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RIESZ TRANSFORM AND VERTICAL OSCILLATION IN THE HEISENBERG GROUP

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We study the L^2-boundedness of the 3-dimensional (Heisenberg) Riesz transform on intrinsic Lipschitz graphs in the first Heisenberg group $\mathbb{H}$. Inspired by the notion of vertical perimeter, recently defined and studied by Lafforgue, Naor, and Young, we first introduce new scale and translation invariant coefficients $\operatorname{osc}_{\Omega}(B(q,r))$. These coefficients quantify the vertical oscillation of a domain $\Omega \subset \mathbb{H}$ around a point $q \in \partial\Omega$, at scale $r > 0$. We then proceed to show that if Ω is a domain bounded by an intrinsic Lipschitz graph Γ, and

$$\int_0^{\infty} \operatorname{osc}_{\Omega}(B(q,r))\,\frac{dr}{r} \le C < \infty, \quad q \in \Gamma,$$

then the Riesz transform is L^2-bounded on Γ. As an application, we deduce the boundedness of the Riesz transform whenever the intrinsic Lipschitz parametrisation of Γ is an ϵ better than $\frac{1}{2}$-Hölder continuous in the vertical direction.

We also study the connections between the vertical oscillation coefficients, the vertical perimeter, and the natural Heisenberg analogues of the β-numbers of Jones, David, and Semmes. Notably, we show that the L^p-vertical perimeter of an intrinsic Lipschitz domain Ω is controlled from above by the p-th powers of the L^1-based β-numbers of $\partial\Omega$.

1. Introduction

1A. ***A Euclidean introduction to the Heisenberg Riesz transform.*** A fundamental singular integral operator (SIO) in $\mathbb{R}^d$ is the $(d-1)$-*dimensional Riesz transform*, formally defined by the convolution

$$R_{d-1}\nu(x) = \nu * \frac{x}{|x|^d}.$$

Here $x/|x|^d$ is the $(d-1)$-*dimensional Riesz kernel* which is, up to a constant, the gradient of the fundamental solution of the Laplacian. Through this connection to the Laplace equation, the operator R_{d-1} has many applications to problems concerning analytic and harmonic functions. For instance, whenever R_{d-1} is bounded on $L^2(\mu)$ for a $(d-1)$-regular measure μ, then the support of μ is nonremovable for Lipschitz harmonic functions (or bounded analytic functions in the plane); see [Tolsa 2014] for an in depth introduction to this topic and many more references.

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A second application of the SIO R_{d-1} is the *method of layer potentials* employed to solve the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = g \end{cases} \tag{1.1}$$

on domains $\Omega \subset \mathbb{R}^d$ with Lipschitz boundaries, and with, say, $g \in L^2(\mathcal{H}^{d-1}|_{\partial\Omega})$. As the name suggests, a key component in the method of layer potentials is the study of the *boundary layer potential*

$$D\nu(x) = \text{p.v.}\, \frac{1}{\omega_d} \int_{\partial\Omega} \frac{(y-x)n_{\partial\Omega}(y)}{|y-x|^d}\, d\nu(y).$$

The boundedness of the operator D on $L^2(\mathcal{H}^{d-1}|_{\partial\Omega})$ can be derived from the boundedness of R_{d-1}; see [Fabes et al. 1978; Verchota 1984].

By now, the L^2-boundedness properties of the operator R_{d-1} are well understood. According to a result of David and Semmes [1991], generalising the earlier works of Calderón [1977] and Coifman, McIntosh, and Meyer [Coifman et al. 1982], R_{d-1} is bounded on $L^2(\mathcal{H}^{d-1}|_S)$ whenever $S \subset \mathbb{R}^d$ is *uniformly* $(d-1)$*-rectifiable*. More recently, Nazarov, Tolsa, and Volberg [Nazarov et al. 2014a] proved a converse: if $S \subset \mathbb{R}^d$ is $(d-1)$-regular, then the uniform rectifiability of S is necessary for the boundedness of R_{d-1} on $L^2(\mathcal{H}^{d-1}|_S)$. These results have been used to show that a compact $(d-1)$-set is removable for Lipschitz harmonic functions if and only if it is purely $(d-1)$-unrectifiable [Mattila and Paramonov 1995; Nazarov et al. 2014b] and that the Dirichlet problem (1.1) is solvable in Lipschitz domains with L^2-boundary values [Verchota 1984].

The work here is motivated by aspirations to extend parts of the theory above to the case of a basic hypoelliptic and nonelliptic operator, the *sub-Laplacian* (also known as the *Kohn Laplacian*)

$$\Delta_{\mathbb{H}} = X^2 + Y^2$$

in $\mathbb{R}^3$. Here X and Y are the vector fields

$$X = \partial_x - \tfrac{1}{2} y \partial_t \quad \text{and} \quad Y = \partial_y + \tfrac{1}{2} x \partial_t. \tag{1.2}$$

A first step is to understand the L^2-boundedness of an associated *Riesz transform* operator, which we will soon define.

Whereas the operators X, Y, $\Delta_{\mathbb{H}}$ do not interact particularly nicely with Euclidean translations, they do commute with the following *left translations* $\tau_p : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\tau_p(q) := \big(x + x', y + y', t + t' + \tfrac{1}{2}(xy' - x'y)\big),$$

where $p = (x, y, t) \in \mathbb{R}^3$ and $q = (x', y', t') \in \mathbb{R}^3$. This suggests that it is natural to study questions about $\Delta_{\mathbb{H}}$ in the setting of the first *Heisenberg group* $\mathbb{H} = (\mathbb{R}^3, \cdot)$, where the group law "$\cdot$" is *defined* so that X and Y are (left) invariant:

$$p \cdot q := \tau_p(q).$$

It was shown by Folland [1975] that the operator $\Delta_{\mathbb{H}}$ has a fundamental solution $G : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$, whose formula is given by

$$G(p) = \frac{c}{((x^2+y^2)^2+16t^2)^{1/2}} =: \frac{c}{\|p\|_{\mathrm{Kor}}^2}, \qquad p = (x, y, t) \in \mathbb{H} \setminus \{0\}.$$

Here $c > 0$ is a constant and $\|p\|_{\mathrm{Kor}} := ((x^2+y^2)^2+16t^2)^{1/4}$. This quantity is known as the *Korányi norm* of the point $p \in \mathbb{H}$, and it induces a metric d_{Kor} on $\mathbb{H}$ via the relation

$$d_{\mathrm{Kor}}(p, q) = \|q^{-1} \cdot p\|_{\mathrm{Kor}}. \tag{1.3}$$

The distance d_{Kor} is invariant under the left translations, that is, $d_{\mathrm{Kor}}(p \cdot q_1, p \cdot q_2) = d_K(q_1, q_2)$ for all $p, q_1, q_2 \in \mathbb{H}$.

In analogy with the $(d-1)$-dimensional Riesz transform discussed above, one may now consider the SIO R formally defined by

$$R\nu(p) := \nu * \nabla_{\mathbb{H}} G(p).$$

Here $\nabla_{\mathbb{H}}$ stands for the *horizontal gradient* $\nabla_{\mathbb{H}} G = (XG, YG)$, and the convolution should be understood in the Heisenberg sense:

$$f * g(p) = \int f(q) g(q^{-1} \cdot p)\, dq.$$

The main open question is the following:

Question 1. For which locally finite Borel measures μ on $\mathbb{H}$ (equivalently $\mathbb{R}^3$) is the operator R bounded on $L^2(\mu)$?

Here the boundedness on $L^2(\mu)$ is defined in the standard way via ϵ-truncations; see Section 4 for the precise definition.

1B. ***Previous work.*** To the best of our knowledge, the Heisenberg Riesz transform R was first mentioned in [Chousionis and Mattila 2014], where the following removability question was raised and studied: Which subsets of $\mathbb{H}$ (more generally, of Heisenberg groups of arbitrary dimensions) are removable for Lipschitz harmonic functions? The notions of *Lipschitz* and *harmonic* should be interpreted in the Heisenberg sense: We call a function $u : \mathbb{H} \to \mathbb{R}$ *harmonic* if it solves the sub-Laplace equation $\Delta_{\mathbb{H}} u = 0$. A function $f : \mathbb{H} \to \mathbb{R}$ is *Lipschitz* if $|f(p) - f(q)| \le L d_{\mathrm{Kor}}(p, q)$ for some $L \ge 1$ and all $p, q \in \mathbb{H}$.

It was shown in [Chousionis and Mattila 2014, Theorem 3.13] that the critical exponent for the removability problem in $\mathbb{H}$ is 3 (keeping in mind that $\dim_{\mathrm{H}}(\mathbb{H}, d_{\mathrm{Kor}}) = 4$). More precisely, sets with vanishing 3-dimensional measure are removable, while sets of Hausdorff dimension exceeding 3 are not. In [Chousionis and Mattila 2014, Section 5], the authors formulate (essentially) Question 1 and suggest its connection to the removability problem.

The connection was formalised by Chousionis and the authors in the following theorem:

Theorem 1.4 [Chousionis et al. 2019a, Theorem 1.2]. *If μ is a* 3*-regular measure on* $\mathbb{H}$ (see (1.5) below), *and R is bounded on $L^2(\mu)$, then* spt μ *is nonremovable for Lipschitz harmonic functions in* $\mathbb{H}$.

In [Chousionis et al. 2019a], we also proved the first nontrivial results on the L^2-boundedness of R (and a class of other SIOs). To discuss these results, and also the ones in the present paper, we need the concept of *intrinsic Lipschitz functions and graphs*. A *vertical subgroup* $\mathbb{W} \subset \mathbb{H}$ is, from a geometric point of view, any 2-dimensional subspace of $\mathbb{R}^3$ containing the t-axis. The *complementary horizontal subgroup of* $\mathbb{W}$ is the line $\mathbb{V} = \mathbb{W}^\perp$ in the xy-plane.

We give the definition of *intrinsic Lipschitz functions* $\phi : \mathbb{W} \to \mathbb{V}$ and the associated *intrinsic Lipschitz graphs* $\Gamma_\phi \subset \mathbb{H}$ in Section 2C. These objects were introduced by Franchi, Serapioni, and Serra Cassano [Franchi et al. 2006], and they appear to be fundamental building blocks in the theory of *high-dimensional* rectifiability in the Heisenberg group; see for example [Chousionis et al. 2019b; Mattila et al. 2010]. In particular, intrinsic Lipschitz graphs $\Gamma \subset \mathbb{H}$ are closed 3-regular sets, which means that the measure $\mu = \mathcal{H}^3|_\Gamma$ satisfies

$$\mu(B(p,r)) \sim r^3, \quad p \in \operatorname{spt}\mu, \;\; 0 < r \le \operatorname{diam}(\operatorname{spt}\mu). \tag{1.5}$$

In another paper of Franchi, Serapioni, and Serra Cassano [Franchi et al. 2011], a Rademacher-type theorem was established for intrinsic Lipschitz functions: without delving into detail, we just mention that if $\phi : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz, then for Lebesgue almost every $w \in \mathbb{W}$ there exists an *intrinsic gradient* for ϕ, denoted by $\nabla^\phi \phi(w)$.

Recall that in $\mathbb{R}^d$, Calderón [1977] and Coifman, McIntosh, and Meyer [Coifman et al. 1982] proved that R_{d-1} is bounded on $L^2(\mathcal{H}^{d-1}|_\Gamma)$ if $\Gamma \subset \mathbb{R}^d$ is a Lipschitz graph. In analogy, one can ask:

Question 2. Assume that $\Gamma \subset \mathbb{H}$ is an intrinsic Lipschitz graph. Is R bounded on $L^2(\mathcal{H}^3|_\Gamma)$?

We are not convinced enough to upgrade the question to a conjecture. In [Chousionis et al. 2019a], we obtained a positive answer under a extra regularity:

Theorem 1.6 [Chousionis et al. 2019a, Theorem 1.1]. *Assume* $\alpha > 0$ *and that* $\phi \in C^{1,\alpha}(\mathbb{W})$ *has compact support. Then* R *is bounded on* $L^2(\mathcal{H}^3|_{\Gamma_\phi})$.

The assumption $\phi \in C^{1,\alpha}(\mathbb{W})$ means that the intrinsic gradient of ϕ exists everywhere and satisfies an intrinsic version of α-Hölder regularity (which is weaker than Euclidean α-Hölder regularity). The assumption implies, see [Chousionis et al. 2019a, Proposition 4.1], that the affine approximation of Γ_ϕ at $p \in \Gamma$ improves at a geometric rate as one zooms into p.

1C. *New results.* A novelty of the current paper is to prove the L^2-boundedness of R in some scenarios where there is no *pointwise decay* for the quality of affine approximation of Γ. As a basic example, Theorem 4.1 below applies to graphs of the form

$$\Gamma = \Gamma_{\mathbb{R}^2} \times \mathbb{R} \subset \mathbb{H},$$

where $\Gamma_{\mathbb{R}^2}$ is a (Euclidean) Lipschitz graph in $\mathbb{R}^2$. It turns out that a key feature of these graphs is the following. The two complementary domains $\Omega_1, \Omega_2 \subset \mathbb{H} \setminus \Gamma$ have zero *vertical oscillation*: for $j \in \{1,2\}$, every vertical line $\ell \subset \mathbb{H}$ satisfies

$$\ell \subset \Omega_j \quad \text{or} \quad \ell \cap \Omega_j = \varnothing. \tag{1.7}$$

The condition (1.7) is qualitative, not to mention exceedingly restrictive, so we looked for a way to quantify and relax it. For these purposes, we introduce the *vertical oscillation coefficients* $\operatorname{osc}_\Omega(B(p,r))$. Given a domain $\Omega \subset \mathbb{H}$ and a point $p \in \partial\Omega$, the number $\operatorname{osc}_\Omega(B(p,r))$ quantifies, in a scale and translation invariant way, how far Ω is (locally) from satisfying (1.7). The definition of the coefficients $\operatorname{osc}_\Omega(B(p,r))$ was inspired by the notion of *vertical perimeter* recently introduced in [Lafforgue and Naor 2014, Section 4] and further studied in [Naor and Young 2018]; see Remark 3.2 for the definition. We postpone further details on the vertical oscillation coefficients to Section 3.

Here is the main theorem of the paper.

Theorem 1.8. *Let $\Gamma \subset \mathbb{H}$ be an intrinsic Lipschitz graph, and let Ω be one of the components of $\mathbb{H} \setminus \Gamma$. Assume that there is a finite constant $C > 0$ such that*

$$\int_0^\infty \operatorname{osc}_\Omega(B(p,r))\,\frac{dr}{r} \le C, \qquad p \in \partial\Omega. \tag{1.9}$$

Then R is bounded on $L^2(\mathcal{H}^3|_\Gamma)$.

In general, we do not know how reasonable the assumption (1.9) is. It follows easily from the Rademacher theorem for intrinsic Lipschitz functions (and Corollary 3.34 below) that $\operatorname{osc}_\Omega(B(p,r)) \to 0$ for $\mathcal{H}^3$ almost every $p \in \Gamma$ as $r \searrow 0$. But we have no quantitative estimates for $\operatorname{osc}_\Omega(B(p,r))$ if nothing better than intrinsic Lipschitz regularity is assumed of Γ; see Section 6 for a concrete question in this vein. However, we can complement Theorem 1.8 with the following application.

Theorem 1.10. *Let $\phi : \mathbb{W} \to \mathbb{R}$ be an intrinsic Lipschitz function that satisfies the following Hölder regularity in the vertical direction*:

$$|\phi(y,t) - \phi(y,s)| \le H|t-s|^{(1+\tau)/2}, \qquad |s-t| \le 1 \tag{1.11}$$

and

$$|\phi(y,t) - \phi(y,s)| \le H|t-s|^{(1-\tau)/2}, \qquad |s-t| > 1, \tag{1.12}$$

where $H \ge 1$ and $0 < \tau \le 1$. Then R is bounded on $L^2(\mathcal{H}^3|_{\Gamma_\phi})$.

It is well known that intrinsic Lipschitz functions are always $\frac{1}{2}$-Hölder continuous in the vertical direction. So, Theorem 1.10 states that an ϵ of additional regularity in this one direction yields the L^2-boundedness of R on Γ_ϕ.

1D. *Vertical oscillation and β-numbers.* A fundamental concept in the theory of quantitative rectifiability in $\mathbb{R}^n$ is the β-*number*, first introduced in [Jones 1990], then further developed in [David and Semmes 1991], and later applied by too many authors to begin acknowledging here. It is no surprise that suitable variants of the β-numbers (see Section 3A for definitions) can also be used to study quantitative rectifiability questions in $\mathbb{H}$, as well as higher dimensional Heisenberg groups. A few papers already doing so are [Chousionis and Li 2017; Chousionis et al. 2019a; 2019b; Fässler et al. 2020; Juillet 2010; Li and Schul 2016a; 2016b]. Since we here introduce new coefficients related to the theory of quantitative rectifiability in $\mathbb{H}$, it is natural to ask: is there a connection to β-numbers? We investigate this matter in Sections 3A and 6B.

We only mention the key results here briefly and informally. First, the vertical oscillation coefficients of Ω are bounded from above by the (L^1-based) β-numbers of $\partial\Omega$ — at least if $\partial\Omega$ is 3-regular. This is the content of Corollary 3.34. Second, if $\partial\Omega$ is 3-regular, and if the β-numbers associated to $\partial\Omega$ satisfy an *L^p-Carleson packing condition*, see (6.4), then the L^p-variant of the vertical perimeter of Ω inside balls $B(q, r)$, $q \in \partial\Omega$, is bounded by the usual (horizontal) perimeter of Ω in $B(q, r)$. This is Corollary 6.5.

This result should be contrasted with the work of Naor and Young in higher dimensional Heisenberg groups: in [Naor and Young 2018, Proposition 41], they prove that if $\Omega \subset \mathbb{H}^n$, $n \geq 2$, is an intrinsic Lipschitz domain, then the L^2-vertical perimeter of Ω in balls centred at $\partial\Omega$ is automatically bounded by the horizontal perimeter — without any reference to β-numbers. Then, at the very end of [Naor and Young 2018], see also Remark 4 in the same work, the authors mention showing in a forthcoming paper [Naor and Young 2022] that a similar inequality fails for the L^2-vertical perimeter in $\mathbb{H}^1 = \mathbb{H}$, but holds for the L^p-vertical perimeter for some $p > 2$ (specifically, the authors mention $p = 4$).[1] If this is the case, then, according to Corollary 6.5, one cannot expect the β-numbers of intrinsic Lipschitz graphs to satisfy an L^2-Carleson packing condition. This is in contrast to the situation in $\mathbb{R}^n$, where the β-numbers on Lipschitz graphs do satisfy an L^2-Carleson packing condition; see [David and Semmes 1991, (C3)].

2. Preliminaries

In this section, we collect essential notions related to the algebraic and metric structures of the first Heisenberg group $\mathbb{H}$, and we recall the definition and basic properties of intrinsic Lipschitz graphs over vertical planes in $\mathbb{H}$. For a more thorough introduction to these subjects, we refer the reader to [Capogna et al. 2007; Serra Cassano 2016].

2A. ***Right- and left-invariant vector fields.*** Recall from the Introduction that X and Y denote the standard left-invariant vector fields on $\mathbb{H}$ defined in (1.2). We will also work with their right invariant counterparts

$$\widetilde{X} = \partial_x + \tfrac{1}{2} y \partial_t \quad \text{and} \quad \widetilde{Y} = \partial_y - \tfrac{1}{2} x \partial_t.$$

We define the left and right (horizontal) gradients of $\phi \in \mathcal{C}^1(\mathbb{R}^3)$ as the 2-vectors

$$\nabla_{\mathbb{H}} \phi = (X\phi, Y\phi) \quad \text{and} \quad \widetilde{\nabla}_{\mathbb{H}} \phi = (\widetilde{X}\phi, \widetilde{Y}\phi).$$

For $V = (V_1, V_2) \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^2)$, we define the left and right divergences as the functions

$$\operatorname{div}_{\mathbb{H}} V := XV_1 + YV_2 \in \mathcal{C}^0(\mathbb{R}^3) \quad \text{and} \quad \widetilde{\operatorname{div}}_{\mathbb{H}} V := \widetilde{X}V_1 + \widetilde{Y}V_2 \in \mathcal{C}^0(\mathbb{R}^3).$$

For $V, W \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R}^2)$, we define the *inner product*

$$\langle V, W \rangle := V_1 W_1 + V_2 W_2 \in \mathcal{C}^1(\mathbb{R}^3).$$

Finally, we denote the left and right sub-Laplacians as

$$\Delta_{\mathbb{H}} := XX + YY \quad \text{and} \quad \widetilde{\Delta_{\mathbb{H}}} := \widetilde{X}\widetilde{X} + \widetilde{Y}\widetilde{Y}.$$

[1] While the present paper was under review, the paper [Naor and Young 2022] appeared, and indeed contains the results mentioned here.

2B. ***Metric structure.*** Various left-invariant distance functions on $\mathbb{H}$ are commonly used in the literature, for instance the standard sub-Riemannian distance or the Korányi metric given in (1.3). The choice of metric that we are going to use is motivated by the divergence theorem (Theorem 4.3), which holds for the spherical Hausdorff measure $\mathcal{S}^3$ with respect to the metric

$$d : \mathbb{H} \times \mathbb{H} \to [0, +\infty), \quad d(p, q) := \|q^{-1} \cdot p\|, \tag{2.1}$$

where

$$\|(x, y, t)\| := \max\{|(x, y)|, 2\sqrt{|t|}\}.$$

However, every left-invariant metric on $\mathbb{H}$ that is continuous with respect to the Euclidean topology on $\mathbb{R}^3$ and homogeneous with respect to the one-parameter family of *Heisenberg dilations* $(\delta_\lambda)_{\lambda>0}$,

$$\delta_\lambda : \mathbb{H} \to \mathbb{H}, \quad \delta_\lambda(x, y, t) := (\lambda x, \lambda y, \lambda^2 t),$$

is bi-Lipschitz equivalent to the metric d; this applies in particular to the Korányi distance d_{Kor}. Unless otherwise stated, all metric concepts such as balls $B(p, r)$, diameters, and Hausdorff measures will be defined using the metric d.

2C. ***Intrinsic Lipschitz graphs.*** Let $\mathbb{W}$ be a vertical subgroup with complementary horizontal subgroup $\mathbb{V}$; recall from the paragraph after Theorem 1.4 that, in this paper, the *complementary horizontal subgroup of* $\mathbb{W}$ is the orthogonal complement of $\mathbb{W}$ in $\mathbb{R}^3$. Any point $p \in \mathbb{H}$ can be written as $p = w \cdot v$ for uniquely given $w \in \mathbb{W}$ and $v \in \mathbb{V}$. We write $w =: \pi_{\mathbb{W}}(p)$ and call it the *vertical projection of p to* $\mathbb{W}$; similarly, we denote the *horizontal projection* by $v = \pi_{\mathbb{V}}(p)$. These projections have been studied in connection with uniform rectifiability problems in the Heisenberg group; see for example [Chousionis et al. 2019b; Fässler et al. 2020].

Definition 2.2. A function $\phi : \mathbb{W} \to \mathbb{V}$ is *intrinsic L-Lipschitz* if

$$\|\pi_{\mathbb{V}}(\Phi(w')^{-1}\Phi(w))\| \le L\|\pi_{\mathbb{W}}(\Phi(w')^{-1}\Phi(w))\|, \quad \text{for all } w, w' \in \mathbb{W}, \tag{2.3}$$

where $\Phi : \mathbb{W} \to \mathbb{H}$ denotes the *graph map* $\Phi(w) = w \cdot \phi(w)$. The *intrinsic graph* of ϕ is

$$\Gamma_\phi := \{w \cdot \phi(w) : w \in \mathbb{W}\} = \Phi(\mathbb{W}).$$

The term *intrinsic* refers to the fact that if ϕ is an intrinsic L-Lipschitz function, then, for all $p \in \mathbb{H}$ and $r > 0$, also $\tau_p(\delta_r(\Gamma_\phi))$ is an intrinsic graph of an intrinsic L-Lipschitz function. According to [Chousionis et al. 2019b, Remark 2.6], an intrinsic L-Lipschitz graph over an arbitrary vertical plane can be mapped to an intrinsic L-Lipschitz graph over the (y, t)-plane by an isometry of the form

$$R_\theta : \mathbb{H} \to \mathbb{H}, \quad R_\theta(x, y, t) := (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, t).$$

Since moreover the (complexified) kernel of the Heisenberg Riesz transform satisfies

$$(XG - iYG) \circ R_\theta = e^{i\theta}(XG - iYG),$$

we may without loss of generality assume in the following that $\mathbb{W}$ is the (y,t)-plane and $\mathbb{V}$ is the x-axis. For this choice, we have

$$\pi_{\mathbb{V}}(x,y,t) = (x,0,0) \quad \text{and} \quad \pi_{\mathbb{W}}(x,y,t) = \left(0, y, t + \tfrac{1}{2}xy\right), \qquad \text{for all } (x,y,t) \in \mathbb{H}.$$

Moreover, the map $(x,0,0) \mapsto x$ provides an isometric isomorphism between $(\mathbb{V},\cdot,d)$ and $(\mathbb{R},+,|\cdot|)$, and under this identification of $\mathbb{V}$ with $\mathbb{R}$, the intrinsic Lipschitz condition (2.3) is equivalent to

$$|\phi(0,y,t) - \phi(0,y',t')| \leq L\|\pi_{\mathbb{W}}(\Phi(0,y',t')^{-1}\Phi(0,y,t))\|, \quad \text{for all } (y,t),(y',t') \in \mathbb{R}^2.$$

The subgroup $(\mathbb{W},\cdot)$ is isomorphic to $(\mathbb{R}^2,+)$, and the map $(0,y,t) \mapsto (y,t)$ pushes the measure $\mathcal{H}^3|_{\mathbb{W}}$ forward to $c\mathcal{L}^2$ on $\mathbb{R}^2$, for a constant $0 < c < \infty$. As mentioned in the Introduction, an intrinsic Lipschitz function $\phi : \mathbb{W} \to \mathbb{V}$ possesses an *intrinsic gradient* $\nabla^\phi \phi$ at $\mathcal{H}^3$ almost every point of $\mathbb{W}$. In analogy with the behaviour of Euclidean Lipschitz functions, if $\phi : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz, then

$$\|\nabla^\phi \phi\|_{L^\infty(\mathcal{H}^3|_{\mathbb{W}})} < \infty,$$

by [Citti et al. 2014, Proposition 4.4]. More information about intrinsic gradients is collected for instance in [Chousionis et al. 2019b, Section 4.2; Serra Cassano 2016].

3. Vertical oscillation coefficients

In this section, we define and study the main new concept of the paper, the *vertical oscillation coefficients*. These coefficients are derived from the recent notion of *vertical perimeter*, due to [Lafforgue and Naor 2014, Definition 4.2] (see also [Naor and Young 2018, (28)]).

Definition 3.1 (vertical perimeter). Let $\Omega, U \subset \mathbb{H}$ be Lebesgue measurable sets, and let $s > 0$ be a scale. The *vertical perimeter of* Ω *relative to* U *at scale* s is the quantity

$$\mathrm{v}_\Omega(U)(s) := \int_U |\chi_\Omega(p) - \chi_\Omega(p \cdot (0,0,s^2))|\, dp.$$

Here and in the following, dp refers to integration with respect to Lebesgue measure $\mathcal{L}^3$ on $\mathbb{R}^3$, which agrees up to a multiplicative constant with $\mathcal{H}^4$.

Remark 3.2. Having first defined the vertical perimeter $\mathrm{v}_\Omega(U)(s)$ at a fixed scale $s > 0$, [Lafforgue and Naor 2014, (70)] and [Naor and Young 2018, Section 2.2] proceed to define the L^2*-vertical perimeter of* Ω as the $L^2(ds/s)$-norm of the function $s \mapsto \mathrm{v}_\Omega(\mathbb{H})/s$. More generally, for $p \geq 1$ and an open set $U \subset \mathbb{H}$, one can consider (as in [Naor and Young 2018, (68)] for example) the L^p*-vertical perimeter of* Ω *in* U:

$$\wp_{\Omega,p}(U) := \left\| s \mapsto \frac{\mathrm{v}_\Omega(U)(s)}{s} \right\|_{L^p(ds/s)} = \left(\int_0^\infty \left(\frac{\mathrm{v}_\Omega(U)(s)}{s} \right)^p \frac{ds}{s} \right)^{1/p}.$$

It would be interesting to know if the L^p-vertical perimeter of Ω — for some $p \geq 1$ — can be related to the boundedness of the Heisenberg Riesz transform on $L^2(\mathcal{H}^3|_{\partial\Omega})$.

Definition 3.3 (vertical oscillation coefficients). Let $\Omega \subset \mathbb{H}$ be a Lebesgue measurable (typically open) set, and let $B(p,r) \subset \mathbb{H}$ be a ball. We define

$$\operatorname{osc}_\Omega(B(p,r)) := \frac{1}{r^4} \fint_0^r \mathrm{v}_\Omega(B(p,r))(s)\,ds.$$

Next we examine the basic properties of the oscillation coefficients.

Lemma 3.4. *There is an absolute constant $C \geq 1$ such that $\operatorname{osc}_\Omega(B(p,r)) \leq C$ for all Lebesgue measurable sets $\Omega \subset \mathbb{H}$ and all balls $B(p,r) \subset \mathbb{H}$. The vertical oscillation coefficients are approximately monotone in the following sense: if $B(p_1,r_1) \subset B(p_2,r_2) \subset \mathbb{H}$ are two balls with $r_2 \leq C_1 r_1$, then*

$$\operatorname{osc}_\Omega(B(p_1,r_1)) \lesssim_{C_1} \operatorname{osc}_\Omega(B(p_2,r_2)). \tag{3.5}$$

Finally, the vertical oscillation coefficients are invariant with respect to dilations and left translations in the following sense:

$$\operatorname{osc}_{\delta_t(q\cdot\Omega)}(B(\delta_t(q\cdot p),tr)) = \operatorname{osc}_\Omega(B(p,r)), \quad t>0, \quad q \in \mathbb{H}. \tag{3.6}$$

Proof. To prove the first claim, observe that $\mathrm{v}_\Omega(B(p,r))(s) \leq 2\mathcal{H}^4(B(p,r)) \sim r^4$ for all $0 \leq s \leq r$, so

$$\operatorname{osc}_\Omega(B(p,r)) \lesssim \fint_0^r \frac{r^4}{r^4}\,ds = 1.$$

The approximate monotonicity property (3.5) follows immediately from the inequality $\mathrm{v}_\Omega(B(p_1,r_1))(s) \leq \mathrm{v}_\Omega(B(p_2,r_2))(s)$, valid for all $s>0$.

The left-invariance $\operatorname{osc}_{q\cdot\Omega}(B(q\cdot p,r)) = \operatorname{osc}_\Omega(B(p,r))$ of the vertical oscillation coefficients follows from the evident left-invariance of the vertical perimeter, so we assume that $p=q=0$ and prove that

$$\operatorname{osc}_{\delta_t(\Omega)}(B(0,tr)) = \operatorname{osc}_\Omega(B(0,r)), \quad t>0.$$

To see this, we start by expanding

$$\begin{aligned}\operatorname{osc}_{\delta_t(\Omega)}(B(0,tr)) &= \frac{1}{(tr)^5}\int_0^{tr} \mathrm{v}_{\delta_t(\Omega)}(B(0,tr))(s)\,ds \\ &= \frac{1}{(tr)^5}\int_0^{tr}\int_{B(0,tr)} |\chi_{\delta_t(\Omega)}(p) - \chi_{\delta_t(\Omega)}(p\cdot(0,0,s^2))|\,dp\,ds.\end{aligned}$$

Then, we make the change of variables $p \mapsto \delta_t(q)$, and finally $s \mapsto ut$:

$$\operatorname{osc}_{\delta_t(\Omega)}(B(0,tr)) = \frac{1}{r^5}\int_0^r\int_{B(0,r)} |\chi_\Omega(q) - \chi_\Omega(q\cdot(0,0,u^2))|\,dq\,du = \operatorname{osc}_\Omega(B(0,r)). \qquad \square$$

Remark 3.7. The previous lemma says that $\operatorname{osc}_\Omega(B(p,r)) \lesssim 1$ no matter what Ω looks like. If Ω is the sub- or supergraph of an intrinsic Lipschitz function satisfying better than $\frac{1}{2}$-Hölder regularity in the vertical direction, then the oscillation coefficients of Ω have geometric decay. A more precise statement can be found in Lemma 5.6.

In connection with singular integrals, the vertical oscillation coefficients will enter through the next lemma.

Lemma 3.8. *Let $\Omega \subset \mathbb{H}$ be a Lebesgue measurable set. Let $p \in \mathbb{H}$, $r > 0$, and let $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with* $\operatorname{spt}\psi \subset B(p,r)$. *Then*

$$\left| \frac{1}{r^4} \int_\Omega \partial_t \psi(p)\, dp \right| \lesssim \|\partial_t \psi\|_\infty \operatorname{osc}_\Omega(B(p, 10r)), \tag{3.9}$$

where $\partial_t \psi$ is the derivative of ψ with respect to the third variable.

Proof. We start by reducing to the case $B(p,r) = B(0,1)$. So, assume that (3.9) holds for every Lebesgue measurable set Ω and all $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi \subset B(0,1)$ and with $\operatorname{osc}_\Omega(B(0,10))$ on the right-hand side. Then, if $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi \subset B(p,r)$, we consider the function $\psi_{p,r} = \psi \circ \tau_p \circ \delta_r \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi_{p,r} \subset B(0,1)$. It follows that

$$\begin{aligned} \left| \frac{1}{r^4} \int_\Omega \partial_t \psi(q)\, dq \right| &= \left| \int_{\delta_{1/r}(p^{-1}\cdot\Omega)} \partial_t \psi(p \cdot \delta_r(q))\, dq \right| \\ &= \left| \int_{\delta_{1/r}(p^{-1}\cdot\Omega)} r^{-2} \partial_t \psi_{p,r}(q)\, dq \right| \\ &\lesssim \frac{\|\partial_t \psi_{p,r}\|_\infty}{r^2} \operatorname{osc}_{\delta_{1/r}(p^{-1}\cdot\Omega)}(B(0,10)) \\ &= \|\partial_t \psi\|_\infty \operatorname{osc}_\Omega(B(p,10r)), \end{aligned}$$

using Lemma 3.4 in the last equation.

It remains to prove the case $B(p,r) = B(0,1)$, so fix $\psi \in \mathcal{C}^1(\mathbb{R}^3)$ with $\operatorname{spt}\psi \subset B(0,1)$. By Fubini's theorem, we can write

$$\int_\Omega \partial_t \psi(q)\, dq = \int_{\mathcal{L}} \int_\ell \partial_t \psi(q) \chi_\Omega(q)\, d\mathcal{H}^1_E(q)\, d\eta(\ell), \tag{3.10}$$

where $\mathcal{L}$ stands for the collection of vertical lines, η is the two-dimensional Lebesgue measure on $\mathbb{R}^2$ (which is used to parametrise $\mathcal{L}$), and $\mathcal{H}^1_E$ denotes the 1-dimensional Hausdorff measure with respect to the Euclidean distance. Next, we note that if $\ell \in \mathcal{L}$ is a fixed line, then

$$\int_\ell \partial_t \psi(q)\, d\mathcal{H}^1_E(q) = 0. \tag{3.11}$$

Now, let $Q := [-5,5]^2 \times [-2,-1] \subset B(0,10)$. We note that whenever $\ell \in \mathcal{L}$ is a line with nonzero contribution in (3.10), we have $\ell \cap B(0,1) \neq \varnothing$, and in particular

$$\mathcal{H}^1_E(\ell \cap Q) = 1.$$

Then, use (3.10)–(3.11) to write

$$\begin{aligned} \left| \int_\Omega \partial_t \psi(q)\, dq \right| &= \left| \int_{\mathcal{L}} \int_{\ell \cap Q} \int_\ell \partial_t \psi(q) [\chi_\Omega(q) - \chi_\Omega(p)]\, d\mathcal{H}^1_E(q)\, d\mathcal{H}^1_E(p)\, d\eta(\ell) \right| \\ &\leq \|\partial_t \psi\|_\infty \int_{\mathcal{L}} \int_{\ell \cap Q} \int_{\ell \cap B(0,1)} |\chi_\Omega(q) - \chi_\Omega(p)|\, d\mathcal{H}^1_E(q)\, d\mathcal{H}^1_E(p)\, d\eta(\ell). \end{aligned}$$

Next, for $\ell \in \mathcal{L}$ and $p \in \ell \cap Q$ fixed, we make the change of variable $q \mapsto p \cdot (0, 0, s)$ in the innermost integral: since $q \in \ell \cap B(0, 1)$ and $p \in \ell \cap Q$, we note that $s \in [0, 3]$. This leads to

$$\begin{aligned}\left|\int_\Omega \partial_t \psi(q)\, dq\right| &\leq \|\partial_t \psi\|_\infty \int_{\mathcal{L}} \int_{\ell \cap Q} \int_0^3 |\chi_\Omega(p \cdot (0,0,s)) - \chi_\Omega(p)|\, ds\, d\mathcal{H}^1_E(p)\, d\eta(\ell) \\ &\leq \|\partial_t \psi\|_\infty \int_0^3 \int_{\mathcal{L}} \int_{\ell \cap B(0,10)} |\chi_\Omega(p \cdot (0,0,s)) - \chi_\Omega(p)|\, d\mathcal{H}^1_E(p)\, d\eta(\ell)\, ds \\ &\lesssim \|\partial_t \psi\|_\infty \int_0^{\sqrt{3}} \mathrm{v}_\Omega(B(0,10))(u) u\, du \lesssim \|\partial_t \psi\|_\infty \operatorname{osc}_\Omega(B(0,10)).\end{aligned}$$

In the final inequality, the factor "u" was simply estimated by $\sqrt{3}$. □

3A. ***Vertical oscillation vs. vertical β-numbers.*** Given a set $E \subset \mathbb{H}$ and a ball $B(q, r) \subset \mathbb{H}$, we recall from [Chousionis et al. 2019b, Definition 3.3] the following *vertical β-number of E in $B(q, r)$, $q \in E$*,

$$\beta_{E,\infty}(B(q,r)) := \inf_{\mathbb{W},z} \sup_{x \in B(q,r) \cap E} \frac{\operatorname{dist}(x, z \cdot \mathbb{W})}{r},$$

where the inf runs over all vertical subgroups $\mathbb{W} \subset \mathbb{H}$ and all points $z \in \mathbb{H}$. More generally, one can consider the L^p-variants

$$\beta_{E,p}(B(q,r)) := \inf_{\mathbb{W},z} \left(\frac{1}{r^3} \int_{B(q,r) \cap E} \left(\frac{\operatorname{dist}(x, z \cdot \mathbb{W})}{r} \right)^p d\mathcal{H}^3(x) \right)^{1/p}, \quad 1 \leq p < \infty,$$

assuming that E has locally finite 3-dimensional measure. If E happens to be 3-regular, then the $\beta_{E,p}$-numbers are essentially monotone in p:

$$\beta_{E,p_1}(B(q,r)) \lesssim \beta_{E,p_2}(B(q,r)), \quad q \in E, \quad 1 \leq p_1 \leq p_2 \leq \infty.$$

The next theorem shows that the vertical oscillation coefficients of Ω are always bounded by the $\beta_{E,\infty}$-numbers of $\partial\Omega$, and also *almost* bounded from above by the $\beta_{E,1}$-numbers of $\partial\Omega$. After this statement concerning general domains Ω, we will give a corollary to domains with 3-regular boundaries: in this case the word *almost* above can be omitted.

Theorem 3.12. *Let $\Omega \subset \mathbb{H}$ be an open set such that $\partial\Omega$ has locally finite 3-dimensional measure, and let $r > 0$. Then, for any $p \in \partial\Omega$ and $0 < s \leq r$,*

$$\frac{\mathrm{v}_\Omega(B(p,r))(s)}{r^4} \lesssim_\epsilon \inf_{\mathbb{W},z} \left[\frac{1}{r^3} \int_{B(p,15r) \cap \partial\Omega} \frac{d(q, z \cdot \mathbb{W})}{15r}\, d\mathcal{H}^3(q) + \epsilon\left(\sup_{q \in B(p,15r) \cap \partial\Omega} \frac{d(q, z \cdot \mathbb{W})}{15r} \right) \right] \tag{3.13}$$

for any nondecreasing function $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\epsilon(\delta) \to 0$ as $\delta \to 0$.

The same estimate for the vertical oscillation coefficient $\operatorname{osc}_\Omega(B(p, r))$ follows immediately by taking the average over $s \in (0, r]$ on the left-hand side; we will however need the sharper result later, in Section 6B. Note also that the quantity on the right-hand side of (3.13) looks like

$$\beta_{\partial\Omega,1}(B(p, 15r)) + \epsilon[\beta_{\partial\Omega,\infty}(B(p, 15r))],$$

but can be sometimes larger, as only one choice of $z, \mathbb{W}$ is made on the right-hand side of (3.13). The quantities on both sides of the inequality (3.13) are invariant under scaling and translation, so we may assume that $p = 0$ and $r = 1$. We start the proof of Theorem 3.12 with the following simple lemma.

Lemma 3.14. *Let $\Omega \subset \mathbb{H}$ be an open set. Let $H \subset \mathbb{H}$ be a vertical half-space, that is, a half-space bounded by the translate of some vertical subgroup. Then*

$$\mathrm{v}_\Omega(B(0,1))(s) \le 2\mathcal{H}^4([\Omega \Delta H] \cap B(0,3)), \quad 0 < s \le 1.$$

Proof. Let $0 \le s \le 1$. Note that $\chi_H(q) = \chi_H(q \cdot (0,0,s^2))$ for all $q \in \mathbb{H}$. Hence,

$$\begin{aligned}\mathrm{v}_\Omega(B(0,1))(s) &\le \int_{B(0,1)} |\chi_\Omega(q) - \chi_H(q) + \chi_H(q \cdot (0,0,s^2)) - \chi_\Omega(q \cdot (0,0,s^2))|\, dq \\ &\le 2 \int_{B(0,3)} |\chi_\Omega(q) - \chi_H(q)|\, dq = 2\mathcal{H}^4([\Omega \Delta H] \cap B(0,3)). \qquad \square\end{aligned}$$

Now, to conclude the proof of Theorem 3.12, it suffices to show (after scaling Ω by $\frac{1}{3}$) that there exists a vertical half-space $H \subset \mathbb{H}$ such that

$$\mathcal{H}^4([\Omega \Delta H] \cap B(0,1)) \lesssim_\epsilon \inf_{\mathbb{W},z} \left[\int_{B(0,5)\cap\partial\Omega} d(q, z\cdot\mathbb{W})\, d\mathcal{H}^3(q) + \epsilon\left(\sup_{q \in B(0,5)\cap\partial\Omega} d(q, z \cdot \mathbb{W}) \right) \right]. \tag{3.15}$$

Further, to prove (3.15), we may assume that if $P := z \cdot \mathbb{W}$ is a vertical plane minimising the right-hand side in (3.15), then $d(q, P) \le \delta := 10^{-10}$ for all $q \in B(0,5) \cap \partial\Omega$. Indeed, (3.15) is clear if this fails (with implicit constant $\sim 1/\epsilon(\delta)$). In particular, since $0 = p \in \partial\Omega$, we may write $P = z' \cdot \mathbb{W}$ with $d(0, z') \le \delta$. By left-translating both P and Ω by the inverse of z' and then rotating suitably around the t-axis, we may suppose that $P = \{(0, y, t) : y, t \in \mathbb{R}\}$ and

$$\sup_{q \in B(0,4)\cap\partial\Omega} d(q, P) \le \delta. \tag{3.16}$$

In other words, (3.16) holds for a suitable rotation of $(z')^{-1} \cdot \Omega$, but we keep denoting this set by Ω. We will no longer need the information $0 \in \partial\Omega$ in the sequel. Now, with this new notation, it suffices to prove (3.15) with $[\Omega\Delta H] \cap B(0, 1.1)$ on the left-hand side and, say, $B(0,4) \cap \partial\Omega$ on the right-hand side.

We will, in fact, show that there exists a vertical half-space $H \subset \mathbb{H}$ such that

$$\mathcal{H}^4([\Omega \Delta H] \cap B(0, 1.1)) \lesssim \int_{B(0,4)\cap\partial\Omega} d(q, P)\, d\mathcal{H}^3(q). \tag{3.17}$$

So, the L^1-based β-number of $\partial\Omega$ dominates the vertical oscillation of Ω under the a priori assumption that the L^∞-based β-number is sufficiently small. We now choose H. We denote the (closed) half-spaces bounded by P by

$$\mathbb{H}_+ := \{(x, y, t) : x \ge 0\} \quad \text{and} \quad \mathbb{H}_- := \{(x, y, t) : x \le 0\}.$$

Write U_+, U_- for the connected components of $B(0,4) \setminus P(\delta)$, where $P(\delta)$ is the closed δ-neighbourhood of P, with

$$U_+ \subset \mathbb{H}_+ \quad \text{and} \quad U_- \subset \mathbb{H}_-.$$

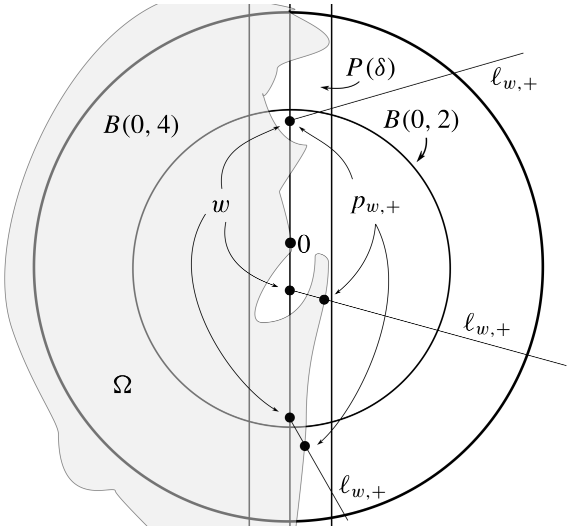

Figure 1. Various concepts in the proof of Theorem 3.12. Scenario (a) is depicted.

By (3.16), we may infer that either $U_+ \subset \Omega$ or $U_+ \cap \Omega = \varnothing$, and similarly either $U_- \subset \Omega$ or $U_- \cap \Omega = \varnothing$. The definition of H depends on which of these cases occur:

(a) If $U_- \subset \Omega$ and $U_+ \cap \Omega = \varnothing$, let $H := \mathbb{H}_-$.

(b) If $U_- \cap \Omega = \varnothing$ and $U_+ \subset \Omega$, let $H := \mathbb{H}_+$.

(c) If $U_+, U_- \subset \Omega$, let H be any vertical half-space containing $B(0,4)$.

(d) If $U_+ \cap \Omega = \varnothing = U_- \cap \Omega$, let H be any vertical half-space with $H \cap B(0,4) = \varnothing$.

The point of these choices is that always

$$[\Omega \Delta H] \cap B(0,4) \subset P(\delta), \tag{3.18}$$

as one may easily verify.

We claim that (3.17) holds for the choice of H above. To see this, we will need additional notation. For $w \in P$, let

$$\ell_w := \{w \cdot (x,0,0) : x \in \mathbb{R}\}$$

be the left-translate of the x-axis passing through w. We also define the half-lines

$$\ell_{w,+} := \ell_w \cap \mathbb{H}_+ \quad \text{and} \quad \ell_{w,-} := \ell_w \cap \mathbb{H}_-,$$

see Figure 1. To prove (3.17), we study separately the parts of $[\Omega \Delta H] \cap B(0,1.1)$ inside $\mathbb{H}_-$ and $\mathbb{H}_+$. These investigations are symmetrical, so we restrict our attention to $\mathbb{H}_+$. For notational convenience, we write $B(0,s) \cap \mathbb{H}_+ := B_+(0,s)$ in the sequel. We will apply the general integration estimate

$$\mathcal{H}^4(A) \sim \int_P \mathcal{H}^1(A \cap \ell_w)\, dw, \qquad A \subset \mathbb{H} \text{ Borel.} \tag{3.19}$$

Here "dw" refers to the 3-dimensional Hausdorff measure on P, which coincides (up to a constant) with the Lebesgue measure on P. In order to establish formula (3.19), recall that $\mathcal{H}^4$ agrees up to a multiplicative constant with the 3-dimensional Lebesgue measure and the transformation $\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{H}$, $\Phi((w_1, w_2), s) = (0, w_1, w_2) \cdot (s, 0, 0)$ has Jacobian determinant equal to 1. Hence,

$$\mathcal{H}^4(A) \sim \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \chi_A(\Phi(w, s))\, ds\, dw. \tag{3.20}$$

Next, for every $w \in P$, the map $s \mapsto \Phi(w, s) = w \cdot (s, 0, 0)$ is an isometry between $(\mathbb{R}, |\cdot|)$ and (ℓ_w, d), and thus we find that

$$\int_{-\infty}^{\infty} \chi_A(\Phi(w, s))\, ds = \int_{\ell_w} \chi_A(q)\, d\mathcal{H}^1(q) = \mathcal{H}^1(A \cap \ell_w). \tag{3.21}$$

These facts together prove (3.19). Applied to the set $A = [\Omega \Delta H] \cap B_+(0, 1.1)$, this formula then yields

$$\mathcal{H}^4([\Omega \Delta H] \cap B_+(0, 1.1)) \lesssim \int_{P \cap B(0,2.2)} \mathcal{H}^1([\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1))\, dw. \tag{3.22}$$

Here, the integration is restricted to $P \cap B(0, 2.2)$ as $\Phi(w, s)$, $w \in P$, can lie in $B(0, 1.1)$ only if $|s| \leq 1.1$, and in that case $d(\Phi(w, s), 0) \geq d(w, 0) - d(0, (s, 0, 0)) > 1.1$ if $w \in P \setminus B(0, 2.2)$; in other words, the lines ℓ_w with $w \in P \setminus B(0, 2.2)$ avoid $B(0, 1.1)$. Now, we fix $w \in P \cap B(0, 2.2)$, and we will establish a suitable pointwise bound for the integrand in (3.22). To this end:

- If $\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0, 4) = \varnothing$, set $p_{w,+} := w$.
- If $\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0, 4) \neq \varnothing$, let

  $$p_{w,+} := \max[\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0, 4)],$$

  where the max refers to the only natural ordering on $\ell_{w,+}$.

Then, by (3.18), we have in both cases

$$p_{w,+} \in \ell_{w,+} \cap P(\delta) \subset P(\delta) \cap B(0, 3), \quad w \in P \cap B(0, 2.2). \tag{3.23}$$

(If w is sufficiently close to $\partial B(0, 2.2)$, then it may happen that $\ell_{w,+} \cap P(\delta) \not\subset B(0, 2.2)$, see Figure 1. However, $\delta > 0$ has been chosen so small that the second inclusion in (3.23) holds.) Next we define

$$h_+(w) := \operatorname{dist}(p_{w,+}, P), \quad w \in P \cap B(0, 2.2).$$

The *suitable pointwise bound* for the integrand in (3.22) is

$$\mathcal{H}^1([\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1)) \leq h_+(w), \quad w \in P \cap B(0, 2.2). \tag{3.24}$$

In proving (3.24), we may evidently assume that

$$[\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1) \neq \varnothing. \tag{3.25}$$

Now, to prove (3.24), we will first argue that also

$$[\Omega \Delta H]^c \cap \ell_{w,+} \cap B(0, 4) \neq \varnothing. \tag{3.26}$$

This will follow immediately once we manage to argue that

$$U_+ \subset [\Omega \Delta H]^c, \tag{3.27}$$

since evidently $\ell_{w,+} \cap U_+ \neq \varnothing$. The proof of (3.27) depends on the scenario (a)–(d):

(a) Here $U_+ \cap \Omega = \varnothing$ and $H = \mathbb{H}_-$, so $U_+ \subset \Omega^c \cap H^c \subset [\Omega \Delta H]^c$.

(b) Here $U_+ \subset \Omega$ and $H = \mathbb{H}_+$, so $U_+ \subset \Omega \cap H \subset [\Omega \Delta H]^c$.

(c) Here $U_+ \subset \Omega$ and $B(0, 4) \subset H$, so $U_+ \subset \Omega \cap H \subset [\Omega \Delta H]^c$.

(d) Here $U_+ \cap \Omega = \varnothing$ and $H \cap B(0, 4) = \varnothing$, so $U_+ \subset \Omega^c \cap H^c \subset [\Omega \Delta H]^c$.

We have now established (3.27), and hence (3.26). Combining (3.25)–(3.26), we see that

$$p_{w,+} = \max[\ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0, 4)]$$

is well-defined, and moreover

$$[\Omega \Delta H] \cap \ell_{w,+} \cap B(0, 1.1) \subset [w, p_{w,+}], \tag{3.28}$$

where $[w, p_{w,+}]$ stands for the (horizontal) line segment connecting w and $p_{w,+}$. The point $p_{w,+}$ can be uniquely expressed as $p_{w,+} = w \cdot v_+$, where $v_+ = (x_+, 0, 0)$ for some $x_+ \geq 0$. Thus we find by the definition of the metric d that

$$x_+ \leq \|\bar{w}^{-1} w v_+\| = d(w v_+, \bar{w}) = d(p_{w,+}, \bar{w}), \quad \text{for all } \bar{w} \in P.$$

On the other hand, it holds that $d(p_{w,+}, w) = x_+$. Hence

$$h_+(w) = \operatorname{dist}(p_{w,+}, P) = d(p_{w,+}, w) = \mathcal{H}^1([w, p_{w,+}]), \tag{3.29}$$

where the last identity follows from the fact that $x \mapsto w \cdot (x, 0, 0)$ is an isometry from $(\mathbb{R}, |\cdot|)$ to (ℓ_w, d). We can now infer (3.24) from (3.28) and (3.29).

Before proceeding further, we record that the function $h_+ : P \cap B(0, 2.2) \to \mathbb{R}$ is Borel, in fact even upper semicontinuous. To see this, note that $p_{w,+}$ is always contained in the compact set

$$K := (P \cup \partial[\Omega \Delta H]) \cap \overline{B(0, 3)}$$

for $w \in P \cap B(0, 2.2)$, and, consequently, also $h_+(P \cap B(0, 2.2))$ is contained in the compact set $K' := \{\operatorname{dist}(p, P) : p \in K\} \subset \mathbb{R}$. If h_+ was not upper semicontinuous, there would exist $w \in P \cap B(0, 2.2)$, $\epsilon > 0$, and a sequence $(w_n)_n \subseteq P \cap B(0, 2.2)$ with

$$\lim_{n \to \infty} w_n = w \quad \text{and} \quad \lim_{n \to \infty} h_+(w_n) > h_+(w).$$

We may assume that the limit on the right exists by the compactness of K'. Reducing to a further subsequence if necessary, we may assume that the sequence of points $p_{w_n,+} = w_n \cdot (h_+(w_n), 0, 0)$ converges to a point $p = w \cdot v \in K$. Moreover,

$$h_+(w) < \lim_{k \to \infty} h_+(w_n) = \lim_{k \to \infty} \operatorname{dist}(p_{w_n,+}, P) = \operatorname{dist}(p, P). \tag{3.30}$$

Since $p \in \ell_{w,+} \cap \partial[\Omega \Delta H] \cap B(0,4)$ (note that $p \notin P$ by (3.30)), this contradicts the maximality in the definition of $p_{w,+}$, and the proof of the upper semicontinuity of h_+ is complete.

We now resume the proof of our goal (3.17). Combining (3.18) and (3.24), we have now established that

$$\mathcal{H}^4([\Omega \Delta H] \cap B_+(0,1.1)) \lesssim \int_{P \cap B(0,2.2)} h_+(w)\,dw = \int_{P \cap B(0,2.2)} \operatorname{dist}(p_{w,+}, P)\,dw. \tag{3.31}$$

Noting that $p_{w,+} \in \partial\Omega \cap B(0,4)$ if $\operatorname{dist}(p_{w,+}, P) \neq 0$, this conclusion is not too far from (3.17) anymore. To arrive at (3.17) from (3.31), we use the vertical projection $\pi := \pi_P$ to the subgroup P, introduced in Section 2C. The most central features of π, for now, are that $\pi^{-1}\{w\} = \ell_w$ for $w \in P$ and that π does not increase the 3-dimensional Hausdorff measure (too much): there exists a constant $C \geq 1$ such that

$$\mathcal{H}^3(\pi(A)) \leq C\mathcal{H}^3(A), \quad A \subset \mathbb{H}. \tag{3.32}$$

For a proof, see [Chousionis et al. 2019b, Lemma 3.6]. To apply these facts, let $F : P \cap B(0,2.2) \to \mathbb{H}$ be the map $F(w) := p_{w,+}$. It follows from the discussion leading to (3.29) that $F(w) = w \cdot (h_+(w), 0, 0)$ and hence F is a Borel function. We deduce that the push-forward measure $\nu := F_\sharp(\mathcal{H}^3|_{B(0,2.2)\cap P})$, defined by $\nu(A) := \mathcal{H}^3(B(0,2.2) \cap P \cap F^{-1}(A))$, is a Borel measure on $\mathbb{H}$, and we have the integration formula

$$\int_{B(0,2.2)\cap P} \operatorname{dist}(p_{w,+}, P)\,dw = \int_{\mathbb{H}} \operatorname{dist}(q, P)\,d\nu(q), \tag{3.33}$$

see for instance [Mattila 1995, Theorem 1.19]. Clearly $\nu(\mathbb{H} \setminus F(P \cap B(0,2.2))) = 0$, which shows that $\operatorname{spt} \nu \subseteq \overline{F(P \cap B(0,2.2))}$. Moreover,

$$\nu \ll \mathcal{H}^3|_{\overline{F(P\cap B(0,2.2))}}$$

with bounded density, because $F^{-1}(A) \subset \pi(A)$ for all $A \subset \mathbb{H}$, and hence

$$\nu(A) = \mathcal{H}^3([B(0,2.2) \cap P] \cap F^{-1}(A)) \leq \mathcal{H}^3(\pi(A)) \leq C\mathcal{H}^3(A), \quad A \subset \mathbb{H},$$

using (3.32). Finally, we observe that

$$\overline{F(P \cap B(0,2.2))} \subseteq \overline{B(0,3)} \cap (P \cup \partial[\Omega \Delta H]) \subseteq B(0,4) \cap (P \cup \partial\Omega).$$

The last inclusion follows from the generalities $\partial[A \cup B], \partial[A \cap B] \subset \partial A \cup \partial B$:

$$\partial[\Omega \Delta H] \subset \partial[\Omega \cap H^c] \cup \partial[\Omega^c \cap H] \subset \partial\Omega \cup \partial H.$$

In cases (a) and (b) we have $\partial H = P$, while in cases (c) and (d) the boundary of H does not intersect $B(0,4)$. Combining these observations with (3.33), we find

$$\int_{B(0,2.2)\cap P} \operatorname{dist}(p_{w,+}, P)\,dw \lesssim \int_{B(0,4)\cap\partial\Omega} \operatorname{dist}(q, P)\,d\mathcal{H}^3(q).$$

Hence the right-hand side of (3.31) is bounded by a constant times the right-hand side of (3.17). The proof of (3.17), and of Theorem 3.12, is complete. □

We conclude the section by strengthening Theorem 3.12 in the case when $\partial\Omega$ is 3-regular.

Corollary 3.34. *Assume that $\Omega \subset \mathbb{H}$ is an open set such that $\partial\Omega$ is 3-regular. Then*

$$\frac{\mathrm{v}_\Omega(B(p,r))(s)}{r^4} \lesssim \beta_{\partial\Omega,1}(B(p,30r)), \quad p \in \partial\Omega, \ \ 0 < s \le r.$$

Proof. As usual, we may assume that $p = 0 \in \partial\Omega$ and $r = 1$. The proof is based on the general observation that if $E \subset \mathbb{H}$ is 3-regular and $P \subset \mathbb{H}$ is a vertical plane with $P \cap B(0,2) \neq \varnothing$, then

$$\operatorname{dist}(q,P) \lesssim \left(\int_{B(0,2)\cap E} d(x,P)\, d\mathcal{H}^3(x) \right)^{1/4}, \quad q \in E \cap B(0,1). \tag{3.35}$$

In Euclidean space, the analogous argument can be found for example in [David and Semmes 1991, (5.4)]. To prove (3.35), denote the right-hand side by $\beta^{1/4}$, and assume to reach a contradiction that there exists a point $q \in B(0,1) \cap E$ with $d(q,P) \ge C\beta^{1/4}$ for some large constant $C \ge 1$. We record that this implies that $\frac{1}{4}C\beta^{1/4} \le 1$, since we assumed $P \cap B(0,2) \neq \varnothing$. Also, clearly

$$\operatorname{dist}(y,P) \ge \tfrac{1}{2}C\beta^{1/4}, \quad y \in E \cap B\big(q, \tfrac{1}{4}C\beta^{1/4}\big) \subset B(0,2).$$

By 3-regularity,

$$(C\beta^{1/4})^3 \lesssim \mathcal{H}^3\big(B\big(q,\tfrac{1}{4}C\beta^{1/4}\big) \cap E\big) \le \frac{2}{C\beta^{1/4}} \int_{B(q,C\beta^{1/4}/4)\cap E} d(x,P)\, d\mathcal{H}^3(x) \le \frac{2\beta^{3/4}}{C},$$

and a contradiction is hence reached for $C \ge 1$ large enough.

From (3.35) (with "1" and "2" replaced by "15" and "30", respectively), choosing $P = z \cdot \mathbb{W}$ to be the best-approximating vertical plane for $\beta_{\partial\Omega,1}(B(0,30))$, we may now infer that

$$\inf_{\mathbb{W},z} \left[\int_{B(0,30)\cap\partial\Omega} d(q, z\cdot\mathbb{W})\, d\mathcal{H}^3(q) + \left(\sup_{q \in B(0,15)\cap\partial\Omega} d(q, z\cdot\mathbb{W}) \right)^4 \right] \lesssim \beta_{\partial\Omega,1}(B(0,30)).$$

In combination with Theorem 3.12 applied to $\epsilon(\delta) := \delta^4$, this inequality completes the proof. ☐

4. Boundedness of the Riesz transform

4A. ***Definitions and restating the main theorem.*** We now begin to relate the vertical oscillation coefficients to the boundedness of the 3-dimensional Riesz transform in $\mathbb{H}$. For technical convenience, we replace the vectorial kernel $\nabla_{\mathbb{H}} G = (XG, YG)$ from the Introduction with the complex kernel

$$K(p) = XG(p) - iYG(p),$$

where $G(p) = c\|p\|_{\mathrm{Kor}}^{-2}$ is still the fundamental solution to the sub-Laplace equation $\Delta_{\mathbb{H}} u = 0$. For the time being, we will only need to know that K is smooth outside the origin and -3-homogeneous with respect to the dilations δ_r:

$$K(\delta_r(q)) = r^{-3}K(q), \quad q \in \mathbb{H} \setminus \{0\}.$$

It follows that $|K(q)| \lesssim \|q\|^{-3}$ for $q \in \mathbb{H} \setminus \{0\}$. To the kernel K we associate the ϵ-truncated SIOs

$$\mathcal{R}_\epsilon(\mu)(p) := \int_{\{q\in\mathbb{H} : \|q^{-1}\cdot p\| \ge \epsilon\}} K(q^{-1}\cdot p)\, d\mu(q),$$

where μ is any complex measure on $\mathbb{H}$ with finite total variation.

Let μ be a locally finite Borel measure on $\mathbb{H}$. We say that $\mathcal{R}$ *is bounded on* $L^2(\mu)$ if the operators $\mathcal{R}_\epsilon$ are bounded on $L^2(\mu)$ uniformly in $\epsilon > 0$:

$$\|\mathcal{R}_\epsilon(f\mu)\|_{L^2(\mu)} \leq A\|f\|_{L^2(\mu)}, \quad f \in L^1(\mu) \cap L^2(\mu), \ \epsilon > 0.$$

The measures μ relevant here are 3-regular measures on intrinsic Lipschitz graphs. For intrinsic Lipschitz graphs $\Gamma \subset \mathbb{H}$ as in Theorem 1.8, we will directly prove the $L^2(\mu)$-boundedness of $\mathcal{R}$ for the particular measure

$$\mu := \mathcal{S}^3|_\Gamma,$$

where $\mathcal{S}^3$ is the 3-dimensional spherical Hausdorff measure defined using the metric d from (2.1). This choice makes it more straightforward to use the divergence theorem, but is otherwise arbitrary. In particular, once the $L^2(\mathcal{S}^3|_\Gamma)$-boundedness of $\mathcal{R}$ has been established, then it is easy to check (or see [Chousionis et al. 2019a, Lemma 3.1]) that $\mathcal{R}$ is bounded on $L^2(\mu)$ with respect to any 3-regular measure μ supported on Γ — in particular $\mathcal{H}^3|_\Gamma$.

Here is more precisely the result we will prove.

Theorem 4.1. *Let $\mathbb{W} \subset \mathbb{H}$ be a vertical subgroup, which we identify with $\{(y,t) : y,t \in \mathbb{R}\}$. Let $\phi : \mathbb{W} \to \mathbb{R}$ be an intrinsic Lipschitz function, let*

$$\Omega := \{(x,y,t) : x > \phi(\pi_\mathbb{W}(x,y,t))\}$$

be the supergraph of ϕ, and assume that

$$\int_0^\infty \operatorname{osc}_\Omega(B(p,r))\,\frac{dr}{r} \leq C < \infty, \quad p \in \Gamma.$$

Then $\mathcal{R}$ is bounded on $L^2(\mathcal{S}^3|_{\Gamma_\phi})$.

It is easy to check that $\mathbb{H} \setminus \Gamma_\phi$ has exactly two connected components, namely the supergraph Ω above and the subgraph $\Omega' := \{(x,y,t) : x < \phi(\pi_\mathbb{W}(x,y,t))\}$. Since

$$\operatorname{osc}_\Omega(B(p,r)) = \operatorname{osc}_{\mathbb{H}\setminus\Omega}(B(p,r)) = \operatorname{osc}_{\Omega'}(B(p,r)), \quad p \in \Gamma, \ r > 0,$$

fixing the complementary component in Theorem 4.1 does not render the statement less general than that of Theorem 1.8 in the Introduction.

4B. ***Test functions and the divergence theorem.*** We will prove Theorem 4.1 by verifying the conditions of Christ's local $T(b)$ theorem [1990]. We first introduce some more notation. From now on the intrinsic Lipschitz graph $\Gamma := \Gamma_\phi$ will be fixed as in Theorem 4.1, and we write $\mu := \mathcal{S}^3|_\Gamma$. We define the complex-valued function ν on Γ as

$$\nu(w \cdot \phi(w)) := \nu_1(w \cdot \phi(w)) + i\nu_2(w \cdot \phi(w)) := \frac{1}{\sqrt{1 + (\nabla^\phi \phi(w))^2}} + i\frac{-\nabla^\phi \phi(w)}{\sqrt{1 + (\nabla^\phi \phi(w))^2}}, \tag{4.2}$$

where $\nabla^\phi \phi$ is the intrinsic gradient of ϕ. Since ϕ is intrinsic Lipschitz, $\nu(p)$ exists for μ almost every $p \in \Gamma$, because $\nabla^\phi \phi(w)$ exists for $\mathcal{S}^3$ almost every $w \in \mathbb{W}$, and the graph map $\Phi(w) = w \cdot \phi(w)$ preserves $\mathcal{S}^3$

null sets by the area formula for intrinsic Lipschitz functions, see [Citti et al. 2014, Theorem 1.6]. By similar reasoning, $\nu \in L^\infty(\mu)$.

We also define the $\mathbb{R}^2$-valued map

$$\nu_H(q) = (\nu_1(q), \nu_2(q)) = \left(\frac{1}{\sqrt{1 + (\nabla^\phi \phi(w))^2}}, \frac{-\nabla^\phi \phi(w)}{\sqrt{1 + (\nabla^\phi \phi(w))^2}} \right) \in \mathbb{R}^2, \quad q = w \cdot \phi(w).$$

Then, by [Citti et al. 2014, Corollary 4.2], ν_H is the inward-pointing horizontal normal of the intrinsic supergraph $\Omega = \{(x, y, t) : x > \phi(\pi_{\mathbb{W}}(x, y, t))\}$, expressed in the frame $\{X, Y\}$. With this notation, we have the following divergence theorem due to Franchi, Serapioni, and Serra Cassano [Franchi et al. 2001].

Theorem 4.3 (divergence theorem). *Let $V \in \mathcal{C}_c^1(\mathbb{R}^3, \mathbb{R}^2)$, and let $\Gamma = \Gamma_\phi$ be an intrinsic Lipschitz graph as above. Then*

$$-\int_\Omega \operatorname{div}_{\mathbb{H}} V(p)\, dp = c \int_\Gamma \langle V, \nu_H \rangle \, d\mathcal{S}^3,$$

where $\Omega = \{(x, y, t) : x > \phi(\pi_{\mathbb{W}}(x, y, t))\}$ and $c > 0$ is an absolute constant.

Remark 4.4. The divergence theorem in [Franchi et al. 2001] looks a little different than Theorem 4.3 above, so a few remarks are in order. First, the sub- and supergraphs of intrinsic Lipschitz graphs are $\mathbb{H}$-Caccioppoli sets by [Franchi et al. 2011, Theorem 4.18], so [Franchi et al. 2001, Corollary 7.6] gives the formula

$$-\int_\Omega \operatorname{div}_{\mathbb{H}} V(p)\, dp = c \int_{\partial_{*,\mathbb{H}}\Omega} \langle V, \nu_H \rangle \, d\mathcal{S}^3, \quad V \in \mathcal{C}_c^1(\mathbb{R}^3, \mathbb{R}^2).$$

Here $\partial_{*,\mathbb{H}}\Omega$ stands for the measure theoretic boundary of Ω; see [Franchi et al. 2001, Definition 7.4]. But for domains Ω bounded by intrinsic Lipschitz graphs Γ, the measure theoretic boundary of Ω equals the topological boundary $\partial\Omega = \Gamma$: the inclusion $\Gamma \subset \partial_{*,\mathbb{H}}\Omega$ follows from basic definitions, and the inclusion $\partial_{*,\mathbb{H}}\Omega \subset \Gamma$ follows from [Franchi et al. 2001, Lemma 7.5 (i)].

We now use the complex function ν to specify a collection of accretive test functions. Let $\psi : \mathbb{H} \to [0, 1]$ be a smooth function with $\chi_{B(0,1/2)} \le \psi \le \chi_{B(0,1)}$, and let

$$\psi_{B(p,r)}(q) := \psi(\delta_{1/r}(p^{-1} \cdot q))$$

be a rescaled version of ψ with $\operatorname{spt} \psi_{B(p,r)} \subset B(p, r)$. We record that

$$|\nabla_{\mathbb{H}} \psi_{B(p,r)}| \lesssim \frac{1}{r} \chi_{B(p,r)} \quad \text{and} \quad |\partial_t \psi_{B(p,r)}| \lesssim \frac{1}{r^2} \chi_{B(p,r)}. \tag{4.5}$$

We set

$$b_{B(p,r)} := \psi_{B(p,r)} \nu, \quad p \in \Gamma, \quad r > 0.$$

Then, recalling the formula (4.2) for ν, we note

$$\|b_{B(p,r)}\|_{L^\infty(\mu)} \lesssim 1 \quad \text{and} \quad \operatorname{Re}\left(\int b_{B(p,r)} \, d\mu \right) \gtrsim \mu(B(p, r))$$

for all $B(p,r)$ with $p \in \Gamma$ and $r > 0$. According to Main Theorem 10 in [Christ 1990], the $L^2(\mu)$ boundedness of $\mathcal{R}$ will follow once we verify the testing conditions

$$\|\mathcal{R}_\epsilon(b_B\mu)\|_{L^\infty(\mu)} \le C \quad \text{and} \quad \|\mathcal{R}_\epsilon^*(b_B\mu)\|_{L^\infty(\mu)} \le C \tag{4.6}$$

for all balls $B = B(p,r)$ centred on Γ, with $C \ge 1$ independent of $\epsilon > 0$. Here $\mathcal{R}_\epsilon^*$ is the adjoint of $\mathcal{R}_\epsilon$ with kernel

$$K^*(p) = K(p^{-1}).$$

In fact, it will be technically more convenient to verify the testing conditions (4.6) for *smooth truncations* of $\mathcal{R}$. By a smooth truncation, we mean the operator $\mathcal{R}_{s,\epsilon}$ associated to the kernel

$$K_\epsilon := \varphi_\epsilon K, \tag{4.7}$$

where φ is smooth and radially symmetric with

$$\chi_{\mathbb{H}\setminus B(0,2)} \le \varphi \le \chi_{\mathbb{H}\setminus B(0,1)},$$

and $\varphi_\epsilon(p) := \varphi(\delta_{1/\epsilon}(p))$ for $p \in \mathbb{H}$. For future reference, we remark that

$$|\nabla_{\mathbb{H}}\varphi_\epsilon| \lesssim \frac{1}{\epsilon}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)} \quad \text{and} \quad |\partial_t\varphi_\epsilon| \lesssim \frac{1}{\epsilon^2}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)}. \tag{4.8}$$

Also, if $\epsilon = 2^{-N}$ for some $N \in \mathbb{N}$, then φ_ϵ can be expanded as a series:

$$\varphi_\epsilon = \varphi_{2^{-N}} = \sum_{j\le N}(\varphi_{2^{-j}} - \varphi_{2^{-j+1}}) =: \sum_{j \le N}\eta_j, \tag{4.9}$$

noting that η_j is supported on the annulus $B(0,2^{-j+2}) \setminus B(0,2^{-j})$. We will assume without loss of generality that ϵ has this form in the sequel.

Now, instead of (4.6), we will check that

$$\|\mathcal{R}_{s,\epsilon}(b_B\mu)\|_{L^\infty(\mu)} \le C \quad \text{and} \quad \|\mathcal{R}_{s,\epsilon}^*(b_B\mu)\|_{L^\infty(\mu)} \le C \tag{4.10}$$

for all balls B centred on Γ, and for some constant $C \ge 1$ independent of $\epsilon > 0$. It is easy to check that

$$|\mathcal{R}_{s,\epsilon}(f)(p) - \mathcal{R}_\epsilon(f)(p)| \lesssim M_\mu(f)(p)$$

for all $f \in L^\infty(\mu)$ and $p \in \Gamma$, where M_μ is the Hardy–Littlewood maximal function

$$M_\mu f(p) = \sup_{r>0} \fint_{B(p,r)} |f(q)|\,d\mu(q).$$

Since $\|M_\mu(b_B\mu)\|_{L^\infty(\mu)} \lesssim \|b_B\|_{L^\infty(\mu)} \lesssim 1$, we see that (4.10) implies (4.6).

4C. *Initial reductions for verifying the testing conditions.* We start by verifying the first condition in (4.10), that is, proving

$$|\mathcal{R}_{s,\epsilon}(b_B\mu)(p)| \le C, \quad p \in \Gamma. \tag{4.11}$$

The arguments concerning the second testing condition in (4.10) will be very similar. To prove (4.11), we make a few reductions, which show that it suffices to verify (4.11) for $p = 0 \in \Gamma$ and for a ball B with $\mathrm{dist}(0, B) \leq \mathrm{diam}(B) = 1$.

As a first step, we argue that it suffices to consider $p \in \Gamma$ with

$$\mathrm{dist}(p, B) \leq \mathrm{diam}(B). \tag{4.12}$$

Indeed, (4.11) follows from standard kernel estimates if $\mathrm{dist}(p, B) > \mathrm{diam}(B)$. To see this, we write $B = B(p_0, r)$ and fix $p \in \Gamma$ with $\mathrm{dist}(p, p_0) \geq 2r$. Then $d(p, q) \geq r$ for all $q \in B$, and consequently

$$|\mathcal{R}_{s,\epsilon}(b_B)(p)| \lesssim \|b_B\|_{L^\infty(\mu)} \int_B \frac{d\mu(q)}{d(p,q)^3} \lesssim \frac{\mu(B)}{r^3} \sim 1.$$

So, in the sequel we may assume that (4.12) holds.

Next, we argue that it suffices to consider the case $p = 0 \in \Gamma$. Indeed, note first that

$$\tilde{\mu} := \mathcal{S}^3|_{p^{-1}\cdot\Gamma} = (\tau_{p^{-1}})_\sharp \mathcal{S}^3|_\Gamma = (\tau_{p^{-1}})_\sharp \mu.$$

Then, write

$$\tilde{b}_{p^{-1}\cdot B} := \psi_{p^{-1}\cdot B} \nu_{p^{-1}\cdot\Gamma},$$

where $\nu_{p^{-1}\cdot\Gamma}$ is the analogue of ν (recall (4.2)) for the left-translated intrinsic Lipschitz graph $p^{-1} \cdot \Gamma$. In particular,

$$\nu_{p^{-1}\cdot\Gamma}(p^{-1} \cdot q) = \nu(q), \quad q \in \Gamma,$$

so that

$$\tilde{b}_{p^{-1}\cdot B}(p^{-1} \cdot q) = \psi_B(q)\nu(q) = b_B(q), \quad q \in \Gamma.$$

Using this equation, we infer that

$$\begin{aligned}
\mathcal{R}_{s,\epsilon}(\tilde{b}_{p^{-1}\cdot B}\tilde{\mu})(0) &= \int_{p^{-1}\cdot\Gamma} K_\epsilon(q^{-1})\tilde{b}_{p^{-1}\cdot B}(q)\, d\mathcal{S}^3(q) \\
&= \int K_\epsilon(q^{-1})\tilde{b}_{p^{-1}\cdot B}(q)\, d[(\tau_{p^{-1}})_\sharp\mu](q) \\
&= \int_\Gamma K_\epsilon((p^{-1}\cdot q)^{-1})\tilde{b}_{p^{-1}\cdot B}(p^{-1}\cdot q)\, d\mathcal{S}^3(q) \\
&= \int_\Gamma K_\epsilon(q^{-1}\cdot p) b_B(q)\, d\mathcal{S}^3(q) = \mathcal{R}_{s,\epsilon}(b_B\mu)(p).
\end{aligned}$$

This shows that, to find a bound for $\mathcal{R}_{s,\epsilon}(b_B\mu)(p)$, it suffices to do so for $\mathcal{R}_{s,\epsilon}(\tilde{b}_{p^{-1}\cdot B}\tilde{\mu})(0)$. But the intrinsic Lipschitz graph $p^{-1} \cdot \Gamma$ has all the same properties as we assumed from Γ in Theorem 4.1: the intrinsic Lipschitz constants do not change, nor do the bounds for the vertical oscillation numbers, recalling Lemma 3.4, so we may assume that $p = 0 \in \Gamma$.

Finally, we argue that we may assume $\mathrm{diam}(B) = 1$. For this purpose, we first note that

$$r^3 \cdot \delta_{r\sharp}\mu = S^3|_{\delta_r(\Gamma)} =: \tilde{\mu}. \tag{4.13}$$

Indeed, if $A \subset \delta_r(\Gamma)$, then $\delta_{1/r}(A) \subset \Gamma$, hence

$$r^3 \cdot (\delta_{r\sharp}\mu)(A) = r^3 \mathcal{S}^3(\Gamma \cap \delta_{1/r}(A)) = \mathcal{S}^3(\delta_r(\Gamma) \cap A) = \tilde{\mu}(A),$$

which proves (4.13). Now, let $r := \mathrm{diam}(B)$, and let $\tilde{b}_{\delta_{1/r}(B)} := \psi_{\delta_{1/r}(B)} \cdot \nu_{\delta_{1/r}(\Gamma)}$, where $\nu_{\delta_{1/r}(\Gamma)}$ stands for the analogue of ν for the dilated intrinsic Lipschitz graph $\delta_{1/r}(\Gamma)$. In particular, it is easy to check that

$$\tilde{b}_{\delta_{1/r}(B)}(\delta_{1/r}(q)) = b_B(q), \quad q \in \Gamma.$$

We also record the equation

$$K_\epsilon(\delta_r(q)) = \varphi_\epsilon(\delta_r(q))K(\delta_r(q)) = r^{-3} \cdot \varphi_{\epsilon/r}(q)K(q) = r^{-3}K_{\epsilon/r}(q),$$

using the definition of the kernel K_ϵ from (4.7) and the -3-homogeneity of K. Then we may use (4.13) and the equations above to get

$$\begin{aligned}
\mathcal{R}_{s,\epsilon/r}(\tilde{b}_{\delta_{1/r}(B)}\tilde{\mu})(0) &= \int_{\delta_{1/r}(\Gamma)} K_{\epsilon/r}(q^{-1})\tilde{b}_{\delta_{1/r}(B)}(q)\, d\mathcal{S}^3(q) \\
&= r^{-3}\int_\Gamma K_{\epsilon/r}(q^{-1})\tilde{b}_{\delta_{1/r}(B)}(q)\, d\delta_{(1/r)\sharp}\mu(q) \\
&= r^{-3}\int_\Gamma K_{\epsilon/r}([\delta_{1/r}(q)]^{-1})\tilde{b}_{\delta_{1/r}(B)}(\delta_{1/r}(q))\, d\mathcal{S}^3(q) \\
&= \int_\Gamma K_\epsilon(q^{-1})b_B(q)\, d\mathcal{S}^3(q) = \mathcal{R}_{s,\epsilon}(b_B\mu)(0).
\end{aligned}$$

So, to estimate $\mathcal{R}_{s,\epsilon}(b_B\mu)(0)$ it suffices to estimate $\mathcal{R}_{s,\epsilon/r}(\tilde{b}_{\delta_{1/r}(B)}\tilde{\mu})(0)$. But, arguing as in the previous reduction, $\delta_{1/r}(\Gamma)$ is an intrinsic Lipschitz graph with the same properties as Γ. So in the sequel we assume that $\mathrm{diam}(B) = 1$.

Summarising, we have reduced the proof of (4.11) to the case

$$p = 0 \in \Gamma \quad \text{and} \quad \mathrm{dist}(0, B) \leq \mathrm{diam}(B) = 1. \tag{4.14}$$

4D. ***Verifying the testing conditions.*** With the above reductions in mind, we start the proof of (4.11). We record that

$$K(q^{-1}) = -\widetilde{X}G(q) + i\widetilde{Y}G(q), \quad q \in \mathbb{H} \setminus \{0\}, \tag{4.15}$$

as a straightforward computation shows. Hence, we may write

$$\begin{aligned}
&\mathcal{R}_{s,\epsilon}(b_B\mu)(0) \\
&\quad = \int_\Gamma \varphi_\epsilon(q)(-\widetilde{X}G(q) + i\widetilde{Y}G(q))b_B(q)\, d\mathcal{S}^3(q) \\
&\quad = -\int_\Gamma \langle \psi_B(q)\varphi_\epsilon(q)\widetilde{\nabla}_{\mathbb{H}}G(q), \nu_H(q)\rangle\, d\mathcal{S}^3(q) + i\int_\Gamma \langle \psi_B(q)\varphi_\epsilon(q)(\widetilde{Y}G(q), -\widetilde{X}G(q)), \nu_H(q)\rangle\, d\mathcal{S}^3(q) \\
&\quad =: I_1 + iI_2,
\end{aligned}$$

recalling the notation from Section 2A. In order to evaluate I_1 and I_2, we will apply the divergence theorem (Theorem 4.3) to the vector fields

$$V_1 := (\psi_B\varphi_\epsilon \widetilde{X}G, \psi_B\varphi_\epsilon \widetilde{Y}G) \in \mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R}^2) \quad \text{and} \quad V_2 := (\psi_B\varphi_\epsilon \widetilde{Y}G, -\psi_B\varphi_\epsilon \widetilde{X}G) \in \mathcal{C}_c^\infty(\mathbb{R}^3,\mathbb{R}^2),$$

respectively.

4D1. *Estimate for I_1.* After an application of Theorem 4.3, I_1 becomes

$$\begin{aligned} I_1 &= -c\int_\Omega \operatorname{div}_{\mathbb{H}}(\psi_B(q)\varphi_\epsilon(q)\widetilde{\nabla}_{\mathbb{H}}G(q))\,dq \\ &= -c\int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), \widetilde{\nabla}_{\mathbb{H}}G(q)\rangle\,dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q)\operatorname{div}_{\mathbb{H}}\widetilde{\nabla}_{\mathbb{H}}G(q)\,dq \\ &=: -cI_1^1 - cI_1^2. \end{aligned}$$

For I_1^1, we infer from (4.5), (4.8), and the product rule that

$$|\nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)| \lesssim \frac{1}{\epsilon}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)} + \chi_B.$$

Since moreover $|\widetilde{\nabla}_{\mathbb{H}}G(q)| \lesssim \|q\|^{-3}$ (this follows from (4.15) for instance), we get

$$\left|\int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), \widetilde{\nabla}_{\mathbb{H}}G(q)\rangle\,dq\right| \lesssim \frac{1}{\epsilon}\int_{B(0,2\epsilon)\setminus B(0,\epsilon)} \|q\|^{-3}dq + \int_B \|q\|^{-3}\,dq \lesssim 1. \tag{4.16}$$

To handle the term I_1^2, we observe the following general relationship between left and right divergence:

$$\operatorname{div}_{\mathbb{H}}(V_1,V_2) = \widetilde{\operatorname{div}}_{\mathbb{H}}(V_1,V_2) + \partial_t(-yV_1 + xV_2), \quad (V_1,V_2)\in\mathcal{C}^1(\mathbb{R}^3,\mathbb{R}^2). \tag{4.17}$$

It follows that

$$I_1^2 = \int_\Omega (\psi_B\varphi_\epsilon)(q)\widetilde{\operatorname{div}}_{\mathbb{H}}\widetilde{\nabla}_{\mathbb{H}}G(q)\,dq + \int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t(-y\widetilde{X}G(q) + x\widetilde{Y}G(q))\,dq.$$

Here

$$\widetilde{\operatorname{div}}_{\mathbb{H}}\widetilde{\nabla}_{\mathbb{H}}G(q) = \widetilde{\Delta}_{\mathbb{H}}G(q) = 0, \quad q\in \operatorname{spt}\varphi_\epsilon,$$

since G is simultaneously the fundamental solution for both operators $\Delta_{\mathbb{H}}$ and $\widetilde{\Delta}_{\mathbb{H}}$. So the first term vanishes. Consequently,

$$I_1^2 =: \int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t\widetilde{K}(q)\,dq = \int_\Omega \partial_t(\psi_B\varphi_\epsilon\widetilde{K})(q)\,dq - \int_\Omega \partial_t(\psi_B\varphi_\epsilon)(q)\widetilde{K}(q)\,dq, \tag{4.18}$$

where $\widetilde{K}$ is the -2-homogeneous kernel

$$\widetilde{K}(z,t) = -y\widetilde{X}G(z,t) + x\widetilde{Y}G(z,t) = \frac{8t|z|^2}{\|(z,t)\|_{\mathrm{Kor}}^6}, \quad z=(x,y).$$

The main term in (4.18) is the first one, because the second one can be treated in the same fashion as I_1^1 above. Indeed, simply notice from (4.5), (4.8), and the product rule that

$$|\partial_t(\psi_B\varphi_\epsilon)(q)| \lesssim \frac{1}{\epsilon^2}\chi_{B(0,2\epsilon)\setminus B(0,\epsilon)} + \chi_B,$$

so that

$$\begin{aligned}\left|\int_\Omega \partial_t(\psi_B\varphi_\epsilon)(q)\widetilde{K}(q)\,dq\right| &\lesssim \frac{1}{\epsilon^2}\int_{B(0,2\epsilon)\setminus B(0,\epsilon)} |\widetilde{K}(q)|\,dq + \int_B |\widetilde{K}(q)|\,dq\\ &\lesssim \frac{1}{\epsilon^4}\mathcal{H}^4(B(0,2\epsilon)) + 1 \sim 1.\end{aligned}$$

Finally, the first term in (4.18) is handled using (4.9) and Lemma 3.8 (noting that $\operatorname{spt}(\psi_B\eta_j\widetilde{K}) \subset B(0,s)$ for any $s \in [2^{-j+2}, 2^{-j+3}]$) to yield

$$\begin{aligned}\left|\int_\Omega \partial_t(\psi_B\varphi_\epsilon\widetilde{K})(q)\,dq\right| &\le \sum_{j\le N}\left|\int_\Omega \partial_t(\psi_B\eta_j\widetilde{K})(q)\,dq\right|\\ &\lesssim \sum_{j\le N} 2^{-4j}\|\partial_t(\psi_B\eta_j\widetilde{K})\|_\infty \int_{2^{-j+2}}^{2^{-j+3}} \operatorname{osc}_\Omega(B(0,10s))\,\frac{ds}{s}.\end{aligned}$$

From the product rule, noting that

- $\operatorname{spt}\eta_j \subset B(0,2^{-j+2}) \setminus B(0,2^{-j})$,
- $\operatorname{spt}\psi_B \subset B \subset B(0,2)$ by (4.14),
- $\widetilde{K}$ is -2-homogeneous, and
- $\partial_t\widetilde{K}$ is -4-homogeneous,

we see that

$$\|\partial_t(\psi_B\eta_j\widetilde{K})\|_\infty \lesssim \begin{cases} 2^{4j}, & j \ge -1,\\ 0, & j < -1.\end{cases}$$

To verify the last bullet point, one can simply compute that $\partial_t\widetilde{K}$ is the kernel

$$\partial_t\widetilde{K}(z,t) = 8\frac{|z|^2(|z|^4 - 32t^2)}{\|(z,t)\|_{\mathrm{Kor}}^{10}}, \qquad z = (x,y).$$

Summarising the estimate above, we have now shown that

$$\begin{aligned}|I_1| &\lesssim 1 + \sum_{-1\le j\le N}\int_{2^{-j+2}}^{2^{-j+3}} \operatorname{osc}_\Omega(B(0,10s))\,\frac{ds}{s}\\ &\lesssim 1 + \int_0^\infty \operatorname{osc}_\Omega(B(0,s))\,\frac{ds}{s}\\ &\le 1 + C.\end{aligned}$$

4D2. *Estimate for I_2.* We move to the term

$$\begin{aligned}I_2 &= \int_\Gamma \langle \psi_B(q)\varphi_\epsilon(q)(\widetilde{Y}G(q), -\widetilde{X}G(q)), \nu_H(q)\rangle \, d\mathcal{S}^3(q)\\ &= -c\int_\Omega \operatorname{div}_{\mathbb{H}}(\psi_B\varphi_\epsilon(\widetilde{Y}G, -\widetilde{X}G))(q)\, dq\\ &= -c\int_\Omega \langle \nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), (\widetilde{Y}G(q), -\widetilde{X}G(q))\rangle\, dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q)\operatorname{div}_{\mathbb{H}}(\widetilde{Y}G, -\widetilde{X}G)(q)\, dq\\ &=: -cI_2^1 - cI_2^2,\end{aligned}$$

where the divergence theorem was applied. The term I_2^1 can be handled precisely as I_1^1 above; see (4.16). So we concentrate on the term I_2^2. Once again, due to the presence of the right-invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$, it is useful to consider the right divergence instead of the left one. Recalling (4.17) and setting $p = (x, y, t)$, we write

$$\begin{aligned}\operatorname{div}_{\mathbb{H}}(\widetilde{Y}G, -\widetilde{X}G)(p) &= \widetilde{\operatorname{div}}_{\mathbb{H}}(\widetilde{Y}G, -\widetilde{X}G)(p) + \partial_t(-y\widetilde{Y}G - x\widetilde{X}G)(p)\\ &= (\widetilde{X}\widetilde{Y}G - \widetilde{Y}\widetilde{X}G)(p) + \partial_t\hat{K}(p)\\ &= -\partial_t G(p) + \partial_t\hat{K}(p).\end{aligned}$$

Here $\hat{K}$ is yet another -2-homogeneous kernel with explicit expression

$$\hat{K}(z,t) = \frac{2|z|^4}{\|(z,t)\|_{\mathrm{Kor}}^6}, \qquad (z,t) \in \mathbb{H}\setminus\{0\}.$$

In other words,

$$I_2^2 = -\int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t G(q)\, dq + \int_\Omega (\psi_B\varphi_\epsilon)(q)\partial_t\hat{K}(q)\, dq. \tag{4.19}$$

From this point on, the treatment of both terms can be continued as on line (4.18) above. The only facts we needed about the kernel $\widetilde{K}$ there was that it is -2-homogeneous and its t-derivative is -4-homogeneous. These properties are also satisfied for G and $\hat{K}$. In fact, the t-derivatives are given by

$$\partial_t G(z,t) = \frac{16t}{\|(z,t)\|_{\mathrm{Kor}}^6} \quad\text{and}\quad \partial_t\hat{K}(z,t) = -\frac{96|z|^4 t}{\|(z,t)\|_{\mathrm{Kor}}^{10}}.$$

Continuing as in (4.18), and afterwards, we obtain

$$|I_2^2| \lesssim 1 + \int_0^\infty \operatorname{osc}_\Omega(B(0,s))\,\frac{ds}{s} \le 1 + C.$$

This concludes the proof of (4.11) as we have shown that

$$\|\mathcal{R}_{s,\epsilon}(b_B\mu)\|_{L^\infty(\mu)} \le C. \tag{4.20}$$

4D3. *The adjoint.* To prove Theorem 4.1, it remains to establish the bound analogous to (4.20) for the adjoint $\mathcal{R}_{s,\epsilon}^*$. Arguing as in Section 4C, we may assume that the conditions in (4.14) are in force. In other words, it suffices to show that

$$|\mathcal{R}_{s,\epsilon}^*(b_B\mu)(0)| \le C,$$

where $B \subset \mathbb{H}$ is a ball with $\operatorname{dist}(0, B) \leq 1 = \operatorname{diam}(B)$, and $0 \in \Gamma$. By definition,

$$\begin{aligned}\mathcal{R}^*_{s,\epsilon}(b_B\mu)(0) &= \int_\Gamma \varphi_\epsilon(q)(XG(q) - iYG(q))b_B(q)\,d\mathcal{S}^3(q)\\ &= \int_\Gamma \langle(\psi_B\varphi_\epsilon)(q)\nabla_{\mathbb{H}}G(q), \nu_H(q)\rangle\,d\mathcal{S}^3 + i\int_\Gamma \langle(\psi_B\varphi_\epsilon)(q)(-YG, XG)(q), \nu_H(q)\rangle\,d\mathcal{S}^3(q)\\ &=: J_1 + iJ_2.\end{aligned}$$

The situation is now similar to, but slightly simpler than, the one we have already treated. After we apply the divergence theorem and use the product rule, we have

$$J_1 = -c\int_\Omega \langle\nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), \nabla_{\mathbb{H}}G(q)\rangle\,dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q)\operatorname{div}_{\mathbb{H}}\nabla_{\mathbb{H}}G(q)\,dq.$$

The second term vanishes, as $\operatorname{div}_{\mathbb{H}}\nabla_{\mathbb{H}}G(q) = \Delta_{\mathbb{H}}G(q) = 0$ for $q \in \operatorname{spt}\varphi_\epsilon$. The first term can be estimated as in (4.16).

Concerning J_2, the divergence theorem gives

$$J_2 = -c\int_\Omega \langle\nabla_{\mathbb{H}}(\psi_B\varphi_\epsilon)(q), (-YG, XG)(q)\rangle\,dq - c\int_\Omega (\psi_B\varphi_\epsilon)(q)\operatorname{div}_{\mathbb{H}}(-YG, XG)(q)\,dq.$$

Once more, the first term is estimated using the argument from (4.16). In the second term, we find that

$$\operatorname{div}_{\mathbb{H}}(-YG, XG)(q) = -XYG(q) + YXG(q) = -\partial_t G(q), \quad q \in \mathbb{H}\setminus\{0\}.$$

From this point on, the estimates are the same as for the term I_2^2 above; see (4.19). We have now established that

$$\|\mathcal{R}^*(b_B\mu)\|_{L^\infty(\mu)} \leq C,$$

and the proof of Theorem 4.1 is complete. □

5. Application: intrinsic Lipschitz graphs with extra vertical regularity

In this section, we prove Theorem 1.10, which we restate below.

Theorem 5.1. *Let $\phi : \mathbb{W} \to \mathbb{R}$ be an intrinsic Lipschitz function which satisfies the following Hölder regularity in the vertical direction*:

$$|\phi(y,t) - \phi(y,s)| \leq H|t-s|^{(1+\tau)/2}, \quad |s-t| \leq 1 \tag{5.2}$$

and

$$|\phi(y,t) - \phi(y,s)| \leq H|t-s|^{(1-\tau)/2}, \quad |s-t| > 1, \tag{5.3}$$

where $H \geq 1$ and $0 < \tau \leq 1$. Then R is bounded on $L^2(\mathcal{H}^3|_{\Gamma_\phi})$.

As a corollary, we recover the main theorem of [Chousionis et al. 2019a] for the Riesz transform.

Corollary 5.4. *Let $\mathbb{W} \subset \mathbb{H}$ be a vertical plane, let $\alpha > 0$, and let $\phi : \mathbb{W} \to \mathbb{V}$ be a compactly supported $C^{1,\alpha}(\mathbb{W})$ in the sense of [Chousionis et al. 2019a]. Then R is bounded on $L^2(\mathcal{H}^3|_{\Gamma_\phi})$.*

Proof. By [Chousionis et al. 2019a, Proposition 4.2], an intrinsic $C^{1,\alpha}$-function ϕ satisfies (5.2) with exponent $\tau = \alpha$. Since ϕ is continuous and compactly supported, (5.3) is also satisfied if the constant H is chosen large enough. To apply Theorem 5.1, we still need to argue that ϕ is intrinsic Lipschitz: this is the content of [Chousionis et al. 2019a, Remark 2.18]. □

Besides the compact support assumption, a notable difference between Theorem 5.1 and the main theorem of [Chousionis et al. 2019a] is that the intrinsic $C^{1,\alpha}$-condition implies extra regularity in both vertical and horizontal directions. The conditions (5.2)–(5.3), on the other hand, imply nothing about the horizontal behaviour of ϕ. To emphasise this, we give another corollary of Theorem 5.1.

Corollary 5.5. *Let $\phi_0 : \mathbb{R} \to \mathbb{R}$ be a (Euclidean) Lipschitz function, and let $\phi(0, y, t) := \phi_0(y)$. Then $\mathcal{R}$ is bounded on $L^2(\mu)$, where μ is $\mathcal{H}^3$ restricted to Γ_ϕ.*

Proof. We first note that ϕ is intrinsic Lipschitz because

$$|\phi(0, y, t) - \phi(0, y', t')| \lesssim |y - y'| \leq \|\pi_{\mathbb{W}}(\Phi(0, y', t')^{-1} \cdot \Phi(0, y, t))\|,$$

where $\Phi(0, y, t) = (0, y, t) \cdot (\phi(0, y, t), 0, 0)$ is the graph map parametrising Γ_ϕ. Conditions (5.2)–(5.3) are trivially satisfied, so the claim follows from Theorem 5.1. □

5A. *Proof of Theorem 5.1.* The proof is based on the following lemma.

Lemma 5.6. *Assume $\phi : \mathbb{W} := \{(0, y, t) : y, t \in \mathbb{R}\} \to \mathbb{R}$ is intrinsic Lipschitz and satisfies (5.2)–(5.3). Then*

$$\operatorname{osc}_\Omega(B(p, r)) \lesssim H^4 \min\{r^\tau, r^{-\tau}\}, \quad p \in \Gamma_\phi, \ \ 0 < r < \infty, \tag{5.7}$$

where $\Omega = \{(x, y, t) : x > \phi(\pi_{\mathbb{W}}(x, y, t))\}$, and the implicit constants depend on the intrinsic Lipschitz constants of ϕ.

By Theorem 4.1, the lemma above will prove Theorem 5.1.

Proof of Lemma 5.6. The plan is to first use (5.2) to establish the bound

$$\operatorname{osc}_\Omega(B(p, r)) \lesssim H^4 r^\tau, \quad p \in \Gamma_\phi, \ \ 0 < r \leq 1. \tag{5.8}$$

The second bound in (5.7) will follow by a similar argument from (5.3) for $r > 1$.

Write $\Gamma := \Gamma_\phi$, and fix $0 < r \leq 1$ and $0 < s \leq r$. We claim

$$\mathrm{v}_\Omega(B(p, r))(s) = \int_{B(p,r)\cap\Gamma(Hr^{1+\tau})} |\chi_\Omega(q) - \chi_\Omega(q \cdot (0, 0, s^2))| \, dq, \tag{5.9}$$

where $\Gamma(Hr^{1+\tau})$ denotes the $(Hr^{1+\tau})$-neighbourhood of Γ. To prove this, it suffices to show that if $q \in B(p, r)$ with $\operatorname{dist}(q, \Gamma) > Hr^{1+\tau}$, then

$$\chi_\Omega(q) = \chi_\Omega(q \cdot (0, 0, s^2)).$$

Indeed, assume to the contrary that $q = (x, y, t) \in B(p, r)$ can be found with $\mathrm{dist}(q, \Gamma) > Hr^{1+\tau}$ and $\chi_\Omega(q) \neq \chi_\Omega(q \cdot (0, 0, s^2))$. This has two consequences: First, in particular,

$$\begin{aligned} |x - \phi(\pi_{\mathbb{W}}(x, y, t))| &= d((x, 0, 0), \phi(\pi_{\mathbb{W}}(q))) \\ &= d(\pi_{\mathbb{W}}(q) \cdot (x, 0, 0), \pi_{\mathbb{W}}(q) \cdot \phi(\pi_{\mathbb{W}}(q))) \\ &= d(q, \Phi(\pi_{\mathbb{W}}(q))) > Hr^{1+\tau}, \end{aligned}$$

where $\Phi(w) = w \cdot \phi(w)$ is the graph map parametrising Γ. Second, there exists $0 \leq u \leq s$ such that $(x, y, t + u^2) = q \cdot (0, 0, u^2) \in \Gamma$, so in particular,

$$x = \phi(\pi_{\mathbb{W}}(q \cdot (0, 0, u^2))).$$

Combining the information above,

$$|\phi(\pi_{\mathbb{W}}(x, y, t + u^2)) - \phi(\pi_{\mathbb{W}}(x, y, t))| > Hr^{1+\tau}.$$

Spelling out the definition of $\pi_{\mathbb{W}}$, this is equivalent to

$$Hr^{1+\tau} < \left|\phi\big(0, y, t + u^2 + \tfrac{1}{2}xy\big) - \phi\big(0, y, t + \tfrac{1}{2}xy\big)\right| \leq Hu^{1+\tau} \leq Hs^{1+\tau} \leq Hr^{1+\tau}.$$

We have reached a contradiction, and hence proved (5.9).

It follows from (5.9) that

$$\mathrm{osc}_\Omega(B(p, r)) = \frac{1}{r^4} \int_0^r \mathrm{v}_\Omega(B(p, r))(s)\, ds \lesssim \frac{\mathcal{H}^4(B(p, r) \cap \Gamma(Hr^{1+\tau}))}{r^4}.$$

To conclude the proof, we find a maximal $Hr^{1+\tau}$-separated set $S \subset B(p, 2Hr) \cap \Gamma$; note that this step uses the assumption $r \leq 1$, so that $r^{1+\tau} \leq r$. Since Γ is 3-regular,

$$\mathrm{card}\, S \lesssim r^{-3\tau}. \tag{5.10}$$

On the other hand, the balls $B(q, 10Hr^{1+\tau})$, $q \in S$, cover $B(p, r) \cap \Gamma(Hr^{1+\tau})$, whence

$$\mathrm{osc}_\Omega(B(p, r)) \lesssim \frac{\mathcal{H}^4(B(p, r) \cap \Gamma(Hr^{1+\tau}))}{r^4} \lesssim (\mathrm{card}\, S) \frac{(Hr^{1+\tau})^4}{r^4} \lesssim H^4 r^\tau.$$

This proves (5.8).

To prove the second bound in (5.7), one fixes $r \geq 1$ and proceeds as above, using (5.3) instead of (5.2). One first obtains

$$\mathrm{v}_\Omega(B(p, r))(s) = \int_{B(p,r) \cap \Gamma(Hr^{1-\tau})} |\chi_\Omega(q) - \chi_\Omega(q \cdot (0, 0, s^2))|\, dq$$

This leads to $\mathrm{osc}_\Omega(B(p, r)) \lesssim \mathcal{H}^4(B(p, r) \cap \Gamma(Hr^{1-\tau}))/r^4$. Since $r \geq 1$, one has $r^{1-\tau} \leq r$. One finally chooses a maximal $Hr^{1-\tau}$-separated set $S \subset B(p, 2Hr) \cap \Gamma$, and finds that (5.10) gets replaced by $\mathrm{card}\, S \lesssim r^{3\tau}$. This gives $\mathrm{osc}_\Omega(B(p, r)) \lesssim H^4 r^{-\tau}$, as desired. □

6. Problems and remarks

6A. ***Carleson packing conditions for the vertical oscillation coefficients?*** Theorem 1.8 guarantees the L^2-boundedness of R on intrinsic Lipchitz graphs $\Gamma = \partial\Omega \subset \mathbb{H}$ satisfying the uniform condition

$$\int_0^\infty \operatorname{osc}_\Omega(B(p,r))\,\frac{dr}{r} \lesssim 1, \qquad p \in \partial\Omega. \tag{6.1}$$

A comparison with analogous results in Euclidean space, in particular those in [David and Semmes 1991], suggests that it might be possible to relax (6.1) to a *Carleson packing condition* for the vertical oscillation coefficients, such as

$$\int_{\partial\Omega\cap B(p_0,R)} \int_0^R \operatorname{osc}_\Omega(B(p,r))^\eta\,\frac{dr}{r}\,d\mathcal{H}^3(p) \lesssim R^3, \qquad p_0 \in \partial\Omega,\ \ 0 < R \le \operatorname{diam}\partial\Omega. \tag{Car(η)}$$

Here $\eta \ge 1$ is a parameter, and evidently the condition (Car(η)) gets weaker as η increases. Two questions now arise:

Question 3. For which parameters $\eta \ge 1$ — if any — does the following hold? Assume that $\Gamma = \partial\Omega \subset \mathbb{H}$ is an intrinsic Lipschitz graph satisfying (Car(η)). Then R is bounded on $L^2(\mathcal{H}^3|_\Gamma)$.

Question 4. For which parameters $\eta \ge 1$ — if any — does the following hold? Every intrinsic Lipschitz graph $\Gamma \subset \mathbb{H}$ satisfies (Car(η)).

We have no further insight on either of the questions at the moment. We conjecture that every intrinsic Lipschitz graph $\Gamma \subset \mathbb{H}$ satisfies (Car(η)) for $\eta \ge 4$.

6B. ***A connection between vertical perimeter and β-numbers.*** Let $\Omega \subset \mathbb{H}$ be an open set with 3-regular boundary, and let $1 \le p < \infty$. Recall from Remark 3.2 that the L^p-vertical perimeter of Ω in a ball $B(q,r)$, $q \in \partial\Omega$, is the quantity

$$\wp_{\Omega,p}(B(q,r)) := \left(\int_0^\infty \left(\frac{\mathrm{v}_\Omega(B(q,r))(s)}{s}\right)^p \frac{ds}{s}\right)^{1/p}.$$

Given Corollary 3.34, it is reasonable to expect an inequality between $\wp_{\Omega,p}$ and some quantity defined via the vertical β-numbers $\beta_{\partial\Omega,1}$. Such an inequality is given by the following proposition.

Proposition 6.2. *Let $\Omega \subset \mathbb{H}$ be a nonempty open set with 3-regular boundary, and let $p_0 \in \partial\Omega$ and $0 < R \le \operatorname{diam}\partial\Omega$. Then*

$$\wp_{\Omega,p}(B(p_0,R)) \lesssim R^3 + \int_{\partial\Omega\cap B(p_0,CR)} \left(\int_0^R \beta_{\partial\Omega,1}(B(q,Cr))^p\,\frac{dr}{r}\right)^{1/p} d\mathcal{H}^3(q),$$

where $C \ge 1$ is an absolute constant.

Proof. Fix $0 < r \le R$. We start by arguing that

$$\frac{\mathrm{v}_\Omega(B(p_0,R))(r)}{r} \lesssim \int_{\partial\Omega\cap B(p_0,CR)} \beta_{\partial\Omega,1}(B(p,Cr))\,d\mathcal{H}^3(p). \tag{6.3}$$

To this end, let $\mathcal{B}_r$ be a finite family of balls of radius r covering $B(p_0, R)$ such that the concentric balls of radius $r/2$ are disjoint. Note that if $\mathrm{dist}(B, \partial\Omega) > 2r$, then

$$|\chi_\Omega(q) - \chi_\Omega(q \cdot (0, 0, r^2))| = 0, \quad q \in B,$$

because $d(q, q \cdot (0, 0, r^2)) = 2r$ with our choice of metric d; recall (2.1). Whenever $B \in \mathcal{B}_r$ with $\mathrm{dist}(B, \partial\Omega) \leq 2r$, we pick some ball $\hat{B} \supset B$ which is centred on $\partial\Omega$ and has radius at most $5r$. By the 3-regularity of the boundary, we then have

$$\mathcal{H}^3(\hat{B} \cap \partial\Omega) \sim r^3, \quad B \in \mathcal{B}_r, \ \mathrm{dist}(B, \partial\Omega) \leq 2r.$$

Then, by the bounded overlap of the balls $\hat{B}$ and applying Corollary 3.34, we can estimate

$$\begin{aligned}
\frac{\mathrm{v}_\Omega(B(p_0, R))(r)}{r} &= \int_{B(p_0,R)} \frac{|\chi_\Omega(q) - \chi_\Omega(q \cdot (0, 0, r^2))|}{r} \, dq \\
&\leq \sum_{\substack{B \in \mathcal{B}_r \\ \mathrm{dist}(B, \partial\Omega) \leq 2r}} \int_B \frac{|\chi_\Omega(q) - \chi_\Omega(q \cdot (0, 0, r^2))|}{r} \, dq \\
&\lesssim \sum_{\substack{B \in \mathcal{B}_r \\ \mathrm{dist}(B, \partial\Omega) \leq 2r}} \frac{\mathrm{v}_\Omega(\hat{B})(r)}{r^4} \mathcal{H}^3(\hat{B} \cap \partial\Omega) \\
&\lesssim \sum_{\substack{B \in \mathcal{B}_r \\ \mathrm{dist}(B, \partial\Omega) \leq 2r}} \beta_{\partial\Omega,1}(30\hat{B}) \mathcal{H}^3(\hat{B} \cap \partial\Omega) \\
&\lesssim \int_{B(p_0,CR)} \beta_{\partial\Omega,1}(B(q, Cr)) \, d\mathcal{H}^3(q).
\end{aligned}$$

This is (6.3). Applying Minkowski's integral inequality, we infer the bound

$$\begin{aligned}
\left(\int_0^R \left(\frac{\mathrm{v}_\Omega(B(p_0, R))(r)}{r} \right)^p \frac{dr}{r} \right)^{1/p} &\lesssim \left(\int_0^R \left(\int_{\partial\Omega \cap B(p_0,CR)} \beta_{\partial\Omega,1}(B(q, Cr)) \, d\mathcal{H}^3(q) \right)^p \frac{dr}{r} \right)^{1/p} \\
&\leq \int_{\partial\Omega \cap B(p_0,CR)} \left(\int_0^R \beta_{\partial\Omega,1}(B(q, Cr))^p \, \frac{dr}{r} \right)^{1/p} d\mathcal{H}^3(q).
\end{aligned}$$

Finally, it remains to note

$$\left(\int_R^\infty \left(\frac{\mathrm{v}_\Omega(B(p_0, R))(r)}{r} \right)^p \frac{dr}{r} \right)^{1/p} \lesssim \left(\int_R^\infty \frac{R^{4p}}{r^{p+1}} \, dr \right)^{1/p} \sim R^3,$$

and the proposition follows by combining the two estimates above. □

As an immediate corollary, we infer that if the $\beta_{\partial\Omega,1}$-numbers satisfy a Carleson packing condition similar to (Car(η)), namely

$$\int_{\partial\Omega \cap B(p_0,R)} \int_0^R \beta_{\partial\Omega,1}(B(q, r))^p \, \frac{dr}{r} \, d\mathcal{H}^3(q) \lesssim R^3, \quad p_0 \in \partial\Omega, \ 0 < R \leq \mathrm{diam}\, \partial\Omega, \tag{6.4}$$

then the L^p-vertical perimeter is bounded by (a constant times) the horizontal perimeter.

Corollary 6.5. *Let $1 \le p < \infty$. Assume that $\Omega \subset \mathbb{H}$ is a nonempty open set with 3-regular boundary, and assume that (6.4) holds. Then*

$$\wp_{\Omega,p}(B(q,r)) \lesssim r^3, \quad q \in \partial\Omega, \;\; 0 < r \le \operatorname{diam} \partial\Omega.$$

Proof. Apply Proposition 6.2, then Hölder's inequality, and finally (6.4). □

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