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ORIGINAL ARTICLE

On Positivity Sets for Helmholtz Solutions

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Abstract
We address the question of finding global solutions of the Helmholtz equation that are positive in a given set. This question arises in inverse scattering for penetrable obstacles. In particular, we show that there are solutions that are positive on the boundary of a bounded Lipschitz domain.

Keywords Helmholtz equation · Acoustic equation · Lipschitz domain · Inverse scattering problem

Mathematics Subject Classification (2010) 35J05 · 35J15 · 35J20 · 35R30 · 35R35

1 Introduction

The objective in this short note is to consider the following problem.

Question 1.1 Let $k > 0$ and let E be a subset of $\mathbb{R}^n$ $(n \geq 2)$. Does there exist a solution of $(\Delta + k^2)u = 0$ in $\mathbb{R}^n$ with $u|_E > 0$?

Note that any solution of the Helmholtz equation $(\Delta + k^2)u = 0$ is C^∞, and thus the condition $u|_E > 0$ can be understood pointwise. There is a substantial literature on zero sets of solutions of elliptic equations and eigenfunctions, as discussed in the review [11]. In our setting, any real valued solution of $(\Delta + k^2)u = 0$ in $\mathbb{R}^n$ must have a zero in any closed ball

Dedicated to Carlos E. Kenig on the occasion of his 70$^{\text{th}}$ birthday

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of radius $j_{\frac{n-2}{2},1}k^{-1}$ where $j_{\frac{n-2}{2},1}$ is the first zero of the Bessel function $J_{\frac{n-2}{2}}$ (see e.g. [14, Lemma 3.1]). Question 1.1 above is related to producing a global solution whose zero set avoids a given set E.

Our motivation comes from inverse scattering theory and the works [2, 9, 14]. In these works, one considers a bounded open set $D \subset \mathbb{R}^n$ (penetrable obstacle) together with a coefficient $h \in L^\infty(\mathbb{R}^n)$ with $|h| \geq c > 0$ a.e. near ∂D (contrast), and asks whether it is possible to find a solution $u_0 \not\equiv 0$ of $(\Delta + k^2)u_0 = 0$ in $\mathbb{R}^n$ (incident wave) such that the obstacle D with contrast h does not produce any scattering response. The last condition can be precisely formulated as the existence of a function u solving

$$\begin{aligned} &(\Delta + k^2 + h\chi_D)u = 0 \ \text{ in } \mathbb{R}^n, \\ &u = u_0 \ \text{ outside some ball.} \end{aligned}$$

If this happens for some contrast h, then the obstacle D is called a *non-scattering domain* and it will be invisible with respect to probing with the incident wave u_0.

It was proved in [14, Theorem 2.1] that if D has real-analytic boundary and if there is an incident wave u_0 with $u_0|_{\partial D} > 0$, then D is a non-scattering domain. Similarly, the work [9] introduced the notion of quadrature domains for the Helmholtz operator $\Delta + k^2$ and proved that if D is such a domain, and if there is an incident wave u_0 with $u_0|_{\partial D} > 0$, then D is a non-scattering domain. On the other hand, the works [2, 14] show that under a nonvanishing condition for u_0 on ∂D, the boundary of a non-scattering domain can be interpreted as a free boundary in an obstacle-type problem and hence such a domain must be either regular or have thin complement near any boundary point.

It was also proved in [14] that one may be able to find incident waves that are positive on the boundary of a bounded C^1 domain (Lipschitz if $n = 2, 3$). Our first main result extends this to Lipschitz domains in any dimension.

Theorem 1.1 *Let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain such that $\mathbb{R}^n \setminus \overline{D}$ is connected. Suppose that $k^2 > 0$ is not a Dirichlet eigenvalue of $-\Delta$ in D. Then there exists a Herglotz wave function u_0 (see Definition 2.1) satisfying*

$$(\Delta + k^2)u_0 = 0 \ \text{ in } \ \mathbb{R}^n \ \text{ and } \ u_0|_{\partial D} > 0.$$

The proof of Theorem 1.1 is done in two steps. One first constructs a solution v of $(\Delta + k^2)v = 0$ in D with $v|_{\partial D} > 0$ by solving a Dirichlet problem. Then one approximates v in D by a suitable Herglotz wave u_0 in $\mathbb{R}^n$ via a Runge approximation argument. This approximation needs to be done in a suitable norm to obtain the pointwise condition $u_0|_{\partial D} > 0$, but since D only has Lipschitz boundary the solution v is not very regular and this limits the choice of possible norms. We will work with fractional Sobolev spaces $H^{s,p}$ and invoke the theory of boundary value problems in Lipschitz domains.[1]

We remark that the assumption in Theorem 1.1 that k^2 is not an eigenvalue is necessary, at least when D is a ball (see Example 2.5). For the first eigenvalue this was pointed out in [14, Remark 3.2].

Another instance of subsets $E \subset \mathbb{R}^n$ where one can arrange $u_0|_E > 0$ is given in the following result.

Theorem 1.2 *Let $k > 0$, and let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain such that $\mathbb{R}^n \setminus \overline{D}$ is connected and $|D| \leq |B_r|$ where $r = j_{\frac{n-2}{2},1}k^{-1}$. If $E \subset D$ is compact, then there*

[1] This is one of the areas where Carlos Kenig has made pioneering contributions.

exists a Herglotz wave function u_0 (*see Definition* 2.1) *satisfying*

$$(\Delta + k^2)u_0 = 0 \ \ in \ \ \mathbb{R}^n \ \ and \ \ u_0|_E > 0. \tag{1.1}$$

The proof is similar to that of Theorem 1.1, except that in the first step we use the Faber–Krahn inequality to produce a solution v that is positive near E.

Remark 1.3 If E is sufficiently nice and low dimensional, it may be possible to use Theorem 1.2 to find solutions that are positive on E. For example, let E be a smooth compact manifold with $\dim(E) = m \leq n-2$ embedded in $\mathbb{R}^n$, which is homeomorphic to a compact submanifold E_1 of $\mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. This holds e.g. when $m < n/2$ by the Whitney embedding theorem, or when E is homeomorphic to S^m. Since $\mathbb{R}^n \setminus E_1$ is connected, by [12, Corollary 7.9] one sees that $\mathbb{R}^n \setminus E$ is (pathwise) connected. One can construct a tubular neighborhood $D = \{x \in \mathbb{R}^n : d(x, E) < \varepsilon\}$ of E having smooth boundary ∂D and arbitrarily small measure [12, Theorem 9.23 and Remark 9.24] (see also [10, Theorem 6.24]). Since $\mathbb{R}^n \setminus E$ is connected, one can connect any two points in $\mathbb{R}^n \setminus D$ by a curve γ in $\mathbb{R}^n \setminus E$. By considering the curve $F(\gamma)$ where F is a continuous map on $\mathbb{R}^n$ that fixes $\mathbb{R}^n \setminus D$ and collapses $D \setminus E$ to ∂D, we see that $\mathbb{R}^n \setminus D$ is connected. Since D has smooth boundary, also $\mathbb{R}^n \setminus \overline{D}$ is connected. (See [5, pp. 61–62] for a related discussion.) Thus we may apply Theorem 1.2 to find a Herglotz wave function u_0 satisfying (1.1). Note that the connectedness of $\mathbb{R}^n \setminus E$ can fail when E has dimension $n-1$.

2 Solutions Satisfying the Positivity Condition

In this section we will prove Theorems 1.1 and 1.2. We begin with some preparations.

2.1 Fractional Sobolev Spaces

For each $s \in \mathbb{R}$ and $1 < p < \infty$, the fractional Sobolev space $H^{s,p}(\mathbb{R}^n)$ is the Banach space equipped with the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} := \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)},$$

where $\langle D\rangle^s$ is the the *Bessel potential* of order s, i.e. the Fourier multiplier corresponding to $\langle\xi\rangle^s = (1 + |\xi|^2)^{\frac{s}{2}}$. In particular when $s = k \geq 1$ is an integer, we also have $H^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$, where

$$W^{k,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) | D^\alpha u \in L^p(\mathbb{R}^n) \text{for all multi-indices } \alpha \text{ with } |\alpha| \leq k\}.$$

From [1, Corollary 6.2.8], we have the duality statement

$$(H^{s,p}(\mathbb{R}^n))^* = H^{-s,p'}(\mathbb{R}^n) \quad \text{for all } s \in \mathbb{R} \text{ and } 1 < p < \infty, \tag{2.1}$$

where $(p')^{-1} + p^{-1} = 1$. We also recall the Sobolev embedding ([1, Theorem 6.5.1]):

$$H^{s,p}(\mathbb{R}^n) \subset H^{s_1,p_1}(\mathbb{R}^n)$$

whenever $1 < p \leq p_1 < \infty$, $-\infty < s_1 \leq s < \infty$, and $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$.

Let D be an open set in $\mathbb{R}^n$. We define

$$H^{s,p}(D) := \{u|_D \mid u \in H^{s,p}(\mathbb{R}^n)\} \quad \text{for all } s \in \mathbb{R} \text{ and } 1 < p < \infty.$$

This is a Banach space equipped with the quotient norm

$$\|v\|_{H^{s,p}(D)} := \inf\{\|u\|_{H^{s,p}(\mathbb{R}^n)} \mid u|_D = v\}.$$

When D is a bounded Lipschitz domain, from [8, Theorem 2.3] we know that there exists a bounded linear extension operator

$$E : H^{s,p}(D) \to H^{s,p}(\mathbb{R}^n) \quad \text{with } Eu = u \text{ in } D \text{ for all } u \in H^{s,p}(D).$$

If $F \subset \mathbb{R}^n$ is closed, we define

$$H^{s,p}_F(\mathbb{R}^n) := \{u \in H^{s,p}(\mathbb{R}^n) \mid \operatorname{supp}(u) \subset F\}.$$

If D is a bounded Lipschitz domain, the following result can be found in [8, Remark 2.7]:

$$C^\infty_c(D) \text{ is dense in } H^{s,p}_{\overline{D}}(\mathbb{R}^n) \text{ for each } s \in \mathbb{R} \text{ and } 1 < p < \infty. \tag{2.2}$$

2.2 Runge–Herglotz Approximation

The next objective is to prove a result stating that solutions in $H^{s,p}(D)$ can be approximated in D by Herglotz waves. We first give a definition.

Definition 2.1 Let $k > 0$ and consider the operator $P_k : C^\infty(\mathcal{S}^{n-1}) \to C^\infty(\mathbb{R}^n)$ defined by

$$(P_k f)(x) := \int_{\mathcal{S}^{n-1}} e^{ikx\cdot\hat{z}} f(\hat{z})\, d\hat{z}, \qquad x \in \mathbb{R}^n.$$

The functions $u = P_k f$ with $f \in C^\infty(\mathcal{S}^{n-1})$ are called *Herglotz waves*, and they are particular solutions of $(\Delta + k^2)u = 0$ in $\mathbb{R}^n$.

Proposition 2.2 *Let $k > 0$, $0 < s \le 1$, $1 < p < \infty$, and let $D \subset \mathbb{R}^n$ $(n \ge 2)$ be a bounded Lipschitz domain such that $\mathbb{R}^n \setminus \overline{D}$ is connected. Given any $v \in H^{s,p}(D)$ with $(\Delta + k^2)v = 0$ in D, there exist Herglotz waves $u_j \in C^\infty(\mathbb{R}^n)$ such that*

$$\|u_j - v\|_{H^{s,p}(D)} \to 0 \quad \text{as} \quad j \to \infty.$$

If v is real-valued, then so are u_j.

The proof of Proposition 2.2 is very similar to [14, Proposition 3.4] that considered approximation in $W^{1,p}(D)$. Here we need to work with fractional Sobolev spaces instead.

Proof In view of the Hahn–Banach theorem, it is enough to prove that any bounded linear functional $\ell : H^{s,p}(D) \to \mathbb{C}$ that vanishes on $\{P_k f|_D \mid f \in C^\infty(\mathcal{S}^{n-1})\}$ must also vanish on $\{v \in H^{s,p}(D) \mid -(\Delta + k^2)v = 0 \text{ in } D\}$. Let ℓ be such a linear functional, and define a bounded linear functional $\ell_1 : H^{s,p}(\mathbb{R}^n) \to \mathbb{C}$ by $\ell_1(u) := \ell(u|_D)$. By duality (2.1), there exists a unique $\mu \in H^{-s,p'}(\mathbb{R}^n)$ such that

$$\ell_1(u) = (u, \mu) \quad \text{for all } u \in H^{s,p}(\mathbb{R}^n),$$

where $(\cdot, \cdot)$ is the sesquilinear distributional pairing in $\mathbb{R}^n$. It is easy to see that $\mu = 0$ in $\mathbb{R}^n \setminus \overline{D}$, and the condition $\ell(P_k f|_D) = 0$ for all $f \in C^\infty(\mathcal{S}^{n-1})$ implies that

$$(P_k f, \mu) = 0 \quad \text{for all } f \in C^\infty(\mathcal{S}^{n-1}). \tag{2.3}$$

We now define the distribution $w := \Phi_k * \mu$, where

$$\Phi_k(x) = \frac{ik^{\frac{n-2}{2}}}{4(2\pi)^{\frac{n-2}{2}}}|x|^{-\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(k|x|)$$

is the outgoing fundamental solution of the Helmholtz operator $-(\Delta + k^2)$ and $H^{(1)}_\alpha$ is the Hankel function (see [15, §1.2.3]). Then w is a distributional solution of

$$-(\Delta + k^2)w = \mu \quad \text{in } \mathbb{R}^n. \tag{2.4}$$

Elliptic regularity yields $w \in H^{2-s,p'}_{\mathrm{loc}}(\mathbb{R}^n)$, and since $\operatorname{supp}(\mu) \subset \overline{D}$ we also have that w is C^∞ in $\mathbb{R}^n \setminus \overline{D}$.

Given any $f \in C^\infty(\mathcal{S}^{n-1})$, we write $u = P_k f \in C^\infty(\mathbb{R}^n)$. Using (2.3) and the fact that μ has compact support, we have

$$0 = (u, \mu) = \lim_{r\to\infty}(u, \mu)_{B_r}, \tag{2.5}$$

where $(\cdot, \cdot)_{B_r}$ is the sesquilinear distributional pairing in the ball B_r. We now consider a cut-off function $\chi \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \le \chi \le 1$ and $\chi = 1$ near $\overline{D}$. Using (2.4), we can write (2.5) as

$$\begin{aligned} 0 &= \lim_{r\to\infty}\left[(\chi u, (\Delta + k^2)w)_{B_r} + ((1-\chi)u, (\Delta + k^2)w)_{B_r}\right] \\ &= \lim_{r\to\infty}\Big[((\Delta + k^2)(\chi u), w)_{B_r} + ((\Delta + k^2)((1-\chi)u), w)_{B_r} \\ &\qquad + \int_{\partial B_r}(u\overline{\partial_{|x|}w} - (\partial_{|x|}u)\overline{w})\, dS\Big] \\ &= \lim_{r\to\infty}\int_{\partial B_r}(u\overline{\partial_{|x|}w} - (\partial_{|x|}u)\overline{w})\, dS, \end{aligned} \tag{2.6}$$

where $\partial_{|x|} = \hat{x}\cdot\nabla$ denotes the radial derivative. Here we also used the fact that $(\Delta + k^2)u = 0$ in $\mathbb{R}^n$.

Using [13, Lemma 1.2 and equation (1.18)], we know that the Herglotz function $u = P_k f$ has the following asymptotics as $|x| \to \infty$:

$$u(x) = c'_{n,k}\,|x|^{-\frac{n-1}{2}}\left(e^{ik|x|}f(\hat{x}) + i^{n-1}e^{-ik|x|}f(-\hat{x})\right) + O(|x|^{-\frac{n+1}{2}}), \tag{2.7a}$$

$$\partial_{|x|}u(x) = c'_{n,k}\,|x|^{-\frac{n-1}{2}}\,ik\left(e^{ik|x|}f(\hat{x}) - i^{n-1}e^{-ik|x|}f(-\hat{x})\right) + O(|x|^{-\frac{n+1}{2}}), \tag{2.7b}$$

where $c'_{n,k} = k^{\frac{n-1}{2}}e^{\frac{\pi(n-1)i}{4}}(2\pi)^{-\frac{n-1}{2}}$. On the other hand, from [15, equation (2.27)], we know that w has the asymptotics

$$w(x) = c''_{n,k}|x|^{-\frac{n-1}{2}}e^{ik|x|}\hat{\mu}(k\hat{x}) + O(|x|^{-\frac{n+1}{2}}) \quad \text{as } |x| \to \infty, \tag{2.7c}$$

$$\partial_{|x|}w(x) = c''_{n,k}|x|^{-\frac{n-1}{2}}ike^{ik|x|}\hat{\mu}(k\hat{x}) + O(|x|^{-\frac{n+1}{2}}) \quad \text{as } |x| \to \infty, \tag{2.7d}$$

where $c''_{n,k} = 2^{-1}e^{-\frac{\pi(n-3)i}{4}}(2\pi)^{-\frac{n-1}{2}}k^{\frac{n-3}{2}}$ and $\hat{\mu} \in C^\infty(\mathbb{R}^n)$ is the Fourier transform of the compactly supported distribution μ.

Combining (2.6) with (2.7a)–(2.7d), we obtain

$$\int_{\mathcal{S}^{n-1}} f(\hat{x})\overline{\hat{\mu}(k\hat{x})}\, d\hat{x} = 0.$$

By the fact that $f \in C^\infty(\mathcal{S}^{n-1})$ was arbitrary, we conclude $\hat{\mu}(k\hat{x}) = 0$ for all $\hat{x} \in \mathcal{S}^{n-1}$. Consequently, (2.7c) becomes

$$w(x) = O(|x|^{-\frac{n+1}{2}}) \quad \text{as} \quad |x| \to \infty.$$

In other words, the far-field pattern of w is vanishing. By the Rellich uniqueness theorem [4, 7], the unique continuation principle and the connectedness of $\mathbb{R}^n \setminus \overline{D}$, we conclude that

$$w = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \overline{D}.$$

Since $w \in H^{2-s,p'}_{\text{loc}}(\mathbb{R}^n)$, we also conclude that $w \in H^{2-s,p'}_{\overline{D}}(\mathbb{R}^n)$.

Now let $v \in H^{s,p}(D)$ be any solution of $(\Delta + k^2)v = 0$ in D, and let $\tilde{v} \in H^{s,p}(\mathbb{R}^n)$ be such that $\tilde{v}|_D = v$. We see that

$$\ell(v) = \ell_1(\tilde{v}|_D) = (\tilde{v}, \mu) = (\tilde{v}, (\Delta + k^2)w).$$

From (2.2), we know that there are $w_j \in C_c^\infty(D)$ with $w_j \to w$ in $H^{2-s,p'}(\mathbb{R}^n)$. Since $(\Delta + k^2)\tilde{v} = 0$ in D, we finally conclude that

$$\ell(v) = \lim_{j\to\infty} (\tilde{v}, (\Delta + k^2)w_j) = \lim_{j\to\infty} ((\Delta + k^2)\tilde{v}, w_j) = 0,$$

which is our desired result. □

2.3 Proof of the main result

Theorem 1.1 is an immediate consequence of the following result:

Theorem 2.3 *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ ($n \geq 2$) such that $\mathbb{R}^n \setminus \overline{D}$ is connected. Suppose that $k^2 > 0$ is not a Dirichlet eigenvalue of $-\Delta$ in D. Given any constant $c_0 \in \mathbb{R}$, there exist Herglotz wave functions $u_j \in C^\infty(\mathbb{R}^n)$ solving $(\Delta + k^2)u_j = 0$ in $\mathbb{R}^n$ such that*

$$\lim_{j\to\infty} \|u_j - c_0\|_{L^\infty(\partial D)} = 0.$$

Before we prove Theorem 2.3 we need the following result, which is a special case of [8, Theorems 1.1 & 1.3].

Proposition 2.4 *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ ($n \geq 2$). If $2 \leq p < \infty$ and $f \in H^{s-2,p}(D)$ where*

$$\frac{1}{p} < s < \frac{3}{p},$$

then there exists a unique $u \in H^{s,p}(D)$ satisfying $-\Delta u = f$ in D and $u = 0$ on ∂D.

Proof We first consider the case when $n \geq 3$. Let p_0 be as in [8, Theorem 1.1] (with $\Omega = D$). If $p_0' \leq p < \infty$, the result follows from [8, Theorem 1.1(c)]. On the other hand, if $2 \leq p < p_0'$, the result follows from [8, Theorem 1.1(a)] since $s < \frac{3}{p} \leq 1 + \frac{1}{p}$. The case when $n = 2$ can be proved using identical reasoning using [8, Theorem 1.3] and the observation $\frac{3}{p} \leq \frac{2}{p} + \frac{1}{2}$. □

Proof of Theorem 2.3 Since k^2 is not a Dirichlet eigenvalue in D, there exists a unique solution $v \in H^{1,2}(D)$ such that

$$(\Delta + k^2)v = 0 \;\text{ in }\; D \quad \text{and} \quad v = c_0 \;\text{ on }\; \partial D.$$

If $v \in H^{s,p}(D)$ for some $0 < s \leq 1$ and $p > n/s$, using Proposition 2.2, we know that there exist Herglotz waves $u_j \in C^\infty(\mathbb{R}^n)$ such that

$$\|u_j - c_0\|_{L^\infty(\partial D)} = \|u_j - v\|_{L^\infty(\partial D)} \leq \|u_j - v\|_{C(\overline{D})} \leq C\|u_j - v\|_{H^{s,p}(D)} \to 0,$$

where we used the Sobolev embedding.

It remains to show that $v \in H^{s,p}(D)$ for some s, p with $s > n/p$, and this follows from a standard bootstrap argument based on Proposition 2.4. We claim that

$$v \in H^{\frac{2}{p_j},p_j}(D) \quad \text{for } 0 \leq j < \frac{n-2}{4}, \tag{2.5}$$

where

$$\frac{1}{p_j} = \frac{1}{2} - j\frac{2}{n-2}.$$

The case $j = 0$ follows since $v \in H^{1,2}(D)$. We argue by induction and assume that this holds for j. Define $w := v - c_0$ and note that w solves

$$-\Delta w = k^2 v \in H^{\frac{2}{p_j},p_j}(D), \qquad w|_{\partial D} = 0.$$

We next use the Sobolev embedding $H^{\frac{2}{p_j},p_j}(D) \subset H^{\frac{2}{q}-2,q}(D)$ where $\frac{2}{p_j} > \frac{2}{q} - 2$ and

$$\frac{2}{p_j} - \frac{n}{p_j} = \frac{2}{q} - 2 - \frac{n}{q}.$$

It follows that $q = p_{j+1}$ and then indeed $\frac{2}{p_j} > \frac{2}{q} - 2$. In particular $-\Delta w \in H^{\frac{2}{p_{j+1}}-2,p_{j+1}}(D)$ with $w|_{\partial D} = 0$, and we may use Proposition 2.4 to conclude that $w \in H^{\frac{2}{p_{j+1}},p_{j+1}}(D)$. This completes the induction step and proves (2.5).

We have proved that $v \in H^{\frac{2}{p_j},p_j}(D)$ where j is the largest integer $< \frac{n-2}{4}$. Using the above notation, we have $\Delta w \in H^{\frac{2}{p_j},p_j}(D)$ and $w|_{\partial D} = 0$. By Sobolev embedding we have $\Delta w \in H^{s-2,p}(D)$ whenever $p \geq p_j$ and

$$\frac{2}{p_j} - \frac{n}{p_j} = s - 2 - \frac{n}{p}.$$

The last condition implies that

$$s - \frac{n}{p} = 2 + \frac{2-n}{p_j} = 2 + \frac{2-n}{2} + 2j \geq 0$$

since $j \geq \frac{n-2}{4} - 1$. If $j > \frac{n-2}{4} - 1$, using Proposition 2.4 once again we obtain that w and hence v is in $H^{s,p}$ for some $s > n/p$. On the other hand, if $j = \frac{n-2}{4} - 1$ we iterate the argument once more to get $v \in H^{s,p}$ for some $s > n/p$. This concludes the proof. □

The next simple example shows that the condition that k^2 is not an eigenvalue is necessary at least for balls.

Example 2.5 Let $v(x) := |x|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(|x|)$. We see that $v \in C^\infty(\mathbb{R}^n)$ and $(\Delta + 1)v = 0$ in $\mathbb{R}^n$. Suppose that u_1 is a real-valued function satisfying $(\Delta + 1)u_1 = 0$ in $\mathbb{R}^n$. Since

$$v(x) = 0 \quad \text{when } |x| = j_{\frac{n-2}{2},m} \quad \text{for any } m \geq 1,$$

where $j_{\frac{n-2}{2},m}$ denotes the mth positive zero of $J_{\frac{n-2}{2}}$, we have

$$\int_{|x|=j_{\frac{n-2}{2},m}} u_1 \frac{\partial v}{\partial r}\, dS = \int_{|x|<j_{\frac{n-2}{2},m}} (u_1 \Delta v - v \Delta u_1)\, dx = 0.$$

Since

$$(-1)^m \frac{\partial v}{\partial r}(x) > 0 \quad \text{when } |x| = j_{\frac{n-2}{2},m},$$

it follows that u_1 must change sign on $|x| = j_{\frac{n-2}{2},m}$.

Similarly, if $R > 0$ and if u_0 solves $(\Delta + k_m^2)u_0 = 0$ in $\mathbb{R}^n$ where $k_m = R^{-1} j_{\frac{n-2}{2},m}$, define u_1 via the rescaling

$$u_0(x) = u_1\left(R^{-1} j_{\frac{n-2}{2},m} x\right) \quad \text{for } x \in \mathbb{R}^n.$$

We see that $(\Delta + 1)u_1 = 0$ in $\mathbb{R}^n$. The above discussion shows that u_0 must change sign on ∂B_R.

The following strong maximum principle can be found in [9, Appendix A]. However, for readers' convenience, here we exhibit the statement as well as its proof.

Lemma 2.6 (Strong maximum principle) *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ ($n \geq 2$), and let $k^2 < \lambda_1(D)$, where $\lambda_1(D) > 0$ denotes the smallest $H_0^1(D)$-eigenvalue of $-\Delta$. If the solution $u \in H^1(D)$ satisfies*

$$(\Delta + k^2)u = 0 \text{ in } D, \quad u \geq 0 \text{ on } \partial D,$$

then for each open component G of D we have either $u \equiv 0$ in G or $u > 0$ in G (note that $u \in C^\infty(G)$ by elliptic regularity).

Proof It is easy to see that for each component G of D we have $k^2 < \lambda_1(G)$ and

$$(\Delta + k^2)u = 0 \text{ in } G, \quad u \geq 0 \text{ on } \partial G.$$

Testing the equation above by $u_- \in H_0^1(G)$ and using Poincaré inequality, we have

$$\int_G |u_-|^2\, dx \leq \frac{1}{\lambda_1(G)} \int_G |\nabla u_-|^2\, dx = \frac{k^2}{\lambda_1(G)} \int_G |u_-|^2\, dx.$$

Since $\frac{k^2}{\lambda_1(G)} < 1$, then $u_- \equiv 0$ in G, that is,

$$u \geq 0 \text{ in } G. \tag{2.6}$$

Let $x_0 \in G$ such that $u(x_0) = 0$. The mean value theorem for Helmholtz equation (see e.g. [9, Appendix A]) gives that

$$\int_{B_\varepsilon(x_0)} u(x)\, dx = 0 \tag{2.7}$$

for all sufficiently small $\varepsilon > 0$ so that $\overline{B_\varepsilon(x_0)} \subset G$. Since u is continuous in G, combining (2.6) and (2.7) we know that $u = 0$ in $B_\varepsilon(x_0)$, and this shows that $\{x \in G \mid u(x) = 0\}$ is both open and closed in G. Since G is connected, then we have either

$$\{x \in G \mid u(x) = 0\} = G \quad \text{or} \quad \{x \in G \mid u(x) = 0\} = \emptyset,$$

which concludes our lemma. □

Finally, we give the proof of Theorem 1.2.

***Proof of Theorem* 1.2** Since $|D| \leq |B_r|$ where $r = j_{\frac{n-2}{2},1}k^{-1}$, the Faber–Krahn inequality (see e.g. [3, Theorem III.3.1]) implies that each connected component G of D satisfies

$$\lambda_1(G) \geq \lambda_1(B_r) = k^2.$$

Case 1. If $\lambda_1(G) = k^2$, we choose v to be an eigenfunction corresponding to the first eigenvalue with $v > 0$ in G, i.e. v solves $(\Delta + k^2)v = 0$ in G with $v \in H_0^1(G)$, see e.g. [6, Theorem 2(ii) in Section 6.5.1].
Case 2. If $\lambda_1(G) > k^2$, then there exists a unique solution $v \in H^1(G)$ such that

$$(\Delta + k^2)v = 0 \text{ in } G, \quad v = 1 \text{ on } \partial G.$$

Using the strong maximum principle in Lemma 2.6, we know that $v > 0$ in G.

Next we choose a bounded Lipschitz domain D_1 that satisfies $E \subset D_1$, $\overline{D}_1 \subset D$, and $\mathbb{R}^n \setminus \overline{D}_1$ is connected. The function $v|_{D_1}$ is in $H^{1,p}(D_1)$ for any $p > n$ and satisfies $v|_{\overline{D}_1} > 0$. The approximation result in Proposition 2.2 yields a sequence of Herglotz waves u_j satisfying

$$\|u_j|_{D_1} - v\|_{H^{1,p}(D_1)} \to 0 \quad \text{as } j \to \infty.$$

If j is sufficiently large, the Sobolev embedding ensures that $u_j|_E > 0$. □

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Data Availability All data needed are contained in the manuscript.

Declaration

Conflicts of interest The authors declare that there are no competing or conflict of interests.

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