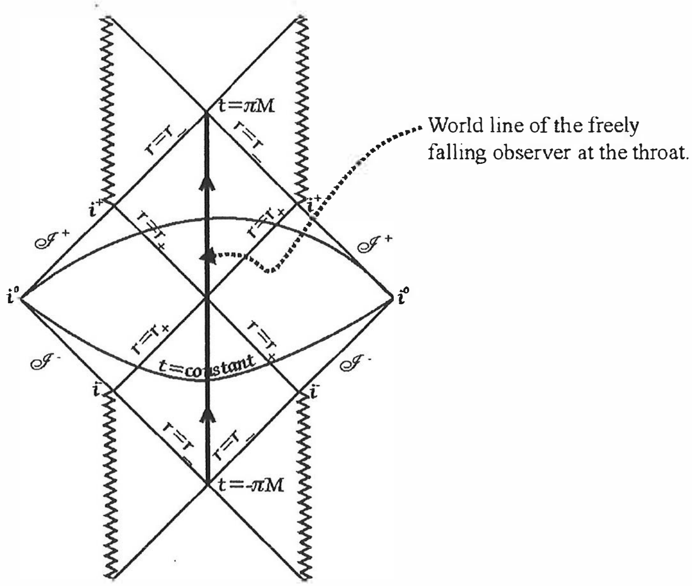

Figure 3.3: The world line of an observer in free fall at the throat is a vertical line going through the bifurcation point in the conformal diagram. The proper time of such an observer is identified with the asymptotic Minkowski time.

It is even possible to show that our construction gives the Novikov coordinate system in the Schwarzschild spacetime when one sets $q = a = 0$ in Eq. (3.231). This result is given by Eqs. (3.231) and (3.232). We get an analogous coordinate system for the Reissner-Nordström spacetime when $a = 0$. It can be shown that then the relation between r_{max} and R^* is

$$r = r_{max} = \left(M + \sqrt{M^2 - q^2(1 + R^{*2})^{-1}}\right)(1 + R^{*2}) . \tag{3.237}$$

It should be noted, however, that all foliations in which the proper time on the throat and the asymptotic Minkowski time are identified are incomplete since such foliations, in addition to failing to cover the regions outside the past and the future $r = r_-$ hypersurfaces, also fail to cover the exterior regions of the hole. More precisely, these foliations are valid only when $-\pi M \leq t \leq \pi M$ (see Fig. 6.2).

3.4.4 Hamiltonian Dynamics with Charge and Angular Momentum as Dynamical Variables

The next task is to complete the classical Hamiltonian (3.224) such that q and $\tilde{a} := \iota/m$ are replaced by functions of appropriate phase space variables describing the dynamics of Kerr-Newman spacetimes in a natural manner. To this end, we must find, for constant M, a canonical transformation from the phase space coordinates (q, p_q) and (ι, p_ι) to some new phase space coordinates which we shall denote by u and v, and their canonical momenta p_u and p_v.

We shall perform such a transformation in two steps. At the first stage we replace Q and a by canonical momenta conjugate to yet some unknown coordinates w_1 and w_2 of the configuration space:

$$p_{w_1} := q \, , \tag{3.238}$$

$$p_{w_2} := \bar{a} \, . \tag{3.239}$$

Then the classical Hamiltonian of Eq. (3.224) takes the form

$$H = \frac{1}{2R}\left(p_R^2 + p_{w_1}^2 + p_{w_2}^2 + R^2\right) . \tag{3.240}$$

The next task is to find w_1 and w_2. One expects that w_1 and w_2 are related in one way or another to the momenta p_q and p_ι conjugate to q and ι, respectively. Because we see from Eq. (3.213) that p_q determines the electromagnetic gauge and from Eq. (3.214) that p_ι determines the angular velocity of far-away coordinate systems we first write the classical Hamiltonian in a general electromagnetic gauge when far-away coordinate systems rotate with arbitrary angular velocities:

$$H = \frac{1}{2R}\left(p_R^2 + p_{w_1}^2 + p_{w_2}^2 + R^2\right) + (A_0^+ - A_0^-)p_{w_1} + m(\omega_+ - \omega_-)p_{w_2} \, , \tag{3.241}$$

which follows from Eq. (3.199). Using Eqs. (3.213) and (3.214) and the fact that m is a constant when the classical equations of motion are satisfied, we get for the Hamiltonian equations of motion for w_1 and w_2

$$\dot{w}_1 := \frac{\partial H}{\partial p_{w_1}} = \frac{p_{w_1}}{R} - \dot{p}_q \, , \tag{3.242}$$

$$\dot{w}_2 := \frac{\partial H}{\partial p_{w_2}} = \frac{p_{w_2}}{R} - m\dot{p}_\iota \, . \tag{3.243}$$

An expression for p_q and p_ι in terms of R, p_R, w_1, w_2, p_{w_1} and p_{w_2} can be gained by integrating both sides of Eqs. (3.242) and (3.243) along the classical trajectory in phase space:

$$p_q := \int \frac{p_{w_1}}{R\dot{R}} dR - w_1 \, , \tag{3.244}$$

$$p_\iota := \frac{1}{m}\int \frac{p_{w_2}}{R\dot{R}} dR - w_2 \, , \tag{3.245}$$

where we have substituted

$$\dot{R} = -\mathrm{sgn}(p_m)\sqrt{\frac{2M}{R} - 1 - \frac{p_{w_1}^2 + p_{w_2}^2}{R}} \, . \tag{3.246}$$

This substitution involves choosing $\dot{p}_q = \dot{p}_\iota = 0$. When the electric potentials are assumed to vanish at infinities, and the asymptotic coordinate systems are assumed to be non-rotating, this kind of a choice can be made. With an appropriate choice of the integration constant we get

$$p_q = \mathrm{sgn}(p_m)p_{w_1} \times \left[\sin^{-1}\left(\frac{p_R^2 + p_{w_1}^2 + p_{w_2}^2 - R^2}{\sqrt{\left(p_R^2 + p_{w_1}^2 + p_{w_2}^2 + R^2\right)^2 - 4R^2\left(p_{w_1}^2 + p_{w_2}^2\right)}}\right) + \frac{\pi}{2}\right]$$

$$
\begin{aligned}
& -w_1 \ , & (3.247) \\
p_\iota & = \mathrm{sgn}(p_m) \frac{2Rp_{w_2}}{p_R^2 + p_{w_1}^2 + p_{w_2}^2 - R^2} \\
& \times \left[\sin^{-1} \left(\frac{p_R^2 + p_{w_1}^2 + p_{w_2}^2 - R^2}{\sqrt{\left(p_R^2 + p_{w_1}^2 + p_{w_2}^2 + R^2\right)^2 - 4R^2\left(p_{w_1}^2 + p_{w_2}^2\right)}} \right) + \frac{\pi}{2} \right] \\
& -w_2 \ , & (3.248)
\end{aligned}
$$

where we have made the substitution

$$
m = \frac{1}{2R}\left(p_R^2 + p_{w_1}^2 + p_{w_2}^2 + R^2\right) \ . \tag{3.249}
$$

Equations (3.221), (3.238), (3.239), (3.247) and (3.248) constitute a transformation from the phase space coordinates m, p_m, q, p_q, ι and p_ι to the phase space coordinates R, p_R, w_1, p_{w_1}, w_2, and p_{w_2}. One can easily show that this transformation is well-defined and canonical. Moreover, the transformation is one-to-one provided that we impose the restrictions

$$
\left| \frac{p_q + w_1}{p_{w_1}} \right| \leq \pi \ , \tag{3.250}
$$

$$
\left| \frac{mp_\iota - w_2}{p_{w_2}} \right| \leq \pi \ . \tag{3.251}
$$

These restrictions are related to the fact that we are considering the spacetime between two successive $r = r_-$ hypersurfaces. Since both $\dot{p}_q$ and $\dot{p}_\iota$ vanish when the electric potentials are assumed to vanish at asymptotic infinities and the asymptotic coordinate systems are assumed to be non-rotating, we find that classically w_1 and w_2 have the following properties: At the past $r = r_-$ hypersurface $w_1 = -q\pi + p_q$ and $w_2 = -\tilde{a}\pi + mp_\iota$, at the bifurcation surface $w_1 = p_q$ and $w_2 = mp_\iota$, and at the future $r = r_-$ hypersurface $w_1 = q\pi + p_q$, and $w_2 = \tilde{a}\pi + mp_\iota$. In other words, the classical domains of w_1 and w_2 are bounded by the fact that the $t = constant$ hypersurfaces cannot be pushed beyond the $r = r_-$ hypersurfaces.

As the last step we perform a canonical transformation from the variables w_1, p_{w_1}, w_2 and p_{w_2} to the variables u, p_u, v and p_v [1]. We define

$$
\begin{aligned}
u & := p_{w_1} \sin\left(\frac{w_1}{p_{w_1}}\right) \ , & (3.252) \\
p_u & := p_{w_1} \cos\left(\frac{w_1}{p_{w_1}}\right) \ , & (3.253) \\
v & := p_{w_2} \sin\left(\frac{w_2}{p_{w_2}}\right) \ , & (3.254) \\
p_v & := p_{w_2} \cos\left(\frac{w_2}{p_{w_2}}\right) \ . & (3.255)
\end{aligned}
$$

This transformation is well-defined, canonical and, with the restrictions (3.250) and (3.251), one-to-one as well. We find that

$$
\begin{aligned}
p_{w_1}^2 & = p_u^2 + u^2 \ , & (3.256) \\
p_{w_2}^2 & := p_v^2 + v^2 \ . & (3.257)
\end{aligned}
$$

[1] u and v should not be confused with light cone coordinates or anything like that!

In other words, we may identify $p_u^2 + u^2$ as the square of the electric charge q, and $p_v^2 + v^2$ as the square of the angular momentum per unit mass of the hole $\tilde{a}$. Because of that, the classical Hamiltonian of Kerr-Newman black holes finally takes a very simple form

$$H = \frac{1}{2R}\left(p_R^2 + p_u^2 + p_v^2 + R^2 + u^2 + v^2\right) . \tag{3.258}$$

3.5 Hamiltonian Dynamics of Schwarzschild Spacetimes

In this chapter we shall give a short review of the classical Hamiltonian theory of spacetimes containing a Schwarzschild black hole. The Hamiltonian dynamics of spherically symmetric vacuum spacetimes can be obtained from the Hamiltonian dynamics of the corresponding electrovacuum spacetimes in the absence of the electromagnetic field. In other words, the results of this section can be obtained from the results of Sec. 3.3 by setting the electromagnetic field to zero. Originally, the Hamiltonian dynamics of spherically symmetric vacuum spacetimes was investigated, among others, by K. Kuchař [61].

The only spherically symmetric, asymptotically flat vacuum solution to Einstein's field equations is the Schwarzschild solution. When the spacelike hypersurfaces, where $t = constant$, are chosen to go from the left- to the right-hand asymptotic infinities in the Kruskal diagram, crossing both horizons, Kuchař [61] found that the total action of spherically symmetric vacuum spacetimes containing a primordial black hole takes the form

$$\begin{aligned} S[\Lambda, P_\Lambda, R, P_R; N, N^r] &= \int dt \int_{-\infty}^{\infty} dr\left(P_\Lambda\dot{\Lambda} + P_R\dot{R} - N\mathcal{H} - N^r\mathcal{H}_r\right) \\ &\quad - \int dt\,(N_+M_+ + N_-M_-) \quad , \end{aligned} \tag{3.259}$$

where Λ, P_Λ, R and P_R are the configuration coordinates and the corresponding canonical momenta (compare to Eqs. (3.36), (3.46) and (3.47)). N and N^r are the lapse function and the radial component of the shift vector, respectively. $\mathcal{H}$ and $\mathcal{H}_r$ give the Hamiltonian and the diffeomorphism constraints in the spherically symmetric vacuum spacetimes. N_+ and N_- are the fixed lapse functions at the infinities, and M_- and M_+ are the ADM energies of spherically symmetric vacuum spacetimes. Note that this action is, indeed, a special case of the action (3.76) in the absence of the electromagnetic field.

When one follows the procedure given in Sec. 3.3 by reconstructing of the mass and time, and performing the appropriate canonical transformation from the old phase space $\Lambda, P_\Lambda, R, P_R$ to the new phase space $M, P_M, \mathsf{R}, P_\mathsf{R}$ and solving the constraints, only two canonical degrees of freedom are left. If these two degrees of freedom are chosen to be the Schwarzschild mass m, and its conjugate momentum p_m, the classical action of the system is

$$S_\mathrm{K} = \int dt\,[p_m\dot{m} - m\,(N_+ + N_-)] \quad , \tag{3.260}$$

which is given by Eq. (3.111) when the electromagnetic degrees of freedom are not present. The classical Hamiltonian of the whole maximally extended Schwarzschild black hole spacetime found by Kuchař can therefore be written in terms of the two physical phase space coordinates m and p_m

$$H_{\text{whole}} = m\,(N_+ + N_-) \quad . \tag{3.261}$$

Classically, H_{whole} may be understood as the total energy of the whole spacetime. To choose a specific observer, who measures the energy of the gravitational field, we fix the values of the lapse functions at asymptotic infinities. From the point of view of an observer at the right-hand-side infinity at rest with respect to the hole, we can set $N_- = 0$ and $N_+ = 1$. On the other hand, one may view the Schwarzschild mass m as the total energy of the Schwarzschild spacetime, measured by a distant observer. Hence, we may write $H_{\text{whole}} = m$.

The classical Hamiltonian theory of the right-hand-side *exterior* region of the Schwarzschild black hole was investigated by Louko and Whiting [67]. Louko and Whiting considered a spacetime foliation where the spacelike hypersurfaces begin from the bifurcation two-sphere, and end at a right-hand-side timelike three-surface, i.e. at a "box wall" in the Kruskal diagram. With this choice, the spatial slices are entirely contained within the right-hand-side exterior region of the Kruskal spacetime. One of the main observations, and the only difference with the approach of Sec. 3.3, was that such foliations bring along an additional boundary term into the classical action. Hence, the Louko-Whiting boundary action $S_{\partial\Sigma}$ consists of terms resulting from the initial and the final spatial surfaces, that is, from the bifurcation two-sphere and from the "box wall". After solving the classical constraints, Louko and Whiting found that when the physical degrees of freedom are identified, the true Hamiltonian action is

$$S_{\text{LW}} = \int dt\,(p_m \dot{m} - h(t)) \quad , \tag{3.262}$$

where $h(t)$ is the reduced Hamiltonian such that, when the radius of the initial boundary two-sphere does not change in time t. The Hamiltonian $h(t)$ is defined as

$$h(t) := \left(1 - \sqrt{1 - \frac{2m}{R}}\right) R\sqrt{-g_{tt}} - 2N_0(t)m^2 \quad , \tag{3.263}$$

where R is the time independent value of the radial coordinate of general spherically symmetric, asymptotically flat vacuum spacetime at the final timelike boundary, i.e., at the "box wall". Here g_{tt} is the $tt-$component of the metric tensor expressed as a function of the canonical variables after performing a canonical transformation and of Lagrange's multipliers. N_0 is a function of the global time t at the bifurcation two-sphere such that

$$\Theta := \int_{t_1}^{t_2} dt N_0(t) \tag{3.264}$$

is the boost parameter elapsed at the bifurcation two-sphere during the time interval $[t_1, t_2]$. Details can be seen in Ref. [67].

It is easy to see that if one transfers the "box wall" to the asymptotic infinity by taking the limit $R \to \infty$, the Hamiltonian $h(t)$ of Eq. (3.263) reduces to the the classical Hamiltonian

$$H_{\text{ext}} = mN_{+} - \frac{1}{2}R_h^2 N_0 \quad , \tag{3.265}$$

where $R_h = 2m$ is the Schwarzschild radius, and, as before, N_{+} is the lapse function at the right-hand-side asymptotic infinity. Classically, the Hamiltonian H_{ext} describes the exterior region of the Schwarzschild black hole spacetime and may be interpreted, in a certain foliation, as the total energy of the exterior region of the hole. Another interpretation has been suggested by Bose et al. [68] H_{ext}. According to those authors H_{ext} is the free energy of the whole black hole spacetime.

Part II

Quantum Mechanics of Black Holes

Chapter 4

Semiclassical Results: Hawking Radiation and Black Hole Entropy

4.1 Brief Introduction to Semiclassical General Relativity

In semiclassical general relativity one investigates the properties of quantized matter fields in curved spacetime geometry which is kept as a non-dynamical background. Since the spacetime geometry is considered as a rigid background, the quantum effects of the spacetime itself are assumed to be unimportant for the quantum phenomena taking place in spacetime. However, in the full theory of quantum gravity one should take into account the quantized gravitational field and its effects as well. The semiclassical theory of quantized matter fields in general relativity is called *quantum field theory in curved spacetime* [69, 70, 71]. This theory is extensively used to investigate the influence of the classical gravitational field on quantized matter fields.

The exact region of validity of semiclassical gravity is not known until the complete theory of quantized gravitational field is available. However, when one treats the gravitational field as a small perturbation in a flat spacetime background and tries to quantize it in a usual manner, one notices that the gravitational effects become very significant at the distances of the order of the *Planck length* $l_{\text{Pl}} := (G\hbar/c^3)^{1/2} \sim 10^{-35}$ m. Therefore, one suspects that the semiclassical approximation is valid as far as to the Planck length scale. On the other hand, it is possible to combine the universal constants G, $\hbar$ and c such that they produce the so-called *Planck time* $t_{\text{Pl}} := (G\hbar/c^5)^{1/2} \sim 10^{-43}$ s. For this reason, strong gravitational effects are believed to occur also when the gravitational field suffers rapid time-dependent changes taking place in the Planck timescale. Thus one expects that semiclassical approximation fails only when we consider either microscopic black holes or the very early epoch of the universe.

4.2 Hawking Effect

By using the methods of quantum field theory in curved spacetime it was found in 1974 by Hawking [17] that black holes are not black, but they emit radiation with a perfect thermal spectrum of a black body. Hawking's original analysis [72, 73] considered quantized real-valued scalar fields in a spacetime geometry produced by a collapsing spherically symmetric mass distribution in vacuum. In this section we follow Hawking's treatment, and we calculate the particle production caused by a continously changing gravitational field surrounding collapsing matter. For a good review of the field of particle production in curved spacetime see Ref. [74] and references quoted therein. Essential features of the Hawking effect are briefly discribed in Ref. [75, 76]

It is known that the unique spacetime metric outside the collapsing spherically symmetric mass distribution is the Schwarzschild metric. The Schwarzschild spacetime, in turn, is asymptotically flat and stationary. Therefore, far from the surface of the collapsing matter distribution spacetime can be regarded as the Minkowski spacetime with well-defined concepts of energy and particles. In Minkowski spacetime one can construct a unique vacuum of particles. As the matter distribution collapses the spacetime geometry changes and the quantum vacuum of the resulting spacetime geometry no longer corresponds to the particle vacuum in Minkowski spacetime. As the spacetime geometry changes the changes in the resulting quantum vacua produce particles.

4.2.1 Quantum Field Theory in Curved Spacetime

The equation of motion of a real scalar field $\phi(x)$ in a flat Minkowski spacetime with signature $(+,-,-,-)$ is the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0 \; , \tag{4.1}$$

which in a curved spacetime takes the form

$$\left(g^{\mu\nu} D_\mu D_\nu + m^2\right)\phi = 0 \; , \tag{4.2}$$

where D_μ is the covariant derivative compatible with the metric $g_{\mu\nu}$ and m is the mass of the field quantum. If the spacetime is taken to be globally hyperbolic then the Klein-Gordon equation (4.2) has a well-posed initial-value formulation. The *inner product* between any two solutions ϕ_1 and ϕ_2 of the Klein-Gordon equation (4.2) is

$$\langle\phi_1|\phi_2\rangle := -i \int_\Sigma d\Sigma^\mu \left[\phi_1 \left(\partial_\mu \phi_2^*\right) - \left(\partial_\mu \phi_1\right)\phi_2^*\right] \; , \tag{4.3}$$

where * denotes the complex conjugate, $d\Sigma^\mu := n^\mu d\Sigma$ such that n^μ is a future-directed timelike unit vector orthogonal to the spacelike hypersurface Σ and it means the volume element on the hypersurface Σ. The hypersurface Σ is taken to be a Cauchy surface. It is possible to show that the value of the inner product (4.3) is independent of the the choice of the hypersurface Σ, and therefore the solutions u_i and their complex conjugates must satisfy

$$\langle u_i|u_j\rangle \;=\; \delta_{ij} \; , \tag{4.4}$$

$$\langle u_i^* | u_j^* \rangle = -\delta_{ij} , \qquad (4.5)$$
$$\langle u_i | u_j^* \rangle = 0 . \qquad (4.6)$$

The solutions u_i are called *orthonormal wavemodes.*

A general solution to the classical Klein-Gordon equation (4.2) can be written in the form of the *Fourier series*:

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right] , \qquad (4.7)$$

where a_i and $a_i^\dagger \in \mathbb{C}$ are the *Fourier coefficients* which can be obtained from

$$a_i = \langle u_i | \phi \rangle , \qquad (4.8)$$
$$a_i^\dagger = -\langle u_i^* | \phi \rangle . \qquad (4.9)$$

The covariant quantization procedure in curved spacetime is based on the canonical commutation relations between the quantized field operator $\hat{\phi}$ and its canonical momentum operator $\hat{p}$ conjugate to the field on the Cauchy surface Σ such that

$$\left[\hat{\phi}(\Sigma, P), \hat{p}(\Sigma, P')\right] = i\hbar\delta^3(P, P') , \qquad (4.10)$$
$$\left[\hat{\phi}(\Sigma, P), \hat{\phi}(\Sigma, P')\right] = 0 , \qquad (4.11)$$
$$\left[\hat{p}(\Sigma, P), \hat{p}(\Sigma, P')\right] = 0 , \qquad (4.12)$$

where P and $P' \in \Sigma$, and the classical canonical momentum p is related to the field configuration variable ϕ by the definition

$$p := \frac{\partial \mathcal{L}}{\partial(n^\mu \partial_\mu \phi)} , \qquad (4.13)$$

where the classical Lagrangian density $\mathcal{L}$ for the real scalar field ϕ in curved spacetime can be chosen to be

$$\mathcal{L} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \left(\partial_\mu \phi \partial_\mu \phi - m^2 \phi \right) . \qquad (4.14)$$

When the scalar field expansion (4.7) is replaced by its operator counterpart

$$\hat{\phi}(x) = \sum_i \left[\hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^*(x) \right] , \qquad (4.15)$$

the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ satisfy commutation relations

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = i\hbar\delta_{ij} , \qquad (4.16)$$
$$\left[\hat{a}_i, \hat{a}_j\right] = 0 , \qquad (4.17)$$
$$\left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0 , \qquad (4.18)$$

which are equivalent to (4.12).

From the quantum field theory in flat Minkowski spacetime we know how to construct normalized basis vectors from the standard Minkowski space vacuum state $|0\rangle$ by using the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$. In flat spacetime the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ annihilate and create particles. In curved space on a Cauchy surface the construction of the Hilbert space proceeds exactly in the same manner as in the Minkowski spacetime: One just operates by the linear operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ to a vacuum state $|0\rangle$ for all i. However, in flat and curved spacetimes there is one major difference in the interpretation of the basis states of the system. While in the flat spacetime the basis states can be interpreted as the particle or many-particle states, the basis states do not necessarily have such an interpretation in curved spacetime. This difference is a manifestation of missing the notion of global inertial observer in curved spacetime, whereas Minkowski spacetime has global inertial observers that are related to each other by Poincarè transformations. Poincarè symmetry of the global inertial observers ensures the existence of a unique vacuum state and a unique definition of energy. As the curved spacetime has no global inertial observers, it has no Poincarè symmetry, and therefore the notions of energy vacuum cannot be defined uniquely, but they depend on the observer and on the geometry of spacetime. For this reason the basis states in curved spacetime cannot be necessarily interpreted as particle states. However, asymptotically flat and stationary spacetimes have the standard particle interpretation in the asymptotic regions, since asymptotically flat regions possess Poincarè symmetry, and the stationary spacetime, in turn, has a time coordinate in which the spacetime geometry is constant.

Because of the non-unique vacuum state in curved spacetime we shall define the so-called *Bogolubov transformations* between any two complete orthonormal sets of solutions $\{u_j : j \in \mathcal{I}\}$ and $\{u'_j : j \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set. In terms of the primed basis, the Klein-Gordon field operator $\hat{\phi}$ can be expanded as

$$\hat{\phi}(x) = \sum_i \left[\hat{a}'_i u'_i(x) + \hat{a}'^\dagger_i u'^*_i(x)\right] , \tag{4.19}$$

and therefore we may define a new vacuum $|0'\rangle$ and a new Fock space. The Bogolubov transformations between the two sets are

$$u'_j = \sum_i \left(A_{ji} u_i + B_{ji} u^*_i\right) , \tag{4.20}$$

$$u_j = \sum_j \left(A^*_{ji} u'_j - B_{ji} u'^*_j\right) . \tag{4.21}$$

The numbers A_{ji} and B_{ji} are complex-valued and they are called *Bogolubov coefficients.*

As we transform from one complete set of modes $\{u_j : j \in \mathcal{I}\}$ to another complete set $\{u'_j : j \in \mathcal{I}\}$, the annihilation and creation operators $\hat{a}'_j$ and $\hat{a}'^\dagger_j$ must change also. The relations between the two sets of annihilation and creation operators are

$$\hat{a}_j = \sum_i \left(A_{ij} \hat{a}'_i + B^*_{ij} \hat{a}'^\dagger_i\right) , \tag{4.22}$$

$$\hat{a}'_j = \sum_i \left(A^*_{ji} \hat{a}_i - B^*_{ji} \hat{a}^\dagger_i\right) . \tag{4.23}$$

Therefore the two Fock spaces related to the two different sets of modes are clearly different only if $B_{ij} \neq 0$. This can be seen from Eq. (4.22).

By the definition, the operator $\hat{a}_j$ annihilates the vacuum state $|0\rangle$ and, correspondingly, the operator $\hat{a}'_j$ annihilates the vacuum state $|0'\rangle$, but for example the operator $\hat{a}'_j$ will not annihilate the vacuum $|0\rangle$:

$$\hat{a}'_j|0\rangle = \sum_i \left(A^*_{ji}\hat{a}_i - B^*_{ji}\hat{a}^\dagger_i\right)|0\rangle = \sum_i A^*_{ji}\hat{a}_i|0\rangle - \sum_i B^*_{ji}\hat{a}^\dagger_i|0\rangle \quad (4.24)$$

$$= -\sum_i B^*_{ji}|1_i\rangle \neq 0 \ . \quad (4.25)$$

As the vacuum state $|0\rangle$ defined at the asymptotically Minkowski spacetime changes to another vacuum state $|0\rangle$ at the asymptotic infinity, it is easy to show that the number of particles in the state corresponding to the mode j is given by the expectation value of the operator $\hat{N}_j := \hat{a}^\dagger_j\hat{a}_j$. The number of u_j-mode particles in the state vacuum $|0'\rangle$ is

$$n_j = \langle 0'|\hat{N}_j|0'\rangle = \langle 0'|\hat{a}^\dagger_j\hat{a}_j|0'\rangle = \sum_i B^*_{ij}B_{ij} \ . \quad (4.26)$$

4.2.2 Quantum Scalar Field in a Collapsing Spacetime Geometry

The orthonormal set of modes change circumstances that are very natural. For example, consider the spacetime geometry during the evolution of the gravitational collapse of a spherically symmetric mass distribution. Initially, the spacetime geometry at the locus of the far-away observer is stationary and asymptotically flat, and hence, according to Birkhoff's theorem, the geometry is uniquely given by the Schwarzschild line element (2.3). During the collapse, the interior geometry of the collapsing mass distribution suffers from continuous changes, and these changes, in turn, alter the solutions to the Klein-Gordon equation (4.2). Finally, when the late time solutions of the Klein-Gordon field are once more considered far-away from the collapsing mass distribution the spacetime geometry may be taken stationary. In this manner the spacetime geometry changes from a stationary state to another. During some time interval, the ingoing solutions to the Klein-Gordon equation travel through the continuously changing interior geometry, and therefore the vacuum state of the outgoing solutions is different from the ingoing vacuum state.

Our next task is to reproduce the Bogolubov coefficients between the ingoing and outgoing vacuum states and calculate the particle production of the Schwarzschild black hole. As the particle production seems to be involved in the details of the collapse, one might expect black hole radiation to be related to the collapse only. However, as the time coordinate used in the analysis is the Schwarzschild time coordinate, we know that the particle production is due to the presence of the event horizon and is independent of the details of the collapse, since otherwise it would take an infinite amount of time to escape to infinity. Further research has shown that the particles are created by eternal black holes as well.

Let us consider the Klein-Gordon equation for the massless particles ($m = 0$) in the Schwarzschild spacetime geometry. As we write the positive frequency $\omega > 0$ mode solutions u in the Schwarzschild coordinates in the form

$$u_{\omega l m_l}(t, r, \theta, \phi) = \frac{1}{r}\rho(r)Y_{lm_l}(\theta, \phi)\exp(-i\omega t) \ , \quad (4.27)$$

where Y_{lm_l} is a spherical harmonic, the Klein-Gordon equation (4.2) can be separated and the radial function $\rho(r)$ satisfies the radial equation

$$\left[-\frac{\partial^2}{\partial r_*^2} + V(r)\right]\rho(r) = \omega^2\rho(r) , \tag{4.28}$$

where

$$V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right] \tag{4.29}$$

and r_* is the Regge-Wheeler tortoise coordinate defined in Eq. (2.5).

Because the 'potential' $V(r)$ vanishes at the event horizon $r = R_s$ and in the asymptotic region $r \longrightarrow \infty$, the ingoing and the outgoing positive frequency mode solutions $u^{\text{in}}_{\omega l m_l}$ and $u^{\text{out}}_{\omega l m_l}$ right in the vicinity of the event horizon and in the asymptotic region are

$$\begin{aligned} u^{\text{in}}_{\omega l m_l} &= N_{\omega l m_l} Y_{l m_l} \frac{1}{r} \exp(-i\omega\mathcal{V}) , &(4.30)\\ u^{\text{out}}_{\omega l m_l} &= N_{\omega l m_l} Y_{l m_l} \frac{1}{r} \exp(-i\omega\mathcal{U}) , &(4.31)\end{aligned}$$

where $\mathcal{U}, r$ and $\mathcal{V}, r$ are the Eddington-Finkelstein coordinates defined in Eqs. (2.6) and (2.7), and $N_{\omega l m_l}$ is a normalization constant. Note that the solutions near the event horizon and in the asymptotic infinity are the same, because we have already chosen $m = 0$.

In order to see that these mode solutions describe particles moving along lightlike geodesics, we perform a coordinate transformation from the Kruskal coordinates u, v (definition given in Eqs. (2.10) and (2.11)) to a set of null coordinates $\tilde{u}, \tilde{v}$ such that

$$\begin{aligned} v &= \frac{1}{2}(\tilde{v} + \tilde{u}) , &(4.32)\\ u &= \frac{1}{2}(\tilde{v} - \tilde{u}) . &(4.33)\end{aligned}$$

In terms of the null coordinates $\tilde{u}, \tilde{v}$ the Schwarzschild line element takes the form

$$ds^2 = -\frac{32M^3\exp(-r/2M)}{r} d\tilde{v}d\tilde{u} + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{4.34}$$

which tells us that in the vicinity of the event horizon the coordinates $\tilde{u}$ and $\tilde{v}$ give affine parameters to the lightlike geodesics. Moreover, at the horizon, where $v = u$, is $\tilde{u}$ zero, and at the horizon, where $v = -u$, is $\tilde{v}$ zero.

In the exterior spacetime region the null coordinates $\tilde{u}$ and $\tilde{v}$ are related to Eddington-Finkelstein coordinates $\mathcal{U}$ and $\mathcal{V}$ by

$$\begin{aligned} \tilde{v} &= \exp\left(\frac{\mathcal{V}}{4M}\right) , &(4.35)\\ \tilde{u} &= -\exp\left(\frac{-\mathcal{U}}{4M}\right) , &(4.36)\end{aligned}$$

and therefore the outgoing and the ingoing mode solutions (4.31) and (4.30) to the Klein-Gordon equation, in terms of $\tilde{v}$ and $\tilde{u}$, take in the vicinity of the event horizon the form

$$\begin{aligned} u^{\text{out}}_{\omega l m_l} &= N_{\omega l m_l} Y_{l m_l} \frac{1}{r} \exp(-i4M\omega\log(-\tilde{u})) , &(4.37)\\ u^{\text{in}}_{\omega l m_l} &= N_{\omega l m_l} Y_{l m_l} \frac{1}{r} \exp(-i4M\omega\log(\tilde{v})) . &(4.38)\end{aligned}$$

As the null coordinates $\tilde{u}$ and $\tilde{v}$ approach zero at the event horizon, the outgoing mode solution suffers an infinite decrease in frequency, i.e., infinitely increasing redshift whereas the ingoing solution suffers an infinite increase of frequency, i.e., infinite blueshift. This observation allows us to approximate the lightlike geodesics of particles as classical light rays with constant phase reflecting from mirrors and propagating as straight lines in ordinary manner. This approximation is called the *geometric optics approximation.* We will use the geometric optics approximation as we trace the worldline of light like particle backwards in time from $\mathfrak{I}^+$ to $\mathfrak{I}^-$ (see Fig. (1.4) in Sec. 1.5.1).

Let us consider the worldline of a light ray in the spacetime geometry of a collapsing spherically symmetric, non-charged and non-rotating star. A Penrose-Carter diagram of the collapsing star that collapses to a black hole is represented in Fig. 4.1. As it can be noticed from Fig. 4.1 the light rays

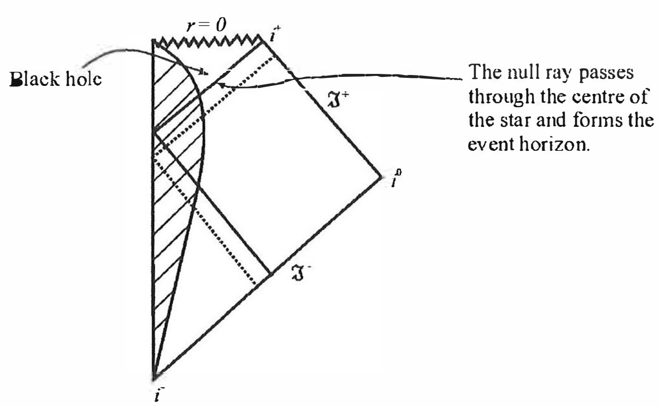

Figure 4.1: Collapsing mass distribution is represented by the shaded region and the exterior region is a portion of the regions I and II given in Fig. 2.2. The figure includes two worldlines of light rays.

that leave the past null infinity $\mathfrak{I}^-$ 'early enough' go through the collapsing star and reach the future null infinity $\mathfrak{I}^+$, but the light rays that leave 'too late' are either emerged to form the event horizon or destined to go to the black hole singularity $r = 0$. On the other hand, we are interested in how the ingoing mode solutions (4.38) change to outgoing mode solutions (4.37) near the event horizon written in the future null infinity $\mathfrak{I}^+$. Therefore we shall trace back to the past the worldlines of such light rays that propagate close but exterior to the event horizon that is formed during the collapse. Let us denote the distance of the light rays which have $\mathcal{U} = constant$ from the event horizon $\mathcal{U} = \mathcal{U}_0$ by ε such that the distance is given by the affine parameter $\tilde{u}$ (see Fig. 4.2):

$$\varepsilon = -\tilde{u} \,, \tag{4.39}$$

and therefore the outgoing mode solution (4.37) near the horizon can be expressed as

$$u^{\text{out}}_{\omega l m_l} \propto \exp[i4M\omega \log(\varepsilon)] \,, \tag{4.40}$$

which represent very densely propagating wavefronts. Next we transport these wave forms back to the $\mathfrak{I}^-$.

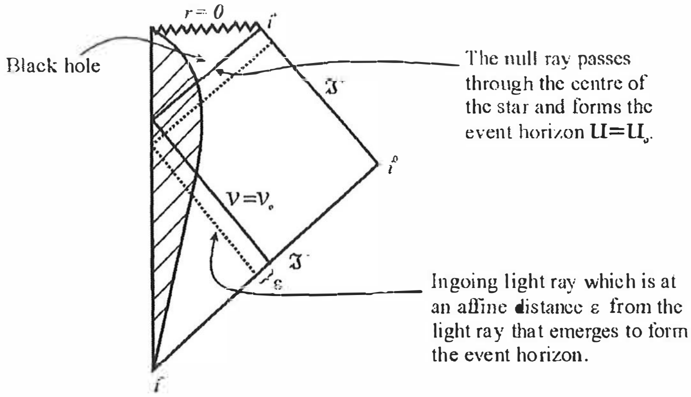

Figure 4.2: In the figure the light ray of interest is represented by the dashed line. Its affine distance is ε from the light rays that form the horizon $\mathcal{U} = \mathcal{U}_0$.

At the asymptotic infinity $\mathfrak{I}^-$ far from i^- the spacetime is flat and, as shown, the 'potential' $V(r)$ vanishes. Therefore the ingoing mode solutions $u^{\text{in}}_{\omega l m_l}$ are, in terms of the Eddington-Finkelstein coordinates, of the form (4.30), whereas near the event horizon the corresponding outgoing mode solutions $u^{\text{out}}_{\omega l m_l}$ are of the form (4.31). In what follows, we shall simply denote

$$u^{\text{in}}_{\omega l m_l} := u_\omega , \tag{4.41}$$

$$u^{\text{out}}_{\omega l m_l} := u'_\omega . \tag{4.42}$$

The transformation from the solutions u_ω to the solutions u'_ω is given by the Bogolubov transformation (4.20), and our task is to calculate the Bogolubov coefficients $A_{\omega\omega'}$ and $B_{\omega\omega'}$. The key idea in calculating the Bogolubov coefficients is that the surface $\mathcal{V} = \mathcal{V}_0$ (see Fig. 4.2) corresponds to the event horizon in the following technical sense: the light rays that are emitted from $\mathfrak{I}^-$ and obey $\mathcal{V} = \mathcal{V}_0$ form the horizon of the black hole. Therefore, the light rays that are at an affine distance ε from the horizon correspond to the surface $\mathcal{V} = constant$, which is at an affine distance ε from the surface $\mathcal{V} = \mathcal{V}_0$. Because the spacetime is flat at the asymptotic infinity, the line element for constant θ and ϕ in terms of the Eddington-Finkelstein coordinates is

$$ds^2 = -d\mathcal{U}d\mathcal{V} , \tag{4.43}$$

and therefore $\mathcal{V}_0 - \mathcal{V}$ corresponds to an affine distance from the surface $\mathcal{V} = constant$ to the surface $\mathcal{V} = \mathcal{V}_0$. Furthermore, affine distances $\mathcal{V}_0 - \mathcal{V}$ and ε are related to each other:

$$\mathcal{V}_0 - \mathcal{V} = C\varepsilon , \tag{4.44}$$

where C is a constant. Hence, at $\mathfrak{I}^-$, when $\mathcal{V}_0 - \mathcal{V}$ is taken to be very small, we may write the outgoing mode solution (4.31) as

$$u'_\omega = N_{\omega l m_l} Y_{l m_l} \frac{1}{r} \exp\{i4M\omega \log[(\mathcal{V}_0 - \mathcal{V})/C]\} , \tag{4.45}$$

and thus for $\mathcal{V}_0 - \mathcal{V} > 0$ the Bogolubov transformation (4.20) at $\mathfrak{I}^-$ takes the form

$$\exp\{i4M\omega \log[(\mathcal{V}_0 - \mathcal{V})/C]\} = \sum_{\omega'} \left[A_{\omega\omega'} \exp(-i\omega'\mathcal{V}) + B_{\omega\omega'} \exp(i\omega'\mathcal{V})\right] . \tag{4.46}$$

If $\mathcal{V}_0 - \mathcal{V} \leq 0$, the left hand side of Eq. (4.46) vanishes since the particles that follow the surfaces where $\mathcal{V}_0 - \mathcal{V} \leq 0$ cannot escape from the black hole. The coefficients $A_{\omega\omega'}$ and $B_{\omega\omega'}$ in Eq. 4.46) are, in fact, *Fourier transforms* of the function

$$f_\omega(\mathcal{V}) := \begin{cases} 0 \ , \ \text{when } \mathcal{V}_0 - \mathcal{V} \leq 0 \\ \exp\{i4M\omega \log[(\mathcal{V}_0 - \mathcal{V})/C]\} \ , \ \text{when } \mathcal{V}_0 - \mathcal{V} > 0 \ . \end{cases} \tag{4.47}$$

Hence, by the definition of the Fourier transforms, the coefficients $A_{\omega\omega'}$ and $B_{\omega\omega'}$ may be obtained from the integrals

$$A_{\omega\omega'} = \frac{1}{2\pi} \int_{-\infty}^{\mathcal{V}_0} d\mathcal{V} \exp(i\omega'\mathcal{V}) \exp\{i4M\omega \log[(\mathcal{V}_0 - \mathcal{V})/C]\} \ , \tag{4.48}$$

$$B_{\omega\omega'} = \frac{1}{2\pi} \int_{-\infty}^{\mathcal{V}_0} d\mathcal{V} \exp(-i\omega'\mathcal{V}) \exp\{i4M\omega \log[(\mathcal{V}_0 - \mathcal{V})/C]\} \ . \tag{4.49}$$

When we substitute $\mathcal{V}_0 - \mathcal{V} = x$ into the integrals (4.48) and (4.49), and consider the absolute values $|A_{\omega\omega'}|$ and $|B_{\omega\omega'}|$ only, they take the form

$$|A_{\omega\omega'}| = \frac{1}{2\pi} \left| \int_{-\infty}^{0} dx \exp(i\omega' x) \exp[i4M\omega \log(-x)] \right| \ , \tag{4.50}$$

$$|B_{\omega\omega'}| = \frac{1}{2\pi} \left| \int_{-\infty}^{0} dx \exp(-i\omega' x) \exp[i4M\omega \log(-x)] \right| \ . \tag{4.51}$$

These integrals may be integrated by using the *residue theorem* of complex variable functions. We shall not explicitly calculate these integrals, but we just give the results. After some residue calculus one finds that $|A_{\omega\omega'}|$ is related to $|B_{\omega\omega'}|$ such that

$$|A_{\omega\omega'}| = \exp(4\pi M\omega)|B_{\omega\omega'}| \ . \tag{4.52}$$

Since the mode solutions u'_ω are orthonormal, the Bogolubov coefficients satisfy the relation

$$\sum_{\omega'} (A_{\omega\omega'} A^*_{\omega''\omega'} - B_{\omega\omega'} B^*_{\omega''\omega'}) = \delta_{\omega\omega''} \ , \tag{4.53}$$

and therefore

$$\sum_{\omega'} \left(|A_{\omega\omega'}|^2 - |B_{\omega\omega'}|^2\right) = 1 \ . \tag{4.54}$$

Because of the relation (4.52), Eq. (4.54) implies that

$$\sum_{\omega'} [\exp(8\pi M\omega) - 1] \, |B_{\omega\omega'}|^2 = 1 \ . \tag{4.55}$$

Hence, the sum of the squared absolute value of the coefficients $B_{\omega\omega'}$ is

$$\sum_{\omega'} |B_{\omega\omega'}|^2 = \frac{1}{\exp(8\pi M\omega) - 1} \ . \tag{4.56}$$

Hence, even when there are no ingoing particles, i.e., $n_\omega = \langle 0|\hat{N}_\omega|0\rangle = 0$, according to Eqs. (4.26) and (4.56) the expectation value of the number of the outgoing particles of frequency ω at $\mathfrak{I}^+$ is

$$\begin{aligned} n_\omega &= \sum_{\omega'} B^*_{\omega\omega'} B_{\omega\omega'} \\ &= \sum_{\omega'} |B_{\omega\omega'}|^2 = \frac{1}{\exp(8\pi M\omega) - 1} \ , \end{aligned} \tag{4.57}$$

which agrees with the Planck distribution for black body radiation at the *Hawking temperature*

$$T_{\rm H} = \frac{1}{8\pi M} = \frac{\kappa}{2\pi} \tag{4.58}$$

or, in the SI-units

$$T_{\rm H} = \frac{\hbar c^3}{8\pi G k_{\rm B} M} \ , \tag{4.59}$$

where $\hbar = 1,055 \cdot 10^{-34}$ Js is *Planck's constant* and $k_{\rm B} = 1,381 \cdot 10^{-23}$ J/K is *Boltzmann's constant.*

As the classical black holes were totally black, quantum black holes radiate exactly like black bodies. This seems quite contradictory, since classically nothing can come out of the black hole. Heuristically, there are two possible explanations to the Hawking radiation: One can think the positive energy particle at $\mathfrak{I}^+$ having tunneled out through the event horizon. Alternatively, continous spontaneous creation of virtual particle-antiparticle pairs at the vicinity of the event horizon may be used to illustrate the Hawking effect. In normal situations virtual particle-antiparticle pairs would annihilate each other very soon after their emergence. However, when such a pair is produced in the vicinity of the black hole event horizon, sometimes the particle with the positive energy escapes to infinity contributing to the Hawking radiation flux, while the negative energy antiparticle ends up in the black hole. The tunneling explanation arises from the method used extensively by Damour and Ruffini [77], where they consider the possibility that the correct outgoing wave is an adequate superposition of outgoing waves written separately outside and inside the black hole. The authors show that the outgoing wave function inside the hole in fact corresponds to the ingoing negative energy wave. We must emphasize that these are just qualitative attempts to explain the Hawking radiation.

One can notice from the expression of the Hawking temperature (4.58) that small black holes are hotter than large holes. In particular, when the cosmic background temperature is greater that the Hawking temperature the holes are absorbing rather than emitting radiation, and therefore large holes cannot be detected by their characteristic Hawking radiation. Moreover, the specfic heat of the hole is negative – the more they radiate the hotter they become.

According to *Wien's displacement* law the frequency ω_0 corresponding to the maximum of the energy distribution of the thermal radiation is proportional to the temperature T:

$$\omega_0 = aT \ , \tag{4.60}$$

where a is the constant of proportionality. Because the Hawking temperature T_{H} is proportional to $1/M$, Wien's displacement law implies that for the frequency

$$\omega_0 \sim \frac{1}{M} \ , \tag{4.61}$$

or, in SI-units,

$$\omega_0 \sim \frac{c^3}{GM} \ . \tag{4.62}$$

Hence the energy of the radiation increases as the mass of the hole decreases.

The thermal radiation of a black hole carries energy from the gravitational field to infinity, and therefore the black hole must lose its mass. Because of that black holes evaporate. A rough approximation to the energy radiated by a hole is given by the Stefan-Boltzmann law:

$$\frac{dE}{dt} = -\sigma T_{\mathrm{H}}^4 A \ , \tag{4.63}$$

where $\sigma = \pi^2 k_{\mathrm{B}}^2/(60\hbar^3 c^2) = 5,67 \cdot 10^{-8}$ J/(m^{2}sK^4) is the Stefan-Boltzmann constant and A is the event horizon area given in Eq. (2.4). By using $E = Mc^2$ we get

$$\frac{dM}{dt} \approx \frac{\hbar c^4}{15360\pi G^2 M^2} \ , \tag{4.64}$$

which yields for the time of evaporation

$$t \approx \frac{5120\pi G^2}{\hbar c^4} M^3 = 9 \cdot 10^{-16}\ \mathrm{s/kg}^3 M^3 \ . \tag{4.65}$$

Eq. (4.64) implies that the evaporation tends to escalate towards the end of the life of the hole. However, its final state has not yet been resolved and the present theory even does not hold when the lifetime of the hole is of the the order of the Planck time $t_{\mathrm{Pl}} = (\hbar G/c^5)^{1/2} \sim 10^{-43}$ s, since the concept of the fixed classical background geometry cannot be assumed any more due to the severe quantum fluctuations of spacetime. Whatever happens during the last stages of the evolution, we may say within the present theory that the hole emits an energy equivalent to 10^6 megaton thermonuclear bombs during the final tenth of a second of its life.

When we reproduced the Hawking temperature of black holes, we assumed no backscattering nor backreaction from the gravitational field. More precisely, we did not pay any attention to the portion of modes that are scattered back to the hole from the 'potential' $V(r)$. For a more realistic calculation we should have allowed the mass M of the hole change during the process while we kept it constant. The backreaction becomes important not until $dM/dt \sim M$, i.e., in the final stages of the evaporation. These effects have been considered carefully and a short introduction can be found in Ref. [69]. As a result of these additional effects of the gravitational field the Hawking radiation is not purely thermal, but in a certain sence it may be considered as thermal: Like the outgoing particles the ingoing particles are also backscattered when the black hole is in a heat bath, and therefore the ingoing-outgoing flux ratio is independent of the details of the backscattering. Thus, the black hole remains to be in thermal equilibrium with its surroundings and can be seen as producing a black body spectrum.

4.3 Black Hole Entropy

Since black holes radiate with the temperature $T_{\rm H} = \kappa/2\pi$ they may be assigned an entropy as Bekenstein has anticipated. In particular, when the expression for the Hawking temperature is substituted into Eq. (2.66) one notices that

$$\frac{1}{4}\delta A = \delta S_{\rm BH} \ . \tag{4.66}$$

Because of that, the first law of black hole mechanics (2.66) may now be written as the first law of black hole thermodynamics:

$$\delta M = T_{\rm H}\delta S_{\rm BH} + \Omega_{\rm H}\delta J + \Phi_{\rm H}\delta Q \ , \tag{4.67}$$

where $S_{\rm BH}$ is the *Bekenstein-Hawking entropy*

$$S_{\rm BH} = \frac{1}{4}A \ . \tag{4.68}$$

The first law of black hole thermodynamics is now built on fairly firm grounds and it is no longer based on just an analogy. In addition to establishing the proportionality between the horizon area and the entropy, the constant of proportionality γ is also given by Hawking's work and it is equal to $1/4$.

The fact that the black hole looses its mass forces the event horizon area to decrease in violation of the second law of black hole mechanics. On the other hand, when matter enters the black hole and if one does not assign an entropy to the hole, then the entropy $S_{\rm ext}$ in the exterior spacetime region would decrease. Therefore Bekenstein conjectured that the total entropy $S = S_{\rm BH} + S_{\rm ext}$ should always be a non-decresing function of time in any physical process [78]. This statement is the so-called *generalized second law of thermodynamics* and it simply states that

$$\delta S \geq 0 \text{ in any process} \ . \tag{4.69}$$

This law has the status that is reminiscent of the status that the second law of thermodynamics had before statistical mechanics was found. Today one of the most challenging problems is to find a fundamental or "statistical mechanical" explanation for the black hole entropy. One needs to answer the basic question: What are the underlying quantum mechanical microstates of the black hole corresponding to the Bekenstein–Hawking entropy? Are the microstates related to the matter quantum fields on a background geometry or is it possible to assign the notion of a black hole entropy purely to a geometrical entity of the hole? Since the black hole entropy is $\frac{1}{4}A$, one might expect that there are $\exp(\frac{1}{4}A)$ microstates corresponding to the same macrostate of the hole, and the problem is to identify these microstates. This assumption of degeneracy is justified because entropy, in general, can be interpreted as the logarithm of the number of microstates corresponding to the same macrostate.

The classical no-hair theorem states that after the collapse, when a black hole has settled down to a stationary state, its properties are determined by very few parameters observed far from the hole: the mass M, the charge Q and the angular momentum J of the hole. Thus, from the classical point of view, black holes have only three degrees of freedom. What has happened to the enormous amount of the degrees of freedom of the collapsing matter? The no-hair theorem prompts one to believe that these degrees of freedom, and the information contained in them, is lost in the collapse, and that the entropy of the black hole may be understood as a measure of information loss during the gravitational collapse, because between entropy and information there is a well-known relationship given by Brilliouin [79]: the decrease in information increases entropy. This viewpoint is purely quantum mechanical: According to quantum mechanics all the information from the collapsing star is not able to reach an observer exterior to the newly formed event horizon. In other words, all the microstates of the collapsing star can not be measured by the external observer. This results to an increasing entropy S. The question now arises: After the collapse of matter, are the degrees of freedom contained in the matter fields somehow encoded into the quantum states of the black hole spacetime itself, or have they vanished altogether, leaving no trace whatsoever? Of course, it is natural to claim that they are encoded into the quantum states of spacetime itself such that there is a vast $\exp(\frac{1}{4}A)-$fold degeneracy in the quantum states of the hole itself. This leads us to a conclusion that the total number of the unknown quantum states of the black hole must be enormous, too. Thus, from a quantum-mechanical point of view, the number of the physical degrees of freedom of the hole is not limited to just few parameters. The contradiction between quantum and classical black hole is obvious. The number of physical degrees of freedom of the classical hole is three, whereas the number of physical degrees of freedom of the quantum black hole is enormous. The problem with this view is that it is not quite clear how, starting from general relativity, quantization itself might bring along a huge number of additional degrees of freedom. Later in this chapter we shall consider one possible interpretation to the origin of the black hole entropy [15].

4.3.1 Area Spectrum

A viable solution to the question about the origin of the black hole entropy is given by one of Bekenstein's brilliant proposals [22], suggesting that the possible eigenvalues of the event horizon area of the black hole are of the from

$$A_n = \gamma n l_{\rm Pl}^2 \,, \tag{4.70}$$

where γ is a pure number of order one, n ranges over all non-negative integers, and $l_{\rm Pl} := (\hbar G/c^3)^{1/2}$ is the Planck length. This proposal was made in 1974, and since then it has been revived by several authors [16, 14, 66, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]. It is interesting to notice that Bekenstein's proposal immediately leads one to conclude that the angular frequencies of the quanta of the Hawking radiation are of the form

$$\omega = m\omega_0 \ , \tag{4.71}$$

where $\omega_0 := \frac{\gamma}{32\pi}\frac{1}{M}$ is the so-called fundamental angular freguency, and m is a positive integer. This can be understood on the grounds of the fact that the horizon area is quantized and that the difference between any two consecutive area eigenvalues is a constant and equal to

$$A_n - A_{n-1} = \gamma l_{\mathrm{Pl}}^2 \ \forall\ n \ . \tag{4.72}$$

In other words, the area spectrum is uniformly spaced. As the hole emits one quantum of the radiation such that the horizon area of the hole decreases by the amount of γl_{Pl}^2, and since the relation between the horizon area and the mass of the hole is $A = 16\pi M^2$, the total mass of the radiation quantum is then equal to the decrease of the mass of the hole

$$\Delta M = \frac{\gamma M_{\mathrm{Pl}}^2}{32\pi M} \ , \tag{4.73}$$

where $M_{\mathrm{Pl}} = (\hbar c/G)^{1/2}$. According to quantum mechanics the energy change of the hole is $\Delta E = \hbar\omega_0$, where ω_0 is the frequency of the radiation quantum. On the other hand, the change in energy ΔE and the change in mass ΔM are related by the famous result $\Delta E = \Delta M c^2$. Because of that the frequency of the radiation quantum, when the hole performs a transition from one stationary state to the nearest lower area eigenstate, is

$$\omega_0 \propto \Delta M \propto \frac{1}{M}. \tag{4.74}$$

In other words, according to Bekenstein's proposal (4.70) the radiation spectrum of the black hole is discrete and the wavelength of the emitted quanta is $\sim M$, whereas according to Hawking's calculation the spectrum is continuous with the characteristic wavelength of $\sim M$. This discrepancy was investigated in Ref. [90], and the resolution to the apparent contradiction follows from the uncertainty principle of quantum mechanics. Within the limits of the uncertainty principle Bekenstein's proposal and Hawking's semiclassical results can be regarded as special cases of the one and same theory of quantum black holes. The difference between the spectra arises from the fact that Hawking's semiclassical theory describes black holes in the presence of the matter fields, whereas Bekenstein's proposal can be seen to follow from the quantum description of the vacuum black hole geometry itself.

Now, if we adopt Bekenstein's proposal then one may think that the event horizon area is constructed from small patches each having an area γl_{Pl}^2 and carrying one bit of information. Let us imagine 0 or 1 written on each patch. As the area of the horizon is A, the number of the equally probable configurations would be $N = 2^{A/(\gamma l_{\mathrm{Pl}}^2)}$, and the corresponding Boltzmann entropy would be ($k_{\mathrm{B}} = 1$)

$$S = \log N = \frac{\log 2}{\gamma l_{\mathrm{Pl}}^2} A \ , \tag{4.75}$$

which gives the correct result (4.68) provided that $\gamma = 4\log 2$ [101, 102]. The above argumentation was originally intended to demystify the direct proportionality of black hole entropy and horizon area. The key incredient in the argumentation is its dependence on the uniformly spaced area spectrum.

4.4 Path Integral Approach to Black Hole Spacetimes

The essential difference between Hawking's and Bekenstein's approaches seems to be that the black hole entropy in the former case depends on the behaviour of the quantum matter fields and in the latter case the entropy is a purely geometrical quantity not depending on any matter fields present. This difference alludes that the black hole entropy may be an intrinsic property of the black hole and more than just quantum field theory in curved spacetime. To show that the entropy really can be undestood as a geometrical quantity we shall reproduce another calculation made by Hawking, namely, the path integral derivation of the black hole entropy [103, 104].

4.4.1 Path Integrals in Non-Relativistic System

The basic object in the path integral quantization of any system of particles is the *propagator*

$$K(x_f, t_f; x_i, t_i) := \langle x_f, t_f | x_i, t_i \rangle \ , \tag{4.76}$$

which gives the probability amplitude that the particle is propagated from point x_i at time t_i to point x_f at time t_f ($t_f > t_i$). $|x_i, t_i\rangle$ and $|x_f, t_f\rangle$ are the eigenstates of the position operator $\hat{x}$ in the Heisenberg picture at the initial and the final states of the system, i.e., the state $|x, t\rangle := \exp(i/\hbar \hat{H} t)|x\rangle$, where $\hat{H}$ is the Hamiltonian operator of the system. An essential feature of the path integral quantization is that the system could move from the initial spacetime point (t_i, x_i) to the final spacetime point (t_f, x_f) along any smooth classical path $x = x(t)$ in spacetime. By the definition (4.76) the propagator $K(x_f, t_f; x_i, t_i)$ gives the time evolution of the system:

$$\Psi_f(x_f, t_f) = \int dx_i K(x_f, t_f; x_i, t_i) \Psi_i(x_i, t_i) \ , \tag{4.77}$$

where $\Psi_i(x_f, t_f)$ and $\Psi_f(x_f, t_f)$ are the corresponding initial and final states of the system. Thus, in order to get the time evolution of the system in the path integral approach one has to obtain the propagator of the system. It was Feynman's idea that the propagator is [105]

$$K(x_f, t_f; x_i, t_i) = \int_{\text{paths}} \mathcal{D}[x(t)] \exp\left\{ \frac{i}{\hbar} S[x(t)] \right\} \ , \tag{4.78}$$

where the integration is performed over all smooth paths $x = x(t)$ joining the spacetime points (t_i, x_i) and (t_f, x_f), $\mathcal{D}[x(t)]$ is a measure on the space of all smooth paths. $S = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t); t)$ is the classical action of the system corresponding to the path $x = x(t)$ and L is the Lagrangian of the system.

In contrast to the canonical approach which is based on the Hamiltonian formulation, the path integral approach rests on the Lagrangian formulation. However, since the propagator gives the time evolution of the system the path integral approach should be equivalent to the canonical quantization, and indeed, by using the propagator (4.79) one can obtain the Schrödinger equation $\hat{H}\Psi(t, x) = i\hbar \partial/\partial t \Psi(t, x)$, and vice versa.

4.4.2 Path Integrals and Gravitation

As we proceed to quantize the gravity the first task is to find the "gravitational" counterparts of the concepts such as spacetime point, path, propagator and configuration coordinates of the system. In Sec. 1.1 we already found that the concept of time in general relativity corresponds uniquely to a hypersurface Σ_t of globally hyperbolic spacetime manifold $\mathcal{M}$, and the concept of path, in general, is equivalent to the history of the system, which in geometrodynamics corresponds to spacetime geometry between two hypersurfaces Σ_{t_i} and Σ_{t_f}. The configuration coordinates in general relativity are chosen to be the components of the three-dimensional metric tensor q_{ab} of the spacelike hypersurfaces Σ_t. With these gravitational counterparts we can represent the propagator in quantized gravity as

$$K(q_{ab}(f), \Sigma_{t_f}; q_{ab}(i), \Sigma_{t_i}) = \int\limits_{4-\text{geometries}} \mathcal{D}[g_{\mu\nu}] \exp\left\{\frac{i}{\hbar} S[g_{\mu\nu}]\right\} , \qquad (4.79)$$

$S[g_{\mu\nu}]$ is the gravitational action corresponding to the spacetime in between the hypersurfaces Σ_{t_i} and Σ_{t_f} and $g_{\mu\nu}$ is the metric of the spacetime. The hypersurfaces Σ_{t_i} and Σ_{t_f} are usually asymptotically flat or closed. We shall consider the path integral approach in asymptotically flat spacetimes and therefore the four-geometries of our spacetimes are bounded by two asymptotically flat spacelike hypersurfaces.

Normally the gravitational contribution to the action without any matter fields present is taken to be the Einstein-Hilbert action (1.1). This action is zero for vacuum solutions, therefore one might suspect that the classical action for the Schwarzschild spacetime is zero. However, we have seen in Sec. 1.1 that the Ricci scalar R on the spacetime manifold $\mathcal{M}$ contains second derivatives of the metric $g_{\mu\nu}$, and these derivatives, in turn, can be transferred into the first derivatives of the metric by integrating by parts. When the spacetime is bounded or asymptotically flat there is no reason why the first derivatives should automatically cancel each other at the hypersurfaces Σ_{t_i} and Σ_{t_f}, and if the derivatives at the boundaries are not the same then the gravitational action (1.1) is not an extremum. Therefore, in order to get a stationary action and correct field equations by varying the action with respect to the metric which vanishes on the boundaries but which may have non-zero first-order derivatives we must supplement S by the so-called *Gibbons-Hawking boundary term* [106]

$$S_{\rm GH} = \frac{1}{8\pi} \int\limits_{\partial\mathcal{M}} d^3x K\sqrt{-q} + C , \qquad (4.80)$$

where q is the determinant of the induced three-metric on the boundary and C is a term which does not depend on the four-geometry inside the boundaries, the integration is performed over the boundary of the region for which the action is being evaluated and K is the trace of the extrinsic curvature tensor K_{ab} of the boundary given in Eq. (1.17). The nature of the C-term is quite difficult. C is a term that depends only on the boundary $\partial\mathcal{M}$, and in asymptotically flat spacetimes the boundary $\partial\mathcal{M}$ can be chosen to be the t-axis times a sphere of radius $r = r_0$. In other words, the boundary $\partial\mathcal{M}$ can be obtained by joining the initial and final hypersurfaces Σ_{t_i} and Σ_{t_f}

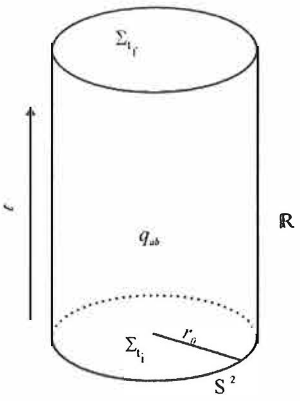

Figure 4.3: In the asymptotically flat spacetime the initial and final spacelike hypersurfaces are joined to form a timelike tube at large radius r_0. Note that this region over which we perform the path integral is compact.

by a timelike tube. The tube is illustrated in Fig. 4.3. Hence the boundary of the asymptotically flat spacetime can then be understood as a timelike three-cylinder $\mathbb{R} \times S^2$ with large radius $r = r_0$. Now, it is natural to choose C such that the contribution to the action S is zero when the spacetime metric $g_{\mu\nu}$ is the flat Minkowski metric $\eta_{\mu\nu}$. In that case the surface term for the flat space is

$$C = -\frac{1}{8\pi} \int\limits_{\partial\mathcal{M}} d^3x\sqrt{-q}K^0 \ , \tag{4.81}$$

where K^0 is the trace of the extrinsic curvature tensor of the boundary imbedded in flat space. As a result, we substract off the surface term (4.81) for flat space and the surface term (4.80) for the surface S^2. Hence the correct vacuum spacetime action is

$$S = \frac{1}{16\pi}\left[\int\limits_{\mathcal{M}} d^4xR\sqrt{-g} + 2\int\limits_{\partial\mathcal{M}} d^3x\sqrt{-q}\left(K - K^0\right)\right] . \tag{4.82}$$

To be able to evaluate the action (4.86) in general one should be able to imbed all kinds of curved boundaries in flat space, but there are surfaces that cannot be imbedded in any-dimensional flat spaces. However, in an asymptotically flat space case one can take the boundary surface asymptotically imbeddable as the radius of the two-sphere gets large enough. Therefore in the path integral approach for the asymptotically flat spacetimes we use non-compact and spatially asymptotically flat metrics, all of which can be written for large r in the form

$$ds^2 = -\left(1 - \frac{2M_t}{r}\right)dt^2 + \left(1 + \frac{2M_r}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \mathcal{O}(r^{-2}) \ . \tag{4.83}$$

Consider the Schwarzschild solution to the Einstein's field equation with mass M, then $M_t = M_r = M$. Furthermore, if we choose the boundary $\partial\mathcal{M}$

to be the one illustrated in Fig. 4.3 then the extrinsic curvature tensor K_{ab} for surfaces of constant r becomes

$$K_{ab} = \frac{1}{2N}\frac{\partial q_{ab}}{\partial r} , \tag{4.84}$$

where the induced three-metric q_{ab} on the *timelike* boundary is represented by the matrix

$$q_{ab} = \begin{pmatrix} -\left(1-\frac{2M_t}{r}\right) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} , \tag{4.85}$$

and $N = (1-2M/r)^{1/2}$. For the two-sphere of radius r imbedded in the flat space the curvature scalar is $K^0 = 2/r$. Thus the surface term for a large radius $r = r_0$ is given by the non-zero integral

$$\begin{aligned}
&\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-q}\left(\frac{1}{2N}q^{ab}\frac{\partial q_{ab}}{\partial r} - \frac{2}{r}\right)\Bigg|_{r=r_0} \\
&= \frac{1}{8\pi}\int dt\int_0^{\pi} d\theta\int_0^{2\pi} d\phi\sqrt{1-\frac{2M}{r_0}}r_0^2\sin\theta \times \\
&\left[\frac{1}{2}\sqrt{1-\frac{2M}{r_0}}\left(\frac{4}{r_0}+\frac{2M}{r_0^2}+\frac{(2M)^2}{r_0^3}+\mathcal{O}(r_0^{-4})\right)-\frac{2}{r_0}\right] \\
&= \frac{r_0^2}{2}\int dt\left[\left(1-\frac{2M}{r_0}\right)\left(\frac{2}{r_0}+\frac{M}{r_0^2}+\frac{2M^2}{r_0^3}+\mathcal{O}(r_0^{-4})\right)\right. \\
&\left. -\sqrt{1-\frac{2M}{r_0}}\frac{2}{r_0}\right] \\
&= -\frac{M}{2}\int dt + \mathcal{O}(r_0^{-1}) .
\end{aligned} \tag{4.86}$$

Hence the action for the asympotically flat spherically symmetric spacetime can be evaluated from the boundary represented by the timelike tube, and in the highest-order approximation it is

$$S = -\frac{M}{2}\int dt + \mathcal{O}(r_0^{-1}) . \tag{4.87}$$

In the path integral approach one has to take into account all the possible metrics. Therefore it is useful to expand the classical action in its Taylor series with respect to the variations of the metric $\delta g_{\mu\nu}$ about the background metric $g_{\mu\nu}^{\text{background}}$ that extremize the action, i.e., which is solution to the classical field equations:

$$S[g_{\mu\nu}] = S[g_{\mu\nu}^{\text{background}}] + S_1[\delta g_{\mu\nu}]\delta g_{\mu\nu} + S_2[\delta g_{\mu\nu}](\delta g_{\mu\nu})^2 + \dots , \tag{4.88}$$

where the first-order term vanishes identically since the background metric is a solution to the field equations. The higher-order terms are expected to be finite but small corrections to the zeroth-order term. For this, as a matter of fact, very subtle reason, which we shall not discuss in length here, the main contribution to the action is given by the Gibbons-Hawking boundary term. When all the possible asymptotically flat metrics are taken into account the path integral (4.79) simply becomes

$$\int_{4-\text{geometries}} \mathcal{D}[g_{\mu\nu}] \exp\left\{\frac{i}{\hbar} S[g_{\mu\nu}]\right\} =$$

$$\int_{4-\text{geometries}} \mathcal{D}[g_{\mu\nu}] \exp\left\{\frac{-iM}{2\hbar}\int dt - \frac{iS_2[\delta g_{\mu\nu}](\delta g_{\mu\nu})^2}{\hbar} - \dots\right\} \quad (4.89)$$

This path integral oscillates and probably will not converge. Because of this we perform the so-called *Wick rotation*, where the time axis is rotated 90° making it imaginary. More presicely, we make the transformation

$$t \longrightarrow -i\tau\ , \qquad \tau \in \mathbb{R}\ . \quad (4.90)$$

As a consequence of the Wick rotation the metric on the purely imaginary timelike boundary becomes positive definite, i.e., our pseudo-Riemannian boundary of the spacetime manifold has become Riemannian, or the so called Euclidean metric with the signature $(+,+,+,+)$. Therefore the action $S[g_{\mu\nu}]$ in Eq. (4.88) becomes an Euclidean action:

$$I[g_{\mu\nu}] = -iS[g_{\mu\nu}] = I[g_{\mu\nu}^{\text{background}}] + I_2[\delta g_{\mu\nu}](\delta g_{\mu\nu})^2 + \dots\ , \quad (4.91)$$

where

$$I[g_{\mu\nu}^{\text{background}}] = \frac{M}{2}\int d\tau\ . \quad (4.92)$$

The corresponding Euclidean integral turns out to be exponentially damping

$$\int_{4-\text{geometries}} \mathcal{D}[g_{\mu\nu}] \exp\left\{\frac{i}{\hbar} S[g_{\mu\nu}]\right\} =$$

$$\int_{4-\text{geometries}} \mathcal{D}[g_{\mu\nu}] \exp\left\{\frac{-M}{2\hbar}\int d\tau - \frac{I_2[\delta g_{\mu\nu}](\delta g_{\mu\nu})^2}{\hbar} - \dots\right\} \ ;(4.93)$$

here the integration is performed over all positive definite four-geometries $g_{\mu\nu}$ that induce the given positive definite metric (4.85) on the timelike imaginary boundary.

It may seem that the Wick rotation is an 'ad-hoc' trick just to guarantee converging path integrals. However, one may find some justification to the trick from other than quantum field theories of gravitation when the path integral approach is applied. When such a quantum field theoretical system possesses non-converging path integrals, one usually introduces a new term of the form $i\epsilon\phi^2$ into the Lagrangian density, if the field ϕ is to be quantized. After performing evaluations of the converging integrals, ϵ may be set to zero. It is possible to show that this procedure is equivalent to a small rotation of the time axis in the complex plane towards the imaginary time axis. Such a rotation is also possible when the angle of rotation is taken to be finite, for example, one may choose a rotation of an amount of 90°, which is equivalent to the Wick rotation we performed.

After all the necessary calculations have been performed in the Euclidean spacetime, the Euclidean results, in principle, could be analytically continued back to the Riemannian spacetime. However, this is not necessary, when one is interested in the thermodynamical properties of the system.

4.4.3 Propagators and Partition Functions

Consider, for example, a one-dimensional, non-relativistic system of particles, which is represented by the complete set $\{\phi_n\}$ of eigenfunctions of the Hamiltonian operator $\hat{H}$ of the system such that the eigenvalue equation

$$\hat{H}\phi_n = E_n\phi_n \tag{4.94}$$

and the orthonormality condition

$$\int \phi_n^* \phi_{n'} = \delta_{nn'} \tag{4.95}$$

are satisfied. Then

$$\begin{aligned}
\Psi_f(x_f, t_f) &= \int dx_i \langle x_f, t_f | x_i, t_i\rangle \Psi_i(x_i, t_i) \\
&= \int dx_i \sum_n \langle x_f, t_f | \phi_n\rangle\langle\phi_n | x_i, t_i\rangle \Psi_i(x_i, t_i) \\
&= \int dx_i \sum_n \langle x_f | \exp(-i/\hbar \hat{H} t_f)|\phi_n\rangle\langle\phi_n| \exp(i/\hbar \hat{H} t_i)|x_i\rangle \Psi_i(x_i, t_i) \\
&= \int dx_i \sum_n \exp(-i/\hbar E_n t_f)\langle x_f|\phi_n\rangle\langle\phi_n|x_i\rangle \exp(i/\hbar E_n t_i)\Psi_i(x_i, t_i) \\
&= \int dx_i \sum_n \phi_n^*(x)\phi_n(x) \exp[-i/\hbar E_n(t_f - t_i)]\Psi_i(x_i, t_i) \\
&= \int dx_i K(x_f, t_f; x_i, t_i)\Psi_i(x_i, t_i) \ ,
\end{aligned} \tag{4.96}$$

where

$$K(x_f, t_f; x_i, t_i) := \sum_n \phi_n^*(x_i)\phi_n(x_f) \exp[-iE_n/\hbar(t_f - t_i)] \tag{4.97}$$

is the propagator of the system given in the basis $\{\phi_n\}$.

The *density matrix* of the system, in the basis $\{\phi_n\}$, is usually defined as

$$\rho(x_f; x_i) = \sum_n \phi_n^*(x_i)\phi_n(x_f) \exp(-\beta E_n) \ , \tag{4.98}$$

where β is a real number. The trace of the density matrix gives the *partition function* of the system field configuration in thermal equilibrium with its surroundings with the temperature T. Let us show this:

$$\begin{aligned}
\mathrm{Tr}\rho(x_f; x_i) &= \int dx \rho(x; x) = \int dx \sum_n \phi_n^*(x)\phi_n(x) \exp(-\beta E_n) \\
&= \sum_n \exp(-\beta E_n) \\
&= Z \ ,
\end{aligned} \tag{4.99}$$

where Z, indeed, is called the partition function of the system [107]. Now, from statistical mechanics it is well known that *Shannon's entropy* S of the system is

$$S = -\sum_n p_n \log p_n \ , \tag{4.100}$$

where $p_n := \exp(-\beta E_n)/Z$ gives the probability that the system is in the state n. Therefore the entropy of the system is

$$\begin{aligned} S &= -\sum_n \frac{\exp(-\beta E_n)}{Z} \log \frac{\exp(-\beta E_n)}{Z} \\ &= \beta \sum_n \frac{E_n \exp(-\beta E_n)}{Z} + \log Z \\ &= \beta \langle E \rangle + \log Z \ , \end{aligned} \tag{4.101}$$

where $\langle E \rangle := \sum_n \frac{E_n \exp(-\beta E_n)}{Z}$ is the expectation value of the energy of the system. Since the inverse of the temperature of the system and the entropy of the system are related by the relation

$$\frac{1}{T} = \frac{\partial S}{\partial \langle E \rangle} \ , \tag{4.102}$$

we get from Eq. (4.101), in natural units, that

$$\frac{1}{T} = \beta \ . \tag{4.103}$$

Hence, the partition function Z for the canonical ensemble consisting of the fields $\{\phi_n\}$ really describes the thermodynamics of the system at the temperature $T = 1/\beta$. Moreover, if one performs the Wick rotation, i.e., sets $t_f - t_i = -i\beta$ and requires the periodicity $\phi_n(x_i) = \phi_n(x_f)$ in the propagator (4.97), and integrates over all possible spatial points one finds that the partition function Z is related to the propagator, which , in turn, is related to the path integral

$$Z = \mathrm{Tr}\rho = \int dx K(x, -i\beta; x, 0) = \int dx \int_{\text{paths}} \mathcal{D}[x(t)] \exp(-I[x(t)]) \ , \tag{4.104}$$

where the path integral is over all classical paths which are periodic with period β in imaginary time.

As we transform this to the case of the gravitational fields $\{g_{\mu\nu}\}$, then the partition function Z of the gravitating system at the temperature T is given by the path integral over all Euclidean metrics which have the period $\beta = 1/T$ in the imaginary time direction. More precisely, we have

$$Z = \int dg_{\mu\nu} \int \mathcal{D}[g_{\mu\nu}] \exp(-I[g_{\mu\nu}])/\hbar) \ , \tag{4.105}$$

and when the initial and final hypersurface are not fixed, the integration $\int dq_{ab}$ may be included in the path integral, and therefore

$$Z = \int \mathcal{D}[g_{\mu\nu}] \exp(-I[g_{\mu\nu}])/\hbar) \ . \tag{4.106}$$

We have already argued that the main contribution to the path integral (4.106) comes from the background metric. Moreover, if the spacetime background settles down to the Schwarzschild geometry, then the background action is zero everywhere but on the asymptotic boundary. Therefore the non-zero contribution to the path integral arises from the boundary of the background spacetime. So the remaining crucial task in the evaluation of the partition function of the Schwarzschild background is to find out what is the period in the imaginary time of the Eucledian Schwarzschild spacetime. The Schwarzschild metric is given in Sec. 2.2 in Eq. (2.3). If we put $t = -i\tau$ then the Schwarzschild metric becomes positive definite for $r > 2M$. Because of the coordinate singularity at the event horizon, we define a new radial coordinate $x = 4M(1 - 2M/r)^{1/2}$ for $r \geq 2M$, and substitute it into the Schwarzschild line element. Then the metric for $x \geq 0$ becomes

$$ds^2 = \left(\frac{x}{4M}\right)^2 d\tau^2 + \left(\frac{r^2}{(2M)^2}\right)^2 dx^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \qquad (4.107)$$

which is singularity free at the point $x = 0$. If one compares the metric (4.107) in the $x\tau$-plane to the flat space metric $ds^2 = d\rho^2 + \rho^2 d\phi^2$ written in the polar coordinates (ρ, ϕ) then one notices that the coordinates x and τ behave like the polar coordinates provided that the coordinate τ is not assigned the period 2π but $8\pi M$. The manifold defined by $x \geq 0$ and $0 \leq \tau \leq 8\pi M$ is called the *Euclidean section of the Schwarzschild solution.* On the Euclidean section the metric is positive definite, asymptotically flat and non-singular.

As the Euclidean section of the Schwarzschild solution is periodic the boundary is not represented by the tube $\mathbb{R} \times S^2$ but by the torus $S^1 \times S^2$ where the time axis is rolled. Hence the topology of the Schwarzschild solution changes when performing the Wick rotation. The path integral (4.106) is performed over all metrics that are periodic in the imginary time direction. Because the period in imaginary time coincides with $8\pi M$, the time interval between the initial and the final metric on the boundary is given by the period $\beta = 8\pi M$. Thus the temperature of such metric configurations is

$$T = \frac{1}{8\pi M} , \qquad (4.108)$$

which agrees with the Hawking temperature (4.58) corresponding to the radiation flux of matter fields caused by the collapsing spherically symmetric mass distribution.

Because the background spacetime boundary is periodic with the period $\beta = 8\pi M$ the Euclidean background boundary term contribution to the action $I[g_{\mu\nu}]$, according to Eq. (4.92), is

$$I_{\text{GH}} = \frac{M}{2}\int_0^\beta d\tau + \mathcal{O}(r_0^{-1}) \approx \frac{\beta^2}{16\pi} = 4\pi M^2 . \qquad (4.109)$$

Then the highest-order contribution to the path integral for the partition function is $\exp[-\beta^2/(16\pi)]$, and the partition function itself takes the form

$$Z = \mathcal{N} \exp\left(\frac{-\beta^2}{16\pi}\right) , \qquad (4.110)$$

where

$$\mathcal{N} := \int_{4-\text{geometries}} \mathcal{D}[g_{\mu\nu}] \exp\left\{-\frac{I_2[\delta g_{\mu\nu}](\delta g_{\mu\nu})^2}{\hbar} - \mathcal{O}\left((\delta g_{\mu\nu})^3\right)\right\} . \qquad (4.111)$$

The first term in Eq. (4.111) is the so-called one-loop term representing the effect of quantum fluctuations around the background metric caused by the quantization of gravitons [106]. The higher-order terms cannot be renormalized—thus their infinite contribution is ignored altogether. We even ignore the one-loop correction to the action since we are interested in the thermodynamical properties of the background spacetime. Because of that the pure background contribution to the partition function (4.110) is

$$\log Z = -\frac{\beta^2}{16\pi} , \tag{4.112}$$

and the entropy S corresponding to the background spacetime geometry, given by Eq. (4.100), is

$$\begin{aligned} S &= \beta\langle E\rangle + \log Z \\ &= -\beta\frac{\partial \log Z}{\partial \beta} + \log Z \\ &= \beta\frac{\beta}{8\pi} - \frac{\beta^2}{16\pi} \\ &= 8\pi M^2 - 4\pi M^2 \\ &= 4\pi M^2 \\ &= \frac{1}{4}A . \end{aligned} \tag{4.113}$$

This is a most remarkable result and it agrees exactly with the Bekenstein-Hawking entropy of black holes obtained by using the first law of black hole thermodynamics and considering the particle-antiparticle pair creation on a fixed background.

What have we learnt from all of this? The lesson is that classical solutions in general relativity contribute to the entropy, and therefore gravity has intrinsic entropy [1]. At first sight this seems rather peculiar since we have only one classical background metric that gives rise to a non-zero entropy. This property is closely connected to the facts that general relativity allows the gravitational field to have different topologies and that the gravitational action is not scale invariant, i.e., the action transforms $I \longrightarrow k^2 I$ under the scale transformation $g_{\mu\nu} \longrightarrow k^2 g_{\mu\nu}$, where k is a constant. For more information about the implications of the scale transformation, see Ref. [103].

The key to the non-zero entropy lies in the fact that the gravitational action of the Euclidean background metric is only $\beta\langle E\rangle/2$ and not $\beta\langle E\rangle$. Since if the action were $\beta\langle E\rangle$ then the entropy of the background metric would be zero. This difference in the actions is explained by the different topologies of the Schwarzschild solution and the Euclidean section of the Schwarzschild solution. Hawking has shown that the total action of the Schwarzschild solution arises from four different surface terms of which the event horizon contributes the missing part $\beta\langle E\rangle/2$ of the action. The total action in that case is equal to $\beta\langle E\rangle$ which yields zero entropy as one would expect. On the other hand, we have just seen that the periodic spacetime has only one boundary two-sphere of radius $r = r_0$ contributing $\beta\langle E\rangle/2$ and, in particular, there is no boundary at the event horizon. The single boundary

[1] This is not a characteristic feature of other classical field theories.

of the periodic Schwarzschild spacetime gives rise to the action equal to $\beta\langle E\rangle/2$, yielding the entropy $\frac{1}{4}A$. Hence, from these qualitative arguments one can see that the entropy of the gravitational field is a consequence of different allowed topologies for the gravitational field. Moreover, the entropy can be seen as a consequnce of the existence of the event horizon, since the Euclidean Schwarzschild section does not include the event horizon nor the region $r < 2GM$, which gives rise to the action equal to $\beta\langle E\rangle/2$. Therefore one may think that neglecting this contribution is equal to as summing over all metrics for $r < 2GM$. If there were not an event horizon present in the background metric the corresponding entropy would then be zero. Therefore, the gravitational entropy $\frac{1}{4}A$ can be regarded as an intrinsic entropy of the event horizon of the Schwarzschild black hole.

Perhaps it is not so remarkable that similar results hold for Kerr-Newman black holes as well. The periodicity of the Euclidean section of stationary spacetimes is related to the surface gravity of the spacetime under consideration. Such a relation is always shown to lead to an entropy equal to $\frac{1}{4}A$.

4.5 Further Properties of the Partition Function of the Schwarzschild Black Hole

In this section we consider partition functions for the whole maximally extended Schwarzschild spacetime, i.e., Kruskal spacetime, and its right-hand-side exterior region in the approach based on the Hamiltonian dynamics of such spacetimes. This problem has been investigated by Kastrup [108], and by Louko and Whiting [67], and by Bose et. al [68], and by Mäkelä and Repo [15], and recently by Gour [109]. Kuchař gave a detailed analysis of the geometrodynamics of the classical Kruskal extension of the Schwarzschild spacetime [61], and Louko and Whiting adapted and applied Kuchař's analysis to the region exterior of a Schwarzschild black hole in a box with timelike boundary (see Sec. 3.5.). It was shown in Ref. [67] that a boundary term associated with the bifurcation two-sphere in the Lorentzian Hamiltonian of the exterior region with the timelike boundary yields the black hole entropy after perfoming the Wick rotation to the Hamiltonian operator of the system. Another way to look at the thermodynamics of these spacetimes is to consider the corresponding *Lorentzian* partition functions.

In general, one of the major difficulties in obtaining a Lorentzian partition function of a thermodynamical system is that one needs to know the density of the energy eigenstates. One may expect that if one assumes that the hole has $\exp(\frac{1}{4}A)$−fold degeneracy in the quantum states of the energy, then the Lorentzian counterpart of the reduced Hamiltonian operator of the Kruskal extension of the black hole would give rise to the correct Bekenstein-Hawking entropy. In fact, this will be shown in this section. Furthermore, we shall see that the Bekenstein–Hawking entropy of a black hole is reproducable from the statistical mechanics of the exterior region of the Schwarzschild black hole spacetime, even if we assume that there is no degeneracy in the mass eigenstates of the hole. That we choose to investigate the thermodynamics of the *exterior* region of the black hole spacetime may be justified on the grounds that the exterior region of the black hole is separated from the interior region by a horizon. Hence, an external observer cannot make any observations on the interior region, and one is justified to take a point of view that, for such an observer, physics of a black hole is physics of its exterior region.

4.5.1 Hamiltonian Thermodynamics

Black holes can be considered as thermodynamical systems in a heat bath of temperature T [59, 110, 111, 112]. Therefore, if $\hat{H}$ is the Lorentzian Hamiltonian operator of a black hole spacetime, the partition function of the system in a thermal equilibrium is

$$Z = \mathrm{Tr}\exp(-\beta\hat{H}) \ , \tag{4.114}$$

where $\beta = (k_B T)^{-1}$, k_B is Boltzmann's constant and T is the temperature of the system in the heat bath.

We first obtain the partition function corresponding to the whole maximally extended Schwarzschild spacetime. To do this, we have to substitute into Eq. (4.114) an operator counterpart $\hat{H}_{\text{whole}}$ of the Hamiltonian H_{whole}.

We saw in Sec. 3.5 that $H_{\text{whole}} = m$. Because of that, we define the Hamiltonian operator of the whole maximally extended Schwarzschild spacetime to be $\hat{H}_{\text{whole}} := \hat{m}$. In Sec. 4.3.1 we saw that during the recent years there has been increasing evidence that the mass spectrum of the black hole spacetime might be discrete. If we denote these discrete mass eigenvalues of the mass operator $\hat{m}$ by m_n $(n = 0, 1, 2, \ldots)$ and the corresponding eigenvectors by $|m_n\rangle$, we obtain the eigenvalue equation

$$\hat{H}_{\text{whole}}|m_n\rangle = \hat{m}|m_n\rangle = m_n|m_n\rangle \ . \tag{4.115}$$

When the discrete energy spectrum is employed, the partition function corresponding to the Kruskal spacetime becomes

$$Z_{\text{whole}}(\beta) = \mathrm{Tr}\exp(-\beta\hat{H}_{\text{whole}}) = \sum_{n=0}^{\infty}\exp(-\beta m_n) \ . \tag{4.116}$$

To actually calculate this partition function, we have to make two assumptions about the density of the mass states of the spacetime and the mass spectrum itself. First, it is natural to assume an $\exp(\frac{1}{4}A)$-fold degeneracy in the possible mass eigenvalues m_n of the hole. Since for a Schwarzschild black hole with mass m, $A = 16\pi m^2$ holds, we are prompted to define $g(m_n)$ as the number of degenerate states corresponding to the same mass eigenvalue m_n such that

$$g(m_n) = \exp(4\pi m_n^2) \ . \tag{4.117}$$

However, this definition is valid only at the semiclassical limit of the full quantum theory of gravity. Hence, our partition function for the Kruskal spacetime is a semiclassical approximation to the full partition function of a Schwarzschild black hole. Secondly, we have to assume that the mass eigenvalues m_n of the hole obey a specific spectrum. Bekenstein made a proposal that the possible eigenvalues of the area of the event horizon of the black hole are of the form given by Eq. (4.70). When imposing this proposal, we find that the partition function of the whole Schwarzschild black hole spacetime is

$$Z_{\text{whole}}(\beta) = \sum_{n=0}^{\infty} \exp\left(-\frac{\beta}{4}\sqrt{\frac{\gamma n}{\pi}} + \frac{\gamma n}{4}\right) , \tag{4.118}$$

where the summation is performed over different mass eigenvalues only. The partition function (4.118) diverges very badly, indeed, and we shall investigate this issue a bit later.

Let us turn our attention to the partition function corresponding to the *exterior* region of the Schwarzschild black hole spacetime. To obtain the partition function for the exterior region, we replace the operator $\hat{H}$ of Eq. (4.114) by an operator counterpart $\hat{H}_{\text{ext}}$ of H_{ext} and we require, as before, that the mass spectrum is discrete. The quantization of H_{ext} is performed simply by replacing m with its operator counterpart $\hat{m}$. Then, from the point of view of the observer at the right-hand-side asymptotic infinity, one gets

$$\hat{H}_{\text{ext}} = \hat{m} - 2N_0\hat{m}^2 , \tag{4.119}$$

where we have, again, fixed the value of the lapse function such that $N_+ = 1$.

In contrast to our discussion concerning the partition function of the whole spacetime, we assume the mass eigenstates of the exterior region of the Schwarzschild spacetime to be non-degenerate. This assumption of non-degeneracy will be justified by its consequences. We assume, again, that the possible eigenvalues of the area of the event horizon of the hole is given by Eq. (4.70). This assumption fixes the spectrum of the ADM mass of the spacetime observed at the asymptotic infinity. When these assumptions are used for the exterior region of the Schwarzschild spacetime, we get the following partition function:

$$Z_{\text{ext}}(\beta) = \sum_{n=0}^{\infty} \exp\left[-\beta\left(\frac{1}{4}\sqrt{\frac{\gamma n}{\pi}} - \frac{N_0}{8}\frac{\gamma n}{\pi}\right)\right] , \tag{4.120}$$

which also diverges.

We have obtained two diverging partition functions (4.118) and (4.120) for the Kruskal spacetime and for its right-hand-side exterior region. Kastrup has suggested some very original and interesting solutions to the divergency problem [108]. Gour, in turn, successfully investigated Schwarzschild black hole as a grand canonical ensemble [109]. Our solution to this problem is to study not the partition functions of spacetime itself but, instead, the partition function of the *radiation* emitted by the hole. First, according to the semiclassical picture, Hawking radiation is created by means of quantum mechanical processes just outside the black hole horizon [72]. To make use of our idea to consider partition functions of the radiation, we must assume that black holes evaporate in a reversible way, i.e., when black holes evaporate in a time-symmetric way they send out radiation in e^S ways [113]. In other words, we assume that the entropy of the Schwarzschild black hole is converted exactly into the entropy of the radiation. The validity of this assumption has been investigated by Zurek [114]. His conclusion was that if the temperature of the heat bath is the same as that of the hole, then the black hole evaporation is a reversible process. This is exactly the case in our approach to black hole thermodynamics, since we assumed that the black hole is in a thermal equilibrium in a heat bath. Let us put this assumption into practise. First, we choose the zero point of the energy emitted by the hole. This could be done in many ways, but we choose the total energy of the radiation emitted to be zero when the hole has evaporated completely leaving nothing but radiation. With this choice of the zero point of the total energy of the radiation, we find that the relationship between the energy E^{rad} emitted by the hole and the mass m of the Schwarzschild black hole measured at the asymptotic right-hand-side infinity is

$$E^{\text{rad}} = -m \quad . \tag{4.121}$$

Note that the zero point of the energy emitted by both the Kruskal and the exterior region spacetime can be chosen to coincide because the distant observer outside the hole observes the same energy E^{rad}.

Since all the entropy of the hole is assumed to be converted into the entropy of the radiation by means of reversible transitions between the energy eigenstates of the hole, the degeneracy of the energy of the radiation is, up to a normalization, the same as is the degeneracy of the black hole energy eigenstates. This means that, in the case of Kruskal spacetime, the number of degenerate states corresponding to the same total energy emitted by the hole since its formation up to the point where the Schwarzschild black hole has totally evaporated, is $\exp(\frac{1}{4}A_0)$, where A_0 is the initial surface area of the black hole event horizon, measured just before the hole has begun its evaporation, whereas in the case of the exterior region of Schwarzschild spacetime the number of degenerate states corresponding to the radiation energy of the hole is always zero. The number of the degenerate states corresponding to the same total energy emitted by the hole since its formation up to the point where the Schwarzschild mass has achieved the value m_n, is given by a function $g^{\text{rad}}(m_n)$. In a reversible proceses, all the entropy of the hole is exactly converted into the entropy of the radiation emitted by the hole. In that case we may choose the function $g^{\text{rad}}(m_n)$ for the Kruskal spacetime to be

$$g^{\text{rad}}(m_n) = \exp(\frac{1}{4}A_0 - 4\pi m_n^2) \quad , \tag{4.122}$$

This expression satisfies the intuitive properties of the degeneracy of the radiation energy eigenstates: $g^{\text{rad}}(m_n)$ increases when m_n decreases and the decrease of the black hole entropy from $\frac{1}{4}A$ to $\frac{1}{4}(A - dA)$ increases the number of degenerate states of the radiation emitted by the hole by a factor $\exp(\frac{1}{4}dA)$. Moreover, this choice reflects the fact that just after the hole has been formed, and not yet radiated, the entropy of the radiation is zero, whereas the entropy is $\frac{1}{4}A$ after the hole has evaporated completely.

Now, since $E^{\text{rad}} = -m$ and $H_{\text{whole}} = m$, we argue that

$$H^{\text{rad}}_{\text{whole}} = -m \quad , \tag{4.123}$$

and, also

$$H^{\text{rad}}_{\text{ext}} = -H_{\text{ext}} = 2N_0 m^2 - m \quad , \tag{4.124}$$

since all the energy of the exterior region is assumed to be converted into the energy of the radiation.

After quantizing the Hamiltonians (4.123) and (4.124), we can use exactly the same procedure as before to obtain our partition functions for the radiation. Eqs. (4.114), (5.61), (4.122), (4.70) and (4.123) yield

$$Z^{\text{rad}}_{\text{whole}}(\beta) = \exp\left(\frac{1}{4}A_0\right)\sum_{n=0}^{\infty}\exp\left(\frac{\beta}{4}\sqrt{\frac{\gamma n}{\pi}} - \frac{\gamma n}{4}\right) . \tag{4.125}$$

This partition function describes the radiation emitted by the Schwarzschild black hole. It is easy to see that $Z^{\text{rad}}_{\text{whole}}$ converges very nicely. The similarly obtained partition function corresponding to the exterior region of the Schwarzschild spacetime is

$$Z^{\text{rad}}_{\text{ext}}(\beta) = \exp\left(\frac{1}{4}A_0\right)\sum_{n=0}^{\infty}\exp\left[\frac{\beta}{4}\left(\sqrt{\frac{\gamma n}{\pi}} - \frac{N_0}{2}\frac{\gamma n}{\pi}\right)\right] , \tag{4.126}$$

which, when keeping N_0 fixed, converges, too. Here we have chosen an appropriate normalization constant to the partition function. This is allowed, since the normalization does not have any effects on the measurable thermodynamical quantities, like the temperature, of the system.

At the semiclassical limit the mass m of the hole is assumed to be very large. This implies, because of Hawking's expressions for the black hole temperature, that β is also very large. Thus, at the semiclassical limit, we may approximate the sums (4.125) and (4.126) by integrals [115]:

$$\begin{aligned} Z^{\text{rad}}_{\text{whole}}(\beta) &\approx \exp\left(\frac{1}{4}A_0\right)\int_0^{\infty} dn \exp\left(\frac{\beta}{4}\sqrt{\frac{\gamma n}{\pi}} - \frac{\gamma n}{4}\right) \\ &= \exp\left(\frac{1}{4}A_0\right)\left\{\frac{4}{\gamma} + \frac{\beta}{\gamma}\left[1 + \operatorname{erf}\left(\frac{\beta}{4\sqrt{\pi}}\right)\right]\right. && (4.127) \\ &\quad \left.\times \exp\left(\frac{\beta^2}{16\pi}\right)\right\} && (4.128) \end{aligned}$$

and

$$\begin{aligned} Z^{\text{rad}}_{\text{ext}}\ (\beta) &\approx \exp\left(\frac{1}{4}A_0\right)\int_0^{\infty} dn \exp\left[\frac{\beta}{4}\left(\sqrt{\frac{\gamma n}{\pi}} - \frac{N_0}{2}\frac{\gamma n}{\pi}\right)\right] \\ &= \exp\left(\frac{1}{4}A_0\right)\left\{\frac{8\pi}{\gamma\beta N_0} + \frac{4}{\gamma}\sqrt{\frac{2}{\beta}}\left(\frac{\pi}{N_0}\right)^{3/2}\left[\frac{1}{2}\right.\right. && (4.129) \\ &\quad \left.\left.+\frac{1}{2}\operatorname{erf}\left(\frac{\beta}{8N_0}\right)^{1/2}\right]\exp\left(\frac{\beta}{8N_0}\right)\right\} , && (4.130) \end{aligned}$$

where $\operatorname{erf}(x)$ is the error function.

If we choose

$$N_0 = \frac{2\pi}{\beta} , \tag{4.131}$$

then

$$Z^{\text{rad}}_{\text{whole}} = Z^{\text{rad}}_{\text{ext}} =: Z^{\text{rad}} . \tag{4.132}$$

This is a very interesting result. It should be noted that this result in not just an artefact of an approximation of a sum by an integral, but it holds even for exact expressions (4.125) and (4.126). We shall shortly discuss some of the possible consequences of our result at the end of this section. Let us, in the meantime, try to justify Eq. (4.131).

It was noted by Bose et al. that when Einstein's field equations are satisfied, the quantity N_0 can be expressed as $N_0 = \kappa\frac{dT}{dt}$ [68], where T is the Schwarzschild time coordinate, i.e., the Killing time t is the global time coordinate, and $\kappa = \frac{1}{4m}$ is the surface gravity of the black hole. If we now foliate the spacetime near the black hole horizon such that the foliation is determined by the Schwarzschild time coordinate T, then $\frac{dT}{dt} = 1$, and $N_0 = \frac{1}{4m}$. This kind of a foliation is justified on the grounds of our aim to describe the black hole thermodynamics from the point of view of a far-away observer at rest: The Schwarzschild time coordinate is just the time coordinate used by our external observer at rest when he makes observations on spacetime properties. This observation prompts us to make two more requirements: If one requires that, at the semiclassical limit,

$$N_0 \approx \frac{1}{4\langle m\rangle} \ , \tag{4.133}$$

and that the the energy expectation value of the radiation is:

$$\langle E^{\rm rad}\rangle := -\frac{\partial}{\partial\beta}\ln Z^{\rm rad}_{\rm ext}(\beta) = -\langle m\rangle \ , \tag{4.134}$$

then – as noted in Ref. [68] – one gets

$$\langle m\rangle \approx \frac{\beta}{4C} - \frac{2}{\beta} \ , \tag{4.135}$$

where C is a constant of integration. At the semiclassical limit this reduces to

$$\beta \approx 4C\langle m\rangle \ , \tag{4.136}$$

where the constant C can be chosen to be 2π, since it is well known from the semiclassical calculations that $\beta = 8\pi\langle m\rangle$. Now, substituting this into Eq. (4.133), we get the N_0 of Eq. (4.131). Hence, if we use a Schwarzschild -type foliation right from the beginning, we can obtain, up to a constant, our choice (4.131). In other words, the meaning of the choice (4.131) is that the spacetime foliation near the horizon of the Schwarzschild black hole is, in the semiclassical limit, determined by the Schwarzschild time coordinate T.

If Eq. (4.131) holds $Z^{\rm rad}_{\rm whole}$ and $Z^{\rm rad}_{\rm ext}$ coincide at the semiclassical limit, and the semiclassical partition function of the radiation observed by an external observer at asymptotic infinity is

$$Z^{\rm rad}(\beta) \approx \exp\left(\frac{1}{4}A_0\right)\frac{2\beta}{\gamma}\exp\left(\frac{\beta^2}{16\pi}\right) \ . \tag{4.137}$$

It is easy to show that the upper bound for the absolute error made, when replacing the sums (4.125) and (4.126) by integrals (4.128) and (4.130) is, in the leading order approximation, $\exp(1/4A_0 + \beta^2/16\pi)$. If one compares the result (4.137) to the absolute error made when replacing the sums by integrals, one notices that, for very large β, the fractional error is much smaller than unity. Hence, in the leading order approximation, the resulting partition function (4.137) approximates the sums (4.125) and (4.126) very well and, most importantly, the effect of the error bars on the thermodynamical quantities is negligibly small.

When calculcating the entropy S^{rad} of the radiation and the first-order correction to S^{rad}, one uses Eqs. (4.130), (4.133) and (4.135). This calculation gives the following corrected semiclassical entropy of radiation:

$$\begin{aligned} S^{\text{rad}} &= \ln Z - \beta \frac{\partial}{\partial \beta} \ln Z \\ &= \frac{1}{4}(A_0 - A) + \frac{1}{2}\ln A + \ln\left(\frac{4\sqrt{\pi}}{\gamma}\right) - \frac{127}{64} + \frac{129}{4096}A^{-1} \\ &\quad + \mathcal{O}\left(A^{-1/2}\right)\exp\left(-\frac{1}{4}A + \frac{1}{64} - \frac{1}{4096}A^{-1}\right) \quad . \end{aligned} \tag{4.138}$$

Hence, when the area of the black hole has shrinked from A_0 to A, the entropy carried away by the radiation is, in the leading order approximation, $\frac{1}{4}(A_0 - A)$. Under the assumption that the black hole radiation is a reversible process, this result is compatible with the Bekenstein-Hawking expression for black hole entropy: A decrease in the area by an amount $A_0 - A$ decreases the entropy of the hole by an amount $\frac{1}{4}(A_0 - A)$. The error made when approximating the sum by an integral causes an error in the entropy which is of the order $\mathcal{O}\left(A^{-1/2}\right)$.

It is most interesting that, in Lorentzian spacetime, the Bekenstein-Hawking entropy of the Schwarzschild black hole can be attained by considering the exterior region of the black hole only, without assuming any degeneracy for the mass eigenstates of the hole. Implications of this result are far from clear to us but they might point to the direction that interpretation of black hole entropy as something else than just as a logarithm of the number of microscopic states of the gravitational field corresponding to the same macrostate could also be worth of at least a tentative consideration.

4.6 Black Hole Thermodynamics in String Theory and Quantum Geometry

Physicists have made many brave attempts to explain the thermodynamical properties of black holes in a precise statistical mechanical manner and, in particular, reproducing the semiclassical Bekenstein-Hawking entropy is usually used as a preliminary test when counting the black hole microstates. Currently there are two ambitious theories under development that have been succesful in counting the black hole microstates and reproducing the correct black hole entropy. These theories are called the *string theory* and the *loop quantum gravity* or *quantum geometry*. In this section we shall shortly discuss these theories and their application to the black hole entropy.

4.6.1 Black Hole Entropy in String Theory

In a conventional field theory particles are considered pointlike fundamental objects, whereas in string theory a one-dimensional object called the *string* takes the role of the fundamental object. When a string moves in a higher-dimensional flat spacetime it forms a two-dimensional "world sheet" which has an intrinsic geometry $g_{\mu\nu}$ $(\mu, \nu = 1, 2)$. The coordinates X_μ of the world sheet points in a higher-dimensional spacetime can be considered as configuration variables of the theory and the string action can be written in terms of these variables. After finding the action for the string and matter fields on the string, the resulting string theory can be quantized as usual. The quantized string theory has certain infinities. When one requires that the infinities should vanish at the one-loop level then, remarkably, one finds that the background geometry of the finite quantized string theory must obey Einstein's field equations. Especially, when one considers the one-loop term in five dimensions, one finds that the string theory has five-dimensional black hole solutions. In other words, this particular model of string theory is two-dimensional quantum field theory possessing five-dimensional black hole solutions. These "stringy" black holes have a notion of event horizon and they emit Hawking radiation as well. In other words, the concept of black hole entropy can be assigned to these black hole solutions also. An interesting question then is, provided that the entropy of the five-dimensional black holes agrees with the Bekenstein-Hawking entropy, what is the microscopic origin of entropy of the five-dimensional black holes. By counting the number of the string states corresponding to the black hole solution Strominger and Vafa [116, 117] managed to derive the Bekenstein-Hawking entropy. While the string theory seems to have an explanation to the origin of the black hole entropy, the explanation was first provided for extremal black holes only. Since then string theorists have managed to count the quantum states of the string corresponding to the non-extremal black holes as well [118]. In spite of the great success, string theoretical approaches have some serious problems. First of all, it seems that the black hole entropy for extremal black holes could be zero [119, 120, 121, 122] and not $\frac{1}{4}A$ since the surface gravity κ is zero for extremal holes. Secondly, there is no physical reason why one should use more than four spacetime dimensions.

4.6.2 Black Hole Entropy in Loop Quantum Gravity

The idea in loop quantum gravity is to quantize general relativity. So loop quantum gravity is a quantum theory of spacetime itself. Therefore the theory cannot be formulated as a quantum field theory on spacetime, because there is no background metric in which spacetime evolves. However, loop quantum gravity is quantum field theory in spacetime without the metric. So far the main merit of loop quantum gravity is that it provides a mathematically consistent formulation of a background independent non-perturbative covariant quantum field theory of gravity. Especially, the theory has certain geometrical Hermitian operators such as the area and volume operator. The area operator is diagonal in the spin network states [13], and the area of a surface is determined by the spins on the lines that puncture the surface. The number of ways to obtain the same area is the exponential of the area times a still unknown constant. Thus, loop quantum gravity succeeds in calculating the microscopical states corresponding to the area, and especially, it succeeds in calculating the microscopical origin of the black hole entropy corresponding to the black hole event horizon [13]. The unknown constant should coincide with 1/4, but to obtain it, one should be able to find the classical limit of the theory. To find this limit one should be able to deal with the dynamics of the loop quantum gravity, but this aspect of the theory is not yet fully understood and therefore the factor $1/4$ is not yet accounted for. The loop quantum gravity approach to quantum gravity appeals relativists more than the string theory, but it has its own problems too. The whole construction of the loop quantum gravity is based on more or less convenient choices. Perhaps the most disturbing choice is made when the basis of the Hilbert space is taken to be the spin network basis [13].

Chapter 5

Quantum-Mechanical Models of Black Holes

5.1 Quantum-Mechanical Model of Reissner-Nordström Black Holes

After finding the classical Hamiltonian (3.141) that reflects the dynamical properties of Reissner-Nordström spacetimes, we are now prepared to go into the Hamiltonian quantization of such spacetimes.

5.1.1 Quantum Theory with Charge as an External Parameter

First we shall consider the electric charge as an external parameter of the theory. In what follows, we shall adopt a particular class of Hamiltonian quantum theories. More precisely, we choose our Hilbert space to be the space $L^2(\mathbb{R}^+, a^s da)$ with the inner product

$$\langle\psi_1|\psi_2\rangle := \int_0^\infty \psi_1^*(a)\psi_2(a)a^s\, da \quad , \tag{5.1}$$

where s is some real number. Through the substitution $p_a \longrightarrow -id/da$ we replace the classical Hamiltonian H of Eq. (3.127) with the corresponding symmetric Hamiltonian operator

$$\hat{H} := -\frac{1}{2}a^{-s}\frac{d}{da}\left(a^{s-1}\frac{d}{da}\right) + \frac{1}{2}a + \frac{Q^2}{2a} \quad . \tag{5.2}$$

Since the numerical value of the classical Hamiltonian H is the total (ADM) energy of the Reissner-Nordström hole, we can view the eigenvalue equation

$$\hat{H}\psi(a) = E\psi(a) \tag{5.3}$$

as an eigenvalue equation for the total energy of the hole, from the point of view of a distant observer at rest.

Before going into the detailed analysis of Eq. (5.3), let us pause for a moment to investigate some qualitative aspects of that equation. One finds, by substituting M for E, that Eq. (5.3) can be written in the form

$$a^{-s}\frac{d}{da}\left(a^{s-1}\frac{d}{da}\right)\psi(a) = \left(\frac{Q^2}{a} + a - 2M\right)\psi(a) \quad . \tag{5.4}$$

As one can see, the function $(\frac{Q^2}{a} + a - 2M)$ is negative, when $r_- < a < r_+$, and positive (or zero) elsewhere. Semiclassically, one may therefore expect oscillating behaviour from the wave function $\psi(a)$, when $r_- < a < r_+$, and exponential behaviour elsewhere. Hence, our system is somewhat analogous to a particle in a potential well such that a is confined, classically, between the outer and inner horizons of the black hole. What happens semiclassically is that the wave packet corresponding to the variable a is reflected from the future inner horizon. As a result we get, when the hole is in a stationary state, a standing wave between the outer and inner horizons. Thus, the classical incompleteness, associated with the fact that our foliation is valid only when $-\pi M \leq t \leq \pi M$, is removed by quantum mechanics: in a stationary state there are no propagating wave packets between the horizons, and our quantum theory is therefore valid in any moment of time.

Let us go into the detailed analysis of the eigenvalue equation (5.3). To begin with, we see, as in Ref. [66], that if we denote

$$x := a^{3/2} , \tag{5.5}$$

$$\psi := x^{-r}\chi(x) , \tag{5.6}$$

where we have defined

$$r := \frac{2s-1}{6}; \; s \geq 2 , \tag{5.7}$$

$$r := \frac{7-2s}{6}; \; s < 2 , \tag{5.8}$$

then Eq. (5.3) takes the form:

$$\frac{9}{8}\left[-\frac{d^2}{dx^2} + \frac{r(r-1)}{x^2} + \frac{4}{9}\left(x^{2/3} + \frac{Q^2}{x^{2/3}}\right)\right]\chi(x) = E\chi(x) . \tag{5.9}$$

The Hilbert space is $L^2(\mathbb{R}^+, dx)$ with the inner product

$$\langle\chi_1|\chi_2\rangle := \int_0^\infty \chi_1^*(x)\chi_2(x)\,dx . \tag{5.10}$$

It was shown in Ref. [14] that the energy spectrum in Eq. (5.9) is discrete, bounded below, and can be made positive. From the physical point of view, the semi-boundedness and positivity (in some cases) of the spectrum are very satisfying results: the semi-boundedness of the spectrum implies that one cannot extract an infinite amount of energy from the system, whereas the positivity of the spectrum is in harmony with the well-known positive energy theorems of general relativity which state, roughly speaking, that the ADM energy of spacetime is always positive or zero when Einstein's equations are satisfied. However, one can prove even more than that, and to show it we have to introduce the so-called *WKB approximation* [123] to the solutions of differential equations. The WKB approximation is reviewed in Appendix A.

5.1.2 Solution of the Energy Eigenvalue Equation for Large Energies and Charges

In this subsection we evaluate the large eigenvalues of the Hamiltonian operator $\hat{H}$ which was written in Eq. (5.2). This leads us to the eigenvalue equation (5.3) as we have already seen. We shall find the large eigenvalue solutions of the eigenvalue equation by using the WKB approximation method when both $|Q|$ and $E^2 - Q^2$ are, in natural units, much greater than unity and, in addition, we demand that $r_- \geq 1$ or, which is the same thing, $(2E-1)/Q^2 \leq 1$. The basic idea is to match the WKB approximation with an expression of the wave function in terms of modified Bessel functions close to the point where $a = 0$. The results used here on the matching of the WKB wave function and the Bessel function approximations close to the turning points are widely known – see for example Ref. [124] – and therefore we shall use them without any special review. Cases $r = 1/2$, $r \geq 3/2$, $r = 7/6$, $7/6 < r < 3/2$ and $1/2 < r < 7/6$ will be discussed separately. First we shall look for the general solution $\psi(a)$, when the argument a is very small, i.e., $|Q|a \ll 1$. After that we shall search for the solution for "slightly bigger" a, i.e., $|Q|a \leq M$, where M is an arbitrary positive number.

To begin with we recast the eigenvalue equation (5.9) in an appropriate manner. If we substitute into Eq. (5.9)

$$a := x^{2/3} , \tag{5.11}$$

$$\chi := a^{-1/4}u(a) , \tag{5.12}$$

we get

$$\left[\frac{d^2}{da^2} - \frac{(\frac{3}{2}r - \frac{1}{4})(\frac{3}{2}r - \frac{5}{4})}{a^2} - a^2 - Q^2 + 2Ea\right]u(a) = 0 , \tag{5.13}$$

where r is defined in Eqs. (5.7) and (5.8). Eq. (5.13) is invariant under the transformation $r \to 1 - r$; thus it is sufficient to consider solutions of the eigenvalue equation (5.13) for $r \geq 1/2$. As a consequence the inner product of Eq. (5.10) becomes

$$\langle u_1 | u_2 \rangle = \int_0^\infty u_1^*(a) u_2(a) a \, da . \tag{5.14}$$

We shall solve Eq. (5.13) when $E^2 - Q^2 \gg 1$, $|Q| \gg 1$ and $(2E-1)/Q^2 \leq 1$. For very small a, the linearly independent solutions to Eq. (5.13) are, when $r > 1/2$ and $r \neq 7/6$,

$$u_1(a) = Aa^{(3/2)r}[a^{-1/4} + \mathcal{O}(a^{7/4})] , \tag{5.15}$$

$$u_2(a) = Ba^{-(3/2)r}[a^{5/4} + \mathcal{O}(a^{13/4})] , \tag{5.16}$$

where A and B are constants. The case $r = 1/2$ will be considered later in this section, and if $r = 7/6$ then the term proportional to $a^{-(3/2)r+13/4}$ in Eq. (5.16) must be multiplied by a term proportional to $\ln(\frac{1}{2}|Q|a)$. The leading term, however, is the same as in Eq. (5.16) when $r > 1/2$.

By writing

$$x = |Q|a , \tag{5.17}$$

we get from Eq. (5.13)

$$\left[\frac{d^2}{dx^2} - \frac{(\frac{3}{2}r - \frac{1}{4})(\frac{3}{2}r - \frac{5}{4})}{x^2} - 1 - \frac{x^2}{Q^4} + \frac{2Ex}{|Q|^3}\right]u(x) = 0 . \tag{5.18}$$

Now the terms proportional to x^2 and x are asymptotically small at large $|Q|$, whenever $x \in (0, M]$, where M is an arbitrary positive constant. Omitting these last terms we get

$$\left[\frac{d^2}{da^2} - \frac{(\frac{3}{2}r - \frac{1}{4})(\frac{3}{2}r - \frac{5}{4})}{a^2} - Q^2\right] u(a) = 0 \ , \tag{5.19}$$

when the substitution (5.17) is inversed. The general linearly independent solutions are, when $\frac{3}{2}(r - 1/2)$ is not an integer, modified Bessel functions of the first kind, up to an overall normalization constant:

$$u_1(a) = a^{1/2} I_{\frac{3}{2}(r-1/2)}(|Q|a) \ , \tag{5.20}$$

$$u_2(a) = a^{1/2} I_{-\frac{3}{2}(2-1/2)}(|Q|a) \ . \tag{5.21}$$

If $\frac{3}{2}(r - 1/2)$ is an integer then the general solutions are, similarly,

$$u_1(a) = a^{1/2} I_{\frac{3}{2}(r-1/2)}(|Q|a) \ , \tag{5.22}$$

$$u_2(a) = a^{1/2} K_{\frac{3}{2}(r-1/2)}(|Q|a) \ , \tag{5.23}$$

where K_p is the modified Bessel function of the second kind of order p.

1. Case $r \geq 3/2$

We first consider the case $r \geq 3/2$. Throughout the discussion we shall assume that $E^2 - Q^2 > 0$. The solutions (5.15) and (5.16) to Eq. (5.13) are normalizable with respect to the inner product (5.10) only if the constant B vanishes. Now, a comparison with the asymptotic behaviour of the modified Bessel functions of Eqs. (5.20) and (5.21) for small a implies that the only normalizable solution for small a is

$$u(a) = C a^{1/2} I_{\frac{3}{2}(r-1/2)}(|Q|a), \tag{5.24}$$

when $\frac{3}{2}(r - 1/2)$ is not an integer. If $\frac{3}{2}(r - 1/2)$ is an integer, a comparison with Eqs. (5.22) and (5.23) gives similarly that the leading term is the same as in Eq. (5.24). To verify this, the Bessel functions must be expanded as their small a series. If we fix $\delta_1, \delta_2 > 0$ such that $\delta_1 \leq a \leq \delta_2$, the asymptotic large $|Q|$ behaviour of $u(a)$ is, up to a normalization constant,

$$u(a) \,\tilde{\propto}\, (2\pi|Q|)^{-1/2} \exp(|Q|a) \tag{5.25}$$

for all $r \geq 3/2$. From now on, the symbol $\tilde{\propto}$ is used for the asymptotic form at large $|Q|$, up to a possibly (E, Q)-dependent coefficient.

After a very small, small, and slightly bigger argument a we enter into the region $a \in (0, a_-)$, where a_- is the smaller turning point that for large E and $|Q|$ satifies $a_- \approx r_-$. Our aim is now to use the WKB approximation method to the wave function in the region in question. The WKB approximation corresponding to such a wave function $u(a)$, which decreases to the left of the turning point a_-, is

$$u_{\text{WKB}}(a) = [p_1(a)]^{-1/2} \exp\left[-\int_a^{a_-} p_1(a')da' + \eta_1\right] \ , \tag{5.26}$$

where

$$p_1(a) = \sqrt{a^2 + \frac{(3/2r - 1/4)(3/2r - 5/4)}{a^2} - 2Ea + Q^2} \quad . \tag{5.27}$$

The major problem in the WKB approximation involves the evaluation of the integral (5.26). It turns out, however, that it is not necessary to evaluate the WKB integral (5.26) at all: We are interested in solutions for small a, i.e., $|Q|a \ll 1$ and such solutions can be achieved easily by Taylor series. In the evaluation of the series it should be clear that $E^2 - Q^2$ and $|Q|$ are assumed to be very large and $(2E-1)/Q^2 \le 1$. Furthermore, we assume that $a \ge \delta_1$ such that δ_1^2 is negligible. The integral in the exponent in Eq. (5.26) is

$$S(a) := -\int_a^{a_-} da' \sqrt{a'^2 + Q^2 - 2Ea' - \frac{(3/2r - 1/4)(3/2r - 5/4)}{a'^2}} \quad . \tag{5.28}$$

With the help of Eq. (5.28) the exponent for small a in Eq. (5.26) can be written as

$$S'(\delta_1)(a - \delta_1) + \frac{1}{2}S''(\delta_1)(a - \delta_1)^2 + \mathcal{O}(\delta_1^2) \quad , \tag{5.29}$$

where S' denotes $\frac{dS}{da}$. Hence for small a, we have for the WKB wave function, given by Eqs. (5.26) and (5.29),

$$u_{\mathrm{WKB}}(a) \,\tilde{\propto}\, |Q|^{-1/2} \exp(|Q|a) \quad , \tag{5.30}$$

when the constant η_1 in Eq. (5.26) takes the form

$$\eta_1 = Q\delta_1 \quad . \tag{5.31}$$

This connects the WKB solution with the asymptotic solution in Eq. (5.25). In other words, the WKB solution decreases exponentially to the left of the turning point a_-.

We next enter into a region with oscillations. We let the energy E to be so large that the eigenvalue equation has two turning points. We denote them like before as a_- and a_+. It should be clear that $a_- \approx r_-$ and $a_+ \approx r_+$, when $E^2 - Q^2$ and $|Q|$ are large enough. The region of oscillations is far right of r_- and far left of r_+. As the wave function decreases exponentially right of the larger turning point and left of the smaller turning point, the WKB approximation to the solution, according to Eqs. (A.17) and (A.18), can be written far right of r_- as

$$u^{r_-}_{\mathrm{WKB}}(a) = C_1[p_2(a)]^{-1/2} \cos[\int_{a_-}^{a} da' p_2(a') - \frac{\pi}{4}] \quad , \tag{5.32}$$

and far left of r_+ as

$$u^{r_+}_{\mathrm{WKB}}(a) = C_1[p_2(a)]^{-1/2} \cos[\int_{a}^{a_+} da' p_2(a') - \frac{\pi}{4}] \quad , \tag{5.33}$$

where

$$p_2(a) := \sqrt{-a^2 + 2Ea - Q^2}\sqrt{1 - \frac{(3/2r - 1/4)(3/2r - 5/4)}{a'^2(-a'^2 - Q^2 + 2Ea')}} \quad . \tag{5.34}$$

The wave functions above are equal if

$$S := \int_{a_-}^{a_+} da\, p_2(a) = (n + \frac{1}{2})\pi \quad , \tag{5.35}$$

where $n \geq 0$ is an integer. This integral fixes the levels of the spectrum of the Reissner–Nordström black hole.

In the evaluation of the WKB integral (5.34), $E^2 - Q^2$ and $|Q|$ are assumed to be very large. We can expand the second square root in its Taylor series as the first square root is of order $\mathcal{O}(E^2 - Q^2)$ and the second term in the second square root is of order $\mathcal{O}(1/(a_-{}^2\sqrt{E^2 - Q^2}))$. Thus we can write the integral as

$$S = \int_{a_-}^{a_+} da \left[\sqrt{-a^2 - Q^2 + 2Ea} - \frac{(3/2r - 1/4)(3/2r - 5/4)}{2a^2\sqrt{-a^2 - Q^2 + 2Ea}} - \ldots \right] \quad . \tag{5.36}$$

Now that $a_- = r_- + \mathcal{O}(1/r_-{}^3)$ and $a_+ = r_+ - \mathcal{O}(1/r_+{}^3)$, the evaluation of the integral S can be done by parts, and by replacing the limits a_- and a_+ by r_- and r_+ the second integral gives us a term of the order of $\mathcal{O}(E/Q^3)$, which, on the grounds of the requirement $(2E - 1)/Q^2 \leq 1$, is small compared to the first term. Thus the second term can be omitted from S. In the evaluation of the third term from r_- to r_+ the integral does not converge. We therefore have to alter the integration region near the turning points by cho osing a couple of constants, namely, δ_3, $\delta_4 > 0$ such that we are able to restrict the argument a in the region $a_- < r_- + \delta_3/(E^2 - Q^2)^{1/8} \leq a \leq r_+ - \delta_4/(E^2 - Q^2)^{1/8} < a_+$ for large enough $E^2 - Q^2$. Then the limits can be replaced by $r_- + \delta_3/(E^2 - Q^2)^{1/8}$ and $r_+ - \delta_4/(E^2 - Q^2)^{1/8}$. Now the third term in the integral S gives us a term, which is at most of the order of $\mathcal{O}(r_-{}^{-4}(E^2 - Q^2)^{-1/4})$, which is small compared to the first term and can thus be omitted. The remaining integral is elementary and we obtain

$$S \approx \int_{r_-}^{r_+} da\, \sqrt{-a^2 + 2Ea - Q^2} = \frac{\pi}{2}(E^2 - Q^2) = (n + \frac{1}{2})\pi \quad . \tag{5.37}$$

This yields for large energies and charges, when $r \geq 3/2$ and $r_- \geq 1$, the WKB estimate

$$E^2 - Q^2 \sim 2n + 1 + o(1) \quad . \tag{5.38}$$

2. Cases $7/6 < r < 3/2$ and $1/2 < r < 7/6$

Now we are in a situation where we cannot just exclude either of the integration constants A or B in Eqs. (5.15) and (5.16) on the grounds of the normalizability of the wave function. However, the self–adjointness [125] of the Hamiltonian operator implies the following boundary condition for the solutions $u_{1,2}(a)$:

$$\lim_{a \to 0} \left[u_1^*(a)\frac{du_2(a)}{da} - \frac{du_1^*(a)}{da}u_2(a) \right] = 0 \quad . \tag{5.39}$$

Here u_1 and u_2 are two linearly independent, non–degenerate eigenfunctions. As shown in Eqs. (5.15) and (5.16) the differential equation $\hat{H}u(a) = Eu(a)$ has two small a solutions which satisfy all those conditions stated above

– at least when $E^2 - Q^2 > 0$. It is easy to show that, for very small a, the eigenfunctions of a self–adjoint Hamiltonian operator behave, up to normalization, as

$$u(a) \approx \cos(\theta) a^{(3/2)r-1/4} + \sin(\theta) a^{-(3/2)r+5/4} \quad , \tag{5.40}$$

where $\theta \in [0, \pi)$ is a parameter to be fixed later. Comparing the small a expansions of Eq. (5.40) with Eqs. (5.20) and (5.21) we can adjust the constants A and B such that $u(a)$ behaves asymptotically the following manner:

$$u(a) \tilde{\propto} a^{1/2} \Big[2^{\frac{3}{2}(r-1/2)} \Gamma(3/2r + 3/4) |Q|^{-\frac{3}{2}(r-1/2)} \cos(\theta) I_{\frac{3}{2}(r-1/2)}(|Q|a) \\ + 2^{-\frac{3}{2}(r-1/2)} \Gamma(-3/2r + 5/4) |Q|^{\frac{3}{2}(r-1/2)} \sin(\theta) I_{-\frac{3}{2}(r-1/2)}(|Q|a) \Big] \tag{5.41}$$

When $\theta = 0$, the second term in Eq. (5.41) vanishes and we can proceed just as in the case $r \geq 3/2$ from which it follows that the WKB estimate is given by Eq. (5.38). When $\theta \neq 0$ the second term in Eq. (5.41) dominates at large $|Q|$ and the asymptotic behaviour is as in Eq. (5.25). Therefore the WKB estimate is again given by Eq. (5.38).

3. Case $r = 7/6$

When $r = 7/6$, the number $\frac{3}{2}(r - 1/2)$ becomes an integer and the general solution of Eq. (5.19) includes modified Bessel functions of the second kind as shown before. Furthermore, we cannot rule out either of the adjustable constants A or B and therefore we have to keep both the solutions in Eqs. (5.22) and (5.23). As before we get from the boundary condition (5.39) that at least when $E^2 - Q^2 > 0$ the eigenfunction of a self–adjoint Hamiltonian operator is, for small a,

$$u(a) \approx \sin(\theta) a^{-1/2} + \cos(\theta) a^{3/2} \quad , \tag{5.42}$$

where again $\theta \in [0, \pi)$ is a parameter.

After expanding the general solution of Eq. (5.19) when a is small we have that $u(a)$ is asymptotically

$$u(a) \tilde{\propto} a^{1/2} [(2\cos(\theta)|Q|^{-1} - \sin(\theta)|Q|\gamma) I_1(|Q|a) + \sin(\theta)|Q| K_1(|Q|a)] \quad , \tag{5.43}$$

where γ is Euler's constant. When $\theta = 0$ the term proportional to $a^{1/2} K_1(|Q|a)$ vanishes and we get the same WKB estimate as before in Eq. (5.38). On the other hand, when $\theta \neq 0$, the term proportional to $a^{1/2} I_1(|Q|a)$ dominates for large $|Q|$ and the situation is quite the same as before. The resulting WKB estimate is therefore given by Eq. (5.38).

4. Case $r = 1/2$

When $r = 1/2$ we can no more write the solutions of Eq. (5.19) as powers of small a, because of the loss of linear independence of the solutions (5.15) and (5.16). By using the boundary condition (5.39), however, and expanding the general solution of Eq. (6.37) as $u(a) = a^{1/2}[C\, I_0(|Q|a) + D\, K_0(|Q|a)]$ for small a, we notice that Eq. (5.41) can be replaced by

$$u(a) \tilde{\propto} |Q|^{1/4} a^{1/2} [(\cos(\theta) - \sin(\theta)(\gamma + \ln(\frac{1}{2}|Q|))) I_0(|Q|a) - \sin(\theta) K_0(|Q|a)] \ . \tag{5.44}$$

For any θ the term proportional to $a^{1/2} I_0(|Q|a)$ dominates the term proportional to $a^{1/2} K_0(|Q|a)$ for large charges, and the asymptotic behaviour is given by Eq. (5.25). Thus the WKB result is, again, given by Eq. (5.38).

It should be noted that we have not investigated what happens when the condition $(2E-1)/Q^2 \leq 1$ following from the requirement $r_- \geq 1$ does not hold; i.e., when Q is arbitrarily small when compared to E. In that case, however, one expects that the WKB eigenenergies given by Eq. (5.38) should be replaced by those given in Ref. [66] for the Schwarzschild black hole. On the grounds of the results of Ref. [66] it is likely that the eigenvalues of the quantity $\sqrt{E^2 - Q^2}$ are of the form $\sqrt{2n}$ even when $|Q|$ is arbitrarily small compared to the black hole energy E.

We have shown that the analysis of Eq. (5.61) will yield the result that when $|Q| \gg 1$ and $E^2 - Q^2 \gg 1$, such that $r_- \geq 1$, the large eigenenergies E_n have a property

$$E_n^2 - Q^2 \sim 2n + 1 + o(1) \ , \tag{5.45}$$

where n is an integer and $o(1)$ denotes a term that vanishes asymptotically for large n. We have tested the accuracy of this WKB estimate numerically, and we have found that, up to the term 1 on the right hand side of Eq. (5.45), the WKB estimate (5.45) gives fairly accurate results even when $|Q|$ and n are relatively small (i.e., of order ten).[1] In other words, it seems that the eigenvalues of the quantity $\sqrt{E^2 - Q^2}$ are of the form $\sqrt{2n}$ in the semiclassical limit.

5.1.3 Positiveness of the Spectrum of the Quantity $E^2 - Q^2$

In this subsection we shall investigate the possible positiveness of the spectrum of the quantity $E^2 - Q^2$. Cases $r = 1/2$ and $r \geq 3/2$ and $1/2 < r < 3/2$ will be considered separately.

1. Case $r \geq 3/2$

Let $u(a)$ be an eigenfunction of Eq. (5.13) with any eigenvalue $E^2 - Q^2$. Now, when $r \geq 3/2$, we have, up to a well-chosen normalization constant, a small a expansion to the eigenfunction $u(a)$, given by Eq. (5.15) as $u(a) = a^{(3/2)r}[a^{-\frac{1}{4}} + \mathcal{O}(a^{7/4})]$. It is clear that $u(a)$ and $u'(a)$ are both real-valued and positive. Therefore the eigenfunction is positive and real-valued for sufficiently small a. It is easy to see that the eigenvalue equation (5.13) can be written as

$$u''(a) = \left[\frac{(\frac{3}{2}r - \frac{1}{4})(\frac{3}{2}r - \frac{5}{4})}{a^2} + (a - E)^2 - (E^2 - Q^2) \right] u(a) \ . \tag{5.46}$$

Let us assume that $E^2 - Q^2$ is not strictly positive, i.e., $E^2 - Q^2 \leq 0$. In that case Eq. (5.46) implies that $u''(a) > 0$ for all a such that $u(a) > 0$.

[1] We thank Dr. Matias Aunola for performing this numerical analysis to us.

Since both $u(a)$ and $u'(a)$ are positive for sufficiently small a, the positivity of $u''(a)$ whenever $u(a)$ is positive implies that $u'(a)$, and hence $u(a)$, are increasing functions of a. Because of that, we have $\lim_{a\to\infty} u(a) > 0$ and $u(a)$ is not normalizable. Hence we must have $E^2 - Q^2 > 0$.

2. Case $1/2 < r < 3/2$

Here we shall show that it is possible to find self-adjoint extensions of the Hamiltonian operator such that the spectrum is strictly positive. This corresponds to an appropriate choice of the parameter θ introduced in Eq. (5.40). We have already shown that when $7/6 < r < 3/2$ and $1/2 < r < 7/6$ the extensions take, up to an overall normalization constant, for small a the form given in Eq. (5.40), where the parameter θ specifies the self-adjoint extensions.

We consider first an extension with $\theta \in [0, \pi/2]$. By taking $E^2 - Q^2$ in Eq. (5.46) to be negative or zero, we get, when $u(a)$ is positive,

$$u''(a) \geq \frac{(\frac{3}{2}r - \frac{1}{4})(\frac{3}{2}r - \frac{5}{4})}{a^2} u(a) \quad \forall a > 0 \ . \tag{5.47}$$

To find a possible lower bound for $u(a)$ we therefore consider an equation

$$f''(a) = \frac{(\frac{3}{2}r - \frac{1}{4})(\frac{3}{2}r - \frac{5}{4})}{a^2} f(a) \ . \tag{5.48}$$

Now when $r > 1/2$ the general solution of Eq. (5.48) is

$$f(a) = A a^{(3/2)r - 1/4} + B a^{-(3/2)r + 5/4} \ , \tag{5.49}$$

and we see that if we choose $A = \cos\theta$ and $B = \sin\theta$ then $f(a)$ coincides with the small a solution (5.40). Since $u(a)$ is positive for sufficiently small a when both $\cos\theta$ and $\sin\theta$ are positive, we find that the solution $u(a)$ is equal or greater than the solution $f(a)$ for all $a \geq 0$ when $0 \leq \theta \leq \pi/2$. Hence the solution $u(a)$ does not vanish exponentially as a goes to infinity. Yet, the potential increases without bound as a increases towards infinity; therefore any eigenfunction must vanish exponentially at large a. Hence the spectrum must be strictly positive.

As to the remaining range $\theta \in (\pi/2, \pi)$ we refer to the results of Ref. [66], which state that the energy spectrum is bounded from below.

3. Case $r = 1/2$

For $r = 1/2$, Eqs. (5.47) and (5.48) still hold and Eq. (5.49) must be replaced by the solution

$$u(a) = (\cos\theta - \sin\theta) a^{1/2} - \sin\theta a^{1/2} \ln a \ . \tag{5.50}$$

When $\theta = 0$, $f(a)$ is positive for all $a \geq 0$ and one can proceed as before.

In the remaining range $0 < \theta < \pi$, the energy spectrum is bounded from below on the grounds of the results presented in Ref. [66].

We have shown that the eigenvalue equation (5.9) implies that when $r \geq 3/2$, the eigenvalues of the quantity

$$E^2 - Q^2$$

are *strictly positive*, and when $1/2 \leq r < 3/2$, the eigenvalues of the quantity $E^2 - Q^2$ can, again, be made positive by means of an appropriate choice of the boundary conditions of the wave function $\chi(x)$ at the point $x = 0$ or, more precisely, by means of an appropriate choice of a self-adjoint extension.

5.1.4 Quantum Theory with Charge as a Dynamical Variable

We proceed to quantize the charge, and we choose our Hilbert space to be the space $L^2(\mathbb{R}^+ \times \mathbb{R}, a^s\, da\, db)$ with the inner product

$$\langle\psi_1|\psi_2\rangle := \int_0^\infty da\, a^s \int_{-\infty}^\infty db\, \psi_1^*(a,b)\psi_2(a,b) \quad . \tag{5.51}$$

Through the substitutions $p_a \to -i\frac{\partial}{\partial a}$ and $p_b \to -i\frac{\partial}{\partial b}$ we replace the classical Hamiltonian H of Eq. (3.127) with the corresponding symmetric operator

$$\hat{H} := -\frac{1}{2}a^{-s}\frac{\partial}{\partial a}\left(a^{s-1}\frac{\partial}{\partial a}\right) - \frac{1}{2a}\frac{\partial^2}{\partial b^2} + \frac{1}{2}a + \frac{b^2}{2a} \quad . \tag{5.52}$$

As in Sec. 5.1.1, we can view the corresponding eigenvalue equation $\hat{H}\psi = E\psi$ as an eigenvalue equation for the total ADM energy of the Reissner-Nordström black hole, from the point of view of a distant observer at rest.

The eigenvalue equation $\hat{H}\psi = E\psi$ can be separated if we write

$$\psi(a,b) := \varphi(a)\beta(b) \quad , \tag{5.53}$$

and we get

$$\left[-\frac{1}{2}a^{-s}\frac{d}{da}\left(a^{s-1}\frac{d}{da}\right) + \frac{1}{2}a + \frac{Q^2}{2a}\right]\varphi(a) = E\varphi(a) \quad , \tag{5.54}$$

$$\left(-\frac{d^2}{db^2} + b^2\right)\beta(b) = Q^2\beta(b) \quad . \tag{5.55}$$

Eq. (5.54) is identical to Eq. (5.61), which is an eigenvalue equation for the total energy of the hole when Q is treated as an external parameter, whereas Eq. (5.55) can be understood as an eigenvalue equation for the square of the electric charge Q of the hole. When the eigenfunctions $\beta(b)$ are chosen to be the harmonic oscillator eigenfunctions, we find that the possible eigenvalues of Q^2 are of the form

$$Q_k^2 = 2k + 1 \quad , \tag{5.56}$$

or, in SI units

$$Q_k^2 = (2k+1)\frac{e^2}{\alpha} \quad , \tag{5.57}$$

where $k = 0, 1, 2, 3, \ldots$. In this equation, e is the elementary charge, and

$$\alpha := \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \tag{5.58}$$

is the fine structure constant. In other words, our theory implies that the electric charge of the Reissner-Nordström black hole has a *discrete spectrum*.

5.2 Quantum-Mechanical Model of Kerr-Newman Black Holes

After discussing the classical Hamiltonian theory of stationary spacetimes containing a Kerr-Newman black hole in Sec. 3.4, we are prepared to consider the canonical quantum theory of such spacetimes. In what follows, we shall concentrate on a specific class of canonical quantum theories. More precisely, we define the Hilbert space to be the space $L^2(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, R^s dR du dv)$ with the inner product

$$\langle \Psi_1 | \Psi_2 \rangle := \int_0^\infty dR R^s \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \Psi_1^*(R,u,v) \Psi_2(R,u,v) . \tag{5.59}$$

Through the substitutions $p_R \to -i\frac{\partial}{\partial R}$, $p_u \to -i\frac{\partial}{\partial u}$ and $p_v \to -\frac{\partial}{\partial v}$ we replace the classical Hamiltonian of Eq. (3.258) by the corresponding symmetric operator

$$\hat{H} := -\frac{1}{2}R^{-s}\frac{\partial}{\partial R}\left(R^{s-1}\frac{\partial}{\partial R}\right) - \frac{1}{2R}\frac{\partial^2}{\partial u^2} - \frac{1}{2R}\frac{\partial^2}{\partial v^2} + \frac{1}{2}R + \frac{u^2}{2R} + \frac{v^2}{2R} . \tag{5.60}$$

This operator may be viewed as the Hamiltonian operator of Kerr-Newman black holes. Its eigenvalues are eigenvalues of the ADM energy E of such a hole, from the point of view of a far-away observer at rest. The eigenvalue equation in question takes the form

$$\left[-\frac{1}{2}R^{-s}\frac{\partial}{\partial R}\left(R^{s-1}\frac{\partial}{\partial R}\right) - \frac{1}{2R}\frac{\partial^2}{\partial u^2} - \frac{1}{2R}\frac{\partial^2}{\partial v^2} + \frac{1}{2}R + \frac{u^2}{2R} + \frac{v^2}{2R}\right]\Psi(R,u,v) = E\Psi(r,u,v) \tag{5.61}$$

This equation is one of the main results of this thesis. In a certain sense, it can be considered as a sort of a "time-independent Schrödinger equation of all the possible black holes", and $\Psi(R,u,v)$ as the wave function of black holes. Specifying to the quantum theories, where $s=1$, we find that Eq. (5.61) takes a particularly simple and beautiful form

$$\frac{1}{2R}\left(-\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + R^2 + u^2 + v^2\right)\Psi(R,u,v) = E\Psi(R,u,v) . \tag{5.62}$$

If we write

$$\Psi(R,u,v) = \psi(R)\varphi_1(u)\varphi_2(v) , \tag{5.63}$$

we find that Eq. (5.61) can be separated into eigenvalue equations for M, Q^2 and a^2:

$$\left[-\frac{1}{2}R^{-s}\frac{d}{dR}\left(R^{s-1}\frac{d}{dR}\right) + \frac{1}{2}R + \frac{Q^2}{2R} + \frac{a^2}{2R}\right]\psi(R) = M\psi(R) , \tag{5.64}$$

$$\left(-\frac{d^2}{du^2} + u^2\right)\varphi_1(u) = Q^2\varphi_1(u) , \tag{5.65}$$

$$\left(-\frac{d^2}{dv^2} + v^2\right)\varphi_2(v) = a^2\varphi_2(v) . \tag{5.66}$$

Consider Eq. (5.64), the eigenvalue equation for the ADM mass M of the hole, in more details. It can be written as

$$\left[R^{-s}\frac{d}{dR}\left(R^{s-1}\frac{d}{dR}\right)\right]\psi(R) = \left(\frac{Q^2}{R} + \frac{a^2}{R} + R - 2M\right)\psi(R) . \tag{5.67}$$

As one can see, the function

$$\frac{Q^2}{R} + \frac{a^2}{R} + R - 2M$$

is negative when $r_- < R < r_+$ and positive (or zero) elsewhere. Semiclassically, one may therefore expect oscillating behaviour from the wave function when $r_- < R < r_+$ and exponential behaviour elsewhere. Hence, our system is somewhat analogous to a particle in a potential well such that R is confined, classically, between the outer and inner horizons of the Kerr-Newman black hole. What happens semiclassically is that the wave packet corresponding to the variable R is reflected from the inner horizon. As a result, we get, when the hole is in a stationary state, a standing wave between the outer and inner horizons. Thus the classical incompleteness, associated with the fact that our foliation is valid only when $-\pi M \leq t \leq \pi M$, is removed by quantum mechanics: In a stationary state there are no propagating wave packets between the horizons and our quantum theory is therefore valid at any moment of time.

When $a = Q = 0$, we have a Schwarzschild black hole, and the inner horizon is replaced by the black hole singularity: The wave packets are no longer reflected from the inner horizon but from the singularity. Again, we have a standing wave in a stationary state and the quantum theory is valid at any moment of time, but the wave lies between the Schwarzschild horizon and the singularity. As such there is an interesting resemblance between the properties of Eq. (5.61) and those of the Schrödinger equation of a hydrogen atom: When the hydrogen atom is in an s-state where the orbital angular momentum of the electron orbiting the proton vanishes, the electron should, classically, collide with the proton in a very short time. Quantum mechanically, however, the wave packet representing the electron is reflected from the proton, and finally the electron is represented by a standing wave, which makes the quantum theory of the hydrogen atom valid at any moment of time. In a Schwarzschild black hole, the proton is replaced by the black hole singularity, and the distance of an electron from the proton is replaced by the throat radius R of the hole. Nevertheless, the solution provided by quantum theory to the problems of the classical one is similar for both black holes and hydrogen atoms.

We shall now enter the detailed analysis of the eigenvalue equation (5.64). To begin with, we see that if we denote

$$x := R^{3/2} , \tag{5.68}$$

$$\psi := x^{(1-2s)/6}\chi(x) , \tag{5.69}$$

and define

$$\rho := \frac{2s-1}{6} , s \geq 2 , \tag{5.70}$$

$$\rho := \frac{7-2s}{6} , s < 2 , \tag{5.71}$$

then Eq. (5.64) takes the form

$$\frac{9}{8}\left[-\frac{d^2}{dx^2} + \frac{\rho(\rho-1)}{x^2} + \frac{4}{9}\left(x^{2/3} + \frac{Q^2+a^2}{x^{2/3}}\right)\right]\chi(x) = M\chi(x) . \tag{5.72}$$

This equation has been analyzed in details in Sec. 5.1. The only difference between Eq. (5.72) and Eq. (5.9) is that Q^2 has been replaced by $Q^2 + a^2$. Hence one just replaces Q^2 by $Q^2 + a^2$ in the results obtained for Eq. (5.9) in Sec. 5.1.

As in Ref. [66], one can show that the spectrum of M is discrete, bounded from below, and can be made positive. Furthermore, one can show that the eigenvalue equation (5.72) implies that when $\rho \geq 3/2$, the eigenvalues of the quantity

$$M^2 - Q^2 - a^2$$

are strictly positive, and when $1/2 \leq \rho < 3/2$, the eigenvalues of the quantity $M^2 - Q^2 - a^2$ can again be made positive by means of an appropriate choice of the boundary conditions of the wave function $\chi(x)$ at the point $x = 0$ or, more precisely, by means of an appropriate choice of a self-adjoint extension. Moreover, the WKB analysis of Eq. (5.72) yields the result that when $Q^2 + a^2 \gg 1$, and $M^2 - Q^2 - a^2 \gg 1$ such that $r \gg 1$, the WKB eigenvalues M_n have a property

$$M_n^2 - Q^2 - a^2 \sim 2n + 1 + o(1) , \tag{5.73}$$

where n is an integer and $o(1)$ denotes a term that vanishes asymptotically for large n. A numerical analysis of Eq. (5.72) yields the result that, up to the term 1 on the right hand side, Eq. (5.73) gives fairly accurate results even when $\sqrt{Q^2 + a^2}$ and n are relatively small (i.e., of order 10). In other words, it seems that the eigenvalues of the quantity $\sqrt{M^2 - Q^2 - a^2}$ are of the form $\sqrt{2n}$ in the semiclassical limit.

Before considering the implications of Eq. (5.73), let us calculate the spectra of Q and a from Eqs. (5.65) and (5.66). As one can see, Eq. (5.65) is exactly the same as Eq. (5.55). Hence the eigenvalues of Q^2 are given by Eq. (5.56).

Let us turn our attention to Eq. (5.66) which gives the spectrum of a^2. As for the electric charge, we find that the possible eigenvalues of a^2 are

$$a_l^2 = 2l + 1 \tag{5.74}$$

or, in SI units,

$$a_l^2 = (2l + 1)\frac{\hbar G}{c} , \tag{5.75}$$

where $l = 0, 1, 2, \ldots$. Again, one observes that the quantity under consideration is quantized in such a way that $\hbar$ is multiplied by an integer. Putting Eqs. (5.56), (5.73) and (5.74) together we find that, semiclassically, the mass eigenvalues of the black hole are

$$M_m \sim \sqrt{2m} \tag{5.76}$$

or, in SI units,

$$M_m \sim \sqrt{2m} M_{\mathrm{P}\,l} , \tag{5.77}$$

where

$$m := n + l + k = 0, 1, 2, \ldots , \tag{5.78}$$

and

$$M_{\mathrm{P}\,l} = \sqrt{\frac{\hbar c}{G}} \tag{5.79}$$

is the Planck mass.

Consider the angular momentum spectrum of black holes. We observe from Eqs. (5.75), (5.77) and (5.78) that the possible eigenvalues of the angular momentum $J = Ma$ of the hole are, semiclassically, of the form

$$J_{n,l,k} \sim \pm 2\sqrt{l(l+n+k)}\hbar \ . \tag{5.80}$$

For an uncharged black hole where $k = 0$ we therefore find, in the limit of extremality where $l \gg n$, that the angular momentum eigenvalues are of the form

$$J_{m_j} \sim m_j \hbar \ , \tag{5.81}$$

where $m_j = 0, \pm 2, \pm 4, \ldots$.

5.3 Concluding Remarks

In this chapter we have considered two particular quantum-mechanical models concerning Reissner-Nordström and Kerr-Newman black holes. Our analysis of Kerr-Newman spacetimes produced Eq. (5.61) which, in a certain very restricted sense, may be considered as a sort of "Schrödinger equation of black holes". That equation gives, in the context of our Kerr-Newman black hole model, the mass, the electric charge and the angular momentum spectra of Kerr-Newman black holes. When the angular momentum is zero, the "Schrödinger equation" gives the mass and the electric charge spectra of Reissner-Nordström black holes. In fact the results presented in Sec. 5.1 can be obtained from the results given in Sec. 5.2. Moreover, our "Schrödinger equation of black holes" gives the results found in Ref. [66] for Schwarzschild black holes.

Eq. (5.61) implied that the mass, the electric charge and the angular momentum spectra of black holes are discrete. Moreover, it implied that the mass spectrum is bounded from below and can be made positive. By means of a choice of an appropriate self-adjoint extension we showed that the spectrum of the quantity

$$M^2 - Q^2 - a^2 \ ,$$

where a is the angular momentum per unit mass of the hole, is always positive.

Now, how should we understand these results? The positivity of the spectrum of the quantity $M^2 - Q^2 - a^2$ has an interesting consequence regarding Hawking radiation: If one thinks of Hawking radiation as an outcome of a chain of transitions from higher- to lower-energy eigenstates, the positivity of the spectrum of $M^2 - Q^2 - a^2$ implies that a non-extreme Kerr-Newman black hole can never become, through Hawking radiation, an extreme black hole with zero temperature, as well as in the case when $a = 0$ the positivity of the spectrum of $M^2 - Q^2$ implies that a non-extreme Reissner-Nordström black hole can never become, through Hawking radiation, an extreme Reissner-Nordström black hole with zero temperature. These results are in agreement with both the third law of thermodynamics and the qualitative difference between extreme and non-extreme black holes. One may consider this as a strong argument in favor of our models.

The spectra of the quantities M, Q and a have interesting consequences regarding the area spectrum of black holes. As it is well known, the area of the outer horizon of a Kerr-Newman black hole is

$$A^{+} = 4\pi(r_{+}^{2} + a^{2}) \ , \tag{5.82}$$

whereas the area of the inner horizon is

$$A^{-} = 4\pi(r_{-}^{2} + a^{2}) \ . \tag{5.83}$$

Using Eqs. (5.73), (5.74) and (5.56) we observe that the semiclassical eigenvalues of the quantity

$$A^{tot} := A_{+} + A_{-} \ , \tag{5.84}$$

which we shall call, for the sake of convenience, the *total area* of a black hole, are of the form

$$A^{tot}_{n,l,k} \sim 16\pi(2n + 2l + k) \tag{5.85}$$

or, in SI units,

$$A^{tot}_{n,l,k} \sim 16\pi(2n + 2l + k)l_{Pl}^{2} \ , \tag{5.86}$$

where

$$l_{Pl} := \sqrt{\frac{\hbar G}{c^{3}}} \tag{5.87}$$

is the Planck length. In other words, we have obtained a result which is closely related, although not quite identical to Bekenstein's proposal (4.70). In other words, we have obtained a result which states that the total area of the hole, with $\gamma = 16\pi$, instead of the area of its inner horizon, is quantized as in Eq. (4.70). (In contrast to our result (5.86) and to Bekenstein's proposal (4.70), in [126] C. Vaz and L. Witten interestingly found that the *difference* between the outer and inner horizon areas is quantized in integer Planck units.) Later in this section we shall consider in more details the possibility that it is perhaps the total area, and not the area of the outer horizon nor the difference between the outer and inner horizon areas, which should be an integer in Planck units. A similar result holds for the Reissner-Nordström black hole horizon areas: The area of the apparent horizon of the Reissner-Nordström hole is, in natural units,

$$A_{+} := 4\pi(M + \sqrt{M^{2} - Q^{2}})^{2} \ , \tag{5.88}$$

and

$$A_{-} := 4\pi(M - \sqrt{M^{2} - Q^{2}})^{2} \tag{5.89}$$

is the area of the inner horizon. We get

$$A^{tot} = 16\pi(M^{2} - Q^{2}) + 2A^{ext} \ , \tag{5.90}$$

where

$$A^{ext} := 4\pi Q^{2} \tag{5.91}$$

is the area of an extremal black hole. Eq. (5.38) now gives the spectrum of the total area of the horizons:

$$A^{tot}_{n} \sim 32\pi(n + \frac{1}{2})l_{Pl}^{2} + 2A^{ext} + o(1) \ . \tag{5.92}$$

Although our result about an equal spacing for the spectrum of the sum of the horizon areas may have certain esthetic merits, it also involves some problems. For instance, the fact that the mass eigenvalues are of the form $\sqrt{2m}$ which, together with the fact that Q and a have similar spectra, implied the area spectrum under consideration, also implies that the angular frequencies of quanta of Hawking radiation emitted in transitions between nearby states is

$$\omega \approx \frac{1}{M} . \tag{5.93}$$

For Schwarzschild black holes this is something one might expect because the Hawking temperature of such a hole is given by Eq. (4.58), and therefore it follows from Wien's displacement law that the maximum of the thermal spectrum of black hole radiation corresponds to the angular frequency

$$\omega_{max} \propto \frac{1}{M} . \tag{5.94}$$

In other words, the angular frequency associated with the discrete spectrum of Hawking radiation as predicted by our model, behaves, as a function of M, in the same way as does the angular frequency corresponding to the maximum of the thermal spectrum as predicted by Hawking and others (see Eq. (4.61)).

Unfortunately, this nice correspondence between Hawking's result and our model breaks down when Q or a are different from zero. In that case the Hawking temperature (4.58) of the black hole is

$$T_{\mathrm{H}} = \frac{\sqrt{M^2 - Q^2 - a^2}}{2\pi \left[\left(M + \sqrt{M^2 - Q^2 - a^2} \right)^2 + a^2 \right]} , \tag{5.95}$$

and one finds that the maximum of the thermal spectrum corresponds to the angular frequency

$$\omega_{max} \propto \frac{\sqrt{M^2 - Q^2 - a^2}}{\left(M + \sqrt{M^2 - Q^2 - a^2} \right)^2 + a^2} . \tag{5.96}$$

In other words, the angular frequency (5.93) predicted by our model corresponds, when the hole is near extremality, to a temperature which is much *higher* than the Hawking temperature.

However, there may be a possible way out of this problem. In all our investigations we have emphasized the importance of the dynamics of the intermediate region between the horizons of the black hole. The dynamics of this intermediate region is, in our model, responsible for the discrete eigenvalues of the mass, electric charge and angular momentum of the hole. Now, if we take this point of view to its extreme limits we are tempted to speculate that both of the horizons of the hole, acting as the boundaries of the intermediate region, may participate, in one way or another, in the radiation process of the black hole. In other words, both of the horizons may radiate. The radiation emitted by the inner horizon is probably emitted inside the inner horizon, and is therefore not observed by the external observer. Nevertheless, an emission of this radiation is likely to reduce considerably the number of quanta, and hence the temperature, of the radiation coming out from the hole: The more the inner horizon radiates, the less quanta are left for the outer horizon.

Let us give up for a moment our resistance to this most charming temptation and have a play with the thought that both of the horizons have an important role in black hole radiation. For instance, one might consider one quarter of the total area of the hole as a sort of "total entropy" of the hole:

$$S_{tot} = \frac{1}{4}(A^+ + A^-) . \tag{5.97}$$

Moreover, one might be inclined to define a temperature T corresponding to this entropy (whatever that means) such that

$$\frac{1}{T} := \frac{\partial S_{tot}}{\partial E} , \tag{5.98}$$

and one finds, quite unexpectedly, that

$$T = \frac{1}{8\pi M} . \tag{5.99}$$

In other words, we have recovered the Hawking temperature of the Schwarzschild black hole (see Eq. (4.58)). This expression is the same for all black holes, and it is inversely proportional to the mass M of the hole. It may well be that all this is just meaningless play with symbols, without any physical content, but nevertheless the idea that it is the total area, and not the area of the outer horizon, which is of fundamental importance in black hole quantum mechanics, appears to possess remarkable internal consistency: If the total area of the hole has equal spacing in its spectrum, one expects the temperature of the hole to be inversely proportional to the mass M, and this result is recovered if the total entropy of the hole is taken to be one quarter of not the area of the outer horizon but of the total area of the hole. These ideas might be worth of a more detailed investigation in the future.

The WKB spectrum (5.73) has yet another property which is of some interest. It follows from the mass formula of black holes that the ADM mass of a rotating electrovacuum black hole is, from the point of view of a distant observer at rest [127, 124],

$$M = \frac{\kappa}{4\pi}A + \Phi Q + \Omega J . \tag{5.100}$$

In this equation, κ is the surface gravity, A is the area of the (outer) horizon, Φ is the electric potential of the hole, and Ω is the angular velocity of the horizon. For a Kerr-Newman black hole we have

$$\kappa = \frac{2\pi(r_+ - r_-)}{A} , \tag{5.101}$$

$$\Phi = \frac{Qr_+}{r_+^2 + a^2} , \tag{5.102}$$

$$\Omega = \frac{a}{r_+^2 + a^2} . \tag{5.103}$$

Eq. (5.73) implies that the WKB eigenvalues of the quantity $\frac{\kappa}{4\pi}A$ are of the form $\sqrt{2n}$. The physical interest of this result lies in its charge independence.

In other words, the spectrum of the quantity $\kappa A/4\pi$ is the same for the Kerr-Newman, Reissner-Nordström and Schwarzschild black holes. Moreover, one may be tempted to regard the quantity ΦQ as a sort of electromagnetic energy, ΩJ as a sort of rotational energy, and $\kappa A/4\pi$ as a sort of gravitational energy of the hole. Hence the gravitational energy appears to have the same spectrum whether or not the hole is charged or rotating or neither. Such a feature is not posed by Bekenstein's proposal (4.70).

Consider the charge spectrum (5.56). It is interesting that the electric charge of the primordial Reissner-Nordström and Kerr-Newman black holes is quantized in terms of the "Planck charge" $e/\sqrt{\alpha}$ in exactly the same way as the quantity $\sqrt{M^2 - Q^2 - a^2}$ is quantized in terms of the Planck mass $m_{Pl} := (\hbar c/G)^{1/2}$, i.e., it is proportional to $\sqrt{2n}$. In fact, one may have very mixed feelings about the physical validity of the charge spectrum of Eq. (5.56): For elementary particles at least, the electric charge Q itself, instead of its square Q^2, is an integer. Because of that it might appear at the first sight that the charge spectrum we have just obtained contradicts all the possible observations and expectations, and should therefore be rejected on physical grounds.

Such a conclusion, however, would be much too rapid. Firstly, elementary particles are certainly not black holes because for them $|Q| \gg M$. Secondly, a dimensional investigation reveals that the charge spectrum (5.56) is exactly what one expects for primordial black holes: when one writes the electric charge in terms of the natural constants ϵ_0, $\hbar$ and c, one finds that the natural unit of electric charge is the so called "Planck charge"

$$Q_{Pl} := \sqrt{4\pi\epsilon_0 \hbar c}\,. \tag{5.104}$$

One observes that the square Q_{Pl}^2 of the Planck charge Q_{Pl}, instead of the Planck charge Q_{Pl} itself, is proportional to $\hbar$. Now, for bounded systems, the observed quantities usually tend to be quantized in such a manner that when we write that quantity in terms of the natural constants relevant to the system under consideration, then $\hbar$ must be multiplied by an integer in the spectrum. In a hydrogen atom, for instance, the relevant natural constants are ϵ_0, $\hbar$, e and the mass m_e of the electron. From these quantities one may construct a natural unit of energy in a hydrogen atom:

$$\frac{m_e e^4}{(4\pi\epsilon_0)^2\hbar^2}\,,$$

and one expects the energy to be quantized such that the energy eigenvalues are of the form

$$E_n = -\gamma\frac{m_e e^4}{(4\pi\epsilon_0)^2\hbar^2 n^2}\,, \tag{5.105}$$

where γ is some pure number and n is an integer. Indeed, if we take $\gamma = 1/2$, we get exactly the correct energy spectrum for the hydrogen atom. Now, for black holes the only natural constants we are allowed to use are, in SI units, $\hbar$, c, G and ϵ_0. Hence, the Planck charge Q_{Pl} of Eq. (5.104) is the natural unit of charge for black holes, and therefore one expects that the square of the electric charge, instead of the charge itself, must be an integer. In other words, the charge spectrum (5.56) is exactly what one expects for primordial black holes.

In addition to the dimensional arguments, there is yet another reason why the electric charge of the black hole does not necessarily have the same spectrum as ordinary matter. Consider the conformal diagram of Fig. 3.2 of the Reissner-Nordström black hole. It is easy to see that the spacelike hypersurfaces $t = constant$ never touch the singularity $R = 0$ of the black hole. From this it follows that the lines of force of the electric field on these hypersurfaces neither begin nor end anywhere (if they did, they would do so at the singularity $R = 0$). Beacuse of that it is not possible to talk about the electric charge of a primordial black hole in the same sense as we talk about electric charge of ordinary matter: For ordinary matter the charge lies at the point where the lines of force of the electric field either begin or end but for primordial black holes no such point exists. Hence it appears that what an external observer observes as the "electric charge" of the black hole is a consequence from the geometrical and causal properties of a black hole spacetime, rather than from the properties of matter. Since the electric charge of the black hole is not necessarily connected with the electric charge of ordinary matter, it does not necessarily have the same spectrum, either.

Consider the angular momentum spectrum (5.74). As one can see, the angular momentum spectrum of black holes, as predicted by our theory is, at least in the limit of extremality, exactly what one might expect. Even the fact that the angular momentum J is an even number, is in harmony with our expectations: When the black hole performs a transition from one angular momentum eigenstate to another, a graviton is emitted or absorbed. Because the spin of a graviton is two, one might expect that the angular momentum of the black hole could change only by an even number. For instance, one may show, quite rigorously, that when a system consisting of two mass points revolving around their common center of mass emits or absorbs a graviton, the angular momentum quantum number of the system can change only by an even number (see the next chapter). Because of that, the angular momentum spectrum given by Eq. (5.81) for extremal black holes may be used as a very strong argument in favor of the physical validity of our quantum-mechanical model of black holes.

Unfortunately, our model also appears to contain a very serious problem regarding the angular momentum spectrum: According to Eq. (5.80) the angular momentum of a black hole is not in general an integer times the Planck constant $\hbar$. Should we be worried because of this result?

The answer to this question is: Not necessarily. The usual rules for the quantum mechanics of angular momentum follow from the symmetries of *flat spacetime*, and spacetime containing a Kerr-Newman black hole is certainly not flat. In curved spacetime the angular momentum eigenvalues of a system do not necessarily have the same properties as they would have in flat spacetime.

To illustrate this fact by a simple example, consider a particle moving in a conelike spacetime geometry (see Fig. 5.1). The z-component L_z of the angular momentum eigenvalues may be calculated from the equation

$$-i\hbar\frac{\partial}{\partial\phi}\Psi(\phi) = L_z\Psi(\phi) , \tag{5.106}$$

from which it follows that the angular momentum eigenfunctions are of the

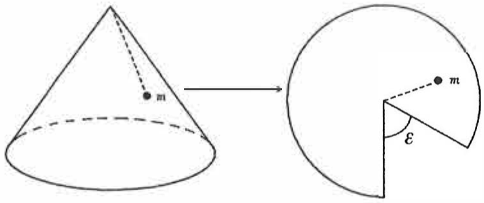

Figure 5.1: A particle moving in a conelike spacetime geometry. When the cone is stretched on a plane, the deficit angle ε appears. As a result of the appearance of this deficit angle, the periodic boundary condition for the angular momentum eigenfunction $\psi(\phi)$ is $\psi(\phi + 2\pi - \varepsilon) = \psi(\phi)$.

form

$$\Psi(\phi) = Ce^{\frac{i}{\hbar}L_z\phi} , \tag{5.107}$$

where C is a constant. In flat spacetime the period of $\Psi(\phi)$ is 2π, producing the usual angular momentum spectrum. In a conelike spacetime geometry, however, the period of Ψ is *not* 2π but $2\pi - \epsilon$, where ϵ is the deficit angle of the cone (see Fig. 6.3). In other words, we must have

$$\Psi(\phi + 2\pi - \epsilon) = \Psi(\phi) , \tag{5.108}$$

and therefore the angular momentum eigenvalues are of the form

$$L_z = m_z \frac{1}{1 - \frac{\epsilon}{2\pi}} \hbar , \tag{5.109}$$

where $m_z = 0, \pm 1, \pm 2, \ldots$. In other words, the angular momentum of a system is not necessarily an integer times the Planck constant in curved spacetime.

As a final test of our model, we find that the possible eigenvalues of the area $A^{ext} := 4\pi(Q^2 + 2a^2)$ of an extreme Kerr-Newman black hole are of the form

$$A_k^{ext} = 8\pi(k + \frac{1}{2})l_{Pl}^2 . \tag{5.110}$$

In other words, we have recovered Bekenstein's proposal (4.70), with $\gamma = 8\pi$, for a black hole near extremality. One can also see from Eq. (5.85) that even when the electric charge and the angular momentum are made dynamical variables, the spectrum of the total area of the horizons of the Kerr-Newman hole is the same as, according to Bekenstein, is the area spectrum of the outer horizon of the hole. However, it should be noted that when k is very big, the difference between two successive charge eigenvalues is

$$Q_{k+1} - Q_k \approx \frac{e^2}{\alpha Q_k} , \tag{5.111}$$

and we find that we have, in practice, continuous charge spectra.

To conclude, our quantum-mechanical models of Reissner-Nordström and Kerr-Newman black holes appear to involve several physically sensible properties but also some problems. For instance, the claim that Kerr-Newman spacetime and our phase space variables satisfy the assumptions of Regge's and Teitelboim's theorem has been left unproved. The proper analysis of the Hamiltonian dynamics of Kerr-Newman spacetimes along the lines shown by Kuchař for Schwarzschild spacetime should therefore be performed [61]. However, we have shown that the assumptions of Regge's and Teitelboim's theorem hold for the Reissner-Nordström spacetimes.

Another problem is, whether the quantum mechanics of black holes can be described with a sufficient accuracy by means of a model having just two or three independent degrees of freedom. In other words, are the mass, the electric charge and the angular momentum spectra obtained from our model reliable? When answering this question one can just say that at least the spectra are such as one might expect on semiclassical and dimensional grounds. As to the problems related to the statistical origin of black hole entropy and things like that our model says nothing.

A somewhat analogous situation can already be met with in ordinary quantum mechanics. Consider a hydrogen atom. In elementary textbooks, the only degrees of freedom under consideration are the three degrees of freedom associated with the electron going around the proton. In more advanced textbooks, however, a student is revealed that not only should one quantize the degrees of freedom associated with the electron but also the degrees of freedom associated with the electromagnetic field. As a result, one gets an enormous number of degrees of freedom associated with the virtual photons and electron-positron pairs appearing as an outcome of the quantization of the electromagnetic field. In other words, although classically we have, in effect, only the degrees of freedom associated with the electron, the full quantum theory with quantized electromagnetic field reveals an enormous number of particles and an enormous number of degrees of freedom. However, the whole contribution of all these additional degrees of freedom to the energy levels of the hydrogen atom is very small. Now, something similar may happen with black holes: classically, the number of relevant degrees of freedom is very small, but when the full quantum theory of gravity is employed, an enormous amount of degrees of freedom are likely to appear. Hence, one may feel tempted to regard the relationship between our model and the full quantum theory of black holes as somewhat analogous to the relationship between the treatments of a hydrogen atom in elementary and advanced textbooks: quantization of the three degrees of freedom of the classical Kerr-Newman hole corresponds to the quantization of the three degrees of freedom associated with the electron going around the proton in a hydrogen atom. Whether the additional degrees of freedom appearing as a likely outcome of the full quantum theory of black holes have great or small effects to the energy levels of the hole is an open question. However, given the enormous pace of progress in the current research in black hole physics, one may hope for a definite answer in the not so distant future.

Chapter 6

Microscopic Black-hole Pairs

Some years ago Hawking and others introduced an interesting idea about a possible spontaneous creation of virtual black-hole pairs [113, 119, 128, 129, 130]. This process would be analogous to the spontaneous creation of electron-positron pairs in quantum electrodynamics, and the members of the black-hole pairs would presumably be Planck-size objects. As it is well known, electrons and positrons may sometimes, in very favorable conditions, form a system called positronium, in which an electron and a positron revolve around their common center of mass. The possibility of a spontaneous creation of black-hole pairs, then, gives rise to some very interesting questions: Could the black hole pairs sometimes form systems, analogous to positronium, where two microscopic black holes revolve around each other? What are the possible quantum states of such systems? What happens when the system performs a transition from one quantum state to another? Could one observe these transitions?

These are the questions which will be addressed in this chapter. Actually, such questions were raised already, among others, by Novikov and Frolov in their book [127]. In this chapter, however, we shall consider these questions in details. We shall consider the simplest possible case where the members of the pair are Schwarzschild black holes, both having the same mass M. In general, the problem of quantization of a system containing a black-hole pair presents immense difficulties: Indeed, even the classical solution to Einstein's equations describing a spacetime containing two black holes is unknown. However, if the two microscopic black holes are sufficiently far away from each other, they can be considered, to a very good approximation, as point-like particles moving with non-relativistic speeds. Moreover, the gravitational interaction between the holes can be described by the good old Newtonian theory of gravitation. Hence, the quantum mechanics of the black-hole pair is described by the non-relativistic Schrödinger equation

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{GM^2}{r}\right)\psi = E\psi\,, \tag{6.1}$$

where $\mu := \frac{1}{2}M$ is the reduced mass of the system, and E is its total energy. r is the distance between the black holes.

Our first task is to check whether the approximations we just made are valid. As it is well known from elementary quantum mechanics, for stationary states the energy eigenvalues calculated from Eq. (6.1) are

$$E_n = -\frac{1}{4}\frac{G^2M^5}{\hbar^2n^2}\,, \tag{6.2}$$

where $n = 1, 2, 3, \ldots$ is the principal quantum number of the system. In these states the expectation values of r are

$$\langle r\rangle_{n,l} = \frac{2\hbar^2}{GM^3}n^2\left\{1+\frac{1}{2}\left[1-\frac{l(l+1)}{n^2}\right]\right\}\,, \tag{6.3}$$

where $l = 0, 1, 2, \ldots, n-1$ is the angular momentum quantum number of the system. Our approximations are valid if

$$\frac{2\hbar^2}{GM^3}n^2 \gg R_S\,, \tag{6.4}$$

where

$$R_S := \frac{2GM}{c^2} \tag{6.5}$$

is the Schwarzschild radius of a black hole with mass M. Combining Eqs. (6.4) and (6.5) we find that we must have

$$\frac{1}{\sqrt{n}}M \ll M_{Pl}\,, \tag{6.6}$$

where

$$M_{Pl} := \sqrt{\frac{\hbar c}{G}} \approx 2.2 \times 10^{-8}\ \mathrm{kg} \tag{6.7}$$

is the Planck mass. Hence, either n must be very big or M must be much smaller than the Planck mass. The average velocity on the orbit is

$$\langle v_{ave}\rangle_n = \frac{1}{2}\left(\frac{M}{M_{Pl}}\right)^2\frac{1}{n}c\,. \tag{6.8}$$

As one can see, the black holes move with non-relativistic speeds if Eq. (6.6) holds. In what follows, we shall always assume that M is smaller than, or equal to, the Planck mass, and n is very large. In other words, we shall consider microscopic black-hole pairs in highly-excited states.

Consider next the transitions between highly-excited stationary states. It follows from Eq. (6.2) that the energy released when the system performs a transition where n is reduced by one is

$$E_n - E_{n-1} \approx \frac{G^2M^5}{2\hbar^2n^3} = \frac{1}{2n^3}\left(\frac{M}{M_{Pl}}\right)^4 Mc^2\,. \tag{6.9}$$

For instance, if n is around ten and M is the Planck mass, the energy released is 0,0005 times the Planck energy, or 10^6 J, which is about the same as the energy needed when an automobile is accelerated from rest to the speed of 100 miles per hour. As one can see, enormous energies could, in principle, be stored in systems containing microscopic black-hole pairs.

Since the holes are assumed to be uncharged, one may expect that the main reason for transitions between stationary states are quantum fluctuations of the gravitational field between the holes: quantum fluctuations of the gravitational field perturb the stationary states, a transition occurs, and a graviton is emitted or absorbed spontaneously. Since the black holes are assumed to be in a highly-excited state, and therefore relatively far away from each other, we may use the linear field approximation when investigating the perturbative effects caused by the quantum fluctuations of the gravitational field.

The Lagrangian of a point particle moving in a weak gravitational field $h_{\mu\nu}$ is

$$L = -M\sqrt{(\eta_{\mu\nu} + h_{\mu\nu})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\,. \tag{6.10}$$

Under the assumptions that $|h_{\mu\nu}| \ll 1$, and the particle moves very slowly, we may write Eq. (6.10), in SI units:

$$L \approx -Mc^2 + \frac{1}{2}M(\delta_{jk} - h_{jk})\dot{x}^j\dot{x}^k - Mch_{0j}\dot{x}^j - \frac{1}{2}Mc^2h_{00}\,, \tag{6.11}$$

where $j,k = 1,2,3$ and $\dot{x}^j := dx^j/dt$. Dropping the term $-Mc^2$, which is a mere constant, we may infer that, in the center of mass coordinates, the Lagrangian of the black-hole pair can be written, in effect, as

$$\begin{aligned} L &= \frac{1}{2}\mu\left[\delta_{jk} - \frac{1}{2}h_{jk}(\frac{\vec{r}}{2},t) - \frac{1}{2}h_{jk}(-\frac{\vec{r}}{2},t)\right]\dot{x}^j\dot{x}^k \\ &- \mu c\left[h_{0j}(\frac{\vec{r}}{2},t) - h_{0j}(-\frac{\vec{r}}{2},t)\right]\dot{x}^j \\ &- \mu c^2\left[h_{00}(\frac{\vec{r}}{2},t) + h_{00}(-\frac{\vec{r}}{2},t)\right]\,, \end{aligned} \tag{6.12}$$

where $\vec{r}$ is the vector joining the hole 1 to the hole 2, and the x^j's are defined such that $\vec{r} = x^j\hat{e}_j$, where the $\hat{e}_j$'s are orthonormal basis vectors.

Now, it is easy to see that the terms proportional to h_{00} represent the gravitational potential energy of the system. These terms should give, when the system is in a highly-excited state, the Newtonian potential energy between the holes. The terms proportional to h_{0j} and h_{jk}, in turn, are related to the quantum fluctuations of the gravitational field. If we assume that transitions from one stationary state to another are associated with spontaneous emissions or absorptions of gravitons, the only remaining terms, in addition to the Newtonian potential energy, are the terms proportional to h_{jk}. That is because the "scalar" and the "vector" gravitons proportional to h_{00} and h_{0j}, respectively, can be gauged away, and the physical gravitons correspond to the (jk)-components of the field $h_{\mu\nu}$. Therefore, the Lagrangian of our system interacting with spontaneously emitted or absorbed gravitons is

$$L = \frac{1}{2}\mu\left[\delta_{jk} - \frac{1}{2}h_{jk}(\frac{\vec{r}}{2},t) - \frac{1}{2}h_{jk}(-\frac{\vec{r}}{2},t)\right]\dot{x}^j\dot{x}^k + \frac{GM^2}{r}\,. \tag{6.13}$$

Hence we find, under the assumption that h_{jk} is small, that the Hamiltonian operator of the black-hole pair interacting with gravitational radiation is

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2\mu} - \frac{GM^2}{\hat{r}} + \hat{H}_{int}\,, \tag{6.14}$$

where

$$\hat{H}_{int} := \frac{1}{4\mu}\left[\hat{h}^{jk}(\frac{\vec{r}}{2},t) + \hat{h}^{jk}(-\frac{\vec{r}}{2},t)\right]\hat{p}_j\hat{p}_k\,, \tag{6.15}$$

and p_j is the canonical momentum conjugate to x^j. In these equations, objects equipped with hats are operators replacing the corresponding classical quantities.

The transition rate $\mathrm{R}_{i\to f}$ from the initial state $|i\rangle$ to the final state $|f\rangle$ can be evaluated by means of the Golden Rule [131]. Evaluation of the transition rate involves quantization of the linearized gravitational field in the radiation gauge where $h_{0\mu} = 0$ for every $\mu = 0, 1, 2, 3$. This has been performed in Appendix B, and we find that if exactly one graviton with angular frequency ω is emitted in a transition, then

$$\mathrm{R}_{i\to f} = \frac{G\omega}{2\pi c^5\hbar\mu^2}|\langle f|\cos(\frac{1}{2}\vec{k}\cdot\vec{r})\epsilon^{ab}\hat{p}_a\hat{p}_b|i\rangle|^2\,d\Omega\,, \tag{6.16}$$

where $\vec{k}$ is the wave vector of the graviton, ϵ^{ab} its polarization tensor, and $d\Omega$ is the solid angle into which the graviton emerges. The polarization tensors $\epsilon^{(1)ab}$ and $\epsilon^{(2)ab}$ corresponding to the two physical polarizations of the graviton have been chosen such that

$$\epsilon^{(\lambda')ab}\epsilon^{(\lambda)}_{ab} = 2\delta^{\lambda'\lambda} \tag{6.17}$$

for every $\lambda', \lambda = 1, 2$. In the lowest order approximation we can write

$$\mathrm{R}_{i\to f} = \frac{G\omega}{2\pi c^5\hbar\mu^2}|\langle f|\epsilon^{ab}\hat{p}_a\hat{p}_b|i\rangle|^2\,d\Omega\,. \tag{6.18}$$

Transitions corresponding to this expression may be viewed as the gravitational analogues of E1 transitions in atomic physics. We shall therefore call them as G1 transitions.

The next task is to integrate Eq. (6.16) over all the possible directions into which the gravitons may emerge. This has been done in Appendix C. If the initial state $|i\rangle$ and the final state $|f\rangle$ are taken to be solutions to the Schrödinger equation (6.1) of the black-hole pair, and, moreover, one takes into account the fact that gravitons have exactly two independent polarizations, one finds that the integrated transition rate from the state $|i\rangle$ to the state $|f\rangle$ in G1 transitions is

$$R_{i\to f} = \frac{4G\omega}{3c^5\hbar\mu^2}|\langle f|\hat{p}_1^2|i\rangle|^2\,. \tag{6.19}$$

Transition rates in processes associated with spontaneous emissions of gravitons have also been investigated by Weinberg in Chapter 10 of his book [132]. Weinberg's idea was to consider first the power emitted as gravitational radiation by an arbitrary classical system, which he expressed in terms of the energy-momentum-stress tensor of that system. He then assumed that gravitational radiation consists of gravitons with energies $\hbar\omega$, and he also replaced the classical energy-momentum-stress tensor by a matrix whose elements correspond to transitions between different quantum states of the system under consideration. As a result, he got an expression for the transition rates in spontaneous emissions of gravitons. It is far

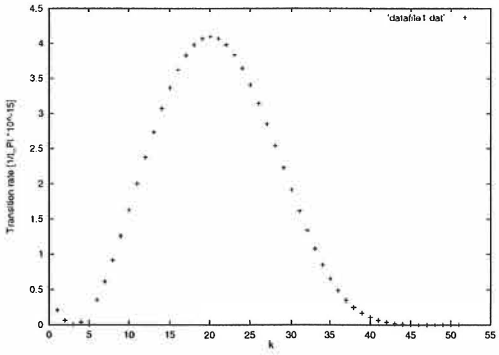

Figure 6.1: The integrated transition rate $R_{i \to f}$ as a function of k, which is the difference between the initial value n_i and the final value n_f of the principal quantum number n. In this figure we have $\Delta l = \Delta m_l = -2$. The initial values of n, l and m_l are, respectively, $n_i = 100$, $l_i = 50$ and $m_{l_i} = 50$.

from clear, however, what is the precise relationship, if there is any, between Weinberg's semiclassical reasoning, and our systematic quantum-mechanical derivation of the expression (6.18) for transition rates.

From the expression (6.19) one can obtain the G1 selection rules for the black-hole pair. In other words, one finds the transitions for which the integrated G1 transition rate (6.19) is non-zero. Since gravitons may be viewed as spin-two particles, one might expect that for allowed G1 transitions the angular momentum quantum number l as well as the corresponding azimuthal quantum number m_l could change only by zero, or by plus or minus two. A detailed investigation, which has been performed in Appendix D, shows that this is indeed the case: the G1 selection rules are

$$\Delta l = 0, \pm 2, \tag{6.20}$$

$$\Delta m_l = 0, \pm 2. \tag{6.21}$$

It is straightforward, although very laborious, to calculate the transition rates in allowed G1 transitions from Eq. (6.19). This arduous task, with explicit expressions for transition rates in different G1 transitions, has been performed in Appendix E. For instance, if the principal quantum number n is of the order of 100, then it turns out that the transition rates in transitions between nearby states are of the order of 10^{28} 1/s, if $\Delta l = 0, -2$, and of the order of 10^{24} 1/s, if $\Delta l = +2$, provided that we assume that the mass M is equal to the Planck mass M_{Pl}. Transitions between nearby states, however, are not necessarily the most favored ones. In Figs. 6.1, 6.2, and 6.3 we have plotted the transition rates as functions of k, the difference between the initial and the final values of the principal quantum number n. In all of the figures $R_{i \to f}$ has been plotted in the units of $\frac{1}{2t_{\rm Pl}} \times 10^{-15}$, where $t_{\rm Pl} := \sqrt{\frac{\hbar G}{c^5}} \approx 5.4 \times 10^{-44}$ s is the Planck time. One finds, for instance, that when $\Delta l = \Delta m_l = -2$, and $l_i = m_{l_i} = 50$, then the most favoured transitions are those where $k \approx 20$, and if, for the same values of l_i and m_{l_i}, $\Delta l = 0$ and $\Delta m_l = -2$, then for the most favoured transitions $k \approx 10$. If $\Delta l = +2$ and $\Delta m_l = 0$, then $k = 3$ for the most favoured transitions.

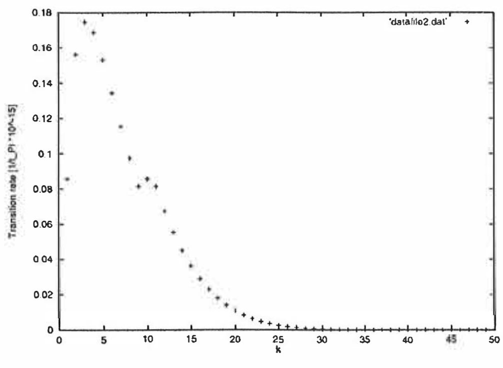

Figure 6.2: $R_{i \to f}$ as a function of k, when $n_i = 100$, $l_i = m_{l_i} = 50$, $\Delta l = +2$, and $\Delta m_l = 0$.

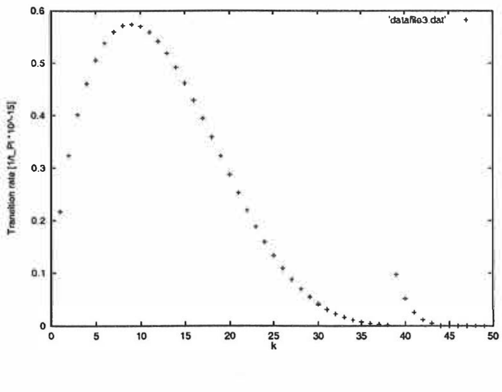

Figure 6.3: $R_{i \to f}$ as a function of k, when $n_i = 100$, $l_i = m_{l_i} = 50$, $\Delta l = 0$, and $\Delta m_l = -2$.

As we have seen, Planck-size black-hole pairs perform extremely rapid transitions from one stationary state to another. However, the problem is, whether the transitions are probably too rapid so that one cannot meaningfully talk about discrete energy eigenstates at all. To answer this question, one must calculate the lifetime of an initial state with large n, when all the allowed G1 transitions are taken into account. The lifetime of an initial state, in turn, is the inverse of the sum of the transition rates of all the allowed G1 transitions. Using the formulas of Appendix E, and assuming that transitions from higher to lower states are dominant, one finds that the lifetime of an initial state with n of the order of 100, is of the order of

$$\tau_{100} \sim 10^{-30}\ \mathrm{s}\,, \tag{6.22}$$

if $M = M_{Pl}$. This result settles, within the approximations made in this chapter, the question about the existence of sharp energy eigenstates. According to Heisenberg's uncertainty principle the natural linewidth of a state with lifetime τ_n is

$$\delta E_n \sim \frac{\hbar}{\tau_n}\,, \tag{6.23}$$

and therefore

$$\delta E_{100} \sim 10^{-4}\ \mathrm{J}\,. \tag{6.24}$$

However, the energy difference between nearby states is, according to Eq. (6.9),

$$E_{100} - E_{99} \approx 10^{3}\ \mathrm{J}\,, \tag{6.25}$$

and therefore

$$\frac{\delta E_{100}}{E_{100} - E_{99}} \sim 10^{-7}\,. \tag{6.26}$$

In other words, the uncertainty of the energy eigenvalues of the system is much smaller than the energy difference between nearby states. Hence, the energy spectrum appears to be discrete, at least as far as one can trust in the approximations made in this chapter. The transitions between discrete energy eigenstates are extremely rapid, and the energies released are enormous.

In this chapter we have investigated microscopic Schwarzschild black-hole pairs revolving around their common center of mass. Considering the holes as point-like objects interacting with the Newtonian gravitational force, we quantized the system and studied the stationary energy levels when the system is in a highly-excited state. We then calculated, by means of perturbative methods, the transition rates and lifetimes in a certain class of transitions. These calculations were based on the quantization of the linearized gravitational field in the radiation gauge. We obtained, in the lowest-order approximation, explicit expressions for the transition rates and evaluated the transition rates numerically when for the principal quantum number $n \sim 100$, and the mass of a microscopic black hole is assumed to be one Planck mass. We found that the transition rates are of the order of $10^{24} - 10^{28}$ 1/s for allowed transitions, and the lifetimes of energy eigenstates are of the order of 10^{-30} s. Gravitons with energies of the order of 10^{22} eV, or even higher, are emitted in these very rapid transitions between discrete energy eigenstates.

No doubt, one might be justified to have some feelings of suspicion towards the validity of the approximations on which our investigation is based. Indeed, it may well be somewhat daring to apply Newton's ancient theory of gravitation to microscopic black holes! However, if one accepts the view that, at the classical level, Einstein's general theory of relativity is the correct theory of gravitation (so far we have no experimental evidence suggesting that the things could be otherwise), having Newton's theory as its non-relativistic limit, one is also forced to accept the "Newtonian approximation" made in this chapter: Holes with $M = M_{Pl}$ move, if $n \sim 100$, with velocities which are of the order of $0.01c$ and the expectation value of their mutual distance is about 10^4 times their Schwarzschild radius. Of course, one could calculate general-relativistic corrections to energy levels and transition rates but as far as one is interested in mere order-of-magnitude estimates, Newton's theory should be sufficiently accurate. Quite another matter is, whether the first-order perturbative quantum theory of linearized general relativity is a sufficiently accurate approximation of quantum gravity for the evaluation of transition rates and lifetimes. As far as the energies of the emitted or absorbed gravitons are well below the Planck energy, which is the case at least for transitions between nearby states when $n \sim 100$, however, one might be inclined to rely on the approximations made in this chapter.

One of the basic lessons one can learn from this chapter is that enormous energies could be released by means of the quantum effects of the gravitational field: The energies released in transitions between the energy eigenstates of the microscopic black-hole pair are, even in the transitions between nearby states when $n \sim 100$, about fourteen orders of magnitude greater that the energies typically released in nuclear phenomena, and yet we are talking about a microscopic system.

As a whole, our investigation has been based on the assumptions that Planck-size black-hole pairs really exist, and that the lifetimes of the members of the pair are much longer than the lifetimes of the discrete energy eigenstates of the system. The object of this investigation has been to answer the question: If black-hole pairs with those properties do exist, then what will happen? It seems to us that the question about the existence or nonexistence of such black-hole pairs remains unsettled at the present state of research. To be able to give a definitive answer to that question we should have a complete quantum theory of gravity. Until then all bets are there.

For the sake of completeness, however, one may consider appropriate to list the main options. The first option is that Planck-size black holes evaporate within Planck time which is much smaller than 10^{-30} s, the typical lifetime of the energy eigenstate considered in this chapter, and therefore our calculation has no physical content whatsoever. When making such a conclusion, however, we are actually extrapolating the results originally obtained for the evaporation of macroscopic black holes by means of semi-classical methods down to Planck-size black holes – an extrapolation which may seem somewhat objectionable from the physical point of view.

Another option is that primordial black holes actually have a certain ground state where the mass is positive and of the order of one Planck mass. In that case primordial black holes leave Planck-size remnants as the end products of their evaporation. The existence or nonexistence of remnants, again, is a very controversial question [70] and remains completely unsettled until the complete quantum theory of gravity is found, altough it is possible to construct mathematically consistent quantum theories of black holes where the ground state energy of the hole is positive (see Refs. [66, 14]). If such remnants exist, however, it is possible that some of the them form pairs. Our calculation shows that if the masses of the remnants are of the order of the Planck mass, extremely energetic gravitons are emitted in very rapid transitions from one state to another. Some of these very energetic gravitons, in turn, may materialize into observable particles producing cosmic rays with very high energies. Measuring the amount of very energetic cosmic rays could therefore be used when trying to estimate the abundance of the Planck-size black-hole remnants in the universe. If there are black-hole pairs formed by black-hole remnants of the order of the Planck mass, one expects to observe cosmic rays with energies of the order of 10^{20} eV, or even higher.

Epilogue

Let us make some general and final remarks in the same spirit as we did in Prologue. Perhaps it is correct to state that the greatest enterprice of modern theoretical physics is to find a quantum theory of gravity. Although this enterprice has been recognized for several decades, I feel that one has just managed to find some hints about the right questions that would lead the way to the correct formulation. Perhaps one should try to learn from the history of the development of physics and question the old underlying principles of physics. This is often a hard and a lengthy journey but usually worth the pay. Although the old principles of physics are not thrown away in this thesis and the so-called right questions remain disguised, the moral of this work is that very simple models of black holes produce results that should be applicable in the semiclassical limit of the final formulation of quantum gravity.

Appendix A: WKB approximation

The purpose of the WKB approximation [123] is to find an approximative solution to differential equations. We shall concentrate our attention on the one-dimensional time-independent Schrödinger equation

$$\left\{\frac{d^2}{dx^2}+\frac{2m}{\hbar^2}[E-U(x)]\right\}\Psi(x)=0 \quad . \tag{A.1}$$

Let us now write a trial function for the stationary solution to Eq. (A.1) as

$$\Psi(x) = \exp\left[\frac{i}{\hbar}W(x)\right] \quad , \tag{A.2}$$

$$W(x) = S(x)+\frac{\hbar}{i}T(x) \quad , \tag{A.3}$$

$$A(x) = \exp[T(x)] \quad , \tag{A.4}$$

where S and T are required to be even functions of x. A and Ψ are uniquely defined in terms of S and T. Note that A and S are not necessarily real.

Substituting the expression (A.2) into the Schrödinger equation and separating real and imaginary parts, one obtains two equations equivalent to the Schrödinger equation

$$S'^2-2m[E-U] = \hbar^2\frac{A''}{A} \quad , \tag{A.5}$$

$$2A'S'+AS'' = 0 \quad . \tag{A.6}$$

Eq. (A.6) can be integrated to yield

$$A=C\cdot(S')^{-1/2} \quad , \tag{A.7}$$

where C is a constant of integration. When substituting Eq. (A.7) into Eq. (A.5) one obtains an equivalent equation to the Eq. (A.1):

$$S'^2=2m[E-U]+\hbar^2\left[\frac{3}{4}\left(\frac{S''}{S'}\right)^2-\frac{1}{2}\frac{S'''}{S'}\right] \quad . \tag{A.8}$$

The key idea in the WKB approximation is to expand S in $\hbar^2$ such that

$$S=S_0+S_1\hbar^2+\ldots \quad . \tag{A.9}$$

In general, this expansion does not converge but it gives, for small $\hbar$ and for a finite numbers of terms, a good approximation to S. In that case, substituting the expansion (A.9) into Eq. (A.8) and keeping the zero-order terms only, one obtains

$$S'^2 \approx S_0'^2 = 2m[E - U] \quad . \tag{A.10}$$

For the future development one must divide the investigation into two cases: in the first case we consider $E > U$, which is a classically allowed region of space. In the second case we consider $E < U$, which is a classically forbidden region of space.

$E > U$:

Inside the allowed region for classical particles the solutions are oscillating. Using a wavelength

$$\lambda(x) := \frac{\hbar}{\sqrt{2m[E - U(x)]}} \tag{A.11}$$

Eq. (A.10) will be satisfied if $S' \approx \pm\hbar/\lambda$. Then

$$\Psi(x) = \alpha\sqrt{\lambda}\cos\left(\int^x \frac{dx}{\lambda} + \varphi\right) \quad , \tag{A.12}$$

where α and φ are constants, is the *WKB solution* to the time-independent Schrödinger equation.

$E < U$:

Inside the forbidden region for classical particles one may define a wavelength

$$l(x) := \frac{\hbar}{\sqrt{2m[U(x) - E]}} \quad . \tag{A.13}$$

Now, Eq. (A.10) is satisfied if $S' \approx \pm\hbar/l$, and the linear combination

$$\Psi(x) = \sqrt{l}\left[\gamma\exp\left(\int^x \frac{dx}{l}\right) + \delta\exp\left(-\int^x \frac{dx}{l}\right)\right] \quad , \tag{A.14}$$

where γ and δ are constants, is the *WKB solution* to the time-independent Schrödinger equation.

The WKB approximation is valid if

$$\lambda'(x) \ll 1 \quad \text{when} \quad E > U(x) \quad , \tag{A.15}$$
$$l'(x) \ll 1 \quad \text{when} \quad E < U(x) \quad . \tag{A.16}$$

These criteria can be found by calculating the effect of the next-to-leading-order term $S_1\hbar^2$ of the expansion (A.9). In most cases the criteria are satisfied. However, there are the so-called *turning points* of the classical motion which satisfy $E = U(x)$. At the turning points the wavelength becomes infinite and the WKB approximation is not legimate. In order to get the complete solution, one must solve the Schrödinger equation in a small neighbourhood of the turning points, and then smoothly join these solutions to the exponential and oscillatory WKB solutions. However, this joining problem is a difficult task and we only give the so-called *connection formulae* between the exponential and the oscillatory WKB solution. The connection formulae are [124]:

$$\Psi_1(x) = \sqrt{l(x)}\exp\left(\int_{x}^{E=V(x)} \frac{dx}{l(x)}\right)$$

$$\updownarrow$$

$$\Psi_1(x) = -\sqrt{\lambda}\sin\left(\int_{E=V(x)}^{x} \frac{dx}{\lambda} - \frac{\pi}{4}\right) \quad , \tag{A.17}$$

$$\Psi_2(x) = \frac{\sqrt{l(x)}}{2}\exp\left(-\int_{x}^{E=V(x)} \frac{dx}{l(x)}\right)$$

$$\updownarrow$$

$$\Psi_2(x) = -\sqrt{\lambda}\cos\left(\int_{E=V(x)}^{x} \frac{dx}{\lambda} - \frac{\pi}{4}\right) \quad , \tag{A.18}$$

i.e., if the solution Ψ is exponentially increasing in the classically forbidden region it must be joined with the solution Ψ that is proportional to sine function inside the classically allowed region. If the solution Ψ is exponentially decreasing in the classically forbidden region it must be joined with the solution Ψ that is proportional to cosine function inside the classically allowed region. Note that the general solution is a linear combination of two solutions $\Psi_1(x)$ and $\Psi_2(x)$ in each region. These formulae are used in the next subsection in the special case of Reissner-Nordström black holes. Finally, joining the oscillatory solutions inside the classically allowed region one obtains the rule that fixes the energy levels of the discrete spectrum.

Appendix B: Quantization of Linearized Gravity in the Radiation Gauge

The Lagrangian density

$$\mathcal{L} = \frac{1}{64\pi G}\left(\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - 2\partial^\mu h_{\mu\nu}\partial_\lambda h^{\nu\lambda} + 2\partial^\mu h_{\mu\nu}\partial^\nu h - \partial_\nu h \partial^\nu h\right) . \quad \text{(B.1)}$$

of the linearized gravitational field $h_{\mu\nu}$ can be written, when the Hilbert gauge condition

$$\partial_\mu \left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h\right) = 0 \quad \text{(B.2)}$$

is used, as ($c = 1$) [133]

$$\mathcal{L} = \frac{1}{64\pi G}\left(\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{2}\partial_\nu h \partial^\nu h\right) . \quad \text{(B.3)}$$

In the radiation gauge we have, in addition,

$$h^{\mu 0} = 0 \ \forall \mu = 0, 1, 2, 3 , \quad \text{(B.4)}$$

and the Lagrangian density can be written in the form

$$\mathcal{L} = \frac{1}{64\pi G}\left(\dot{h}_{mn}\dot{h}^{mn} + \partial_l h_{mn}\partial^l h^{mn} - \frac{1}{2}\partial_n h \partial^n h\right) , \quad \text{(B.5)}$$

where $\dot{h}_{mn} := \partial h_{mn}/\partial t$, and the Latin indices take the values 1,2,3. The canonical momentum conjugate to h_{ab} is therefore

$$p^{ab} := \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}} = \frac{1}{32\pi G}\dot{h}^{ab} . \quad \text{(B.6)}$$

As the first step towards quantization we write the field h_{ab} as a Fourier expansion:

$$h_{ab}(t, \vec{r}) = \sum_{\vec{k}} \sum_{\lambda=1}^{2} \epsilon_{ab}^{(\lambda)} \left(u_{\vec{k}}(t, \vec{r}) a_{\vec{k}}^{(\lambda)} + u_{\vec{k}}^*(t, \vec{r}) a_{\vec{k}}^{(\lambda)\dagger}\right) . \quad \text{(B.7)}$$

In this equation, the sum is taken over the wave vectors $\vec{k}$ and the polarizations λ. $\epsilon_{ab}^{(\lambda)}$ is the polarization tensor, $a_{\vec{k}}^{(\lambda)}$ and $a_{\vec{k}}^{(\lambda)\dagger}$ are the Fourier coefficients, and the functions $u_{\vec{k}}$ are orthonormal wave modes:

$$u_{\vec{k}}(t, \vec{r}) = N_{\vec{k}} e^{i(\vec{k}\cdot\vec{r} - \omega t)} , \quad \text{(B.8)}$$

where $N_{\vec{k}}$ is a normalization constant, and ω is the angular frequency of the graviton.

Introducing periodic boundary conditions in a box with edge L,

$$h_{ab}(0,y,z,t) = h_{ab}(L,y,z,t)\,, \tag{B.9}$$
$$h_{ab}(x,0,z,t) = h_{ab}(x,L,z,t)\,, \tag{B.10}$$
$$h_{ab}(x,y,0,t) = h_{ab}(x,y,L,t)\,, \tag{B.11}$$

we find that the possible values of k_x, k_y and k_z are

$$k_x = n_x\frac{2\pi}{L}\,, \tag{B.12}$$
$$k_y = n_y\frac{2\pi}{L}\,, \tag{B.13}$$
$$k_z = n_z\frac{2\pi}{L}\,, \tag{B.14}$$

where n_x, n_y and n_z are integers. Introducing, moreover, an inner product between wave modes,

$$\langle u_{\vec{k}'}|u_{\vec{k}}\rangle := -i\int_0^L dx\int_0^L dy\int_0^L dz\,\left(u_{\vec{k}'}\dot{u}^*_{\vec{k}} - \dot{u}_{\vec{k}'}u^*_{\vec{k}}\right)\,, \tag{B.15}$$

together with the requirement that the wave modes are orthonormal,

$$\langle u_{\vec{k}'}|u_{\vec{k}}\rangle = \delta_{\vec{k}'\vec{k}}\,, \tag{B.16}$$

indicates that

$$u_{\vec{k}}(t,\vec{r}) = \left(2L^3\omega\right)^{-1/2}e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{B.17}$$

Consider now the polarizations of gravitons. The Hilbert gauge condition (B.2), together with the radiation gauge condition (B.4) implies

$$\epsilon^{(\lambda)}_{ab}k^b = 0\,, \tag{B.18}$$

for all $a = 1,2,3$ and $\lambda = 1,2$. To satisfy this condition, we may choose the polarization tensors $\epsilon^{(\lambda)}_{ab}$ such that

$$\epsilon^{(\lambda)}_{ab}\epsilon^{(\lambda')ab} = 2\delta^{\lambda\lambda'}\,, \tag{B.19}$$

for all $\lambda, \lambda' = 1,2$. For instance, we may choose

$$\left(\epsilon^{(1)}_{ab}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.20}$$
$$\left(\epsilon^{(2)}_{ab}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.21}$$

In that case the gravitons propagate to the z-direction. Using Eqs. (B.16) and (B.19) we can write the Fourier coefficients $a^{(\lambda)}_{\vec{k}}$ and $a^{(\lambda)\dagger}_{\vec{k}}$ as

$$a^{(\lambda)\dagger}_{\vec{k}} = -\frac{i}{2}\epsilon^{(\lambda)ab}\int_0^L dx\int_0^L dy\int_0^L dz\,\left(u_{\vec{k}}\dot{h}_{ab} - \dot{u}_{\vec{k}}h_{ab}\right)\,, \tag{B.22}$$
$$a^{(\lambda)}_{\vec{k}} = \frac{i}{2}\epsilon^{(\lambda)ab}\int_0^L dx\int_0^L dy\int_0^L dz\,\left(u^*_{\vec{k}}\dot{h}_{ab} - \dot{u}^*_{\vec{k}}h_{ab}\right)\,. \tag{B.23}$$

We now proceed to quantization. It follows from Eq. (B.6) that the canonical equal time commutation relations between the field operators $\hat{h}_{ab}$ and their conjugates $\hat{p}^{ab}$,

$$\left[\hat{h}_{ab}(t,\vec{r}'),\hat{p}^{cd}(t,\vec{r})\right] = i\hbar\delta^3(\vec{r}',\vec{r})\delta_a^c\delta_b^d\,, \tag{B.24}$$

$$\left[\hat{h}_{ab}(t,\vec{r}'),\hat{h}_{cd}(t,\vec{r})\right] = \left[\hat{p}^{ab}(t,\vec{r}'),\hat{p}^{cd}(t,\vec{r})\right] = 0\,, \tag{B.25}$$

can be written as

$$\left[\hat{h}_{ab}(t,\vec{r}'),\dot{\hat{h}}^{cd}(t,\vec{r})\right] = i32\pi G\hbar\delta^3(\vec{r}',\vec{r})\delta_a^c\delta_b^d\,, \tag{B.26}$$

$$\left[\hat{h}_{ab}(t,\vec{r}'),\hat{h}_{cd}(t,\vec{r})\right] = \left[\dot{\hat{h}}^{ab}(t,\vec{r}'),\dot{\hat{h}}^{cd}(t,\vec{r})\right] = 0\,. \tag{B.27}$$

Using Eqs. (B.19), (B.23) and (B.23) we find that the commutation relations between the operators $\hat{a}_{\vec{k}}^{(\lambda)\dagger}$ and $\hat{a}_{\vec{k}}^{(\lambda)}$ are

$$\left[\hat{a}_{\vec{k}'}^{(\lambda')},\hat{a}_{\vec{k}}^{(\lambda)\dagger}\right] = 16\pi G\hbar\delta^{\lambda'\lambda}\delta_{\vec{k}'\vec{k}}\,, \tag{B.28}$$

$$\left[\hat{a}_{\vec{k}'}^{(\lambda')},\hat{a}_{\vec{k}}^{(\lambda)}\right] = \left[\hat{a}_{\vec{k}'}^{(\lambda')\dagger},\hat{a}_{\vec{k}}^{(\lambda)\dagger}\right] = 0\,. \tag{B.29}$$

Hence, $\hat{a}_{\vec{k}}^{(\lambda)\dagger}$ creates and $\hat{a}_{\vec{k}}^{(\lambda)}$ annihilates a graviton with wave vector $\vec{k}$ and polarization λ. More precisely, they act on the vacuum $|0\rangle$ such that

$$\hat{a}_{\vec{k}}^{(\lambda)\dagger}|0\rangle = \sqrt{16\pi G\hbar}|1\rangle\,, \tag{B.30}$$

$$\hat{a}_{\vec{k}}^{(\lambda)}|0\rangle = 0\,, \tag{B.31}$$

where $|1\rangle$ is the one-graviton state. If we put everything here derived in together, we find that the operator $\hat{H}_{int}$ of Eq. (6.15) takes the form

$$\hat{H}_{int} = \frac{1}{2\mu}\sum_{\vec{k}}\sum_{\lambda=1}^{2}\left(2L^3\omega\right)^{-1/2}\epsilon^{\lambda)ab}\cos\left(\frac{1}{2}\vec{k}\cdot\vec{r}\right)\left(e^{-i\omega t}\hat{a}_{\vec{k}}^{(\lambda)} + e^{i\omega t}\hat{a}_{\vec{k}}^{(\lambda)\dagger}\right)\hat{p}_a\hat{p}_b\,. \tag{B.32}$$

Now, the Golden Rule [131] implies that if in a transition from the state $|i\rangle$ to the state $|f\rangle$, a graviton with energy $\hbar\omega$, wave vector $\vec{k}$ and polarization λ is emitted, the corresponding transition rate is

$$\mathrm{R}_{i\to f} = \frac{2\pi}{\hbar}|\langle f|\frac{1}{2\mu}\left(2L^3\omega\right)^{-1/2}\epsilon^{(\lambda)ab}\cos\left(\frac{1}{2}\vec{k}\cdot\vec{r}\right)\hat{a}_{\vec{k}}^{(\lambda)\dagger}\hat{p}_a\hat{p}_b|i\rangle|^2\rho(E_f)\,, \tag{B.33}$$

where

$$\rho(E_f) := \frac{\omega^2L^3}{8\pi^3\hbar}\,d\Omega \tag{B.34}$$

is the density of states close to the final state, and $d\Omega$ is the solid angle into which the graviton emerges. If the state $|i\rangle$ represents a zero-graviton state and the state $|f\rangle$ a one-graviton state, Eq. (A20) implies that

$$\mathrm{R}_{i\to f} = \frac{G\omega}{2\pi\hbar\mu^2}|\langle f|\cos\left(\frac{1}{2}\vec{k}\cdot\vec{r}\right)\epsilon^{(\lambda)ab}\hat{p}_a\hat{p}_b|i\rangle|^2\,d\Omega \tag{B.35}$$

or, in SI units,

$$\mathrm{R}_{i\to f} = \frac{G\omega}{2\pi c^5\hbar\mu^2}|\langle f|\cos\left(\frac{1}{2}\vec{k}\cdot\vec{r}\right)\epsilon^{(\lambda)ab}\hat{p}_a\hat{p}_b|i\rangle|^2\,d\Omega\,, \tag{B.36}$$

which is Eq. (6.16).

Appendix C: Integration Over All Directions of the Gravitational Radiation

In this appendix we shall calculate the integrated transition rate by integrating over all the possible directions into which the graviton may emerge. To evaluate the integral, we assume first that all the gravitons are the so-called h^{12}-gravitons propagating into the z-direction. Then the only non-zero components of the polarization tensor ϵ^{ab} are

$$\epsilon^{12} = \epsilon^{21} = 1 . \tag{C.1}$$

Now, the idea is here that the components of the polarization tensor change when the direction of propagation of the emitting graviton changes. When the gravitons no more propagate into the z-direction, we introduce a new Cartesian coordinate system K' which is rotated such that the gravitons always propagate, in frame K', along z-axis, and the original coordinate system K is kept fixed. The relationship between the orthonormal basis vectors $\hat{e}_a$ and $\hat{e}'_a$ of the frames K and K' is

$$\begin{aligned}
\hat{e}'_1 &= \sin\phi\ \hat{e}_1 + \cos\phi\ \hat{e}_2 , &\text{(C.2)}\\
\hat{e}'_2 &= -\cos\theta\cos\phi\ \hat{e}_1 + \cos\theta\sin\phi\ \hat{e}_2 + \sin\theta\ \hat{e}_3 , &\text{(C.3)}\\
\hat{e}'_3 &= \sin\theta\cos\phi\ \hat{e}_1 + \sin\theta\sin\phi\ \hat{e}_2 + \cos\theta\ \hat{e}_3 , &\text{(C.4)}
\end{aligned}$$

where θ and ϕ are polar spherical angles. These relations imply that the polarization tensor corresponding to h^{12}-gravitons propagating into an arbitrary direction is, in frame K, represented by the matrix

$$\epsilon = \begin{pmatrix} -\cos\theta\sin 2\phi & \cos\theta(2\sin^2\phi - 1) & \sin\theta\sin\phi \\ \cos\theta(2\sin^2\phi - 1) & \cos\theta\sin 2\phi & \sin\theta\cos\phi \\ \sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} . \tag{C.5}$$

To begin with, we perform the summation in Eq. (6.18). Because the momentum operators commute with each other, we get

$$\begin{aligned}
\int |\langle f|\epsilon^{ab}\hat{p}_a\hat{p}_b|i\rangle|^2 d\Omega = & \int |\epsilon^{11}\langle f|\hat{p}_1^2|i\rangle + \epsilon^{22}\langle f|\hat{p}_2^2|i\rangle + \epsilon^{33}\langle f|\hat{p}_3^2|i\rangle \\
& +2\epsilon^{12}\langle f|\hat{p}_1\hat{p}_2|i\rangle + 2\epsilon^{13}\langle f|\hat{p}_1\hat{p}_3|i\rangle \\
& +2\epsilon^{23}\langle f|\hat{p}_2\hat{p}_3|i\rangle|^2 d\Omega . \qquad \text{(C.6)}
\end{aligned}$$

After squaring and integrating the expression (C.6) over all directions, we have

$$\begin{aligned}\int |\langle f|\epsilon^{ab}\hat{p}_a\hat{p}_b|i\rangle|^2 d\Omega \quad = \quad & \frac{2\pi}{3}\Big(|\langle f|\hat{p}_1^2|i\rangle|^2 + |\langle f|\hat{p}_2^2|i\rangle|^2 + 4|\langle f|\hat{p}_1\hat{p}_2|i\rangle|^2 \\ & + 8|\langle f|\hat{p}_1\hat{p}_3|i\rangle|^2 + 8|\langle f|\hat{p}_2\hat{p}_3|i\rangle|^2 - \langle f|\hat{p}_2^2|i\rangle^*\langle f|\hat{p}_1^2|i\rangle \\ & - \langle f|\hat{p}_1^2|i\rangle^*\langle f|\hat{p}_2^2|i\rangle + 8\langle f|\hat{p}_1\hat{p}_3|i\rangle^*\langle f|\hat{p}_2\hat{p}_3|i\rangle \\ & + 8\langle f|\hat{p}_2\hat{p}_3|i\rangle^*\langle f|\hat{p}_1\hat{p}_3|i\rangle\Big) . \qquad \text{(C.7)}\end{aligned}$$

In Appendix E we shall calculate the above transition amplitudes in all details. If we now use the results (E.16)–(E.18), we get for the integrated transition rate a simple and nice expression, by taking into account that gravitons have two independent polarization states:

$$R_{i\to f} = \frac{16\omega}{3M^2}|\langle f|\hat{p}_1^2|i\rangle|^2 , \qquad \text{(C.8)}$$

in the units where $G = c = \hbar = 1$.

Appendix D: G1 Selection Rules

In this appendix we shall consider, in the lowest order approximation, the selection rules for the spontaneous emissions and absorptions of gravitons by a microscopic black-hole pair. These selection rules we shall call G1 selection rules, in analogy to the E1 selection rules in atomic physics. The most convenient way of deriving these rules is to use the so-called Wigner–Eckart theorem [131]. That theorem concerns irreducible tensor operators on a spherical basis. The Cartesian components $\hat{p}_{1,\,2,\,3}$ of the momentum operator can be written in terms of the standard components $\hat{p}_{-1,\,+1,\,0}$ of the irreducible momentum operator $\hat{\vec{p}}$ of rank 1 as

$$\hat{p}_1 = -\frac{1}{\sqrt{2}}\left(\hat{p}_{+1} - \hat{p}_{-1}\right), \tag{D.1}$$

$$\hat{p}_2 = \frac{i}{\sqrt{2}}\left(\hat{p}_{+1} + \hat{p}_{-1}\right), \tag{D.2}$$

$$\hat{p}_3 = \hat{p}_0 . \tag{D.3}$$

To find the G1 selection rules, we must find all the allowed changes in the angular momentum of the microscopic black-hole pair between the initial and final states. To determine these changes one has to consider the transition amplitude given in Eq. (C.8). The transition amplitude can be written in terms of the standard components as

$$\langle f|\hat{p}_1^2|i\rangle = \langle f|\frac{1}{2}\left(\hat{p}_{+1}^2 - \hat{p}_{+1}\hat{p}_{-1} - \hat{p}_{-1}\hat{p}_{+1} + \hat{p}_{-1}^2\right)|i\rangle, \tag{D.4}$$

where all the products between the standard components can be written, by using the definition of the tensor product, as

$$\begin{aligned}
\langle f|\hat{p}_1^2|i\rangle = \frac{1}{2}\Big[& \langle f|\sum_{J,M}(1,1,1,1|J,M)[\hat{\vec{p}}\hat{\vec{p}}]_{JM}|i\rangle \\
& - \langle f|\sum_{J,M}(1,1,1,-1|J,M)[\hat{\vec{p}}\hat{\vec{p}}]_{JM}|i\rangle \\
& - \langle f|\sum_{J,M}(1,-1,1,1|J,M)[\hat{\vec{p}}\hat{\vec{p}}]_{JM}|i\rangle \\
& + \langle f|\sum_{J,M}(1,-1,1,-1|J,M)[\hat{\vec{p}}\hat{\vec{p}}]_{JM}|i\rangle\Big],
\end{aligned} \tag{D.5}$$

where $[\hat{\vec{p}}\hat{\vec{p}}]_{JM}$ denotes the JM-component of the irreducible tensor of rank 2, J corresponds to the eigenvalue of the total angular momentum operator $\hat{\vec{J}}$, and M is the corresponding eigenvalue of the projection operator $\hat{J}_z$. After taking the sum over J and M, Eq. (D.5) reduces to

$$\langle f|\hat{p}_1^2|i\rangle = \frac{1}{2}\langle f|[\hat{\vec{p}}\hat{\vec{p}}]_{22}|i\rangle - \frac{1}{\sqrt{3}}\langle f|[\hat{\vec{p}}\hat{\vec{p}}]_{00}|i\rangle - \frac{1}{6\sqrt{5}}\langle f|[\hat{\vec{p}}\hat{\vec{p}}]_{20}|i\rangle + \frac{1}{2}\langle f|[\hat{\vec{p}}\hat{\vec{p}}]_{2\ -2}|i\rangle\,, \tag{D.6}$$

which, by the Wigner-Eckart theorem, is equal to

$$\begin{aligned}\langle f|\hat{p}_1^2|i\rangle \;=\; (-1)^{l_f-m_f}\Bigg[&\frac{1}{2}\begin{pmatrix} l_f & 2 & l_i \\ -m_f & 2 & m_i\end{pmatrix}\Big(n_f\ l_f||[\hat{\vec{p}}\hat{\vec{p}}]_{22}||n_i\ l_i\Big)\\ &-\frac{1}{\sqrt{3}}\begin{pmatrix} l_f & 0 & l_i \\ -m_f & 0 & m_i\end{pmatrix}\Big(n_f\ l_f||[\hat{\vec{p}}\hat{\vec{p}}]_{00}||n_i\ l_i\Big)\\ &-\frac{1}{6\sqrt{5}}\begin{pmatrix} l_f & 2 & l_i \\ -m_f & 0 & m_i\end{pmatrix}\Big(n_f\ l_f||[\hat{\vec{p}}\hat{\vec{p}}]_{20}||n_i\ l_i\Big)\\ &+\frac{1}{2}\begin{pmatrix} l_f & 2 & l_i \\ -m_f & -2 & m_i\end{pmatrix}\Big(n_f\ l_f||[\hat{\vec{p}}\hat{\vec{p}}]_{2\ -2}||n_i\ l_i\Big)\Bigg]\,,\end{aligned} \tag{D.7}$$

where m_i and m_f are the eigenvalues of the z-component of the angular momentum operator at the final and the initial states of the microscopic black-hole pair, respectively, $\begin{pmatrix} l_f & L & l_i \\ m_f & M & m_i\end{pmatrix}$ denotes a 3j-symbol, and $(f||\hat{T}_L||i)$ is the reduced matrix element of the irreducible tensor operator of rank L.

It is well known that the 3j-symbol vanishes unless the following two conditions hold for the eigenvalues of the angular momenta:

$$|l_f - L| \leq l_i \leq \ l_f \ +L\,, \tag{D.8}$$

$$m_i - m_f + M \ = \ 0\,. \tag{D.9}$$

These conditions, together with Eq. (D.7), imply that the allowed transitions are the ones where $\Delta l = 0,\ \pm 1,\ \pm 2$ and $\Delta m_l = 0,\ \pm 2$. However, there is a further constraint which comes from the parity conservation: Since the position operator $\hat{x}^a$ is odd in reflections for every $a = 1, 2, 3$, then $\hat{p}^a$ is also odd for all a. Therefore $\hat{p}^a\hat{p}^b$ must behave as an even linear operator in reflections, and the states of the microscopic black-hole pair must not change parity in G1 transitions. Since the parity of any state is given by $(-1)^l$, the value of l cannot be ± 1. As the final G1 selection rules for the microscopic black-hole pairs we find that the only allowed transitions in the lowest-order approximation are the ones where

$$\Delta l = 0,\ \ \pm 2\,, \tag{D.10}$$

$$\Delta m_l = 0,\ \ \pm 2\,. \tag{D.11}$$

Appendix E: Transition Rates

In this appendix we shall derive explicit expressions for the transition rates in G1 transitions for microscopic black hole pairs. In addition, we shall estimate the transition rates numerically in some well-chosen cases. Furthermore, we shall estimate the lifetime of the initial quantum state of the black-hole pair.

In position representation the initial and final quantum states, $|i\rangle$ and $|f\rangle$, respectively, are represented by the "hydrogenic wave functions", where the factor $q^2/4\pi\epsilon_0$ is replaced by GM^2, where G is the gravitational constant and M is the mass of each hole [134]:

$$\begin{aligned}\Psi_{nlm_l}(r,\theta,\phi) &= \left[\left(\frac{GM^3}{n\hbar^2}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}\right]^{1/2} \exp\left(\frac{-GM^3}{2n\hbar^2}r\right)\left(\frac{GM^3}{n\hbar^2}r\right)^l \\ &\quad \times L^{2l+1}_{n-l-1}\left(\frac{GM^3}{n\hbar^2}r\right) Y_{lm_l}(\theta,\phi)\,, \qquad \text{(E.1)}\end{aligned}$$

where

$$L^s_k(x) := \sum_{m=0}^{s}(-1)^m \frac{(s+k)!}{(s-m)!(k+m)!m!}x^m \qquad \text{(E.2)}$$

is the associated Laguerre polynomial, and

$$Y_{lm_l}(\theta,\phi) := (-1)^{m_l}\left[\frac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}\right]^{1/2} P^{m_l}_l(\cos\theta)\exp(im_l\phi) \qquad \text{(E.3)}$$

is the spherical harmonic function. $P^{m_l}_l(x)$ is the associated Legendre function. When calculating an expression for the transition rate, we will mostly be interested in the transitions where the initial and final states are in their maximal projection state where $m_{l_i} = l_i$ and $m_{l_f} = l_f$. In that case we may use the result

$$P^l_l(\cos\theta) = (2l-1)!!\sin^l\theta\,, \qquad \text{(E.4)}$$

where the double factorial is defined by $n!! := n(n-2)(n-4)\cdots$ and $0!! = 1$.

In Appendix D we derived the G1 selection rules (D.10) and (D.11). Because of these rules we must consider four different cases: We calculate the three transition rates $R_{i\to f}$ where either $\Delta l = 0$ or $\Delta l = \pm 2$ such that the initial and final states are maximal projection states, and, moreover, we consider the case where $\Delta l = 0$ and $\Delta m_l = -2$.

Case $\Delta l = -2$

We first calculate the transition rate in the case where $\Delta l = -2 = \Delta m_l$. In other words we take the initial and final states to be $|i\rangle = |n, l, l\rangle$ and $|f\rangle = |n-k, l-2, l-2\rangle$ ($k = 1, 2, 3, \ldots$) such that $l > 2$ and $n \gg 1$. We can write these states in the position representation using Eq. (E.1). To evaluate the integrals in Eq. (C.7), we must express the momentum operators $\hat{p}_a$ in the spherical coordinates:

$$\hat{p}_1 = -i\hbar\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right), \tag{E.5}$$

$$\hat{p}_2 = -i\hbar\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right), \tag{E.6}$$

$$\hat{p}_3 = -i\hbar\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right). \tag{E.7}$$

After employing the expressions for the wave functions and for the momentum operators one gets a rather messy expression for the first transition amplitude in Eq. (C.7), which we still have to integrate (in the units where $c = G = \hbar = M = 1$):

$$\begin{aligned}
\langle f|\hat{p}_1^2|i\rangle &= \langle\hat{p}_1^\dagger f|\hat{p}_1 i\rangle = \langle\hat{p}_1 f|\hat{p}_1 i\rangle = \frac{(2l-1)!!(2l-5)!!}{8\pi n^{l+2}(n-k)^l} \\
&\times \left[\frac{(2l+1)(2l-3)(n-l-1)!(n-k-l+1)!}{(n+l)!(2l)!(n-k+l-2)!(2l-4)!}\right]^{1/2} \\
&\times \sum_{m=0}^{n-l-1} a_m \sum_{m'=0}^{n-k-l+1} b_{m'} \int_0^\infty dr r^2 \int_0^\pi d\theta\sin\theta \int_0^{2\pi} d\phi\exp(i2\phi) \\
&\times \Big[\sin^{2l}\theta\cos^2\phi R_i'(r)R_f'(r) + l\sin^{2l-2}\theta\cos^2\theta\cos^2\phi\frac{1}{r}R_i(r)R_f'(r) \\
&- \frac{il}{2r}\sin^{2l-2}\theta\sin 2\phi R_i(r)R_f'(r) \\
&+ \frac{l-2}{r}\sin^{2l-4}\theta\cos^2\theta\cos^2\phi R_i'(r)R_f(r) \\
&+ \frac{l(l-2)}{r^2}\sin^{2l-4}\theta\cos^4\theta\cos^2\phi R_i(r)R_f(r) \\
&- \frac{i(l-2)}{2r}\sin^{2l-2}\theta\sin 2\phi R_i'(r)R_f(r) \\
&+ \frac{l(l-2)}{r^2}\sin^{2l-4}\theta\sin^2\phi R_i(r)R_f(r)\Big],
\end{aligned} \tag{E.8}$$

where we have denoted

$$a_m := (-1)^m\frac{(n+l)!}{(n-l-m-1)!(2l+m+1)!m!n^m}, \tag{E.9}$$

$$b_{m'} := (-1)^{m'}\frac{(n-k+l-2)!(m'!(n-k)^{m'})^{-1}}{(n-k-l-m'+1)!(2l+m'-3)!}, \tag{E.10}$$

$$R_i(r) := r^{l+m}\exp\left(-\frac{r}{2n}\right), \tag{E.11}$$

$$R_f(r) := r^{l+m'-2}\exp\left(-\frac{r}{2(n-k)}\right), \tag{E.12}$$

and the prime $'$ denotes the partial derivative with respect to r. The ϕ-part of the integral is very easy to perform, and the radial part gives four different integrals which can be integrated separately by using the well-known result

$$\int_0^\infty dx x^n \exp(-ax) = \frac{n!}{a^{n+1}}. \tag{E.13}$$

Moreover, it is straightforward to show that the θ-integrals in Eq. (E.8) can be reduced to the following integral:

$$\int_0^\pi d\theta \sin^{2l+1}\theta = \frac{2^{2l+1}l!^2}{(2l+1)!}. \tag{E.14}$$

When substituting all these integrals into Eq. (E.8), we get, for the transitions where $\Delta l = -2$, such that the system always remains in its maximal projection state,

$$\begin{aligned}
\langle f|\hat{p}_1^2|i\rangle &= \frac{1}{8n^{l+2}(n-k)^l} \\
&\quad \left[\frac{2l(2l-2)(n+l)!(n-k+l-2)!(n-l-1)!(n-k-l+1)!}{(2l+1)(2l-1)}\right]^{1/2} \\
&\quad \times \sum_{m=0}^{n-l-1} \frac{(-1)^m}{(n-l-m-1)!(2l+m+1)!m!n^m} \\
&\quad \times \sum_{m'=0}^{n-k-l+1} \frac{(-1)^{m'}}{(n-k-l-m'+1)!(2l+m'-3)!m'!(n-k)^{m'}} \\
&\quad \times \left[\frac{2n(n-k)}{2n-k}^{2l+m+m'}\right] \left\{(2l+m+m'-2)!(2l+m+2)m'\right. \\
&\quad \times \frac{2n-k}{2n(n-k)} \\
&\quad - (2l+m+m'-1)!\left(\frac{2l+m'-4}{2n} + \frac{2l+m-1}{2(n-k)}\right) \\
&\quad \left. + \frac{(2l+m+m')!}{2(2n-k)}\right\}.
\end{aligned} \tag{E.15}$$

In a very similar manner one can show that the rest of the transition amplitudes in Eq. (C.7) in this particular case are given by

$$\begin{aligned}
\langle f|\hat{p}_2^2|i\rangle &= -\langle f|\hat{p}_1^2|i\rangle & \text{(E.16)}\\
\langle f|\hat{p}_j\hat{p}_3|i\rangle &= 0 \ \forall j = 1,2,3, & \text{(E.17)}\\
\langle f|\hat{p}_1\hat{p}_2|i\rangle &= i\langle f|\hat{p}_1^2|i\rangle. & \text{(E.18)}
\end{aligned}$$

The only difference between these amplitudes and the first amplitude (E.15) comes from the ϕ-integration.

The angular frequency ω depends on the quantum number n and on $\Delta n = -k$ as follows:

$$\omega = -\frac{1}{2}\left(\frac{1}{n^2} - \frac{1}{(n-k)^2}\right). \tag{E.19}$$

As the final result, when the n-dependence of the angular frequency is taken into account, the transition rate for the microscopic black-hole pairs becomes, in the case where $\Delta l = -2 = \Delta m_l$,

$$\begin{aligned} R_{i\to f} &= -\frac{1}{24}\left(\frac{1}{n^2} - \frac{1}{(n-k)^2}\right) \\ &\times \frac{2l(2l-2)(n+l)!(n-k+l-2)!(n-l-1)!(n-k-l+1)!}{(2l+1)(2l-1)n^{2l+4}(n-k)^{2l}} \\ &\times \left\{ \sum_{m=0}^{n-l-1} \frac{(-1)^m}{(n-l-m-1)!(2l+m+1)!m!n^m} \right. \\ &\times \sum_{m'=0}^{n-k-l+1} \frac{(-1)^{m'}}{(n-k-l-m'+1)!(2l+m'-3)!m'!(n-k)^{m'}} \\ &\times \left[\frac{2n(n-k)}{2n-k}^{2l+m+m'}\right] \\ &\times \left[(2l+m+m'-2)!(2l+m+2)m'\frac{2n-k}{2n(m-k)}\right. \\ &- (2l+m+m'-1)!\left(\frac{2l+m'-4}{2n} + frac2l+m-12(n-k)\right) \\ &+ \left.\left.\frac{(2l+m+m')!}{2(2n-k)}\right]\right\}^2 . \end{aligned} \tag{E.20}$$

If one prefers SI-units to the natural units, then the above transition rate should be multiplied by the factor $\frac{G^7 M^{15}}{c^5 \hbar^8}$.

As an example, let us now consider a binary black-hole system consisting of two microscopic black holes with equal masses $M = M_{Pl} \approx 22\ \mu\text{g}$. In that case the transition rate between two neighbouring states from state $|i\rangle = |100, 50, 50\rangle$ to state $|f\rangle = |99, 48, 48\rangle$ is approximately

$$R_{i\to f} \approx 2.25 \times 10^{27}\ \frac{1}{\text{s}}, \tag{E.21}$$

and the corresponding lifetime of the initial state in this transition is

$$\tau = \frac{1}{R_{i\to f}} \approx 4.46 \times 10^{-28}\ \text{s}. \tag{E.22}$$

When $\Delta m_l \neq -2$, we may calculate transition rates from Eq. (D.7), since the reduced matrix elements do not depend on the value of the z-component of the angular momentum operator. If $\Delta m_l = +2$ one can easily show that the transition rate is given by

$$\tilde{R}_{i\to f} = \begin{pmatrix} l_f & 2 & l_i \\ -\tilde{m}_f & 2 & \tilde{m}_i \end{pmatrix}^2 \begin{pmatrix} l_f & 2 & l_i \\ -m_f & -2 & m_i \end{pmatrix}^{-2} R_{i\to f}, \tag{E.23}$$

where $R_{i\to f}$ is the transition rate related to the case $\Delta m_l = -2$ and the tilde $\tilde{}$ denotes the case $\Delta m_l = +2$.

In the same way one can show that the case $\Delta m_l = 0$ is related to the case $\Delta m_l = -2$ such that

$$\tilde{R}_{i\to f} = \begin{pmatrix} l_f & 2 & l_i \\ -\tilde{m}_f & 0 & \tilde{m}_i \end{pmatrix}^2 \begin{pmatrix} l_f & 2 & l_i \\ -m_f & -2 & m_i \end{pmatrix}^{-2} R_{i\to f}, \tag{E.24}$$

where $R_{i\to f}$ is the transition rate related to the case $\Delta m_l = -2$ and now the tilde $\tilde{}$ denotes the case $\Delta m_l = 0$.

Case $\Delta l = 0$

Let us next consider the case where $\Delta l = 0$ such that the system always stays in its maximal projection state. In other words the initial state is $|i\rangle = |n, l, l\rangle$ and the final state is $|f\rangle = |n-k, l, l\rangle$. The position representation of these states can be read from Eq. (E.1). In fact, the whole procedure to derive an expression for the transition rate in this case is analogous to the previous case where $\Delta l = -2$. Therefore we just give the final result:

$$\begin{aligned} R_{i\to f} &= -\frac{1}{3}\left(\frac{1}{n^2} - \frac{1}{(n-k)^2}\right)\left(\frac{l+1}{2l+3}\right)^2 \\ &\times \frac{(n+l)!(n-k+l)!(n-l-1)!(n-k-l-1)!}{n^{2l+4}(n-k)^{2l+4}} \\ &\times \left\{\sum_{m=0}^{n-l-1} \frac{(-1)^m}{(n-l-m-1)!(2l+m+1)!m!n^m}\right. \\ &\times \sum_{m'=0}^{n-k-l-1} \frac{(-1)^{m'}}{(n-k-l-m'-1)!(2l+m'+1)!m'!(n-k)^{m'}} \\ &\times \left[\frac{2n(n-k)}{2n-k}\right]^{2l+m+m'+2} \left[(2l+m+m')!\left((l(m+m'+l)+mm')\right.\right. \\ &- \left.\frac{6l^3-4l^2+3l+(m+m')(2l^2-4l)}{2l+2}\right)\frac{2n-k}{2n(n-k)} \\ &+ (2l+m+m'+1)!\left(\frac{l-m'+1}{2n} - \frac{l+m-1}{2(n-k)}\right) \\ &+ \left.\left.\frac{(2l+m+m'+2)!(n-k)}{2n-k}\right]\right\}^2 . \end{aligned} \tag{E.25}$$

For instance, if the initial and the final states are fixed such that the initial state is $|i\rangle = |100, 50, 50\rangle$ and the final state is $|f\rangle = |99, 50, 50\rangle$, then the transition rate between these states is

$$R_{i\to f} \approx 1.44 \times 10^{28}\ \frac{1}{\text{s}}\ , \tag{E.26}$$

and the corresponding lifetime of the initial state in this transition is

$$\tau \approx 6.94 \times 10^{-29}\ \text{s}\,. \tag{E.27}$$

When $\Delta m_l \neq 0$, we cannot use Eq. (D.7) to derive expressions for the transition rates; instead we have to perform another lengthy calculation. The difference between these cases comes from the definition of the associated Legendre function, since we cannot use Eq. (E.4), which holds only for the maximal projection states, in the cases where $\Delta m_l \neq 0$. Because our system is not necessarily in a maximal projection state we have to use the general definition for the associated Legendre functions:

$$P_l^{m_l}(x) := \frac{(1-x^2)^{m_l/2}}{2^l l!} \sum_{i=0}^{l} \frac{(-1)^{l-i} l!}{i!(l-i)!} \prod_{j=0}^{l+m_l-1} |2i-j| x^{2i-l-m_l}\,. \tag{E.28}$$

As this definition comes to have an effect on Eq. (E.1), the transition rate is still given by (C.8). One can show, after performing some integrals, that when $\Delta l = 0$ and $\Delta m_l = -2$, the transition rate of our system obeys, for large n, the following expression:

$$\begin{aligned} R_{i\to f} &= -\frac{1}{24}\left(\frac{1}{n^2}-\frac{1}{(n-k)^2}\right) \\ &\times \frac{(2l+1)^2 2^{2m_l} m_l!^2 (l-m_l)!(l-m_l+2)!(n+l)!}{n^{2l+4}(n-k)^{2l+4}(l+m_l)!(l+m_l-2)!} \\ &\times (n-k+l)!(n-l-1)!(n-k-l-1)! \\ &\times \Bigg\{\sum_{m=0}^{n-l-1}\sum_{m'=0}^{n-k-l-1}\sum_{s=0}^{l}\sum_{s'=0}^{l} a_m b_{m'}\Bigg[I_1 K_{ss'}T^1_{ss'} \\ &+ I_3\Big(-K_{ss'}T^2_{ss'}+K_{ss'}T^3_{ss'}\Big) + I_2\Big(-K_{ss'}T^4_{ss'}+K_{ss'}T^5_{ss'}\Big) \\ &+ I_4\Big(K_{ss'}T^6_{ss'}-K_{ss'}T^7_{ss'}+K_{ss'}T^8_{ss'}\Big)\Bigg]\Bigg\}^2, \end{aligned} \tag{E.29}$$

where we have denoted

$$a_m := \frac{(-1)^m}{(n-l-m-1)!(2l+m+1)!m!n^m}, \tag{E.30}$$

$$b_{m'} := \frac{(-1)^{m'}}{(n-k-l-m'-1)!(2l+m'+1)!m'!(n-k)^{m'}}, \tag{E.31}$$

$$I_1 := \int_0^\infty dr r^2 R_i'(r)R_f'(r), \tag{E.32}$$

$$I_2 := \int_0^\infty dr r R_i'(r)R_f(r), \tag{E.33}$$

$$I_3 := \int_0^\infty dr r R_i(r)R_f'(r), \tag{E.34}$$

$$I_4 := \int_0^\infty dr R_i(r)R_f(r), \tag{E.35}$$

$$R_i(r) := r^{l+m}\exp\left(-\frac{r}{2n}\right), \tag{E.36}$$

$$R_f(r) := r^{l+m'}\exp\left(-\frac{r}{2(n-k)}\right), \tag{E.37}$$

$$K_{ss'} := \frac{(-1)^{2l-s-s'}}{2^{2l}s!(l-s)!s'!(l-s')!}\prod_{i=0}^{l+m_l-1}|2s-i|\prod_{j=0}^{l+m_l-3}|2s'-j|, \tag{E.38}$$

$$T^1_{ss'} := \frac{2^{s+s'-l}(s+s'-l)!}{(2s+2s'-2l+3)(2s+2s'-2l+1)!}, \tag{E.39}$$

$$\begin{aligned} T^2_{ss'} &:= \frac{2^{s+s'-l}(s+s'-l)!}{(2s+2s'-2l+1)!}\times \\ &\left(\frac{1}{2(2s+2s'-2l+3)}+2s'-m_l-l\right), \end{aligned} \tag{E.40}$$

$$T^3_{ss'} := \frac{m_l(2m_l)!}{2^{s+s'-l+2m_l}(s+s'-l-1)!(2m_l-1)m_l!^2}, \tag{E.41}$$

$$T^4_{ss'} := \frac{2^{s+s'-l}(s+s'-l)!}{2m_l(2m_l-2)(2s+2s'-2l)!} \times \left(\frac{2-m_l}{2m_l-4} + 2s' - m_l - l + 2\right), \tag{E.42}$$

$$T^5_{ss'} := \frac{(2-m_l)(2m_l)!2^{s+s'-l-2m_l}(s+s'-l)!}{m_l!^2(2s+2s'-2l+1)!}, \tag{E.43}$$

$$T^6_{ss'} := \frac{2^{s+s'-l}(s+s'-l)!}{2(2s+2s'-2l)!}\left(\frac{m_l-2}{(2s+2s'-2l+1)(2m_l-2)} + \frac{2s'-l-m_l+2}{(2s+2s'-2l+3)(2s+2s'-2l+1)} - \frac{(m_l-2)(2s-l-m_l)}{m_l(2m_l-2)} + \frac{(2s-l-m_l)(2s'-l-m_l+2)}{m_l(2s+2s'-2l+1)}\right), \tag{E.44}$$

$$T^7_{ss'} := \frac{(2m_l)!2^{s+s'-l-2m_l}(s+s'-l)!}{m_l!^2} \times \left(\frac{m_l(2-m_l)}{(2s+2s'-2l)!(2m_l-5)(2m_l-3)(2m_l-1)} + \frac{(2-m_l)(2s-l-m_l)}{(s+s'-l+)!^2 2^{2s+2s'-2l}} + \frac{m_l(2s'-l-m_l+2)}{(2m_l-3)(2m_l-1)(2s+2s'-2l)!} + \frac{m_l(2-m_l)}{(2m_l-1)2^{2s+2s'-2l}(s+s'-l)!^2(2s+2s'-2l+2)}\right), \tag{E.45}$$

$$T^8_{ss'} := \frac{(2-m_l)2^{s+s'-l}(s+s'-l)!}{2(2s+2s'-2l+1)!}. \tag{E.46}$$

When $\Delta m_l = +2$, we may use Eq. (D.7) to give an expression for the transition rate

$$\tilde{R}_{i\to f} = \begin{pmatrix} l_f & 2 & l_i \\ -\tilde{m}_f & 2 & \tilde{m}_i \end{pmatrix}^2 \begin{pmatrix} l_f & 2 & l_i \\ -m_f & -2 & m_i \end{pmatrix}^{-2} R_{i\to f}, \tag{E.47}$$

where $R_{i\to f}$ corresponds to the case $\Delta l = 0$ and $\Delta m_l = -2$.

Case $\Delta l = +2$

The only case we are left with is $\Delta l = +2$. If, for now, we only allow the transitions where $\Delta m_l = +2$, then all the calculations, except for the angular part, proceed in a manner very similar to the first case considered in this appendix. However, the angular integrals are straightforward to evaluate. Therefore we just present the result of the calculation. The transition rate in the case $\Delta n = -k$, $\Delta l = +2$ and $\Delta m_l = +2$ for large n is

$$R_{i\to f} = -\frac{1}{24}\left(\frac{1}{n^2} - \frac{1}{(n-k)^2}\right)\frac{(2l+2)(2l+4)(n+l)!(n-k+l+2)!}{(2l+3)(2l+5)n^{2l+4}(n-k)^{2l+8}} \times (n-l-1)!(n-k-l-3)! \times \left\{\sum_{m=0}^{n-l-1}\frac{(-1)^m}{(n-l-m-1)!(2l+m+1)!m!n^m} \times \sum_{m'=0}^{n-k-l-3}\frac{(-1)^{m'}}{(n-k-l-m'-3)!(2l+m'+5)!m'!(n-k)^{m'}}\right.$$

$$\times \left[\frac{2n(n-k)}{2n-k}^{2l+m+m'+4}\right]\left[\frac{(2l+m+m'+4)!}{4n-k}\right.$$

$$-\left(\frac{l+m'+2}{2n}+\frac{l+m}{2n-2k}+\frac{1}{2}\right)(2l+m+m'+3)!+\frac{2n-k}{2n(n-k)}$$

$$\times \left.\left[(l+m'+2)(l+m)-\frac{1}{2}\right](2l+m+m'+2)!\right]\Bigg\}^2 . \qquad \text{(E.48)}$$

As an example, let us now consider a transition between two neighbouring states from state $|i\rangle = |100, 50, 50\rangle$ to state $|f\rangle = |99, 52, 52\rangle$. We obtain

$$R_{i\to f} \approx 8.30 \times 10^{23}\ \frac{1}{\mathrm{s}}\,, \qquad \text{(E.49)}$$

and the corresponding lifetime is

$$\tau \approx 1.20 \times 10^{-24}\ \mathrm{s}\,. \qquad \text{(E.50)}$$

For $\Delta m_l \neq +2$, we may calculate transition rates from Eq. (D.7). If $\Delta m_l = -2$ one can easily show that the transition rate is given by

$$\tilde{R}_{i\to f} = \begin{pmatrix} l_f & 2 & l_i \\ -\tilde{m}_f & -2 & \tilde{m}_i \end{pmatrix}^2 \begin{pmatrix} l_f & 2 & l_i \\ -m_f & 2 & m_i \end{pmatrix}^{-2} R_{i\to f}\,, \qquad \text{(E.51)}$$

and in the same way one finds that the case $\Delta m_l = 0$ is related to the case where $\Delta l = +2$ and $\Delta m_l = -2$ in the following manner:

$$\tilde{R}_{i\to f} = \begin{pmatrix} l_f & 2 & l_i \\ -\tilde{m}_f & 0 & \tilde{m}_i \end{pmatrix}^2 \begin{pmatrix} l_f & 2 & l_i \\ -m_f & 2 & m_i \end{pmatrix}^{-2} R_{i\to f}\,. \qquad \text{(E.52)}$$

These relations conclude our considerations in these particular cases.

To answer the problem concerning the discreteness of the energy spectrum of microscopic black-hole pairs, we have calculated the lifetime τ of the initial state of the system when all the possible transitions from a fixed initial state are taken into account. We have chosen the initial state to be $|i\rangle = |100, 50, 50\rangle$. This initial state yields a summation of about three hundred transition rates, when all the allowed G1 transitions are taken into account. A numerical estimate for each transition rate can be calculated by using the expressions for transition rates given in this appendix. The lifetime of the initial state is obtained as an inverse of the sum of all the possible transition rates. We have found that when all the transitions are taken into account, the lifetime of the initial state is

$$\tau \approx 1.14 \times 10^{-30}\mathrm{s}\,. \qquad \text{(E.53)}$$

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The independent variables of the Hamiltonian mechanics are the generalized coordinates q_a and the conjugate momenta p_a $(a = 1, \ldots, n)$. The *phase space* is a $2n-$dimensional space where a point of the phase space is given by $2n$ numbers $q_1, \ldots, q_n, p_1, \ldots, p_n$.

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$$\begin{aligned} q_a &= f_a(Q_1, \ldots, Q_n; P_1, \ldots, P_n) \quad , \\ p_a &= g_a(Q_1, \ldots, Q_n; P_1, \ldots, P_n) \quad , \end{aligned}$$

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