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6

Mika Laitinen

Mathematical Modelling of Conductive-Radiative Heat Transfer

UNIVERSITY OF JYVÄSKYLÄ

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ABSTRACT

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This study focuses on mathematical models combining conductive and radiative heat transfer. In these models, radiation appears as a source or flux term in the heat conduction equation, and, on the other hand, for radiation we need to solve simultaneously an integral or transport equation depending on temperature.

The intention of this work is to find the general mathematical properties of conductive-radiative equations that guarantee the well-posedness of a model. First, we show that the equations are in general coercive and pseudomonotone and therefore they have a solution. Then, we establish a comparison principle which implies that the solution is unique. Further, we prove the boundedness of a solution. The general theory is then demonstrated by analysing the most common cases encountered in practice: opaque bodies with diffuse-grey surfaces and semitransparent materials with either diffuse or specular boundary reflections.

Finally, we study heat transfer in optically thick (i.e. highly absorptive) materials. Intuitively, in such materials radiation propagates diffusively, and in very thick materials radiation concentrates on surfaces. We justify these conceptions rigorously using asymptotic analysis. We also propose a simple and effective diffusion approximation accurate for optically thick materials.

The thesis is restricted to materials whose radiative properties are independent of wavelength (grey materials).

Keywords: conductive-radiative heat transfer, Stefan-Boltzmann law, comparison principle, diffuse-grey surfaces, semitransparent material, optical thickness, diffusion approximation

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My sincerest appreciation goes to my wife Mari for support and putting up with my absent-mindedness.

Jyväskylä, 18 August, 2000

Mika Laitinen

CONTENTS

LIST OF INCLUDED ARTICLES

[A] M. Laitinen and T. Tiihonen, *Heat transfer in conducting and radiating bodies*, Appl. Math. Lett., 10 (1997), pp. 5–8.

[B] M. Laitinen and T. Tiihonen, *Integro-differential equation modelling heat transfer in conducting, radiating and semitransparent materials*, Math. Methods Appl. Sci., 21 (1998), pp. 375–392.

[C] M. Laitinen and T. Tiihonen, *Conductive-radiative heat transfer in grey materials*, Report B 6/2000, University of Jyväskylä, Department of Mathematical Information Technology. Submitted to Quart. Appl. Math.

[D] M. Laitinen, *Asymptotic analysis of conductive-radiative heat transfer*, Report B 10/2000, University of Jyväskylä, Department of Mathematical Information Technology. Submitted to Asymptot. Anal.

INTRODUCTION

This thesis is devoted to the analysis of heat transfer models, where heat radiation is combined with conduction. Heat radiation is a significant factor in heat transfer, especially if temperature is high or conductivity is low. Therefore, heat radiation is important in understanding various heating, melting, cooling, drying and combustion phenomena, and it plays an important role, for example, in space technology [5, 18], glass industry [23] and silicon crystal growth [7].

Conduction and radiation are very different by nature and, for this reason their coupling is nontrivial. On the other hand, if the interaction between conduction and radiation is understood, it is relatively easy to include also convection [20]. Conduction is characterized by a local second order differential operator, whereas radiation is usually presented by a nonlocal integral operator or a first order direction dependent transport equation. Then, due to the Stefan-Boltzmann radiation law, the coupling is nonlinear leading to somewhat nonstandard function spaces. As a result of nonlinearity and nonlocality, the coupled problem is also nonmonotone. Finally, since radiation can carry heat through a vacuum, geometries in applications are often nonconnected.

To illustrate these features in more detail, let us next consider heat transfer in a furnace which is used, for example, to produce silicon crystals [7]. This problem has been our favourite since it contains all the essential mathematical difficulties of conductive-radiative heat transfer and yet is relatively easy to write down. The designers of the furnace would like to simulate the temperature distribution in the furnace in order to heat and design the furnace in an optimal way. The furnace walls conduct heat and the interior surfaces of the furnace exchange heat by radiation. The furnace interiors are usually quite complex consisting of several disjoint components (heaters, thermal shields, heated objects) often lacking symmetry. Frequently, also other complicated phenomena have to be modelled simultaneously, such as fluid flow, phase changes and thermal stresses. Therefore, a thorough mathematical understanding of heat transfer is important.

For illustrative purposes, it suffices to describe the furnace by a container consisting of two conducting components $\Omega_1, \Omega_2 \subset \mathbb{R}^3$, as sketched in Figure 1. Assuming for simplicity that heat transfer is stationary and that the medium between Ω_1 and Ω_2 behaves like a vacuum, the heat equation for absolute temperature T reads as

$$
\begin{aligned}
-\nabla \cdot (k\nabla T) &= f \qquad && \text{in } \Omega := \Omega_1 \cup \Omega_2, \\
-k\frac{\partial T}{\partial n} &= q_c && \text{on } \Gamma, \\
-k\frac{\partial T}{\partial n} &= q_r && \text{on } \Sigma,
\end{aligned}
$$

where n is the outward unit normal, k is the coefficient of heat conduction, f is a

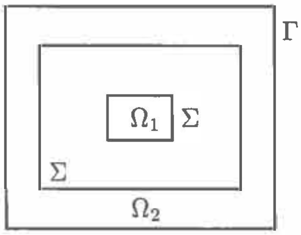

FIGURE 1 A "furnace" consisting of two components.

given heat source, q_c is the convective heat flux and q_r is the radiative heat flux. We assume that the convective heat flux is proportional to the temperature difference, $q_c = \alpha(T - T_0)$, with a heat transfer coefficient α and the exterior temperature T_0.

To model the radiative flux, we make two typical assumptions: We assume that Σ is *diffuse* and *grey*, that is, the emissivity coefficient ϵ of Σ depends neither on direction nor on wavelength of the radiation. Then, the radiative heat flux can be modelled be means of the total incoming radiation $u(x)$ and the total outgoing radiation $\rho(x)$,

$$q_r(x) = \rho(x) - u(x), \qquad x \in \Sigma.$$

The outgoing radiation ρ is composed of emission and reflected part of u. According to Stefan-Boltzmann law emission is proportional to T^4, so that

$$\rho(x) = \epsilon\sigma T^4(x) + (1-\epsilon)u(x), \qquad x \in \Sigma,$$

σ denoting Stefan-Boltzmann constant. The incoming radiation $u(x)$ at $x \in \Sigma$, on the other hand, is a "sum" of radiation leaving other parts of Σ, described by an integral operator K,

$$u(x) = (K\rho)(x) := \int_\Sigma \frac{n(s)\cdot(x-s)\, n(x)\cdot(s-x)}{\pi|x-s|^4}\Xi(x,s)\rho(s)\,ds,$$

where $\Xi(x,s)$ is the visibility factor: $\Xi(x,s) = 1$ if x and s see each other (that is $\overline{xs} \cap \Omega = \emptyset$) and $\Xi(x,s) = 0$ otherwise. The above relations can be gathered as

$$q_r = (I-K)\rho, \qquad (I-(1-\epsilon)K)\rho = \epsilon\sigma T^4.$$

By showing that K maps $L^p(\Sigma)$ to itself and that $I-(1-\epsilon)K$ is invertible, we can write our problem in variational form as: Find $T \in V := H^1(\Omega) \cap L^5(\Sigma)$ such that

$$\int_\Omega k\nabla T\cdot\nabla\varphi\,dx + \int_\Sigma (G\sigma T^4)\varphi\,ds + \int_\Gamma \alpha T\varphi\,ds = \langle \tilde{f}, \varphi\rangle \quad \forall\varphi \in V \qquad (\star)$$

where the operator G is defined by $Gv := (I-K)(I-(1-\epsilon)K)^{-1}\epsilon v$ and $\tilde{f} \in V^*$ is a data term due to external heat source and convective heat flux.

The existence and uniqueness of this problem have been studied independently in [15, 16, 19] and the basic theory has also been extended to include several conducting bodies [17, 20] and time dependence [14]. However, these articles are either confined to geometries where Σ is not an enclosed surface, pose restrictions to ϵ or assume that there exist a pair of sub and super solutions. Note that if the radiating surface Σ were convex, the problem would be simpler [5, 13].

One of our goals in [C] was to analyse problem $(\star)$ without such restrictions. First we needed to resolve the properties of the operator G, particularly the following facts

1. G is a linear operator from $L^p(\Sigma)$ to itself, $1 \leq p \leq \infty$.
2. $H := I - G$ is positive, i.e. $Hu \geq 0$ if $u \geq 0$.
3. G can be factorized as $G = E - F$ where E is a multiplication operator and F is a compact operator.
4. $\int_\Sigma (GT^4)T\,ds \geq 0$ for all $T \in L^5(\Sigma)$.
5. If $\int_\Sigma (GT^4)T\,ds = 0$, then T is a constant on Σ.

Note that here Σ is allowed to be nonconnected enclosure and the emissivity ϵ is restricted only by physics $0 \leq \epsilon(x) \leq 1$, $\epsilon \not\equiv 0$. Owing to these properties, we can infer that problem $(\star)$ is a special case of the general theory presented in [C] and, thus, it satisfies the following

(i) For each $\bar{f} \in V^*$, there exists a unique solution to $(\star)$.

(ii) Let T_1, T_2 be solutions to $(\star)$ corresponding to data $f_1, f_2 \in V^*$ and suppose $\langle f_1 - f_2, \varphi\rangle \geq 0$ for all $\varphi \geq 0$. Then $T_1 \geq T_2$ a.e. in Ω and a.e on $\partial\Omega$.

(iii) If $f \in L^{3/2+\delta}(\Omega) + L^{2+\delta}(\partial\Omega)$ for some $\delta > 0$ then the solution of $(\star)$ is in $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$.

The existence of a solution follows from coercivity and pseudomonotonicity [22]. Pseudomonotonicity is not so surprising since from property 3 of G we see that G can divided into monotone and compact part. The proof of coercivity is more difficult as in nonconnected geometries the null spaces of both conduction operator and radiative operator are nontrivial. The argumentation in this proof is somewhat similar than in the proofs of Poincare and Friedrichs inequalities. The uniqueness follows from the comparison principle (ii), which was proved following the work of Křížek & Liu [11] concerning nonlinear heat conduction equation. The boundedness of solutions is proved utilizing Moser iteration [6]. The main tools in determining properties 1–5 of G were the theory of positive operators [10] and the interpolation theory of operators [3].

Let us now briefly consider the time dependent counterpart of $(\star)$. We denote the stationary conductive-radiative operator by Q

$$\langle QT, \varphi\rangle := \int_\Omega k\nabla T \cdot \nabla\varphi\,dx + \int_\Sigma (G\sigma T^4)\varphi\,ds + \int_\Gamma \alpha T\varphi\,ds,$$

and introduce the function spaces

$$X = L^2(0,\tau; H^1(\Omega)) \cap L^5(0,\tau; L^5(\Sigma)),$$
$$W = \{T \,:\, T \in X, T' \in X^*\}.$$

Then, setting heat capacity equal to one, the time dependent problem can be posed as: For $f \in X^*$ and $T_0 \in L^2(\Omega)$, find $T \in W$ such that

$$\langle T'(t), \varphi\rangle_V + \langle QT(t), \varphi\rangle_V = \langle f(t), \varphi\rangle_V,$$
$$T(0) = T_0,$$

for all $\varphi \in V$ and almost all $t \in [0,\tau]$. This problem is nontrivial since for Galerkin method [22] we need a priori estimates in the space $L^5(0,\tau; L^5(\Sigma))$ which does not follow from the coercivity of the stationary operator Q. We found two ways to proceed: First, if we assume that Σ is not an enclosure as in [14], then we have additionally an inequality $\int_\Sigma (GT^4)T\,ds \geq C\|T\|^5_{L^5(\Sigma)}$ which saves the game. Less restrictive option is to assume for some $\delta > 0$ that

$$T_0 \in L^5(\Omega), \qquad f(t) \in L^2((0,\tau) \times \Omega) + L^{5+\delta}(0,\tau; L^{5/3+\delta}(\Sigma \cup \Gamma)),$$

which enables us to derive the desired a priori estimate using Moser iteration. If either of these two assumptions hold, then from the general theory presented in [C] it follows that the above problem has at least one solution. The time dependent comparison principle is proved in [C] without additional assumptions and, therefore, the solution is always unique.

In the above example we implicitly assumed that radiation is a surface phenomenon. However, in *semitransparent* materials such as glass or gas, radiation travels a significant distance before being absorbed and therefore radiation gives rise to a volumetric heat source or sink. To give a rough idea of this process, let us modify the above example such that Ω_1 is made of semitransparent material. For simplicity we assume that the absorption coefficient κ of Ω_1 is a constant and we ignore the scattering. Then, the heat equation in Ω_1 is reads as

$$-\nabla \cdot (k\nabla T) = f - \kappa(4\sigma T^4 - \int_S v(x,\omega)\,d\omega)$$

where $v(x,\omega)$ is the radiation intensity describing flow of photons in a point $x \in \Omega_1$ to direction $\omega \in S$, S denoting the unit sphere of $\mathbb{R}^3$. The intensity v is governed by a transport equation depending on temperature

$$\omega \cdot \nabla_x v + \kappa v = \kappa \frac{\sigma}{\pi} T^4 \quad \text{in } \Omega_1 \times S,$$
$$v|_{\Gamma_-} = R(v|_{\Gamma_+}, u|_{\Gamma_-}),$$

where Γ_+, Γ_- denote the outflow and inflow boundaries

$$\Gamma_+ = \{(x,\omega) \in \partial\Omega_1 \times S \,:\, \omega \cdot n(x) > 0\},$$
$$\Gamma_- = \{(x,\omega) \in \partial\Omega_1 \times S \,:\, \omega \cdot n(x) < 0\},$$

and $R(v|_{\Gamma_+}, u|_{\Gamma_-})$ is a reflection operator which gives the fraction of the intensity v on the outflow boundary that is reflected back and the fraction of incoming radiation u that is reflected into Ω_1. In applications the reflections are usually either *diffuse* (uniformly distributed) or *specular* (mirror-like). In [A] and [B] and we studied semitransparent materials with isotropic scattering and diffuse reflections whereas in [C] we examined general scattering laws and specular reflections.

Although surface and volume radiation are modelled using somewhat different principles, they have similar mathematical structure. With techniques set forth in [B] and [C] we can solve the intensity v by means of temperature and write the heat equation in Ω_1 as

$$-\nabla \cdot (k\nabla T) = f - \tilde{G}\sigma T^4 + U \qquad \text{in } \Omega_1,$$

where $\tilde{G}$ is an operator from $L^p(\Omega_1)$ to itself and U characterizes the contribution of the incoming radiation u. Following [B] and [C] we can also show that $\tilde{G}$ has properties analogous to 1–5 of the operator G described earlier. Thus, the existence, uniqueness, comparison principle and boundedness of a solution for this problem (as specified by assertions (i)–(iii)) can deduced from the general theory presented in [C]. Of course, we must first specify the reflection operator more precisely and modify the boundary conditions on Σ.

Apart from our works [A], [B], [C], the mathematical theory of volumetric conductive-radiative heat transfer is almost nonexistent. To our attention has come only a single work investigating the existence and uniqueness of a solution in one dimensional case with nonreflecting surfaces [8]. On the other hand, transport equations alone are extensively studied in astrophysics and neutron transport; see [1, 4] and the references therein.

Our main achievement [C] focuses on generalized conductive-radiative model

$$\langle AT, \varphi\rangle + \int_{\Lambda\cup\Sigma} (GT^4)\, \varphi \, d\mu = \langle f, \varphi\rangle \qquad \forall \varphi \in H^1(\Omega) \cap L^5_\mu,$$

where A is an elliptic operator describing heat conduction and local heat transfer on surfaces (convection on Γ in the above example). Here μ measures both semitransparent volumes and opaque surfaces; hence we no longer need to distinguish between surface and volume radiation. Also models with several disjoint components and combined volume and surface radiation can be analysed within this framework. We define the abstract model by postulating a set (minimal, we hope) of general mathematical properties which the operator $G : L^p_\mu \to L^p_\mu$ has to satisfy in order to guarantee the coercivity and pseudomonotonicity of the conductive-radiative operator and, thus, provide the existence of a weak solution. These postulates are generalizations of the properties 1–5 described above. Within this abstract framework we also establish comparison principle and boundedness of solutions as well as analyse time dependent models. To apply the general theory in some specific application, we need to formulate the measure μ and the operator G as well as check that G has the desired features.

The final article of my thesis [D] focuses on heat transfer in optically thick (i.e. highly absorptive) materials. Intuitively, in such materials radiation propagates in a diffusive manner and in very thick materials radiation concentrates on surfaces. From the mathematical point of view, however, this is by no means obvious. Therefore, the first goal of this article was to prove that the sequence of solutions of volume radiation problems converge to a solution of surface radiation problem as absorption coefficient goes to infinity. In view of above examples, this means that we recover the original surface radiation problem ($\star$) by letting $\kappa \to \infty$ in volumetric problem ($\star\star$). The main tools in this analysis were singular perturbation theory, boundary layer analysis as well as certain stability and a priori estimates, which had to be derived uniformly in κ.

The second goal was to derive rigorously a diffusion equation (called diffusion or Rosseland approximation in the literature) which effectively approximates the volumetric problem for large absorption coefficients and is fairly easy to solve numerically. For example, when κ is large the problem ($\star\star$) could approximated by a local nonlinear diffusion equation

$$-\partial_i(k + 4T^3(x)a_{ij}(x)\partial_j T(x)) + c_i(x)\partial_i T^4(x) + d(x)T^4(x) = f \qquad \text{in } \Omega_1,$$

where the coefficients a_{ij}, c_i, d depend on κ and shape of Ω_1. Such approximations are widely used in practice [9, 12, 18, 21], but they tend to be formally derived and tuned for some special application. In [23] a method based on two-scale asymptotics is introduced and also theoretical analysis for the radiative transport equation is carried out, but it seems that the analysis including the heat equation has not been performed before. In astrophysics and neutron transport the asymptotic analysis of transport equations is well studied [1, 2, 4] but from a slightly different perspective.

Conclusions

We showed that a large class of mathematical models combining conductive and radiative heat transfer are mathematically well defined. Furthermore, we characterized general mathematical properties of conductive-radiative models and concluded that the surface radiation models and the volume radiation models have similar mathematical structure, although these two models are derived from different physical principles. The volume and surface radiation models were also related by proving that a volume radiation model converge to a surface radiation model as the volumetric absorption coefficient approaches infinity. This result shows that conductive-radiative models behave physically with respect to absorption coefficient and justifies the use of a surface radiation model for a highly absorptive material. We also proposed a systematic and rigorous method to derive diffusion approximations for optically thick materials, which appears to be of practical and engineering interest. In my opinion, these facts form a good foundation for numerical analysis and efficient numerical simulation.

There are several limitations to this study. First, we only investigated materials whose radiative properties are independent of wavelength (grey materials). This allowed us to factorize the radiative part into a linear operator G and a nonlinear law T^4, and thus we could utilize theory of linear operators. When radiative coefficients depend on wavelength, this factorization is possible only for each wavelength and after integration over wavelengths the radiative operator becomes fully nonlinear. We observed some similarities between these two cases using theory of nonlinear operators, but more research in this direction is required.

Concerning the asymptotic analysis, the most interesting question would be to see how the diffusion approximations derived in [D] perform in the two and three dimensional geometries, especially, around a corner point. The smoothness assumptions in [D] seem to be intrinsic character of the problem and additional error is introduced if the boundary is not smooth. Whether this error is negligible or additional corner layer terms are required, needs to be resolved. Another interesting question would be to extend the theory in [D] to include scattering, and particularly find an accurate and yet simple diffusion approximation.

Author's Contribution

Finally, I report my role in the articles [A], [B], [C] written with professor Timo Tiihonen. The first work [B] employed and extended the ideas that Tiihonen had already discovered for surface radiation [19]. The most laborious part was to formulate the volumetric problem such that the previous ideas could be used. This was inseparably teamwork. The main guidelines were drawn by Tiihonen whereas I did all the proofs. I discovered that the interpolation theory of operators is a useful tool in analysing radiative operators. The next article [A] combined the efforts of [19] and [B] and announced the idea of Tiihonen that both surface and volume radiation can be studied within the same framework.

The core of [C], the abstract framework and the existence theory (Sections 2 and 4), were derived mostly together, a notable exception being Lemma 9 which I discovered independently. I established independently the comparison principle (Section 5), the boundedness of solutions (Section 6) and majority of the time dependence (Section 7). Apart from the time dependent comparison principle by Tiihonen, I provided all the proofs to [C].

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YHTEENVETO (FINNISH SUMMARY)

Väitöskirjassa tarkastellaan lämmön johtumista ja säteilyä kuvaavia matemaattisia malleja. Lämmönjohtumisyhtälössä säteily esiintyy lähde- tai vuoterminä. Säteilyn selvittämiseksi täytyy samanaikaisesti ratkaista integraali- tai kuljetusyhtälö, joka riippuu lämpötilasta.

Tutkimuksen tavoitteena on määrittää lämmönsiirtoyhtälöiden yleiset matemaattiset ominaisuudet. Työssä osoitetaan yhtälöiden pseudomonotonisuus ja koersiivisuus. Näistä kahdesta ominaisuudesta voidaan päätellä, että lämmönsiirtoyhtälöllä on ratkaisu. Toiseksi väitöskirjassa todistetaan vertailuperiaate lämmönsiirtoyhtälöille. Vertailuperiaatteesta seuraa, että ratkaisu on yksikäsitteinen. Kolmanneksi työssä osoitetaan lämmönsiirtoyhtälöiden ratkaisujen rajoittuneisuus.

Teoriaa havainnollistetaan analysoimalla tyypillisimpiä teollisuudessa esiintyviä tilanteita. Näistä toinen on säteilylämmönsiirto diffuusiivisten sekä harmaiden pintojen välillä ja toinen säteilylämmönsiirto puoliläpäisevässä materiaalissa diffuusi- tai peiliheijastavilla reunoilla.

Lopuksi työssä tarkastellaan lämmönsiirtoa optisesti tiheissä materiaaleissa, joissa absorptiokertoimet ovat suuria. Väitöskirjassa osoitetaan, että tällaisissa materiaaleissa lämmönsiirtoyhtälöitä voidaan approksimoida diffuusioyhtälöillä. Samalla todistetaan, että optisen tiheyden kasvaessa puoliläpäisevän materiaalin lämmönsiirtomalli lähestyy pintasäteilymallia.

Tutkimus rajoittuu tapauksiin, joissa materiaaliominaisuudet ovat riippumattomia säteilyn aallonpituudesta (harmaat materiaalit).

INCLUDED ARTICLES

A

HEAT TRANSFER IN CONDUCTING AND RADIATING BODIES

M. LAITINEN AND T. TIIHONEN, Appl. Math. Lett., 10 (1997), pp. 5–8.

https://doi.org/10.1016/S0893-9659(97)00074-8

B

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