Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 372 (2023) 458–504

Journal of Differential Equations

www.elsevier.com/locate/jde

Classical flows of vector fields with exponential or sub-exponential summability

Luigi Ambrosio [a], Sebastiano Nicolussi Golo [b,*], Francesco Serra Cassano [c]

[a] *Scuola Normale Superiore, Pisa, Italy*
[b] *Department of Mathematics and Statistics, University of Jyväskylä, Finland*
[c] *Dipartimento di Matematica, Università di Trento, Italy*

Received 2 August 2022; revised 23 May 2023; accepted 3 July 2023

Abstract

We show that vector fields b whose spatial derivative $D_x b$ satisfies a Orlicz summability condition have a spatially continuous representative and are well-posed. For the case of sub-exponential summability, their flows satisfy a Lusin (N) condition in a quantitative form, too. Furthermore, we prove that if $D_x b$ satisfies a suitable exponential summability condition then the flow associated to b has Sobolev regularity, without assuming boundedness of $\mathrm{div}_x b$. We then apply these results to the representation and Sobolev regularity of weak solutions of the Cauchy problem for the transport and continuity equations.

MSC: 35F10; 35A01; 35A02

Keywords: Vector fields; Flow; Sobolev–Orlicz spaces; Transport equation; Continuity equation

* Corresponding author.
E-mail addresses: luigi.ambrosio@sns.it (L. Ambrosio), sebastiano.s.nicolussi-golo@jyu.fi (S. Nicolussi Golo), francesco.serracassano@unitn.it (F. Serra Cassano).

https://doi.org/10.1016/j.jde.2023.07.005

Contents

1. Introduction

In this paper we are concerned with the study of the existence and uniqueness of classical solutions of the Cauchy problem for the ODE system

$$\begin{cases} \dot\gamma(t) = b(t, \gamma(t)) \\ \gamma(s) = x\,, \end{cases} \tag{1}$$

with $x \in \Omega$, an open domain in $\mathbb{R}^n$, $s \in I$, an open interval in $\mathbb{R}$, and $b : I \times \Omega \to \mathbb{R}^n$ a continuous, possibly non-autonomous vector field Even though we will mostly deal with the case when b is continuous, we will point out which proofs can easily be adapted to the case when b is only measurable with respect to t. If solutions to (1) exist and are unique for every s and x, we say that the vector field b is *well-posed* in $I \times \Omega$ (or in $\overline{I} \times \Omega$, see Definition 3.1 for a more precise statement). For every well posed vector field $b : I \times \Omega \to \mathbb{R}^n$ we have a *flow*, that is, a map $X : I \times I \times \Omega \to \Omega$, defined as $X(t, s, x) := \gamma(t)$ where γ is the unique absolutely continuous solution of (1). More precisely, for each $t, s \in I$ we denote by $\Omega_{(t,s)} \subset \Omega$ the open set of all $x \in \Omega$ such that the path starting at x at time s can be extended until time t (see Section 3.2 and Remark 3.3). Then $X(t, s, \cdot)$ is a well defined homeomorphism $\Omega_{(t,s)} \to \Omega_{(s,t)}$ (see Remark 3.3).

Di Perna–Lions [17] carried out a pioneering and far-reaching theory by introducing a generalized notion of flow for vector fields $b \in L^1_{\mathrm{loc}}((0, T); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n))$ with important applications to the existence and uniqueness of weak solutions for the Cauchy problem of the *transport equation* associated to a weakly differentiable vector field b, that is,

$$\begin{cases} \partial_t u + b \cdot D_x u = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ u(0,\cdot) = \bar{u}\,. \end{cases} \tag{2}$$

The theory was later remarkably extended by the first author [2] to vector fields $b(t,\cdot)$ with BV regularity. In these works, the regularity of b is paired with the boundedness of its spatial divergence, that is

$$\mathrm{div}_x b \in L^1((0,T); L^\infty(\mathbb{R}^n)), \tag{3}$$

which ensures the existence and uniqueness of the generalized flow of b. If (3) does not hold, then uniqueness of the flow may fail, as it was already shown in [17, Section IV.1]. The existence and uniqueness of a generalized flow associated to a weakly regular vector field b has been the object of an intensive study with applications to the Cauchy problem for the transport equation as well as for the *continuity equation* associated to b, that is,

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(b\rho) = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ \rho(0,\cdot) = \bar{\rho} & \text{in } \mathbb{R}^n\,. \end{cases} \tag{4}$$

Existence, uniqueness and regularity of solutions of these three problems (1), (2) and (4) are connected with each other. In particular, the existence of a unique flow X with enough regularity implies existence and uniqueness of solutions to both the transport equation and the continuity equation. A fairly complete account of the development in this topic can be found in [5] and references therein. A sample of the literature on this subject is [4,8–10,13,11,12,14–16,27,29,28].

Our contribution focuses on two problems. First, we want to weaken the boundedness assumption on the divergence (3). We will show in Theorem A that sub-exponential summability of $\|D_x b\|$ guarantees the existence of a unique *classical* flow (in the Di Perna–Lions–Ambrosio theory, flows have a weaker definition).

Second, we want to find conditions on b for the flow to have Sobolev regularity, instead of just L^p integrability. It is well-known that high L^p integrability of matrix Jacobian $D_x b$, even coupled with (3), is not enough in order to provide Sobolev regularity of the flow X (see, for instance, [26]). A strategy used in the recent papers [13,8] was to strengthen the hypotheses by requiring exponential summability of $\|D_x b\|$. We refer in particular to the recent paper [8], where it has been shown that b has a unique flow with Sobolev regularity under the condition

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{T}^n} \exp(\beta \|D_x b(t,x)\|)\,\mathrm{d}x < \infty \quad \text{and} \quad \mathrm{div}_x b \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{T}^n), \tag{5}$$

for some $\beta > 0$, where $\mathbb{T}^n$ is the n-dimensional torus. We prove analogous results without conditions on the divergence of b in Theorem D, see also Remark 6.3.

Our results are of three types. We first provide integral conditions of sub-exponential type on Db that ensure well-posedness. Then, we study the Sobolev regularity of the homeomorphisms

$X(t, s, \cdot)$. Finally, we apply these results to both the transport equation and the continuity equation.

1.1. Well-posedness

Let us focus, first, on the well-posedness. If $\|D_x b\|$ satisfies an *exponential summability*, that is, conditions of the form

$$\int_I \int_\Omega \exp\left(\beta \|D_x b(t,x)\|\right) \mathrm{d}x \, \mathrm{d}t < +\infty \tag{6}$$

for some $\beta > 0$, then it is well-known that b is well-posed. Indeed, in this case, the vector field $b(t, \cdot)$ satisfies a so-called Log-Lipschitz condition; see, for instance, [8,32]. However, reformulating the condition of exponential summability in a Orlicz-like form, we extend the result to some sub-exponential cases.

Theorem A. *Let $\Theta : [0, +\infty) \to (0, +\infty)$ be a non decreasing locally Lipschitz function. Assume that*

(A.I) if $n > 1$, there exists $\alpha \in (1, \frac{n}{n-1})$ such that $\Theta^{\frac{\alpha-1}{\alpha}}$ is convex, while, if $n = 1$, Θ is convex;
(A.II) there exists $C_\Theta \geq 1$ such that $\Theta : [C_\Theta, +\infty) \to [\Theta(C_\theta), +\infty)$ is bijective and

$$\Theta(s_1)\Theta(s_2) \leq \Theta(C_\Theta \, s_1 \, s_2) \text{ for all } s_1, s_2 \geq C_\Theta\,; \tag{7}$$

(A.III) $\displaystyle\int_1^\infty \frac{\Theta'(s)}{s\Theta(s)} \, \mathrm{d}s = +\infty.$

Let $b \in \mathrm{L}^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n))$ and assume that for every $o \in \Omega$ there exist $c > 0$, $R > 0$ such that $B(o, 2R) \subset \Omega$ and the function

$$t \mapsto \psi(t) := \int_{B(o,2R)} \Theta(c\|D_x b(t,z)\|) \, \mathrm{d}z \tag{8}$$

belongs to $\mathrm{L}^1_{\mathrm{loc}}(I)$. Then $b(t, \cdot) : \Omega \to \mathbb{R}^n$ has a continuous representative $\tilde{b}(t, \cdot)$ for a.e. $t \in I$ and $\tilde{b}$ is well-posed in $I \times \Omega$. Moreover, if there exists $m \in L^1(I)$ such that

$$|b(t,x)| \leq m(t) \quad \text{for a.e. } t \in I, \text{ for a.e. } x \in B(o, R)\,, \tag{9}$$

then $\tilde{b}$ is also well-posed in $\overline{I} \times \Omega$.

Notice that a byproduct of the proof of Theorem A is that the Sobolev-Orlicz space $W^1 L_\Theta(\Omega)$ embeds in $C^0(\Omega)$, with modulus of continuity that depends only on Θ. See [7, Section 2.6] for the definition of $W^1 L_\Theta(\Omega)$.

The proof is inspired from [32]. In fact, we shall prove that a continuous representative $\tilde{b}(t,\cdot)$ of $b(t,\cdot)$ satisfies the Osgood's criterion (see [23, Chap. III, Corollary 6.2] or Proposition 3.2 below). The proof of this result is given in Section 4.

We note in Proposition 4.4 that if an increasing function Θ satisfies condition (A.III) of Theorem A, then it does not have polynomial growth. Examples of functions Θ satisfying the properties (A.I)-(A.III) are of the form

$$\mathscr{E}_{k,\beta}(s) = \exp\left(\frac{s}{\log(s)\log\log(s)\ldots(\underbrace{\log\ldots\log}_{k\text{-times}} s)^{\beta}}\right) \quad \text{for } s \geq \bar{s}, \tag{10}$$

with $\mathscr{E}_{k,\beta}(s) = \mathscr{E}_{k,\beta}(\bar{s})$ for $s < \bar{s}$, where $\bar{s}$ is large enough, $k \geq 1$ is an integer and $0 \leq \beta \leq 1$, see Proposition 4.6. Notice that the asymptotic behavior of $\mathscr{E}_{k,\beta}$ as $s \to \infty$ is almost sharp in order that assumption (A.III) holds, see Remark 4.7. If $\mathscr{E}_{k,1}(c\,\|D_x b\|) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ for some $c > 0$, we say that $\|D_x b\|$ satisfies a *subexponential summability* of order k. Therefore, Theorem A shows that, if $\|D_x b\|$ has *subexponential summability*, then b has a *classical* unique flow. However, we stress that, under the hypothesis of Theorem A, $\|D_x b\|$ does not need to be in $L^p_{\mathrm{loc}}(I \times \Omega)$ for each $p > 1$ (see Remark 4.5).

1.2. Regularity

Moving on to the regularity of the flow, we can prove that, if $D_x b$ satisfies a subexponential summability of order 1, the associated flow $X(t,s,\cdot)$ satisfies a weak regularity property, namely it maps the Lebesgue measure into absolutely continuous measures. Notice that, in this case, $\|D_x b(t,\cdot)\|$ does belong to $L^p_{\mathrm{loc}}(\Omega)$ for every $p > 1$, for almost every $t \in I$. A quantitative version, that we obtain adapting [11], is the following.

Theorem B. *Let $b \in \mathrm{L}^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n))$. Suppose that*

$$\frac{|b(t,x)|}{1 + |x|\,\log^+|x|} \in L^1(I; L^\infty(\mathbb{R}^n)), \tag{11}$$

and

$$\int_I \int_{\mathbb{R}^n} \exp\left(\frac{\|D_x b\|}{\log^+\|D_x b\|}\right)(t,x)\, d\gamma_n(x) dt < +\infty, \tag{12}$$

where γ_n is the Gaussian measure on $\mathbb{R}^n$, namely

$$\gamma_n := \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|x|^2}{2}\right)\mathcal{L}^n$$

with $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$.

Then the space continuous representative $\tilde{b}$, granted by the Sobolev embedding, is well-posed in $\overline{I} \times \Omega$ and the associated flow X of $\tilde{b}$ is globally defined, that is, $X : \overline{I} \times \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$. Moreover, for every $t, s \in \overline{I}$, the image measure $X(t, s, \cdot)_\# \mathcal{L}^n$ is absolutely continuous with respect to $\mathcal{L}^n$ and there exists a positive constant $\alpha_0(t, s) > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{L}^n}(X(t, s, \cdot)_\# \mathcal{L}^n) \in \mathrm{L}^{\Phi_\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \tag{13}$$

for each $0 < \alpha < \alpha_0(s, t)$, where $\mathrm{L}^{\Phi_\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ is the Orlicz space with

$$\Phi_\alpha : [0, +\infty) \to [0, +\infty), \qquad \Phi_\alpha(w) := w \exp((\log^+ w)^\alpha),$$

and $\alpha_0 : I \times I \to \mathbb{R}$ is continuous with $\alpha_0(t, t) = 1$ for every $t \in I$.

See Section 5 for the proof. By trivial considerations, in the one-dimensional case, we can improve Theorem B to absolutely continuity of the flow.

Theorem C. *If $n = 1$ and b satisfies the conditions of Theorem B, then, for every $t, s \in I$, the map $X(t, s, \cdot)$ is an absolutely continuous homeomorphism between intervals of $\mathbb{R}$.*

Sobolev regularity stated in Theorem C is sharp, as we show in Example 8.2. We don't know whether Theorem C can be extended to the case of sub-exponential summability of order $k > 1$.

In higher dimensions, for subexponential summability, we have a partial negative result with an example in Section 8.2, or the example constructed in [8], see also Remark 7.2: *there are vector fields satisfying a subexponential summability of order 1 whose flow is not in $W^{1,p}_{\mathrm{loc}}$ for any $p > n$*. However, it remains open whether vector fields satisfying a subexponential summability of order 1 can fail to have the flow in $W^{1,1}_{\mathrm{loc}}$.

On the other hand, in higher dimensions, for exponential summability we have a positive result:

Theorem D. *Let $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be bounded open sets and let $b \in \mathrm{L}^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n))$ be bounded. Assume that for some $p > 2n$ the vector field b satisfies the global geometric condition*

$$\Lambda_p := \int_\Omega \int_I \max\left\{ \ell^{\frac{n}{n-p}}, \frac{(\mathrm{dist}(x, \partial\Omega))^{\frac{n}{n-p}}}{(\sup|b|)^{\frac{n}{n-p}}} \right\} \exp\left(\frac{\ell p^2}{p-n} \|D_x b(s, x)\| \right) \mathrm{d}s \, \mathrm{d}x < +\infty, \tag{14}$$

with ℓ equal to the length of I. Then the space-continuous representative $\tilde{b}$ is well-posed in $\overline{I} \times \Omega$ thanks to Theorem A and, in addition, for every $t \in I$ and for almost every $s \in I$, one has

$$\text{for a.e. } s \in I, \quad X(t, s, \cdot) \in W^{1,p}(\Omega_{(t,s)}; \mathbb{R}^n) \quad \text{and} \quad X(s, t, \cdot) \in W^{1,p}(\Omega_{(s,t)}; \mathbb{R}^n), \tag{15}$$

with

$$\int_I \int_{\Omega_{(t,s)}} \|D_x X(t, s, x)\|^p \, \mathrm{d}x \, \mathrm{d}s \le \ell^{\frac{n}{p-n}} \Lambda_p. \tag{16}$$

With regards to (15), a lower Sobolev regularity for the flow can be proved for all pairs of times $s, t \in I$, see Corollary 6.9. For $b \in C^0(I; C^1(\Omega; \mathbb{R}^n))$, we have a more detailed statement in Theorem 6.1. We may also consider the case when $b : I \times \mathbb{R}^n \to \mathbb{R}^n$ and the support of $b(t, \cdot)$ is contained in a compact set independent of t. See also Remarks 6.3 and 6.2.

Theorem E. *Let $I \subset \mathbb{R}$ be a bounded open interval and let $b \in \mathrm{L}^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n))$ be a bounded vector field. Assume that there exist a bounded open set $\Omega \subset \mathbb{R}^n$ and $p > n$ such that*

$$\mathtt{spt}(b(t, \cdot)) \subset \Omega \quad \textit{for each } t \in I \tag{17}$$

and (6) holds with $\beta = \ell p^2/(p - n)$ and ℓ equal to the length of I.

Then the space-continuous representative $\tilde{b}$, understood as vector field in $I \times \Omega$, is well-posed in $\overline{I} \times \Omega$ thanks to Theorem A, $\Omega_{(t,s)} = \Omega$ and for every $t, s \in \overline{I}$ one has

$$\begin{aligned} &X(t, s, \cdot) \in W^{1,p}(\Omega; \mathbb{R}^n) \quad \textit{with} \\ &\int_\Omega \|D_x X(t, s, x)\|^p \,\mathrm{d}x \le \frac{1}{\ell} \int_I \int_\Omega \exp\left(\frac{\ell p^2}{p - n} \|D_x b(v, y)\|\right) \mathrm{d}y \,\mathrm{d}v. \end{aligned} \tag{18}$$

Notice also that, in the previous theorem, we obviously have that the flow map $X : \overline{I} \times \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$ of b is identically equal to the identity on $\overline{I} \times \overline{I} \times (\mathbb{R}^n \setminus \overline{\Omega})$.

Remark 1.1. (1) Since $\mathrm{dist}(x, \partial\Omega)$ is bounded, (14) can be stated in the equivalent form

$$\int_\Omega \int_I (\mathrm{dist}(x, \partial\Omega))^{n/(n-p)} \exp\left(\frac{\ell p^2}{p - n} \|D_x b(s, x)\|\right) \mathrm{d}s \,\mathrm{d}x < +\infty.$$

We used that specific form for the purpose of the estimate (15).

(2) If $\alpha = n/(p - n)$ is smaller than 1 (i.e., $p > 2n$) and $\partial\Omega$ is regular, then the geometric condition (14), thanks to the Hölder inequality, is implied by the simpler condition

$$\int_\Omega \int_I \exp\left(c\|D_x b(s, x)\|\right) \mathrm{d}s \,\mathrm{d}x < +\infty \tag{19}$$

provided $c > (1/\alpha)' p^2/(p - n)$, where $(1/\alpha)' = (1 - \alpha)^{-1}$ is the dual exponent. ♦

We also point out that the Sobolev exponent p, related to the constant in the exponential integrability condition in (14), may not be sharp. In other words, we can have $\Lambda_p = \infty$ but $X(t, s, \cdot) \in W^{1,p}(\Omega_{(t,s)}; \mathbb{R}^n)$, see Example 8.3.

Several regularity properties of X follow from Theorem D, see Corollaries 6.7, 6.8 and 6.10. In particular we show that, as in Theorem B, $X(t, s, \cdot)_\# \mathcal{L}^n \ll \mathcal{L}^n$, for each $t, s \in \overline{I}$.

Remark 1.2. Suppose that $b \in \mathrm{L}^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n))$ has compact support and (19) holds for some $c > 0$. For every $p > 2n$, we get (18) on subintervals of I containing s and t of length ℓ such that $\frac{\ell p^2}{p-n} = c$. Since $\frac{p^2}{p-n} > 4n$ for $p > 2n$, then $\ell < \frac{c}{4n}$. ♦

1.3. Transport and continuity equations

Finally, we apply Theorem B and Theorem D to the transport equation (2) and the continuity equation (4), respectively. By standard methods, we provide existence, uniqueness, representation and regularity of solutions.

In [12, Theorem 1], uniqueness of weak solutions $u \in L^\infty((0,T); L^\infty(\mathbb{R}^n))$ has been proved, provided that the spatial derivatives of b satisfy a sub-exponential summability and $\bar{u} \in L^\infty(\mathbb{R}^n)$. Our next result provides even in this context the classical representation of this unique solution in terms of the flow map, granted by Theorem B.

Theorem F *(Representation of the solutions for the transport equation). Let b be a vector field as in Theorem B, and let X be the flow associated to the continuous representative b. Then, for each $\bar{u} \in L^\infty(\mathbb{R}^n)$, for each $t \in [0,T]$, the function*

$$v(t,x) := \bar{u}\left(X(t,0,\cdot)^{-1}(x)\right) \text{ for a.e. } x \in \mathbb{R}^n\,, \tag{20}$$

is the unique weak solution in $L^\infty((0,T); L^\infty(\mathbb{R}^n))$ of the Cauchy problem for the transport equation (2), understood in the sense of distributions.

By means of the theory of maps with finite distortion, see for instance [25], we can also show Sobolev regularity of the solution v to the transport equation, see Corollary 7.1.

By applying the well-posedness and representation results proved in [3,14] (see Theorem 7.4 and Remark 7.5), we make more explicit the representation of the weak solutions of (4), under the assumptions of Theorem B and Theorem E. It is useful to deal with the case where $\rho(t,\cdot)$ belongs to the space of signed Borel measures on $\mathbb{R}^n$ with finite total variation, which we will denote by $\mathcal{M}(\mathbb{R}^n)$.

Theorem G *(Representation of the solutions for the continuity equation). Let $I = (0,T)$, let $b : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector field with $b(t,\cdot)$ continuous for a.e. $t \in I$.*

(i) Suppose that b satisfies the assumptions of Theorem B and let X be the flow associated to b. Then, for each signed measure $\bar{\rho} \in \mathcal{M}(\mathbb{R}^n)$,

$$\rho_t = \rho(t,\cdot) = X(t,0,\cdot)_\# \bar{\rho} \quad t \in [0,T] \tag{21}$$

is the unique weak solution for the Cauchy problem (4) in $L^\infty((0,T); \mathcal{M}(\mathbb{R}^n))$.

Moreover, if $\bar{\rho} \in L^1(\mathbb{R}^n)$, ρ_t can be represented as

$$\rho_t := (\bar{\rho} J_{X,t}) \circ X(0,t,\cdot)\,, \tag{22}$$

where $J_{X,t}(y) = \frac{\mathrm{d}X(t,0,\cdot)_\#\mathcal{L}^n}{\mathrm{d}\mathcal{L}^n}(X(t,0,y))$. In particular we have $\rho \in L^\infty((0,T); L^1(\mathbb{R}^n))$.

(ii) If b satisfies the stronger hypotheses of Theorem E and $\bar{\rho} \in L^1(\mathbb{R}^n)$, then the unique weak solution for the Cauchy problem (4) can also be represented as

$$\rho_t = \frac{\bar{\rho}}{J_{X(t,0,\cdot)}} \circ X(0,t,\cdot)\,, \tag{23}$$

where $J_{X(t,0,\cdot)}$ *denotes the Jacobian of homeomorphism* $X(t,0,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ *(see* (25)*).*

Notice that existence and uniqueness of distributional solutions of continuity equation (4) has been obtained in [15] by weakening assumption (3). More precisely, the authors showed that if $\bar{\rho}\in L^\infty(\mathbb{R})$ and

$$\operatorname{div}_x b\in(BMO\cap L^1)(\mathbb{R}^n), \tag{24}$$

then there exists a unique distributional solution $\rho\in L^\infty((0,T)\times\mathbb{R}^n)$ of the Cauchy problem (4) However, we point out that condition (6) does not imply (24), see Example 8.3.

1.4. *Structure of the paper*

We start in Sections 2 and 3 by listing known results about homeomorphisms of finite distortion and classical flows of vector fields. In Section 4 we prove Theorem A. Theorems B and C are proven in Section 5. The proof of Theorem D is given in Section 6. Theorems F and G are shown in Section 7. Finally, Section 8 is devoted to the illustration of a few examples.

Acknowledgments

We thank A. Clop for useful suggestions and discussions on this topic. We are also grateful to R. Serapioni for several discussions about Section 4. We want to thank also the anonymous referee for their useful comments and corrections.

L.A. and F.S.C. have been supported by the PRIN 2017 project “Gradient flows, Optimal Transport and Metric Measure Structures.” S.N.G. has been supported by University of Padova STARS Project “Sub-Riemannian Geometry and Geometric Measure Theory Issues: Old and New” (SUGGESTION) and by the Academy of Finland (grant 328846 “Singular integrals, harmonic functions, and boundary regularity in Heisenberg groups”, grant 322898 “Sub-Riemannian Geometry via Metric-geometry and Lie-group Theory”, grant 314172 “Quantitative rectifiability in Euclidean and non-Euclidean spaces”). S.N.G. and F.S.C. have been supported by the INdAM – GNAMPA Project 2019 “Rectifiability in Carnot groups”.

2. Preliminaries on homeomorphisms

In this section we present some results about homeomorphisms that we will need later. Most of the material comes from the book [25].

2.1. *Weak derivatives of homeomorphisms*

If $\Omega\subset\mathbb{R}^n$ is open and $\Psi:\Omega\to\mathbb{R}^n$, we denote by $D\Psi$ the weak differential of Ψ. When a point-wise analysis is needed, we assume that $D\Psi(x)$ is equal to the classical differential of Ψ at every $x\in\Omega$ where Ψ is differentiable.[1] The *Jacobian* of Ψ is

[1] Notice that the distributional derivative of a Sobolev function Ψ agrees with the classical derivative at a.e. x where Ψ is differentiable. If the reader needs an argument, let us point out that the approximate differential of Ψ agrees with the distributional derivative almost everywhere by [18, Theorem 6.1.4, page 233], and that the classical derivative is equal to the approximate differential wherever it exists.

$$J_\Psi(x) = \det(D\Psi(x))\,. \tag{25}$$

A map $\Psi : \Omega \to \mathbb{R}^n$ satisfies the *Lusin (N) condition* if for all $E \subset \Omega$ the vanishing of $\mathcal{L}^n(E)$ implies $\mathcal{L}^n(\Psi(E)) = 0$. An *embedding* is a homeomorphism onto its image.

Lemma 2.1 *([25, Lemma A.29]). Let $\Omega \subset \mathbb{R}^n$ be open and let $\Psi : \Omega \to \mathbb{R}^n$ be an embedding that is differentiable at x. The map Ψ^{-1} is differentiable at $\Psi(x)$ if and only if $\det(D\Psi(x)) \neq 0$. In this case, we have*

$$D(\Psi^{-1})(\Psi(x)) = (D\Psi(x))^{-1}.$$

Lemma 2.2 *([25, Theorem A.35]: Area Formula). If $\Omega \subset \mathbb{R}^n$ is open, $\Psi \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $\eta : \mathbb{R}^n \to [0, +\infty]$ is a Borel function, then*

$$\int_\Omega \eta(\Psi(x))|J_\Psi(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}^n} \eta(y)N(\Psi, \Omega, y)\,\mathrm{d}y, \tag{26}$$

where $N(\Psi, \Omega, y) = \#(\Psi^{-1}(y) \cap \Omega)$ is the cardinality of the set $\Psi^{-1}(y) \cap \Omega$. If Ψ satisfies also the Lusin (N) condition then equality holds in (26).

Remark 2.3. Observe that the Lusin (N) condition for a homeomorphism $\Psi : \Omega \to \Omega'$ is equivalent to assume that the push-forward measure $(\Psi^{-1})_\#(\mathcal{L}^n \llcorner \Omega')$ is absolutely continuous with respect to $\mathcal{L}^n$. An abstract area-type formula with respect to a general measure μ is also introduced in Lemma 7.6. ♦

Lemma 2.4 *([25, Theorem 4.2 and Theorem 6.2.1]). If $\Omega \subset \mathbb{R}^n$ is open, $\Psi \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $p > n$, then the continuous representative $\tilde{\Psi}$ is differentiable at a.e. $x \in \Omega$ and $\tilde{\Psi}$ satisfies the Lusin (N) condition.*

2.2. Mappings of finite distortion

We denote by $\|M\|$ the operator norm of a linear map $\mathbb{R}^n \to \mathbb{R}^n$ or $m \times n$-matrix, that is,

$$\|M\| := \sup\left\{|Mv| : v \in \mathbb{R}^n,\ |v| = 1\right\},$$

where $|\cdot|$ is the Euclidean norm. With this choice of the norm, one can easily show the following inequalities (the first one is called *Hadamard's inequality*):

$$|\det M| \leq \Pi_{i=1}^n |Me_i| \leq \|M\|^n. \tag{27}$$

Definition 2.5. A *map of finite distortion* on an open set $\Omega \subset \mathbb{R}^n$ is a function $\Psi \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ such that there exists $K : \Omega \to [1, +\infty)$ with

$$\|D\Psi(x)\|^n \leq K(x)J_\Psi(x) \qquad \text{for a.e. } x \in \Omega. \tag{28}$$

For $q \geq 1$, the *q-distortion function* of a map of finite distortion Ψ is

$$K_q^\Psi(x) := \begin{cases} \dfrac{\|D\Psi(x)\|^q}{J_\Psi(x)} & \text{if } J_\Psi(x) \neq 0, \\ 1 & \text{otherwise .} \end{cases}$$

Our main reference on maps of finite distortion is [25]. Notice that K_n^Ψ is the optimal distortion function for the inequality (28) to hold.

Lemma 2.6. *If $\Omega \subset \mathbb{R}^n$ is open and $\Psi \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ is an embedding with $J_\Psi > 0$ almost everywhere in Ω, then Ψ has finite distortion.*

Proof. The function K_n^Ψ is finite almost everywhere and (28) holds whenever $J_\Psi(x) > 0$. Since $J_\Psi > 0$ almost everywhere in Ω, then Ψ has finite distortion. $\square$

Lemma 2.7. *Let $\Psi : \Omega_1 \to \Omega_2$ be a homeomorphism between open subsets of $\mathbb{R}^n$, $q > 0$ and $p > n$. Suppose that*

$$\Psi \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n), \quad \Psi^{-1} \in W^{1,p}_{\mathrm{loc}}(\Psi(\Omega); \mathbb{R}^n) \quad \text{and } J_\Psi(x) > 0 \text{ for a.e. } x \in \Omega.$$

Then Ψ is a homeomorphism of finite distortion and

$$K_q^\Psi \in \mathrm{L}^r_{\mathrm{loc}}(\Omega)$$

with $r = \left(\frac{q}{p} + \frac{n}{p-n}\right)^{-1}$, which may be smaller than 1.

Proof. The map Ψ has finite distortion by Lemma 2.6. Let $r > 0$ and $U \Subset \Omega$. By Hölder inequality, we have

$$\int_U |K_q^\Psi|^r \,\mathrm{d}x \le \left(\int_U \|D\Psi\|^{rq\alpha} \,\mathrm{d}x\right)^{1/\alpha} \left(\int_U |J\Psi|^{-r\beta} \,\mathrm{d}x\right)^{1/\beta} \tag{29}$$

for $\alpha, \beta \ge 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. If $rq\alpha = p$, then the first term of the right-hand side is finite.

By Lemma 2.4, both Ψ and Ψ^{-1} are differentiable almost everywhere. Therefore, using Lemma 2.1, it follows that, for almost every $x \in U$,

$$|J\Psi(x)|^{-1} = |J\Psi^{-1}(\Psi(x))|.$$

Using the area inequality (26), we have for all $\gamma > 1$

$$\begin{aligned} \int_U |J\Psi(x)|^{-\gamma} \,\mathrm{d}x &= \int_U |J\Psi^{-1}(\Psi(x))|^{\gamma+1} |J\Psi(x)| \,\mathrm{d}x \\ &\le \int_{\Psi(U)} |J\Psi^{-1}(y)|^{\gamma+1} \,dy \le \int_{\Psi(U)} \|D\Psi^{-1}(y)\|^{n(\gamma+1)} \,\mathrm{d}x, \end{aligned}$$

where we used (27) in the last step. Now, if $\gamma = r\beta$, then

$$\int_U |J\Psi|^{-r\beta}\,\mathrm{d}x \le \int_{\Psi(U)} \|D\Psi^{-1}(y)\|^{n(r\beta+1)}\,\mathrm{d}x,$$

which is finite when $n(r\beta+1) = p$.

Finally, solving the three equations $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $rq\alpha = p$ and $n(r\beta+1) = p$ in α, β and r, we get

$$r = \left(\frac{q}{p} + \frac{n}{p-n}\right)^{-1}$$

and $\alpha = \frac{p}{rq} = 1 + \frac{pn}{q(p-n)} > 1$, by the assumption $p > n$. Therefore, we get from (29) that $\int_U |K_\Psi|^r\,\mathrm{d}x < \infty$. $\square$

Proposition 2.8 *([25, Theorem 5.13 and equation (5.18)]: Regularity of composition). Let $\Psi : \Omega_1 \to \Omega_2$ be a homeomorphism between open subsets of $\mathbb{R}^n$ and let $1 \le p < q < \infty$. Define the composition operator $T_\Psi u = u \circ \Psi$ for $u : \Omega_2 \to \mathbb{R}$. Suppose that Ψ has finite distortion and*

$$K_q^\Psi \in \mathrm{L}_{\mathrm{loc}}^{\frac{p}{q-p}}(\Omega_1). \tag{30}$$

Then T_Ψ is continuous from $W_{\mathrm{loc}}^{1,q}(\Omega_2)$ to $W_{\mathrm{loc}}^{1,p}(\Omega_1)$ for every $1 \le q \le \infty$, where the operator norm on T_Ψ is controlled by $\|K_q^\Psi\|_{\mathrm{L}^{\frac{p}{q-p}}}^{1/q}$ on balls.

3. Preliminaries on flows of vector fields

In this section we present a part of the classical theory of flows of vector fields that we will need later.

3.1. Well-posedness of vector fields

We denote by $B(x,r)$ the Euclidean ball of radius r and center x. Recall that a vector field $b(t,x)$ is said to be *autonomous* if it does not depend on t, *non-autonomous* if it may depend on t.

Definition 3.1 *(Well posedness).* Let $\Omega \subset \mathbb{R}^n$ open, $I \subset \mathbb{R}$ an open interval and $b : I \times \Omega \to \mathbb{R}^n$ a function. We say that b is *well-posed in* $I \times \Omega$ if for every $(s,x) \in I \times \Omega$ there exist $\epsilon > 0$ and, in the interval $(s-\epsilon, s+\epsilon) \subset I$, a unique absolutely continuous solution $\gamma : (s-\epsilon, s+\epsilon) \to \Omega$ of (1), that is, $\dot\gamma(t) = b(t,\gamma(t))$ for a.e. $t \in (s-\epsilon, s+\epsilon)$ and $\gamma(s) = x$. Analogously, we say that b is *well-posed in* $\overline{I} \times \Omega$ if for every $(s,x) \in \overline{I} \times \Omega$ there exist $\epsilon > 0$ and, in the interval $I \cap (s-\epsilon, s+\epsilon)$, a unique absolutely continuous solution $\gamma : I \cap (s-\epsilon, s+\epsilon) \to \Omega$ of (1).

The following classical result provides a sufficient condition for well-posedness, in local and global form with respect to the time variable. The proof is rather classical and well known, see for instance [19,23].

Proposition 3.2 *(Well-posedness with Osgood condition). Let $\omega : [0, +\infty) \to [0, +\infty)$ be non-decreasing with $\omega(0) = 0$, $\omega(t) > 0$ for $t > 0$ and*

$$\int_0^1 \frac{1}{\omega(t)}\,\mathrm{d}t = +\infty. \tag{31}$$

Let $I \subset \mathbb{R}$ be an interval, and $\Omega \subset \mathbb{R}^n$ an open set. Let $b : I \times \Omega \to \mathbb{R}^n$ be a function such that:

(a) there exists a positive function $\phi \in L^1_{\mathrm{loc}}(I)$ such that

$$|b(t,x) - b(t,y)| \le \phi(t)\omega(|x-y|) \tag{32}$$

for all $t \in I$ and $x,\ y \in \Omega$;

(b) for every $x \in \Omega$, the function $t \mapsto b(t,x)$ is measurable;

(c) there exists $m \in L^1_{\mathrm{loc}}(I)$ such that $|b(t,x)| \le m(t)$ for all $(t,x) \in I \times \Omega$.

Then the following hold:

(i) The vector field b is well posed in $I \times \Omega$. For $(s,x) \in I \times \Omega$, denote by $\gamma_{(s,x)}$ the maximal solution of (1) with $\gamma_{(s,x)}(s) = x$ and by $I_{(s,x)}$ its maximal domain of existence. Define also $\alpha(s,t) = \inf I_{(s,x)}$ and $\beta(s,t) = \sup I_{(s,x)}$.

(ii) If $\beta(s,x) \neq \sup I$, then $\lim_{t\to\beta(s,x)} \gamma_{(s,x)}$ exists and belongs to $\partial\Omega$. Similarly for $\alpha(s,x) \neq \inf I$.

(iii) If $(s_j, x_j) \in I \times \Omega$ is a sequence converging to $(s_\infty, x_\infty) \in I \times \Omega$, with $\gamma_j = \gamma_{(s_j,x_j)} : I_{(s_j,x_j)} \to \Omega$ the corresponding maximal solutions, $j \in \mathbb{N} \cup \{\infty\}$, and if $s_\infty \in J \Subset I_{(s_\infty,x_\infty)}$, then $J \subset I_{(s_j,x_j)}$ for j large enough and γ_j converge to γ_∞ uniformly on J.

In particular, α is upper semicontinuous and β is lower semicontinuous.

(iv) If $m \in L^1(I)$, then $\gamma_{(s,x)}$ is well defined also for $(s,x) \in \partial I \times \Omega$ and the limits

$$\lim_{t\to\beta(s,x)} \gamma_{(s,x)}(t) \quad \text{and} \quad \lim_{t\to\alpha(s,x)} \gamma_{(s,x)}(t)$$

exist in $\bar{\Omega}$. In particular, b is well posed in $\overline{I} \times \Omega$.

3.2. The flow of a well-posed vector field

Fix an open set $\Omega \subset \mathbb{R}^n$, an open interval $I \subset \mathbb{R}$ and a well-posed vector field $b : I \times \Omega \to \mathbb{R}^n$. For every $(s,x) \in I \times \Omega$, let $I_{(s,x)} = (\alpha(s,x), \beta(s,x)) \subset I$ be the maximal open interval of existence of a solution $\gamma_{(s,x)} : I_{(s,x)} \to \Omega$ to the Cauchy problem (1) with initial datum $\gamma_{(s,x)}(s) = x$. Define

$$\mathscr{D}_b := \left\{(t,s,x) : (s,x) \in I \times \Omega,\ t \in I_{(s,x)}\right\} \subset I \times I \times \Omega,$$

and the *flow* $X : \mathscr{D}_b \to \Omega$ of b as $X(t,s,x) = \gamma_{(s,x)}(t)$. For every $(t,s) \in I \times I$, we define $\Omega_{(t,s)} = \{x \in \Omega : (t,s,x) \in \mathscr{D}_b\} = \{x \in \Omega : t \in I_{(s,x)}\}$ (which is possibly empty) and $X_{(t,s)} : \Omega_{(t,s)} \to \Omega$ as $X_{(t,s)}(x) = X(t,s,x)$.

An analogous notation can be used for the case of vector fields well posed in $\overline{I} \times \Omega$, just allowing s and t to belong also to ∂I.

We recall several well-known facts about flows of vector fields, referring to fix the ideas to the weaker case of well posedness in $I \times \Omega$. See for instance [23] for reference.

Remark 3.3. If $b : I \times \Omega \to \mathbb{R}^n$ satisfies the hypothesis of Proposition 3.2, then the sets $\mathscr{D}_b$ and $\Omega_{(t,s)}$ are open and X is continuous, thanks to Proposition 3.2(iii).

Uniqueness gives the semigroup-type property

$$X(t_3, t_2, X(t_2, t_1, x)) = X(t_3, t_1, x) , \tag{33}$$

whenever the expressions make sense. Moreover, the map $X_{(t,s)}$ is a homeomorphism $\Omega_{(t,s)} \to \Omega_{(s,t)}$ with inverse $X_{(s,t)}$. Finally, if $k \geq 1$ and $b \in C^k(I \times \Omega; \mathbb{R}^n)$, then $X \in C^{k+1}(\mathscr{D}_b; \Omega)$. ♦

Lemma 3.4 *([23, Theorem 3.1, Chap. V]). Any $b \in C^0(I; C^1(\Omega; \mathbb{R}^n))$ is well posed in $I \times \Omega$ and the homeomorphisms $X_{(t,s)} : \Omega_{(t,s)} \to \Omega_{(s,t)}$ are of class C^1.*

Fix $(s, x) \in I \times \Omega$ and set the following functions defined on $I_{(s,x)}$

$$\begin{aligned}
y(t) &:= D_x X(t, s, x),\\
B(t) &:= (D_x b)(t, X(t, s, x)),\\
J(t) &:= \det(D_x X(t, s, x)),\\
\beta(t) &:= \operatorname{div}_x(b)(t, X(t, s, x)) = \operatorname{trace}(B(t)).
\end{aligned}$$

Then y and J are the unique C^1 solutions of the following initial value problems:

$$\begin{cases} \dot{y}(t) &= B(t)y(t), \\ y(s) &= \mathrm{Id}, \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{J}(t) &= \beta(t)J(t), \\ J(s) &= 1. \end{cases}$$

When considering smooth approximations of vector fields, we need to control the convergence of the corresponding flows, as in Lemma 3.5 below. The following results are fairly well-known, but in absence of a reference precisely adapted to our purposes we provide sketch of proofs for the reader's convenience. An analogous statement holds for vector fields well posed in $\overline{I}$, considering the additional points $(s_\infty, x_\infty) \in \partial I \times \Omega$ and the stronger convergence in $L^1(I; C(\Omega'; \mathbb{R}^n))$ for any $\Omega' \Subset \Omega$.

Lemma 3.5. *Let $\{b_h\}_{h \in \mathbb{N} \cup \{\infty\}}$ be well posed vector fields in I and denote by $I^h_{(s,x)}$ their maximal existence times and by X^h their flows. If $b_h \to b_\infty$ in $L^1_{\mathrm{loc}}(I; C(\Omega'; \mathbb{R}^n))$ for any $\Omega' \Subset \Omega$ and $(s_h, x_h) \to (s_\infty, x_\infty) \in I \times \Omega$ as $h \to \infty$, then for any compact interval $J \subset I^\infty_{(s_\infty, x_\infty)}$ one has*

$$J \subset I^h_{(s_h, x_h)} \qquad \text{for } h \text{ large enough}$$

and $X^h(\cdot, s_h, x_h)$ converge uniformly to $X^\infty(\cdot, s_\infty, x_\infty)$ on J.

Proof. Let us consider the compact curve $\Gamma = \{(v, X^{\infty}(v, s_\infty, x_\infty)) : \ v \in J\}$ and let U be the open set $J' \times \Omega'$, with $J \Subset J' \Subset I$ and $\Omega' \Subset \Omega$ chosen in such a way that $\Gamma \subset U$. By assumption, $\int_{J'} \sup_{\Omega'} |b_h(t, \cdot) - b(t, \cdot)| \, \mathrm{d}t$ is infinitesimal as $h \to \infty$. Let us consider the maximal solutions $\gamma_h(v)$ for the ODE relative to b_h, starting from x_h at s_h and with graph remaining in U (in particular, restrictions of $X^h(\cdot, s_h, x_h)$), and let $J_h \subset I^h_{(s_h,x_h)}$ their maximal existence intervals. Since the curves γ_h are equicontinuous, it is clear that limit points of the graph of these curves exist, and that any limit point is the graph of a solution γ_∞ to the ODE relative to b_∞, with $\gamma_\infty(s_\infty) = x_\infty$, defined on an interval J_∞ and with the property that, as v tends to one, if any, of the extreme points of J_∞ different from ∂I, $(v, \gamma_\infty(v))$ tends to ∂U. Since $U \Subset I \times \Omega$, the well-posedness of b_∞ yields that the curve γ_∞ is not only a restriction of the maximal curve $X^{\infty}(v, s_\infty, x_\infty)$, $v \in I^{\infty}_{(s_\infty,x_\infty)}$, but also contains the curve $X^{\infty}(v, s_\infty, x_\infty)$, $v \in J$, since this restricted curve does not touch the boundary of U. $\square$

The previous lemma grants, in particular, the lower semicontinuity of $(s, x) \mapsto \mathrm{length}(I_{(s,x)})$ in $I \times \Omega$. Moreover, a simple contradiction argument gives

$$
\begin{aligned}
&\text{for any compact interval } J \subset I^{\infty}_{(s_\infty,x_\infty)} \text{ and any open set } A \Subset \Omega, \text{ one has}\\
&\exists \bar{h} \text{ such that } J \subset I^h_{(s,x)} \qquad \text{for each } h > \bar{h} \text{ and } (s,x) \in J \times A\,;
\end{aligned} \tag{34}
$$

$$
A \Subset \Omega^{\infty}_{(t,s)} \quad \text{implies} \quad A \Subset \Omega^h_{(t,s)} \quad \text{for } h \text{ large enough} \tag{35}
$$

and the uniform convergence of $X_h(t, s, \cdot)$ to $X_\infty(t, s, \cdot)$ on A.

4. Well-posedness with Orlicz condition

In this section we are going to prove Theorem A. We fix some dimensional constants for $n \geq 1$. Let $\omega_n = |B(0, 1)|$ be the volume of the unit Euclidean ball in $\mathbb{R}^n$ and $\sigma_{n-1} = n\omega_n$ the perimeter of $B(0, 1)$; let $\tau_n = |B(0, 1) \cap B(q, 1)|$ for any $q \in \partial B(0, 1)$; finally, C_n is the constant from Lemma 4.1 and $\kappa_n = 2\tau_n^{-1}\omega_n C_n \sigma_{n-1}$ will appear in Lemma 4.3.

Lemma 4.1. *Let $b \in W^{1,1}(B(x, r); \mathbb{R}^n)$ and assume that x is a Lebesgue point of b. Then, for some dimensional constant C_n, one has*

$$
\fint_{B(x,r)} |b(x) - b(y)| \, \mathrm{d}y \leq C_n \int_{B(x,r)} \frac{\|Db(y)\|}{|x - y|^{n-1}} \, \mathrm{d}y. \tag{36}
$$

Proof. We assume for simplicity $n \geq 2$. From the same argument of [18, Lemma 1, Sec. 4.5.2], based on a radial integration, for all $x \in \mathbb{R}^n$, $r > 0$, $\epsilon \in (0, 1)$ and $f \in C^1(B(x, r))$ we have

$$
\begin{aligned}
\fint_{B(x,r)\setminus B(x,\epsilon r)} |f(x) - f(y)| \, \mathrm{d}y &\leq \int_{\epsilon r}^{r} \fint_{B(x,s)} |y - x||Df(y)| \, \mathrm{d}y \, \mathrm{d}s\\
&\leq \int_{\epsilon r}^{r} s \fint_{B(x,s)} |Df(y)| \, \mathrm{d}y \, \mathrm{d}s\,.
\end{aligned} \tag{37}
$$

Assume now that $f \in L^1_{\mathrm{loc}}(\Omega)$, x is a Lebesgue point of f, $B(x,r) \Subset \Omega$ and $f \in W^{1,1}(B(x,r))$. Let $\delta := \mathrm{dist}(B(x,r), \mathbb{R}^n \setminus \Omega)$, $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega) > \delta/2\}$ and $(\rho_h)_h$ be a sequence of mollifiers in $\mathbb{R}^n$. Then it is well-defined the sequence of regularized functions $f_h := f * \rho_h : \Omega_\delta \to \mathbb{R}$ if $h > 1/\delta$ and it satisfies the following properties: $f_h \in C^\infty(\Omega_\delta)$,

$$f_h(z) \to f(z), \text{ for each } z \in \Omega_\delta \text{ Lebegue point of } f, \text{ as } h \to \infty, \tag{38}$$

$$f_h \to f \text{ in } W^{1,1}(B(x,r))\,. \tag{39}$$

Applying (37) with $f \equiv f_h$ and let $h \to \infty$, by (38) and (39), we get the same inequality when $f \in W^{1,1}(B(x,r))$ and x is a Lebesgue point of f. Now we can let $\epsilon \to 0$ and use Fubini's theorem in the right hand side to get

$$\fint_{B(x,r)} |f(x) - f(y)|\,\mathrm{d}y \le \frac{1}{n} \int_{B(x,r)} \frac{|Df(y)|}{|x-y|^{n-1}}\,\mathrm{d}y\,. \tag{40}$$

Finally, notice that there is a constant c_n such that, if M is a $n \times n$ matrix with rows M_j, then $\sum_j |M_j| \le c_n \|M\|$. Therefore,

$$\begin{aligned}\fint_{B(x,r)} |b(x) - b(y)|\,\mathrm{d}y &\le \sum_{j=1}^{n} \fint_{B(x,r)} |b_j(x) - b_j(y)|\,\mathrm{d}y \\ &\le \frac{1}{n}\sum_{j=1}^{n} \int_{B(x,r)} \frac{|Db_j(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \le \frac{c_n}{n} \int_{B(x,r)} \frac{\|Db(y)\|}{|x-y|^{n-1}}\,\mathrm{d}y,\end{aligned}$$

which proves (36) with $C_n = c_n/n$. □

Recall Jensen's inequality

$$\Phi\left(\int_X \chi(x)\,\mathrm{d}\mu(x)\right) \le \int_X \Phi(\chi(x))\,\mathrm{d}\mu(x), \tag{41}$$

where $\Phi : [0,+\infty) \to [0,+\infty)$ is a convex function, μ is a probability measure on X and $\chi : X \to \mathbb{R}$ is a μ-measurable non-negative function. The following lemma is a direct application of Jensen's inequality.

Lemma 4.2. *Let $\Phi : [0,+\infty) \to [0,+\infty)$ be a convex function. If $n \ge 1$, $x \in \mathbb{R}^n$, $r > 0$, $\chi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ is non-negative, then*

$$\Phi\left(\frac{1}{r}\int_{B(x,r)} \frac{\chi(z)}{|z-x|^{n-1}}\,\mathrm{d}z\right) \le \frac{1}{r\sigma_{n-1}} \int_{B(x,r)} \frac{\Phi(\sigma_{n-1}\chi(z))}{|x-z|^{n-1}}\,\mathrm{d}z. \tag{42}$$

Proof. Using polar coordinates, one easily checks that

$$\int_{B(0,r)} \frac{1}{|z|^{n-1}}\,\mathrm{d}z = r\sigma_{n-1}.$$

Therefore, the measure $\mathrm{d}\mu := \frac{\chi_{B(x,r)}(z)}{r\sigma_{n-1}} \frac{\mathrm{d}z}{|x-z|^{n-1}}$ is a probability measure on $\mathbb{R}^n$ and Jensen's inequality (41) applies. □

Lemma 4.3. *Let $\Phi : [0,+\infty) \to [0,+\infty)$ be a convex non-decreasing function. Let $o \in \mathbb{R}^n$, $R > 0$ and $b \in W^{1,1}(B(o,2R);\mathbb{R}^n)$. Then, for all $\alpha \in (1, \frac{n}{n-1})$, and all x, $y \in B(o,R)$ distinct Lebesgue points of b, one has*

$$\Phi\left(\frac{|b(x)-b(y)|}{|x-y|}\right) \le \frac{1}{\sigma_{n-1}} \left(\int_{B(o,2R)} \frac{1}{|z|^{\alpha(n-1)}}\,\mathrm{d}z \right)^{1/\alpha} \times \tag{43}$$
$$\times \frac{1}{|x-y|} \left(\int_{B(o,2R)} \Phi\left(\kappa_n \|Db(z)\|\right)^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}z \right)^{\frac{\alpha-1}{\alpha}}.$$

In the case $n = 1$, the estimate (43) is understood to hold for $\alpha \in (1,+\infty)$ and we also have

$$\Phi\left(\frac{|b(x)-b(y)|}{|x-y|}\right) \le \frac{1}{2|x-y|} \int_{B(o,2R)} \Phi\left(\kappa_n \|Db(z)\|\right)\mathrm{d}z. \tag{44}$$

Proof. Set $r := |x-y| > 0$ and $W = B(x,r) \cap B(y,r)$. Notice that $|W| = \tau_n r^n$. Then

$$\begin{aligned} |b(x)-b(y)| &\le \fint_W |b(x)-b(z)|\,\mathrm{d}z + \fint_W |b(y)-b(z)|\,\mathrm{d}z \\ &\le \frac{\omega_n}{\tau_n} \left(\fint_{B(x,r)} |b(x)-b(z)|\,\mathrm{d}z + \fint_{B(y,r)} |b(y)-b(z)|\,\mathrm{d}z \right) \\ &\le \frac{2\omega_n}{\tau_n} \sup_{w\in B(o,R)} \fint_{B(w,r)} |b(w)-b(z)|\,\mathrm{d}z. \end{aligned}$$

Since Φ is non-decreasing, we have:

$$\Phi\left(\frac{|b(x)-b(y)|}{|x-y|}\right) \le \sup_{w\in B(o,R)} \Phi\left(\frac{2\omega_n}{r\tau_n} \fint_{B(w,r)} |b(w)-b(z)|\,\mathrm{d}z \right)$$

$$\overset{\text{by (36)}}{\leq} \sup_{w\in B(o,R)} \Phi\left(\frac{2\omega_n C_n}{r\tau_n} \int_{B(w,r)} \frac{\|Db(z)\|}{|w-z|^{n-1}}\,\mathrm{d}z\right)$$

$$\overset{\text{by (42)}}{\leq} \frac{1}{r\sigma_{n-1}} \sup_{w\in B(w,r)} \int_{B(w,r)} \Phi\left(\frac{2\omega_n C_n \sigma_{n-1}}{\tau_n}\|Db(z)\|\right)\frac{1}{|w-z|^{n-1}}\,\mathrm{d}z\,.$$

If $n=1$, we have already obtained (44). However, if $\alpha>1$, we apply the Hölder inequality to obtain

$$\int_{B(w,r)} \Phi\left(\kappa_n\|Db(z)\|\right)\frac{1}{|w-z|^{n-1}}\,\mathrm{d}z$$

$$\leq \left(\int_{B(w,r)} \Phi\left(\kappa_n\|Db(z)\|\right)^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}z\right)^{\frac{\alpha-1}{\alpha}} \times \left(\int_{B(w,r)} \frac{1}{|w-z|^{\alpha(n-1)}}\,\mathrm{d}z\right)^{1/\alpha}$$

$$\leq \left(\int_{B(o,2R)} \Phi\left(\kappa_n\|Db(z)\|\right)^{\frac{\alpha}{\alpha-1}}\,\mathrm{d}z\right)^{\frac{\alpha-1}{\alpha}} \times \left(\int_{B(0,2R)} \frac{1}{|z|^{\alpha(n-1)}}\,\mathrm{d}z\right)^{1/\alpha}.$$

Notice that $\int_{B(0,2R)} \frac{1}{|z|^{\alpha(n-1)}}\,\mathrm{d}z<\infty$ if and only if $\alpha(n-1)<n$, i.e., $\alpha<\frac{n}{n-1}$ when $n>1$ or $\alpha<\infty$ when $n=1$. Applying this estimate to the former inequality, we obtain (43). $\square$

Proof of Theorem A. Notice that assumption (A.I) implies that Θ is also convex, and condition (7) is equivalent to

$$\Theta^{-1}(s_1 s_2) \leq C_\Theta\, \Theta^{-1}(s_1)\Theta^{-1}(s_2) \qquad \forall s_1,\, s_2 \geq \Theta(C_\Theta). \tag{45}$$

Let $n>1$, fix $\alpha\in(1,\frac{n}{n-1})$ as in the assumption (A.I), so that $\Phi(s):=\Theta(s/\kappa_n)^{\frac{\alpha-1}{\alpha}}$ is convex, where κ_n is the constant defined at the beginning of the section.

Fix $o\in\Omega$ and the corresponding $R>0$ and $c>0$ as in (8). We claim that the space continuous representative $\tilde{b}$ is well posed in $I\times B(o,R)$. Since $b: I\times B(o,R)\to\mathbb{R}^n$ is well posed and satisfies (8) if and only if cb does, in the proof of the claim we can assume with no loss of generality $c=1$. Applying Lemma 4.3 with the above $\alpha\in(1,\frac{n}{n-1})$ and Φ, we have, for almost every t and all $x,\, y\in B(o,R)$ distinct Lebegue points of $b(t,\cdot)$, one has

$$\Theta\left(\frac{|b(t,x)-b(t,y)|}{\kappa_n|x-y|}\right)^{\frac{\alpha-1}{\alpha}} \leq C(n,R,\alpha)\frac{1}{|x-y|}\left(\int_{B(o,2R)} \Theta\left(\|Db(t,z)\|\right)\mathrm{d}z\right)^{\frac{\alpha-1}{\alpha}}.$$

We write $s_1\vee s_2$ for $\max\{s_1,s_2\}$ and $s_1\wedge s_2$ for $\min\{s_1,s_2\}$. Then, by applying (45) twice we obtain that

$$\frac{|b(t,x)-b(t,y)|}{\kappa_n|x-y|} \le C_\Theta^2 \Theta^{-1}(\Theta(C_\Theta) \vee C(n,R,\alpha)^{\frac{\alpha}{\alpha-1}}) \times$$
$$\times\, \Theta^{-1}\left(\Theta(C_\Theta) \vee \int\limits_{B(o,2R)} \Theta\left(\|Db(t,z)\|\right) \mathrm{d}z\right) \times$$
$$\times\, \Theta^{-1}\left(\Theta(C_\Theta) \vee \left(\frac{1}{|x-y|}\right)^{\frac{\alpha}{\alpha-1}}\right).$$

If we set

$$\omega(\delta) := \delta\Theta^{-1}\left(\Theta(C_\Theta) \vee \left(\frac{1}{\delta}\right)^{\frac{\alpha}{\alpha-1}}\right) \tag{46}$$

and

$$\varphi(t) := \kappa_n C_\Theta^2 \Theta^{-1}(\Theta(C_\Theta) \vee C(n,R,\alpha)^{\frac{\alpha}{\alpha-1}})\, \Theta^{-1}\left(\Theta(C_\Theta) \vee \int\limits_{B(o,2R)} \Theta\left(\|Db(t,z)\|\right) \mathrm{d}z\right), \tag{47}$$

we obtain that $\omega \in C^0((0,\infty))$ and b satisfies

$$|b(t,x)-b(t,y)| \le \varphi(t)\,\omega(|x-y|) \tag{48}$$

for each $x,\ y \in B(o,R)$ Lebesgue points of $b(t,\cdot)$. From (48), it follows that $b(t,\cdot)$ is uniformly continuous on the set of Lebesgue points of $b(t,\cdot)$ contained in $B(o,R)$. Since this set is dense in $B(o,R)$, it follows that there exists a unique continuous extension $\tilde{b}(t,\cdot) : B(o,R) \to \mathbb{R}^n$ still satisfying (48) on the whole $B(o,R)$. Therefore we can conclude that $\tilde{b}(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, condition holds (48) on $B(o,R)$ and

$$\tilde{b} = b \text{ a.e. on } I \times B(o,R)\,. \tag{49}$$

Moreover, it is clear that $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(I;\mathbb{R})$, because Θ^{-1} is concave and the function in (8) belongs to $\mathrm{L}^1_{\mathrm{loc}}(I;\mathbb{R})$ by assumption.

We claim that condition (A.III) implies $\int_0^1 \frac{1}{\omega(\delta)}\,\mathrm{d}\delta = \infty$ and $\lim_{\delta\to 0}\omega(\delta) = 0$. Indeed, on the one hand, using the monotonicity of Θ^{-1} and the change of variables $s = \Theta^{-1}((1/\delta)^{\frac{\alpha}{\alpha-1}})$, we obtain, if $\bar{\delta} := \Theta(C_\Theta)^{\frac{1-\alpha}{\alpha}} \wedge 1$ and $\bar{s} = \Theta^{-1}((1/\bar{\delta})^{\frac{\alpha}{\alpha-1}})$,

$$\int\limits_0^1 \frac{1}{\omega(\delta)}\,\mathrm{d}\delta \ge \int\limits_0^{\bar{\delta}} \frac{1}{\delta\Theta^{-1}((1/\delta)^{\frac{\alpha}{\alpha-1}})}\,\mathrm{d}\delta = \frac{\alpha-1}{\alpha}\int\limits_{\bar{s}}^{\infty} \frac{\Theta'(s)}{s\Theta(s)}\,\mathrm{d}s = +\infty.$$

On the other hand, to prove $\lim_{\delta\to 0}\omega(\delta) = \lim_{\delta\to 0}\delta\Theta^{-1}((1/\delta)^{\frac{\alpha}{\alpha-1}}) = 0$, we only need to show that the function $\delta \mapsto \delta\Theta^{-1}((1/\delta)^{\frac{\alpha}{\alpha-1}})$ is monotone for δ small enough, because the above integral is not bounded. Again with the change of variables $s = \Theta^{-1}((1/\delta)^{\frac{\alpha}{\alpha-1}})$, we see that this monotonicity is equivalent to the monotonicity of the function $\phi : s \mapsto s^{-1}\Theta(s)^{-\frac{\alpha-1}{\alpha}}$ for s large. Inspecting the derivative of ϕ, we see that

$$\phi'(s) = \frac{\Theta(s)^{-\frac{\alpha-1}{\alpha}}}{s^2}\left(-\frac{\alpha-1}{\alpha}\frac{\Theta'(s)}{s\Theta(s)} - 1\right) < 0,$$

and so ϕ is monotone for s sufficiently large. We conclude that, for each $o \in \Omega$ and t such that $\varphi(t) < +\infty$ there exists $R > 0$ such that $B(o, R) \subset \Omega$ and $\tilde b(t, \cdot) : B(o, R) \to \mathbb{R}^n$ satisfying Osgood's condition

$$|\tilde b(t, x) - \tilde b(t, y)| \le \varphi(t)\,\omega(|x - y|) \text{ for each } x, y \in B(o, R)\,. \tag{50}$$

$$\begin{aligned} |\tilde b(t, x)| &\le |\tilde b(t, x) - \tilde b(t, x_0)| + |\tilde b(t, x_0)| \le \varphi(t)\,\omega(|x - x_0|) + |\tilde b(t, x_0)| \\ &\le \omega(2R)\,\varphi(t) + |\tilde b(t, x_0)| \text{ for a.e. } t \in I \text{ and for each } x,\ x_0 \in B(o, R)\,. \end{aligned} \tag{51}$$

Now observe that, since $b \in L^1_{\mathrm{loc}}(I \times \Omega; \mathbb{R}^n)$, by (49), $\tilde b \in L^1_{\mathrm{loc}}(I \times B(o, R); \mathbb{R}^n)$. Thus there exists $x_0 \in B(o, R)$ such that $|\tilde b(t, x_0)| \in L^1_{\mathrm{loc}}(I)$. By (50), (51) and Proposition 3.2 (i), we obtain that $\tilde b : I \times B(o, R) \to \mathbb{R}^n$ is well-posed. As a consequence also the vector field $\tilde b : I \times \Omega \to \mathbb{R}^n$ is well-posed.

Moreover, if (9) holds, it also follows that

$$|\tilde b(t, x)| \le m(t) \text{ for a.e. } t \in I, \text{ for a.e. } x \in B(o, R)\,.$$

Thus, by applying now Proposition 3.2 (iv), we can conclude that the vector field $\tilde b : \overline{I} \times B(o, R) \to \mathbb{R}^n$ is well-posed, which implies that $\tilde b : \overline{I} \times \Omega \to \mathbb{R}^n$ is well-posed, too.

When $n = 1$, let $\Phi(s) := \Theta(s/\kappa_n)$. Then we can repeat the same arguments of the previous case, showing that the space continuous representative of b satisfies Osgood's condition. □

The next proposition shows why we could not use a standard Sobolev embedding in the proof of Theorem A to obtain a stronger Sobolev regularity for b. See also Remark 4.5 below.

Proposition 4.4. *Condition (A.III) in Theorem A implies that Θ cannot have polynomial growth, that is,*

$$\forall m \in \mathbb{N} \qquad \limsup_{s\to\infty} \frac{\Theta(s)}{s^m} = +\infty, \tag{52}$$

but it does not imply that Θ has more than polynomial growth.

Proof. On the one hand, we have

$$\begin{aligned}
\infty &= \int_1^\infty \frac{\Theta'(s)}{s\Theta(s)}\,\mathrm{d}s = \sum_{k=1}^\infty \int_k^{k+1} \frac{\Theta'(s)}{s\Theta(s)}\,\mathrm{d}s \\
&\le \sum_{k=1}^\infty \frac{1}{k} \int_k^{k+1} \frac{\Theta'(s)}{\Theta(s)}\,\mathrm{d}s = \sum_{k=1}^\infty \frac{1}{k}\left(\log(\Theta(k+1)) - \log(\Theta(k))\right) \\
&= -\log(\Theta(1)) + \sum_{k=2}^\infty \frac{1}{k^2-k}\log(\Theta(k)) + \limsup_{k\to\infty} \frac{\log(\Theta(k))}{k}
\end{aligned}$$

If the series in the last row is infinite, then for every $0<\alpha<1$ there exists a sequence $k_j \to \infty$ such that

$$\frac{1}{k_j^2 - k_j}\log(\Theta(k_j)) \ge \frac{1}{k_j^{1+\alpha}},$$

that is, for all j with $k_j \ge 2$,

$$\Theta(k_j) \ge \exp\left(\frac{1}{2}k_j^{1-\alpha}\right).$$

If instead the series in the last row is finite, then $\limsup_{k\to\infty} \frac{\log(\Theta(k))}{k} = \infty$, that is, there exists a sequence $k_j \to \infty$ such that

$$\Theta(k_j) \ge \exp\left(k_j\right).$$

On the other hand, we cannot improve the lim sup in (52) with a lim inf (or a lim). Indeed, with a similar estimate as above, we obtain

$$\int_1^\infty \frac{\Theta'(s)}{s\Theta(s)}\,\mathrm{d}s \ge -\frac{1}{2}\log(\Theta(1)) + \sum_{k=2}^\infty \frac{1}{k^2+k}\log(\Theta(k)).$$

Thus, taking a sequence $\{k_j\}_j$ sparse enough, one can construct a piece-wise linear increasing convex function $\Theta : [0,+\infty) \to [0,+\infty)$ such that $\Theta(k_j) = \exp(k_j^2 + k_j)$ (thus $\int_1^\infty \frac{\Theta'(s)}{s\Theta(s)}\,\mathrm{d}s = \infty$) and $\Theta(x_j) = x_j^2$ for some $k_j < x_j < k_{j+1}$ (hence $\liminf_{s\to\infty} \frac{\Theta(s)}{s^2} < \infty$).

Let us give more details about such construction. As a start, define $k_1 = 1$, $\Theta(1) = \exp(2)$ and $\alpha_1 = \exp(2)$. Then define $\Theta(t) = \alpha_1 t$ for $t \in [0, x_1]$, where x_1 is so that $\Theta(x_1) = x_1^2$, that is, $x_1 = \alpha_1$. Iteratively, given x_j and α_j, and the function Θ defined in $[0, x_j]$ with $\Theta(x_j) = x_j^2$, define

$$
\begin{aligned}
&k_{j+1} = \lceil 2x_j \rceil \\
&\alpha_{j+1} = \frac{\exp(k_{j+1}^2 + k_{j+1}) - x_j^2}{k_{j+1} - x_j} \\
&\Theta(t) = \Theta(x_j) + \alpha_{j+1}(t - x_j) \text{ for } t \in [x_j, x_{j+1}] \\
&\qquad \text{where } x_{j+1} \text{ is so that } \Theta(x_j) + \alpha_{j+1}(x_{j+1} - x_j) = x_{j+1}^2
\end{aligned}
$$

By construction, we have $\Theta(k_{j+1}) = \exp(k_{j+1}^2 + k_{j+1})$ and $\Theta(x_{j+1}) = x_{j+1}^2$. Moreover, since $\exp(x^2 + x) > x^2$ for $x \geq 0$, we have $\alpha_{j+1} > \alpha_j$ and thus the resulting function Θ is convex; in fact, Θ is the sup of the linear functions $t \mapsto \Theta(x_j) + \alpha_{j+1}(t - x_j)$, $j = 1, \ldots,$ and thus it is convex. $\square$

Remark 4.5. Using the function Θ constructed in the previous Proposition 4.4, we can give an example of *a continuous vector field $b : \mathbb{R} \to \mathbb{R}$ that is well posed, even though $b \notin W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ for all $p > 2$.*

Indeed, define $b(t) = \int_0^t \chi(s)\,\mathrm{d}s$, where $\chi \in L^1(\mathbb{R})$ is the function

$$
\chi(s) = \sum_{j=1}^{\infty} \chi_{I_j}(s) \cdot x_j \text{ where } I_j := \left(\sum_{k=1}^{j-1} \frac{1}{k^2 x_k^2}, \sum_{k=1}^{j} \frac{1}{k^2 x_k^2} \right).
$$

Notice that, by construction,

$$
\exp(4x_j^2 + x_j) \leq \exp(k_j^2 + k_j) = \Theta(k_{j+1}) < \Theta(x_{j+1}) = x_{j+1}^2,
$$

and thus $x_{j+1} > \exp(2x_j^2 + x_j) > \exp(x_j) > \exp(j)$ for all $j \geq 1$. It follows that $\bigcup_{j=1}^{\infty} I_j$ is bounded and that

$$
\int_{\mathbb{R}} \chi(s)^p \,\mathrm{d}s = \sum_{j=1}^{\infty} \frac{x_j^p}{j^2 x_j^2} = \sum_{j=1}^{\infty} \frac{x_j^{p-2}}{j^2}
$$

is finite if and only if $p \leq 2$. ♦

4.1. A class of subexponential summability types

Examples of functions Θ satisfying the properties listed in Theorem A are of the form $\mathscr{E}_{k,\beta}(s)$ as in (10), as we will show in this section. It is clear that the subexponential summability of type $\mathscr{E}_{k,\beta}$ implies the subexponential summability of type $\mathscr{E}_{k',\beta'}$ for all $\beta \leq \beta' \leq 1$ and $k' \geq k$.

If $\|D_x b\|$ satisfies an exponential summability, then it is well-known that b is well-posed. Indeed, in this case, the vector field b satisfies a so-called Log-Lipschitz condition (see, for instance, [8,32]). Theorem A extends the well-posedness to subexponential summability order $k \geq 1$. We will show in Section 8.2 that the upper bound on β for the subexponential summability order 1 is in fact necessary; see also [12, Section 6], [11, p. 1240].

Let us now show that $\mathscr{E}_{k,\beta}$ satisfies the assumptions of Theorem A. Let us introduce some notation in order to better represent $\mathscr{E}_{k,\beta}$. Let us denote by $E_k : \mathbb{R} \to \mathbb{R}$ the *kth-iterated exponential function*, that is, by induction on k,

$$E_1(s) := \exp(s), \quad E_{k+1}(s) := \exp(E_k(s)) \text{ if } s \in \mathbb{R},\ k \geq 1.$$

Since $\lim_{s\to-\infty} E_1(s) = 0$, notice that $\lim_{s\to-\infty} E_k(s) = E_{k-1}(0) =: s_k$ for all $k > 1$, and that $\lim_{k\to\infty} s_k = \infty$. Denote by $L_k : E_k(\mathbb{R}) \to \mathbb{R}$ the *kth-iterated logarithm function* as the inverse of E_k, that is

$$L_k(s) := E_k^{-1}(s) \text{ if } s \in E_k(\mathbb{R}) = (s_k, +\infty).$$

Define then $P_k(s) := \prod_{j=1}^k L_j(s)$, so that one can easily check that

$$L'_{k+1}(s) = \frac{1}{s P_k(s)}, \text{ so that } P'_k(s) = \frac{P_k(s)}{s} \sum_{j=1}^{k} \frac{1}{P_j(s)}.$$

With this notation, we have

$$\mathscr{E}_{k,\beta}(s) = \exp\left(\frac{s}{P_{k-1}(s) L_k^{\beta}(s)}\right).$$

A direct computation shows that

$$\mathscr{E}'_{k,\beta} = \mathscr{E}_{k,\beta} H_{k,\beta} \quad \text{where} \quad H_{k,\beta} = \frac{1}{P_{k-1} L_k^{\beta}} \left(1 - \sum_{j=1}^{k-1} \frac{1}{P_j} - \frac{\beta}{P_k}\right), \tag{53}$$

and that

$$H'_{k,\beta} = \frac{1}{s P_{k-1} L_k^{\beta}} \Big(-\Big(\sum_{j=1}^{k-1} \frac{1}{P_j} + \frac{\beta}{P_k}\Big)\Big(1 - \sum_{j=1}^{k-1} \frac{1}{P_j} - \frac{\beta}{P_k}\Big) + \sum_{j=1}^{k-1} \frac{1}{P_j} \Big(\sum_{i=1}^{j} \frac{1}{P_i} + \frac{\beta}{P_k}\Big)\Big). \tag{54}$$

Proposition 4.6. *The function $\mathscr{E}_{k,\beta} : \mathbb{R} \to \mathbb{R}$ satisfies the assumptions of Theorem A for each integer $k \geq 1$ and $0 \leq \beta \leq 1$.*

Proof. Monotonicity of $\mathscr{E}_{k,\beta}$: From (53) it is evident that $\mathscr{E}'_{k,\beta}$ is positive for s large enough and thus $\mathscr{E}_{k,\beta}$ is strictly increasing in $[\alpha, +\infty)$ for α large enough.

Verification of (A.I): If $\gamma > 0$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathscr{E}_{k,\beta}(s)^{\gamma} = \gamma \mathscr{E}_{k,\beta}(s)^{\gamma} \left(\gamma H_{k,\beta}^2 + H'_{k,\beta}\right).$$

Because of the presence of the factor $1/s$ in (54), one can see that $\lim_{s\to\infty} \frac{H'_{k,\beta}}{H^2_{k,\beta}} = 0$ and thus $\frac{\mathrm{d}^2}{\mathrm{d}s^2} \mathscr{E}_{k,\beta}(s)^{\gamma}$ is positive for s large enough. Thus we obtain that $\mathscr{E}_{k,\beta}(s)^{\gamma}$ is convex for s large.

Verification of (A.II): Firstly, notice that $\log(st) = \log(s) + \log(t) \le \log(s)\log(t)$ for all $s, t > e$. Therefore, by induction on k, we have $L_k(st) \le L_k(s)L_k(t)$ for all $s, t > E_n(1)$. Hence, $P_k(st) = \prod_{j=1}^k L_j(st) \le \prod_{j=1}^k L_j(s)L_j(t) = P_k(s)P_k(t)$, or all $s, t > E_n(1)$.

Secondly, condition (7), i.e., $\mathscr{E}_{k,\beta}(ts) \le \mathscr{E}_{k,\beta}(s)\mathscr{E}_{k,\beta}(t)$ for s, t large enough, is equivalent to

$$\frac{s}{P_{k-1}(s)L_k(s)^\beta} + \frac{t}{P_{k-1}(t)L_k(t)^\beta} \le \frac{st}{P_{k-1}(st)L_k(st)^\beta},$$

that is

$$\frac{P_{k-1}(t)L_k(t)^\beta}{t} + \frac{P_{k-1}(s)L_k(s)^\beta}{s} \le \frac{P_{k-1}(s)L_k(s)^\beta P_{k-1}(t)L_k(t)^\beta}{P_{k-1}(st)L_k(st)^\beta}. \tag{55}$$

Thirdly, on the one hand we have $\lim_{t\to\infty} \frac{P_{k-1}(t)L_k(t)^\beta}{t} = 0$, and thus the left-hand side of (55) smaller than 1 for s and t large. On the other hand, by the initial observation,

$$\frac{P_{k-1}(s)L_k(s)^\beta P_{k-1}(t)L_k(t)^\beta}{P_{k-1}(st)L_k(st)^\beta} \ge \frac{P_{k-1}(s)L_k(s)^\beta P_{k-1}(t)L_k(t)^\beta}{P_{k-1}(s)P_{k-1}(t)L_k(s)^\beta L_k(t)^\beta} = 1.$$

So, inequality (55) holds true and so condition (7).

Verification of (A.III): Notice that, by (53), for $s > E_k(1)$,

$$\frac{\mathscr{E}'_{k,\beta}(s)}{s\mathscr{E}_{k,\beta}(s)} \sim_{s\to\infty} \frac{1}{sP_{k-1}(s)L_k(s)^\beta}.$$

Thus, since $\int_{E_k(1)}^\infty \frac{1}{sP_{k-1}(s)L_k(s)^\beta}\,\mathrm{d}s = \infty$ if and only if $\beta \le 1$, we also have $\int_1^\infty \frac{\mathscr{E}'_{k,\beta}(s)}{s\mathscr{E}_{k,\beta}(s)}\,\mathrm{d}s = +\infty$ if and only if $\beta \le 1$. $\square$

Remark 4.7. Observe that, if (A.III) holds, then

$$\limsup_{s\to\infty} \frac{P_{k-1}(s)L_k(s)^{1+\alpha}\Theta'(s)}{\Theta(s)} = \infty \text{ for every } \alpha > 0 \text{ and } k \ge 1\,. \tag{56}$$

♦

5. Regularity of the flow with subexponential summability

This section is devoted to the proof of Theorem B and its consequence in dimension 1 as written in Theorem C. Let us recall that, if $\Phi : [0, +\infty) \to [0, +\infty)$ is an increasing homeomorphism, so that $\Phi(0) = 0$ and $\lim_{t\to+\infty}\Phi(t) = +\infty$, the *Orlicz space* $\mathrm{L}^\Phi(\mathbb{R}^n)$ is the space of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which the *Luxembourg norm*

$$\|f\|_{\mathrm{L}^\Phi} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{f(x)}{\lambda}\right)\mathrm{d}x \le 1\right\}$$

is finite. $L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ will denote, as usual, the space of measurable functions f such that $f\chi_K \in \mathrm{L}^{\Phi}(\mathbb{R}^n)$ for each compact set $K \subset \mathbb{R}^n$. If

$$\Phi(t) = \exp\left(\frac{t}{\log^+ t}\right) - 1, \tag{57}$$

then we will denote the obtained $\mathrm{L}^{\Phi}(\mathbb{R}^n)$ (respectively $\mathrm{L}^{\Phi}_{\rm loc}(\mathbb{R}^n)$) by $\mathrm{Exp}\big(L/\log L\big)$ (respectively $\mathrm{Exp}_{\rm loc}\big(L/\log L\big)$). When changing the reference measure from the Lebegues measure to a Radon measure μ on $\mathbb{R}^n$, we will denote $\mathrm{L}^{\Phi}(\mathbb{R}^n,\mu)$ (respectively $L^{\Phi}_{\rm loc}(\mathbb{R}^n,\mu)$) the associated Orlicz space and, if Φ is as in (57), by $\mathrm{Exp}_{\mu}\big(L/\log L\big)$ (respectively $\mathrm{Exp}_{\mu,\rm loc}\big(L/\log L\big)$).

Proof of Theorem B. We assume with no loss of generality that b coincides with its space continuous representative. It is easy to see that, from assumptions (11) and (12), it follows that, for a given $R > 0$ and

$$m(t) := (1 + R\log^+ R) \sup_{x\in B(0,R)} \frac{|b(t,x)|}{1+|x|\log^+|x|} \text{ for a.e. } t \in I\,,$$

then

$$m \in L^1(I) \text{ and } |b(t,x)| \le m(t) \text{ for a.e. } t \in I,\ \text{ for each } x \in B(0,R) \tag{58}$$

and

$$\exp\left(\frac{\|D_x b\|}{\log^+\|D_x b\|}\right) \in L^1(I; L^1_{\rm loc}(\mathbb{R}^n))\,. \tag{59}$$

Thus, by (58) and (59), the vector field b is well-posed in $\overline{I}\times\mathbb{R}^n$ by Theorem A and Proposition 4.6. Moreover the flow X associated to b is globally defined thanks to the growth condition (11), so that $X \in C^0(\overline{I}\times\overline{I}\times\mathbb{R}^n;\mathbb{R}^n)$ (see, for instance, [23, Theorem 5.1, Chap. III]). In particular, b satisfies the assumptions of [11, Main Theorem]. Therefore the push-forward measure $X(t,s,\cdot)_{\#}\gamma_n$ is absolutely continuous with respect to the Gaussian measure γ_n with density $w(t,s)$ belonging to $\mathrm{L}^{\Phi_\alpha}(\mathbb{R}^n,\gamma_n)$ for each $0<\alpha<\alpha_0(s,t)$, where

$$\alpha_0(t,s) = \exp\left(-16\,e^2\int_s^t \|\mathrm{div}_{\gamma_n} b(v,\cdot)\|_{\mathrm{Exp}_{\gamma_n}\left(\frac{L}{\log L}\right)}\,\mathrm{d}v\right),$$

where

$$\mathrm{div}_{\gamma_n} b(v) = \mathrm{div} b(v) - v\cdot b(v),$$

that is, div_{γ_n} is the adjoint of the gradient operator with respect to the measure γ_n. Notice that α_0 is a continuous function with $\alpha_0(t,t)=1$ for every $t \in I$.

To complete the proof, it is enough to show that

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{L}^n}(X(t,s,\cdot)_{\#}\mathcal{L}^n)(x) = D(x)w(t,s)(x) \quad \text{for a.e. } x \in \mathbb{R}^n\,, \tag{60}$$

where $D(x) = \exp\left(\frac{|X(s,t,x)|^2 - |x|^2}{2}\right)$ and $\frac{d}{d\mathcal{L}^n}$ denotes the Radon-Nikodym derivative with respect to $\mathcal{L}^n$. Indeed, since the map $D : \mathbb{R}^n \to [0, \infty)$ is continuous and since $\mathrm{L}^{\Phi_\alpha}(\mathbb{R}^n, \mu) \subset \mathrm{L}^{\Phi_\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, we can conclude (13).

In order to prove (60), fix $t,\ s \in I$ and set $\psi := X(s, t, \cdot)$. By the differentiation theorem for Radon measures (see, for instance, [6]), we have

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{L}^n}\psi_\#\mathcal{L}^n = \lim_{r\to 0^+} \frac{\mathcal{L}^n(\psi(B(x,r)))}{\mathcal{L}^n(B(x,r))},$$

for almost every $x \in \mathbb{R}^n$. Notice that

$$\frac{\mathcal{L}^n(\psi(B(x,r)))}{\mathcal{L}^n(B(x,r))} = \frac{\mathcal{L}^n(\psi(B(x,r)))}{\gamma_n(\psi(B(x,r)))}\frac{\gamma_n(\psi(B(x,r)))}{\gamma_n(B(x,r))}\frac{\gamma_n(B(x,r)))}{\mathcal{L}^n(B(x,r))},$$

where, again by differentiation theorem for Radon measures,

$$\begin{aligned}
\lim_{r\to 0^+} \frac{\gamma_n(B(x,r)))}{\mathcal{L}^n(B(x,r))} &= \frac{1}{(2\pi)^{n/2}}\exp\left(-\frac{|x|^2}{2}\right),\\
\lim_{r\to 0^+} \frac{\gamma_n(\psi(B(x,r)))}{\gamma_n(B(x,r))} &= w(t,s)(x),\\
\lim_{r\to 0^+} \frac{\gamma_n(\psi(B(x,r)))}{\mathcal{L}^n(\psi(B(x,r)))} &= \frac{1}{(2\pi)^{n/2}}\exp\left(-\frac{|\psi(x)|^2}{2}\right).
\end{aligned}$$

A short computation leads to (60). □

When the spatial dimension n equals 1, we have stronger results.

Proof of Theorem C. By Theorem B, b is well-posed and both maps $X_{(t,s)} : \Omega_{(t,s)} \to \Omega_{(s,t)}$ and $X^{-1}_{(t,s)} = X_{(s,t)} : \Omega_{(s,t)} \to \Omega_{(t,s)}$ satisfy the Lusin (N) condition. Thus, by a well-known result of real analysis (see, for instance, [20, Theorem 7.45]) they must be locally absolutely continuous. □

Remark 5.1. We do not know whether, if $n \geq 2$, there exists a flow X associated to a vector field b satisfying (12), but $X(t,s,\cdot) \notin W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ for some $t, s \in \mathbb{R}$. Recall it has been proven in [21] that there exists an almost everywhere approximately differentiable, orientation and measure preserving homeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, whose Jacobian is equal to -1 almost everywhere on the unit n-dimensional cube Q, $\Psi(x) = x$ if $x \in \mathbb{R}^n \setminus Q$ and $\Psi \notin W^{1,1}(Q;\mathbb{R}^n)$. In particular it holds that

$$\Psi_\#\mathcal{L}^n \ll \mathcal{L}^n \text{ and } \frac{\mathrm{d}}{\mathrm{d}\mathcal{L}^n}(\Psi_\#\mathcal{L}^n)(x) = 1 \text{ for a.e. } x \in \mathbb{R}^n.$$

We are not aware whether such a homeomorphism Ψ could be induced by the flow of a suitable vector field. ♦

Remark 5.2. Sobolev regularity stated in Theorem C is sharp, as we show in Example 8.2 below. Moreover it can be compared with the result in [27], which proves existence and uniqueness of a unique flow X, with $D_x X(t,s,\cdot) \in A_\infty(\mathbb{R})$, provided that $D_x b \in L^1((0,T); BMO(\mathbb{R}))$ (see [22, Chap. 7] for the definition of the class of functions $BMO(\mathbb{R}^n)$). Notice that a function in BMO admits exponential summability, as defined in Section 4.1. In particular, (12) holds, too. Whereas there are vector fields b with $D_x b$ exponentially summable, but $D_x b \notin \mathrm{L}^1((0,T); BMO(\mathbb{R}))$ (see Example 8.3 below). Notice also that there are flows associated to well-posed vector fields, which are not absolutely continuous (see Example 8.1 below). ♦

6. Regularity of the flow with exponential summability

This section is devoted to the proof of Theorem D. Under the stronger regularity assumption

$$b \in C^0(I; C^1(\Omega; \mathbb{R}^n)) \tag{61}$$

we are able to derive a more precise a priori estimate, involving the quantity $\ell(s,x)$, namely the length of the maximal interval $I_{(s,x)}$. Notice that we clearly have $\ell(s,x) \le \ell$, with ℓ the length of I. Moreover, when b is a well posed vector field, the maximal integral curves of b stop either at the boundary of I or of Ω and thus, for all $s \in I$,

$$\ell(s,x) \ge \min\{\ell, \mathrm{dist}(x, \partial\Omega)/\sup|b|\}. \tag{62}$$

From this follows that, $\Lambda'_p \le \Lambda_p$, where Λ'_p is defined in (66) below and Λ_p in (14). In particular, the finiteness condition (63) implies the finiteness of Λ'_p as stated in (66).

Theorem 6.1. *Let $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be bounded open sets and let $b \in C^0(I; C^1(\Omega; \mathbb{R}^n))$ be a bounded vector field. Assume that for some $p > 2n$ one has*

$$\int_I \int_\Omega \mathrm{dist}(x, \partial\Omega)^{\frac{n}{n-p}} \exp\left(\frac{\ell p^2}{p-n} \|D_x b(s,x)\|\right) \mathrm{d}x\,\mathrm{d}s < +\infty. \tag{63}$$

Then b is a well-posed vector field in $\overline{I} \times \Omega$ thanks to Theorem A.

For all $t \in \overline{I}$

$$\text{for almost every } s \in I,\ X(t,s,\cdot) \in W^{1,p}(\Omega_{(t,s)}; \mathbb{R}^n) \quad \text{and} \quad X(s,t,\cdot) \in W^{1,p}(\Omega_{(s,t)}; \mathbb{R}^n). \tag{64}$$

Moreover, we have that, for all $t \in \overline{I}$,

$$\int_I \int_{\Omega_{(t,s)}} \|D_x X(t,s,x)\|^p \,\mathrm{d}x\,\mathrm{d}s \le \Lambda'_p, \tag{65}$$

where

$$\Lambda'_p := \int_I \int_\Omega \left(\frac{\ell}{\ell(s,x)} \right)^{\frac{n}{p-n}} \exp\left(\frac{\ell p^2}{p-n} \| D_x b(s,x) \| \right) \mathrm{d}x \, \mathrm{d}s < \infty. \tag{66}$$

If we also assume that there exists $\hat{\ell} > 0$ such that

$$\ell(t,x) \geq \hat{\ell} \text{ for all } t \in I \text{ and } x \in \Omega, \tag{67}$$

then, for every $s, t \in I$,

$$\int_{\Omega_{(t,s)}} \| D_x X(t,s,x) \|^p \, \mathrm{d}x \leq \frac{1}{\hat{\ell}} \left(\frac{\ell}{\hat{\ell}} \right)^{\frac{n^2}{p(p-n)}} \Lambda''_p \tag{68}$$

where

$$\Lambda''_p := \int_I \int_\Omega \exp\left(\frac{\ell p^2}{p-n} \| D_x b(v,y) \| \right) \mathrm{d}y \, \mathrm{d}v.$$

Moreover, if there exists an open set $\Omega' \Subset \Omega$ such that

$$\mathtt{spt}(b(t,\cdot)) \subset \Omega' \text{ for each } t \in I, \tag{69}$$

then (63) implies (68) also for $p > n$.

Remark 6.2. Typical cases where (67) holds are the following. First, as we already observed in the statement above, if there exists an open set $\Omega' \Subset \Omega$ such that (69) holds then it turns out that $\Omega_{(t,s)} = \Omega$ and $\ell(t,x) = \ell$ for every $t,\ s \in \overline{I}$ and $x \in \Omega$.

Second, if $\partial\Omega$ is smooth and compact and b is tangent to $\partial\Omega$, then we have again $\Omega_{(t,s)} = \Omega$ and $\ell(t,x) = \ell$ for every $t,\ s \in \overline{I}$ and $x \in \Omega$.

Third, one can easily have $\hat{\ell} < \ell$: for example, if on the plane $\mathbb{R}^2$ with coordinates (x,y), we take $b(t,(x,y)) = \partial_x$ and $\Omega = \{(x,y) : -2 + y < x < 2 - y,\ 0 < y < 1\}$. In this case we have $\ell(t,(x,y)) = 4 - 2y$. ♦

The proof of Theorem 6.1 is based on the well-known estimate (70) below and a bootstrap argument in three steps. The proof gives also an intermediate estimate (75) on subdomains $A \Subset \Omega$ that involves the function $\ell(s,x)$ in place of $\mathrm{dist}(x, \partial\Omega)$.

Remark 6.3. The proof of Sobolev regularity of the map $X(t,s,\cdot)$ becomes even simpler when the spatial domain Ω is replaced by a compact Riemannian manifold M without boundary, as for instance the n-dimensional torus $\mathbb{T}^n$ considered in [8] (see also [13,11,12,14,10]). Indeed, in this case the quantity $\ell(s,x)$ equals the length of I. The extension to the case of Sobolev regularity with respect to the space variable, along the lines of Theorem D (i.e., dropping assumption (61)), simply requires a global approximation of b by more regular vector fields. In the case $M = \mathbb{T}^n$, one may argue by convolution with respect to the space variable, viewing the vector field as a spatially periodic one defined in $I \times \mathbb{R}^n$. ♦

The first lemma is one of the many variants of Gronwall's lemma, whose simple proof is omitted.

Lemma 6.4. *Let $f : [s,t] \to [0,+\infty)$ be an absolutely continuous function and $\beta \in L^1(s,t)$ non-negative. If $f' \le \beta f$ a.e. in (s,t) and $f(s) = 1$, then $f(t) \le \exp(\int_s^t \beta(v)\,\mathrm{d}v)$.*

Lemma 6.5. *Let $b \in C^0(I; C^1(\Omega; \mathbb{R}^n))$ and let $X : \mathscr{D}_b \to \Omega$ the corresponding flow. Then, for every $(t,s,x) \in \mathscr{D}_b$ one has*

$$\|D_x X(t,s,x)\| \le \exp\left|\int_s^t \|D_x b(v, X(v,s,x))\|\,\mathrm{d}v\right|. \tag{70}$$

Proof. Set $y(v) = D_x X(v,s,x)$, $B(v) = (D_x b)(v, X(v,s,x))$ and assume, to fix the ideas, $s \le t$ (the proof in the other case is similar). Since, thanks to Lemma 3.4, y is of class C^1, the function $[s,t] \ni v \mapsto |y(v)|$ is absolutely continuous. Using once more Lemma 3.4, we have

$$\frac{\mathrm{d}}{\mathrm{d}v}|y(v)| = \langle \frac{y(v)}{|y(v)|}, \dot{y}(v)\rangle = \langle \frac{y(v)}{|y(v)|}, B(v)y(v)\rangle \le \|B(v)\||y(v)| \text{ for a.e. } v \in (s,t)\,.$$

Lemma 6.4 implies (70), where the absolute value in the argument of the exponential is necessary when $t < s$. □

Proof of Theorem 6.1. Fix $t \in I$ and $A \subset \Omega$ open; first, we use Lemma 6.5 and Jensen's inequality (41) to get

$$\begin{aligned}\int_I \int_{A_{(t,s)}} \|D_x X(t,s,x)\|^p \,\mathrm{d}x\,\mathrm{d}s &\le \int_I \int_{A_{(t,s)}} \exp\left(p\left|\int_s^t \|D_x b(v, X(v,s,x))\|\,\mathrm{d}v\right|\right)\mathrm{d}x\,\mathrm{d}s \\ &\le \int_I \int_A \int_I \frac{\chi_{A(v,s)}(x)}{\ell_A(s,x)} \exp\left(\ell p \|D_x b(v, X(v,s,x))\|\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{71}$$

where we write $\ell_A(s,x)$ for the length of the interval of the maximal integral curve of b in A that starts from x at time s. Setting $J(s,v,\cdot) = J_X(s,v,\cdot) = \det D_x X(s,v,\cdot)$, we apply the change of variable $y = X(v,s,x)$, i.e., $X(s,v,y) = x$ and $\chi_{A(v,s)}(x)\,\mathrm{d}x = J(s,v,y)\chi_{A(s,v)}(y)\,\mathrm{d}y$; notice in particular that $x \in A(v,s)$ if and only if $y = X(v,s,x) \in A(s,v)$. Together with Hadamard's inequality (27) and the Hölder inequality with exponents $q = p/n$ and $q' = p/(p-n)$, we get

$$
\begin{aligned}
&\int_I \int_I \int_A \frac{\chi_{A(v,s)}(x)}{\ell_A(s,x)} \exp\left(\ell p \|D_x b(v, X(v,s,x))\|\right) \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}s \\
&\le \int_I \int_I \int_A \frac{\chi_{A(s,v)}(y)}{\ell_A(s, X(s,v,y))} \exp\left(\ell p \|D_x b(v,y)\|\right) \|D_y X(s,v,y)\|^n \, \mathrm{d}y\, \mathrm{d}v\, \mathrm{d}s \\
&\le \left(\int_I \int_I \int_A \frac{\chi_{A(s,v)}(y)}{\ell_A(s, X(s,v,y))^{q'}} \exp\left(\ell p q' \|D_x b(v,y)\|\right) \mathrm{d}y\, \mathrm{d}v\, \mathrm{d}s \right)^{1/q'} \times \\
&\qquad \times \left(\int_I \int_I \int_A \|D_y X(s,v,y)\|^{nq} \, \mathrm{d}y\, \mathrm{d}v\, \mathrm{d}s \right)^{1/q} .
\end{aligned} \tag{72}
$$

The identity $\ell_A(s, X(s,v,y)) = \ell_A(v,y)$ gives us

$$
\int_I \frac{\chi_{A(s,v)}(y)}{\ell_A(s, X(s,v,y))^{q'}} \, \mathrm{d}s = \ell_{A(v,y)}(y)^{1-q'},
$$

and we conclude that

$$
\begin{aligned}
&\int_I \int_{A_{(t,s)}} \|D_x X(t,s,x)\|^p \, \mathrm{d}x\, \mathrm{d}s \\
&\le \left(\int_I \int_A \ell_A^{1-q'}(v,y) \exp\left(\ell p q' \|D_x b(v,y)\|\right) \mathrm{d}y\, \mathrm{d}v \right)^{1/q'} \times \\
&\qquad \times \left(\int_I \int_I \int_A \|D_y X(s,v,y)\|^{nq} \, \mathrm{d}y\, \mathrm{d}v\, \mathrm{d}s \right)^{1/q} .
\end{aligned} \tag{73}
$$

If we take $A \Subset \Omega$, the latter triple integral is finite. By integrating this inequality with respect to t, and then rearranging the terms, we obtain that

$$
\int_I \int_I \int_{A_{(t,s)}} \|D_x X(t,s,x)\|^p \, \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t \le \ell^{q'} \int_I \int_A \ell_A^{1-q'}(v,y) \exp\left(\ell p q' \|D_x b(v,y)\|\right) \mathrm{d}y\, \mathrm{d}v \tag{74}
$$

If we plug the estimate (74) into (73), we obtain

$$
\int_I \int_{A_{(t,s)}} \|D_x X(t,s,x)\|^p \, \mathrm{d}x\, \mathrm{d}s \le \ell^{\frac{n}{p-n}} \int_I \int_A \ell_A^{\frac{n}{n-p}}(v,y) \exp\left(\frac{\ell p^2}{p-n} \|D_x b(v,y)\|\right) \mathrm{d}y\, \mathrm{d}v. \tag{75}
$$

To prove (65), we need to take a limit in (75) along a sequence of sets $A \Subset \Omega$ that fills Ω. To this aim, we define

$$A^\epsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\}$$

for $\epsilon > 0$, plug A^ϵ in (75) and take $\epsilon \to 0$.

For the left-hand side of (75), we observe that $A^{\epsilon_1}_{(t,s)} \subset A^{\epsilon_2}_{(t,s)}$ whenever $\epsilon_1 > \epsilon_2$. Thus, we have

$$\lim_{\epsilon\to 0} \int\limits_I \int\limits_{A^\epsilon_{(t,s)}} \|D_x X(t,s,x)\|^p \,\mathrm{d}x\,\mathrm{d}s = \int\limits_I \int\limits_{\Omega_{(t,s)}} \|D_x X(t,s,x)\|^p \,\mathrm{d}x\,\mathrm{d}s.$$

The right-hand side of (75) is more tricky. Define $f(v,y) := \exp\left(\ell p q' \|D_x b(v,y)\|\right)$, $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, and $\alpha = \frac{n}{n-p}$. Notice that, since $p \geq 2n$, we have $\alpha \in (-1, 0)$.

We claim that

$$\liminf_{\epsilon\to 0} \int\limits_{A^\epsilon} \int\limits_I \ell^\alpha_{A^\epsilon}(v,y) f(v,y) \,\mathrm{d}v\,\mathrm{d}y \leq \int\limits_\Omega \int\limits_I \ell^\alpha_\Omega(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y. \tag{76}$$

If $K \Subset \Omega$, the monotone convergence theorem implies that

$$\lim_{\epsilon\to 0} \int\limits_K \int\limits_I \ell^\alpha_{A^\epsilon}(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y = \int\limits_K \int\limits_I \ell^\alpha_\Omega(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y.$$

Thus, (76) is shown if we can prove that

$$\inf_{K\Subset\Omega} \liminf_{\epsilon\to 0} \int\limits_{A^\epsilon\setminus K} \int\limits_I \ell^\alpha_{A^\epsilon}(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y = 0. \tag{77}$$

Indeed, thanks to (66), for every $K \Subset \Omega$ we have

$$\begin{aligned} &\left| \liminf_{\epsilon\to 0} \int\limits_{A^\epsilon} \int\limits_I \ell^\alpha_{A^\epsilon}(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y - \int\limits_\Omega \int\limits_I \ell^\alpha_\Omega(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y \right| \\ &\qquad \leq \liminf_{\epsilon\to 0} \int\limits_{A^\epsilon\setminus K} \int\limits_I \ell^\alpha_{A^\epsilon}(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y + \int\limits_{\Omega\setminus K} \int\limits_I \ell^\alpha_\Omega(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y. \end{aligned}$$

If there is a sequence $K_j \Subset \Omega$ such that

$$\lim_{j\to\infty} \liminf_{\epsilon\to 0} \int\limits_{A^\epsilon\setminus K_j} \int\limits_I \ell^\alpha_{A^\epsilon}(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y = 0,$$

then $\lim_{j\to\infty} |\Omega \setminus K_j| = 0$, and $\lim_{j\to\infty} \int_{\Omega\setminus K_j} \int_I \ell^\alpha_\Omega(v,y) f(v,y)\,\mathrm{d}v\,\mathrm{d}y = 0$, and so (76) holds.

To prove (77) we use the inequality (62). If K is large enough, so that $\mathrm{dist}(y, \partial A^\epsilon) < \ell$ for all $y \in A_\epsilon \setminus K$, the inequality (62) simplifies to

$$\ell_{A^\epsilon}(v, y)\|b\|_{L^\infty} \geq \mathrm{dist}(y, \partial A^\epsilon).$$

We use the latter inequality to compute the following averaged integral

$$\begin{aligned}
&\frac{1}{\eta}\int_0^\eta \int_{A^\epsilon\setminus K} \int_I \ell^\alpha_{A^\epsilon}(v, y) f(v, y)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\epsilon \\
&\quad\leq \frac{1}{\eta}\int_0^\eta \int_{A^\epsilon\setminus K} \int_I \frac{\mathrm{dist}(y, \partial A^\epsilon)^\alpha}{\|b\|^\alpha_{L^\infty}} f(v, y)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\epsilon \\
&\quad= \frac{1}{\|b\|^\alpha_{L^\infty}} \int_{\Omega\setminus K} \int_I \left(\frac{1}{\eta}\int_0^\eta \chi_{A^\epsilon}(y)\mathrm{dist}(y, \partial A^\epsilon)^\alpha\,\mathrm{d}\epsilon \right) f(v, y)\,\mathrm{d}v\,\mathrm{d}y.
\end{aligned}$$

Since $\delta(y) \leq \mathrm{dist}(y, \partial A^\epsilon) + \epsilon$, i.e., $\mathrm{dist}(y, \partial A^\epsilon) \geq \delta(y) - \epsilon$, we compute

$$\begin{aligned}
\int_0^\eta \chi_{A^\epsilon}(y)\mathrm{dist}(y, \partial A^\epsilon)^\alpha\,\mathrm{d}\epsilon &\leq \int_0^\eta \chi_{A^\epsilon}(y)(\delta(y) - \epsilon)^\alpha\,\mathrm{d}\epsilon \\
&= \int_0^{\min\{\eta,\delta(y)\}} (\delta(y) - \epsilon)^\alpha\,\mathrm{d}\epsilon \\
&\overset{(*)}{\leq} \min\left\{2^{-\alpha}, \frac{2}{\alpha+1}\right\}\eta\delta(y)^\alpha.
\end{aligned}$$

In $(*)$, we considered two cases: first, when $\eta < \delta(y)/2$, the integral is bounded by $2^{-\alpha}\delta(y)^\alpha\eta$; second, when $\eta \geq \delta(y)/2$, using the fact that $\alpha > -1$, the integral is bounded by $\frac{2}{\alpha+1}\eta\delta(y)^\alpha$.

Therefore, there exists $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$, we can estimate the averaged integral with

$$\frac{1}{\eta}\int_0^\eta \int_{A^\epsilon\setminus K} \int_I \mathrm{dist}(y, \partial A^\epsilon)^\alpha f(v, y)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}\epsilon \leq 2^{-\alpha} \int_{\Omega\setminus K} \int_I \delta(y)^\alpha f(v, y)\,\mathrm{d}v\,\mathrm{d}y.$$

It follows that

$$\liminf_{\epsilon\to 0} \int_{A^\epsilon\setminus K} \int_I \mathrm{dist}(y, \partial A^\epsilon)^\alpha f(v, y)\,\mathrm{d}v\,\mathrm{d}y \leq 2^{-\alpha} \int_{\Omega\setminus K} \int_I \delta(y)^\alpha f(v, y)\,\mathrm{d}v\,\mathrm{d}y.$$

Then, since we assumed $\int_\Omega \int_I \delta(y)^\alpha f(v, y)\,\mathrm{d}v\,\mathrm{d}y < \infty$ in (63), we obtain the estimate (77) and thus our claim (76).

We have thus completed the proof of (65), and thus (64). Next we work on the second part of the theorem. If (67) holds, i.e., $\hat{\ell} \le \ell(t,x) \le \ell$ for all $t \in I$ and $x \in \Omega$, then

$$\Lambda''_p \le \Lambda'_p \le \left(\frac{\ell}{\hat{\ell}}\right)^{\frac{n}{p-n}} \Lambda''_p, \tag{78}$$

since $p \ge 2n$. We can repeat the steps in (71) and (72) without the integral in s, and then apply (65) and (78) to obtain

$$\begin{aligned}
&\int\limits_{\Omega_{(t,s)}} \|D_x X(t,s,x)\|^p \, \mathrm{d}x \\
&\le \left(\int\limits_I \int\limits_\Omega \frac{\chi_{\Omega(s,v)}(y)}{\ell_\Omega(s, X(s,v,y))^{q'}} \exp\left(\ell p q' \|D_x b(v,y)\|\right) \mathrm{d}y \, \mathrm{d}v \right)^{1/q'} \times \\
&\qquad \times \left(\int\limits_I \int\limits_\Omega \|D_y X(s,v,y)\|^{nq} \, \mathrm{d}y \, \mathrm{d}v \right)^{1/q} \le \frac{1}{\hat{\ell}} \left(\frac{\ell}{\hat{\ell}}\right)^{\frac{n^2}{p(p-n)}} \Lambda''_p,
\end{aligned}$$

and this proves (68).

If (69) holds, then the triple integral in (73) is finite also for $A = \Omega$ and $p > n$. In this case, we can obtain directly (75) for $A = \Omega$ without passing through the approximation A^ϵ, where we used the stronger hypothesis $p > 2n$. □

The extension of Theorem 6.1 to the case of Sobolev spatial regularity requires a global approximation of b by more regular vector fields that seems to be not trivial, also because of the weight function $\ell(s,x)$ depending on the vector field itself. In addition, the non-doubling property of the exponential function is source of extra difficulties, when performing standard convolution arguments. In the case of a weight independent of t, we addressed this problem in the note [7], from which we extract the following result.

Theorem 6.6 *([7, Theorem 3 and Remark 25]). Let Ω be a bounded open set and let $w : \Omega \to (0, +\infty)$, with $w + w^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$.*

(i) If $b \in \mathrm{L}^1(I; W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n))$ satisfies

$$\int\limits_I \int\limits_\Omega w(x) \exp(c \|D_x b(s,x)\|) \, \mathrm{d}x \, \mathrm{d}s < +\infty, \tag{79}$$

for some $c > 0$, then there exist $b_h \in C^\infty(I \times \Omega; \mathbb{R}^n)$ satisfying, whenever $\Omega' \Subset \Omega$,

$$b_h \text{ converge to } \tilde{b} \text{ in } L^1(I; C(\Omega'; \mathbb{R}^n)) \text{ as } h \to \infty\,, \tag{80}$$

where $\tilde{b}$ is the space continuous representative of b,

$$D_x b_h \to D_x b \text{ in } L^1(I; L^1(\Omega'; \mathbb{R}^n)), \text{ as } h \to \infty, \tag{81}$$

and

$$\lim_{h\to\infty} \int_I \int_\Omega w(x) \exp(c\|D_x b_h(s,x)\|)\,\mathrm{d}x\,\mathrm{d}s = \int_I \int_\Omega w(x) \exp(c\|D_x b(s,x)\|)\,\mathrm{d}x\,\mathrm{d}s. \tag{82}$$

(ii) If $b \in L^1(I; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n))$ and there exists a bounded open set Ω such that

$$\operatorname{spt}(b(t,\cdot)) \subset \Omega \text{ for each } t \in I, \tag{83}$$

and b satisfies (79) on Ω, then there exist $b_h \in C^\infty(I \times \mathbb{R}^n; \mathbb{R}^n)$ still satisfying (80), (81), (82) and also

$$\operatorname{spt}(b_h(t,\cdot)) \subset \Omega \text{ for each } t \in I \text{ and } h \in \mathbb{N}. \tag{84}$$

Proof of Theorem D. Let

$$w(x) := \max\left\{\ell^{n/(n-p)}, \frac{(\operatorname{dist}(x,\partial\Omega))^{n/(n-p)}}{(\sup|b|)^{n/(n-p)}}\right\},$$

so that, with $c = \ell p^2/(p-n)$, one has

$$\int_I \int_\Omega w(x) \exp(c\|Db(s,x)\|)\,\mathrm{d}x\,\mathrm{d}s < +\infty$$

and we may apply Theorem 6.6 (i) to b. Since b is bounded, we can also assume, by a truncation argument, that $\sup|b_h| \le \sup|b|$. In addition, the quantity Λ'_p in (66) is finite and satisfies $\Lambda'_p \le \Lambda_p$, using the inequality (62).

It follows from these considerations that

$$\limsup_{h\to\infty} \Lambda'_{p,h} \le \limsup_{h\to\infty} \Lambda_{p,h} \le \Lambda_p < +\infty,$$

where

$$\Lambda'_{p,h} := \int_I \int_\Omega (\ell_h(s,x))^{n/(n-p)} \exp\left(\frac{\ell p^2}{p-n}\|D_x b_h(s,x)\|\right)\mathrm{d}x\,\mathrm{d}s, \tag{85}$$

and similarly $\Lambda_{p,h}$ is defined with b_h.

Now, in order to apply (65) from Theorem 6.1 to b_h and the pass to the limit $h \to \infty$, it suffices by Fatou's lemma to prove that

$$\int_{\Omega_{(t,s)}} \|D_x X(t,s,x)\|^p\,\mathrm{d}x \le \liminf_{h\to\infty} \int_{\Omega^h_{(t,s)}} \|D_x X^h(t,s,x)\|^p\,\mathrm{d}x, \tag{86}$$

where $\Omega^h_{(t,s)}$ and X^h are relative to the vector fields b_h. Now, (35), derived from Lemma 3.5, yields that for any open domain $A \Subset \Omega_{(t,s)}$ one has $A \Subset \Omega^h_{(t,s)}$ for h large enough, and that $X^h(t,s,\cdot)$ converge to $X(t,s,\cdot)$ uniformly on A. Hence the lower semicontinuity of $w \mapsto \int_A |D_x w|^p \, dx$ yields

$$\int\limits_A \|D_x X(t,s,x)\|^p \, dx \le \liminf_{h\to\infty} \int\limits_A \|D_x X^h(t,s,x)\|^p \, dx \le \liminf_{h\to\infty} \int\limits_{\Omega^h_{(t,s)}} \|D_x X^h(t,s,x)\|^p \, dx \, .$$

Letting $A \uparrow \Omega_{(t,s)}$ we obtain the claimed semicontinuity property (86). $\square$

Proof of Theorem E. First assume that b is understood as vector field $b : I \times \Omega \to \mathbb{R}^n$. Let $w \equiv 1$ so that, with $c = \ell\, p^2/(p-n)$, one has

$$\int\limits_I \int\limits_\Omega \exp(c\|Db(s,x)\|) \, dx \, ds < +\infty \, .$$

From Theorem 6.6 (ii), let $b_h : I \times \Omega \to \mathbb{R}^n$ be the regular sequence of vector fields satisfying (80), (81), (82) and (84).

It follows from (83) that $\ell_h(t,x) = \ell$ and thus

$$\limsup_{h\to\infty} \Lambda'_{p,h} \le \frac{1}{\ell} \int\limits_I \int\limits_\Omega \exp(c\|Db(s,x)\|) \, dx \, ds < +\infty \, ,$$

where $\Lambda'_{p,h}$ is the quantity in (85).

Now, in order to apply (68) with $p > n$ from Theorem 6.1 to b_h and then pass to the limit $h \to \infty$, we only need to use (86) again.

Assume now that b is understood as vector field $b : I \times \mathbb{R}^n \to \mathbb{R}^n$. It is clear that the flow map $X : I \times I \times \mathbb{R}^n \to \mathbb{R}^n$ is identically equal to the identity on $I \times I \times (\mathbb{R}^n \setminus \overline{\Omega'})$. Since (18) holds for any bounded open set $\Omega \Supset \Omega'$, then $X(t,s,\cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$. $\square$

Corollary 6.7. *Under the assumptions of Theorem D, we have, for every $t \in I$,*

$$\text{for almost every } s \in I$$
$$X(t,s,\cdot)_\# \mathcal{L}^n \llcorner \Omega_{(t,s)} = J_X(s,t,\cdot)\mathcal{L}^n \llcorner \Omega_{(s,t)} = \frac{1}{J_{X(t,s,X(s,t,\cdot))}} \mathcal{L}^n \llcorner \Omega_{(s,t)} \, , \tag{87}$$

where $J_X(t,s,\cdot) = \det(D_x X(t,s,\cdot)) \in L^{p/n}(\Omega_{(t,s)})$ is non-zero almost everywhere. Under the assumptions of Theorem E, then (87) holds for every $s, t \in I$ replacing both $\Omega_{(t,s)}$ and $\Omega_{(s,t)}$ by $\mathbb{R}^n$.

In fact, J_X is strictly positive, as we will show in the next corollary.

Proof. Since $X(t,s,\cdot)^{-1} = X(s,t,\cdot)$ is also in $W^{1,p}_{\mathrm{loc}}(\Omega_{(s,t)}; \mathbb{R}^n)$ with $p > 2n$, then both maps $X(t,s,\cdot)$ and $X(t,s,\cdot)^{-1}$ are differentiable almost everywhere by Lemma 2.4. Therefore, we have $J_X(t,s,x) \neq 0$ by Lemma 2.1. By Lemma 2.4, $X(t,s,\cdot)$ satisfies Lusin's (N) condition, and thus, by Lemma 2.2, the area formula holds, that is, (87) holds. Moreover, by the Laplace expansion of the determinant, one can easily prove, by induction on the order matrix and Hölder inequality, that $J_X(t,s,\cdot) = \det(D_x X(t,s,\cdot)) \in \mathrm{L}^{p/n}(\Omega_{(t,s)})$. The same argument applies under the assumptions of Theorem E. □

Corollary 6.8. *Under the assumptions of Theorem D, for every $(s,x) \in I \times \Omega$, if we set*

$$y(t) := D_x X(t,s,x), \qquad B(t) := (D_x b)(t, X(t,s,x)),$$
$$J(t) := \det(D_x X(t,s,x)), \qquad \beta(t) := \mathrm{div}_x b(t, X(t,s,x)) = \mathrm{trace}(B(t)),$$

then y and J are absolutely continuous solutions to the initial value problems

$$\begin{cases} \dot{y}(t) = B(t)y(t), \\ y(s) = \mathrm{Id}. \end{cases} \tag{88}$$

$$\begin{cases} \dot{J}(t) = \beta(t)J(t), \\ J(s) = 1. \end{cases} \tag{89}$$

Moreover, for almost every $(t,s,x) \in \mathscr{D}_b$ we have

$$D_x X(t,s,x) = \exp\left(\int_s^t D_x b(v, X(v,s,x))\,\mathrm{d}v\right), \tag{90}$$

$$J_X(t,s,x) = \exp\left(\int_s^t \mathrm{div}_x b(v, X(v,s,x))\,\mathrm{d}v\right). \tag{91}$$

In particular, $J_X > 0$ almost everywhere.

Proof. First, we claim that, given s, for almost every x the matrix B belongs to $\mathrm{L}^1(I_{(s,x)}; \mathbb{R}^{n^2})$, where $I_{(s,x)}$ is defined in Section 3.2. Indeed, the change of variables $z = X(t,s,x)$ and the identity $J_X(s,t,X(t,s,x)) = 1/J_X(t,s,x)$ give

$$\int_I \int_{\Omega_{(s,t)}} \|(D_x b)(t, X(t,s,x))\|\,\mathrm{d}x\,\mathrm{d}t = \int_I \int_{\Omega_{(t,s)}} \|(D_x b)(t,z)\| J_X(s,t,z)\,\mathrm{d}z\,\mathrm{d}t.$$

The latter integral is finite because $D_x b \in \mathrm{L}^q$ for all q and $J_X \in \mathrm{L}^{p/n}$. Therefore, the claim is true.

Second, using the same approximation of b as in the proof of Theorem D, we know that $D_x X^h \to D_x X$ weakly in $\mathrm{L}^p(\mathscr{D}_b)$, since $D_x X^h$ are uniformly bounded in $\mathrm{L}^p(\mathscr{D}_b)$.

Third, we see that the distributional derivative $\partial_t D_x X$ has the following form: for every $\phi \in C_c^\infty(\mathscr{D}_b; \mathbb{R}^n)$,

$$\begin{aligned}\partial_t D_x X[\phi] &= -\int\limits_{\mathscr{D}_b} D_x X \partial_t \phi \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x \\ &= -\lim_{\epsilon\to 0}\int\limits_{\mathscr{D}_b} D_x X_\epsilon \partial_t \phi \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x \\ &= \lim_{\epsilon\to 0}\int\limits_{\mathscr{D}_b} D_x b_\epsilon(t, X_\epsilon(t,s,x)) D_x X_\epsilon(t,s,x)\phi \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

Now, $X_\epsilon \to X$ uniformly on compact sets and $D_x b_\epsilon \to D_x b$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathscr{D}_b)$ for all q. So, using Hölder inequality and the Lebesgue dominated convergence theorem, we obtain that the limit above is

$$\partial_t D_x X[\phi] = \int\limits_{\mathbb{R}^{2+n}} D_x b(t, X(t,s,x)) D_x X(t,s,x)\phi \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x.$$

In other words, $\partial_t D_x X = D_x b(t, X(t,s,x)) D_x X(t,s,x)$. This shows that y is solution to the Cauchy system (88). Since B is integrable, then we get (90) by integrating this Cauchy system.

Finally, the validity of (89) follows in a standard way from (88) and (89) implies (91). □

Corollary 6.9. *Under the assumptions of Theorem D, for every $1 < r < p - n$ and for every $s, t \in \overline{I}$, we have that*

$$X(t,s,\cdot) \in W^{1,r}(\Omega_{(t,s)}; \mathbb{R}^n). \tag{92}$$

Proof. The proof is an improvement of (15) through an application of Corollary 6.8, Lemma 2.7 and Proposition 2.8. Indeed, for every $1 < r < p - n$ there exists $0 < q < p^2 - \frac{p^2}{p-n}$ such that

$$r = \frac{p^2(p-n)}{q(p-n) + p^2}, \qquad \text{that is,} \qquad \frac{r}{p-r} = \left(\frac{q}{p} + \frac{n}{p-n}\right)^{-1}.$$

So, given $s, t \in I$, (15) implies that there exists $u \in I$ such that

$$X(s,t,\cdot) = X(s,u,X(u,t,\cdot))$$

and both maps $X(s,u,\cdot)$ and $X(u,t,\cdot)$ belong to $W^{1,p}$ on their domains, with non-zero Jacobian by Corollary 6.8. We then apply Lemma 2.7 and Proposition 2.8 to prove that their composition is of class $W^{1,r}$ on its domain. □

Corollary 6.10. *Under the assumptions of Theorem D, we have*

$$X \in W^{1,p}(\mathscr{D}_b; \mathbb{R}^n).$$

Proof. We know that X is continuous and, from (16), we have that $D_x X \in \mathrm{L}^p(\mathscr{D}_b; \mathbb{R}^{n^2})$. Moreover, by the identity $X(t,s,x) = x + \int_s^t b(v, X(v,s,x))\,\mathrm{d}v$, we have

$$\partial_t X(t,s,x) = b(t, X(t,s,x)),$$

and thus $\partial_t X$ is continuous. Finally, by differentiating with respect to s the semigroup identity (33) in the form $X(t,v,x) = X(t,s,X(s,v,x))$, we get

$$\partial_s X(t,s,X(s,v,x)) + D_x X(t,s,X(s,v,x)) b(s, X(s,v,x)) = 0,$$

for all $t, s, v \in \mathbb{R}$ and $x \in \mathbb{R}^n$ for which the expression makes sense. Therefore, $\partial_s X(t,s,y) = -D_x X(t,s,y) b(s,y)$ and so $\partial_s X \in \mathrm{L}^p(\mathscr{D}_b; \mathbb{R}^n)$. □

Remark 6.11. Corollary 6.10 improves [13, Theorem 4] and [8, Corollary 1.8], by dropping assumption (3), that is $\mathrm{div}_x b \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{L}^\infty(\mathbb{R}^n))$. Notice that, if $n = 1$, assumption (3) reduces to the classical Lipschitz condition of b with respect to x, uniformly in t. Corollary 6.10 also applies to a non Lipschitz one-dimensional vector field b (see Example 8.3 below). ♦

Remark 6.12. Corollary 6.10 looks almost sharp. Indeed, it was proved in [26] that no Sobolev regularity can be expected for the flow, when assuming only that $b \in \mathrm{L}^1(I; W^{1,p}(\mathbb{R}^n; \mathbb{R}^n))$ for all finite $p \in [1, +\infty)$, even when b is compactly supported and divergence-free. See also Remark 5.1. ♦

7. Applications to PDEs

In this section we apply the Sobolev regularity of flows for getting the representation of weak solutions of the Cauchy problems both for the transport and continuity equations.

Proof of Theorem F. By [12, Theorem 1], we can infer the uniqueness of weak solutions $u \in L^\infty((0,T); L^\infty(\mathbb{R}^n))$ for (2), provided that the spatial derivative of b satisfies a sub-exponential summability and $\bar{u} \in L^\infty(\mathbb{R}^n)$. Therefore we have only to show that the function v in (20) is a weak solution of (2), that is,

$$\int_0^T \int_{\mathbb{R}^n} v\,(\partial_t \varphi + \mathrm{div}(b\,\varphi))\,dt dx = -\int_{\mathbb{R}^n} \bar{u}\,\varphi(0,\cdot)\,\mathrm{d}x \tag{93}$$

for each $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^n)$. We divide the proof in two steps.

1st step. Let us assume that $\bar{u} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $b_\epsilon : I \times \mathbb{R}^n \to \mathbb{R}^n$ be the family of vector fields defined as

$$b_\epsilon(t,x) := (\tilde{b}_\epsilon(t,\cdot) * \rho_\epsilon)(x) \text{ if } (t,x) \in I \times \mathbb{R}^n\,,$$

where

$$\tilde{b}_\epsilon(t,x) := (\tilde{b}_{\epsilon,1}(t,x), \ldots, \tilde{b}_{\epsilon,n}(t,x))\,,$$

$$\tilde{b}_{\epsilon,i}(t,x) := \max\{\{\min\{b_i(t,x), 1/\epsilon\}, -1/\epsilon\} \text{ if } (t,x) \in I \times \mathbb{R}^n,\ i = 1,\ldots,n,$$

and $(\rho_\epsilon)_\epsilon$ denotes a family of mollifiers depending on the space variable x. Then, standard properties of convolutions yield

$$|b_\epsilon(t,x)| \le \frac{\sqrt{n}}{\epsilon} \text{ for each } (t,x) \in I \times \mathbb{R}^n, \tag{94}$$

$$b_\epsilon(t,\cdot) \in C^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ and } D_x b_\epsilon(t,x) = \int_{\mathbb{R}^n} D_x\rho_\epsilon(x-y)\,\tilde{b}_\epsilon(t,y)\,dy \quad \forall (t,x) \in I \times \mathbb{R}^n, \tag{95}$$

$$b_\epsilon(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n \text{ is Lipschitz continuous, uniformly with respect to } t \in I, \tag{96}$$

$$b_\epsilon \to b \quad \text{in } L^1(I; L^1(\Omega;\mathbb{R}^n)) \text{ as } \epsilon \to 0, \text{ whenever } \Omega \Subset \mathbb{R}^n. \tag{97}$$

Then, from (96) we get that b_ϵ is well-posed and the flow maps X_ϵ associated to b_ϵ is globally defined, that is, $X_\epsilon : \overline{I} \times \mathbb{R}^n \to \mathbb{R}^n$ and is Lipschitz regular. Define

$$v_\epsilon(t,x) = \bar{u}(X_\epsilon(0,t,x)).$$

By (97), Lemma 3.5 and the subsequent remark, $X_\epsilon \to X$ uniformly on compact subsets of $\overline{I} \times \mathbb{R}^n$. Since $\bar{u}$ is assumed to be continuous, we obtain that $v_\epsilon \to v$ uniformly on compact sets of $\overline{I} \times \mathbb{R}^n$. Moreover, since $\bar{u} \in L^\infty(\mathbb{R}^n)$, by the dominated convergence theorem, we can also assume that $v_\epsilon \to v$ in L^2 on each compact set of $\overline{I} \times \mathbb{R}^n$.

By the classical Cauchy-Lipschitz theory (see, for instance, [5, Section 2]), it is well-known that v_ϵ is a classical solution to (2) with b_ϵ in place of b and, in particular, a weak solution, i.e.,

$$\int_0^T \int_{\mathbb{R}^n} v_\epsilon\,(\partial_t\varphi + \operatorname{div}(b_\epsilon\varphi))\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^n} \bar{u}(x)\varphi(0,x)\,\mathrm{d}x,$$

for each $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^n)$. Since φ has compact support in $[0,T) \times \mathbb{R}^n$, it is easy to check that $\operatorname{div}(b_\epsilon\varphi) \to \operatorname{div}(b\varphi)$ in L^2. Hence

$$\lim_{\epsilon\to 0^+} \int_0^T \int_{\mathbb{R}^n} v_\epsilon\,(\partial_t\varphi + \operatorname{div}(b_\epsilon\varphi))\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^n} v\,(\partial_t\varphi + \operatorname{div}(b\varphi))\,\mathrm{d}x\,\mathrm{d}t,$$

that is, v is a weak solution to (2).

2nd step. Let $\bar{u} \in \mathrm{L}^\infty(\mathbb{R}^n)$ and, by mollification in $\mathbb{R}^n$, let $\bar{u}_j \in \mathrm{L}^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, with $j \in \mathbb{N}$, be a sequence of functions that converges to $\bar{u}$ almost everywhere and such that $\|\bar{u}_j\|_{\mathrm{L}^\infty} \le \|\bar{u}\|_{\mathrm{L}^\infty}$ for every j.

Define

$$v_j(t,x) = \bar{u}_j(X(0,t,x)).$$

By Theorem B, for every t the homeomorphism $X(0,t,\cdot)$ satisfies the Lusin (N) condition, and therefore $v_j(t,\cdot) \to v(t,\cdot)$ almost everywhere in $\mathbb{R}^n$. Since this is true for every t, we get that $v_j \to v$ almost everywhere in $[0,T]\times\mathbb{R}^n$. Moreover, it is clear that $\|v_j\|_{\mathrm{L}^\infty} \le \|\bar u\|_{\mathrm{L}^\infty}$ for all j.

By the previous step, we have, for all j,

$$\int_0^T \int_{\mathbb{R}^n} v_j\,(\partial_t\varphi + \operatorname{div}(b\varphi))\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^n} \bar u_j(x)\varphi(0,x)\,\mathrm{d}x\,,$$

for each $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^n)$. We can now apply the Dominated Convergence Theorem and pass to the limit $j\to\infty$ to obtain that v is a weak solution to (2). □

Corollary 7.1 *(Sobolev regularity of the solutions of the transport equation). Let $b \in C^0(I\times\mathbb{R}^n;\mathbb{R}^n)$ as in Theorem E and let X be the flow of b. Let $p > 2n$ and $1 \le \tilde q \le q < \infty$ be such that*

$$\frac{\tilde q}{q-\tilde q} = \left(\frac{q}{p} + \frac{n}{p-n}\right)^{-1}, \quad i.e.,\quad \tilde q = \frac{pq(p-n)}{q(p-n)+p^2}\,. \tag{98}$$

If $\bar u \in \mathrm{L}^\infty(\mathbb{R}^n)\cap W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$, then the function v in (20) satisfies

$$v \in L^\infty([0,T]; W^{1,\tilde q}_{\mathrm{loc}}(\mathbb{R}^n))\,.$$

Proof. By Theorem E, the function $\Psi := X(0,t,\cdot)$ and its inverse are in $W^{1,p}_{\mathrm{loc}}$ with $p > 2n$. By Lemma 2.7 and Corollary 6.8, Ψ is of finite distortion and $K_q^\Psi \in \mathrm{L}^r_{\mathrm{loc}}$ for $r = \left(\frac{q}{p} + \frac{n}{p-n}\right)^{-1}$. By Proposition 2.8, the composition operator T_Ψ is continuous from $W^{1,q}_{\mathrm{loc}}(\Omega_2)$ to $W^{1,\tilde q}_{\mathrm{loc}}(\Omega_1)$. Since $v(t,\cdot) = T_\Psi(\bar u)$, the proof is concluded. □

Remark 7.2. Notice that the propagation of regularity, in the spirit of Corollary 7.1, may fail below the exponential summability of $D_x b$, even though $\bar u \in C_c^\infty(\mathbb{R}^n)$. Indeed, in [8, Theorem 2.1], the authors constructed a divergence-free vector field $b : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ $(n \ge 2)$ satisfying the subexponential summability condition (12), and a weak bounded, compactly supported solution $u(t,x)$ of (2) such that $\bar u := u(0,\cdot) \in C_c^\infty(\mathbb{R}^n)$ but $u(t,\cdot) \notin \dot W^{s,p}(\mathbb{R}^n)$ for all $t > 0$, $s > 0$ and $p \ge 1$, where $\dot W^{s,p}(\mathbb{R}^n)$ denotes the so-called *homogeneous Sobolev space*. The example is based on the work [1]. Let us recall that when $s = 1$ and $1 < p < \infty$, then $\dot W^{1,p}(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ coincides with the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$ (see [1, Section 2]).

Let us point out that, although b satisfies the hypothesis of Theorem F, $u(t,\cdot) \notin W^{1,p}(\mathbb{R}^n)$ for each $t \in (0,\infty)$ and $p \in (1,\infty)$. This implies that the flow of b has not Sobolev regularity $W^{1,p}$ for some $p > n$, otherwise the same proof of Corollary 7.1 could be repeated. Our Example 8.2 shows the same phenomenon. Notice that we don't know whether $u(t,\cdot) \notin W^{1,1}(\mathbb{R}^n)$. ♦

Remark 7.3. Corollary 7.1 shows that the Sobolev regularity of the flow X implies Sobolev regularity for solutions of the transport equation in the form (20). We notice that the converse implication is almost true. Indeed, suppose that for every $1 \le \tilde q \le q < \infty$ satisfying (98), and for every $\bar u \in \mathrm{L}^\infty(\mathbb{R}^n)\cap W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$, the function v in (20) satisfies $v \in L^\infty((-T,T); W^{1,\tilde q}_{\mathrm{loc}}(\mathbb{R}^n))$.

Then, if we take $\bar{u} = \phi x_j$, where $\phi \in C_c^\infty(\mathbb{R}^n)$ and x_j is the j-th coordinate function, we see that $x_j(X(0,t,\cdot)) \in W^{1,\tilde{q}}_{\mathrm{loc}}(\mathbb{R}^n)$ for almost every t and for all $\tilde{q} < p$. ♦

Before proving Theorem G, let us introduce some notation and recall some existence and uniqueness results as concern as the Cauchy problem (4) for the continuity equation. It is well-known that the non trivial issue in the well-posedness turns out to be the uniqueness of weak solutions. Nonnegative measure-valued solutions of the continuity equation are uniquely determined by their initial condition if the characteristic ODE associated to the vector field b has a unique solution (see [5, Theorem 9]). A partial extension of this result to signed measures was given in [3], under a quantitative two-sided Osgood condition on b. For the proof of Theorem G we will use the following more general result.

Theorem 7.4 *([14, Theorem 3]). Let $b : (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ be such that the following two conditions hold: There exists a continuous and nondecreasing function $G : [0,\infty) \to [0,\infty)$ satisfying $\int_r^\infty \frac{ds}{G(s)} ds = \infty$ for some $r > 0$ such that*

$$\sup_{x \in \mathbb{R}^n} \frac{|b(t,x)|}{G(|x|)} \in L^\infty(0,T), \tag{99}$$

and there exists a continuous and nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ satisfying $\int_0^r \frac{ds}{\omega(s)} ds = \infty$ for some $r > 0$ such that

$$\sup_{x,\, y \in B(0,R),\, x \neq y} \frac{|b(t,x) - b(t,y)|}{\omega(|x-y|)} \in L^\infty(0,T)\,, \tag{100}$$

for any radius $R > 0$. Then for any $\rho_0 \in \mathcal{M}(\mathbb{R}^n)$ there exists a unique solution ρ in $L^1((0,T);\mathcal{M}(\mathbb{R}^n))$ for the Cauchy problem (4), that is,

$$\int_0^T \int_{\mathbb{R}^n} \left(\partial_t \varphi(t,x) + \langle b(t,x), D_x \varphi(t,x)\rangle\right) d\rho_t(x) dt = -\int_{\mathbb{R}^n} \varphi(0,x)\, d\rho_0(x) \tag{101}$$

for each $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^n)$, with $\rho_t := \rho(t,\cdot)$. Moreover this solution representable as $\rho_t = X(t,0,\cdot)_\# \rho_0$, where $X(t,s,\cdot)$ denotes the flow of the vector field b.

Remark 7.5. As pointed out in [14, p. 49], by assuming $\rho \in L^\infty((0,T);\mathcal{M}(\mathbb{R}^n))$, the uniqueness still holds if one replaces $L^\infty(0,T)$ by $L^1(0,T)$ in both (99) and (100). ♦

We will also use the following abstract area-type formula, which can be proved in a standard way.

Lemma 7.6. *Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, $\mu : \mathcal{M}_n \to [0,\infty]$ and $\bar{\rho} : \mathbb{R}^n \to [-\infty,\infty]$ be a homeomorphism, a Radon measure on $\mathbb{R}^n$ and a function in $L^1(\mathbb{R}^n,\mu)$, respectively. Suppose that $\Psi_\#\mu$ is absolutely continuous with respect to μ and denote*

$$w := \frac{d\Psi_\#\mu}{d\mu} \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mu)\,.$$

Then

$$\Psi_\#(\bar\rho\mu) = \bar\rho(\Psi^{-1})\, w\, \mu\,.$$

In particular it also follows that

$$\bar\rho(\Psi^{-1})\, w \in L^1(\mathbb{R}^n, \mu)\,.$$

Proof of Theorem G. (i) In view of the use of Remark 7.5, let us begin to show that (99) and (100) hold with $L^1((0,T))$ in place of $L^\infty((0,T))$. It is immediate that, by choosing

$$G(s) := s\, \log^+ s \text{ if } s \geq 0\,,$$

(99) in the weaker form follows from (11). As for (100), observe that, from (59), it follows that

$$\int_0^T \int_{B(0,R)} \exp\left(\frac{\|D_x b\|}{\log^+ \|D_x b\|}\right) dxdt < +\infty \text{ for each } R > 0\,. \tag{102}$$

Let $\theta : [0,\infty) \to [0,\infty)$ be the function defined as

$$\theta(s) := \begin{cases} \exp\left(\dfrac{\bar s}{\log \bar s}\right) & \text{if } 0 \leq s \leq \bar s\,; \\ \exp\left(\dfrac{s}{\log s}\right) & \text{if } s \geq \bar s\,. \end{cases}$$

Then, by Proposition 4.6, θ satisfies the assumption of Theorem A, for $\bar s$ large enough. In particular, we can deduce, as in the proof of Theorem A, the estimate

$$\sup_{x,\, y \in B(0,R),\, x \neq y} \frac{|b(t,x) - b(t,y)|}{\omega(|x-y|)} \leq \varphi(t)$$

where ω and φ are the modulus of continuity defined in (46) and the function in (47), respectively. From (102), it follows that $\varphi \in L^1(0,T)$. Thus (100) in the weaker form follows, too. Hence, the uniqueness of weak solutions for (4) in $L^\infty((0,T); \mathcal{M}(\mathbb{R}^n))$ is granted by Theorem 7.4 and Remark 7.5.

The existence of weak solutions for (4) can be proved as in [3,14]. More precisely, if $X : [0,T] \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the flow associated to b and ρ is defined as in (21), then one can verbatim repeat the same proof, by showing that ρ is a weak solution of (4) in $L^\infty((0,T); \mathcal{M}(\mathbb{R}^n))$, that is, (101) holds.

If now $\bar\rho \in L^1(\mathbb{R}^n)$, we can apply the previous result with initial value measure $\bar\rho\, \mathcal{L}^n$ and we get that the unique weak solution for (4) is given by

$$\rho_t = X(t,0,\cdot)_\#(\bar\rho\, \mathcal{L}^n) \text{ for each } t \in [0,T]\,.$$

From Theorem B, $\rho_t = X(t,0,\cdot)_\# \mathcal{L}^n$ is absolutely continuous with respect to $\mathcal{L}^n$. Therefore, by Lemma 7.6 with $\Psi = X(t,0,\cdot)$ and $\mu = \mathcal{L}^n$, (22) follows.

(ii) Observe that in this case the assumptions of Theorem B are still satisfied. Thus, by applying the previous claim (i), it follows that (22) holds. We have only to prove that

$$J_{X,t} = J_{X(0,t,\cdot)} \text{ a.e. in } \mathbb{R}^n, \text{ for each } t \in [0,T] \tag{103}$$

in order to conclude the proof. From Theorem E, the flow $X(t,s,\cdot) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ with $p > n$, for each t, $s \in [0,T]$. In particular, from Corollary 6.7, we get that

$$X(t,0,\cdot)_\#(\mathcal{L}^n) = J_{X(0,t,\cdot)}\mathcal{L}^n\,.$$

Thus (103) follows. □

8. Examples

Most of the examples we provide are in low dimension. However, notice that we can always extend the examples to higher dimensions as follows. If $b(t,x)$ is a vector field on $\mathbb{R}^n$ and $m > 0$, then define the vector field $h(t,(x,y)) := (b(t,x),0)$ on $\mathbb{R}^n \times \mathbb{R}^m$. If $X(t,s,x)$ is the flow of b, then $Z(t,s,(x,y)) := (X(t,s,x),y)$ is the flow of h. Notice that $Z(t,s) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if $X(t,s) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$.

8.1. Well-posedness does not imply absolute continuity of the flow

Here we show that *there is a vector field $b \in C^0(\mathbb{R}\times\mathbb{R})$ that is well-posed, but whose flow on $\mathbb{R}$ is not absolutely continuous.* This shows that Theorems B and C are not a direct consequence only of the well-posedness.

A first example is given as follows. It is well-known that a quasisymmetric homeomorphism $\Psi : \mathbb{R} \to \mathbb{R}$ need not be absolutely continuous (see, for instance, [24, p. 107]). Moreover, by [31, p. 250], for each quasisymmetric homeomorphism $\Psi : \mathbb{R} \to \mathbb{R}$ there is a well-posed continuous vector field $b : \mathbb{R}^2 \to \mathbb{R}$ such that, if $X(t,x)$ denote the (unique) flow $X(t,0,x)$ associated to b, then, for some $t > 0$, $X(t,\cdot) = \Psi$. We conclude that there exists a well-posed vector field b whose flow is not absolutely continuous at some time.

We provide a second more explicit example from [30, Section 8]. (Be aware that what is denoted by t in [30] is actually the spatial variable.) In the plane $\mathbb{R}\times\mathbb{R}$ with coordinates (t,x), consider the family of parabolas of the form $x = a(s)t^2 + s$, where $a : \mathbb{R} \to \mathbb{R}$ is 0 for $s \le 0$, 1 for $s \ge 1$ and coincides with the Cantor Staircase Function on $[0,1]$. These parabolas foliate the plane and are the integral curves of a vector field of the form $(1, b(t,x))$. One can show that $b \in W^{1,2}_{\text{loc}}(\mathbb{R}) \cap C^0(\mathbb{R}) \cap \text{Lip}_{\text{loc}}(\mathbb{R}\setminus\{0\})$ and that $(1,b)$ is a well-posed non-autonomous vector field on $\mathbb{R}^2$.[2] However, the flow X of b is not absolutely continuous. Indeed, for $t \neq 0$, the function $X(t,0,\cdot)$ maps the Cantor set to a set of positive measure: with the notation of [30, page 141], if $C \subset \mathbb{R}$ is the Cantor set, then $\mathcal{L}^1(X(t,0,C)) = \mathcal{L}^1(C_t) = t^2/2$.

[2] The uniqueness of integral curves is clear, because b is locally Lipschitz outside the axis $\mathbb{R}\times\{0\}$ and its unique integral curves are then the parabolas $x = a(s)t^2 + s$.

8.2. Sub-exponential condition does not imply high Sobolev regularity

Here we show that *there is a vector field $b \in C^0(\mathbb{R})$ that satisfies* (12)*, but the flow of b is no better that $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$*. The same example allows us to show that *the upper bound on β in Proposition 4.6 is necessary for the well-posedness.*

Given $\alpha \geq 1$, let $b : \mathbb{R} \to \mathbb{R}$ be the autonomous vector field on $\mathbb{R}$ given by

$$b(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x \geq e^{-e}, \\ \beta x \log \frac{1}{x} (\log\log \frac{1}{x} - 1)^{\alpha} & \text{if } 0 < x < e^{-e}. \end{cases}$$

We clearly have that $b \in C^0(\mathbb{R}) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ for all $p \in [1, \infty)$, and that $\mathtt{spt}(b) = [0, e^{-e}]$. Moreover,

$$D_x b(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > e^{-e}, \\ \beta \log \frac{1}{x} (\log\log \frac{1}{x})^{\alpha} (1 + R(x)) & \text{if } 0 < x < e^{-e}, \end{cases}$$

where R is a remainder satisfying $\lim_{x \to 0^+} R(x) = 0$. It is easy to see that

$$\exp\left(\frac{|D_x b|}{\left(\log^+ |D_x b|\right)^{\alpha}} \right) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}). \tag{104}$$

Let us consider now the case $\alpha = 1$. Then b satisfies (12) and thus, by Theorem C, b is well-posed and the unique flow X of b satisfies $X(t, s) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ for every $t, s \in \mathbb{R}$. In fact, one can explicitly compute X by separation of variables: for every $t, s \in \mathbb{R}$,

$$X(t, s, x) = \begin{cases} x & \text{if } x \leq 0, \\ \exp\left(-e \left(\frac{1}{e} \log \frac{1}{x}\right)^{k(t,s)}\right) & \text{if } 0 < x < e^{-e}, \\ x & \text{if } x \geq e^{-e}, \end{cases}$$

where $k(t, s) = \exp(s - t)$. The spatial derivative of X for $0 < x < e^{-e}$ is

$$D_x X(t, s, x) = \exp\left(-e \left(\frac{1}{e} \log \frac{1}{x}\right)^{k(t,s)}\right) k(t, s) \exp\left(-k(t, s) + 1\right) \left(\log \frac{1}{x}\right)^{k(t,s)-1} \frac{1}{x}.$$

If we now choose $s = 0$ and $t > 0$, then it is easy to see that, for every $p > 1$,

$$\int_0^{e^{-e}} |D_x X(t, 0, x)|^p dx = +\infty;$$

in particular, $X(t, 0, \cdot) \notin W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ whenever $p > 1$. Notice also that $X(t, 0, \cdot) \notin C^{0,\gamma}_{\mathrm{loc}}(\mathbb{R}_x)$ for all $\gamma \in (0, 1)$.

In the other case, when $\alpha > 1$, the vector field b still satisfies (104) and (8) with Θ of the form as in (10) (but with $\beta > 1$). However, b is not well-posed any more. For instance, it is easy to see that both functions $\gamma_1 \equiv 0$ and

$$\gamma_2(t) = \begin{cases} 0 & \text{if } t \le 0, \\ \exp\left(-\exp\left(-\left(1+\frac{1}{(\alpha-1)t}\right)^{\frac{1}{\alpha-1}}\right)\right) & \text{if } t > 0, \end{cases}$$

satisfy $\gamma' = b(\gamma)$ and $\gamma(0) = 0$.

8.3. Exponential summability does not imply the divergence in BMO and it is only sufficient for the Sobolev regularity

Here we show that *there is a vector field b satisfying the assumptions of Theorem E such that* $\mathrm{div}_x b \notin BMO(\mathbb{R})$. On the other hand *there exist $\ell > 0$ and an exponent $p \in (1, \infty)$ such that, for each $t, s \in [-\ell/2, \ell/2]$, $X(t, s, \cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$, even though* $\exp(\beta|D_x b|) \notin L^1_{\mathrm{loc}}(\mathbb{R})$ *if* $\beta := \frac{\ell p^2}{p-1}$.

Let $b : \mathbb{R} \to \mathbb{R}$ be the autonomous vector field on $\mathbb{R}$ given by

$$b(x) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x \ge e, \\ x \log \frac{e}{x} & \text{if } 0 < x < e. \end{cases}$$

It is easy to see that $b \in C^0(\mathbb{R}) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ for every $p \in [1, \infty)$, and that $\mathtt{spt}(b) = \mathbb{R} \times [0, e]$. Moreover,

$$D_x b(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > e, \\ \log \frac{1}{x} & \text{if } 0 < x < e. \end{cases}$$

Let $I := (-\ell/2, \ell/2)$ and $\Omega := (-1, 3)$. Observe that, in this case, condition (6) amounts to

$$\beta := \frac{\ell\, p^2}{p-1} < 1. \tag{105}$$

It is also easy to check (see [22, Example 7.1.4]) that

$$D_x b(\cdot) \notin BMO(\mathbb{R})\,.$$

Using (105) and Theorem E, one easily proves that $b : I \times \mathbb{R} \to \mathbb{R}$ is well-posed. In fact, we can integrate b by separation of variables and obtain

$$X(t, s, x) = \begin{cases} x & \text{if } x \le 0, \\ e\left(\frac{x}{e}\right)^{k(t,s)} & \text{if } 0 < x < e, \\ x & \text{if } x \ge e, \end{cases}$$

where $k(t, s) = \exp(s - t)$. Moreover

$$D_x X(t,s,x) = \begin{cases} 1 & \text{if } x < 0, \\ k(t,s)\left(\frac{x}{e}\right)^{k(t,s)-1} & \text{if } 0 < x < e, \\ 1 & \text{if } x > e. \end{cases}$$

It follows that $X(t,s,\cdot) \in W^{1,q}_{\mathrm{loc}}(\mathbb{R})$ if and only if

$$(\exp(s-t)-1)q > -1, \quad \text{i.e.,} \quad \begin{cases} q < \frac{1}{1-\exp(s-t)} & \text{if } s < t, \\ q \geq 1 & \text{if } s > t. \end{cases} \tag{106}$$

This example shows how the Sobolev regularity of $X(t,0,\cdot)$ can deteriorate with time. It also shows that condition (6) is only sufficient: indeed, if $\ell \geq 1/4$ then (105) is not satisfied by any p, but $X(t,s,\cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$, for each $t,\ s \in [-\ell/2, \ell/2]$, if $1 < p < \frac{1}{1-\exp(-\ell)}$. Thus Theorem E does not apply. Notice also that, by (106), if $t > 0$, $X(t,0,\cdot) \notin W^{1,q}_{\mathrm{loc}}(\mathbb{R})$ for any $q \geq \frac{1}{1-\exp(-t)}$.

Data availability

No data was used for the research described in the article.

References

[1] G. Alberti, G. Crippa, A.L. Mazzucato, Loss of regularity for the continuity equation with non-Lipschitz velocity field, Ann. PDE 5 (1) (2019) 9, 19.
[2] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Invent. Math. 158 (2) (2004) 227–260.
[3] L. Ambrosio, P. Bernard, Uniqueness of signed measures solving the continuity equation for Osgood vector fields, Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. 19 (3) (2008) 237–245.
[4] L. Ambrosio, M. Colombo, A. Figalli, Existence and uniqueness of maximal regular flows for non-smooth vector fields, Arch. Ration. Mech. Anal. 218 (2) (2015) 1043–1081.
[5] L. Ambrosio, G. Crippa, Continuity equations and ODE flows with non-smooth velocity, Proc. R. Soc. Edinb., Sect. A 144 (6) (2014) 1191–1244.
[6] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.
[7] L. Ambrosio, S. Golo Nicolussi, F. Cassano Serra, Optimal C^∞-approximation of functions with exponentially or sub-exponentially integrable derivative, Calc. Var. Partial Differ. Equ. 62 (1) (2023) 24, 21.
[8] E. Brué, Q.-H. Nguyen, Sobolev estimates for solutions of the transport equation and ODE flows associated to non-Lipschitz drifts, Math. Ann. 380 (1–2) (2021) 855–883.
[9] F. Cipriano, A.B. Cruzeiro, Flows associated with irregular $\mathbb{R}^d$-vector fields, J. Differ. Equ. 219 (1) (2005) 183–201.
[10] A. Clop, R. Jiang, J. Mateu, J. Orobitg, A note on transport equation in quasiconformally invariant spaces, Adv. Calc. Var. 11 (2) (2018) 193–202.
[11] A. Clop, R. Jiang, J. Mateu, J. Orobitg, Flows for non-smooth vector fields with sub-exponentially integrable divergence, J. Differ. Equ. 261 (2) (2016) 1237–1263.
[12] A. Clop, R. Jiang, J. Mateu, J. Orobitg, Linear transport equations for vector fields with subexponentially integrable divergence, Calc. Var. Partial Differ. Equ. 55 (1) (2016) 21, 30.
[13] A. Clop, H. Jylhä, Sobolev regular flows of non-Lipschitz vector fields, J. Differ. Equ. 266 (8) (2019) 4544–4567.
[14] A. Clop, H. Jylhä, J. Mateu, J. Orobitg, Well-posedness for the continuity equation for vector fields with suitable modulus of continuity, J. Funct. Anal. 276 (1) (2019) 45–77.
[15] M. Colombo, G. Crippa, S. Spirito, Logarithmic estimates for continuity equations, Netw. Heterog. Media 11 (2) (2016) 301–311.
[16] B. Desjardins, A few remarks on ordinary differential equations, Commun. Partial Differ. Equ. 21 (11–12) (1996) 1667–1703.
[17] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (3) (1989) 511–547.

[18] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, revised, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015, pp. xiv+299.
[19] A.F. Filippov, Differential Equations with Discontinuous Righthand Sides, Mathematics and Its Applications (Soviet Series), vol. 18, Kluwer Academic Publishers Group, Dordrecht, 1988, pp. x+304. Translated from the Russian.
[20] R. Gariepy, W. Ziemer, Modern Real Analysis, PWS Pub., 1995.
[21] P. Goldstein, P. Hajłasz, A measure and orientation preserving homeomorphism with approximate Jacobian equal -1 almost everywhere, Arch. Ration. Mech. Anal. 225 (1) (2017) 65–88.
[22] L. Grafakos, Modern Fourier Analysis, third, Graduate Texts in Mathematics, vol. 250, Springer, New York, 2014, pp. xvi+624.
[23] P. Hartman, Ordinary Differential Equations, Classics in Applied Mathematics, vol. 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002, pp. xx+612.
[24] J. Heinonen, Lectures on Analysis on Metric Spaces, Universitext, Springer-Verlag, New York, 2001, pp. x+140.
[25] S. Hencl, P. Koskela, Lectures on Mappings of Finite Distortion, Lecture Notes in Mathematics, vol. 2096, Springer, Cham, 2014, pp. xii+176.
[26] P.-E. Jabin, Critical non-Sobolev regularity for continuity equations with rough velocity fields, J. Differ. Equ. 260 (5) (2016) 4739–4757.
[27] R. Jiang, K. Li, J. Xiao, Flow with $A_\infty(\mathbb{R})$ density and transport equation in BMO($\mathbb{R}$), Forum Math. Sigma 7 (2019) e43.
[28] C. Le Bris, P.-L. Lions, Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications, Ann. Mat. Pura Appl. (4) 183 (1) (2004) 97–130.
[29] P.B. Mucha, Transport equation: extension of classical results for $\operatorname{div} b \in \mathrm{BMO}$, J. Differ. Equ. 249 (8) (2010) 1871–1883.
[30] S. Nicolussi, F. Serra Cassano, The Bernstein problem for Lipschitz intrinsic graphs in the Heisenberg group, Calc. Var. Partial Differ. Equ. 58 (4) (2019) 141, 28.
[31] H.M. Reimann, Ordinary differential equations and quasiconformal mappings, Invent. Math. 33 (3) (1976) 247–270.
[32] E. Zuazua, Log-Lipschitz regularity and uniqueness of the flow for a field in $(W^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$, C. R. Math. Acad. Sci. Paris 335 (1) (2002) 17–22.