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Sobolev homeomorphic extensions

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Abstract. Let $\mathbb{X}$ and $\mathbb{Y}$ be ℓ-connected Jordan domains, $\ell \in \mathbb{N}$, with rectifiable boundaries in the complex plane. We prove that any boundary homeomorphism $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a Sobolev homeomorphic extension $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,1}(\mathbb{X},\mathbb{C})$. If instead $\mathbb{X}$ has s-hyperbolic growth with $s > p-1$, we show the existence of such an extension in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$ for $p \in (1,2)$. Our examples show that the assumptions of rectifiable boundary and hyperbolic growth cannot be relaxed. We also consider the existence of $\mathscr{W}^{1,2}$-homeomorphic extensions with given boundary data.

Keywords. Sobolev homeomorphisms, Sobolev extensions, Douglas condition

1. Introduction

Throughout this text $\mathbb{X}$ and $\mathbb{Y}$ are ℓ-connected Jordan domains, $\ell = 1, 2, \ldots$, in the complex plane $\mathbb{C}$. Their boundaries $\partial\mathbb{X}$ and $\partial\mathbb{Y}$ are thus disjoint unions of ℓ simple closed curves or points. If $\ell = 1$, these domains are simply connected and will just be called Jordan domains. In the simply connected case, the Jordan–Schönflies theorem states that every homeomorphism $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a continuous extension $h\colon \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ which takes $\mathbb{X}$ homeomorphically onto $\mathbb{Y}$. In the first part of this paper we focus on a Sobolev variant of the Jordan–Schönflies theorem. The most pressing demand for studying such variants comes from the variational approach to geometric function theory [3, 19, 33] and nonlinear elasticity [2, 5, 8]. Both theories share the ideas associated to determining the infimum of a given energy functional

$$\mathsf{E}_{\mathbb{X}}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x \tag{1.1}$$

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among orientation preserving homeomorphisms $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in the Sobolev space $\mathcal{W}^{1,p}(\mathbb{X},\mathbb{Y})$ with given boundary data $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$. We denote that class of mappings by $\mathcal{H}^{1,p}_\varphi(\mathbb{X},\mathbb{Y})$. Naturally, a fundamental question is whether the class $\mathcal{H}^{1,p}_\varphi(\mathbb{X},\mathbb{Y})$ is non-empty.

Question 1.1. Under what conditions does a boundary homeomorphism $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admit a homeomorphic extension $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathcal{W}^{1,p}(\mathbb{X},\mathbb{C})$?

A necessary condition is that φ is the Sobolev trace of some (possibly nonhomeomorphic) mapping in $\mathcal{W}^{1,p}(\mathbb{X},\mathbb{C})$. Hence to solve Question 1.1 one could first study the following natural subquestion:

Question 1.2. Suppose that a homeomorphism $\varphi : \partial\mathbb{X} \to \partial\mathbb{Y}$ admits a $\mathcal{W}^{1,p}$-extension to $\mathbb{X}$. Does it then follow that φ also admits a homeomorphic $\mathcal{W}^{1,p}$-extension?

Our main results, Theorem 1.8 and its multiply connected variant (Theorem 1.11), give an answer to these questions when $p \in [1,2)$. The construction of such extensions is important not only to ensure the well-posedness of the related variational questions, but also for example due to the fact that various types of extensions were used to provide approximation results for Sobolev homeomorphisms [16, 18]. We touch upon variational topics in Section 7, where we provide an application of one of our results. Apart from Theorem 1.11 and its proof (§6), the rest of the paper deals with the simply connected case.

Let us start by considering the above questions in the well-studied setting of the Dirichlet energy, corresponding to $p = 2$ above. The Radó [32], Kneser [26] and Choquet [7] theorem asserts that if $\mathbb{Y} \subset \mathbb{R}^2$ is a convex domain then the harmonic extension of a homeomorphism $\varphi\colon \partial\mathbb{X} \to \partial\mathbb{Y}$ is a univalent map from $\mathbb{X}$ onto $\mathbb{Y}$. Moreover, by a theorem of Lewy [29], this univalent harmonic map has a nonvanishing Jacobian and is therefore a real analytic diffeomorphism in $\mathbb{X}$. However, such an extension is not guaranteed to have finite Dirichlet energy in $\mathbb{X}$. The class of boundary functions which admit a harmonic extension with finite Dirichlet energy was characterized by Douglas [9]. The *Douglas condition* for a function $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ reads

$$\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \left|\frac{\varphi(\xi)-\varphi(\eta)}{\xi-\eta}\right|^2 |\mathrm{d}\xi|\,|\mathrm{d}\eta| < \infty. \tag{1.2}$$

The mappings satisfying this condition are exactly the ones that admit an extension with finite $\mathcal{W}^{1,2}$-norm. Among these extensions is the harmonic extension of φ, known to have the smallest Dirichlet energy.

Note that the Dirichlet energy is also invariant with respect to a conformal change of variables in the domain $\mathbb{X}$. Therefore thanks to the Riemann Mapping Theorem, when considering Question 1.1 in the case $p = 2$, we may assume that $\mathbb{X} = \mathbb{D}$ without loss of generality. Now, there is no problem to answer Question 1.1 when $p = 2$ and $\mathbb{Y}$ is Lipschitz. Indeed, for any Lipschitz domain there exists a global bi-Lipschitz change of variables $\Phi\colon \mathbb{C} \to \mathbb{C}$ for which $\Phi(\mathbb{Y})$ is the unit disk. Since the finiteness of the Dirichlet energy is preserved under a bi-Lipschitz change of variables in the target, we may reduce

Question 1.1 to the case when $\mathbb{X} = \mathbb{Y} = \mathbb{D}$, for which the Radó–Kneser–Choquet theorem and the Douglas condition provide an answer. In other words, if $\mathbb{Y}$ is Lipschitz then the following are equivalent for a boundary homeomorphism $\varphi\colon \partial\mathbb{D} \to \partial\mathbb{Y}$:

(1) φ admits a $\mathscr{W}^{1,2}$-Sobolev homeomorphic extension $h\colon \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$.

(2) φ admits $\mathscr{W}^{1,2}$-Sobolev extension to $\mathbb{D}$.

(3) φ satisfies the Douglas condition (1.2).

In the case when $1 \leq p < 2$, the problem is not invariant under a conformal change of variables in $\mathbb{X}$. However, when $\mathbb{X}$ is the unit disk and $\mathbb{Y}$ is a convex domain, a complete answer to Question 1.1 was provided by the following result of Verchota [38].

Proposition 1.3. *Let $\mathbb{Y}$ be a convex domain, and let $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ be any homeomorphism. Then the harmonic extension of φ lies in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{D}, \mathbb{C})$ for all $1 \leq p < 2$.*

This result was further generalized in [15, 20, 24]. The case $p > 2$ will be discussed in Subsection 2.3. Our main purpose is to provide a general study of Question 1.1 in the case when $1 \leq p < 2$.

Considering now the endpoint case $p = \infty$, we find that Question 1.1 is equivalent to the question of finding a homeomorphic Lipschitz map extending the given boundary data φ. In this case the Kirszbraun extension theorem [25] shows that a boundary map $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a Lipschitz extension if and only if φ is a Lipschitz map itself. When $\mathbb{X}$ is the unit disk, a positive answer to Question 1.2 is given by the following recent result by Kovalev [28].

Theorem 1.4 ($p = \infty$). *Let $\varphi\colon \partial\mathbb{D} \to \mathbb{C}$ be a Lipschitz embedding. Then φ admits a homeomorphic Lipschitz extension to the whole plane $\mathbb{C}$.*

Let us return to the case of the Dirichlet energy (see (1)–(3) above). The equivalence of a $\mathscr{W}^{1,2}$-Sobolev extension and a $\mathscr{W}^{1,2}$-Sobolev homeomorphic extension for non-Lipschitz targets is a more subtle question. In this perspective, a slightly more general class of domains is the class of inner chordarc domains studied in geometric function theory [17, 31, 35–37]. By definition [36], a Jordan domain $\mathbb{Y}$ with rectifiable boundary is *inner chordarc* if there exists a constant C such that for every pair of points $y_1, y_2 \in \partial\mathbb{Y}$ one has $|y_1 - y_2| \leq C \cdot \lambda_{\mathbb{Y}}(y_1, y_2)$, where $\lambda_{\mathbb{Y}}(y_1, y_2)$ denotes the infimal length of curves contained in $\overline{\mathbb{Y}}$ with endpoints y_1 and y_2. For example, an inner chordarc domain may have inward cusps on the boundary, as opposed to Lipschitz domains. According to a result of Väisälä [36], the inner chordarc condition is equivalent to the requirement that there exists a homeomorphism $\Psi\colon \overline{\mathbb{Y}} \xrightarrow{\text{onto}} \overline{\mathbb{D}}$, $\mathscr{C}^1$-diffeomorphic in $\mathbb{Y}$, such that the norms of both the gradient matrices $D\Psi$ and $(D\Psi)^{-1}$ are bounded from above.

Surprisingly, the following example shows that, unlike for Lipschitz targets, the answer to Question 1.2 for $p = 2$ is in general negative when the target is only inner chordarc.

Example 1.5. There is an inner chordarc domain $\mathbb{Y}$ and a homeomorphism $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfying the Douglas condition (1.2) which does not admit a homeomorphic extension $h\colon \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,2}(\mathbb{D},\mathbb{Y})$.

In [4], as a part of studies of mappings with smallest mean distortion, it was proved that for $\mathscr{C}^1$-smooth $\mathbb{Y}$ the Douglas condition (1.2) can be equivalently formulated in terms of the inverse mapping $\varphi^{-1}\colon \partial\mathbb{Y} \xrightarrow{\text{onto}} \partial\mathbb{D}$:

$$\int_{\partial\mathbb{Y}}\int_{\partial\mathbb{Y}} \big|\log|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)|\big|\,|\mathrm{d}\xi|\,|\mathrm{d}\eta| < \infty. \tag{1.3}$$

It was recently shown that for inner chordarc targets this condition is necessary and sufficient for φ to admit a $\mathscr{W}^{1,2}$-homeomorphic extension [27]. We extend this result both to cover rectifiable targets and to give a global homeomorphic extension as follows.

Theorem 1.6 ($p=2$). *Let $\mathbb{Y}$ be a Jordan domain with $\partial\mathbb{Y}$ rectifiable. Every $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfying (1.3) admits a homeomorphic extension $h\colon \mathbb{C}\to\mathbb{C}$ of class $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$.*

Without the rectifiability of $\partial\mathbb{Y}$, Question 1.2 will in general have a negative answer for all $p\le 2$. This follows from the following example of Zhang [40].

Example 1.7. There exists a Jordan domain $\mathbb{Y}$ and a homeomorphism $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ which has a $\mathscr{W}^{1,2}$-Sobolev extension to $\mathbb{D}$ but has no homeomorphic extension in the class $\mathscr{W}^{1,1}(\mathbb{D},\mathbb{C})$.

We now return to the case when $1\le p<2$. In this case it is natural to ask under which conditions on the domains $\mathbb{X}$ and $\mathbb{Y}$, any homeomorphism $\varphi : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ admits a $\mathscr{W}^{1,p}$-Sobolev homeomorphic extension. Proposition 1.3 already implies that this is the case for $\mathbb{X}=\mathbb{D}$ and $\mathbb{Y}$ convex. Example 1.7, however, implies that this result does not hold in general for nonrectifiable targets $\mathbb{Y}$. A general characterization is provided by the following theorem.

Theorem 1.8 ($1\le p<2$). *Let $\mathbb{X}$ and $\mathbb{Y}$ be Jordan domains in the plane with $\partial\mathbb{Y}$ rectifiable. Let $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ be a homeomorphism. Then there is a homeomorphic extension $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ such that*

(1) $h\in\mathscr{W}^{1,1}(\mathbb{X},\mathbb{C})$*, provided $\partial\mathbb{X}$ is rectifiable, and*

(2) $h\in\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$ *for $1<p<2$, provided $\mathbb{X}$ has* s-hyperbolic growth *with $s>p-1$.*

Definition 1.9. Let $\mathbb{X}$ be a domain in the plane. Choose a point $x_0\in\mathbb{X}$. We say that $\mathbb{X}$ has *s-hyperbolic growth*, $s\in(0,1)$, if

$$h_{\mathbb{X}}(x_0,x)\le C\left(\frac{\operatorname{dist}(x_0,\partial\mathbb{X})}{\operatorname{dist}(x,\partial\mathbb{X})}\right)^{1-s} \qquad \text{for all } x\in\mathbb{X}. \tag{1.4}$$

Here $h_{\mathbb{X}}$ stands for the quasihyperbolic metric on $\mathbb{X}$ and $\operatorname{dist}(x,\partial\mathbb{X})$ is the Euclidean distance from x to the boundary. The constant C is allowed to depend on s, x_0, and the domain $\mathbb{X}$ but not on the point x.

It is easily verified that this definition does not depend on the choice of x_0. Recall that if Ω is a domain, the quasihyperbolic metric h_Ω is defined by [13]

$$h_\Omega(x_1, x_2) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{1}{\operatorname{dist}(x, \partial \mathbb{X})} \, |\mathrm{d}x|, \qquad x_1, x_2 \in \Omega, \tag{1.5}$$

where Γ is the family of all rectifiable curves in Ω joining x_1 and x_2.

Definition 1.9 is motivated by the following example. For $s \in (0, 1)$ we consider the Jordan domain $\mathbb{X}_s$ whose boundary is given by the curve

$$\Gamma_s = \{(x, y) \in \mathbb{C} \colon \, -1 \le x \le 1, \; y = |x|^s\} \cup \{z \in \mathbb{C} \colon |z - i| = 1, \; \operatorname{Im}(z) \ge 1\}.$$

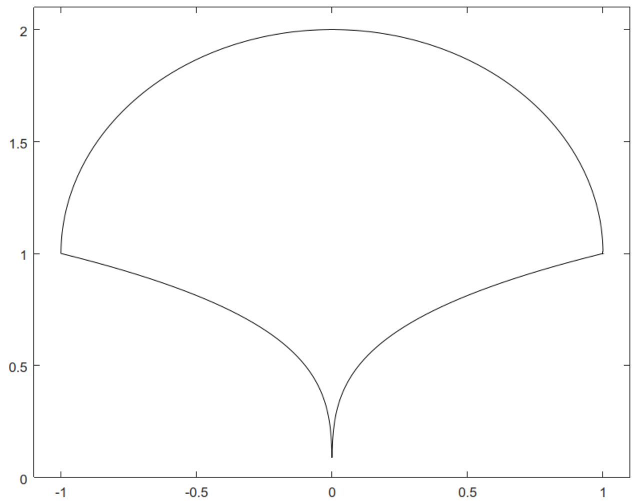

Fig. 1. The Jordan domain $\mathbb{X}_s$.

In particular, the boundary of $\mathbb{X}_s$ is locally Lipschitz except at the origin. Near the origin the boundary of $\mathbb{X}_s$ behaves like the graph of the function $|x|^s$. Then one can verify that the boundary of $\mathbb{X}_s$ has t-hyperbolic growth for every $t \ge s$. Note that the smaller the number s, the sharper the cusp is.

The results of Theorem 1.8 are sharp, as described by the following result.

Theorem 1.10.

(1) *There exists a Jordan domain $\mathbb{X}$ with nonrectifiable boundary and a homeomorphism $\varphi \colon \partial \mathbb{X} \to \partial \mathbb{D}$ such that φ does not admit a continuous extension to $\mathbb{X}$ in the Sobolev class $\mathscr{W}^{1,1}(\mathbb{X}, \mathbb{C})$.*

(2) *For every $p \in (1, 2)$ there exists a Jordan domain $\mathbb{X}$ which has s-hyperbolic growth, with $p - 1 = s$, and a homeomorphism $\varphi \colon \partial \mathbb{X} \to \partial \mathbb{D}$ such that φ does not admit a continuous extension to $\mathbb{X}$ in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$.*

To conclude, as promised earlier, we extend our main result to the case where the domains are not simply connected. The following generalization of Theorem 1.8 holds.

Theorem 1.11. *Let $\mathbb{X}$ and $\mathbb{Y}$ be multiply connected Jordan domains with $\partial\mathbb{Y}$ rectifiable. Let $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ be a homeomorphism which maps the outer boundary component of $\mathbb{X}$ to the outer boundary component of $\mathbb{Y}$. Then there is a homeomorphic extension $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ such that*

(1) $h \in \mathscr{W}^{1,1}(\mathbb{X},\mathbb{C})$*, provided $\partial\mathbb{X}$ is rectifiable, and*

(2) $h \in \mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$ *for $1 < p < 2$, provided $\mathbb{X}$ has s-hyperbolic growth with $s > p-1$.*

2. Preliminaries

2.1. The Dirichlet problem

Let Ω be a bounded domain in the complex plane. A function $u\colon \Omega \to \mathbb{R}$ in the Sobolev class $\mathscr{W}^{1,p}_{\text{loc}}(\Omega)$, $1 < p < \infty$, is called *p-harmonic* if

$$\operatorname{div} |\nabla u|^{p-2}\nabla u = 0. \tag{2.1}$$

We call 2-harmonic functions simply *harmonic*.

There are two formulations of the Dirichlet boundary value problem for the p-harmonic equation (2.1). We first consider the variational formulation.

Lemma 2.1. *Let $u_\circ \in \mathscr{W}^{1,p}(\Omega)$ be a given Dirichlet data. There exists precisely one function $u \in u_\circ + \mathscr{W}^{1,p}_\circ(\Omega)$ which minimizes the p-harmonic energy:*

$$\int_\Omega |\nabla u|^p = \inf\left\{\int_\Omega |\nabla w|^p \colon w \in u_\circ + \mathscr{W}^{1,p}_\circ(\Omega)\right\}.$$

Here $\mathscr{W}^{1,p}_\circ(\Omega)$ denotes the completion of compactly supported smooth functions in Ω with respect to the $\mathscr{W}^{1,p}(\Omega)$ Sobolev norm. The variational formulation coincides with the classical formulation of the Dirichlet problem.

Lemma 2.2. *Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain and $u_\circ \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$. Then there exists a unique p-harmonic function $u \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ such that $u|_{\partial\Omega} = u_\circ|_{\partial\Omega}$.*

For the proofs of these facts we refer to [18].

2.2. The Radó–Kneser–Choquet Theorem

Lemma 2.3. *Consider a Jordan domain $\mathbb{X} \subset \mathbb{C}$ and a bounded convex domain $\mathbb{Y} \subset \mathbb{C}$. Let $h\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ be a homeomorphism and $H\colon \mathbb{X} \to \mathbb{C}$ its harmonic extension. Then H is a $\mathscr{C}^\infty$-diffeomorphism of $\mathbb{X}$ onto $\mathbb{Y}$.*

For the proof of this lemma we refer to [11, 21]. The following p-harmonic analogue of the Radó–Kneser–Choquet Theorem is due to Alessandrini and Sigalotti [1] (see also [22]).

Proposition 2.4. *Let $\mathbb{X}$ be a Jordan domain in $\mathbb{C}$, $1 < p < \infty$, and $h = u + iv\colon \overline{\mathbb{X}} \to \mathbb{C}$ a continuous mapping whose coordinate functions are p-harmonic. Suppose that $\mathbb{Y}$ is convex and $h\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is a homeomorphism. Then h is a diffeomorphism from $\mathbb{X}$ onto $\mathbb{Y}$.*

2.3. Sobolev homeomorphic extensions onto a Lipschitz target

Combining the results in this section allows us to easily solve Question 1.2 for convex targets.

Proposition 2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Jordan domains in the plane with $\mathbb{Y}$ convex, and let $1 < p < \infty$. Suppose that $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is a homeomorphism. Then there exists a continuous $g\colon \overline{\mathbb{X}} \to \mathbb{C}$ in $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$ such that $g(x) = \varphi(x)$ on $\partial\mathbb{X}$ if and only if there exists a homeomorphism $h\colon \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ in $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$ such that $h(x) = \varphi(x)$ on $\partial\mathbb{X}$.*

Proof. The "if" part is immediate. For the "only if" part we write $g = u_\circ + i v_\circ \in \mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C}) \cap \mathcal{C}(\overline{\mathbb{X}}, \mathbb{C})$ and consider the unique p-harmonic functions u and v which coincide with $u_\circ = \operatorname{Re}\varphi$ and $v_\circ = \operatorname{Im}\varphi$ respectively on $\partial\mathbb{X}$. First, these classical solutions agree with the variational ones (see Lemmas 2.1 and 2.2). In particular, we have

$$\int_{\mathbb{X}} |\nabla u|^p \leq \int_{\mathbb{X}} |\nabla u_\circ|^p \quad \text{and} \quad \int_{\mathbb{X}} |\nabla v|^p \leq \int_{\mathbb{X}} |\nabla v_\circ|^p .$$

Second, according to Proposition 2.4 the mapping $h \in \mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$ is a homeomorphism. ∎

Now, replacing the convex $\mathbb{Y}$ by a Lipschitz domain offers no challenge. Indeed, this follows from a global bi-Lipschitz change of variables $\Phi\colon \mathbb{C} \to \mathbb{C}$ for which $\Phi(\mathbb{Y})$ is the unit disk. If the domain in Proposition 2.5 is the unit disk $\mathbb{D}$, then the existence of a finite p-harmonic extension can be characterized in terms of a Douglas type condition. If $1 < p < 2$, then such an extension exists for an arbitrary boundary homeomorphism (Proposition 1.3) and if $2 \leq p < \infty$ the extension exists if and only the boundary homeomorphism $\varphi\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfies

$$\int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} \left| \frac{\varphi(\xi) - \varphi(\eta)}{\xi - \eta} \right|^p |d\xi|\, |d\eta| < \infty. \tag{2.2}$$

For the proof of this last fact we refer to [34, pp. 151–152].

2.4. Carleson measures and the Hardy space H^p

Roughly speaking, a Carleson measure on a domain $\mathbb{G}$ is a measure that is bounded from above by the Hausdorff 1-measure on $\partial\mathbb{G}$ near the boundary of $\mathbb{G}$. We will need the notion of a Carleson measure only on the unit disk $\mathbb{D}$.

Definition 2.6. Let μ be a Borel measure on $\mathbb{D}$. Then μ is a *Carleson measure* if there is a constant $C > 0$ such that

$$\mu(S_\epsilon(\theta)) \leq C\epsilon$$

for every $\epsilon > 0$. Here

$$S_\epsilon(\theta) = \{re^{i\alpha} \colon 1-\epsilon < r < 1,\ \theta - \epsilon < \alpha < \theta + \epsilon\}.$$

Carleson measures have many applications in harmonic analysis. A celebrated result by L. Carleson [6] (see also [10, Theorem 9.3]) tells us that a Borel measure μ on $\mathbb{D}$ is a bounded Carleson measure if and only if the injective mapping from the Hardy space $H^p(\mathbb{D})$ into the measurable space $L^p_\mu(\mathbb{D})$ is bounded.

Proposition 2.7. *Let μ be a Borel measure on the unit disk $\mathbb{D}$. Let $0 < p < \infty$. Then there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f(z)|^p \, d\mu(z)\right)^{1/p} \leq C\|f\|_{H^p(\mathbb{D})} \quad \textit{for all } f \in H^p(\mathbb{D})$$

if and only if μ is a Carleson measure.

Recall that the Hardy space $H^p(\mathbb{D})$, $0 < p < \infty$, is the class of holomorphic functions f on the unit disk satisfying

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0 \leq r < 1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} < \infty.$$

Note that $\|\cdot\|_{H^p(\mathbb{D})}$ is a norm when $p \geq 1$, but not when $0 < p < 1$.

3. Sobolev integrability of the harmonic extension

At the end of this section we prove our main result in the simply connected case, Theorem 1.8. The proof will be based on a suitable reduction of the target domain to the unit disk, and the following auxiliary result which concerns the regularity of harmonic extensions.

Theorem 3.1. *Let $\mathbb{X}$ be a Jordan domain and $\varphi \colon \partial\mathbb{X} \to \partial\mathbb{D}$ an arbitrary homeomorphism. Let h denote the harmonic extension of φ to $\mathbb{X}$, which is a homeomorphism from $\overline{\mathbb{X}}$ to $\overline{\mathbb{D}}$. Then the following hold.*

(1) *If the boundary of $\mathbb{X}$ is rectifiable, then $h \in \mathcal{W}^{1,1}(\mathbb{X}, \mathbb{C})$.*

(2) *If $\mathbb{X}$ has s-hyperbolic growth, then $h \in \mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$ for $p < s + 1$.*

This theorem will be a direct corollary of the following theorem and the two propositions after it.

Theorem 3.2. *Let $\mathbb{X}$ be a Jordan domain, and denote by $g\colon \mathbb{D} \to \mathbb{X}$ a conformal map onto $\mathbb{X}$. Let $1 \le p < 2$. Suppose that*

$$\sup_{\omega\in\partial\mathbb{D}} \int_{\mathbb{D}} \frac{|g'(z)|^{2-p}}{|\omega - z|^p}\, dz \le M < \infty. \tag{3.1}$$

Then the harmonic extension $h\colon \mathbb{X} \to \mathbb{D}$ of any boundary homeomorphism $\varphi\colon \partial\mathbb{X} \to \partial\mathbb{D}$ lies in the Sobolev space $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$, with the estimate

$$\|h\|_{\mathcal{W}^{1,p}(\mathbb{X},\mathbb{C})} \le cM. \tag{3.2}$$

Proposition 3.3. *Let $\mathbb{X}$ be a Jordan domain with rectifiable boundary and let $g\colon \mathbb{D} \to \mathbb{X}$ be conformal. Then condition (3.1) holds with $p = 1$.*

Proposition 3.4. *Let $\mathbb{X}$ be a Jordan domain which has s-hyperbolic growth with $s \in (0, 1)$ and let $g\colon \mathbb{D} \to \mathbb{X}$ be conformal. Then condition (3.1) holds for all $p > 1$ with $p - 1 < s$.*

Proof of Theorem 3.2. First, since $\mathbb{X}$ is a Jordan domain, according to the classical Carathéodory's theorem the conformal mapping $g\colon \mathbb{D} \to \mathbb{X}$ extends continuously to a homeomorphism from the unit circle onto $\partial\mathbb{X}$. Second, since a conformal change of variables preserves harmonicity, the map $H := h \circ g\colon \mathbb{D} \to \mathbb{D}$ is a harmonic extension of the boundary homeomorphism $\psi := \varphi \circ g|_{\partial\mathbb{D}}$.

We will now assume that H is smooth up to the boundary of $\mathbb{D}$. The general result will then follow by an approximation argument. Indeed, for each $r < 1$, we may take the preimage of the disk $B(0, r)$ under H, and letting $\psi_r : \mathbb{D} \to H^{-1}(B(0, r))$ be the conformal map onto this preimage we may define $H_r := H \circ \psi_r$. Then H_r is harmonic, smooth up to the boundary of $\mathbb{D}$, and will converge to H locally uniformly along with its derivatives as $r \to 1$. Hence the general result will follow once we obtain uniform estimates for the Sobolev norm under the assumption of smoothness up to the boundary.

The harmonic extension $H := h \circ g\colon \mathbb{D} \to \mathbb{D}$ of $\psi := \varphi \circ g|_{\partial\mathbb{D}}$ is given by the Poisson integral formula [11],

$$(h \circ g)(z) = H(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|z - \omega|} \psi(\omega)\, d\omega.$$

Differentiating this, we find

$$2\pi i \cdot (h \circ g)_z = \int_{\partial\mathbb{D}} \frac{\psi(\omega)}{(z - \omega)^2}\, d\omega = \int_0^{2\pi} \frac{\psi(e^{it})}{(z - e^{it})^2} i e^{it}\, dt = -\int_0^{2\pi} \frac{\psi'(e^{it})}{z - e^{it}} i e^{it}\, dt,$$

where we have used integration by parts to arrive at the last equality [20, p. 147]. The change of variables formula now gives

$$\begin{aligned} \int_{\mathbb{X}} |h_z(\tilde{z})|^p\, d\tilde{z} &= \int_{\mathbb{D}} |(h \circ g)_z(z)|^p |g'(z)|^{2-p}\, dz \\ &= \frac{1}{(2\pi)^p} \int_{\mathbb{D}} \left| \int_0^{2\pi} \frac{\psi'(e^{it})}{z - e^{it}} i e^{it}\, dt \right|^p |g'(z)|^{2-p}\, dz. \end{aligned}$$

We now apply Minkowski's integral inequality to find that

$$\left(\int_{\mathbb{D}}\left|\int_0^{2\pi}\frac{\psi'(e^{it})}{z-e^{it}}ie^{it}\,dt\right|^p|g'(z)|^{2-p}\,dz\right)^{1/p}$$
$$\leq \int_0^{2\pi}|\psi'(e^{it})|\left(\int_{\mathbb{D}}\frac{|g'(z)|^{2-p}}{|z-e^{it}|^p}\,dz\right)^{1/p}dt \leq M\int_0^{2\pi}|\psi'(e^{it})|\,dt = 2\pi M.$$

This gives the uniform bound $\|h_z\|_{L^p(\mathbb{X})} \leq M$. An analogous estimate for the L^p-norm of $h_{\bar{z}}$ now proves the theorem. ■

Proof of Proposition 3.3. Since $\partial\mathbb{X}$ is rectifiable, the derivative g' of a conformal map from $\mathbb{D}$ onto $\mathbb{X}$ lies in the Hardy space $H^1(\mathbb{D})$ by [10, Theorem 3.12]. By rotational symmetry it is enough to verify condition (3.1) for $\omega = 1$ and $g\colon \mathbb{D} \to \mathbb{X}$ an arbitrary conformal map. By Proposition 2.7, it suffices to verify that the measure $\mu(z) = \frac{dz}{|1-z|}$ is a Carleson measure (see Definition 2.6), to obtain the estimate

$$\int_{\mathbb{D}}\frac{|g'(z)|}{|1-z|}\,dz \leq C\|g'\|_{H^1(\mathbb{D})},$$

which will imply that the proposition holds. Therefore, let us for each ϵ define $S_\epsilon(\theta) = \{re^{i\alpha}\colon 1-\epsilon < r < 1,\ \theta-\epsilon < \alpha < \theta+\epsilon\}$. We then estimate for small ϵ that

$$\mu(S_\epsilon(0)) \leq \mu(B(1,2\epsilon)) = \int_{B(1,2\epsilon)}\frac{dz}{|1-z|} = \int_0^{2\pi}\int_0^{2\epsilon}\frac{1}{r}\,r\,dr\,d\alpha = 4\pi\epsilon.$$

It is clear that for any other angles θ the μ-measure of $S_\epsilon(\theta)$ is smaller than for $\theta = 0$. Hence μ is a Carleson measure and our proof is complete. ■

Proof of Proposition 3.4. Recall that g denotes the conformal map from $\mathbb{D}$ onto $\mathbb{X}$. Since $\mathbb{X}$ has s-hyperbolic growth, we may apply Definition 1.9 with $x_0 = g(0)$ to find that

$$h_{\mathbb{X}}(g(0), g(z)) \leq C\left(\frac{1}{\operatorname{dist}(g(z),\partial\mathbb{X})}\right)^{1-s} \qquad \text{for all } z \in \mathbb{D}. \tag{3.3}$$

Since $\mathbb{X}$ is simply connected, the quasihyperbolic distance is comparable to the hyperbolic distance $\rho_{\mathbb{X}}$. By conformal invariance of the hyperbolic distance we find that

$$C_1 h_{\mathbb{X}}(g(0), g(z)) \geq \rho_{\mathbb{X}}(g(0), g(z)) = \rho_{\mathbb{D}}(0, z) = \log\frac{1}{1-|z|^2}.$$

Now by the Koebe $\frac{1}{4}$-theorem we know that the expression $\operatorname{dist}(g(z),\partial\mathbb{X})$ is comparable to $(1-|z|)|g'(z)|$ with a universal constant. Combining these observations with (3.3) leads to the estimate

$$\log\frac{1}{1-|z|^2} \leq C\left(\frac{1}{(1-|z|)|g'(z)|}\right)^{1-s},$$

which we transform into

$$|g'(z)| \leq \frac{C}{(1-|z|)\log^{1/(1-s)}\frac{1}{1-|z|}}. \tag{3.4}$$

Set $\beta = (2-p)/(1-s)$, so that $\beta > 1$ by assumption. We now apply the estimate (3.4) to find that

$$\int_{\mathbb{D}} \frac{|g'(z)|^{2-p}}{|1-z|^p}\,dz \leq C\int_{\mathbb{D}\setminus\frac{1}{2}\mathbb{D}} \frac{1}{(1-|z|)^{2-p}|1-z|^p\log^{\beta}\frac{1}{1-|z|}}\,dz + \int_{\frac{1}{2}\mathbb{D}} \frac{|g'(z)|^{2-p}}{|1-z|^p}\,dz. \tag{3.5}$$

It is enough to prove that the quantity on the right hand side above is finite as then rotational symmetry will imply that the estimate (3.1) holds for all ω. The second term is easily seen to be finite, as the integrand is bounded on $\frac{1}{2}\mathbb{D}$. To estimate the first integral we will cover the annulus $\mathbb{D}\setminus\frac{1}{2}\mathbb{D}$ by three sets defined by

$$S_1 = \{1 + re^{i\theta} : r \leq 3/4,\ 3\pi/4 \leq \theta \leq 5\pi/4\},$$
$$S_2 = \{(x,y) \in \mathbb{D} : -1/\sqrt{2} \leq y \leq 1/\sqrt{2},\ x \leq 1,\ x \geq 1-|y|\},$$
$$S_3 = \{re^{i\theta} : 1/2 \leq r \leq 1,\ \pi/4 \leq \theta \leq 7\pi/4\}$$

(see Figure 2). Since the sets S_1, S_2 and S_3 cover the annulus in question, it will be enough to see that the first integral on the right hand side of (3.5) is finite when taken over each of these sets. On S_1, one may find by geometry that $1-|z| \geq c|1-z|$ for some constant c.

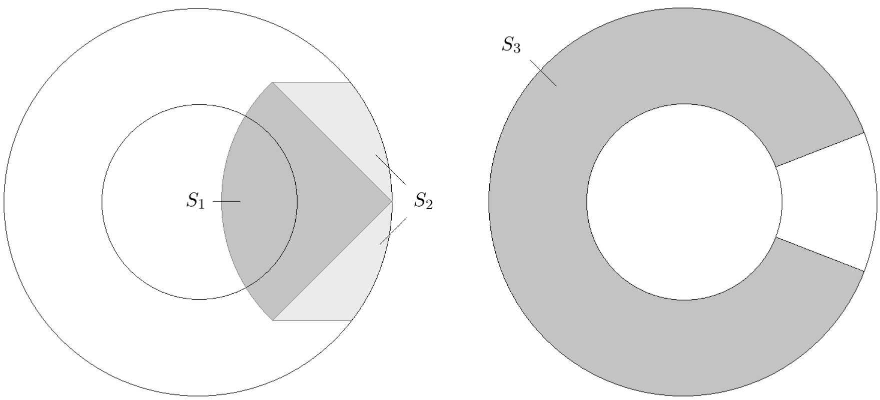

Fig. 2. The sets S_i, $i = 1, 2, 3$.

Hence we may apply polar coordinates around $z = 1$ to find that

$$\int_{S_1} \frac{1}{(1-|z|)^{2-p}|1-z|^p \log^\beta \frac{1}{1-|z|}}\, dz \leq C \int_{3\pi/4}^{5\pi/4} \int_0^{3/4} \frac{1}{r \log^\beta \frac{1}{r}}\, dr\, d\theta < \infty.$$

On S_3, the expression $|1-z|$ is bounded away from zero. Hence bounding this term and the logarithm from below and changing to polar coordinates around the origin yields

$$\int_{S_3} \frac{1}{(1-|z|)^{2-p}|1-z|^p \log^\beta \frac{1}{1-|z|}}\, dz \leq C \int_{\pi/4}^{7\pi/4} \int_{1/2}^{1} \frac{r}{(1-r)^{2-p}}\, dr\, d\theta < \infty.$$

On S_2, we change to polar coordinates around the origin. For each angle θ, we let R_θ denote the intersection of the ray with angle θ starting from the origin and the set S_2. On each such ray, the expression $|1-z|$ is comparable to $|\theta|$. Since $1-|z| < |1-z|$, we may also replace $1-|z|$ by $|1-z|$ inside the logarithm, finally giving

$$\frac{1}{|1-z|^p \log^\beta \frac{1}{1-|z|}} \leq \frac{C}{|\theta|^p \log^\beta \frac{1}{|\theta|}}, \qquad z \in R_\theta. \tag{3.6}$$

On each of the segments R_θ for small enough θ, the modulus $r = |z|$ ranges from a certain distance ρ to 1. This distance $\rho := \rho(\theta)$ is found by applying the sine theorem to the triangle with vertices $0, 1$ and $\rho(\theta)e^{i\theta}$, giving us the equation

$$\frac{\rho(\theta)}{\sin(\pi/4)} = \frac{1}{\sin(\pi - \pi/4 - \theta)} = \frac{1}{\sin(\pi/4+\theta)}.$$

From this one finds that the expression $1-\rho(\theta) = \frac{\sin(\pi/4+\theta)-\sin(\pi/4)}{\sin(\pi/4+\theta)}$, which is also the length of the segment R_θ, is comparable to $|\theta|$. Using this and (3.6) we now obtain

$$\begin{aligned}\int_{S_2} \frac{1}{(1-|z|)^{2-p}|1-z|^p \log^\beta \frac{1}{1-|z|}}\, dz &\leq C \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta|^p \log^\beta \frac{1}{|\theta|}} \int_{\rho(\theta)}^{1} \frac{1}{(1-r)^{2-p}}\, dr\, d\theta \\ &= C \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta|^p \log^\beta \frac{1}{|\theta|}} \frac{(1-\rho(\theta))^{p-1}}{p-1}\, dr\, d\theta \leq C \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta| \log^\beta \frac{1}{|\theta|}}\, dr\, d\theta < \infty.\end{aligned}$$

This finishes the proof. ∎

Proof of Theorem 1.8. Since $\mathbb{Y}$ is a Jordan domain with rectifiable boundary, there exists a constant speed parametrization $\gamma : \partial\mathbb{D} \to \partial\mathbb{Y}$. Such a parametrization is then automatically a Lipschitz embedding of $\partial\mathbb{D}$ to $\mathbb{C}$, and hence Theorem 1.4 implies that γ has a homeomorphic Lipschitz extension $G : \overline{\mathbb{D}} \to \overline{\mathbb{Y}}$.

Let now $\varphi : \partial\mathbb{X} \to \partial\mathbb{Y}$ be a given boundary homeomorphism. We define a boundary homeomorphism $\varphi_0 : \partial\mathbb{X} \to \partial\mathbb{D}$ by setting $\varphi_0 := \gamma^{-1} \circ \varphi$. Let h_0 denote the harmonic extension of φ_0 to $\mathbb{X}$, so that by the RKC Theorem (Lemma 2.3) the composed map $h := G \circ h_0 : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ gives a homeomorphic extension of φ. If h_0 lies in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$, then so does h, since the Sobolev integrability is preserved under composition with a Lipschitz map. Hence Theorem 1.8 now follows from Theorem 3.1. ∎

4. Sharpness of Theorem 1.8

In this section we prove Theorem 1.10. We handle the two claims of the theorem separately.

Example (1). In this example we construct a Jordan domain $\mathbb{X}$ with nonrectifiable boundary and a boundary map $\varphi : \partial\mathbb{X} \to \partial\mathbb{D}$ which does not admit a continuous extension in $\mathcal{W}^{1,1}(\mathbb{X}, \mathbb{C})$. The domain $\mathbb{X}$ will be the following "spiral" domain.

Let R_k, $k = 1, 2, \ldots$, be a set of disjoint rectangles in the plane with bottom sides on the x-axis. Each rectangle has width w_k such that $\sum_{k=1}^{\infty} w_k < \infty$ and the rectangles are sufficiently close to each other so that the collection stays in a bounded set. The heights h_k satisfy $\lim_{k\to\infty} h_k = 0$ and $\sum_{k=1}^{\infty} h_k = \infty$.

We now join these rectangles into a spiral domain as in Figure 3, and add a small portion of boundary to the bottom side of R_1. The exact way these rectangles are joined is not significant, but it is clear that it may be done in such a way as to produce a Jordan domain $\mathbb{X}$ with nonrectifiable boundary, for any sequence of rectangles R_k as described above.

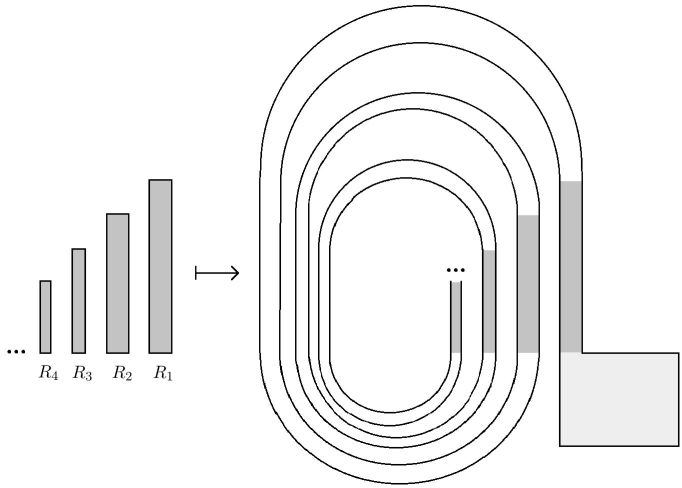

Fig. 3. The rectangles R_k joined into the spiral domain $\mathbb{X}$.

Let us now define the boundary homeomorphism φ. The map φ shall map the "endpoint" (i.e. the point on the x-axis to which the rectangles R_k converge) of the spiral domain $\mathbb{X}$ to the point $1 \in \partial\mathbb{D}$. Furthermore, we choose disjoint arcs A_k^+ on the unit circle so that the endpoints of A_k^+ are $e^{i\alpha_k}$ and $e^{i\beta_k}$ with

$$\pi/2 > \alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \cdots$$

and $\lim_{k\to\infty} \alpha_k = 0$. We mirror the arcs A_k^+ in the x-axis to produce another set of arcs A_k^-. The arcs are chosen in such a way that the minimal distance between A_k^+ and

A_k^- is greater than a given sequence of numbers d_k with $\lim_{k\to\infty} d_k = 0$. It is clear that for any such sequence we can make a choice of arcs as described here.

We now define φ to map the left side of the rectangle R_k to the arc A_k^+, and the right side to A_k^-. On the rest of the boundary $\partial\mathbb{X}$ we define φ in an arbitrary way so as to produce a homeomorphism $\varphi\colon \partial\mathbb{X} \to \partial\mathbb{D}$.

Let now H be a continuous $\mathcal{W}^{1,1}$-extension of φ. Let I_k denote any horizontal line segment with endpoints on the vertical sides of R_k. Then by the above construction, H must map the segment I_k to a curve of length at least d_k, as this is the minimal distance between A_k^+ and A_k^-. Hence we find that

$$\int_{R_k} |DH|\, dz \geq \int_0^{h_k} d_k\, dz = h_k d_k.$$

Adding up, we obtain the estimate

$$\int_{\mathbb{X}} |DH|\, dz \geq \sum_{k=1}^{\infty} h_k d_k.$$

We may now choose, for example, $h_k = 1/k$ and $d_k = 1/\log(1+k)$ to make the above sum diverge, showing that H cannot belong to $\mathcal{W}^{1,1}(\mathbb{X}, \mathbb{C})$. This finishes the proof.

Example (2). Let $1 < p < 2$. Here we construct a Jordan domain $\mathbb{X}$ whose boundary has $(p-1)$-hyperbolic growth and a boundary map $\varphi\colon \partial\mathbb{X} \to \partial\mathbb{D}$ which does not admit a continuous extension in the Sobolev class $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$. In fact, we take the domain $\mathbb{X}_s$ described after Definition 1.9 for $s = p - 1$.

The construction of the boundary map φ is as follows.

We set $\varphi(0) = 1$. Furthermore, we choose two sequences of points p_k^+ and p_k^- belonging to the graph $\Gamma := \{(x, |x|^s)\colon -1 \leq x \leq 1\}$ as follows. The points p_k^+ all have positive x-coordinates, their y-coordinates are decreasing in k with limit zero and the difference between the y-coordinates of p_{k-1}^+ and p_k^+ is $\epsilon_k = k^{-2}/10$. We then let p_k^- be the reflection of p_k^+ in the y-axis.

Similarly, we choose points a_k^+ on the unit circle so that $a_k^+ = e^{i\theta_k}$ for a sequence of angles $\theta_k \in (0, \pi/2)$ decreasing to zero. Letting a_k^- be the reflection of a_k^+ in the x-axis, we choose the sequence in such a way that the line segment between a_k^+ and a_k^- has length $d_k = (\log(100+k))^{-1/p}$ so that d_k is decreasing and $\lim_{k\to\infty} d_k = 0$.

Let Γ_k^+ denote the part of the graph Γ between p_{k-1}^+ and p_k^+. We define the map φ to map Γ_k^- to the smaller arc of the unit circle between a_{k-1}^- and a_k^- with constant speed. We define Γ_k^- and $\varphi|_{\Gamma_k^-}$ similarly. Let now H denote any continuous $\mathcal{W}^{1,p}$-extension of φ to $\mathbb{X}$. By the above definition, any horizontal line segment with endpoints on Γ_k^+ and Γ_k^- is mapped into a curve of length at least d_k under H. Such a line segment is of length at most the distance from p_{k-1}^+ to p_{k-1}^-, which is comparable to $(\sum_{j=k}^{\infty} \epsilon_j)^{1/s}$. If S_k denotes the domain which is the union of all the horizontal line segments between Γ_k^+

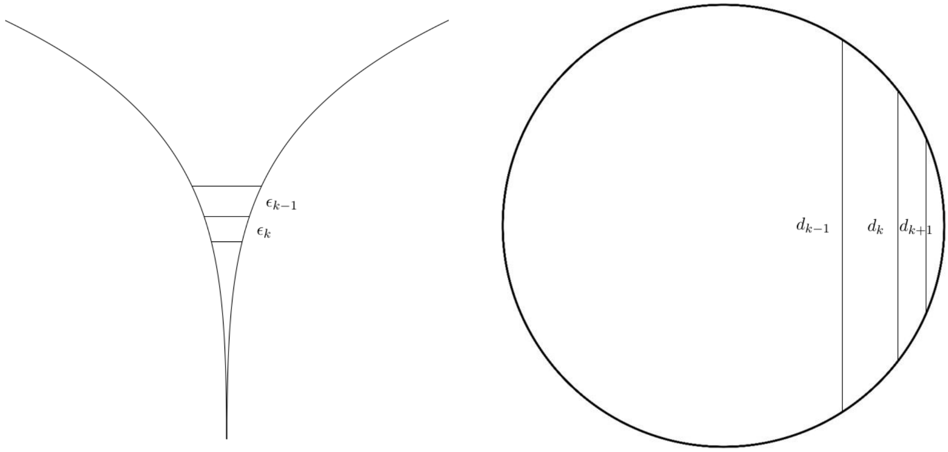

Fig. 4. The portions of height ϵ_k get mapped onto slices with side length d_k.

and Γ_k^-, this gives the estimate

$$\int_{S_k} |DH|^p \, dz \geq \frac{(\int_{S_k} |DH| \, dz)^p}{|S_k|^{p-1}} \geq \frac{c(\int_0^{\epsilon_k} d_k \, dy)^p}{\epsilon_k^{p-1}(\sum_{j=k}^\infty \epsilon_j)^{(p-1)/s}} = \frac{c d_k^p \epsilon_k}{\sum_{j=k}^\infty \epsilon_j}.$$

Now by our choice of $\epsilon_k = k^{-2}/10$, we see that $\sum_{j=k}^\infty \epsilon_j$ is comparable to $1/k$, so by adding up we obtain the estimate

$$\int_{\bigcup_k S_k} |DH|^p \, dz \geq c \sum_{k=1}^\infty \frac{d_k^p}{k}. \tag{4.1}$$

However, our choice of $d_k = (\log(100 + k))^{-1/p}$ ensures that the right hand side of (4.1) diverges. It follows that H cannot lie in $\mathcal{W}^{1,p}(\mathbb{X}_s, \mathbb{C})$, which completes the proof.

5. The case $p = 2$

In this section we address Theorem 1.6 as well as Examples 1.5 and 1.7.

Example 1.5. For this example, let first Φ_τ for any $\tau \in (0, 1]$ denote the conformal map

$$\Phi_\tau(z) = \log^{-\tau}\left(\frac{1-z}{3}\right)$$

defined on the unit disk and having target $\mathbb{Y}_\tau := \Phi_\tau(\mathbb{D})$. In fact, $\mathbb{Y}_\tau$ is a domain with smooth boundary apart from one point at which it has an outer cusp of degree $\tau/(1+\tau)$ (i.e. it is bi-Lipschitz equivalent to the domain $\mathbb{X}_{\tau/(1+\tau)}$ pictured in Figure 1).

Since Φ_τ is conformal and maps the unit disk into a set of finite measure, it lies in the Sobolev space $\mathcal{W}^{1,2}(\mathbb{D},\mathbb{C})$. However, it does not admit a homeomorphic extension to the whole plane in the Sobolev class $\mathcal{W}^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$. The reason is a modulus of continuity estimate for any homeomorphism in $\mathcal{W}^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$. Indeed, let $\omega_z(t)$ denote the modulus of continuity of $g\colon \mathbb{C}\to\mathbb{C}$ at a point z,

$$\omega_z(t) = \operatorname*{osc}_{B(z,t)} g = \sup\{|g(x_1)-g(x_2)|\colon x_1,x_2\in B(z,t)\}.$$

If g is a homeomorphism in $\mathcal{W}^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$, then

$$\int_0^r \frac{\omega_z(t)^2}{t}\,dt < \infty. \tag{5.1}$$

Proof of (5.1). Since g is a homeomorphism, we have

$$\operatorname*{osc}_{B(z,t)} g \le \operatorname*{osc}_{\partial B(z,t)} g.$$

According to Sobolev's inequality on spheres, for almost every $t>0$ we obtain

$$\operatorname*{osc}_{\partial B(z,t)} g \le C\int_{\partial B(z,t)} |Dg|.$$

These together with Hölder's inequality imply

$$\omega_z(t) = \operatorname*{osc}_{B(z,t)} g \le \operatorname*{osc}_{\partial B(z,t)} g \le C\left(t\int_{\partial B(z,t)} |Dg|^2\right)^{1/2},$$

and therefore for almost every $t>0$ we have

$$\frac{\omega_z(t)^2}{t} \le C\int_{\partial B(z,t)} |Dg|^2,$$

where C is independent of z. Integrating this from 0 to $r>0$ yields the claim (5.1). ∎

Now, since the map Φ_τ for $\tau\le 1$ does not satisfy the modulus of continuity estimate (5.1) at the boundary point $z=1$, it is not possible to extend Φ_τ even locally as a $\mathcal{W}^{1,2}$-homeomorphism around the point $z=1$.

To address the exact claim of Example 1.5, we now define an embedding $\varphi\colon\partial\mathbb{D}\to\mathbb{C}$ as follows. Fixing $\tau\in(0,1]$, in the set $\{z\in\partial\mathbb{D}\colon\ \mathrm{Re}(z)\ge 0\}$ we let $\varphi(z)=\Phi_\tau(z)$. We also map the complementary set $\{z\in\partial\mathbb{D}\colon\ \mathrm{Re}(z)<0\}$ smoothly into the complement of $\overline{\mathbb{Y}_\tau}$, and in such a way that $\varphi(\partial\mathbb{D})$ becomes the boundary of a Jordan domain $\tilde{\mathbb{Y}}$ (see Figure 5). It is now easy to see that φ satisfies the Douglas condition (1.2). Indeed, since Φ_τ is in the Sobolev space $\mathcal{W}^{1,2}(\mathbb{D},\mathbb{C})$, its restriction to the boundary must necessarily satisfy the Douglas condition. Since φ agrees with this boundary map in a neighborhood of $z=1$, verifying the finiteness of the integral in (1.2) poses no difficulty in this neighborhood. On the rest of $\partial\mathbb{D}$ we may choose φ to be locally Lipschitz, which shows that (1.2) is

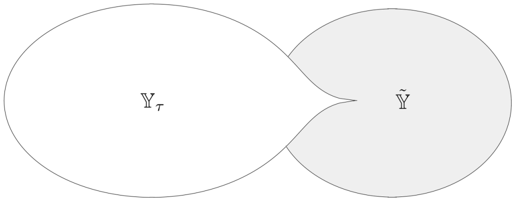

Fig. 5. The Jordan domains $\mathbb{Y}_\tau$ and $\tilde{\mathbb{Y}}$.

necessarily satisfied for φ. Hence we have found a map from $\partial\mathbb{D}$ into the boundary of the chordarc domain $\tilde{\mathbb{Y}}$ which admits a $\mathscr{W}^{1,2}$-extension to $\mathbb{D}$ but not a homeomorphic one.

Example 1.7. In [40], Zhang constructed an example of a Jordan domain, which we shall denote by $\mathbb{Y}$, such that the conformal map $g\colon \mathbb{D} \to \mathbb{Y}$ does not admit a $\mathscr{W}^{1,1}$-homeomorphic extension to the whole plane. We shall not repeat this construction here, but will instead briefly show how it relates to our questions.

The domain $\mathbb{Y}$ is constructed in such a way that there is a boundary arc $\Gamma \subset \partial\mathbb{Y}$ over which one cannot extend the conformal map g even locally as a $\mathscr{W}^{1,1}$-homeomorphism. The complementary part of the boundary, $\partial\mathbb{Y} \setminus \Gamma$, is piecewise linear. Hence we may employ the same argument as in the previous example. We choose a Jordan domain $\tilde{\mathbb{Y}}$ in the complement of $\mathbb{Y}$ whose boundary consists of the arc Γ and, say, a piecewise linear curve. We then define a boundary map $\varphi\colon \partial\mathbb{D} \to \partial\tilde{\mathbb{Y}}$ so that it agrees with g in a neighborhood of the set $g^{-1}(\Gamma)$ and is locally Lipschitz everywhere else. With the same argument as before, this boundary map must satisfy the Douglas condition (1.2). Hence this boundary map admits a $\mathscr{W}^{1,2}$-extension to $\mathbb{D}$ but not even a $\mathscr{W}^{1,1}$-homeomorphic extension. Naturally the boundary of the domain $\tilde{\mathbb{Y}}$ is quite ill-behaved, in particular nonrectifiable (though its Hausdorff dimension is still 1).

Proof of Theorem 1.6. Let $\gamma\colon \partial\mathbb{D} \to \partial\mathbb{Y}$ denote a constant speed parametrization of the rectifiable curve $\partial\mathbb{Y}$. Let $G\colon \mathbb{C} \to \mathbb{C}$ be the homeomorphic Lipschitz extension of γ given by Theorem 1.4. Denoting $f := \varphi^{-1} \circ \gamma$, we find by a change of variables that

$$\int_{\partial\mathbb{Y}}\int_{\partial\mathbb{Y}} \big|\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|\big|\,|\mathrm{d}\xi|\,|\mathrm{d}\eta| = \int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \big|\log|f(z) - f(\omega)|\big|\,|\mathrm{d}z|\,|\mathrm{d}\omega|.$$

Now the result of Astala, Iwaniec, Martin and Onninen [4, Theorems 11.4 and 9.1] shows that the inverse map $f^{-1}\colon \partial\mathbb{D} \to \partial\mathbb{D}$ satisfies the Douglas condition (1.2). Thus f^{-1} extends to a harmonic $\mathscr{W}^{1,2}$-homeomorphism H_1 from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$ by the RKC Theorem (Lemma 2.3). Letting $h := G \circ H_1$, we find that h lies in $\mathscr{W}^{1,2}(\mathbb{D}, \mathbb{C})$ since G is Lipschitz. Moreover, the boundary values of h are equal to $\gamma \circ (\varphi^{-1} \circ \gamma)^{-1} = \varphi$, giving us a homeomorphic extension of φ in $\mathscr{W}^{1,2}(\mathbb{D}, \mathbb{C})$.

To further extend φ into the complement of $\mathbb{D}$, assume first without loss of generality that $0 \in \mathbb{Y}$. We now let $\tau(z) = 1/\overline{z}$ denote the inversion with respect to the unit circle, which is a diffeomorphism in $\mathbb{C} \setminus \{0\}$. The map $\psi := \tau \circ \varphi \circ \tau$ is then a homeomorphism from $\partial\mathbb{D}$ to $\partial\tau(\mathbb{Y})$. Note that since τ is the identity on $\partial\mathbb{D}$, we also have $\psi = \tau \circ \varphi$. Since τ is locally bi-Lipschitz in $\mathbb{C} \setminus \{0\}$, there is $L > 1$ such that τ is L-bi-Lipschitz in a neighborhood of $\partial\tau(\mathbb{Y})$. Hence we may estimate that

$$\begin{aligned}
&\int_{\partial\tau(\mathbb{Y})} \int_{\partial\tau(\mathbb{Y})} \big|\log|\psi^{-1}(\alpha) - \psi^{-1}(\beta)|\big| \, |\mathrm{d}\alpha| \, |\mathrm{d}\beta| \\
&\qquad = \int_{\partial\tau(\mathbb{Y})} \int_{\partial\tau(\mathbb{Y})} \big|\log|\varphi^{-1}(\tau(\alpha)) - \varphi^{-1}(\tau(\beta))|\big| \, |\mathrm{d}\alpha| \, |\mathrm{d}\beta| \\
&\qquad = \int_{\partial\mathbb{Y}} \int_{\partial\mathbb{Y}} \big|\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|\big| \, |\tau'(\xi)| \, |\tau'(\eta)| \, |\mathrm{d}\xi| \, |\mathrm{d}\eta| \\
&\qquad \leq L^2 \int_{\partial\mathbb{Y}} \int_{\partial\mathbb{Y}} \big|\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|\big| \, |\mathrm{d}\xi| \, |\mathrm{d}\eta| < \infty.
\end{aligned}$$

This shows that ψ satisfies condition (1.3), and hence the earlier part of the proof shows that we may extend ψ as a $\mathcal{W}^{1,2}$-homeomorphism $\tilde{h}$ from $\mathbb{D}$ to the Jordan domain bounded by $\partial\tau(\mathbb{Y})$. Hence $\tau \circ \tilde{h} \circ \tau$ is a homeomorphism from $\mathbb{C} \setminus \mathbb{D}$ to $\mathbb{C} \setminus \mathbb{Y}$, equal to φ on the boundary, and in $\mathcal{W}^{1,2}(U, \mathbb{C})$ for any bounded subset $U \subset \mathbb{C} \setminus \mathbb{D}$ due to the bi-Lipschitz bounds on τ in $\mathbb{C} \setminus \{0\}$. This concludes the proof. ∎

6. The multiply connected case: Proof of Theorem 1.11

In this section we consider multiply connected Jordan domains $\mathbb{X}$ and $\mathbb{Y}$ of the same topological type. Any such domain can be equivalently obtained by removing from a simply connected Jordan domain the same number, say $0 \leq \ell < \infty$, of closed disjoint topological disks or single points. Throughout what follows, we will assume that none of the boundary components of $\mathbb{X}$ and $\mathbb{Y}$ are single points – this case will only be addressed at the very end of the proof.

If $\ell = 1$, the resulting doubly connected domain is conformally equivalent to a circular annulus $\mathbb{A} = \{z \in \mathbb{C} : r < |z| < 1\}$ with some $0 < r < 1$. In fact, if $\ell \geq 1$ then every $(\ell + 1)$-connected Jordan domain can be mapped by a conformal mapping onto a *circular domain* [14]. An $(\ell + 1)$-connected circular domain consists of the domain bounded by the boundary of the unit disk $\mathbb{D}$ and k other circles in the interior of $\mathbb{D}$. This conformal equivalence to circular domains will be used in certain parts of the proof. The conformal mappings between multiply connected Jordan domains extend continuously up to the boundaries.

The idea of the proof of Theorem 1.11 is simply to split the multiply connected domains $\mathbb{X}$ and $\mathbb{Y}$ into simply connected parts and apply Theorem 1.8 in each of these parts. Let us consider first the case where $\mathbb{X}$ and $\mathbb{Y}$ are doubly connected.

6.1. *Doubly connected $\mathbb{X}$ and $\mathbb{Y}$*

Case 1: $p = 1$. Suppose that the boundary of $\mathbb{X}$ is rectifiable. We split $\mathbb{X}$ into two rectifiable simply connected domains as follows. Take a line L passing through any point in the bounded component of $\mathbb{C} \setminus \mathbb{X}$. Then there exist two open line segments I_1 and I_2 on L that are contained in $\mathbb{X}$ and have endpoints on different components of the boundary of $\mathbb{X}$. These segments split the domain $\mathbb{X}$ into two Jordan domains $\mathbb{X}_1$ and $\mathbb{X}_2$ with rectifiable boundaries.

For $k = 1, 2$, let p_k denote the endpoint of I_k lying on the inner boundary of $\mathbb{X}$ and P_k the endpoint on the outer boundary. We let $q_k = \varphi(p_k)$ and $Q_k = \varphi(P_k)$, where φ denotes the given boundary map from the statement of Theorem 1.11. We would now simply like to connect q_k to Q_k by a rectifiable curve γ_k inside $\mathbb{Y}$ in such a way that γ_1 and γ_2 do not intersect. It is quite obvious this can be done but we provide a proof regardless.

Let $\mathbb{Y}_+$ denote the Jordan domain bounded by the outer boundary of $\mathbb{Y}$. Take a conformal map $g_+ \colon \mathbb{D} \to \mathbb{Y}_+$. Then g_+' is in the Hardy space H^1 since $\partial\mathbb{Y}_1$ is rectifiable, and by [10, Theorem 3.13] we find that g_+ maps the segment $[0, g_+^{-1}(Q_k)]$ into a rectifiable curve in $\mathbb{Y}_+$. Let γ_k^+ denote the image of the segment $[(1-\epsilon)g_+^{-1}(Q_k), g_+^{-1}(Q_k)]$ under g_+ for a sufficiently small ϵ. Then γ_k^+ is a rectifiable curve connecting Q_k to an interior point Q_k^+ of $\mathbb{Y}$ if ϵ is small enough. With a similar argument, possibly adding a Möbius transformation to the argument to invert the order of the boundaries, one finds a rectifiable curve γ_k^- connecting q_k to an interior point q_k^-. For ϵ small enough the four curves constructed here do not intersect.

If Γ denotes the union of these four curves, we may now use the path-connectedness of the domain $\mathbb{Y} \setminus \Gamma$ to join the points Q_1^+ and q_1^- with a smooth simple curve inside $\mathbb{Y}$ that does not intersect Γ. By combining the curves γ_1^+ and γ_1^- one obtains a rectifiable simple curve γ_1 connecting Q_1 and q_1. Using the fact that $\mathbb{Y} \setminus \Gamma$ is doubly connected, we may now join Q_2^+ and q_2^- with a smooth curve that does not intersect γ_1 or Γ. This yields a rectifiable simple curve γ_2 connecting Q_2 and q_2. This proves the existence of the curves γ_k with the desired properties. These curves split $\mathbb{Y}$ into two simply connected Jordan domains $\mathbb{Y}_1$ and $\mathbb{Y}_2$.

We may now extend the homeomorphism φ to map the boundary of $\mathbb{X}_k$ to the boundary of $\mathbb{Y}_k$ homeomorphically. The exact parametrization which maps the segments I_k to the curves γ_k does not matter. The rest of the claim follows directly from the first part of Theorem 1.8, giving us a homeomorphic extension of φ in the Sobolev class $\mathcal{W}^{1,1}(\mathbb{X}, \mathbb{C})$, as claimed.

Case 2: $1 < p < 2$. Suppose that $\mathbb{X}$ has s-hyperbolic growth. Then we take an annulus $\mathbb{A}$ centered at the origin such that there exists a conformal map $g \colon \mathbb{A} \to \mathbb{X}$. By a result of Gehring and Osgood [12], the quasihyperbolic metrics $h_{\mathbb{X}}$ and $h_{\mathbb{A}}$ are comparable via the conformal map g. This shows that for any fixed $x_0 \in \mathbb{A}$ and all $x \in \mathbb{A}$ we have

$$h_{\mathbb{A}}(x_0, x) \le C h_{\mathbb{X}}(g(x_0), g(x)) \le \frac{C}{\operatorname{dist}(g(x), \partial\mathbb{X})^{1-s}}. \tag{6.1}$$

Let now $\mathbb{A}_+$ denote the simply connected domain obtained by intersecting $\mathbb{A}$ and the upper half-plane. We claim that the domain $\mathbb{X}_+ := g(\mathbb{A}_+)$ has s-hyperbolic growth as well.

To prove this claim, fix $x_0 \in \mathbb{A}_+$ and take an arbitrary $x \in \mathbb{A}$. Let $d = \operatorname{dist}(x, \partial\mathbb{A}_+)$. We aim to establish the inequality

$$h_{\mathbb{A}_+}(x_0, x) \leq \frac{C}{\operatorname{dist}(g(x), \partial\mathbb{X}_+)^{1-s}}. \tag{6.2}$$

Note that $\mathbb{A}_+$ is bi-Lipschitz equivalent to the unit disk, implying that $h_{\mathbb{A}_+}(x_0, x)$ is comparable to $\log(1/d)$. The boundary of $\mathbb{A}_+$ contains two line segments on the real line; let us denote them by I_1 and I_2. Note that we have the estimate

$$\operatorname{dist}(g(x), \partial\mathbb{X}_+) \leq \operatorname{dist}(g(x), \partial\mathbb{X}). \tag{6.3}$$

If it happened that $d = \operatorname{dist}(x, \partial\mathbb{A})$, meaning that the closest point to x on $\partial\mathbb{A}_+$ is not on I_1 or I_2, then the hyperbolic distances $h_{\mathbb{A}_+}(x_0, x)$ and $h_{\mathbb{A}}(x_0, x)$ are comparable and by the inequalities (6.1) and (6.3) the inequality (6.2) holds. Hence it is enough to prove (6.2) when $d = \operatorname{dist}(x, I_1 \cup I_2)$. We may also assume that d is small. Due to the geometry of the half-annulus $\mathbb{A}_+$, the projection of x to the real line lies on either I_1 or I_2, and the vertical line segment L_x between x and its projection lies in $\mathbb{A}_+$ and has length d. Letting D denote the distance from x to $\partial\mathbb{A}_+ \setminus (I_1 \cup I_2)$, we see that $D \geq d$.

We may now reiterate the proof of (3.4) to find that

$$|g'(z)| \leq \frac{C}{\operatorname{dist}(z, \partial\mathbb{A}) \log^{\frac{1}{1-s}}(\operatorname{dist}(z, \partial\mathbb{A})^{-1})}$$

for $z \in \mathbb{A}$. We should mention that the simply connectedness assumption used in the proof of (3.4) may be circumvented by using the equivalence of the quasihyperbolic metrics under g instead of passing to the hyperbolic metric. Hence

$$\operatorname{dist}(g(x), \partial\mathbb{X}_+) \leq \int_{L_x} |g'(z)|\, |dz| \leq \frac{Cd}{D \log^{\frac{1}{1-s}}(1/D)}.$$

From this we find that (6.2) is equivalent to

$$\log(1/d) \leq C \frac{D^{1-s} \log(1/D)}{d^{1-s}},$$

which is true since $D \geq d$. Hence (6.2) holds, and this implies that $\mathbb{X}_+$ has s-hyperbolic growth by reversing the argument that gives (6.1).

We define $\mathbb{X}_-$ similarly. Hence we have split $\mathbb{X}$ into two simply connected domains with s-hyperbolic growth. On the image side, we may split $\mathbb{Y}$ into two simply connected domains with rectifiable boundary as in Case 1. Extending φ in an arbitrary homeomorphic way between the boundaries of these domains and applying part 2 of Theorem 1.8(2) gives a homeomorphic extension of φ in $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$ whenever $s > p - 1$.

6.2. *The general case*

Case 3: $p = 1$. Assume that $\mathbb{X}$ and $\mathbb{Y}$ are ℓ-connected Jordan domains with rectifiable boundaries. By induction, we may assume that the result of Theorem 1.11 holds for $(\ell - 1)$-connected Jordan domains. Hence we are only required to split $\mathbb{X}$ and $\mathbb{Y}$ into two domains with rectifiable boundary, one which is doubly connected and the other is $(\ell - 1)$-connected.

We hence describe how to 'isolate' a given inner boundary component X_0 from an ℓ-connected Jordan domain $\mathbb{X}$. Let $X_{\text{outer}} \neq X_0$ denote the outer boundary component of $\mathbb{X}$. Take a small neighborhood of X_0 inside $\mathbb{X}$. Let γ_0 be a piecewise linear Jordan curve contained in this neighborhood and separating X_0 from the other boundary components of $\mathbb{X}$. Let also γ_1 be a piecewise linear Jordan curve inside $\mathbb{X}$ and in a small enough neighborhood of X_{outer} so that all the inner boundary components of $\mathbb{X}$ are inside γ_1. Take y_0 and y_1 on γ_0 and γ_1 respectively, and connect them with a piecewise linear curve α_y not intersecting any boundary components of $\mathbb{X}$. Choose z_0 on γ_0 close to y_0 and z_1 on γ_1 close to y_1 so that we may connect z_0 and z_1 by a piecewise linear curve α_z arbitrarily close to α_y but intersecting neither α_y nor any boundary components of $\mathbb{X}$. Since the region bounded by X_{outer} and γ_1 is doubly connected, by the construction in Case 1 we may connect y_1 and z_1 to any two given points y_2 and z_2 on the boundary X_{outer} via nonintersecting rectifiable curves β_y and β_z lying inside this region.

Let now Γ denote the union of the curves β_y, β_z, α_y, α_z, and the curve γ_0' obtained by taking γ_0 and removing the part between y_0 and z_0. By construction Γ contains two arbitrary points on X_{outer} and separates the domain $\mathbb{X}$ into a doubly connected domain with inner boundary component X_0 and an $(n - 1)$-connected Jordan domain. Since Γ is rectifiable, both of these domains are also rectifiable.

Applying the same construction for $\mathbb{Y}$, we may separate the boundary component $\varphi(X_0)$ of $\mathbb{Y}$ by a rectifiable curve Γ'. Since the boundary points y_2 and z_2 above were arbitrary, we may assume that Γ' intersects the outer boundary of $\mathbb{Y}$ at the points $\varphi(y_2)$ and $\varphi(z_2)$. Extending φ to a homeomorphism from Γ onto Γ' and applying the induction assumptions now gives a homeomorphic extension in the class $\mathcal{W}^{1,1}(\mathbb{X}, \mathbb{C})$.

Case 4: $1 < p < 2$. We still have to deal with the case where $\mathbb{X}$ has s-hyperbolic growth and is ℓ-connected. By the same arguments as in the previous case, it will be enough to split $\mathbb{X}$ into a doubly connected and an $(\ell - 1)$-connected domain with s-hyperbolic growth.

Since $\mathbb{X}$ is ℓ-connected, there exists a domain Ω such that every boundary component of Ω is a circle and there is a conformal map $g \colon \Omega \to \mathbb{X}$. Let $\Gamma \subset \Omega$ be a piecewise linear simple curve with both endpoints on the outer boundary of Ω such that Γ separates one of the inner boundary components of $\partial\Omega$ from the others, which implies that the curve Γ splits Ω into a doubly connected set Ω_1 and an $(\ell - 1)$-connected set Ω_2. We claim that the domains $\mathbb{X}_1 = g(\Omega_1)$ and $\mathbb{X}_2 = g(\Omega_2)$ have s-hyperbolic growth.

The proof of this claim is nearly identical to the arguments in Case 2, so we will summarize it briefly. For $\mathbb{X}_2$, we aim to establish the inequality

$$h_{\Omega_2}(x_0, x) \leq \frac{C}{\operatorname{dist}(g(x), \partial\mathbb{X}_2)^{1-s}} \tag{6.4}$$

for fixed $x_0 \in \Omega_2$ and $x \in \Omega_2$. For this inequality, it is only essential to consider x close to $\partial\Omega_2$. If x is closer to the boundary of the original set $\partial\Omega$ than to Γ, then the hyperbolic distance between x_0 and x in Ω_2 is comparable to the distance inside the larger set Ω. Then the s-hyperbolic growth of Ω implies (6.4) as in Case 2. If x is closer to Γ but a fixed distance away from the boundary of Ω, then the smoothness of g in compact subsets of Ω implies the result. If x is closest to a line segment in Γ which has its other endpoint on $\partial\Omega$, then we may employ a similar estimate to that in Case 2, using the bound for $|g'(z)|$ in terms of $\operatorname{dist}(z, \partial\Omega)$, to conclude that (6.4) also holds here. This implies that $\mathbb{X}_2$ satisfies (6.4), and hence it has s-hyperbolic growth. The argument for $\mathbb{X}_1$ is the same.

After splitting $\mathbb{X}$ into two domains of smaller connectivity and s-hyperbolic growth, we split the target $\mathbb{Y}$ accordingly into rectifiable parts using the argument from Case 3. Applying induction on ℓ now proves the result in this case.

6.3. *Punctured domains*

We now address the case where $\mathbb{X}$ and $\mathbb{Y}$ are ℓ-connected and where some of the inner boundary components of $\mathbb{X}$ and $\mathbb{Y}$ may be single points. Let these points be $x_1, \dots, x_N \in \mathbb{X}$ and $y_1, \dots, y_N \in \mathbb{Y}$. Without loss of generality we may assume $\varphi(x_j) = y_j$ for all j. Let $\tilde{\mathbb{X}}$ denote the $(\ell - N)$-connected domain $\mathbb{X} \cup \{x_1, \dots, x_N\}$ and define $\tilde{\mathbb{Y}}$ similarly.

We now consider the boundary map $\varphi|_{\partial\tilde{\mathbb{X}}} : \partial\tilde{\mathbb{X}} \to \partial\tilde{\mathbb{Y}}$ and let $\tilde{h} : \tilde{\mathbb{X}} \to \tilde{\mathbb{Y}}$ denote the $\mathcal{W}^{1,p}$-homeomorphic extension of this boundary map. If such a map satisfied $\tilde{h}(x_j) = y_j$ for all j then we would be done. If not, let $\mathbb{U} \subset \tilde{\mathbb{Y}}$ be a smooth simply connected domain large enough to contain all the points $\tilde{h}(x_1), \dots, \tilde{h}(x_N)$ and $y_1, \dots, y_N$. Then consider a diffeomorphic change of variables $\tau : \overline{\mathbb{U}} \to \overline{\mathbb{U}}$ that is the identity map on the boundary and sends the point $\tilde{h}(x_j)$ to y_j for every j. Now the map $h := \tau \circ \tilde{h}|_{\mathbb{X}} : \mathbb{X} \to \mathbb{Y}$ is the desired Sobolev homeomorphic extension of φ.

This finishes the proof of Theorem 1.11.

7. Monotone Sobolev minimizers

The classical *harmonic mapping problem* deals with the question of whether there exists a harmonic homeomorphism between two given domains. Of course, when the domains are Jordan such a mapping problem is always solvable. Indeed, according to the Riemann Mapping Theorem there is a conformal mapping $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$. Finding a harmonic homeomorphism which coincides with a given boundary homeomorphism $\varphi : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is a more subtle question. If $\mathbb{Y}$ is convex, then there always exists a harmonic homeomorphism $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ with $h(x) = \varphi(x)$ on $\partial\mathbb{X}$ by Lemma 2.3. For a nonconvex target $\mathbb{Y}$, however, there always exists at least one boundary homeomorphism whose harmonic extension takes points in $\mathbb{X}$ beyond $\overline{\mathbb{Y}}$. To find a deformation $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ which resembles harmonic homeomorphisms Iwaniec and Onninen [23] applied the direct method of the calculus of variations and considered minimizing sequences in $\mathcal{H}^{1,2}_\varphi(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$. They called such minimizers *monotone Hopf-harmonics* and proved the existence and uniqueness result in the case when $\mathbb{Y}$ is a Lipschitz domain and the boundary data φ satisfies the Douglas condition. Note that by the Riemann Mapping Theorem one may always assume that $\mathbb{X} = \mathbb{D}$. Theorem 1.6 allows one to go beyond the Lipschitz targets. Indeed, under the assumptions of Theorem 1.6, the class $\mathcal{H}^{1,2}_\varphi(\overline{\mathbb{D}}, \overline{\mathbb{Y}})$ is non-empty. Furthermore, if $h_\circ \in \mathcal{H}^{1,2}_\varphi(\overline{\mathbb{D}}, \overline{\mathbb{Y}})$, then $h_\circ$ satisfies the uniform modulus of continuity estimate

$$|h_\circ(x_1) - h_\circ(x_2)|^2 \le C \frac{\int_{\mathbb{D}} |Dh_\circ|^2}{\log\left(\frac{1}{|x_1 - x_2|}\right)}$$

for $x_1, x_2 \in \mathbb{D}$ such that $|x_1 - x_2| < 1$. This follows by taking the global $\mathcal{W}^{1,2}_{\mathrm{loc}}$-homeomorphic extension given by Theorem 1.6 and applying a standard local modulus of continuity estimate for $\mathcal{W}^{1,2}$-homeomorphisms [19, Corollary 7.5.1, p. 155]. Now, applying the direct method of the calculus of variations allows us to find a minimizing sequence in $\mathcal{H}^{1,2}_\varphi(\overline{\mathbb{D}}, \overline{\mathbb{Y}})$ for the Dirichlet energy converges weakly in $\mathcal{W}^{1,2}(\mathbb{D}, \mathbb{C})$ and uniformly in $\overline{\mathbb{D}}$. Being a uniform limit of homeomorphisms the limit mapping $H : \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ becomes *monotone*. Indeed, the classical Youngs approximation theorem [39] asserts that a continuous map between compact oriented topological 2-manifolds (surfaces) is monotone if and only if it is a uniform limit of homeomorphisms. Monotonicity, the concept of Morrey [30], simply means that for a continuous $H : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ the preimage $H^{-1}(y_\circ)$ of a point $y_\circ \in \overline{\mathbb{Y}}$ is a continuum in $\overline{\mathbb{X}}$. We have thus proved the following result.

Theorem 7.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Jordan domains and assume that $\partial\mathbb{Y}$ is rectifiable. If $\varphi\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ satisfies (1.3), then there exists a monotone Sobolev mapping $H : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathcal{W}^{1,2}(\mathbb{X}, \mathbb{C})$ such that H coincides with φ on $\partial\mathbb{X}$ and*

$$\int_{\mathbb{X}} |DH(x)|^2 \, \mathrm{d}x = \inf_{h \in \mathcal{H}^{1,2}_\varphi(\overline{\mathbb{X}}, \overline{\mathbb{Y}})} \int_{\mathbb{X}} |Dh(x)|^2 \, \mathrm{d}x.$$

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References

[1] Alessandrini, G., Sigalotti, M.: Geometric properties of solutions to the anisotropic p-Laplace equation in dimension two. Ann. Acad. Sci. Fenn. Math. **26**, 249–266 (2001)
Zbl 1002.35044 MR 1816571

[2] Antman, S. S.: Nonlinear Problems of Elasticity. Appl. Math. Sci. 107, Springer, New York (1995) Zbl 0820.73002 MR 1323857

[3] Astala, K., Iwaniec, T., Martin, G.: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton Math. Ser. 48, Princeton Univ. Press, Princeton, NJ (2009) Zbl 1182.30001 MR 2472875

[4] Astala, K., Iwaniec, T., Martin, G. J., Onninen, J.: Extremal mappings of finite distortion. Proc. London Math. Soc. (3) **91**, 655–702 (2005) Zbl 1089.30013 MR 2180459

[5] Ball, J. M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. **63**, 337–403 (1976/77) Zbl 0368.73040 MR 475169

[6] Carleson, L.: Interpolations by bounded analytic functions and the corona problem. Ann. of Math. (2) **76**, 547–559 (1962) Zbl 0112.29702 MR 141789

[7] Choquet, G.: Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques. Bull. Sci. Math. (2) **69**, 156–165 (1945) Zbl 0063.00851 MR 16973

[8] Ciarlet, P. G.: Mathematical Elasticity. Vol. I. Studies Math. Appl. 20, North-Holland, Amsterdam (1988) Zbl 0648.73014 MR 936420

[9] Douglas, J.: Solution of the problem of Plateau. Trans. Amer. Math. Soc. **33**, 263–321 (1931) JFM 57.1542.03 MR 1501590

[10] Duren, P. L.: Theory of H^p Spaces. Pure Appl. Math. 38, Academic Press, New York (1970) Zbl 0215.20203 MR 0268655

[11] Duren, P.: Harmonic Mappings in the Plane. Cambridge Tracts in Math. 156, Cambridge Univ. Press, Cambridge (2004) Zbl 1055.31001 MR 2048384

[12] Gehring, F. W., Osgood, B. G.: Uniform domains and the quasihyperbolic metric. J. Anal. Math. **36**, 50–74 (1980) (1979) Zbl 0449.30012 MR 581801

[13] Gehring, F. W., Palka, B. P.: Quasiconformally homogeneous domains. J. Anal. Math. **30**, 172–199 (1976) Zbl 0349.30019 MR 437753

[14] Goluzin, G. M.: Geometric Theory of Functions of a Complex Variable. Transl. Math. Monogr. 26, Amer. Math. Soc., Providence, RI (1969) Zbl 0183.07502 MR 0247039

[15] Greco, L., Schiattarella, R.: An embedding theorem for BV-functions. Comm. Contemp. Math. **22**, art. 1950032, 13 pp. (2020) Zbl 1440.26012 MR 4106822

[16] Hencl, S., Pratelli, A.: Diffeomorphic approximation of $W^{1,1}$ planar Sobolev homeomorphisms. J. Eur. Math. Soc. **20**, 597–656 (2018) Zbl 1393.26013 MR 3776275

[17] Herron, D. A., Sullivan, T. S.: Fractal inner chordarc disks. J. Anal. Math. **84**, 173–205 (2001) Zbl 0989.30012 MR 1849201

[18] Iwaniec, T., Kovalev, L. V., Onninen, J.: Diffeomorphic approximation of Sobolev homeomorphisms. Arch. Ration. Mech. Anal. **201**, 1047–1067 (2011) Zbl 1260.46023 MR 2824471

[19] Iwaniec, T., Martin, G.: Geometric Function Theory and Non-Linear Analysis. Oxford Math. Monogr., Clarendon Press, Oxford Univ. Press, New York (2001) Zbl 1045.30011 MR 1859913

[20] Iwaniec, T., Martin, G., Sbordone, C.: L^p-integrability & weak type L^2-estimates for the gradient of harmonic mappings of $\mathbb{D}$. Discrete Contin. Dynam. Systems Ser. B **11**, 145–152 (2009) Zbl 1168.31001 MR 2461814

[21] Iwaniec, T., Onninen, J.: Radó–Kneser–Choquet theorem. Bull. London Math. Soc. **46**, 1283–1291 (2014) Zbl 1307.31001 MR 3291265

[22] Iwaniec, T., Onninen, J.: Radó–Kneser–Choquet theorem for simply connected domains (p-harmonic setting). Trans. Amer. Math. Soc. **371**, 2307–2341 (2019) Zbl 1410.31002 MR 3896082

[23] Iwaniec, T., Onninen, J.: Monotone Hopf-harmonics. Arch. Ration. Mech. Anal. **237**, 743–777 (2020) Zbl 1439.30039 MR 4097328

[24] Kalaj, D.: Estimates of gradient and of Jacobian of harmonic mappings defined in the unit disk. Proc. Amer. Math. Soc. **139**, 2463–2472 (2011) Zbl 1223.31001 MR 2784812

[25] Kirszbraun, M. D.: Über die zusammenziehende und Lipschitzsche Transformationen. Fund. Math. **22**, 77–108 (1934) JFM 60.0532.03

[26] Kneser, H.: Lösung der Aufgabe 41. Jahresber. Deutsch. Math.-Verein. **35**, 123–124 (1926) JFM 37.0656.04

[27] Koskela, P., Wang, Z., Xu, H.: Controlled diffeomorphic extension of homeomorphisms. Nonlinear Anal. **177**, 586–600 (2018) Zbl 1405.31001 MR 3886592

[28] Kovalev, L. V.: Optimal extension of Lipschitz embeddings in the plane. Bull. London Math. Soc. **51**, 622–632 (2019) Zbl 1435.30027 MR 3990380

[29] Lewy, H.: On the non-vanishing of the Jacobian in certain one-to-one mappings. Bull. Amer. Math. Soc. **42**, 689–692 (1936) Zbl 0015.15903 MR 1563404

[30] Morrey, C. B., Jr.: The topology of (path) surfaces. Amer. J. Math. **57**, 17–50 (1935) Zbl 0011.03701 MR 1507053

[31] Pommerenke, C.: One-sided smoothness conditions and conformal mapping. J. London Math. Soc. (2) **26**, 77–88 (1982) Zbl 0464.30012 MR 667246

[32] Radó, T.: Aufgabe 41. Jahresber. Deutsch. Math.-Verein. **35**, 49 (1926)

[33] Reshetnyak, Y. G.: Space Mappings with Bounded Distortion. Transl. Math. Monogr. 73, Amer. Math. Soc., Providence, RI (1989) Zbl 0667.30018 MR 994644

[34] Stein, E. M.: Singular Integrals and Differentiability Properties of Functions. Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, NJ (1970) Zbl 0207.13501 MR 0290095

[35] Tukia, P.: The planar Schönflies theorem for Lipschitz maps. Ann. Acad. Sci. Fenn. Ser. A I Math. **5**, 49–72 (1980) Zbl 0411.57015 MR 595177

[36] Väisälä, J.: Homeomorphisms of bounded length distortion. Ann. Acad. Sci. Fenn. Ser. A I Math. **12**, 303–312 (1987) Zbl 0653.30009 MR 951979

[37] Väisälä, J.: Quasiconformal maps of cylindrical domains. Acta Math. **162**, 201–225 (1989) Zbl 0674.30017 MR 989396

[38] Verchota, G. C.: Harmonic homeomorphisms of the closed disc to itself need be in $W^{1,p}$, $p < 2$, but not $W^{1,2}$. Proc. Amer. Math. Soc. **135**, 891–894 (2007) Zbl 1109.31007 MR 2262887

[39] Youngs, J. W. T.: Homeomorphic approximations to monotone mappings. Duke Math. J. **15**, 87–94 (1948) Zbl 0030.41603 MR 24623

[40] Zhang, Y. R.-Y.: Schoenflies solutions of conformal boundary values may fail to be Sobolev. Ann. Acad. Sci. Fenn. Math. **44**, 791–796 (2019) Zbl 1437.30009 MR 3973541