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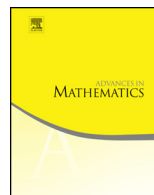
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# Vertical projections in the Heisenberg group via cinematic functions and point-plate incidences <sup>☆</sup>

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## ABSTRACT

Let  $\{\pi_e: \mathbb{H} \rightarrow \mathbb{W}_e : e \in S^1\}$  be the family of vertical projections in the first Heisenberg group  $\mathbb{H}$ . We prove that if  $K \subset \mathbb{H}$  is a Borel set with Hausdorff dimension  $\dim_{\mathbb{H}} K \in [0, 2] \cup \{3\}$ , then

$$\dim_{\mathbb{H}} \pi_e(K) \geq \dim_{\mathbb{H}} K$$

for  $\mathcal{H}^1$  almost every  $e \in S^1$ . This was known earlier if  $\dim_{\mathbb{H}} K \in [0, 1]$ .

The proofs for  $\dim_{\mathbb{H}} K \in [0, 2]$  and  $\dim_{\mathbb{H}} K = 3$  are based on different techniques. For  $\dim_{\mathbb{H}} K \in [0, 2]$ , we reduce matters to a Euclidean problem, and apply the method of cinematic functions due to Pramanik, Yang, and Zahl.

To handle the case  $\dim_{\mathbb{H}} K = 3$ , we introduce a point-line duality between horizontal lines and conical lines in  $\mathbb{R}^3$ . This allows us to transform the Heisenberg problem into a point-plate incidence question in  $\mathbb{R}^3$ . To solve the latter, we apply a Kakeya inequality for plates in  $\mathbb{R}^3$ , due to Guth, Wang, and

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Zhang. This method also yields partial results for Borel sets  $K \subset \mathbb{H}$  with  $\dim_{\mathbb{H}} K \in (5/2, 3)$ .

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**1. Introduction**

Fix  $e \in S^1 \times \{0\} \subset \mathbb{H}$ , and consider the *vertical plane*  $\mathbb{W}_e := e^\perp$  in the first Heisenberg group  $\mathbb{H}$ , see Section 2 for the definitions. Every point  $p \in \mathbb{H}$  can be uniquely decomposed as  $p = w \cdot v$ , where

$$w \in \mathbb{W}_e \quad \text{and} \quad v \in \mathbb{L}_e := \text{span}(e).$$

This decomposition gives rise to the *vertical projection*  $\pi_e := \pi_{\mathbb{W}_e}: \mathbb{H} \rightarrow \mathbb{W}_e$ , defined by  $\pi_e(p) := w$ . A good way to visualise  $\pi_e$  is to note that the fibres  $\pi_e^{-1}\{w\}$ ,  $w \in \mathbb{W}_e$ , coincide with the horizontal lines  $w \cdot \mathbb{L}_e$ . These lines foliate  $\mathbb{H}$ , as  $w$  ranges in  $\mathbb{W}_e$ , but are not parallel. Thus, the projections  $\pi_e$  are non-linear maps with linear fibres. For example, in the special cases  $e_1 = (1, 0, 0)$  and  $e_2 = (0, 1, 0)$  we have the concrete formulae

$$\pi_{e_1}(x, y, t) = \left(0, y, t + \frac{xy}{2}\right) \quad \text{and} \quad \pi_{e_2}(x, y, t) = \left(x, 0, t - \frac{xy}{2}\right). \tag{1.1}$$

From the point of view of geometric measure theory in the Heisenberg group, the vertical projections are the Heisenberg analogues of orthogonal projections to  $(d - 1)$ -planes in  $\mathbb{R}^d$ . One of the fundamental theorems concerning orthogonal projections in  $\mathbb{R}^d$  is the *Marstrand-Mattila projection theorem* [19,20]: if  $K \subset \mathbb{R}^d$  is a Borel set, then

$$\dim_{\mathbb{E}} \pi_V(K) = \min\{\dim_{\mathbb{E}} K, d - 1\} \tag{1.2}$$

for almost all  $(d - 1)$ -planes  $V \subset \mathbb{R}^d$ . Here  $\dim_{\mathbb{E}}$  refers to Hausdorff dimension in Euclidean space – in contrast to the notation “ $\dim_{\mathbb{H}}$ ” which will refer to Hausdorff dimension in the Heisenberg group. In  $\mathbb{R}^d$ , orthogonal projections are Lipschitz maps, so the upper bound in (1.2) is trivial, and the main interest in (1.2) is the lower bound.

The vertical projections  $\pi_e$  are not Lipschitz maps  $\mathbb{H} \rightarrow \mathbb{W}_e$  relative to the natural metric  $d_{\mathbb{H}}$  in  $\mathbb{H}$  and  $\mathbb{W}_e$ . Indeed, they can increase Hausdorff dimension: an easy example is a horizontal line, which is 1-dimensional to begin with, but gets projected to a 2-dimensional set – a parabola – in almost all directions. For general (sharp) results on how much  $\pi_e$  can increase Hausdorff dimension, see [1, Theorem 1.3]. We note that the vertical planes  $\mathbb{W}_e$  themselves are 3-dimensional, and  $\mathbb{H}$  is 4-dimensional.

Can the vertical projections lower Hausdorff dimension? In some directions they can, and the general (sharp) universal lower bound was already found in [1, Theorem 1.3]:

$$\dim_{\mathbb{H}} \pi_e(K) \geq \max\{0, \frac{1}{2}(\dim_{\mathbb{H}} K - 1), 2 \dim_{\mathbb{H}} K - 5\}, \quad e \in S^1.$$

Our main result states that the dimension drop cannot occur in a set of directions of positive measure for sets of dimension in  $[0, 2] \cup \{3\}$ :

**Theorem 1.3.** *Let  $K \subset \mathbb{H}$  be a Borel set with  $\dim_{\mathbb{H}} K \in [0, 2] \cup \{3\}$ . Then  $\dim_{\mathbb{H}} \pi_e(K) \geq \dim_{\mathbb{H}} K$  for  $\mathcal{H}^1$  almost every  $e \in S^1$ .*

The result is sharp for all values  $\dim_{\mathbb{H}} K \in [0, 2] \cup \{3\}$ , and new for  $\dim_{\mathbb{H}} K \in (1, 2] \cup \{3\}$ . It makes progress in [1, Conjecture 1.5] which proposes that

$$\dim_{\mathbb{H}} \pi_e(K) \geq \min\{\dim_{\mathbb{H}} K, 3\} \tag{1.4}$$

for  $\mathcal{H}^1$  almost every  $e \in S^1$ . The cases  $\dim_{\mathbb{H}} K \in [0, 1]$  were established around a decade ago by Balogh, Durand-Cartagena, the first author, Mattila, and Tyson [1, Theorem 1.4]. For  $\dim_{\mathbb{H}} K > 1$ , the strongest previous partial result is due to Harris [14] who in 2022 proved that

$$\dim_{\mathbb{H}} \pi_e(K) \geq \min \left\{ \frac{1 + \dim_{\mathbb{H}} K}{2}, 2 \right\} \quad \text{for } \mathcal{H}^1 \text{ a.e. } e \in S^1.$$

Other partial results, also higher dimensions, are contained in [2,4,13,15].

The “disconnected” assumption  $\dim_{\mathbb{H}} K \in [0, 2] \cup \{3\}$  is due to the fact that Theorem 1.3 is a combination of two separate results, with different proofs. Perhaps surprisingly, the cases  $\dim_{\mathbb{H}} K \in [0, 2]$  are a consequence of a “1-dimensional” projection theorem. Namely, consider the (nonlinear) projections  $\rho_e: \mathbb{R}^3 \rightarrow \mathbb{R}$  obtained as the  $t$ -coordinates of the projections  $\pi_e$ :

$$\rho_e = \pi_T \circ \pi_e, \quad \pi_T(x, y, t) = (0, 0, t). \tag{1.5}$$

Since the  $t$ -axis in  $\mathbb{H}$  is 2-dimensional, it is conceivable that the maps  $\rho_e$  do not a.e. lower the Hausdorff dimension of Borel sets of dimension at most 2. This is what we prove:

**Theorem 1.6.** *Let  $K \subset \mathbb{R}^3$  be a Borel set. Then*

$$\dim_{\mathbb{E}} \rho_e(K) = \min\{\dim_{\mathbb{E}} K, 1\} \quad \text{and} \quad \dim_{\mathbb{H}} \rho_e(K) \geq \min\{\dim_{\mathbb{H}} K, 2\}$$

for  $\mathcal{H}^1$  almost every  $e \in S^1$ . In fact, the following sharper conclusion holds: for  $0 \leq s < \min\{\dim_{\mathbb{H}} K, 2\}$ , we have  $\dim_{\mathbb{E}}\{e \in S^1 : \dim_{\mathbb{H}} \rho_e(K) \leq s\} \leq \frac{s}{2}$ .

Theorem 1.6 implies the cases  $\dim_{\mathbb{H}} K \in [0, 2]$  of Theorem 1.3, because the map  $\pi_T$  is Lipschitz when restricted to any plane  $\mathbb{W}_e$ , thus  $\dim_{\mathbb{H}} \pi_e(K) \geq \dim_{\mathbb{H}} \rho_e(K)$  for all  $e \in S^1$ .

The proof of Theorem 1.6 is a fairly straightforward application of recently developed technology to study the *restricted projections* problem in  $\mathbb{R}^3$  (see [8,9,11,17,22]). Even though the maps  $\rho_e$  are nonlinear, Theorem 1.6 falls within the scope of the *cinematic function* framework introduced by Pramanik, Yang, and Zahl [22]. In Theorem 3.2, we apply this framework to record a more general version of Theorem 1.6 which simultaneously generalises [22, Theorem 1.3] and Theorem 1.6. The details can be found in Section 3.

The case  $\dim_{\mathbb{H}} K = 3$  of Theorem 1.3 is the harder result. This time we do not know how to deduce it from a purely Euclidean statement. Instead, it is deduced from the following “mixed” result between Heisenberg and Euclidean metrics:

**Theorem 1.7.** *Let  $K \subset \mathbb{H}$  be a Borel set with  $\dim_{\mathbb{H}} K \geq 2$ . Then,*

$$\dim_{\mathbb{E}} \pi_e(K) \geq \min\{\dim_{\mathbb{H}} K - 1, 2\} \tag{1.8}$$

for  $\mathcal{H}^1$  almost every  $e \in S^1$ , and consequently

$$\dim_{\mathbb{H}} \pi_e(K) \geq \min\{2 \dim_{\mathbb{H}} K - 3, 3\} \tag{1.9}$$

for  $\mathcal{H}^1$  almost every  $e \in S^1$ .

Theorem 1.7 will further be deduced from a  $\delta$ -discretised result which may have independent interest. We state here a simplified version (the full version is Theorem 5.11):

**Theorem 1.10.** *Let  $0 \leq t \leq 3$  and  $\eta > 0$ . Then, the following holds for  $\delta, \epsilon > 0$  small enough, depending only on  $\eta$ . Let  $\mathcal{B}$  be a non-empty  $(\delta, t, \delta^{-\epsilon})$ -set of Heisenberg balls of radius  $\delta$ , all contained in  $B_{\mathbb{H}}(1)$ . Then, there exists  $e \in S^1$  such that*

$$\text{Leb}(\pi_e(\cup \mathcal{B})) \geq \delta^{3-t+\eta}. \tag{1.11}$$

Here  $\text{Leb}$  denotes Lebesgue measure on  $\mathbb{W}_e$ , identified with  $\mathbb{R}^2$ . For the definition of  $(\delta, t)$ -sets of  $\delta$ -balls, see Definition 5.1. Theorems 1.7 and 1.10 are proved in Sections 5-7.

**Remark 1.12.** It seems likely that the lower bound (1.11) remains valid under the alternative assumptions that  $|\mathcal{B}| = \delta^{-t}$  and

$$|\{B \in \mathcal{B} : B \subset B_{\mathbb{H}}(p, r)\}| \leq \delta^{-\epsilon} \cdot \left(\frac{r}{\delta}\right)^3, \quad p \in \mathbb{H}, r \geq \delta. \tag{1.13}$$

This is because the estimate (1.11) ultimately follows from Proposition 6.7 which works under the non-concentration condition (1.13). We will not need this version of Theorem 1.10, so we omit the details.

### 1.1. Sharpness of the results

Theorem 1.3 is sharp for all values  $\dim_{\mathbb{H}} K \in [0, 2] \cup \{3\}$ . The “mixed” inequality (1.8) in Theorem 1.7 is sharp for all values  $\dim_{\mathbb{H}} K \geq 2$ , even though the Heisenberg corollary (1.9) is unlikely to be sharp for any value  $\dim_{\mathbb{H}} K < 3$  (in fact, Theorem 1.3 shows that (1.9) is **not** sharp for  $\dim_{\mathbb{H}} K < 5/2$ ).

The sharpness examples are as follows: if  $s := \dim_{\mathbb{H}} K \leq 2$ , take an  $s$ -dimensional subset of the  $t$ -axis, and note that the  $t$ -axis is preserved by the projections  $\rho_e$  and  $\pi_e$ . If  $s > 2$ , take  $K$  to be a union of translates of the  $t$ -axis, thus  $K := K_0 \times \mathbb{R}$ . The  $\pi_e$ -projections send vertical lines to vertical lines, so  $\pi_e(K)$  is a union of vertical lines on  $\mathbb{W}_e$ ; more precisely  $\pi_e(K) = \bar{\pi}_e(K_0) \times \mathbb{R}$ , where  $\bar{\pi}_e$  is an orthogonal projection in  $\mathbb{R}^2$ . These observations lead to the sharpness of (1.8), and the sharpness of conjecture (1.4).

Theorem 1.10 is sharp for all values of  $t \in [0, 3]$ . Indeed, it is possible that  $|\mathcal{B}| = \delta^{-t}$ , and then  $\text{Leb}(\pi_e(\cup \mathcal{B})) \lesssim \delta^{3-t}$  for every  $e \in S^1$ . It also follows from (1.11) that the smallest number of  $d_{\mathbb{H}}$ -balls of radius  $\delta$  needed to cover  $\pi_e(\cup \mathcal{B})$  is  $\gtrsim \delta^{-t+\eta}$ . One might think that this solves Conjecture 1.4 for all  $\dim_{\mathbb{H}} K \in [0, 3]$ , but we were not able to make this deduction rigorous: the difficulty appears when attempting to  $\delta$ -discretise Conjecture 1.4, and is caused by the non-Lipschitz behaviour of  $\pi_e: (\mathbb{H}, d_{\mathbb{H}}) \rightarrow (\mathbb{W}_e, d_{\mathbb{H}})$ . This problem will be apparent in the proof of Theorem 1.7 in Section 7. Another, more heuristic, way of understanding the difference between Theorem 1.10 and Conjecture 1.4 is this:  $\text{Leb}(\pi_e(K))$  is invariant under left-translating  $K$ , but  $\dim_{\mathbb{H}} \pi_e(K)$  is generally not.

As we already explained, the proof of Theorem 1.6, therefore the cases  $\dim_{\mathbb{H}} K \in [0, 2]$  of Theorem 1.3, follow from recent developments in the theory of *restricted projections* in  $\mathbb{R}^3$ , notably the *cinematic function* framework in [22]. The proof of Theorem 1.7 does not directly overlap with these results (see Section 1.2 for more details), and for example does not use the  $\ell^2$ -decoupling theorem, in contrast with [8,9,11]. That said, the argument was certainly inspired by the recent developments in the restricted projection problem.

### 1.2. Proof outline for Theorem 1.7

The proof of Theorem 1.7 is mainly based on two ingredients. The first one is a *point-line duality* principle between horizontal lines in  $\mathbb{H}$ , and  $\mathbb{R}^3$ . To describe this principle, let  $\mathcal{L}_{\mathbb{H}}$  be the family of all horizontal lines in  $\mathbb{H}$ , and let  $\mathcal{L}_{\mathcal{C}}$  be the family of all lines in  $\mathbb{R}^3$  which are parallel to some line contained in a conical surface  $\mathcal{C}$ . In Section 4, we show that there exist maps  $\ell: \mathbb{R}^3 \rightarrow \mathcal{L}_{\mathbb{H}}$  and  $\ell^*: \mathbb{H} \rightarrow \mathcal{L}_{\mathcal{C}}$  (whose ranges cover almost all of  $\mathcal{L}_{\mathbb{H}}$  and  $\mathcal{L}_{\mathcal{C}}$ ) which preserve *incidence relations* in the following way:

$$q \in \ell(p) \iff p \in \ell^*(q), \quad p \in \mathbb{R}^3, q \in \mathbb{H}.$$

Thus, informally speaking, incidence-geometric questions between points in  $\mathbb{H}$  and lines in  $\mathcal{L}_{\mathbb{H}}$  can always be transformed into incidence-geometric questions between points in  $\mathbb{R}^3$  and lines in  $\mathcal{L}_{\mathcal{C}}$ . The point-line duality principle described here was used implicitly by Liu [18] to study *Keakeya sets* (formed by horizontal lines) in  $\mathbb{H}$ . However, making the principle explicit has already proved very useful since the first version of this paper appeared: we used it in [5] to study *Keakeya sets* associated with  $SL(2)$ -lines in  $\mathbb{R}^3$ , and Harris [12] used it to treat the case  $\dim_{\mathbb{H}} K > 3$  of Theorem 1.3 (in this case the projections  $\pi_e(K)$  turn out to have positive measure almost surely).

The question about vertical projections in  $\mathbb{H}$  can – after suitable discretisation – be interpreted as an incidence geometric problem between points in  $\mathbb{H}$  and lines in  $\mathcal{L}_{\mathbb{H}}$ . It can therefore be transformed into an incidence-geometric problem between points in  $\mathbb{R}^3$  and lines in  $\mathcal{L}_{\mathcal{C}}$ . Which problem is this? It turns out that while the dual  $\ell^*(p)$  of a point  $p \in \mathbb{H}$  is a line in  $\mathcal{L}_{\mathcal{C}}$ , the dual  $\ell^*(B_{\mathbb{H}})$  of a Heisenberg  $\delta$ -ball resembles an  $\delta$ -plate in  $\mathbb{R}^3$  – a rectangle of dimensions  $1 \times \delta \times \delta^2$  tangent to  $\mathcal{C}$ . So, the task of proving Theorem 1.10 (hence Theorem 1.7) is (roughly) equivalent to the task of solving an incidence-geometric problem between points in  $\mathbb{R}^3$ , and family of  $\delta$ -plates.

Moreover: the plates in our problem appear as duals of certain Heisenberg  $\delta$ -balls, approximating a  $t$ -dimensional set  $K \subset \mathbb{H}$ , with  $0 \leq t \leq 3$ . Consequently, the plates can be assumed to satisfy a  $t$ -dimensional “non-concentration condition” relative to the metric  $d_{\mathbb{H}}$ . In common jargon, the plate family is a  $(\delta, t)$ -set relative to  $d_{\mathbb{H}}$ .

In [10], Guth, Wang, and Zhang proved the *sharp (reverse) square function estimate for the cone in  $\mathbb{R}^3$* . A key component in their proof was a new incidence-geometric (“Keakeya”) estimate [10, Lemma 1.4] for points and  $\delta$ -plates in  $\mathbb{R}^3$  (see Section 6 for the details). While this was not relevant in [10], it turns out that the incidence estimate in [10, Lemma 1.4] interacts perfectly with a  $(\delta, 3)$ -set condition relative to  $d_{\mathbb{H}}$ . This allows us to prove, roughly speaking, that the vertical projections of 3-Frostman measures on  $\mathbb{H}$  have  $L^2$ -densities. See Corollary 5.6 for a more precise statement.

For  $0 \leq t < 3$ , the  $(\delta, t)$ -set condition relative to  $d_{\mathbb{H}}$  no longer interacts so well with [10, Lemma 1.4]. However, we were able to (roughly speaking) reduce Theorem 1.10 for  $(\delta, t)$ -sets,  $0 \leq t \leq 3$ , to the special case  $t = 3$ . This argument is explained in Section 5, so we omit the discussion here.

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**2. Preliminaries on the Heisenberg group**

We briefly introduce the Heisenberg group and relevant related concepts. A more thorough introduction to the geometry of the Heisenberg group can be found in many places, for instance in the monograph [3].

The *Heisenberg group*  $\mathbb{H} = (\mathbb{R}^3, \cdot)$  is the set  $\mathbb{R}^3$  equipped with the non-commutative group product defined by

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(xy' - yx')).$$

The *Heisenberg dilations* are the group automorphisms  $\delta_\lambda$ ,  $\lambda > 0$ , defined by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

The group product gives rise to projection-type mappings onto subgroups that are invariant under Heisenberg dilations. For  $e \in S^1$ , we define the *horizontal subgroup*

$$\mathbb{L}_e := \{(se, 0) : s \in \mathbb{R}\}.$$

The *vertical subgroup*  $\mathbb{W}_e$  is the Euclidean orthogonal complement of  $\mathbb{L}_e$  in  $\mathbb{R}^3$ ; in particular it is a plane containing the vertical axis. Every point  $p \in \mathbb{H}$  can be written in a unique way as a product  $p = p_{\mathbb{W}_e} \cdot p_{\mathbb{L}_e}$  with  $p_{\mathbb{W}_e} \in \mathbb{W}_e$  and  $p_{\mathbb{L}_e} \in \mathbb{L}_e$ . The *vertical Heisenberg projection* onto the vertical plane  $\mathbb{W}_e$  is the map

$$\pi_e : \mathbb{H} \rightarrow \mathbb{W}_e, \quad p = p_{\mathbb{W}_e} \cdot p_{\mathbb{L}_e} \mapsto p_{\mathbb{W}_e}.$$

The vertical projection to the  $xt$ -plane  $\{(x, 0, t) : x, t \in \mathbb{R}\}$  will play a special role; this projection will be denoted  $\pi_{xt}$ , and it has the explicit formula stated in (1.1). Preliminaries about Heisenberg projections can be found for instance in [21,2,1]. These mappings have turned out to play an important role in geometric measure theory of the Heisenberg group endowed with a left-invariant non-Euclidean metric. The *Korányi metric*  $d_{\mathbb{H}}$  is defined by

$$d_{\mathbb{H}}(p, q) := \|q^{-1} \cdot p\|,$$

where  $\|\cdot\|$  is the *Korányi norm* given by

$$\|(x, y, t)\| = \sqrt[4]{(x^2 + y^2)^2 + 16t^2}.$$



We will use the symbol  $B_{\mathbb{H}}(p, r)$  to denote the ball centered at  $p$  with radius  $r$  with respect to the Korányi metric. Balls centred at the origin are denoted  $B_{\mathbb{H}}(r)$ . All vertical planes  $\mathbb{W}_e, e \in S^1$ , equipped with  $d_{\mathbb{H}}$  are isometric to each other via rotations of  $\mathbb{R}^3$  about the vertical axis. The Heisenberg dilations are similarities with respect to  $d_{\mathbb{H}}$ , and it is easy to see that  $(\mathbb{H}, d_{\mathbb{H}})$  is a 4-regular space, while the vertical subgroups  $\mathbb{W}_e$  are 3-regular with respect to  $d_{\mathbb{H}}$ . Moreover, there exists a constant  $0 < c < \infty$ , independent of  $e$ , such that under the obvious identification of  $\mathbb{W}_e$  with  $\mathbb{R}^2$ , the restriction of the 3-dimensional Hausdorff measure  $\mathcal{H}^3$  to  $\mathbb{W}_e$  agrees with the 2-dimensional Lebesgue measure  $\text{Leb}$  on  $\mathbb{R}^2$  up to the multiplicative constant  $c$ .

Vertical projections are neither group homomorphisms nor Lipschitz mappings with respect  $d_{\mathbb{H}}$ . However, they behave well with respect to the Lebesgue measure on vertical planes. Namely, for every Borel set  $E \subset \mathbb{H}$ , we have that

$$\text{Leb}(\pi_e(p \cdot E)) = \text{Leb}(\pi_e(E)), \quad p \in \mathbb{R}^3, e \in S^1, \tag{2.1}$$

see the formula at the bottom of page 1970 in the proof of [7, Lemma 2.20].

### 3. Proof of Theorem 1.6

In this section, we prove Theorem 1.6, and therefore the cases  $\dim_{\mathbb{H}} K \in [0, 2]$  of Theorem 1.3. Further, Theorem 1.6 will be inferred from a more general statement, Theorem 3.2, modelled after [22, Theorem 1.3]. We first discuss Theorem 3.2, and then explain in Section 3.2 how it can be applied to deduce Theorem 1.6.

#### 3.1. Projections induced by cinematic functions

We start by introducing terminology from [22, Definition 1.6] which will be needed for the formulation of Theorem 3.2.

**Definition 3.1** (*Cinematic family*). Let  $I \subset \mathbb{R}$  be a compact interval, and let  $\mathcal{F} \subset C^2(I)$  be a family of functions satisfying the following conditions:

- (1)  $I$  is a compact interval, and  $\mathcal{F}$  has finite diameter in  $(C^2(I), \|\cdot\|_{C^2(I)})$ .
- (2)  $(\mathcal{F}, \|\cdot\|_{C^2(I)})$  is a doubling metric space.
- (3) For all  $f, g \in \mathcal{F}$ , we have

$$\inf_{\theta \in I} |f(\theta) - g(\theta)| + |f'(\theta) - g'(\theta)| + |f''(\theta) - g''(\theta)| \gtrsim \|f - g\|_{C^2(I)}.$$

Then,  $\mathcal{F}$  is called a *cinematic family*.

The following projection theorem is modelled after [22, Theorem 1.3]:

**Theorem 3.2.** *Let  $L > 0$ , let  $I \subset \mathbb{R}$  be a compact interval, and let  $\{\rho_\theta\}_{\theta \in I}$  be a family of  $L$ -Lipschitz maps  $\rho_\theta: B \rightarrow \mathbb{R}$ , where  $B \subset \mathbb{R}^3$  is a ball. For  $p \in B$ , define the function  $f_p: I \rightarrow \mathbb{R}$  by  $f_p(\theta) := \rho_\theta(p)$ . Assume that  $p \mapsto f_p$  is a bilipschitz embedding  $B \rightarrow C^2(I)$ , and assume that  $\mathcal{F} = \{f_p : p \in B\}$  is a cinematic family.*

*Then, the projections  $\{\rho_\theta\}_{\theta \in I}$  satisfy (3.8): if  $K \subset \mathbb{R}^3$  is a Borel set, then*

$$\dim_{\mathbb{E}}\{\theta \in I : \dim_{\mathbb{E}} \rho_\theta(K) \leq s\} \leq s, \quad 0 \leq s < \min\{\dim_{\mathbb{E}} K, 1\}.$$

We only sketch the proof of Theorem 3.2 since it is virtually the same as the proof of [22, Theorem 1.3]: this is the special case of Theorem 3.2, where

$$f_p(\theta) = \rho_\theta(p) := \gamma(\theta) \cdot p, \quad p \in \mathbb{R}^3, \tag{3.3}$$

and  $\gamma: I \rightarrow S^2$  parametrises a curve on  $S^2$  satisfying  $\text{span}\{\gamma, \dot{\gamma}, \ddot{\gamma}\} = \mathbb{R}^3$  (this condition is needed to guarantee that the family  $\{f_p : p \in B\}$  is cinematic for every ball  $B \subset \mathbb{R}^3$ , see the proof of [22, Proposition 2.1]).

The proof of [22, Theorem 1.3] is based on a reduction to [22, Theorem 1.7]. This is a ‘‘Kakeya-type’’ estimate concerning  $\delta$ -neighbourhoods of graphs of cinematic functions. More precisely, [22, Theorem 1.7] is only used via [22, Proposition 2.1], a special case of [22, Theorem 1.7] concerning the cinematic family  $\{\theta \mapsto \gamma(\theta) \cdot p\}_{p \in B}$ . We formulate a more general version of this proposition below: the only difference is that the cinematic family  $\{\theta \mapsto \gamma(\theta) \cdot p\}_{p \in B}$  is replaced by the family  $\{\theta \mapsto \rho_\theta(p)\}_{p \in B}$  relevant for Theorem 3.2:

**Proposition 3.4.** *Fix  $\epsilon > 0$  and  $0 < \alpha \leq \zeta \leq 1$ . Let  $I \subset \mathbb{R}$  be a compact interval, let  $B \subset \mathbb{R}^3$  be a ball, and let  $\rho_\theta: B \rightarrow \mathbb{R}$  be a family of uniformly Lipschitz functions with the properties assumed in Theorem 3.2: thus,  $\mathcal{F} = \{f_p : p \in B\}$  is a cinematic family, and the map  $p \mapsto f_p$  is a bilipschitz embedding  $B \rightarrow C^2(I)$ , where  $f_p(\theta) := \rho_\theta(p)$ . Then there exists  $\delta_0 > 0$  such that the following holds for all  $\delta \in (0, \delta_0]$ :*

*Let  $E \subset \mathbb{R}^2$  be a  $(\delta, \alpha; \delta^{-\epsilon})_1 \times (\delta, \alpha; \delta^{-\epsilon})_1$  quasi-product. Let  $Z_\delta \subset B$  be a  $\delta$ -separated set that satisfies*

$$|Z_\delta \cap B(p, r)| \leq \delta^{-\epsilon} (r/\delta)^\zeta, \quad p \in \mathbb{R}^3, r \geq \delta. \tag{3.5}$$

*Then*

$$\int_E \left( \sum_{p \in Z_\delta} \mathbf{1}_{\Gamma_p^\delta} \right)^{3/2} \leq \delta^{2-\alpha/2-\zeta/2-C\epsilon} |Z_\delta|,$$

*where  $C > 0$  is absolute, and  $\Gamma_p^\delta$  is the  $\delta$ -neighbourhood of the graph of  $f_p$ .*

**Proof.** The proof of [22, Proposition 2.1] is easy (given [22, Theorem 1.7]), but the proof of Proposition 3.4 is almost trivial. Indeed, the first part in the proof of [22, Proposition 2.1] is to verify that the family  $\{\theta \mapsto \rho_\theta(p)\}_{p \in B}$  is cinematic in the case

$\rho_\theta(p) = \gamma(\theta) \cdot p$ , but this is already a part of our hypothesis. The second part in the proof of [22, Proposition 2.1] is to verify that  $p \mapsto f_p$  is a bilipschitz embedding  $B \rightarrow C^2(I)$ , and this is – again – part of our hypothesis. In other words, all the work in the proof of [22, Proposition 2.1] has been made part of the hypotheses of Proposition 3.4.  $\square$

The reduction from [22, Theorem 1.3] to [22, Proposition 2.1] (in our case from Theorem 3.2 to Proposition 3.4) is presented in [22, Sections 2.1-2.4], and does not use the special form (3.3) (for example the linearity) of the maps  $\rho_\theta: \mathbb{R}^3 \rightarrow \mathbb{R}$ : it is only needed that

- (1) the maps  $\rho_\theta$  are uniformly Lipschitz, for  $\theta \in I$ ,
- (2)  $\sup_{p \in B} \sup_{\theta \in I} |\partial_\theta \rho_\theta(p)| < \infty$ .

Property (1) is assumed in Theorem 3.2, whereas property (2) follows from the assumption that the family  $\mathcal{F}$  is cinematic (and in particular a bounded subset of  $C^2(I)$ ).

The argument in [22, Sections 2.1-2.4] is extremely well-written, and our notation is deliberately the same, so we will not copy the whole proof. We only make a few remarks, below. If the reader is unfamiliar with the ideas involved, we warmly recommend reading first the heuristic section [22, Section 1.2].

**Proof sketch of Theorem 3.2.** The argument in [22, Section 2.1] can be copied verbatim; nothing changes. The most substantial change occurs in [22, Section 2.2]. Namely, [22, (2.10)] uses the fact (true in [22]) that the  $\rho_\theta$ -image of a  $\delta$ -cube  $Q \subset \mathbb{R}^3$  has length  $|\rho_\theta(Q)| \gtrsim \delta$ . For the general Lipschitz maps  $\rho_\theta$  in Theorem 3.2 this may not be the case; it would be true for the special maps  $\rho_\theta$  needed in Theorem 3.7, so also this part of [22] would work verbatim for these maps. However, even in the generality of Theorem 3.2 the problem can be completely removed: one only needs to replace every occurrence of  $\rho_\theta(Q)$  in [22, Section 2.2] by an interval

$$I_\theta(Q) := [\rho_\theta(z_Q) - \delta, \rho_\theta(z_Q) + \delta]$$

of length  $\sim \delta$  centred at  $\rho_\theta(z_Q)$ , where  $z_Q \in Q$  is the centre of  $Q$ . Since  $\rho_\theta(Q)$  only appears as a “tool” in [22, Section 2.2], the rest of the argument will remain unchanged. Let us, however, discuss what changes in [22, Section 2.2] when  $\rho_\theta(Q)$  is replaced by  $I_\theta(Q)$ . We assume familiarity with the notation in [22].

First and foremost, [22, (2.9)] remains valid: whenever  $Q \in \mathcal{Q}$  is a cube that intersects  $\rho_\theta^{-1}(G_\theta)$ , then  $\text{dist}(\rho_\theta(z_Q), G_\theta) \lesssim L\delta$  by our assumption that the maps  $\rho_\theta$  are  $L$ -Lipschitz. Therefore,

$$I_\theta(Q) \subset G'_\theta := N_{L\delta}(G_\theta).$$

This gives [22, (2.9)] with the slightly modified definition of  $G'_\theta$ , stated above. Consequently, also the version of [22, (2.10)] is true where  $\rho_\theta(Q)$  is replaced by  $I_\theta(Q)$ : here

the length bound  $|I_\delta(Q)| \gtrsim \delta$  is used. Finally, to deduce [22, (2.13)] from [22, (2.10)], we need to know that [22, (2.12)] remains valid when  $\rho_\theta(Q)$  is replaced with  $I_\theta(Q)$ . This is clear: if  $y \in I_\theta(Q)$ , then  $|y - \rho_\theta(z_Q)| \leq \delta$  by definition, and therefore  $(\theta, y) \in \Gamma_{z_Q}^\delta$ , where

$$\Gamma_z = \{(\theta, \rho_\theta(z)) : \theta \in I\}, \quad z \in B,$$

is the analogue of [22, (1.12)], and  $\Gamma_z^\delta$  is the  $\delta$ -neighbourhood of  $\Gamma_z$ . We have now verified [22, (2.13)]. The intervals  $\rho_\theta(Q)$  or  $I_\theta(Q)$  play no further role in the proof. The rest of [22, Section 2.2] works verbatim.

The same is also true for [22, Section 2.3]: the argument is fairly abstract down to [22, (2.19)], where it is needed that  $\sup_{p \in B} \sup_{\theta \in I} |\partial_\theta \rho_\theta(p)| < \infty$ . The maps  $\rho_\theta$  in Theorem 3.2 satisfy this property automatically, as noted in (2) above.

Finally, we arrive at the short [22, Section 2.4]. The only difference is that we need to apply Proposition 3.4 in place of [22, Proposition 2.1]. This completes the proof of Theorem 3.2.  $\square$

### 3.2. From vertical projections to cinematic functions

We explain how the general projection result, Theorem 3.2, can be applied to prove Theorem 3.7, which concerns the special projections  $\rho_e = \pi_T \circ \pi_e$ . Recall that  $\pi_e$  is the vertical projection to the plane  $\mathbb{W}_e = e^\perp$ . For  $e = (e_1, e_2) \in S^1$ , we write  $J(e) := (-e_2, e_1) \in S^1 \cap e^\perp$  is the counterclockwise rotation of  $e$  by  $\pi/2$ . With this notation, the map  $\pi_e$  has the explicit formula

$$\pi_e(z, t) = (\langle z, Je \rangle, t + \frac{1}{2} \langle z, e \rangle \langle z, Je \rangle), \tag{3.6}$$

where  $\langle \cdot, \cdot \rangle$  is the Euclidean dot product in  $\mathbb{R}^2$ . In the formula (3.6), we have also identified each plane  $\mathbb{W}_e$  with  $\mathbb{R}^2$  via the map  $(yJe, t) \cong (y, t)$ . It is worth noting that the distance  $d_{\mathbb{H}}$  restricted to the plane  $\mathbb{W}_e$  (for  $e \in S^1$  fixed) is bilipschitz equivalent to the parabolic distance on  $\mathbb{R}^2$ , namely  $d_{\text{par}}((x, s), (y, t)) = |x - y| + \sqrt{|s - t|}$ .

With the explicit expression (3.6) in hand, the nonlinear projections  $\rho_e = \pi_T \circ \pi_e$  introduced in (1.5) have the following formula:

$$\rho_e(z, t) = t + \frac{1}{2} \langle z, e \rangle \langle z, Je \rangle, \quad (z, t) \in \mathbb{R}^2 \times \mathbb{R}, e \in S^1,$$

By a slight abuse of notation, we write “ $d_{\mathbb{H}}$ ” for the square root metric on  $\mathbb{R}$ : thus  $d_{\mathbb{H}}(s, t) := \sqrt{s - t}$ . The projection  $\pi_T$  restricted to any fixed plane  $\mathbb{W}_e$  is a Lipschitz map  $(\mathbb{W}_e, d_{\mathbb{H}}) \rightarrow (\mathbb{R}, d_{\mathbb{H}})$ , even though  $\pi_T$  is not “globally” a Lipschitz map  $(\mathbb{H}, d_{\mathbb{H}}) \rightarrow (\mathbb{R}, d_{\mathbb{H}})$ . Therefore  $\dim_{\mathbb{H}} \pi_e(K) \geq \dim_{\mathbb{H}} \rho_e(K)$  for all  $e \in S^1$ , and the cases  $\dim_{\mathbb{H}} K \in [0, 2]$  of Theorem 1.3 follow from Theorem 1.6, whose contents are repeated here:

**Theorem 3.7.** *Let  $K \subset \mathbb{R}^3$  be Borel, and let  $0 \leq s < \min\{\dim_{\mathbb{E}} K, 1\}$ . Then,*

$$\dim_{\mathbb{E}} \{e \in S^1 : \dim_{\mathbb{E}} \rho_e(K) \leq s\} \leq s. \tag{3.8}$$

As a consequence, for every  $0 \leq s < \min\{\dim_{\mathbb{H}} K, 2\}$ ,

$$\dim_{\mathbb{E}}\{e \in S^1 : \dim_{\mathbb{H}} \rho_e(K) \leq s\} \leq \frac{s}{2}. \tag{3.9}$$

In particular,  $\dim_{\mathbb{H}} \rho_e(K) \geq \min\{\dim_{\mathbb{H}} K, 2\}$  for  $\mathcal{H}^1$  almost every  $e \in S^1$ .

**Remark 3.10.** We explain why (3.8) implies (3.9). It is well-known that

$$\dim_{\mathbb{H}} K \leq 2 \dim_{\mathbb{E}} K.$$

for all sets  $K \subset \mathbb{H}$ . This simply follows from the fact that the identity map  $(\mathbb{H}, d_{\text{Eucl}}) \rightarrow (\mathbb{H}, d_{\mathbb{H}})$  is locally  $\frac{1}{2}$ -Hölder continuous. Therefore, if  $0 \leq s < \min\{\dim_{\mathbb{H}} K, 2\}$ , as in (3.9), we have  $0 \leq \frac{s}{2} < \min\{\dim_{\mathbb{E}} K, 1\}$ , and (3.8) is applicable. Since

$$\{e \in S^1 : \dim_{\mathbb{H}} \rho_e(K) \leq s\} = \{e \in S^1 : \dim_{\mathbb{E}} \rho_e(K) \leq \frac{s}{2}\}$$

(the square root metric on  $\mathbb{R}$  doubles Euclidean dimension), we have

$$\dim_{\mathbb{E}}\{e \in S^1 : \dim_{\mathbb{H}} \rho_e(K) \leq s\} = \dim_{\mathbb{E}}\{e \in S^1 : \dim_{\mathbb{E}} \rho_e(K) \leq \frac{s}{2}\} \stackrel{(3.8)}{\leq} \frac{s}{2}.$$

This is what we claimed in (3.9).

For the remainder of this section, we focus on proving the Euclidean statement (3.8). This is chiefly based on verifying that the projections  $\rho_e : \mathbb{R}^3 \rightarrow \mathbb{R}$  give rise to a *cinematic family of functions*, as in Definition 3.1. Let us introduce the relevant cinematic family. We re-parametrise the projections  $\rho_e$ ,  $e \in S^1$ , as  $\rho_{\theta}$ ,  $\theta \in \mathbb{R}$ , where

$$\rho_{\theta} := \rho_{e(\theta)}, \quad e(\theta) := (\cos \theta, \sin \theta).$$

With this notation, we define the following functions  $f_p : \mathbb{R} \rightarrow \mathbb{R}$ ,  $p \in \mathbb{R}^3$ :

$$f_p(\theta) := \rho_{\theta}(p) := t + \frac{1}{2} \langle z, e(\theta) \rangle \langle z, J e(\theta) \rangle, \quad p = (z, t) \in \mathbb{R}^3. \tag{3.11}$$

**Proposition 3.12.** *Let  $p_0 \in \mathbb{R}^3 \setminus \{(0, 0, t) : t \in \mathbb{R}\}$ . Then, there exists a radius  $r = r(p_0) > 0$  such that  $\mathcal{F}(B(p_0, r)) := \{f_p : p \in B(p_0, r)\}$  is a cinematic family.*

The compact interval appearing in conditions (1)-(3) of Definition 3.1 can be taken to be  $[0, 2\pi]$  – this makes no difference, since the functions  $f_p$  are  $2\pi$ -periodic. It turns out that the conditions (1)-(2) are satisfied for the family  $\mathcal{F}(B)$ , whenever  $B \subset \mathbb{R}^3$  is an arbitrary ball. To verify condition (3), we will need to assume that  $B$  lies outside the  $t$ -axis; we will return to this a little later. We first compute the derivatives of the functions in  $\mathcal{F}$ . For  $f_p \in \mathcal{F}$ , we have

$$f'_p(\theta) = \frac{1}{2}\langle z, e'(\theta) \rangle \langle z, Je(\theta) \rangle + \frac{1}{2}\langle z, e(\theta) \rangle \langle z, Je'(\theta) \rangle.$$

This expression can be further simplified by noting that  $e'(\theta) = Je(\theta)$ , and  $Je'(\theta) = -e(\theta)$ . Therefore,

$$f'_p(\theta) = \frac{1}{2}\langle z, Je(\theta) \rangle^2 - \frac{1}{2}\langle z, e(\theta) \rangle^2. \tag{3.13}$$

From this expression, we may compute the second derivative:

$$f''_p(\theta) = \langle z, Je(\theta) \rangle \langle z, Je'(\theta) \rangle - \langle z, e(\theta) \rangle \langle z, e'(\theta) \rangle = -2\langle z, e(\theta) \rangle \langle z, Je(\theta) \rangle. \tag{3.14}$$

The formulae (3.11)-(3.14) immediately show that the map  $p \mapsto f_p$  is locally Lipschitz:

$$\sup_{\theta \in \mathbb{R}} |f_p(\theta) - f_q(\theta)| + |f'_p(\theta) - f'_q(\theta)| + |f''_p(\theta) - f''_q(\theta)| \lesssim_B |p - q|, \quad p, q \in B. \tag{3.15}$$

This implies conditions (1)-(2) in Definition 3.1 for the family  $\mathcal{F}(B)$ . Regarding condition (3) in Definition 3.1, we claim the following:

**Proposition 3.16.** *If  $p_0 \in \mathbb{R}^3 \setminus \{(0, 0, t) : t \in \mathbb{R}\}$ , there exists a radius  $r = r(p_0) > 0$  and a constant  $c = c(p_0) > 0$  such that*

$$|f_p(\theta) - f_q(\theta)| + |f'_p(\theta) - f'_q(\theta)| + |f''_p(\theta) - f''_q(\theta)| \geq c|p - q| \tag{3.17}$$

for all  $p, q \in B(p_0, r)$  and  $\theta \in \mathbb{R}$ .

We start with the following lemma:

**Lemma 3.18.** *For every  $p_0 \in \mathbb{R}^3 \setminus \{(0, 0, t) : t \in \mathbb{R}\}$  there exists a constant  $c > 0$  and a radius  $r > 0$  such that the following holds:*

$$|f_p(0) - f_q(0)| + |f'_p(0) - f'_q(0)| + |f''_p(0) - f''_q(0)| \geq c|p - q|, \quad p, q \in B(p_0, r). \tag{3.19}$$

**Proof.** Recall that  $e(0) = (1, 0)$  and  $Je(0) = (0, 1)$ . We then define  $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$F(p) := (f_p(0), f'_p(0), f''_p(0)) = (t + \frac{1}{2}z_1z_2, \frac{1}{2}(z_2^2 - z_1^2), -2z_1z_2), \quad p = (z, t) \in \mathbb{R}^3.$$

Then, we note that  $|\det DF(p)| = 2|z|^2$ , so in particular the Jacobian of  $F$  is non-vanishing outside the  $t$ -axis. Now (3.19) follows from the inverse function theorem.  $\square$

We then prove Proposition 3.16:

**Proof of Proposition 3.16.** To deduce (3.17) from (3.19), we record the following rotation invariance:

$$f_{R_\varphi(p)}^{(k)}(\theta + \varphi) = f_p^{(k)}(\theta), \quad p \in \mathbb{R}^3, \theta, \varphi \in \mathbb{R}. \tag{3.20}$$

Here  $R_\varphi(z, t) := (e^{i\varphi}z, t)$  is a counterclockwise rotation around the  $t$ -axis. The proof is evident from the formulae (3.11)-(3.14), and noting that

$$\langle e^{i\varphi}z, e(\theta + \varphi) \rangle = \langle z, e(\theta) \rangle \quad \text{and} \quad \langle e^{i\varphi}z, Je(\theta + \varphi) \rangle = \langle z, Je(\theta) \rangle.$$

Now we are in a position to conclude the proof of (3.17). Fix  $p_0 \in \mathbb{R}^3 \setminus \{(0, 0, t) : t \in \mathbb{R}\}$  and  $\theta_0 \in \mathbb{R}$ . Then, apply Lemma 3.18 to the point

$$R_{-\theta_0}(p_0) \in \mathbb{R}^3 \setminus \{(0, 0, t) : t \in \mathbb{R}\}.$$

This yields a constant  $c = c(p_0, \theta_0) > 0$  and a radius  $r_0 = r_0(p_0, \theta_0) > 0$  such that

$$|f_p(0) - f_q(0)| + |f'_p(0) - f'_q(0)| + |f''_p(0) - f''_q(0)| \geq c|p - q| \tag{3.21}$$

for all  $p, q \in B(R_{-\theta_0}(p_0), 2r_0)$ . Next, we choose  $I(\theta_0) = [\theta_0 - r_1, \theta_0 + r_1]$  to be a sufficiently short interval around  $\theta_0$  such that the following holds:

$$R_{-\theta}(p), R_{-\theta}(q) \in B(R_{-\theta_0}(p_0), 2r_0), \quad p, q \in B(p_0, r_0), \theta \in I(\theta_0).$$

Then, it follows from a combination of (3.20) and (3.21) that

$$\sum_{k=0}^2 |f_p^{(k)}(\theta) - f_q^{(k)}(\theta)| \stackrel{(3.20)}{=} \sum_{k=0}^2 |f_{R_{-\theta}(p)}^{(k)}(0) - f_{R_{-\theta}(q)}^{(k)}(0)| \stackrel{(3.21)}{\geq} c|R_{-\theta}(p) - R_{-\theta}(q)| = c|p - q|$$

for all  $p, q \in B(p_0, r_0)$  and all  $\theta \in I(\theta_0)$ . This completes the proof of (3.17) of all  $\theta \in I(\theta_0)$ . To extend the argument of all  $\theta \in \mathbb{R}$ , note that the functions  $f_p$ , and all of their derivatives, are  $2\pi$ -periodic. So, it suffices to show that (3.17) holds for  $\theta \in [0, 2\pi]$ . This follows by compactness from what we have already proven, by covering  $[0, 2\pi]$  by finitely many intervals of the form  $I(\theta_0)$ , and finally defining “ $r$ ” and “ $c$ ” to be the minima of the constants  $r(p_0, \theta_0)$  and  $c(p_0, \theta_0)$  obtained in the process.  $\square$

Proposition 3.12 now follows from Proposition 3.16, and the discussion above it (where we verified Definition 3.1(1)-(2)). We then conclude the proof of Theorem 3.7:

**Proof of Theorem 3.7.** Given Remark 3.10, it suffices to prove (3.8), which will be a consequence of Theorem 3.2. Indeed, since the projections  $\rho_\epsilon$  are isometries on the  $t$ -axis, we may assume that

$$\dim_E(K \setminus \{(0, 0, t) : t \in \mathbb{R}\}) = \dim_E K.$$

Consequently, for  $\epsilon > 0$ , we may fix a point  $p_0 \in K$  outside the  $t$ -axis such that

$$\dim_{\mathbb{E}}(K \cap B(p_0, r)) > \dim_{\mathbb{E}} K - \epsilon, \quad r > 0. \tag{3.22}$$

Apply Proposition 3.12 to find a radius  $r > 0$  such that the family of functions  $\mathcal{F} := \mathcal{F}(B(p_0, r))$  is cinematic. It follows from a combination of (3.15) and Proposition 3.16 that  $p \mapsto f_p$  is a bilipschitz embedding  $B \rightarrow C^2(\mathbb{R})$ . Therefore Theorem 3.2 is applicable: for every  $0 \leq s < \min\{\dim_{\mathbb{E}}(K \cap B(p_0, r)), 1\}$  we have

$$\dim_{\mathbb{E}}\{\theta \in [0, 2\pi] : \dim_{\mathbb{E}} \rho_{\theta}(K \cap B(p_0, r)) \leq s\} \leq s.$$

Now (3.8) follows from (3.22) by letting  $\epsilon \rightarrow 0$ .  $\square$

#### 4. Duality between horizontal lines and $\mathbb{R}^3$

This section contains preliminaries to prove Theorem 1.7. Most importantly, we introduce a notion of *duality* that associates to points and horizontal lines in  $\mathbb{H}$  certain lines and points in  $\mathbb{R}^3$ . The lines in  $\mathbb{R}^3$  will be *light rays* – translates of lines on a fixed conical surface. To define these, we let  $\mathcal{C}_0$  be the vertical cone

$$\mathcal{C}_0 = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1^2 + z_2^2 = z_3^2\},$$

and we denote by  $\mathcal{C}$  the (45°) rotated cone

$$\mathcal{C} = R(\mathcal{C}_0) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2^2 = 2z_1z_3\},$$

where  $R(z_1, z_2, z_3) = ((z_1 + z_3)/\sqrt{2}, z_2, (-z_1 + z_3)/\sqrt{2})$ . The cone  $\mathcal{C}$  is foliated by lines

$$L_y = \text{span}_{\mathbb{R}}(1, -y, y^2/2), \quad y \in \mathbb{R}, \tag{4.1}$$

cf. the proof of [18, Theorem 1.2], where a similar parametrization is used. To be accurate, the lines  $L_y$  only foliate  $\mathcal{C} \setminus \{(0, 0, z) : z \in \mathbb{R}\}$ . We will abuse notation by writing  $L_y(s) = (s, -sy, sy^2/2)$  for the parametrisation of the line  $L_y$ .

**Definition 4.2** (*Light rays*). We say that a line  $L$  in  $\mathbb{R}^3$  is a *light ray* if  $L = z + L_y$  for some  $z \in \mathbb{R}^3$  and  $y \in \mathbb{R}$ . In other words,  $L$  is a (Euclidean) translate of a line contained in  $\mathcal{C}$  (excluding the  $t$ -axis).

**Remark 4.3.** Every light ray can be written as  $(0, u, v) + L_y$  for a unique  $(u, v) \in \mathbb{R}^2$ .

**Definition 4.4** (*Horizontal lines*). A line  $\ell$  in  $\mathbb{R}^3$  is *horizontal* if it is a Heisenberg left translate of a horizontal subgroup, that is, there exists  $p \in \mathbb{H}$  and  $e \in S^1$  such that  $\ell = p \cdot \mathbb{L}_e$ .

**Remark 4.5.** Every horizontal line, apart from left translates of the  $x$ -axis, can be written as  $\ell = \{(as + b, s, (b/2)s + c) : s \in \mathbb{R}\}$  for a uniquely determined point  $(a, b, c) \in \mathbb{R}^3$ .



**Definition 4.6.** We define the following correspondence between points and lines:

- To a point  $p = (x, y, t) \in \mathbb{H}$ , we associate the *light ray*

$$\ell^*(p) = (0, x, t - xy/2) + L_y \subset (0, x, t - xy/2) + \mathcal{C} \subset \mathbb{R}^3. \tag{4.7}$$

(This formula will be motivated by Lemma 4.11 below.)

- To a point  $p^* = (a, b, c) \in \mathbb{R}^3$ , we associate the *horizontal line*

$$\ell(p^*) = \{(as + b, s, \frac{b}{2}s + c) : s \in \mathbb{R}\}.$$

Given a set  $\mathcal{P}$  of points in  $\mathbb{H}$ , we define the family of light rays

$$\ell^*(\mathcal{P}) = \bigcup_{p \in \mathcal{P}} \ell^*(p). \tag{4.8}$$

**Remark 4.9.** It is worth observing that the point  $(0, x, t - xy/2)$  appearing in formula (4.7) is nearly the vertical projection of  $(x, y, t)$  to the  $xt$ -plane; the actual formula for this projection would be  $\pi_{xt}(x, y, t) = (x, 0, t - xy/2)$ . It follows from this observation that

$$\ell^*((u, 0, v) \cdot (0, y, 0)) = (0, u, v) + L_y, \quad u, v, y \in \mathbb{R}, \tag{4.10}$$

because  $\pi_{xt}((u, 0, v) \cdot (0, y, 0)) = (u, 0, v)$ .

Under the point-line correspondence in Definition 4.6, incidences between points and horizontal lines in  $\mathbb{H}$  are in one-to-one correspondence with incidences between light rays and points in  $\mathbb{R}^3$ .

**Lemma 4.11** (*Incidences are preserved under duality*). For  $p \in \mathbb{H}$  and  $p^* \in \mathbb{R}^3$ , we have

$$p \in \ell(p^*) \iff p^* \in \ell^*(p).$$

**Proof.** Let  $p = (x, y, t) \in \mathbb{H}$  and  $p^* = (a, b, c) \in \mathbb{R}^3$ . The condition  $p \in \ell(p^*)$  is equivalent to

$$\begin{cases} ay + b = x \\ \frac{b}{2}y + c = t. \end{cases}$$

Recalling the notation  $L_y(s) = (s, -sy, sy^2/2)$ , this is further equivalent to

$$p^* = (a, b, c) = (0, x, t - xy/2) + L_y(a). \tag{4.12}$$

Finally, (4.12) is equivalent to  $p^* \in \ell^*(p)$ .  $\square$

4.1. Measures on the space of horizontal lines

The duality  $p \mapsto \ell(p)$  between points in  $p \in \mathbb{R}^3$  and horizontal lines  $\ell(p)$  in Definition 4.6 allows one to push-forward Lebesgue measure “Leb” on  $\mathbb{R}^3$  to construct a measure “ $\mathfrak{m}$ ” on the set of horizontal lines:

$$\mathfrak{m}(\mathcal{L}) := (\ell_{\#}\text{Leb})(\mathcal{L}) = \text{Leb}(\{p \in \mathbb{R}^3 : \ell(p) \in \mathcal{L}\}).$$

There is, however, a more commonly used measure on the space of horizontal lines. This measure “ $\mathfrak{h}$ ” is discussed extensively for example in [6, Section 2.3]. The measure  $\mathfrak{h}$  is the unique (up to a multiplicative constant) non-zero left invariant measure on the set of horizontal lines. One possible formula for it is the following:

$$\mathfrak{h}(\mathcal{L}) = \int_{S^1} \mathcal{H}^3(\{w \in \mathbb{W}_e : \pi_e^{-1}\{w\} \in \mathcal{L}\}) d\mathcal{H}^1(e). \tag{4.13}$$

Let  $f \in L^1(\mathbb{H})$ , and consider the weighted measure  $\mu_f := f d\text{Leb}$ . Then, starting from the definition (4.13), it is easy to check that

$$\int_{S^1} \|\pi_e \mu_f\|_{L^2}^2 d\mathcal{H}^1(e) = \int Xf(\ell)^2 d\mathfrak{h}(\ell), \tag{4.14}$$

where  $Xf(\ell) := \int_{\ell} f d\mathcal{H}^1$ .

While the measure  $\mathfrak{h}$  is mutually absolutely continuous with respect to  $\mathfrak{m}$ , the Radon-Nikodym derivative is not bounded (from above and below): with our current notational conventions, the lines  $\ell(p)$  are never parallel to the  $x$ -axis, and the  $\mathfrak{m}$ -density of lines making a small angle with the  $x$ -axis is smaller than their  $\mathfrak{h}$ -density. The problem can be removed by restricting our considerations to lines which make a substantial angle with the  $x$ -axis. For example, let  $\mathcal{L}_{\angle}$  be the set of horizontal lines which have slope at most 1 relative to the  $y$ -axis; thus

$$\mathcal{L}_{\angle} = \ell(\{(a, b, c) \in \mathbb{R}^3 : |a| \leq 1\}).$$

Then,  $\mathfrak{m}(\mathcal{L}) \sim \mathfrak{h}(\mathcal{L})$  for all Borel sets  $\mathcal{L} \subset \mathcal{L}_{\angle}$ . The lines in  $\mathcal{L}_{\angle}$  coincide with pre-images of the form  $\pi_e^{-1}\{w\}$ ,  $e \in S \subset S^1$ , where  $S$  consists of those vectors making an angle at most  $45^\circ$  with the  $y$ -axis. Now, (4.14) also holds in the following restricted form:

$$\int_S \|\pi_e \mu_f\|_{L^2}^2 d\mathcal{H}^1(e) = \int_{\mathcal{L}_{\angle}} Xf(\ell)^2 d\mathfrak{h}(\ell) \sim \int_{\mathcal{L}_{\angle}} Xf(\ell)^2 d\mathfrak{m}(\ell). \tag{4.15}$$

This equation will be useful in establishing Theorem 5.2. This will, formally, only prove Theorem 5.2 with “ $S$ ” in place of “ $S^1$ ”, but the original version is easy to deduce from this apparently weaker version.

4.2. *Ball-plate duality*

Recall from (4.8) the definition of the (dual) line set  $\ell^*(P)$  for  $P \subset \mathbb{H}$ . What does  $\ell^*(B_{\mathbb{H}}(p, r))$  look like? The answer is: a *plate* tangent to the cone  $\mathcal{C}$ . Informally speaking, for  $r \in (0, \frac{1}{2}]$ , an  $r$ -plate tangent to  $\mathcal{C}$  is a rectangle of dimensions  $\sim (1 \times r \times r^2)$  whose long side is parallel to a light ray, and whose orientation is such that the plate is roughly tangent to  $\mathcal{C}$ , see Fig. 1. To prove rigorously that  $\ell^*(B_{\mathbb{H}}(p, r))$  looks like such a plate (inside  $B(1)$ ), we need to be more precise with the definitions.

Recall that the cone  $\mathcal{C}$  is a rotation of the “standard” cone  $\mathcal{C}_0 = \{(x, y, z) : z^2 = x^2 + y^2\}$ .

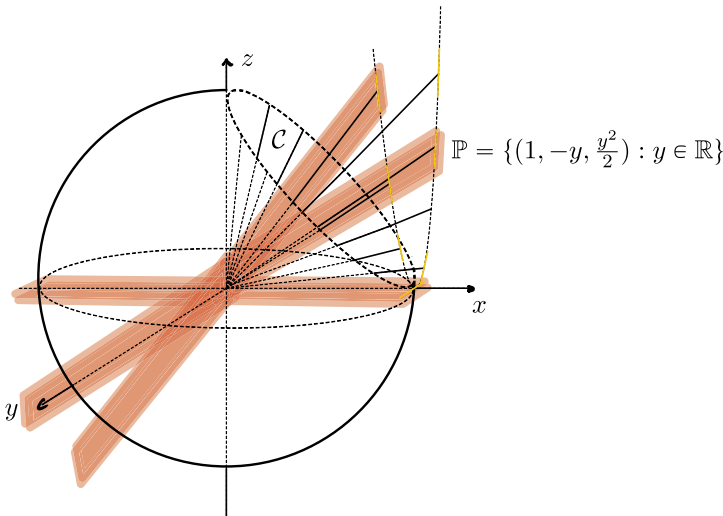


Fig. 1. The cone  $\mathcal{C}$ , the parabola  $\mathbb{P}$ , and three  $r$ -plates.

The intersection of  $\mathcal{C}$  with the plane  $\{x = 1\}$  is the parabola

$$\mathbb{P} = \{(1, -y, y^2/2) : y \in \mathbb{R}\}.$$

For every  $r \in (0, \frac{1}{2}]$  and  $p \in \mathbb{P}$ , choose a rectangle  $\mathcal{R} = \mathcal{R}_r(p)$  of dimensions  $r \times r^2$  in the plane  $\{x = 1\}$ , centred at  $p$ , such that the longer  $r$ -side is parallel to the tangent line of  $\mathbb{P}$  at  $p$ . Then  $\mathbb{P} \cap B(0, cr) \subset \mathcal{R}$  for an absolute constant  $c > 0$ . Now, the  $r$ -plate centred at  $p$  is the set obtained by sliding the rectangle  $\mathcal{R}$  along the light ray containing  $p$  inside  $\{|x| \leq 1\}$ , see Fig. 1. We make this even more formal in the next definition.

**Definition 4.16** ( $r$ -plate). Let  $r \in (0, \frac{1}{2}]$ , and let  $p = (1, -y, y^2/2) \in \mathbb{P} \subset \mathcal{C}$  with  $y \in [-1, 1]$ . Let  $\mathcal{R}_r(0) := [-r, r] \times [-r^2, r^2]$ , and define  $\mathcal{R}_r(y) := M_y(\mathcal{R}_r(0)) \subset \mathbb{R}^2$ , where

$$M_y = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$$

(The rectangle  $\mathcal{R}_r(y)$  is the intersection of an  $r$ -plate with the plane  $\{x = 0\}$ .) Define

$$\mathcal{P}_r(p) := \{(0, \vec{r}) + L_y([-1, 1]) : \vec{r} \in \mathcal{R}_r(y)\},$$

The set  $\mathcal{P}_r(p)$  is called the  $r$ -plate centred at  $p \in \mathbb{P}$ . In general, an  $r$ -plate is any translate of one of the sets  $\mathcal{P}_r(p)$ , for  $p = (1, -y, y^2/2)$  with  $y \in [-1, 1]$ , and  $r \in (0, \frac{1}{2}]$ .

For the  $r$ -plate  $\mathcal{P}_r(p)$ , we also commonly use the notation  $\mathcal{P}_r(y)$ , where  $p = (1, -y, y^2/2)$ .

**Remark 4.17.** Since we require  $y \in [-1, 1]$  in Definition 4.16, it is clear that an  $r$ -plate contains, and is contained in, a rectangle of dimensions  $\sim (1 \times r \times r^2)$ . It is instructive to note that the number of “essentially distinct”  $r$ -plates intersecting  $B(0, 1)$  is roughly  $r^{-4}$ : to see this, take a maximal  $r$ -separated subset of  $\mathbb{P}_r \subset \mathbb{P}$ , and note that for each  $p \in \mathbb{P}_r$ , the plate  $\mathcal{P}_r(p)$  has volume  $r^3$ . Therefore it takes  $\sim r^{-3}$  translates of  $\mathcal{P}_r(p)$  to cover  $B(0, 1)$ . This  $r^{-4}$ -numerology already suggests that the various  $r$ -plates might correspond to Heisenberg  $r$ -balls via duality.

To relate the plates  $\mathcal{P}_r$  to Heisenberg balls, we define a slight modification of the plates  $\mathcal{P}_r$ . Whereas  $\mathcal{P}_r$  is a union of (truncated) light rays in one fixed direction, the following “modified” plates contain full light rays in an  $r$ -arc of directions. These “modified” plates will finally match the duals of Heisenberg balls, see Proposition 4.22.

**Definition 4.18** (*Modified  $r$ -plate*). Let  $r \in (0, \frac{1}{2}]$  and  $y \in [-1, 1]$ . Let  $\mathcal{R}_r(y_0) \subset \mathbb{R}^2$  be the rectangle from Definition 4.16. For  $(u, v) \in \mathbb{R}^2$ , define the *modified  $r$ -plate*

$$\Pi_r(u, v, y) := (0, u, v) + \{(0, \vec{r}) + L_{y'} : \vec{r} \in \mathcal{R}_r(y) \text{ and } |y' - y| \leq r\}. \tag{4.19}$$

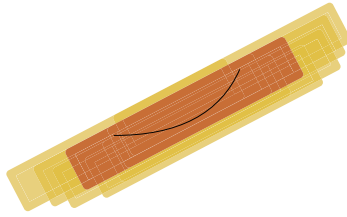
**Remark 4.20.** The relation between the sets  $\mathcal{P}_r$  and  $\Pi_r$  is that the following holds for some absolute constant  $c > 0$ : if  $r \in (0, \frac{1}{2}]$ ,  $y \in [-1, 1]$ , and  $u, v \in \mathbb{R}$ , then

$$\Pi_{cr}(u, v, y) \cap \{(s, y, z) : |s| \leq 2\} \subset (0, u, v) + \mathcal{P}_r(y) \subset \Pi_r(u, v, y). \tag{4.21}$$

(The constant “2” is arbitrary, but happens to be the one we need.) To see this, it suffices to check the case  $u = 0 = v$ . Consider the “slices” of  $\Pi_r(0, 0, y)$  and  $\mathcal{P}_r(y)$  with a fixed plane  $\{x = s\}$  for  $|s| \leq 1$ . If  $s = 0$ , both slices coincide with the rectangle  $\mathcal{R}_y(y)$ . If  $0 < |s| \leq 1$ , the slice  $\Pi_r(0, 0, y) \cap \{x = s\}$  can be written as a sum

$$\Pi_r(0, 0, y) \cap \{x = s\} = \mathcal{R}_r(y) + \{L_{y'}(s) : |y - y'| \leq r\},$$

whereas  $\mathcal{P}_r(y) \cap \{x = s\} = \mathcal{R}_r(y) + \{L_y(s)\}$ . The relationship between these two slices is depicted in Fig. 2. After this, we leave it to the reader to verify that  $\Pi_{cr}(0, 0, y) \cap \{x = s\} \subset \mathcal{P}_r(y) \cap \{x = s\}$  if  $c > 0$  is sufficiently small, and for  $|s| \leq 2$ .



**Fig. 2.** The red box is the slice  $\mathcal{P}_r(y) \cap \{x = s\}$ . The slice  $\Pi_r(0, 0, y) \cap \{x = s\}$  is a union of the yellow boxes centred along the black curve  $\{L_{y'}(s) : |y' - y| \leq r\}$ . All the boxes individually are translates of  $\mathcal{R}_r(y)$ .

We record the following consequence of (4.21):  $\Pi_r(u, v, y) \cap \{(s, y, z) : |s| \leq 1\}$  is contained in a tube of width  $r$  around the line  $(0, u, v) + L_y$ . This is because  $\mathcal{P}_r(y)$  is obviously contained in a tube of width  $\sim r$  around  $L_y$  (this is a very non-sharp statement, using only that the longer side of  $\mathcal{R}_y(r)$  has length  $r$ .)

We then show that the  $\ell^*$ -duals of Heisenberg balls are essentially modified plates:

**Proposition 4.22.** *Let  $p = (u_0, 0, v_0) \cdot (0, y_0, 0)$ ,  $r \in (0, \frac{1}{2}]$ , and  $B := B_{\mathbb{H}}(p, r)$ . Then,*

$$\ell^*(B) \subset \Pi_{2r}(u_0, v_0, y_0) \subset \ell^*(CB), \tag{4.23}$$

where  $C > 0$  is an absolute constant, and  $CB = B_{\mathbb{H}}(p, Cr)$ .

**Remark 4.24.** To build a geometric intuition, it will be helpful to notice the following. The  $y$ -coordinate of the point  $p = (u_0, 0, v_0) \cdot (0, y_0, 0) = (u_0, y_0, v_0 + \frac{1}{2}u_0y_0)$  is “ $y_0$ ”. On the other hand, while the modified plate  $\Pi_{2r}(u_0, v_0, y_0)$  contains many lines, they are all “close” to the “central” line  $(0, u_0, v_0) + L_{y_0}$  (see Definition 4.19). According to the inclusions in (4.23), this means that the “direction”  $L_{y_0}$  of the modified plate containing the dual  $\ell^*(B(p, r))$  is determined by the  $y$ -coordinate of  $p$ . Even less formally: *Heisenberg balls whose centres have the same  $y$ -coordinate are dual to parallel plates.*

**Proof of Proposition 4.22.** To prove the inclusion  $\ell^*(B) \subset \Pi_{2r}(u_0, v_0, y_0)$ , let  $q \in B_{\mathbb{H}}(p, r)$ , and write  $q := (u, 0, v) \cdot (0, y, 0)$  with  $(u, v) \in \mathbb{R}^2$  and  $y \in \mathbb{R}$ . First, we note that

$$|y - y_0| \leq d_{\mathbb{H}}(p, q) \leq r. \tag{4.25}$$

Let  $\pi_{xt}$  be the vertical projection to the  $xt$ -plane  $\{(u', 0, v') : u', v' \in \mathbb{R}\}$ . Then  $(u, 0, v) = \pi_{xt}(q) \in \pi_{xt}(B)$  by the definition of  $\pi_{xt}$ . We now observe that  $B = (u_0, 0, v_0) \cdot B_{\mathbb{H}}((0, y_0, 0), r)$ , so

$$\pi_{xt}(B) = (u_0, 0, v_0) + \pi_{xt}(B_{\mathbb{H}}((0, y_0, 0), r)).$$

We claim that

$$\pi_{xt}(B_{\mathbb{H}}((0, y_0, 0), r)) \subset \{(u', 0, v') : (u', v') \in \mathcal{R}_{2r}(y_0)\}. \tag{4.26}$$

This will prove that

$$(u, 0, v) \in (u_0, 0, v_0) + \{(u', 0, v') : (u', v') \in \mathcal{R}_{2r}(y_0)\}. \tag{4.27}$$

Recalling the definition (4.19), a combination of (4.25) and (4.27) now shows that

$$\ell^*(q) = \ell^*((u, 0, v) \cdot (0, y, 0)) \stackrel{(4.10)}{=} (0, u, v) + L_y \subset \Pi_{2r}(u_0, v_0, y_0).$$

This will complete the proof of the inclusion  $\ell^*(B) \subset \Pi_{2r}(u_0, v_0, y_0)$ .

Let us then prove (4.26). Pick  $(x, y, t) \in B_{\mathbb{H}}((0, y_0, 0), r)$ . Then,

$$\|(x, y - y_0, t + \frac{1}{2}xy_0)\| = d_{\mathbb{H}}((x, y, t), (0, y_0, 0)) \leq r,$$

so

$$|x| \leq r, \quad |y - y_0| \leq r, \quad \text{and} \quad |t + \frac{1}{2}xy_0| \leq r^2. \tag{4.28}$$

Now, to prove (4.26), recall that  $\pi_{xt}(x, y, t) = (x, 0, t - \frac{1}{2}xy)$ . Thus, we need to show that  $(x, t - \frac{1}{2}xy) \in \mathcal{R}_{2r}(y_0) = M_{y_0}(\mathcal{R}_{2r}(0))$ . Equivalently,  $M_{y_0}^{-1}(x, t - \frac{1}{2}xy) \in \mathcal{R}_{2r}(0)$ . Recalling the definition of  $M_y$ , one checks that

$$\begin{aligned} M_{y_0}^{-1}(x, t - \frac{1}{2}xy) &= \begin{pmatrix} 1 & 0 \\ y_0 & 1 \end{pmatrix} (x, t - \frac{1}{2}xy) \\ &= (x, xy_0 + t - \frac{1}{2}xy) \\ &= (x, t + \frac{1}{2}xy_0 + \frac{1}{2}x(y_0 - y)). \end{aligned}$$

Using (4.28), we finally note that the point on the right lies in the parabolic rectangle  $\mathcal{R}_{2r}(0)$ . This concludes the proof of (4.26).

Let us then prove the inclusion  $\Pi_r(u_0, v_0, y_0) \subset \ell^*(CB)$ . The set  $\Pi_r(u_0, v_0, y_0)$  is a union of the lines  $(0, u_0, v_0) + (0, \vec{r}) + L_y$ , where  $\vec{r} \in \mathcal{R}_r(y_0)$  and  $|y - y_0| \leq r$ . We need to show that every such line can be realised as  $\ell^*(q)$  for some  $q \in B_{\mathbb{H}}(p, Cr)$ . In this task, we are aided by the formula

$$\ell^*((u, 0, v) \cdot (0, y, 0)) = (0, u, v) + L_y$$

observed in (4.7). This formula shows that we *need* to define  $q := (u, 0, v) \cdot (0, y, 0)$ , where  $(u, v) := (u_0, v_0) + \vec{r}$ , and  $y$  is as in “ $L_y$ ”. Then we just have to hope that  $q \in B_{\mathbb{H}}(p, Cr)$ .

Recalling that  $p = (u_0, 0, v_0) \cdot (0, y_0, 0)$ , one can check by direct computation that

$$d_{\mathbb{H}}(p, q) = \|(u_0 - u, y_0 - y, v_0 - v + y_0(u_0 - u) + \frac{1}{2}(u - u_0)(y_0 - y))\|. \tag{4.29}$$

On the other hand, one may easily check that  $(u, v) \in (u_0, v_0) + \mathcal{R}_r(y_0)$  is equivalent to

$$(u - u_0, v - v_0 + y_0(u - u_0)) \in \mathcal{R}_r(0),$$

which implies  $|u - u_0| \leq r$  and  $|v - v_0 + y_0(u - u_0)| \leq r^2$ . Since moreover  $|y - y_0| \leq r$  by assumption, it follows from (4.29) and the definition of the norm  $\|\cdot\|$  that  $d_{\mathbb{H}}(p, q) \lesssim r$ . This completes the proof.  $\square$

We close the section with two additional auxiliary results:

**Proposition 4.30.** *Let  $p, q \in \mathbb{H}$  and  $r \in (0, \frac{1}{2}]$ , and assume that  $\|p\| \leq 1/10$ . Assume moreover that  $\ell^*(p) \cap B(1) \subset \ell^*(B_{\mathbb{H}}(q, r))$ . Then  $p \in B_{\mathbb{H}}(q, Cr)$  for some absolute constant  $C > 0$ .*

**Proof.** Write  $p = (u, 0, v) \cdot (0, y, 0)$ , so that  $\ell^*(p) = (0, u, v) + L_y$ . Since  $\|p\| \leq 1/10$ , in particular  $|u| + |v| \leq 1/5$ . By the previous proposition, we already know that

$$[(0, u, v) + L_y] \cap B(1) = \ell^*(p) \cap B(1) \subset \Pi_{2r}(u_0, v_0, y_0),$$

where we have written  $q = (u_0, 0, v_0) \cdot (0, y_0, 0)$ . Since  $(0, u, v) \in B(1)$ , we know that  $(0, u, v) \in \ell^*(p) \cap \Pi_{2r}(u_0, v_0, y_0)$ . But

$$\Pi_{2r}(u_0, v_0, y_0) \cap \{x = 0\} = \{(0, u', v') : (u', v') \in (u_0, v_0) + \mathcal{R}_{y_0}(r)\},$$

so we may deduce that

$$(u, v) \in (u_0, v_0) + \mathcal{R}_{y_0}(r). \tag{4.31}$$

Moreover, in Remark 4.20 we noted that  $\Pi_{2r}(u_0, v_0, y_0) \cap B(1)$  is contained in the  $\sim r$ -neighbourhood  $T$  of the line  $(0, u_0, v_0) + L_{y_0}$ . Therefore also  $(0, u, v) + L_y \cap B(1) \subset T$ . This implies that  $\angle(L_y, L_{y_0}) \lesssim r$ , and hence  $|y - y_0| \lesssim r$ .

Now, we want to use (4.31) and  $|y - y_0| \lesssim r$  to deduce that  $d_{\mathbb{H}}(p, q) \lesssim r$ . We first expand

$$d_{\mathbb{H}}(p, q) = \|(u_0 - u, y_0 - y, v_0 - v + y_0(u_0 - u) + \frac{1}{2}(u - u_0)(y_0 - y)\|. \tag{4.32}$$

Then, using the definition of  $\mathcal{R}_{y_0}(r) = M_y(\mathcal{R}_0(r))$ , we note that (4.31) is equivalent to

$$(u - u_0, v - v_0 + y_0(u - u_0)) \in \mathcal{R}_r(0).$$

Combined with  $|y - y_0| \lesssim r$ , and recalling the definition of  $\|\cdot\|$ , this shows that the right hand side of (4.32) is bounded by  $\lesssim r$ , as claimed.  $\square$

We already noted in Remark 4.24 that the (modified)  $2r$ -plates containing  $\ell^*(B(p_1, r))$  and  $\ell^*(B(p_2, r))$  have (almost) the same direction if the points  $p_1, p_2$  have (almost) the same  $y$ -coordinate. In this case, if  $d_{\mathbb{H}}(p_1, p_2) \geq Cr$ , it is natural to expect that  $\ell^*(B(p_1, r))$  and  $\ell^*(B(p_2, r))$  are disjoint, at least inside  $B(1)$ . The next lemma verifies this intuition.

**Lemma 4.33.** *Let  $p_1 = (u_1, 0, v_1) \cdot (0, y_1, 0) \in B_{\mathbb{H}}(1)$  and  $p_2 = (u_2, 0, v_2) \cdot (0, y_2, 0) \in B_{\mathbb{H}}(1)$  be points with the properties*

$$|y_1 - y_2| \leq r \quad \text{and} \quad \ell^*(B_{\mathbb{H}}(p_1, r)) \cap \ell^*(B_{\mathbb{H}}(p_2, r)) \cap B(1) \neq \emptyset. \tag{4.34}$$

Then,  $d_{\mathbb{H}}(p_1, p_2) \lesssim r$ .

**Proof.** We may reduce to the case  $y_1 = y_2$  by the following argument. Start by choosing a point  $p'_2 \in B_{\mathbb{H}}(p_2, r)$  such that the  $y$ -coordinate of  $p'_2$  equals  $y_1$ . This is possible, because  $|y_1 - y_2| \leq r$ , and the projection of  $B_{\mathbb{H}}(p_2, r)$  to the  $xy$ -plane is a Euclidean disc of radius  $r$ . Then, notice that  $B_{\mathbb{H}}(p_2, r) \subset B_{\mathbb{H}}(p'_2, 2r)$ , so

$$\ell^*(B_{\mathbb{H}}(p_1, 2r)) \cap \ell^*(B_{\mathbb{H}}(p'_2, 2r)) \cap B(1) \neq \emptyset.$$

Now, if we have already proven the lemma in the case  $y_1 = y_2$  (and for “ $2r$ ” in place of “ $r$ ”), it follows that  $d_{\mathbb{H}}(p_1, p'_2) \lesssim r$ , and finally  $d_{\mathbb{H}}(p_1, p_2) \leq d_{\mathbb{H}}(p_1, p'_2) + d_{\mathbb{H}}(p'_2, p_2) \lesssim r$ .

Let us then assume that  $y_1 = y_2 = y$ . It follows from (4.34) and the first inclusion in Proposition 4.22 combined with the first inclusion in (4.21) that

$$((0, u_1, v_1) + \mathcal{P}_{Cr}(y)) \cap ((0, u_2, v_2) + \mathcal{P}_{Cr}(y)) \neq \emptyset$$

for some absolute constant  $C > 0$ . Let “ $x$ ” be a point in the intersection, and (using the definition of  $\mathcal{P}_{Cr}(y)$ ), express  $x$  in the two following ways:

$$(0, u_1, v_1) + (0, \vec{r}_1) + L_y(s) = x = (0, u_2, v_2) + (0, \vec{r}_2) + L_y(s),$$

where  $\vec{r}_1 \in \mathcal{R}_{Cr}(y) = M_y(\mathcal{R}_{Cr}(0))$  and  $\vec{r}_2 \in M_y(\mathcal{R}_r(0))$ , and  $s \in [-1, 1]$ . The terms  $L_y(s)$  conveniently cancel out, and we find that

$$(u_1, v_1) - (u_2, v_2) = \vec{r}_2 - \vec{r}_1 \in M_y(\mathcal{R}_{2Cr}(0)),$$

or equivalently

$$(u_1 - u_2, v_1 - v_2 + y(u_1 - u_2)) = M_y^{-1}(u_1 - u_2, v_1 - v_2) \in \mathcal{R}_{2Cr}(0). \tag{4.35}$$

We have already computed in (4.32) that

$$d_{\mathbb{H}}(p_1, p_2) = \|(u_1 - u_2, 0, v_1 - v_2 + y(u_1 - u_2))\|,$$



and now it follows immediately from (4.35) that  $d_{\mathbb{H}}(p_1, p_2) \lesssim r$ .  $\square$

**5. Discretising Theorem 1.7**

The purpose of this section is to reduce the proof of Theorem 1.7 to Theorem 5.2 which concerns  $(\delta, 3)$ -sets. We start by defining these precisely:

**Definition 5.1** ( $(\delta, t, C)$ -set). Let  $(X, d)$  be a metric space, and let  $t \geq 0$  and  $C, \delta > 0$ . A non-empty bounded set  $P \subset X$  is called a  $(\delta, t, C)$ -set if

$$|P \cap B(x, r)|_{\delta} \leq Cr^t \cdot |P|_{\delta}, \quad x \in X, r \geq \delta.$$

Here  $|A|_{\delta}$  is the smallest number of balls of radius  $\delta$  needed to cover  $A$ . A family of sets  $\mathcal{B}$  (typically: disjoint  $\delta$ -balls) is called a  $(\delta, t, C)$ -set if  $P := \cup \mathcal{B}$  is a  $(\delta, t, C)$ -set.

If  $P \subset \mathbb{H}$ , or  $\mathcal{B} \subset \mathcal{P}(\mathbb{H})$ , the  $(\delta, t, C)$ -set condition is always tested relative to the metric  $d_{\mathbb{H}}$ . We then state a  $\delta$ -discretised version of Theorem 1.7 for sets of dimension 3:

**Theorem 5.2.** *For every  $\eta > 0$ , there exists  $\epsilon > 0$  and  $\delta_0 > 0$  such that the following holds for all  $\delta \in (0, \delta_0]$ . Let  $\mathcal{B}$  be a non-empty  $(\delta, 3, \delta^{-\epsilon})$ -set of  $\delta$ -balls contained in  $B_{\mathbb{H}}(1)$ , with  $\delta$ -separated centres. Let  $\mu = \mu_f$  be the measure on  $\mathbb{H}$  with density*

$$f := (\delta^4 |\mathcal{B}|)^{-1} \sum_{B \in \mathcal{B}} \mathbf{1}_B. \tag{5.3}$$

Then,

$$\int_{S^1} \|\pi_e \mu\|_{L^2}^2 d\mathcal{H}^1(e) \leq \delta^{-\eta}.$$

The proof of Theorem 5.2 will be given in Section 6. Deducing Theorem 1.7 from Theorem 5.2 involves two steps. The first one, carried out in Section 7, is to reduce Theorem 1.7 to a  $\delta$ -discretised version, which concerns  $(\delta, t)$ -sets with all possible values  $t \in [0, 3]$ . This statement is Theorem 5.11 below, a simplified version of which was stated as Theorem 1.10 in the introduction.

The second – and less standard – step, carried out in this section, is to deduce Theorem 5.11 from Theorem 5.2. Heuristically, Theorem 5.2 is nothing but the 3-dimensional case of Theorem 5.11 – although in this case the statement looks more quantitative. We therefore need to argue that *if we already have Theorem 5.11 for sets of dimension 3, then we also have it for sets of dimension  $t \in [0, 3]$* . The heuristic is simple: given a set  $K \subset \mathbb{H}$  of dimension  $t \in [0, 3]$ , we start by “adding” (from the left) to  $K$  another – random – set  $H \subset \mathbb{H}$  of dimension  $3 - t$ . Then, we apply the 3-dimensional version of Theorem 5.11 to  $H \cdot K$ , and this gives the correct conclusion for  $K$ . A crucial point is

that Theorem 5.11 concerns the Lebesgue measure (not the dimension) of  $\pi_e(K)$ . This quantity is invariant under left translating  $K$ . This allows us to control  $\text{Leb}(\pi_e(H \cdot K))$  in a useful way.

We turn to the details. To deduce Theorem 1.7 from Theorem 5.2, we need a corollary of Theorem 5.2, stated in Corollary 5.6, which concerns slightly more general measures than ones of the form  $\mu = \mu_f$  (as in (5.3)):

**Definition 5.4** ( *$\delta$ -measure*). Let  $\delta \in (0, 1]$  and  $C > 0$ . A Borel measure  $\mu$  on  $\mathbb{H}$  is called a  $(\delta, C)$ -measure if  $\mu$  has a density with respect to Lebesgue measure, also denoted  $\mu$ , and the density satisfies

$$\mu(x) \leq C \cdot \frac{\mu(B_{\mathbb{H}}(x, \delta))}{\text{Leb}(B_{\mathbb{H}}(x, \delta))}, \quad x \in \mathbb{H}.$$

If the constant  $C > 0$  irrelevant, a  $(\delta, C)$ -measure may also be called a  $\delta$ -measure.

We will use the following notion of  $\delta$ -truncated Riesz energy:

$$I_s^\delta(\mu) := \iint \frac{d\mu(x) d\mu(y)}{d_{\mathbb{H},\delta}(x, y)^s}, \tag{5.5}$$

where  $\mu$  is a Radon measure,  $0 \leq s \leq 4$ , and  $d_{\mathbb{H},\delta}(x, y) := \max\{d_{\mathbb{H}}(x, y), \delta\}$ .

**Corollary 5.6.** *For every  $\eta > 0$ , there exists  $\delta_0, \epsilon_0 > 0$  such that the following holds for all  $\delta \in (0, \delta_0]$  and  $\epsilon \in (0, \epsilon_0]$ . Let  $\mu$  be a  $(\delta, \delta^{-\epsilon})$ -probability measure on  $B_{\mathbb{H}}(1)$  with  $I_3^\delta(\mu) \leq \delta^{-\epsilon}$ . Then, there exists a Borel set  $G \subset \mathbb{H}$  such that  $\mu(G) \geq 1 - \delta^{\epsilon_0}$ , and*

$$\int_{S^1} \|\pi_e(\mu|_G)\|_{L^2}^2 d\mathcal{H}^1(e) \leq \delta^{-\eta}. \tag{5.7}$$

**Proof.** Fix  $\eta > 0$ ,  $\epsilon \in (0, \epsilon_0]$ , and  $\delta \in (0, \delta_0]$ . The dependence of  $\delta_0, \epsilon_0$  on  $\eta$  will eventually be determined by an application of Theorem 5.2, but we will require at least that  $\epsilon_0 \leq \eta$ .

It follows from  $I_3^\delta(\mu) \leq \delta^{-\epsilon}$  and Chebychev’s inequality that there exists a set  $G_0 \subset \mathbb{H}$  of measure  $\mu(G_0) \geq 1 - 3\delta^{\epsilon_0}$  such that  $\mu(B_{\mathbb{H}}(x, r)) \lesssim \delta^{-\epsilon-\epsilon_0} r^3 \leq \delta^{-2\epsilon_0} r^3$  for all  $x \in G_0$  and  $r \geq \delta$ . Now, for dyadic rationals  $0 < \alpha \lesssim \delta^{3-2\epsilon_0} \leq \delta^2$ , let

$$G_{0,\alpha} := \{x \in G_0 : \frac{\alpha}{2} \leq \mu(B_{\mathbb{H}}(x, \delta)) \leq \alpha\}.$$

We discard immediately the sets  $G_{0,\alpha}$  with  $\alpha \leq \delta^{10}$ : the union of these sets has measure  $\leq \delta^5 \leq \delta^{\epsilon_0}$  for  $\delta > 0$  small enough, so  $\mu(G_1) \geq 1 - 2\delta^{\epsilon_0}$ , where

$$G_1 := G_0 \setminus \bigcup_{\alpha \leq \delta^{10}} G_{0,\alpha}.$$

Now,  $G_1$  is covered by the sets  $G_{0,\alpha}$  with  $\delta^{10} \leq \alpha \lesssim \delta^2$ , and the number of such sets is  $m \lesssim \log(1/\delta)$ . We let  $\{\alpha_1, \dots, \alpha_m\}$  be an enumeration of these values of “ $\alpha$ ”, and we abbreviate  $G^j := G_{0,\alpha_j}$ . We note that the union of the sets  $G^j$  with  $\mu(G^j) \leq \delta^{2\epsilon_0}$  has measure at most  $m \cdot \delta^{2\epsilon_0} \leq \delta^{\epsilon_0}$  (for  $\delta > 0$  small), so finally

$$G := G_1 \setminus \bigcup \{G^j : 1 \leq j \leq m \text{ and } \mu(G^j) \leq \delta^{2\epsilon_0}\}$$

has measure  $\mu(G) \geq 1 - 2\delta^{\epsilon_0} - \delta^{\epsilon_0} \geq 1 - \delta^{\epsilon_0}$ . Moreover,  $G$  is covered by the sets  $G^j$  with  $\mu(G^j) \geq \delta^{2\epsilon_0}$ . Re-indexing if necessary, we now assume that  $\mu(G^j) \geq \delta^{2\epsilon_0}$  for all  $1 \leq j \leq m$ .

For  $1 \leq j \leq m$  fixed, let  $\mathcal{B}_j$  be a finitely overlapping (Vitali) cover of  $G^j$  by balls of radius  $\delta$ , centred at  $G^j$ . Using the facts  $G^j \subset G_0$  and  $\mu(G^j) \geq \delta^{2\epsilon_0}$ , and the uniform lower bound  $\mu(B_{\mathbb{H}}(x, \delta)) \geq \alpha_j/2$  for  $x \in G^j$ , it is easy to check that each  $\mathcal{B}_j$  is a  $(\delta, 3, \delta^{-C\epsilon_0})$ -set with

$$|\mathcal{B}_j| \lesssim \alpha_j^{-1}. \tag{5.8}$$

Thus, writing

$$f_j := (\delta^4 |\mathcal{B}_j|^{-1}) \sum_{B \in \mathcal{B}_j} \mathbf{1}_B \quad \text{and} \quad \mu_j := \mu_{f_j},$$

and assuming that  $\delta_0, \epsilon_0 > 0$  are sufficiently small in terms of  $\eta$ , we may deduce from Theorem 5.2 that

$$\int_{S^1} \|\pi_e(\mu_j)\|_{L^2}^2 d\mathcal{H}^1(e) \leq \delta^{-\eta}, \quad 1 \leq j \leq m.$$

Finally, it follows from the  $(\delta, \delta^{-\epsilon})$ -property of  $\mu$  that

$$\mu(x) \lesssim \delta^{-\epsilon} \cdot \frac{\mu(B_{\mathbb{H}}(x, \delta))}{\delta^4} \leq \delta^{-\epsilon} \cdot \frac{\alpha_j}{\delta^4} \stackrel{(5.8)}{\lesssim} \frac{\delta^{-\epsilon}}{\delta^4 |\mathcal{B}_j|} \leq \delta^{-\epsilon} \cdot \mu_j(x), \quad x \in G^j.$$

Thus, also the density of  $\pi_e(\mu|_{G^j})$  is bounded from above by the density of  $\pi_e(\mu_j)$ :

$$\int_{S^1} \|\pi_e(\mu|_G)\|_{L^2}^2 d\mathcal{H}^1(e) \lesssim \delta^{-\epsilon} \sum_{j=1}^m \int_{S^1} \|\pi_e(\mu_j)\|_{L^2}^2 d\mathcal{H}^1(e) \lesssim \log(1/\delta) \cdot \delta^{-\eta-\epsilon} \leq \delta^{-3\eta}.$$

This completes the proof of (5.7) (with “ $3\eta$ ” in place of “ $\eta$ ”).  $\square$

The concrete  $\delta$ -measures we will consider have the form  $\eta *_{\mathbb{H}} \mu$ , where  $\mu = \mu_f$  has a density of the form (5.3) (these are almost trivially  $\delta$ -measures), and  $\eta$  is a (discrete) probability measure. The notation  $\eta *_{\mathbb{H}} \mu$  refers to the (non-commutative!) Heisenberg

convolution of  $\eta$  and  $\mu$ , that is, the push-forward of  $\eta \times \mu$  under the group product  $(p, q) \mapsto p \cdot q$ . Let us verify that such measures  $\eta *_{\mathbb{H}} \mu$  are also  $\delta$ -measures:

**Lemma 5.9.** *Let  $\mu$  be  $(\delta, C)$  measure, and let  $\eta$  be an arbitrary Borel probability measure on  $\mathbb{H}$ . Then  $\eta *_{\mathbb{H}} \mu$  is again a  $(\delta, C)$ -measure.*

**Proof.** Recall that a  $(\delta, C)$  measure is absolutely continuous by definition, so the notation “ $\mu(p)$ ” is well-defined for Lebesgue almost every  $p \in \mathbb{H}$ . The following formulae are valid, and easy to check, for Lebesgue almost every  $p \in \mathbb{H}$ :

$$(\eta *_{\mathbb{H}} \mu)(p) = \int \mu(q^{-1} \cdot p) d\eta(q)$$

and

$$\frac{(\eta *_{\mathbb{H}} \mu)(B_{\mathbb{H}}(p, r))}{\text{Leb}(B_{\mathbb{H}}(p, r))} = \int \frac{\mu(B_{\mathbb{H}}(q^{-1} \cdot p, r))}{\text{Leb}(B_{\mathbb{H}}(p, r))} d\eta(q). \tag{5.10}$$

Now, if one applies the  $\delta$ -measure assumption to the formula on the left hand side, one obtains

$$(\eta *_{\mathbb{H}} \mu)(p) \leq C \int \frac{\mu(B_{\mathbb{H}}(q^{-1} \cdot p, \delta))}{\text{Leb}(B_{\mathbb{H}}(q^{-1} \cdot p, \delta))} d\eta(q).$$

Lebesgue measure is invariant under left translations, so

$$\text{Leb}(B_{\mathbb{H}}(q^{-1} \cdot p, \delta)) = \text{Leb}(B_{\mathbb{H}}(p, \delta)).$$

Therefore, it follows from equation (5.10) that

$$(\eta *_{\mathbb{H}} \mu)(p) \leq C \cdot \frac{(\eta *_{\mathbb{H}} \mu)(B_{\mathbb{H}}(p, \delta))}{\text{Leb}(B_{\mathbb{H}}(p, \delta))}$$

for Lebesgue almost every  $p \in \mathbb{H}$ . This is what we claimed.  $\square$

We are then ready to state and prove the  $\delta$ -discretised counterpart of Theorem 1.7.

**Theorem 5.11.** *Let  $0 \leq s < t \leq 3$ . Then, there exist  $\epsilon, \delta_0 > 0$ , depending only on  $s, t$ , such that the following holds for all  $\delta \in (0, \delta_0]$ . Let  $\mathcal{B} \neq \emptyset$  be a  $(\delta, t, \delta^{-\epsilon})$  set of  $\delta$ -balls with  $\delta$ -separated centres, all contained in  $B_{\mathbb{H}}(1)$ , and let  $S \subset S^1$  be a Borel set of length  $\mathcal{H}^1(S) \geq \delta^\epsilon$ . Then, there exists  $e \in S$  such that the following holds: if  $\mathcal{B}' \subset \mathcal{B}$  is any sub-family with  $|\mathcal{B}'| \geq \delta^\epsilon |\mathcal{B}|$ , then*

$$\text{Leb}(\pi_e(\cup \mathcal{B}')) \geq \delta^{3-s}.$$

*In particular,  $\pi_e(\cup \mathcal{B}')$  cannot be covered by fewer than  $\delta^{-s}$  parabolic balls of radius  $\delta$ .*

**Proof.** To reach a contradiction, assume that there exists a  $(\delta, t, \delta^{-\epsilon})$ -set  $\mathcal{B}$  of  $\delta$ -balls with  $\delta$ -separated centres, contained in  $B_{\mathbb{H}}(1)$ , and violating the conclusion of Theorem 5.11: there exists  $s < t$ , and for every  $e \in S$  (Borel subset of  $S^1$  of length  $\mathcal{H}^1(S) \geq \delta^\epsilon$ ), there exists a subset  $\mathcal{B}_e \subset \mathcal{B}$  with  $|\mathcal{B}_e| \geq \delta^\epsilon |\mathcal{B}|$  with the property

$$\text{Leb}(\pi_e(\cup \mathcal{B}_e)) \leq \delta^{3-s}. \tag{5.12}$$

We aim for a contradiction if  $\epsilon, \delta$  are sufficiently small. We fix an auxiliary parameter  $0 < \eta < (t - s)/2$ . Then, we apply Corollary 5.6 to find the constant  $\epsilon_0 > 0$  which depends only on  $\eta$ . Finally, we will assume, presently, that  $\epsilon < \epsilon_0/2$ , and  $\eta + 3\epsilon < t - s$ .

Let  $\mu$  be the uniformly distributed probability measure on  $\cup \mathcal{B}$ ; in particular  $\mu$  is a  $\delta$ -measure (with absolute constant), and  $I_t^\delta(\mu) \lesssim \delta^{-\epsilon}$ . Apply Proposition A.1 to find a set  $H \subset B_{\mathbb{H}}(1)$  of cardinality  $|H| \leq \delta^{t-3}$  such that  $I_3^\delta(\tau *_{\mathbb{H}} \mu) \lesssim \delta^{-\epsilon}$ , where  $\tau$  is the uniformly distributed probability measure on  $H$ . Write  $\nu := \tau *_{\mathbb{H}} \mu$ , so  $\nu$  is a  $\delta$ -probability measure by Lemma 5.9. Since  $\epsilon < \epsilon_0/2$  and  $I_3^\delta(\nu) \lesssim \delta^{-\epsilon}$ , it follows from Corollary 5.6 that there exists a set  $G \subset \mathbb{H}$  of measure  $\nu(G) \geq 1 - \delta^{\epsilon_0}$  such that

$$\frac{1}{\mathcal{H}^1(S)} \int_S \|\pi_e(\nu|_G)\|_{L^2}^2 d\mathcal{H}^1(e) \leq \frac{1}{\mathcal{H}^1(S)} \int_{S^1} \|\pi_e(\nu|_G)\|_{L^2}^2 d\mathcal{H}^1(e) \leq \delta^{-\eta-\epsilon}. \tag{5.13}$$

Finally, write  $B_e := H \cdot (\cup \mathcal{B}_e)$  for all  $e \in S^1$ , and note that  $\nu(B_e) \geq \delta^\epsilon$  for all  $e \in S$  (this is a consequence of the general inequality  $(\mu_1 *_{\mathbb{H}} \mu_2)(A \cdot B) \geq (\mu_1 \times \mu_2)(A \times B)$ ). Consequently, also  $\nu(G \cap B_e) \geq \nu(G) + \nu(B_e) - 1 \geq \delta^\epsilon - \delta^{\epsilon_0} \geq \delta^\epsilon/2$ , using  $\epsilon < \epsilon_0/2$ . Therefore,

$$\delta^{2\epsilon}/4 \leq \|\pi_e(\nu|_{G \cap B_e})\|_{L^1}^2 \leq \text{Leb}(\pi_e(B_e)) \cdot \|\pi_e(\nu|_G)\|_{L^2}^2, \quad e \in S^1,$$

using Cauchy-Schwarz, and it follows from (5.13) that  $\text{Leb}(\pi_e(B_e)) \gtrsim \delta^{\eta+3\epsilon}$  for at least one vector  $e \in S$ . On the other hand, note that  $B_e = H \cdot (\cup \mathcal{B}_e)$  is a union of  $\leq \delta^{t-3}$  left translates of  $\cup \mathcal{B}_e$ , and recall from (2.1) that

$$\text{Leb}(\pi_e(p \cdot B)) = \text{Leb}(\pi_e(B)), \quad p \in \mathbb{H}, B \subset \mathbb{H}.$$

Therefore, we have the upper bound

$$\text{Leb}(\pi_e(B_e)) = \text{Leb}(\pi_e(H \cdot (\cup \mathcal{B}_e))) \stackrel{(5.12)}{\leq} \delta^{t-3} \cdot \delta^{3-s} = \delta^{t-s}, \quad e \in S^1.$$

Since  $\eta + 3\epsilon < t - s$  by assumption, the previous lower and upper bounds for  $\text{Leb}(\pi_e(B_e))$  are not compatible for  $\delta > 0$  small enough. A contradiction has been reached.  $\square$

### 6. Keakeya estimate of Guth, Wang, and Zhang

The purpose of this section is to prove Theorem 5.2. This will be based on the duality between horizontal lines and light rays developed in Section 4, and an application of a (reverse) square function inequality for the cone, due to Guth, Wang, and Zhang [10]. To be precise, we will not need the full power of this “oscillatory” statement, but rather only a *Keakeya inequality for plates* in [10, Lemma 1.4]. To introduce the statement, we need to recap some of the terminology and notation in [10]. This discussion follows [10, Section 1], but we prefer a different scaling: more precisely, in our discussion the geometric objects (plates and rectangles) of [10] are dilated by “ $R$ ” on the frequency side and (consequently) by  $R^{-1}$  on the spatial side.

Fix  $R \geq 1$ , and let

$$\Gamma := \Gamma_R := \mathcal{C} \cap \{R/2 \leq |\xi| \leq R\}. \tag{6.1}$$

Let  $\Gamma(1)$  be the 1-neighbourhood of  $\Gamma$ , and let  $\Theta := \Theta_R$  be a finitely overlapping cover of  $\Gamma(1)$  by rectangles of dimensions  $R \times R^{1/2} \times 1$ , whose longest side is parallel to a light ray. The statements in [10] are not affected by the particular construction of  $\Theta$ , but in our application, the relevant rectangles are translates of dual rectangles of the  $\delta$ -plates in Definition 4.16, with  $\delta = R^{-1/2}$ . Indeed,  $\delta$ -plates are rectangles of dimensions  $\sim \delta^2 \times \delta \times 1$  tangent to  $\mathcal{C}$ , so their dual rectangles are plates of dimensions  $\sim R \times R^{1/2} \times 1$ , also tangent to  $\mathcal{C}$  (this is because  $\mathcal{C}$  has opening angle  $\pi/2$ , see Fig. 3). For concreteness, we will use translated duals of  $R^{-1/2}$ -plates (as in Definition 4.16) to form the collection  $\Theta$ .

For each  $\theta \in \Theta$ , let  $f_\theta \in L^2(\mathbb{R}^3)$  be a function with  $\text{spt } \hat{f}_\theta \subset \theta$ , and consider the square function

$$Sf := \left( \sum_{\theta \in \Theta} |f_\theta|^2 \right)^{1/2}.$$

Then, [10, Lemma 1.4] contains an inequality of the following form:

$$\int_{\mathbb{R}^3} |Sf|^4 \lesssim \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau)=s} \sum_{U \parallel U_\tau} \text{Leb}(U)^{-1} \|S_U f\|_{L^2}^4. \tag{6.2}$$

To understand the meaning of the “partial” square functions  $S_U$  we need to introduce more terminology from [10]. Fix a dyadic number  $s \in [R^{-1/2}, 1]$  (an “angular” parameter), and write  $R' := s^2 R \in [1, R]$ . The 1-neighbourhood of the truncated cone  $\Gamma_{R'} = \mathcal{C} \cap \{|\xi| \sim R'\}$  can be covered by a finitely overlapping family  $\Theta_{R'}$  of rectangles of dimensions

$$R' \times (R')^{1/2} \times 1 = s^2 R \times s R^{1/2} \times 1.$$

(Here  $\Theta_R$  agrees with  $\Theta$ , as defined above.) Consequently, the  $(R')^{-1}$ -neighbourhood of  $\Gamma_R$  is covered by the rescaled rectangles

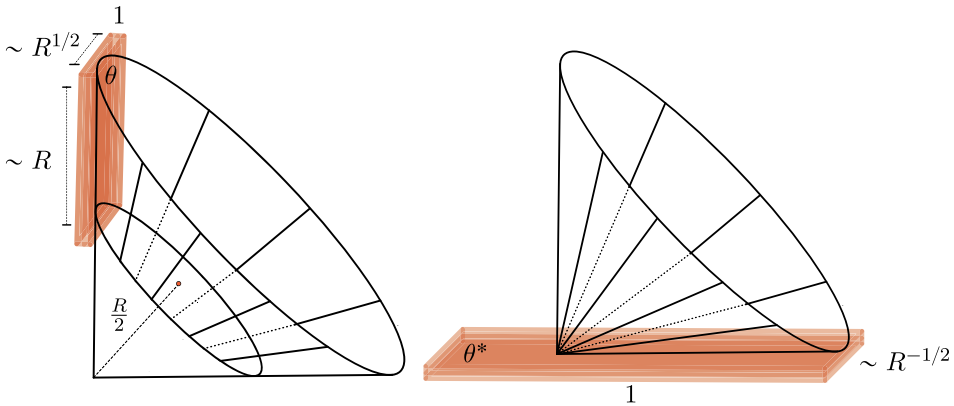
$$\mathcal{T}_s := \{s^{-2}\theta : \theta \in \Theta_{R'}\}$$

of dimensions  $R \times s^{-1}R^{1/2} \times s^{-2}$ . Note that the family  $\mathcal{T}_1$  coincides with  $\Theta_R$  (at least if it is defined appropriately), whereas  $\mathcal{T}_{R^{-1/2}}$  consists of  $\sim 1$  balls of radius  $R$ . For every  $s \in [R^{-1/2}, 1]$ , the rectangles in  $\mathcal{T}_s$  are at least as large as those in  $\Theta_R$ , so we may assume that every  $\theta \in \Theta_R$  is contained in at least one rectangle  $\tau \in \mathcal{T}_s$ .

For  $\theta \in \Theta_R$  and  $\tau \in \mathcal{T}_s$ , let  $\theta^*$  and  $\tau^*$  be the dual rectangles of  $\theta$  and  $\tau$  (here the word “dual” refers to the common notion in Euclidean Fourier analysis, and not the duality in the sense of Proposition 4.22). Then both  $\theta^*$  and  $\tau^*$  are rectangles centred at the origin, with dimensions

$$R^{-1} \times R^{-1/2} \times 1 \quad \text{and} \quad R^{-1} \times sR^{-1/2} \times s^2,$$

respectively. The longest sides of both  $\theta^*$  and  $\tau^*$  remain parallel to a light ray on  $\mathcal{C}$ : this is again the convenient property of the “standard” cone  $\mathcal{C}$  with opening angle  $\pi/2$ , see Fig. 3. Of course,  $\theta^*$  is an  $R^{-1/2}$ -plate in the sense of Definition 4.16, since the elements  $\theta \in \Theta$  were defined as (translates of) duals or  $R^{-1/2}$ -plates.



**Fig. 3.** On the left: the truncated cone  $\Gamma$  and one of the plates  $\theta$ . On the right: the cone  $\mathcal{C}$  and the dual plate  $\theta^*$ .

The set  $\tau^*$  turns out to be (essentially) a dilate of an  $(s^2R)^{-1/2}$ -plate. For every  $\tau \in \mathcal{T}_s$ , consider  $U_\tau := s^{-2}\tau^*$ , which is a rectangle of dimensions

$$s^{-2}R^{-1} \times s^{-1}R^{-1/2} \times 1 = (s^2R)^{-1} \times (s^2R)^{-1/2} \times 1.$$

In particular,  $U_\tau$  is an  $(s^2R)^{-1/2}$ -plate, and hence larger than (or at least as large as)  $\theta^*$ : if  $\theta \subset \tau$ , then every translate of  $\theta^*$  is contained in some translate of  $10U_\tau$ . We let  $\mathcal{U}_\tau$

be a tiling of  $\mathbb{R}^3$  by rectangles parallel to  $U_\tau$ . Now we may finally define the “partial” square function  $S_U f$ :

$$S_U f := \left( \sum_{\theta \subset \tau} |f_\theta|^2 \right)^{1/2} \cdot \mathbf{1}_U, \quad U \in \mathcal{U}_\tau. \tag{6.3}$$

We have now explained the meaning of (6.2), except the sum over “ $d(\tau) = s$ ”. In our notation, this means the same as summing over  $\tau \in \mathcal{T}_s$ .

We are then prepared to prove Theorem 5.2.

**Proof of Theorem 5.2.** Let  $\delta \in (0, \frac{1}{2}]$ , and let  $\mathcal{B}$  be a  $(\delta, 3, \delta^{-\epsilon})$ -set of  $\delta$ -balls with  $\delta$ -separated centres. In the statement of Theorem 5.2, it was assumed that  $\cup \mathcal{B} \subset B_{\mathbb{H}}(1)$ , but for slight technical convenience we strengthen this (with no loss of generality) to  $\cup \mathcal{B} \subset B_{\mathbb{H}}(c)$  for a small absolute constant  $c > 0$ . As in the statement of Theorem 5.2, let  $\mu$  be the measure on  $\mathbb{H}$  with density

$$f := (\delta^4 |\mathcal{B}|)^{-1} \sum_{B \in \mathcal{B}} \mathbf{1}_B.$$

Following the discussion Section 4.1, and in particular recalling equation (4.15), Theorem 5.2 will be proven if we manage to establish that

$$\int_{\mathcal{L}_\angle} X f(\ell)^2 \, d\mathbf{m}(\ell) \leq \delta^{-\eta}, \tag{6.4}$$

assuming that  $\epsilon, \delta > 0$  are small enough, depending on  $\eta$ . Recall that  $\mathcal{L}_\angle = \ell(\{(a, b, c) : |a| \leq 1\})$ . To estimate the quantity in (6.4), notice first that

$$X f(\ell) = \int_{\ell} f \, d\mathcal{H}^1 \lesssim (\delta^3 |\mathcal{B}|)^{-1} \cdot |\{B \in \mathcal{B} : \ell \cap B \neq \emptyset\}|, \quad \ell \in \mathcal{L}_\angle, \tag{6.5}$$

because  $\mathcal{H}^1(B \cap \ell) \lesssim \delta$  for all  $B \in \mathcal{B}$ . Write  $N(\ell) := |\{B \in \mathcal{B} : \ell \cap B \neq \emptyset\}|$ . Then, as we just saw,

$$\begin{aligned} \int_{\mathcal{L}_\angle} X f(\ell)^2 \, d\mathbf{m}(\ell) &\lesssim (\delta^3 |\mathcal{B}|)^{-2} \int_{\mathcal{L}_\angle} N(\ell)^2 \, d\mathbf{m}(\ell) \\ &\leq (\delta^3 |\mathcal{B}|)^{-2} \int_{B(2)} N(\ell(p))^2 \, d\text{Leb}(p). \end{aligned}$$

The second inequality is based on (a) the definition of the measure  $\mathbf{m} = \ell_{\sharp} \text{Leb}$ , and (b) the observation that if  $\ell(p) \in \mathcal{L}_\angle$  and  $N(\ell(p)) \neq 0$ , then  $\ell(p) \cap B_{\mathbb{H}}(c) \neq \emptyset$ , and this forces  $p \in B(2)$  (if  $c > 0$  was taken small enough). Finally, by Lemma 4.11, we have



$$N(\ell(p)) \leq |\{B \in \mathcal{B} : p \in \ell^*(B)\}| = \sum_{B \in \mathcal{B}} \mathbf{1}_{\ell^*(B)}(p).$$

Indeed, whenever  $\ell(p) \cap B \neq \emptyset$  for some  $B \in \mathcal{B}$ , there exists a point  $q \in \ell(p) \cap B$ , and then Lemma 4.11 implies that  $p \in \ell^*(q) \subset \ell^*(B)$ . Therefore, combining (6.4)-(6.5), it will suffice to show that for  $\eta > 0$  fixed, the inequality

$$\int_{B(2)} \left( \sum_{B \in \mathcal{B}} \mathbf{1}_{\ell^*(B)} \right)^2 \leq \delta^{-\eta} \cdot (\delta^3 |\mathcal{B}|)^2 \tag{6.6}$$

holds assuming that we have picked  $\epsilon > 0$  (in the  $(\delta, 3, \delta^{-\epsilon})$ -set hypothesis for  $\mathcal{B}$ ) sufficiently small, depending on  $\eta$ . We formulate a slightly more general version of this inequality in Proposition 6.7 below, and then explain in the remark afterwards why (6.6) is a consequence. This completes the proof of Theorem 5.2.  $\square$

**Proposition 6.7.** *For every  $\epsilon > 0$ , there exists  $\delta_0 > 0$  such that the following holds for all  $\delta \in (0, \delta_0]$ . Let  $\mathcal{B}$  be a family of  $\delta$ -balls contained in  $B_{\mathbb{H}}(1)$  with  $\delta$ -separated centres, and satisfying the following non-concentration condition for some  $\mathbf{C} > 0$ :*

$$|\{B \in \mathcal{B} : B \subset B_{\mathbb{H}}(p, r)\}| \leq \mathbf{C} \cdot \left(\frac{r}{\delta}\right)^3, \quad p \in \mathbb{H}, r \geq \delta. \tag{6.8}$$

Then,

$$\int_{B(2)} \left( \sum_{B \in \mathcal{B}} \mathbf{1}_{\ell^*(B)} \right)^2 \leq \mathbf{C} \cdot \delta^{3-\epsilon} |\mathcal{B}|. \tag{6.9}$$

**Remark 6.10.** Why is (6.6) a consequence of (6.9)? In (6.6), we assumed that  $\mathcal{B}$  is a  $(\delta, 3, \delta^{-\epsilon})$ -set. This implies

$$|\{B \in \mathcal{B} : B \subset B_{\mathbb{H}}(p, r)\}| \lesssim \delta^{-\epsilon} \cdot r^3 |\mathcal{B}|, \quad p \in \mathbb{H}, r \geq \delta.$$

Therefore, (6.8) is satisfied with constant  $\mathbf{C} \sim \delta^{3-\epsilon} |\mathcal{B}|$ . Hence (6.9) implies (6.6) if we choose  $\epsilon < \eta/2$  and then  $\delta > 0$  sufficiently small.

We chose to formulate Proposition 6.7 separately because the “meaning” of (6.9) is easier to appreciate than that of (6.6): namely, if all the sets  $\ell^*(B)$  had a disjoint intersection inside  $B(1)$ , then the left hand side of (6.9) would be roughly  $\delta^3 |\mathcal{B}|$ . Thus, (6.9) tells us that under the non-concentration condition (6.8), the sets  $\ell^*(B)$  are nearly disjoint inside  $B(1)$ , at least at the level of  $L^2$ -norms.

**Proof of Proposition 6.7.** By the discussion in Section 4.2, the intersections  $\ell^*(B) \cap B(2)$  are essentially  $\delta$ -plates – rectangles of dimensions  $1 \times \delta \times \delta^2$  tangent to  $\mathcal{C}$ . More precisely, for every  $B \in \mathcal{B}$ , let  $\mathcal{P}_B \subset \mathbb{R}^3$  be a  $C\delta$ -plate (as in Definition 4.16) with the property

$$\ell^*(B) \cap B(2) \subset \mathcal{P}_B.$$

This is possible by first applying Proposition 4.22 (which yields a modified  $2\delta$ -plate containing  $\ell^*(B)$ ), and then the first inclusion in (4.21), which shows that the intersection of the modified  $2\delta$ -plate with  $B(2)$  is contained in a  $C\delta$ -plate  $\mathcal{P}_B$ . Now, we will prove (6.9) by establishing that

$$\int \left( \sum_{B \in \mathcal{B}} \mathbf{1}_{\mathcal{P}_B} \right)^2 \leq C \cdot \delta^{3-\epsilon} |\mathcal{B}|. \tag{6.11}$$

Every plate  $\mathcal{P}_B$  has a *direction*, denoted  $\theta(\mathcal{P}_B)$ : this is the direction of the longest axis of  $\mathcal{P}_B$ , or more formally the real number “ $y \in [-1, 1]$ ” associated to the line “ $L_y$ ” in Definitions 4.16. By enlarging the plates  $\mathcal{P}_B$  slightly (if necessary), we may assume that their directions lie in the set  $\Theta := (\delta\mathbb{Z}) \cap [-1, 1]$ : this is because if two plates coincide in all other parameters, and differ in direction by  $\leq \delta$ , both are contained in constant enlargements of the other (this is not hard to check). The reason why we may restrict attention to  $[-1, 1]$  is that all the plates  $\mathcal{P}_B$  were associated to the balls  $B \subset B_{\mathbb{H}}(1)$ , and in fact the  $y$ -coordinate of the centre of  $B$  determines the direction of  $\mathcal{P}_B$  (see (4.10)).

We next sort the family  $\{\mathcal{P}_B\}_{B \in \mathcal{B}}$  according to their directions:

$$\{\mathcal{P}_B : B \in \mathcal{B}\} =: \bigcup_{\theta \in \Theta} \mathcal{P}(\theta),$$

where  $\mathcal{P}(\theta) := \{\mathcal{P}_B : \theta(\mathcal{P}_B) = \theta\}$ . Thus, for  $\theta \in \Theta$  fixed, the plates in  $\mathcal{P}(\theta)$  are all translates of each other. Also, the plates in  $\mathcal{P}(\theta)$  for a fixed  $\theta$  have bounded overlap: this follows from the assumption that the balls in  $\mathcal{B}$  have  $\delta$ -separated centres, and uses Lemma 4.33 (the plates with a fixed direction correspond precisely to Heisenberg balls whose  $y$ -coordinates are, all, within “ $\delta$ ” of each other).

Write  $R := \delta^{-2}$ , thus  $\delta = R^{-1/2}$ , and recall the truncated cone  $\Gamma = \Gamma_R$  from (6.1). Since the plates  $\mathcal{P} \in \mathcal{P}(\theta)$  are translates of each other, they all have a common dual rectangle  $\mathcal{P}_\theta^*$  of dimensions  $\sim R \times R^{1/2} \times 1$ . The rectangle  $\mathcal{P}_\theta^*$  is centred at 0, but we may translate it by  $\sim R$  in the direction of its longest  $R$ -side (a light ray depending on  $\theta$ ) so that the translate lies in the  $O(1)$ -neighbourhood of  $\Gamma_R$ . Committing a serious abuse of notation, we will denote this translated dual rectangle again by “ $\theta$ ”, and the collection of all these sets is denoted  $\Theta$ . This notation coincides with the discussion below (6.1). There is a 1-to-1 correspondence between the directions  $\theta \in \Theta = \delta\mathbb{Z} \cap [-1, 1]$  and the rectangles  $\theta \in \Theta$  defined just above, so the notational inconsistency should not cause confusion.

We gradually move towards applying the inequality (6.2) of Guth, Wang, and Zhang. The next task is to define the functions  $f_\theta$  and  $f = \sum_{\theta \in \Theta} f_\theta$ . Fix  $\theta \in \Theta$ ,  $\mathcal{P} \in \mathcal{P}(\theta)$ , and let  $\varphi_{\mathcal{P}} \in \mathcal{S}(\mathbb{R}^3)$  be a non-negative Schwartz function with the properties

$$(1) \quad \mathbf{1}_{\mathcal{P}} \leq \varphi_{\mathcal{P}} \lesssim 1,$$

- (2)  $\varphi_{\mathcal{P}}$  has rapid decay outside  $\mathcal{P}$ ,
- (3)  $\widehat{\varphi_{\mathcal{P}}} \subset \mathcal{P}_{\theta}^*$ .

Here “rapid decay outside  $\mathcal{P}$  has” the usual meaning: if  $\lambda\mathcal{P}$  denotes a  $\lambda$ -times dilated, concentric, version of  $\mathcal{P}$ , then  $\varphi(x) \lesssim_N \lambda^{-N}$  for all  $x \in \mathbb{R}^3 \setminus \lambda\mathcal{P}$  (and for any  $N \in \mathbb{N}$ ). Then, define the function

$$f_{\theta} := \sum_{\mathcal{P} \in \mathcal{P}(\theta)} e_{\theta} \cdot \varphi_{\mathcal{P}}.$$

Here  $e_{\theta}$  is a modulation, depending only on  $\theta$ , such that

$$\widehat{e_{\theta} \cdot \varphi_{\mathcal{P}}} \subset \theta.$$

Now the function  $f = \sum_{\theta \in \Theta} f_{\theta}$  satisfies all the assumptions of the inequality (6.2), so

$$\begin{aligned} \int_{\mathbb{R}^3} \left( \sum_{B \in \mathcal{B}} \mathbf{1}_{\mathcal{P}_B} \right)^2 &= \int_{\mathbb{R}^3} \left( \sum_{\theta \in \Theta} \sum_{\mathcal{P} \in \mathcal{P}(\theta)} \mathbf{1}_{\mathcal{P}_B} \right)^2 \\ &\leq \int_{\mathbb{R}^3} \left( \sum_{\theta \in \Theta} \left| \sum_{\mathcal{P} \in \mathcal{P}(\theta)} e_{\theta} \cdot \varphi_{\mathcal{P}} \right|^2 \right) \\ &= \int_{\mathbb{R}^3} |Sf|^4 \lesssim \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau)=s} \sum_{U \parallel U_{\tau}} \text{Leb}(U)^{-1} \|S_U f\|_{L^2}^4. \end{aligned} \tag{6.12}$$

Recall the notation on the right hand side, in particular that  $\delta = R^{-1/2} \leq s \leq 1$  only runs over dyadic rationals, and the definition of the “partial” square function  $S_U f$  from (6.3). The rectangles  $U$  are  $\Delta$ -plates with  $\Delta = (s^2 R)^{-1/2} = s^{-1} \delta$ . In particular, every  $U$  is essentially the  $\ell^*$ -dual of a Heisenberg  $\Delta$ -ball: this will allow us to control  $\|S_U f\|_{L^2}$  by applying the non-concentration condition (6.8) between scales  $\delta$  and 1.

By definition,

$$\|S_U f\|_2^2 = \int_U \sum_{\theta \subset \tau} |f_{\theta}|^2 = \int_U \sum_{\theta \subset \tau} \left( \sum_{\mathcal{P} \in \mathcal{P}(\theta)} \varphi_{\mathcal{P}} \right)^2 \lesssim \int_U \sum_{\theta \subset \tau} \sum_{\mathcal{P} \in \mathcal{P}(\theta)} \varphi_{\mathcal{P}}. \tag{6.13}$$

Above, and in the sequel, the notation  $A \lesssim B$  means that for every  $\rho > 0$ , there exists a constant  $C_{\rho} > 0$  such that  $A \leq C_{\rho} \delta^{-\rho} B$ . In (6.13), the final “ $\lesssim$ ” inequality follows easily from the rapid decay of the functions  $\varphi_{\mathcal{P}}$ , and the bounded overlap of the plates  $\mathcal{P} \in \mathcal{P}(\theta)$  for  $\theta \in \Theta$  fixed.

For  $\theta \subset \tau$ , each plate  $\mathcal{P} \in \mathcal{P}(\theta)$  is contained in some translate of  $10U_{\tau}$  (this was discussed above (6.3)), but this translates may not be  $U$ . Let  $\mathbf{U} \supset U$  be an  $(R^{\epsilon} \Delta)$ -plate which is concentric with  $U$ . We then decompose the right hand side of (6.13) as

$$\int_U \sum_{\theta \subset \tau} \sum_{\mathcal{P} \in \mathcal{P}(\theta)} \varphi_{\mathcal{P}} \leq \int_U \sum_{\theta \subset \tau} \sum_{\substack{\mathcal{P} \in \mathcal{P}(\theta) \\ \mathcal{P} \subset \mathbf{U}}} \varphi_{\mathcal{P}} + \int_U \sum_{\theta \subset \tau} \sum_{\substack{\mathcal{P} \in \mathcal{P}(\theta) \\ \mathcal{P} \not\subset \mathbf{U}}} \varphi_{\mathcal{P}}. \tag{6.14}$$

Since each  $\mathcal{P} \in \mathcal{P}(\theta)$  is contained in element of the tiling  $\mathcal{U}_\tau$  (consisting of translates of  $U$ ) every plate  $\mathcal{P}(\theta)$  with  $\mathcal{P} \not\subset \mathbf{U}$  is far away from  $U$ : more precisely,  $R^{\epsilon/2}\mathcal{P} \cap U = \emptyset$ . By the rapid decay of  $\varphi_{\mathcal{P}}$  outside  $\mathcal{P}$ , this implies that  $\varphi_{\mathcal{P}} \lesssim_{\epsilon} \delta^{100}$  on  $U$ , and therefore the second term of (6.14) is bounded by, say,  $\lesssim_{\epsilon} \delta^{50}$ .

We then focus on the first term of (6.14), and we first note that

$$\int_U \sum_{\theta \subset \tau} \sum_{\substack{\mathcal{P} \in \mathcal{P}(\theta) \\ \mathcal{P} \subset \mathbf{U}}} \varphi_{\mathcal{P}} \lesssim \delta^3 \cdot |\{\mathcal{P} : \mathcal{P} \subset \mathbf{U}\}|, \tag{6.15}$$

since  $\|\varphi_{\mathcal{P}}\|_{L^1} \sim \text{Leb}(\mathcal{P}) \sim \delta^3$ . So, we need to find out how many  $\delta$ -plates  $\mathcal{P}$  are contained in  $\mathbf{U}$ . Since  $\mathbf{U}$  is an  $(R^\epsilon \Delta)$ -plate, it follows from the second inclusion (4.21), combined with the second inclusion in Proposition 4.22, that

$$\mathbf{U} \subset \ell^*(B_{\mathbb{H}}(p_U, CR^\epsilon \Delta)) =: \ell^*(B_U).$$

for some  $p_U \in \mathbb{H}$ , and for some absolute constant  $C > 0$ . On the other hand, the plates  $\mathcal{P} = \mathcal{P}_B, B \in \mathcal{B}$ , were initially chosen in such a way that  $\ell^*(B) \cap \{(s, y, z) : |s| \leq 1\} \subset \mathcal{P}_B$ . Thus, whenever  $\mathcal{P}_B \subset \mathbf{U}$ , we have

$$\ell^*(B) \cap \{(s, y, z) : |s| \leq 1\} \subset \mathcal{P}_B \subset \mathbf{U} \subset \ell^*(B_U).$$

This implies by Proposition 4.30 that  $B \subset B_U$ , where possibly  $B_U$  was inflated by another constant factor. Thus,

$$|\{\mathcal{P} : \mathcal{P} \subset \mathbf{U}\}| \lesssim |\{B \in \mathcal{B} : B \subset B_U\}|.$$

Using (6.8), this will easily yield useful upper bounds for  $|\{\mathcal{P} : \mathcal{P} \subset \mathbf{U}\}|$ .

To make this precise, we sort the sets “ $U$ ” appearing in (6.12) according to the “richness”

$$\rho(U) := |\{B \in \mathcal{B} : B \subset B_U\}| \stackrel{(6.8)}{\leq} \mathbf{C} \cdot \left(\frac{CR^\epsilon \Delta}{\delta}\right)^3. \tag{6.16}$$

For  $s \in [R^{-1/2}, 1]$  fixed, we choose a (dyadic) value  $\rho = \rho_s$  such that

$$\sum_{d(\tau)=s} \sum_{U \parallel U_\tau} \text{Leb}(U)^{-1} \|S_U f\|_{L^2}^4 \lesssim \sum_{d(\tau)=s} \sum_{\substack{U \parallel U_\tau \\ \rho(U) \sim \rho}} \text{Leb}(U)^{-1} \|S_U f\|_{L^2}^4. \tag{6.17}$$

Here “ $\lesssim$ ” hides a constant of the form  $C \log(1/\delta)$ . Let  $\mathcal{U}(\rho)$  be the collection of sets “ $U$ ” appearing on the right hand side, and let  $\mathcal{B}' \subset \mathcal{B}$  be the subset of the original  $\delta$ -balls which are contained in some ball  $B_U$ ,  $U \in \mathcal{U}(\rho)$ . Then, evidently,

$$|\mathcal{B}'| \lesssim \rho \cdot |\mathcal{U}(\rho)| \lesssim R^{C\epsilon} |\mathcal{B}'|. \tag{6.18}$$

The factor “ $R^{C\epsilon}$ ” arises from the fact that while distinct sets “ $U$ ” are the duals of essentially disjoint Heisenberg  $\Delta$ -balls, the inflated balls  $B_U$  only have bounded overlap, depending on the inflation factor  $R^\epsilon$ .

Now, for  $U \in \mathcal{U}(\rho)$ , we may estimate (6.15) as follows:

$$\|S_U f\|_{L^2}^2 \lesssim_\epsilon \int \sum_{\theta \subset \tau} \sum_{\substack{\mathcal{P} \in \mathcal{P}(\theta) \\ \mathcal{P} \subset U}} \varphi_{\mathcal{P}} \lesssim \delta^3 \cdot \rho \lesssim \delta^3 \cdot R^{C\epsilon} \cdot \frac{|\mathcal{B}'|}{|\mathcal{U}(\rho)|}.$$

(In this estimate, we have omitted the term “ $\delta^{50}$ ” from the second part of (6.14), because this term will soon turn out to be much smaller than the best bounds for what remains.) Plugging this estimate into (6.17), and observing that  $\text{Leb}(U) = \Delta^3$ , we obtain

$$\begin{aligned} \sum_{d(\tau)=s} \sum_{U \parallel U_\tau} \text{Leb}(U)^{-1} \|S_U f\|_{L^2}^4 &\lesssim_\epsilon |\mathcal{U}(\rho)| \cdot \Delta^{-3} \cdot \left( \delta^3 \cdot R^{C\epsilon} \cdot \frac{|\mathcal{B}'|}{|\mathcal{U}(\rho)|} \right)^2 \\ &= \Delta^{-3} \cdot \delta^6 \cdot R^{2C\epsilon} \cdot \frac{|\mathcal{B}'|^2}{|\mathcal{U}(\rho)|} \\ &\stackrel{(6.16)\&(6.18)}{\lesssim} \mathbf{C} \cdot R^{3C\epsilon} \cdot \delta^3 |\mathcal{B}'|. \end{aligned}$$

Notably, this estimate is independent of “ $\Delta$ ” and the parameter “ $s$ ”, so we may finally deduce from (6.12) that

$$\int_{\mathbb{R}^3} \left( \sum_{B \in \mathcal{B}} \mathbf{1}_{\mathcal{P}_B} \right)^2 \lesssim_\epsilon \mathbf{C} \cdot R^{3C\epsilon} \cdot \delta^3 |\mathcal{B}|.$$

Since  $R = \delta^{-2}$  and  $\epsilon > 0$  was arbitrary, this implies (6.9) by renaming variables, and the proof of Proposition 6.7 is complete.  $\square$

**7. Proof of Theorem 1.7**

We recall the statement:

**Theorem 7.1.** *Let  $K \subset \mathbb{H}$  be a Borel set with  $\dim_{\mathbb{H}} K = t \in [2, 3]$ . Then,  $\dim_E \pi_e(K) \geq t - 1$  for  $\mathcal{H}^1$  almost every  $e \in S^1$ . Consequently,  $\dim_{\mathbb{H}} \pi_e(K) \geq 2t - 3$  for  $\mathcal{H}^1$  almost every  $e \in S^1$ .*

**Proof.** The lower bound for  $\dim_{\mathbb{H}} \pi_e(K)$  follows immediately from the lower bound for  $\dim_{\mathbb{E}}(K)$ , combined with a general inequality between Hausdorff dimensions relative to Euclidean and Heisenberg metrics of subsets of  $\mathbb{W}_e$ , see [1, Theorem 2.8]. So, we focus on proving that  $\dim_{\mathbb{E}}(K) \geq t - 1$  for  $\mathcal{H}^1$  almost every  $e \in S^1$ .

The first steps of the proof are standard; similar arguments have appeared, for example the deduction of [16, Theorem 2] from [16, Theorem 1]. So we only sketch the first part of the proof, and provide full details where they are non-standard. First, we may assume that  $K \subset B_{\mathbb{H}}(1)$ , and we may assume, applying Frostman’s lemma, that  $K = \text{spt}(\mu)$  for some Borel probability measure  $\mu$  satisfying  $\mu(B_{\mathbb{H}}(p, r)) \lesssim r^t$  for all  $p \in \mathbb{H}$  and  $r > 0$ .

We make the counter assumption that there exists  $s \in (1, t)$  such that

$$\mathcal{H}^1(\{e \in S^1 : \dim_{\mathbb{E}} \pi_e(K) \leq s - 1\}) > 0.$$

By several applications of the pigeonhole principle, this assumption can be applied to find the following objects for any  $\epsilon > 0$ , and for arbitrarily small  $\delta > 0$ :

- (1) A Borel subset  $S' \subset S^1$  of length  $\mathcal{H}^1(S') \geq \delta^{\epsilon/2}$ .
- (2) For every  $e \in S'$  a collection of  $\leq \delta^{1-s}$  Euclidean  $\delta$ -discs  $\mathcal{W}_e$ , contained in  $\mathbb{W}_e$ .
- (3) If  $W_e := \cup \mathcal{W}_e$  and  $e \in S'$ , then

$$\mu(\pi_e^{-1}(W_e)) \geq \delta^{\epsilon/2}. \tag{7.2}$$

We claim that (1)-(3) violate Theorem 5.11 if  $\delta, \epsilon > 0$  are small enough. To this end, we first need to construct a relevant  $(\delta, t, \delta^{-\epsilon})$ -set of (Heisenberg)  $\delta$ -balls  $\mathcal{B}$  contained in  $B_{\mathbb{H}}(1)$ . Morally, this collection is a  $\delta$ -approximation of  $K = \text{spt}(\mu)$ . More precisely, we need to decompose  $K$  to the following subsets:

$$K_{\alpha} := \{p \in K : \frac{\alpha}{2} \leq \mu(B_{\mathbb{H}}(p, \delta)) \leq \alpha\},$$

where  $\alpha > 0$  runs over dyadic rationals with  $\alpha \lesssim \delta^t$ . By one final application of the pigeonhole principle, and recalling (7.2), one can find a fixed index  $\alpha \in 2^{-\mathbb{N}}$  such that

$$\mu(\pi_e^{-1}(W_e) \cap K_{\alpha}) \geq \delta^{\epsilon} \tag{7.3}$$

for all  $e \in S \subset S'$ , where  $\mathcal{H}^1(S) \geq \delta^{\epsilon}$ . In particular,  $\mu(K_{\alpha}) \geq \delta^{\epsilon}$ . Then, we let  $\mathcal{B}$  be a (Vitali) cover of  $K_{\alpha}$  by finitely overlapping Heisenberg  $\delta$ -balls with  $(\delta/5)$ -separated centres. Note that  $\delta^{\epsilon} \alpha^{-1} \lesssim |\mathcal{B}| \lesssim \alpha^{-1}$ . Using the definition of  $K_{\alpha}$ , and the Frostman condition for  $\mu$ , it is now easy to check that  $\mathcal{B}$  is a  $(\delta, t, C\delta^{-\epsilon})$ -set of  $\delta$ -balls, where  $C$  is roughly the Frostman constant of  $\mu$ .

Finally, from (7.3) and  $\alpha \lesssim |\mathcal{B}|^{-1}$ , we deduce that if  $e \in S$ , then  $\pi_e^{-1}(W_e)$  intersects  $\gtrsim \delta^{\epsilon} |\mathcal{B}|$  elements of  $\mathcal{B}$ , since

$$\delta^{\epsilon} \leq \mu(\pi_e^{-1}(W_e) \cap K_{\alpha}) \leq \alpha \cdot |\{B \in \mathcal{B} : \pi_e^{-1}(W_e) \cap B \neq \emptyset\}|, \quad e \in S.$$

Write  $\mathcal{B}_e := \{B \in \mathcal{B} : \pi_e^{-1}(W_e) \cap B \neq \emptyset\}$ , thus  $|\mathcal{B}_e| \gtrsim \delta^\epsilon |\mathcal{B}|$ . We now arrive at the point where it is crucial that the elements of  $\mathcal{W}_e$  are Euclidean  $\delta$ -discs. Namely, if  $B \in \mathcal{B}_e$ , then  $\pi_e^{-1}(D) \cap B \neq \emptyset$  for some  $D \in \mathcal{W}_e$ . Then, because  $D$  is a Euclidean  $\delta$ -disc, and the Euclidean diameter of  $\pi_e(B)$  is  $\lesssim \delta$ , we may conclude that  $\pi_e(B) \subset 2D$ . This could seriously fail if  $D$  were a disc in the metric  $d_{\mathbb{H}}$ . Now, however, we see that

$$\pi_e(\cup \mathcal{B}_e) \subset \cup \{2D : D \in \mathcal{W}_e\},$$

and in particular  $\text{Leb}(\pi_e(\cup \mathcal{B}_e)) \lesssim \delta^2 \cdot |\mathcal{W}_e| \leq \delta^{3-s}$  for all  $e \in S$ . This violates the conclusion of Theorem 5.11, and the proof of Theorem 7.1 is complete.  $\square$

### Appendix A. Completing $(\delta, t)$ -sets to $(\delta, 3)$ -sets

In this section, we use the following notation for the  $\delta$ -truncated  $s$ -dimensional Riesz energy of a Radon measure  $\nu$  on  $\mathbb{H}$ :

$$I_s^\delta(\nu) := \iint \frac{d\nu(x) d\nu(y)}{d_{\mathbb{H},\delta}(x, y)^{s+t}},$$

where  $d_{\mathbb{H},\delta}(x, y) := \max\{d_{\mathbb{H}}(x, y), \delta\}$ . We also recall that  $\mu *_{\mathbb{H}} \nu$  is the Heisenberg convolution of  $\mu$  and  $\nu$ , that is, the push-forward of  $\mu \times \nu$  under the group operation  $(p, q) \rightarrow p \cdot q$ .

**Proposition A.1.** *Let  $0 \leq s, t \leq 3$  with  $s + t \leq 3$ , and let  $\delta \in (0, \frac{1}{2}]$ . Let  $\mu$  be a Borel probability measure on  $B_{\mathbb{H}}(1)$  with  $I_t^\delta(\mu) \leq C$ . Then, there exists a set  $H \subset B_{\mathbb{H}}(1)$  with  $|H| \leq \delta^{-s}$  such that the uniformly distributed (discrete) measure  $\eta$  on  $H$  satisfies*

$$I_{s+t}^\delta(\eta * \mu) \leq C',$$

where  $C' \leq C \log(1/\delta)^C \cdot C$  for some absolute constant  $C > 0$ .

**Proof.** Let  $Z := \delta \cdot \mathbb{Z}^3 \cap B_{\mathbb{H}}(1)$  be a grid of Euclidean  $\delta$ -separated lattice points in  $B_{\mathbb{H}}(1)$ . Then  $|Z| \sim \delta^{-3}$ . Let  $H_\omega \subset Z$  be a random set, where each point of  $Z$  is included independently with probability  $\delta^{-s}/(2|Z|)$ . In particular,  $\mathbb{E}_\omega |H_\omega| = \delta^{-s}/2$ . While we use the symbol “ $\omega$ ” to index the elements in the underlying probability space, no explicit reference to this space will be needed. Let  $\eta_\omega$  be the random measure

$$\eta_\omega := \delta^s \sum_{p \in H_\omega} \delta_p = \delta^s \sum_{p \in Z} \mathbf{1}_{H_\omega}(p) \cdot \delta_p.$$

We claim that

$$\mathbb{E}_\omega (I_{s+t}^\delta(\eta_\omega *_{\mathbb{H}} \mu)) = \iint \mathbb{E}_\omega \iint \frac{d\eta_\omega(p) d\eta_\omega(q)}{d_{\mathbb{H},\delta}(p \cdot x, q \cdot y)^{s+t}} d\mu(x) d\mu(y) \leq C'. \tag{A.2}$$

for some  $\mathbf{C}' \lesssim \mathbf{C}$ . In this argument, the notation “ $\lesssim$ ” hides a constant of the form  $C \log(1/\delta)^C$ . The inequality (A.2) will complete the proof of the proposition, because  $|H_\omega| \leq \delta^{-s}$  with probability  $\geq \frac{1}{2}$  (for  $\delta > 0$  small enough), and therefore, by Chebychev’s inequality,  $I_{s+t}^\delta(\eta_\omega *_{\mathbb{H}} \mu) \lesssim \mathbf{C}'$  for some “ $\omega$ ” with  $|H_\omega| \leq \delta^{-s}$ .

To prove (A.2), it clearly suffices to establish that

$$\mathbb{E}_\omega \iint \frac{d\eta_\omega(p)d\eta_\omega(q)}{d_{\mathbb{H},\delta}(p \cdot x, q \cdot y)^{s+t}} \lesssim \frac{1}{d_{\mathbb{H},\delta}(x \cdot y)^t}, \quad x, y \in \text{spt}(\mu) \subset B_{\mathbb{H}}(1). \tag{A.3}$$

By definition of  $\eta_\omega$ , we have

$$\begin{aligned} \iint \frac{d\eta_\omega(p)d\eta_\omega(q)}{d_{\mathbb{H},\delta}(p \cdot x, q \cdot y)^{s+t}} &= \delta^{2s} \sum_{p,q \in Z} \frac{\mathbf{1}_{H_\omega}(p)\mathbf{1}_{H_\omega}(q)}{d_{\mathbb{H},\delta}(p \cdot x, q \cdot y)^{s+t}} \\ &= \delta^{2s} \sum_{p \in Z} \frac{\mathbf{1}_{H_\omega}(p)}{d_{\mathbb{H},\delta}(x, y)^{s+t}} + \delta^{2s} \sum_{\substack{p,q \in Z \\ p \neq q}} \frac{\mathbf{1}_{H_\omega}(p)\mathbf{1}_{H_\omega}(q)}{d_{\mathbb{H},\delta}(p \cdot x, q \cdot y)^{s+t}} \\ &=: \Sigma_1(\omega) + \Sigma_2(\omega). \end{aligned}$$

We consider the expectations of  $\Sigma_1(\omega)$  and  $\Sigma_2(\omega)$  separately. The former one is simple, using that  $\mathbb{E}_\omega(\mathbf{1}_{H_\omega}(p)) = \mathbb{P}_\omega\{p \in H_\omega\} = \delta^{-s}/(2|Z|) \sim \delta^{3-s}$ :

$$\mathbb{E}_\omega \Sigma_1(\omega) \sim \delta^{2s} \sum_{p \in Z} \frac{\delta^{3-s}}{d_{\mathbb{H},\delta}(x, y)^{s+t}} = \frac{|Z| \cdot \delta^{3+s}}{d_{\mathbb{H},\delta}(x, y)^{s+t}} \lesssim \frac{\delta^s}{d_{\mathbb{H},\delta}(x, y)^{s+t}} \leq \frac{1}{d_{\mathbb{H},\delta}(x, y)^t},$$

recalling that  $|Z| \lesssim \delta^{-3}$ . To handle the expectation of  $\Sigma_2(\omega)$ , we note that  $\{p \in H_\omega\}$  and  $\{q \in H_\omega\}$  are independent events for  $p \neq q$ , hence

$$\begin{aligned} \mathbb{E}_\omega \Sigma_2(\omega) &\sim \delta^{2s} \sum_{\substack{p,q \in Z \\ p \neq q}} \frac{\delta^{6-2s}}{d_{\mathbb{H},\delta}(p \cdot x, q \cdot y)^{s+t}} \\ &\sim \delta^6 \sum_{p \in Z} \sum_{\delta \leq r \leq 1} r^{-s-t} |\{q \in Z : d_{\mathbb{H},\delta}(p \cdot x, q \cdot y) \sim r\}|, \end{aligned}$$

where “ $r$ ” runs over dyadic rationals. Since the product “ $\cdot$ ” is not commutative, in general  $d_{\mathbb{H},\delta}(p \cdot x, q \cdot y) \neq d_{\mathbb{H},\delta}(p \cdot x \cdot y^{-1}, q)$ , so the set  $\{q \in Z : d_{\mathbb{H},\delta}(p \cdot x, q \cdot y) \sim r\}$  is **not** contained in a  $\mathbb{H}$ -ball of radius  $\sim r$  around  $p \cdot x \cdot y^{-1}$ . This is the key inefficiency in the argument, and causes the restriction  $s + t \leq 3$ : under this restriction, it actually suffices to note that  $\{q \in Z : d_{\mathbb{H},\delta}(p \cdot x, q \cdot y) \sim r\}$  is contained in a Euclidean  $Cr$ -ball. To see this, note that if  $q \in Z$  satisfies  $d_{\mathbb{H},\delta}(p \cdot x, q \cdot y) \lesssim r$  with  $r \geq \delta$ , then

$$q \in B_{\mathbb{H}}(p \cdot x, Cr) \cdot y^{-1}.$$



Here  $B_{\mathbb{H}}(p \cdot x, Cr)$  is contained in a Euclidean ball of radius  $\lesssim r$  (using  $r \leq 1$ ). The same remains true after the right translation by  $y^{-1}$ , because  $|y| \lesssim 1$  (by assumption), and the right translation  $z \mapsto z \cdot y^{-1}$  is Euclidean Lipschitz with constant depending only on  $|y|$ .

Now, since a Euclidean  $r$ -ball contains  $\lesssim (r/\delta)^3$  points of  $Z$ , we see that

$$\mathbb{E}_{\omega} \Sigma_2(\omega) \lesssim \delta^3 \sum_{p \in Z} \sum_{\delta \leq r \leq 1} r^{3-s-t} \lesssim 1 \leq \frac{1}{d_{\mathbb{H},\delta}(x,y)^{s+t}},$$

where in the final inequality we used again that  $x, y \in \text{spt}(\mu) \subset B_{\mathbb{H}}(1)$ . This completes the proof of (A.3), and therefore the proof of the proposition.  $\square$

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