JYU DISSERTATIONS 656

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# On the Boundaries of Sobolev Extension Domains

UNIVERSITY OF JYVÄSKYLÄ
FACULTY OF MATHEMATICS AND SCIENCE

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Jyrki Takanen

## List of included articles

This dissertation consists of an introductory part and the following four publications:

- **[A]** D. Lučić, T. Rajala, and J. Takanen, *Dimension estimates for the boundary of planar Sobolev extension domains*, Adv. Calc. Var. **16** (2023), no. 2, 517–528.
- **[B]** J.Takanen, *Dimension estimates for the two-sided points of planar Sobolev extension domains*, Proc. Amer. Math. Soc. (to appear).
- **[C]** M. García-Bravo T. Rajala, and J. Takanen, *Two-sided boundary points of Sobolev extension domains on Euclidean spaces*, preprint.
- **[D]** M. García-Bravo T. Rajala, and J. Takanen, *A necessary condition for Sobolev extension domains in higher dimensions*, preprint.

The author of this dissertation has actively taken part in the research of the joint articles [A], [C] and [D].

**Abstract**

The main subject of this dissertation is Sobolev extension domains, specially with focus on their boundaries. A fundamental question is to find necessary and sufficient conditions for classifying domains on which each function may be extended to the whole space while preserving inclusion to a given function space. On Euclidean plane, in particular in the simply connected case, characterizations are known for all values of the integralibity of Sobolev functions. In the present work we establish dimension estimates for the boundary and for special subsets of the boundary. On the plane our approach is based on the known characterizations. In higher dimensions the results are proven directly in terms of the operator norm of the extension operator. We also establish a necessary condition for extendability in higher dimensions. The dissertation consists of four articles.

In article [A] we study the size of the boundary of bounded planar simply connected Sobolev extension domains. We show that the boundary is weakly mean porous and establish an upper-bound for the Hausdorff dimension of the boundary. We provide examples showing the sharpness of the result.

In articles [B] and [C] we study the size of the set of two-sided points of Sobolev and BV extension domains. In the simple case, two-sided points are the set of points of where the boundary of the domain intersects itself. In article [B] we establish an upper-bound for the Hausdorff dimension of the set of two-sided points in the case that the domain is bounded simply connected planar extension domain. We provide examples showing the sharpness of the result, and prove equivalence of different definitions of two-sided points. In article [C] we extend the result of [B] for arbitrary Sobolev and BV extension domains in terms of the operator norm of the extension operator. We construct examples that show the sharpness of some of the results and give lower-bounds for others.

In article [D], we give a necessary condition for a domain to have a bounded extension operator between homogeneous Sobolev spaces for $1 < p < 2$ in higher dimensions. We establish a quantitative version of the necessity direction of an earlier characterization in terms of the operator norm, and construct a topologically simple extension domain having a boundary with large Hausdorff dimension.

**Tiivistelmä**

Tämän väitöksen pääaiheena on Sobolev-laajennusalueet, erityisesti niiden reunojen ominaisuudet. Perustavanlaatuinen kysymys on löytää riittäviä ja välttämättömiä ehtoja, jotka luokittelevat alueet, joissa jokainen funktio voidaan jatkaa koko avaruudelle, siten että se kuuluu edelleen samaan funktioavaruuteen. Euklidisessa tasosossa, erityisesti yhdesti yhteinäisen alueen tapauksessa, tunnemme karakterisaatioita kaikille Sobolev-funktioiden integroituvuuden arvoille. Tässä työssä annamme ylärajoja laajennusalueiden reunan sekä erityisten reunan osajoukojen dimensioille. Tasossa sovellamme tunnettuja karakterisaatioita. Korkeammissa ulottuvuuksissa tulokset ovat muotoiltu laajennusoperaattorin normin avulla. Todistamme myös uuden välttämättömän ehdon laajennettavuudelle korkeamassa ulottuvuudessa. Väitöskirja koostuu neljästä artikkelista.

Artikkelissa [A] tutkimme tason yhdesti yhtenäisen rajoitetun laajennusalueen reunan kokoa. Osoitamme, että reuna on heikosti keskiarvohuokoinen sekä annamme ylärajan reunan Hausdorff-dimensiolle.

Artikkeleissa [B] ja [C] tarkastelemme laajennusalueen kaksipuolisten pisteiden joukon kokoa. Yksinkertaisessa tapauksessa kaksipuoliset pisteet muodostuvat reunapisteistä, joissa laajennusalueen reuna leikkaa itseään. Artikkelissa [B] annamme ylärajan kaksipuoleisten pisteiden joukon Hausdorff-dimensiolle tapauksessa, jossa tason alue on rajoitettu ja yhdesti yhtenäinen. Esitämme esimerkkejä jotka näyttävät, että tulos on paras mahdollinen, sekä todistamme erilaisten kaksipuolisten pisteiden määritelmien yhtäpitävyyden. Artikkelissa [C] laajennamme artikkelin [B] tuloksen koskemaan yleisiä Sobolev- ja BV-laajennusalueita laajennusoperaattorin normin avulla muotoiltuna. Esitämme esimerkkejä, jotka näyttävät, että osa tuloksista on tarkkoja toisten antaessa alarajat parhaille mahdollisille tuloksille.

Artikkelissa [D] annamme uuden välttämättömän ehdon lajennusoperaattorin olemassaololle korkeammissa ulottuvuuksissa integroituvuuden ollessa 1 ja 2 välillä. Todistamme kvantitatiivisen version aiemman karakterisaation välttämättömyys suunnasta operaattorinormin avulla lausuttuna. Annamme myös esimerkin topologisesti yksinkertaisesta laajennusalueesta, jonka reunan dimensio on suuri.

# 1. Introduction

An *extension domain* $\Omega \subset X$ is a domain such that there exists an extension operator $E\colon Z(\Omega) \to W(X)$ between (semi)normed function spaces $Z$ and $W$. By an extension operator we mean an operator that is bounded: for all $u \in Z(\Omega)$

$$\|Eu\|_{W(X)} \leq C\|u\|_{Z(\Omega)},$$

where $C$ is independent of $u$, and $Eu|_{\Omega} = u$ for all $u \in Z(\Omega)$. Extendability of functions on a domain depends strongly on the geometric properties of the domain, as well as the topological properties such as connectivity.

In this dissertation we always have $Z = W$, and $X$ is either $\mathbb{R}^2$ or $\mathbb{R}^n$, $n \geq 2$. The spaces we consider are the first order Sobolev spaces $W^{1,p}(\Omega)$, $L^{1,p}(\Omega)$ and the space of functions of bounded variation $BV(\Omega)$. The Sobolev space $W^{1,p}(\Omega)$, for $p \in [1, \infty]$, is the set of all functions $u \in L^p(\Omega)$ whose weak derivatives are in $L^p(\Omega)$. We endow $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

For the (homogeneous) space $L^{1,p}(\Omega)$ we assume only $u \in L^1_{loc}(\Omega)$ and endow $L^{1,p}(\Omega)$ with the seminorm $\|u\|_{L^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$. The space of functions of bounded variation $BV(\Omega)$ is the space of $L^1(\Omega)$ functions with finite bounded total variation

$$\|Du\|(\Omega) = \sup\left\{ \int_{\Omega} u\mathrm{div}(v)\, dx : v \in C_0^{\infty}(\Omega; \mathbb{R}^n), |v| \leq 1 \right\}.$$

The space $BV(\Omega)$ is endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega)$.

Extension of functions has a long history going back at least to Whitney [34, 35]. For Sobolev functions, Calderón [3] proved that Lipschitz domains $\Omega \subset \mathbb{R}^n$ are Sobolev $W^{k,p}$-extension domains for all $k \in \mathbb{N}$ and $1 < p < \infty$. Stein [30] extended the result to the missing cases $p = 1, \infty$. In his seminal paper, Jones [9] defined the much larger class of $(\epsilon, \delta)$-domains and proved that they are $W^{k,p}$-extension domains for all $k \in \mathbb{N}$, $p \in [1, \infty]$. Bugaro and Maz'ya [2] gave a first characterizing result, for the function space $BV_l(\Omega) = \{u \in L^1_{loc}(\Omega) : \|Du\|(\Omega) < \infty\}$, in terms of sets of finite perimeter: A domain $\Omega \subset \mathbb{R}^2$ is a $BV_l$-extension domain if and only if there is a constant $C > 0$ such that whenever $E \subset \Omega$ is a Borel set of finite perimeter in $\Omega$

$$\tau_{\Omega}(E) := \inf\{\ P(F, \mathbb{R}^2 \setminus \Omega) : F \cap \Omega = E\} \leq CP(E, \Omega),$$

where $P(F, \mathbb{R}^2 \setminus \Omega) := \inf\{P(F, U) : U \text{ is open and } \mathbb{R}^2 \setminus \Omega \subset U\}$. We remind the reader that a measurable set $E \subset \mathbb{R}^n$ is called a *set of finite perimeter* in $\Omega$ if $\chi_E \in BV_l(\Omega)$. The *perimeter of $E$ in* $\Omega$ is defined as $P(E, \Omega) := \|D\chi_E\|(\Omega)$. We return to sets of finite perimeter in article [D] of this dissertation. For more recent results on extendability see Section 1.3.

The structure of the introduction is as follows. In Section 1 we give the definitions and introduce the tools needed in the rest of the introduction and in the articles of the dissertation. In Section 2 we look at the concepts of porosity and how they relate to dimension. We also present the results of article [A] where we give an upper bound for Hausdorff dimension of the boundary of a simply connected planar Sobolev extension domain. In Section 3 we study the set of two-sided points of Sobolev extension domains, which is the subject of articles [B] and [C]. Finally, in Section 4 we present a higher dimensional necessary condition for extendability proved in article [D].

1.1. **Properties of domains using curves.** Geometric conditions characterizing Sobolev extension are often represented by different types of curve conditions. A bounded domain $\Omega \subset \mathbb{R}^n$ is called *J-John* if there exists a point $x_0 \in \Omega$ and a constant $J > 0$ so that given $z \in \Omega$ there exists curve parametrized by arc length $\gamma \subset \Omega$ joining $z$ with $x_0$ so that

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq Jt.$$

Another often seen class of domains is that of quasiconvex domains. A domain $\Omega \subset \mathbb{R}^n$ is called *c-quasiconvex* if there exists $c > 0$ so that for each $x, y \in \Omega$ there exists a curve $\gamma \subset \Omega$ connecting $x, y$ such that

$$\ell(\gamma) \leq c|x - y|.$$

Domains satisfying a similar condition where $\ell(\gamma)$ is replaced by $\operatorname{diam}(\gamma)$ are called *c-bounded turning*. Obviously a $c$-quasiconvex domain is $c$-BT. For properties of John, quasiconvex and related domains we refer the reader to [23, 20, 7, 32] and the references therein.

Many other types of domains have been studied in relation with Sobolev extension domains. Apart from the ones mentioned above let us mention a few. There are the so-called cone condition domains which are closely related to Lipschitz domains, QED-domains [11], quasicircle domains [8], slice property domains [1], $(\epsilon, \delta)$ or locally uniform and uniform domains [9], $\alpha$-subhyperbolic domains [5, 19, 1, 26, 14, 27], and so on. Results based on the last property mentioned are discussed in the next subsection, as it turns out that it is the correct condition for characterizing Sobolev extension domains in the plane.

1.2. **Characterizations of extensions.** By now the planar case of Sobolev extension domains is well understood. From the work of Gol'dstein, Latfullin and Vodop'yanov [33], it follows that a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is an $L^{1,2}$-extension domain if and only if $\partial\Omega$ is a quasicircle. By quasicircle we mean the boundary of a domain $\Omega \subset \mathbb{R}^2$ that is an image of unit disk under a global quasiconformal homeomorphism. Notice that by [8] a bounded domain is a $W^{1,p}$-extension domain if and only if it is an $L^{1,p}$-extension domain for $1 < p < \infty$. More generally Jones [9] proved that in the bounded finitely connected case $W^{1,2}$-extendability is equivalent to $\Omega$ being a uniform domain or equivalently that the boundary of $\Omega$ consists of a finite number of points and quasicircles. In [26] Shvartsman proved the following result:

**Theorem 1.1** (Shvartsman). *Let $p \in (2, \infty)$ and let $\Omega \subset \mathbb{R}^2$ be a finitely connected bounded domain. Then $\Omega$ is a Sobolev $W^{1,p}$-extension domain if and only if for some $C > 0$ the following condition is satisfied: for every $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining $x$ to $y$ such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-p}} \, \mathrm{d}s(z) \leq C\|x - y\|^{\frac{p-2}{p-1}}. \tag{1.1}$$

In the simply connected case this is generalized by Shvartsman and Zobin in [27] to hold for $L^{k,p}$-extension domains for every $2 < p < \infty$ and $k \in \mathbb{N}$.

Koskela, Rajala and Zhang [14] characterized the bounded simply connected case for $1 < p < 2$ : a bounded simply connected $\Omega \subset \mathbb{R}^2$ is a Sobolev $W^{1,p}$-extension domain if and only if there exists a constant $C > 1$ such that for every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting $z_1$ and $z_2$ and satisfying

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq C\|z_1 - z_2\|^{2-p}. \tag{1.2}$$

In [15] the same authors gave the following characterization for $W^{1,1}$-extension domains in the planar bounded simply connected case: $\Omega \subset \mathbb{R}^2$ is $W^{1,1}$-extension domain if and only if there is $C > 0$ so that for every $x, y \in \Omega^c$ there exists curve $\gamma \subset \Omega^c$ connecting $x$ and $y$ with

$$\ell(\gamma) \leq C|x-y|, \quad \text{and } \mathcal{H}^1(\gamma \cap \partial\Omega) = 0.$$

In the case of $p = \infty$ for an arbitrary domain $\Omega \subset \mathbb{R}^n$, $\Omega$ is a $W^{1,\infty}$-extension domain if and only if $\Omega$ is uniformly locally quasiconvex [6]. For results for higher order spaces $W^{k,\infty}(\Omega)$ see [38]. Koskela, Miranda and Shanmugalingam [13] proved that simply a connected planar domain $\Omega$ is a $BV$-extension domain if and only if $\Omega^c$ is quasiconvex. For earlier results see [22, 11, 8, 16, 1, 12].

1.3. **Measure density condition and cuspidal domains.** In many proofs we start from weaker geometric information on the domain given in measure theoretic terms. In particular, Hajłasz, Koskela and Tuominen [6] proved that for all $p \in [1, \infty)$ and all $k \in \mathbb{N}$ any $W^{k,p}$-extension domain is a *regular set* or *n-set*, i.e., there exists a constant $c > 0$ such that for all $x \in \Omega$ and all $0 < r \leq 1$

$$|\Omega \cap B(x,r)| \geq cr^n. \tag{1.3}$$

This is often called the measure density condition. The corresponding result for BV-extension domains was proved by García-Bravo and Rajala [4]. Note that (1.3) combined with the Lebesgue differentiation theorem implies that necessarily $|\partial\Omega| = 0$. In the case $p = \infty$ we do not have measure density, since there are obvious examples of quasiconvex domains that do not satisfy the measure density condition, as discussed below.

A typical example of a domain in which we cannot extend all Sobolev functions is one having a cusp on the boundary. Domains with external cusps are not Sobolev extension domains for $p \in [1, \infty)$ as they do not satisfy the measure density condition. In the plane inward cusps are not allowed on the boundary. For $1 < p < 2$ this follows from (1.2), and for $p \in [2, \infty]$ from the fact that the domain is quasiconvex [11, Theorem 3.1]. However for $p = 1$ and for BV functions there may be inward cusps as seen by the characterizations.

Heuristically, we could have a function $f$ with small derivative on $\Omega$ but have $|f(x)-f(y)| \gg |x-y|$ for points which are close together yet separated by the cusp. This would force any extension to have very large derivative between $x$ and $y$. Contrary to the situation in the plane, in higher dimensions points on different sides of a cusp may be connected without going around the vertex of the cusp. Thus we do not have the problem described by the heuristic. Similarly, we see that there are $(\epsilon, \delta)$-domains with inward cusps in $\mathbb{R}^n$ for $n \geq 3$. When extending from $W^{k,p}(\Omega)$ to $W^{k,q}(\mathbb{R}^n)$, $q < p$, the above results do not usually hold. For properties of domains with cusps and about $(p,q)$-extensions we refer reader to thesis [37] and references therein; see also [21, 18]. In what follows we will always have $p = q$.

## 2. Size of the boundary of Sobolev extension domains

In article [A] of the dissertation we give a sharp estimate for the Hausdorff dimension of the boundary of a simply connected planar Sobolev extension domain. This is done for $p \in (1, 2)$ in terms of the constant of the characterizing curve condition (1.2). With (1.2) we show that the complement of the boundary has enough holes to satisfy the weak mean porosity introduced by Nieminen in [24]. For the relations between different concepts of porosity and dimension see the elegant survey [25].

2.1. **Definitions of porosity.** A set $A \subset \mathbb{R}^n$ is called *porous* if

$$\operatorname{por}(A) := \inf_{x \in A} \liminf_{r \searrow 0} \operatorname{por}(A, x, r) > 0,$$

where

$$\operatorname{por}(A, x, r) := \sup\{\alpha \geq 0 : B(y, \alpha r) \subset B(x, r) \setminus A \text{ for some } y \in \mathbb{R}^n\}.$$

A porous set has the dimension upper bound

$$\dim_{\mathrm{H}}(A) \leq n - c \operatorname{por}(A)^n, \tag{2.1}$$

where $c$ is a positive constant depending on $n$ (see [31, proof of Theorem 2]). The estimate (2.1) is asymptotically sharp when $\operatorname{por}(A) \to 0$ ([17], [10, Remark 4.2]).

Koskela and Rohde [17] gave a weaker definition of porosity they called mean porosity. The difference with the definition of porosity is that, instead of requiring balls in the complement of the set for all scales, mean porosity requires balls to be evenly distributed in averaged sense. Nieminen generalized this further with a notion of porosity he called weak mean porosity. The difference is that there are many holes in sparser set of scales. This is the notion of porosity we use, so let us give the precise definition.

Let $E \subset \mathbb{R}^n$ be a compact set. Let $\alpha : (0,1) \to (0,1)$ be a continuous function such that $\alpha(t)/t$ is increasing in $t$, and let $\lambda : \mathbb{Z}^+ \to \mathbb{R}$ be a function. Let $\mathcal{D}$ be a disjoint collection of open cubes in $\mathbb{R}^n \setminus E$. Define

$$\chi_k^{\mathcal{D}}(x) = \begin{cases} 1 & \text{if there exist at least } \lambda(k) \text{ cubes } Q \in \mathcal{D} \text{ with } Q \subset A_k(x) \text{ and } \ell(Q) \geq \alpha(2^{-k}), \\ 0 & \text{otherwise,} \end{cases}$$

where $A_k(x) := B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$. Let

$$S_j^{\mathcal{D}}(x) = \sum_{k=1}^{j} \chi_k^{\mathcal{D}}(x).$$

We say that $E$ is *weakly mean porous with parameters* $(\alpha, \lambda)$ if there exist a collection $\mathcal{D}$ and $j_0 \in \mathbb{Z}^+$ such that

$$\frac{S_j^{\mathcal{D}}(x)}{j} > \frac{1}{2}$$

for all $x \in E$ and for all $j \geq j_0$.

Let $\varepsilon \in (0,1)$ and $c > 0$ be a fixed constant. Using weak mean porosity with parameters

$$\lambda(k) = c\varepsilon^{-1} \quad \text{and} \quad \alpha(t) = \varepsilon t$$

we get as a direct corollary to [24] Theorem 3.3 the following: there exists $C(n,c) > 0$ such that any weakly mean porous set $E \subset \mathbb{R}^n$ with parameters $(\alpha, \lambda)$ as defined above satisfies

$$\dim_{\mathrm{H}}(E) \leq n - C(n,c)\varepsilon^{n-1}. \tag{2.2}$$

2.2. **Porosity of the boundary of a Sobolev extension domain.** Straightforward argument shows that for the boundary of a planar simply connected Sobolev extension domain we have $\operatorname{por}(A) \geq \frac{c}{C}$ where $c > 0$ is universal constant and $C$ is the constant in (1.2). With (2.1) we get the upper bound

$$\dim_{\mathrm{H}}(\partial\Omega) \leq 2 - \frac{c'}{C^2}. \tag{2.3}$$

As a main result of [A] we show the following:

**Theorem 2.1** ([**A**], Theorem 3.2)**.** *There exist universal constants $C', C'' > 0$ so that the following holds. Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected $W^{1,p}$-extension domain. Let $C$ be the constant from the curve condition* (1.2)*. Then $\partial\Omega$ is weakly mean porous with parameters $(\alpha, \lambda)$, where $\lambda(k) = C'C$ and $\alpha(t) = \frac{C''}{C}t$.*

Combining Theorem 2.1 with (2.2) we get a better bound than (2.3)

**Theorem 2.2** ([**A**], Theorem 1.1)**.** *There exists a universal constant $M > 0$ such that for every bounded simply connected domain $\Omega \subset \mathbb{R}^2$ satisfying the curve condition* (1.2) *with some $C \in (1, \infty)$ the following holds:*

$$\dim_{\mathrm{H}}(\partial\Omega) \leq 2 - \frac{M}{C}.$$

Let us remark that the proof of Theorem 2.1 holds true (with necessary modifications) for the assumptions with which curve condition (1.1) of Shvartsman holds. In other words, the conclusion of Theorem 2.1 holds when $2 < p < \infty$ and $\Omega \subset \mathbb{R}^2$ a bounded and finitely connection $W^{1,p}$-extension domain. Let us sketch the differences in the proof in this case. In the case of Theorem 1.1 the curve condition (1.1) holds in the interior of the domain, and $\Omega$ is bounded and finitely connected. To tackle the finite connectedness, denote the maximal connected components of $\partial\Omega$ by $C_j$, $j = 1, \ldots, m$. Trivially we may assume that $\#C_j > 1$ for all $j$. Then, fix the smallest positive integer $k_0$ such that

$$2^{-k_0} < \min(\min\{\,\mathrm{diam}\,(C_j) : j \in \{1, \ldots, m\}\}, \min\{\,\mathrm{dist}\,(C_i, C_j) : i \neq j\}).$$

Replacing $\partial\Omega$ with $C_j$, the proof of Theorem 3.2 holds true in Case 1 (of the proof of Theorem 3.2 in [A]). In Case 2 of the proof, instead of the original selection, we choose points $w_i \in B(y_i, \varepsilon 2^{-k+5}) \cap \Omega$. Since $x \in C_j$ and $C_j \setminus B(x, 2^{-k}) \neq \emptyset$ we have that any curve connecting $w_i$ to $w_{i+1}$ in $\Omega$ must exit $B(w_i, 2^{-k-3})$. The rest of the proof then holds with obvious modifications.

In [A] we show that that there exists a constant $M' > 0$ so that for every $p \in (1, 2)$ and $C \in (\frac{M'}{2-p}, \infty)$ there exists a Jordan domain (the Koch snowflake) $\Omega_C \in \mathbb{R}^2$ satisfying (1.2) with

$$\dim_{\mathrm{H}}(\partial\Omega_C) \geq 2 - \frac{M'}{(2-p)C}.$$

This shows that the theorem is sharp modulo the factor $\frac{1}{2-p}$.

Koskela and Rohde [17] showed that the boundary of a $J$-John domain $\Omega \in \mathbb{R}^2$ has the dimension bound $\dim_{\mathcal{H}}(\partial\Omega) \leq 2 - cJ$ for some $c > 0$. In [A] we show that this bound is sharp and that using the result of Koskela and Rohde with mean porosity (defined also in [17]) it is not possible to get better bound than

$$\dim_{\mathrm{H}}(\partial\Omega) \leq 2 - \frac{M}{((2-p)C)^{1/(2-p)}}.$$

We mention some results in the cases not covered by our results. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected $W^{1,p}$-extension domain with the extension operator $E$. For $p = 1$, the complement of $\Omega$ is $C(\|E\|)$-quasiconvex [15] and in particular $C(\|E\|)$-bounded turning. Then by [23, Theorem 4.5] $\Omega$ is $J(C)$-John.

In the case of $p = \infty$, $\Omega$ is $2\|E\|$-bounded turning [36] and thus $\Omega^c$ is $J(2\|E\|)$-John. Therefore with the estimate given by Koskela and Rohde [17] the boundary of $\Omega$ has the

Hausdorff dimension upper-bound

$$\dim_{\mathrm{H}}(\partial\Omega) \leq 2 - CJ.$$

For $p = 2$, $\Omega$ is a $k(\|E\|)$-quasidisk and a well-known theorem of Smirnov [28] states that the Hausdorff dimension of a $k$-quasicircle is at most $1 + k^2$.

Our method for proving dimension estimates for the boundary does not work in the cases $p \in \{1, 2, \infty\}$. It would be interesting to know the sharp behaviour in terms of the constant in the various characterizations (also in these cases).

For a general extension domain we do not have a non-trivial dimension estimate. Take $\Omega = [0,1]^n \setminus C^n$ with $C$ a Cantor set of $\dim_{\mathrm{H}}(C) = 1$ but Lebesgue measure zero. Then $\Omega$ is a Sobolev $W^{1,p}$ extension domain, but $\dim_{\mathrm{H}}(\partial\Omega) = n$. In the article [D] of the dissertation we show that such dimension bounds for the boundary do not exist for higher dimensions even when the domain is topologically nice:

**Theorem 2.3** (**[D]**, Theorem 1.2)**.** *There exists a domain $\Omega \subset \mathbb{R}^3$ such that $\Omega = h(B(0,1))$ for a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\dim_{\mathrm{H}}(\partial\Omega) = 3$ and $\Omega$ is a $W^{1,p}$-extension domain for all $p \in [1, \infty]$.*

## 3. Two-sided points of an extension domains

In articles [B] and [C] of the dissertation we study the boundary behaviour of Sobolev extension domains. Even in the planar simply connected case the extension domain is not necessarily Jordan. Rather, in some cases the boundary of an extension domain may intersect itself. To quantify this property we give an upper bound for the Hausdorff dimension for the set of two-sided points of the boundary. In [B] this is done in the plane in terms of the constant for the characterizing curve condition (1.2), and in [C] in terms of operator norm $\|E\|$ of the extension operator for $n \geq 2$.

**Definition 3.1** (**[B]**, Two-sided points of the boundary of a domain)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. A point $x \in \partial\Omega$ is called* two-sided*, if there exists $R > 0$ such that for all $r \in (0, R)$ there exist connected components $\Omega_r^1$ and $\Omega_r^2$ of $\Omega \cap B(x, r)$ that are nested: $\Omega_r^i \subset \Omega_s^i$ for $0 < r < s < R$ and $i \in \{1, 2\}$. The set of two-sided points is denoted $\mathcal{T}_\Omega$.*

A slightly different definition of two-sided points was used in [29]. We note that in [B] all the domains we consider are John in which case the two definitions are equivalent. In general our definition implies that of [29] but the converse does not hold. For a counterexample consider a domain where the two-sided point is approached "from one side" by a Topologist's comb on all scales. We note however, that in [C] either of the definitions may be used.

For $W^{1,p}$-extension domains the existence of two-sided points is dependent on the parameter $p$. For $n \leq p < \infty$ there are no two-sided points which is readily seen from the fact that extension domains in this case are quasiconvex. For $p = \infty$ quasiconvexity is known for finitely connected planar domains. In the case of $1 < p < n$ we prove

**Theorem 3.2** (**[C]** Theorem 1.2)**.** *Let $1 < p < n$ and let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^{1,p}$-extension domain. Then there exists a constant $C(n,p) > 0$ so that*

$$\dim_{\mathcal{H}}(\mathcal{T}_\Omega) \leq n - p - \frac{C(n,p)}{\|E\|^n \log(\|E\|)}.$$

In the other direction, we construct a class of extension domains with explicit (Withney type) extension operators such that

$$\dim_{\mathcal{H}}(\mathcal{T}_\Omega) \geq n - p - \frac{C_1(n,p)}{\|E_\lambda\|}, \tag{3.1}$$

where $\mathcal{T}_\Omega = C_\lambda$, $\lambda \in (0, \frac{1}{2})$, is a Cantor set and $\|E_\lambda\| \to \infty$ as $\dim_{\mathrm{H}}(C_\lambda) \to n - p$.

Theorem 3.2 together with the set of examples bounds the asymptotic behaviour for the dimension of the two-sided points in terms of the norm of the extension operator between $n - p - C/\|E\|$ and $n - p - C/(\|E\|^n \log(\|E\|))$.

In the planar simply connected case we have the characterizing curve condition (1.2). We then get the following

**Theorem 3.3** ([**B**], Theorem 1.2)**.** *Let $1 < p < 2$ and $\Omega \subset \mathbb{R}^2$ a simply connected, bounded Sobolev $W^{1,p}$-extension domain. Let $\mathcal{T}$ be the set of two-sided points of $\Omega$. Then*

$$\dim_{\mathcal{H}}(\mathcal{T}) \leq 2 - p + \log_2\left(1 - \frac{2^{p-1}-1}{2^{5-2p}C}\right) \leq 2 - p - \frac{M_1(p)}{C}, \tag{3.2}$$

*where $M_1(p) = \frac{2^{p-1}-1}{2^{5-2p}\log 2}$ and $C \geq 1$ is the constant in* (1.2)*.*

Let us sketch the ideas in the proofs of Theorem 3.2 and Theorem 3.3 while highlighting the similarities and differences. In both cases the fundamental idea is to cover the set of two-sided points in a geometric progression of scales with disjoint balls. As a starting point we prove a lemma which gives a sufficient condition for an upper bound of Hausdorff dimension of a set $F$. The lemma says that, for any $0 < s < \dim_{\mathrm{H}}(F)$ there exists a ball $B$ with radius $\lambda^i$, $0 < \lambda < 1$, for which all maximal $\lambda^j$-nets in $B \cap F$, $j > i$, contain at least $N_j \geq \lambda^{-js}$ points. In both cases we calculate (here $c = c(n,p)$ is a constant and the meaning of variables $A$ and $K$ is described below)

$$cK \geq \sum_{j=0}^{\infty} (\lambda^A)^j N_j \geq \sum_{j=0}^{\infty} (\lambda^{A-s})^j = \frac{1}{1-\lambda^{A-s}}$$

which gives

$$\dim_{\mathrm{H}}(F) \leq A - \frac{\log(1-c/K)}{\log(\lambda)}.$$

The upper bound $K$ for the series is obtained by estimating the series above with a suitable integral, which is then estimated with a constant given by the corresponding condition which holds for Sobolev extension domains. The difference in the estimates comes from the fact that in the plane we have the subhyperbolic distance condition (1.2), while in the general case we do not have such a condition. There, instead, we construct a test function which has values zero and one in components on opposite sides of the two-sided points and estimate the series above with $L^{1,p}$-norm of the test function.

In the planar case we have $K = C$ from (1.2) and $A = 2-p$, and we can choose $\lambda = 1/2$. In the case $n \geq 2$ we have $K = \|E\|^n$, $A = n - p$ and $\lambda \leq c_1(n,p)\|E\|^{c_2(n,p)}$. In the planar case we can choose $\lambda = 1/2$, which stems from the fact that we have better control over relative positions of the covering balls in different scales. In the $n$-dimensional case we cannot control the relative positions of the balls between different scales, and therefore we make the parts of the sets contributing to the $L^{1,p}$-norm disjoint by removing smaller balls for each scale. We then estimate the sum from above with the $L^{1,p}$-norm of the test function. The constant $K$ is obtained by the measure density condition for Sobolev extension domains, which guarantees that in each ball there is enough measure for the sets where the test function has values 0 and 1, giving a lower bound for the $L^{1,p}$-norm in terms of $\lambda$. In order to have the lower bound we need to choose $\lambda$ small enough. The selection of $\lambda$ leads to the factor $\log(\|E\|)$ in the theorem.

We show the sharpness of Theorem 3.3 by constructing a collection of domains (see also [19]) such that the following holds.

**Theorem 3.4** ([**B**], Theorem 1.4). *Let $1 < p < 2$. There exist constants $M_2 > 0$ and $C(p) \geq 1$ such that for each $C > C(p)$ there exists Sobolev $W^{1,p}$-extension domain $\Omega_{p,C}$ satisfying* (1.2) *with $C$, and*

$$\dim_{\mathcal{H}}(\mathcal{T}_{\Omega_{p,C}}) \geq 2 - p - \frac{M_2}{C}. \tag{3.3}$$

For $W^{1,1}$- and $BV$-extension domains we do not obtain better results in the plane in comparison to higher dimensions. The dimension estimates in this case do not have a dependence on the operator norm.

**Theorem 3.5** ([**C**] Theorem 1.2). *Let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^{1,1}$-extension domain. Then $\mathcal{H}^{n-1}(\mathcal{T}_\Omega) = 0$.*

This is also sharp. Sharpness can be shown by replacing the slit in $\Omega = (-1,1)^2 \setminus ([0,1] \times 0)$ with a larger set such that the modified domain $\Omega'$ has two-sided points containing a Cantor set with $\dim_{\mathrm{H}}(C) = 1$ and $\mathcal{H}^1(C) = 0$. Then it is enough to show that $\Omega' \times (-1,1)^{n-2}$ is an extension domain.

As $W^{1,1}$-extension domains are always BV-extension domains, it is natural to extend the result to $BV$-extension domains.

**Theorem 3.6** ([**C**], Theorem 1.3). *Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a $BV$-extension domain. Then $\mathcal{T}_\Omega$ has $\sigma$-finite $(n-1)$-dimensional Hausdorff measure.*

The estimate in Theorem 3.6 is also sharp, as seen by the the slit disk, which is a BV-extension domain since its complement is quasiconvex.

## 4. Necessary condition for extension in Euclidean spaces

As discussed in the beginning of this introduction, in the planar case the search for geometric characterizations of Sobolev extension domains is quite complete. The same is not true in the case $n > 2$. By Jones [9], uniform domains are $W^{1,p}$-extension domains for all $1 \leq p \leq \infty$. This however does not cover all extension domains. See for example [D, Theorem 1.2]. It would be interesting to see how the planar geometric characterizations generalize to higher dimension. In the article [D] we looked into the condition (1.2) and obtained the following version in higher dimensions.

**Theorem 4.1** ([**D**], Theorem 1.1). *Let $\Omega \subset \mathbb{R}^n$ be an $L^{1,p}$-extension domain for some $1 < p < 2$. Then for any $\varepsilon > 0$ and any measurable set $A \subset \Omega$ there exists a set $\widetilde{A} \subset \mathbb{R}^n$ with $A = \widetilde{A} \cap \Omega$ and*

$$\int_{\partial^M \widetilde{A}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) \leq C(n,p,\varepsilon)\|E\|^{n+p+\varepsilon} \int_{\Omega \cap \partial^M A} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z), \tag{4.1}$$

*where $\|E\|$ denotes the norm of the $L^{1,p}$-extension operator, and the constant $C(n,p,\varepsilon)$ depends only on $n$, $p$ and $\varepsilon$.*

Note that in higher dimensions a natural replacement for $\gamma$ in (1.2) is the measure theoretic boundary

$$\partial^M A = \{x \in \mathbb{R}^n : \overline{D}(A,x) > 0 \quad \overline{D}(\mathbb{R}^n \setminus A, x) > 0\},$$

where

$$\overline{D}(A,x) := \limsup_{r \searrow 0} \frac{|A \cap B(x,r)|}{|B(x,r)|}.$$

The reason for the range of exponents is that if $p \geq 2$, then (4.1) is always satisfied by the choice $\widetilde{A} = A$ since the integral on the right-hand side is infinite in the nontrivial cases. Thus, for $p \geq 2$ the conclusion of Theorem 4.1 provides no information.

With the objective of finding characterizations in mind, one may ask whether the condition is also sufficient. The answer to this is negative. To see this, take an arbitrary domain $\Omega \subset \mathbb{R}^n$ and modify $\Omega$ into a new domain $\Omega' = \Omega \setminus \bigcup_{i=1}^{\infty} B(x_i, r_i)$, where the balls $B(x_i, r_i) \subset \Omega$ are selected in a such way that $B(x_i, 2r_i) \setminus B(x_i, r_i) \subset \Omega'$ (so that we have an extension operator $L^{1,p}(\Omega') \to L^{1,p}(\Omega)$), but that $B(x_i, r_i)$ accumulate densely enough to the boundary so that for any $A \subset \Omega'$ we can take $\widetilde{A}$ to be empty outside $\Omega'$ due to the right-hand side of (4.1) being infinite for any $A$ for which we would not be able to take $\widetilde{A} = A$ when considering (4.1) with respect to $\Omega$. Thus condition (4.1) gives us no information on $\Omega'$ as it holds regardless of $\Omega$ and thus $\Omega'$ being an extension domain or not.

Let us now sketch the proof of Theorem 4.1. First, to get better control of the extended set we show that it is enough, instead of $A$, to consider a union of Whitney cubes of the domain which intersect $A$ in a large proportion of the cube. We denote this set by $A'$. The extended set $\widetilde{A}$ is then defined as $A \cup A_0$ where $A_0$ is defined as a union of Whitney cubes $\widetilde{Q}$ of the complementary domain where $\widetilde{Q}$ is included into the extension $A_0$ if the part where scaled cube $c\widetilde{Q}$ intersects $A'$ is larger than the part where it intersects $\Omega \setminus A'$. To be able to apply the extension operator later, we pass from the indicator of $A'$ to a Sobolev function $u$ via a Whitney smoothing operator. By the isoperimetric inequality and properties of the Whitney smoothing operator we obtain an upper bound for the $L^{1,p}$-norm of $u$ by the right-hand side of (4.1).

As a second step we prove that

$$\int_{\partial^M A_0 \setminus \overline{\Omega}} \operatorname{dist}(z, \partial\Omega)^{1-p} \, d\mathcal{H}^{n-1}(z)$$

is bounded above by $C(n,p,\epsilon)\|E\|^{n+p+\epsilon}\|\nabla u\|^p_{L^p(\Omega)}$. Here we are motivated by the ideas of the proof of the necessity direction of (1.2) (see [14, Lemma 3.6]). By the choice of $A_0$, the definition of $u$ and the measure density condition of $\Omega$, we get that $9c\widetilde{Q} \cap \Omega$ contains sets of large enough measure where $u \leq 0$ and $u \geq 1$, so that we get

$$\left( \int_{9c\widetilde{Q}} |\nabla Eu(x)|^s \, dx \right)^{p/s} \geq (C(n,p)\|E\|^{-n} \ell(\widetilde{Q})^{n-p})^{p/s},$$

for $s \in (1,p)$. Since

$$\ell(\widetilde{Q})^{n-p} \approx \mathcal{H}^{n-1}(\partial\widetilde{Q})\ell(\widetilde{Q})^{1-p} \approx \int_{\partial\widetilde{Q}} \operatorname{dist}(z, \partial\Omega)^{1-p} \, d\mathcal{H}^{n-1}(z),$$

by using the Hardy-Littlewood maximal operator, which is bounded when $p/s > 1$, we get the second step in the proof.

What remains to show is that the set where $\partial^M \widetilde{A}$ intersects the boundary of the domain is negligible. More exactly, we show that $\mathcal{H}^{n-p}(\partial^M \widetilde{A} \cap \partial\Omega) = 0$. The idea of the proof is as follows. We cover $F \subset \partial\Omega$, where $D(\widetilde{A}, \mathbb{R}^n, x)$ is neither 0 or 1 for $x \in F$, by a nice cover $F \subset \cup B$ (which is refined by the Vitali covering theorem). Then, using the Sobolev function $u$ defined as a Whitney smoothing of $\chi_{A'}$, in a similar way as in the previous part of the proof, we obtain an upper bound for $\mathcal{H}^{n-p}(F)$ in terms of the norm of $L^{1,p}(\cup B)$. Finally, letting the radii of the covering go to zero and by the fact that $|F| \leq |\partial\Omega| = 0$, which is again given by the measure density condition, we obtain the claim.

As an application of Theorem 4.1 we give a quantitative version in terms of the operator norm of the necessary direction of the curve condition (1.2).

**Theorem 4.2** ([**D**], Theorem 1.3)**.** *Let $\Omega \subset \mathbb{R}^2$ a bounded simply connected $L^{1,p}$-extension domain for some $1 < p < 2$. Then for every $\varepsilon > 0$ there exists a constant $C(p, \varepsilon) > 0$ such that for all $z_1, z_2 \in \partial\Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining $z_1$ and $z_2$ so that*

$$\int_\gamma \operatorname{dist}^{1-p}(z, \partial\Omega)\, ds(z) \leq C(p, \varepsilon) \|E\|^{\frac{4+4p-p^2}{2-p}+\varepsilon} |z_1 - z_2|^{2-p}. \tag{4.2}$$

We note that the combination of the example giving (3.1) and Theorem 3.3 show that $C$ in (1.2) has to grow at least linearly as a function of $\|E\|$, and the combination of Theorem 4.2 with Theorem 3.3 gives a version of Theorem 3.2 in the case $n = 2$. However the bound obtained this way is worse than that of Theorem 3.2.

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# Included articles

[**A**]

# Dimension estimates for the boundary of planar Sobolev extension domains

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Research Article

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# Dimension estimates for the boundary of planar Sobolev extension domains

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**Abstract:** We prove an asymptotically sharp dimension upper-bound for the boundary of bounded simply-connected planar Sobolev $W^{1,p}$-extension domains via the weak mean porosity of the boundary. The sharpness of our estimate is shown by examples.

**Keywords:** Sobolev extension, porosity, Hausdorff dimension

**MSC 2010:** 46E35, 28A75

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## 1 Introduction

A set is porous if it has holes arbitrarily close to any point, and those holes have diameter comparable to the distance to the point. It is easy to see that porous sets in $\mathbb{R}^d$ have zero Lebesgue measure. If the porosity of the set $A \subset \mathbb{R}^d$ is stronger, in the sense that

$$\mathrm{por}(A) := \inf_{x\in A} \liminf_{r\searrow 0} \mathrm{por}(A, x, r) > 0,$$

where we denote the maximal size of a hole of the set $A \subset \mathbb{R}^d$ at $x \in \mathbb{R}^d$ and of scale $r > 0$ by

$$\mathrm{por}(A, x, r) := \sup\{\alpha \geq 0 : \text{there exists } y \in \mathbb{R}^d \text{ such that } B(y, \alpha r) \subset B(x, r) \setminus A\},$$

then the Hausdorff dimension of $A$ is strictly less than $d$. It was shown by Mattila [8] that as $\mathrm{por}(A)$ gets closer to its maximal value $\frac{1}{2}$, the dimension upper-bound for $A$ goes to $d-1$. The sharp asymptotic behavior when $\mathrm{por}(A) \to \frac{1}{2}$ was then established by Salli in [11]. Later, several variants of porosity have been considered. For example, in a variant of porosity called $k$-porosity, one looks at $k$ holes in orthogonal directions, instead of just one, see [3, 4]. For it, the dimension upper-bound approaches $d-k$ as the porosity goes to its maximal value.

In the present paper we are interested in the asymptotic behavior of the dimension upper-bound when $\mathrm{por}(A) \to 0$. In this case, for the usual porosity defined above we have the sharp upper-bound

$$\dim_{\mathcal{H}}(A) \leq d - c\,\mathrm{por}(A)^d,$$

for some constant $c$ depending on the dimension, see for instance [7]. However, sometimes we are in a setting where the porosity condition is not satisfied in the exact form as stated above, but almost. One such instance is the study of growth conditions on the hyperbolic metric, which imply the existence of holes only in a portion of the scales, but not all scales. Motivated by this, Koskela and Rohde introduced a version of porosity called

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*mean porosity* and proved a sharp dimension upper bound for mean porous sets [6] (see also the estimates by Beliaev and Smirnov [1] that deal also with a generalization of Salli's result).

Our aim in this paper is to show sharp dimension bounds for boundaries of Sobolev extension domains. For obtaining these, even the mean porosity of Koskela and Rohde is not flexible enough, because we might have many holes in a more sparse set of scales. Therefore, we use a variant of mean porosity introduced by Nieminen in [10], called *weak mean porosity* (see Section 2.1 for the definition).

Recall that a domain $\Omega \subset \mathbb{R}^d$ is called a Sobolev $W^{1,p}$-extension domain if there exists a constant $C \in (1,\infty)$ so that for every $f \in W^{1,p}(\Omega)$ there exists $F \in W^{1,p}(\mathbb{R}^d)$ so that $F|_\Omega = f$ and $\|F\|_{W^{1,p}(\mathbb{R}^d)} \le C\|f\|_{W^{1,p}(\Omega)}$. When $p > 1$, the operator $f \mapsto F$ can always be assumed to be linear [2]. In [12] and [5], bounded simply-connected Sobolev extension domains $\Omega \subset \mathbb{R}^2$ were characterized by a curve condition, which for the range $1 < p < 2$ is the following: There exists a constant $C > 1$ such that for every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting $z_1$ and $z_2$ and satisfying

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \le C\|z_1 - z_2\|^{2-p}. \tag{1.1}$$

We give an upper bound on the Hausdorff dimension $\dim_{\mathcal{H}}$ of the boundary of $\Omega$ in terms of the constant $C$ in (1.1). This is done by showing the weak mean porosity of the boundary in Theorem 3.2 and by combining it with the dimension estimate proven by Nieminen (Theorem 2.1). The result we obtain is the following.

**Theorem 1.1.** *There exists a universal constant $M > 0$ such that for every bounded simply-connected domain $\Omega \subset \mathbb{R}^2$ satisfying the curve condition* (1.1) *with some $C \in (1,\infty)$ the following holds:*

$$\dim_{\mathcal{H}}(\partial\Omega) \le 2 - \frac{M}{C}.$$

In Section 4, we show that Theorem 1.1 is sharp in the sense that there exists another constant $M' > 0$ so that for every $p \in (1,2)$ and $C \in (\frac{M'}{2-p}, \infty)$ there exists a Jordan domain $\Omega_C \subset \mathbb{R}^2$ satisfying (1.1) with

$$\dim_{\mathcal{H}}(\partial\Omega_C) \ge 2 - \frac{M'}{(2-p)C}.$$

Notice, however, the factor $\frac{1}{2-p}$ difference between Theorem 1.1 and the examples. The curve condition (1.1) implies that $\mathbb{R}^2 \setminus \Omega$ is quasi-convex. Consequently, the domain $\Omega$ is a $J$-John domain [9], meaning that there exists a constant $J > 0$ and a point $x_0 \in \Omega$ so that for every $x \in \Omega$ there exists a unit speed curve $\gamma\colon [0, \ell(\gamma)] \to \Omega$ such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \ge Jt \quad \text{for all } t \in [0, \ell(\gamma)]. \tag{1.2}$$

Koskela and Rohde showed that the boundary of a $J$-John domain $\Omega \subset \mathbb{R}^2$ has the dimension bound

$$\dim_{\mathcal{H}}(\partial\Omega) \le 2 - cJ, \tag{1.3}$$

for some constant $c > 0$. In Section 4 we show that the bound (1.3) is also sharp.

In Section 4 we also show that from the curve condition, via the John condition and the mean porosity of Koskela and Rohde [6], it is not possible to get a better bound than

$$\dim_{\mathcal{H}}(\partial\Omega) \le 2 - \frac{M}{((2-p)C)^{1/(2-p)}}.$$

A reason why the John condition does not give the sharper bound is that using it we consider holes only in the domain (or its complement), whereas by going from the curve condition directly to weak mean porosity, we can use holes on both sides of the boundary.

## 2 Preliminaries

Let us start by introducing some notation and preliminary results. By a *cube* in $\mathbb{R}^d$ we mean an open cube whose sides are parallel to the axes in $\mathbb{R}^d$. The side-length of a cube $Q \subset \mathbb{R}^d$ will be denoted by $\ell(Q)$. By

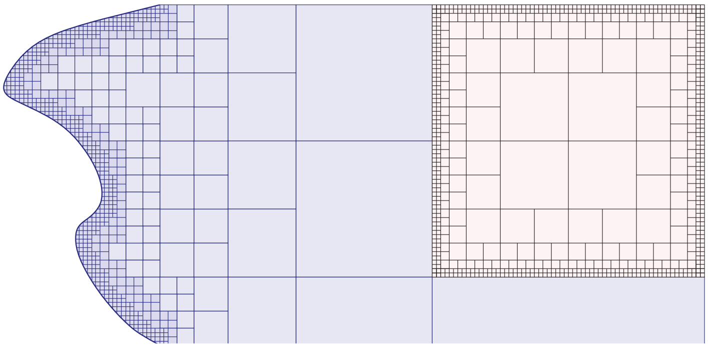

**Figure 1:** In our proof, we will use a double dyadic decomposition similar to the one used in [6]. A domain is first decomposed into its Whitney cubes. Then each Whitney cube is decomposed into its own Whitney cubes, as illustrated here only for the largest cube in the first decomposition.

a dyadic cube $Q$ we mean that it is of the form

$$Q = (i_1 2^{-k}, (i_1 + 1)2^{-k}) \times (i_2 2^{-k}, (i_2 + 1)2^{-k}) \times \cdots \times (i_d 2^{-k}, (i_d + 1)2^{-k})$$

for some $k, i_1, i_2, \dots, i_d \in \mathbb{Z}$. We denote the set of dyadic cubes in $\mathbb{R}^d$ by $\mathcal{D}_d$. Given an open nonempty set $U \subset \mathbb{R}^d$ that is not the whole $\mathbb{R}^d$, we denote by $\mathcal{W}_U$ the *Whitney decomposition* of $U$, defined as

$$\mathcal{W}_U = \{Q \in \widetilde{\mathcal{W}}_U : \text{if } Q' \in \widetilde{\mathcal{W}}_U \text{ with } Q' \cap Q \neq \emptyset, \text{ then } Q' \subset Q\},$$

where

$$\widetilde{\mathcal{W}}_U = \{Q \in \mathcal{D}_d : \text{if } Q' \in \mathcal{D}_d \text{ with } \overline{Q} \cap \overline{Q}' \neq \emptyset \text{ and } \ell(Q) = \ell(Q'), \text{ then } Q' \subset U\}.$$

See Figure 1 for an illustration of the Whitney decomposition.

It readily follows that $\mathcal{W}_U$ is a collection of pairwise disjoint dyadic cubes $Q$ so that

$$U = \bigcup_{Q \in \mathcal{W}_U} \overline{Q}.$$

Moreover, the following condition is satisfied by each $Q \in \mathcal{W}_U$:

$$\ell(Q) \leq \operatorname{dist}(Q, \partial U) \leq 4 \operatorname{diam}(Q) = 4\sqrt{d}\, \ell(Q). \tag{2.1}$$

Moreover, if $Q, Q' \in \mathcal{W}_U$ with $\overline{Q} \cap \overline{Q}' \neq \emptyset$, then

$$\frac{1}{2} \leq \frac{\ell(Q)}{\ell(Q')} \leq 2. \tag{2.2}$$

In the specific case where we take the Whitney decomposition of a dyadic cube $Q \in \mathcal{D}_d$, we have

$$\mathcal{W}_Q = \{Q' \subset Q, : Q' \text{ dyadic cube with } \ell(Q') = \operatorname{dist}(Q', \partial Q)\}.$$

See again Figure 1 for an illustration. It is then easy to check that

$$\#\{Q' \in \mathcal{W}_Q : \ell(Q') = 2^{-j}\ell(Q)\} \geq 2^{(j-1)(d-1)} \quad \text{holds for every } j \geq 2. \tag{2.3}$$

Given any ball $B \subset \mathbb{R}^d$ and any $r > 0$, we denote by $rB$ the ball having the same center as $B$ and the radius $r$ times that of $B$. The ball of radius $r > 0$, centered in $x \in \mathbb{R}^d$ is denoted by $B(x, r)$, while by $B(E, r)$ we denote the $r$-neighborhood of a given set $E \subset \mathbb{R}^d$.

Recall that the *Hausdorff dimension* of a set $E \subset \mathbb{R}^d$ is defined by

$$\dim_{\mathcal{H}}(E) := \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{s > 0 : \mathcal{H}^s(E) = +\infty\},$$

where $\mathcal{H}^s$ stands for $s$-dimensional Hausdorff measure in $\mathbb{R}^d$.

## 2.1 Weakly mean porous sets

In the present subsection, we recall the concept of weak mean porosity introduced in [10]. The weak mean porosity is a variant of mean porosity introduced in [6].

Let $E \subset \mathbb{R}^d$ be a compact set. Let $\alpha\colon ]0,1[ \to ]0,1[$ be a continuous function such that $\alpha(t)/t$ is increasing in $t$, and let $\lambda\colon \mathbb{Z}^+ \to \mathbb{R}$ be a function. Let $\mathcal{D}$ be a disjointed collection of open cubes in $\mathbb{R}^d \setminus E$. Define

$$\chi_k^{\mathcal{D}}(x) = \begin{cases} 1, & \text{if there exist at least } \lambda(k) \text{ cubes } Q \in \mathcal{D} \text{ with } Q \subset A_k(x) \text{ and } \ell(Q) \geq \alpha(2^{-k}), \\ 0, & \text{otherwise,} \end{cases}$$

where $A_k(x) := B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$. Let

$$S_j^{\mathcal{D}}(x) = \sum_{k=1}^{j} \chi_k^{\mathcal{D}}(x).$$

We say that $E$ is *weakly mean porous with parameters* $(\alpha, \lambda)$ if there exist a collection $\mathcal{D}$ and $j_0 \in \mathbb{Z}^+$ such that

$$\frac{S_j^{\mathcal{D}}(x)}{j} > \frac{1}{2}$$

for all $x \in E$ and for all $j \geq j_0$.

We will apply weak mean porosity in the case

$$\lambda(k) = c\varepsilon^{-1} \quad \text{and} \quad \alpha(t) = \varepsilon t, \tag{2.4}$$

for some $\varepsilon \in ]0,1[$ and a fixed constant $c > 0$. In this case, we have the following dimension estimate as a direct corollary of [10, Theorem 3.3].

**Theorem 2.1.** *There exists a constant $C(d,c) > 0$ such that any weakly mean porous set $E \subset \mathbb{R}^d$ with parameters $(\alpha, \lambda)$ defined in* (2.4) *satisfies*

$$\dim_{\mathcal{H}}(E) \leq d - C(d,c)\varepsilon^{d-1}.$$

# 3 Weak mean porosity of the boundary of Sobolev extension domains

In this section we will show that the boundary of a planar bounded simply-connected $W^{1,p}$-extension domain (with $1 < p < 2$) is weakly mean porous with the parameters depending on the constant $C$ appearing in the *curve condition* (3.1) that characterizes $W^{1,p}$-extension domains (cf. Theorem 3.1 below).

The following result has been proven in [5]:

**Theorem 3.1.** *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. Then $\Omega$ is a $W^{1,p}$-extension domain if and only if there exists a constant $C = C(\Omega, p) > 0$ such that every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be joined by a rectifiable curve $\gamma \in \mathbb{R}^2 \setminus \Omega$ satisfying*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \leq C\|z_1 - z_2\|^{2-p}. \tag{3.1}$$

Now we are ready to state our main result.

**Theorem 3.2.** *There exist universal constants $C', C'' > 0$ so that the following holds. Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected $W^{1,p}$-extension domain. Let $C$ be the constant from the curve condition* (3.1)*. Then $\partial\Omega$ is weakly mean porous with parameters $(\alpha, \lambda)$, where $\lambda(k) = C'C$ and $\alpha(t) = \frac{C''}{C}t$.*

In the proof of Theorem 3.2, we use the following result to relate the length of the curve $\gamma$ in (3.1) to the diameter of cubes it intersects.

**Lemma 3.3.** *Let $1 < p < 2$, let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected $W^{1,p}$-extension domain and let two points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ be given. Then there exists a curve $\gamma$ connecting $z_1$ and $z_2$ in $\mathbb{R}^2 \setminus \Omega$ that minimizes*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \tag{3.2}$$

*and satisfies*

$$\mathcal{H}^1(\gamma \cap \overline{Q}) \leq 10\,\ell(Q)$$

*for every $Q \in \mathcal{W}_{\mathbb{R}^2\setminus\overline{\Omega}}$.*

*Proof.* The existence of a minimizer for (3.2) is standard and has been established in the proof of [5, Lemma 2.17]. Let $Q \in \mathcal{W}_{\mathbb{R}^2\setminus\overline{\Omega}}$. Define $t_1 = \min\{t : \gamma(t) \in \overline{Q}\}$ and $t_2 = \max\{t : \gamma(t) \in \overline{Q}\}$. Then, by (2.1) and the minimality of $\gamma$,

$$\begin{aligned}
\mathcal{H}^1(\gamma \cap \overline{Q})(5\sqrt{2}\ell(Q))^{1-p} &\leq \mathcal{H}^1(\gamma \cap \overline{Q})(\operatorname{dist}(Q, \partial\Omega) + \operatorname{diam}(Q))^{1-p} \\
&\leq \int_{\gamma\cap\overline{Q}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \\
&\leq \int_{[\gamma(t_1),\gamma(t_2)]} \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \\
&\leq \operatorname{diam}(Q)\operatorname{dist}(Q, \partial\Omega)^{1-p} \leq \sqrt{2}\ell(Q)^{2-p}.
\end{aligned}$$

Thus, the claim holds. □

*Proof of Theorem 3.2.* Without loss of generality, we may assume that $C \geq 1$. Let $\varepsilon := 2^{-m} \in (\frac{2^{-15}}{C}, \frac{2^{-14}}{C}]$ with $m \in \mathbb{Z}$. We start by constructing the collection $\mathcal{D}$ of cubes in $\mathbb{R}^2 \setminus \partial\Omega$. We decompose every $Q \in \mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}$ into $\mathcal{W}_Q$ and enumerate $\mathcal{W}_Q = \{Q_i(Q)\}_{i\in\mathbb{N}}$. We will show that the family

$$\mathcal{D} := \{Q_i(Q) : i \in \mathbb{N},\ Q \in \mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}\}$$

gives the claimed weak mean porosity of $\partial\Omega$ with the functions $\lambda(k) = \varepsilon^{-1}2^{-10}$ and $\alpha(t) = \varepsilon t$.

Let $k_0$ be the smallest positive integer for which $2^{-k_0} < \operatorname{diam}(\Omega)$. It suffices to show that $\chi_k^{\mathcal{D}}(x) = 1$ for all $k \geq k_0$ and $x \in \partial\Omega$. Let us fix $k \in \mathbb{N}$, with $k \geq k_0$, and $x \in \partial\Omega$.

Case 1. First, let us suppose that the following condition holds true:

$$\text{For every } r \in \left[\frac{2}{3}2^{-k}, \frac{5}{6}2^{-k}\right] \text{ there exists } y \in \partial B(x, r) \text{ so that } B(y, \varepsilon 2^{-k+5}) \cap \partial\Omega = \emptyset. \tag{3.3}$$

Consider the set of radii

$$R := \left\{r : r = \frac{2}{3}2^{-k} + \varepsilon 2^{-k+6} i \leq \frac{5}{6}2^{-k},\ i \in \mathbb{N}\right\}.$$

For each $r \in R$ we select a point $y_r \in \partial B(x, r)$ so that $B(y_r, \varepsilon 2^{-k+5}) \cap \partial\Omega = \emptyset$, as given by (3.3). Now, given any $r \in R$, the set $B(y_r, \varepsilon 2^{-k+5}) \subset \mathbb{R}^2 \setminus \partial\Omega$ contains a dyadic square $Q$ of sidelength $\varepsilon 2^{-k+2}$ with distance at least $\varepsilon 2^{-k+2}$ to $\partial\Omega$. Thus, $\partial B(x, r) \cap Q \neq \emptyset$ for some $Q \in \mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}$ with $\ell(Q) \geq \varepsilon 2^{-k+2}$. Since $x \in \partial\Omega$, $\operatorname{diam}(\Omega) > 2^{-k}$ and $\Omega$ is bounded and simply-connected, we have

$$\partial B(x, r) \cap \partial\Omega \neq \emptyset,$$

and so also arbitrarily small cubes in $\mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}$ intersect $\partial B(x, r)$. Consequently, taking into account (2.2) there exists $Q_r \in \mathcal{W}_{\mathbb{R}^2\setminus\partial\Omega}$ with $\ell(Q_r) = \varepsilon 2^{-k+2}$ and

$$\partial B(x, r) \cap Q_r \neq \emptyset.$$

By the bound (2.3), there exists $Q_r' \in \mathcal{W}_{Q_r} \subset \mathcal{D}$ with $\ell(Q_r') = \varepsilon 2^{-k}$. Then the collection of cubes $\{Q_r' : r \in R\} \subset \mathcal{D}$ is disjointed. A simple calculation shows that we have $\#R \geq \frac{2^{-9}}{\varepsilon}$. Thus, $\chi_k^{\mathcal{D}}(x) = 1$.

Case 2. If condition (3.3) is violated, we argue as follows: Pick $r \in (\frac{2}{3}2^{-k}, \frac{5}{6}2^{-k})$ such that for every $y \in \partial B(x,r)$ it holds that $B(y, \varepsilon 2^{-k+5}) \cap \partial\Omega \neq \emptyset$. Let $\{y_i\}_{i=1}^m$ be a maximal $\varepsilon 2^{-k+5}$-separated net of points in $\partial B(x,r)$ enumerated in a clockwise order around $x$. Since $B(y_i, \varepsilon 2^{-k+5}) \cap \partial\Omega \neq \emptyset$, we can select, for each $i$ a point $w_i \in B(y_i, \varepsilon 2^{-k+5}) \setminus \Omega$. Let us denote $w_{m+1} = w_1$. We claim that for some $i \in \{1, \dots, m\}$,

$$\text{any curve connecting } w_i \text{ to } w_{i+1} \text{ in } \mathbb{R}^2 \setminus \Omega \text{ must exit } B(w_i, 2^{-k-3}). \tag{3.4}$$

Suppose this is not the case. Then we can connect $w_i$ to $w_{i+1}$ by a curve $\sigma_i$ in $B(w_i, 2^{-k-3}) \setminus \Omega$. The concatenation $\sigma$ of $\sigma_1, \dots, \sigma_m$ is then contained in the annulus

$$B(x, r + 2^{-k-3}) \setminus B(x, r - 2^{-k-3}) \subset B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$$

and has winding number $-1$ around $x$. However, since $x \in \partial\Omega$ and $\Omega \setminus B(x, 2^{-k}) \neq \emptyset$, the curve $\sigma$ then disconnects $\Omega$, which is impossible. Thus, we have the existence of $i$ for which (3.4) holds.

Let $\gamma\colon [0,1] \to \mathbb{R}^2 \setminus \Omega$ be a curve connecting $z_1 := w_i$ and $z_2 := w_{i+1}$ which minimizes the integral (3.2). Call $A := \{z \in \gamma : \operatorname{dist}(z, \partial\Omega) > 5\sqrt{2}\varepsilon 2^{-k+2}\}$ and note that (3.1) yields

$$\begin{aligned}(5\sqrt{2}\varepsilon 2^{-k+2})^{1-p}\mathcal{H}^1(\gamma \setminus A) &\le \int_{\gamma\setminus A} \operatorname{dist}(z, \partial\Omega)^{1-p}\,ds(z)\\ &\le \int_{\gamma} \operatorname{dist}(z, \partial\Omega)^{1-p}\,ds(z) \le C(\varepsilon 2^{-k+7})^{2-p}.\end{aligned}$$

Consequently, by the choice of $\varepsilon$, we have that

$$\mathcal{H}^1(\gamma \setminus A) \le 2^{5(2-p)+2}(5\sqrt{2})^{p-1}\varepsilon C 2^{-k} \le 2^{10}\varepsilon C 2^{-k} \le 2^{-k-4}$$

and hence

$$\mathcal{H}^1(A \cap B(w_i, 2^{-k-3})) = \mathcal{H}^1(\gamma \cap B(w_i, 2^{-k-3})) - \mathcal{H}^1(\gamma \setminus A) \ge 2^{-k-3} - 2^{-k-4} \ge 2^{-k-4}. \tag{3.5}$$

Now, notice that by the choice of the radius $r$, the point $w_i$ and the factor $\varepsilon$, we get

$$\operatorname{dist}(\mathbb{R}^2 \setminus A_k(x), B(w_i, 2^{-k-3})) \ge \frac{1}{6}2^{-k} - \varepsilon 2^{-k+5} - 2^{-k-3} \ge 2^{-k-6}. \tag{3.6}$$

Write

$$\mathcal{Q} := \{Q \in \mathcal{W}_{\mathbb{R}^2\setminus\overline{\Omega}} : \ell(Q) \ge \varepsilon 2^{-k+2} \text{ and } Q \cap B(w_i, 2^{-k-3}) \neq \emptyset\}.$$

Suppose first that there exists $Q \in \mathcal{Q}$ with $\ell(Q) \ge 2^{-k-7}$. Then, by the definition of the decomposition $\mathcal{W}_Q$ and by (3.6) a square $Q' \in \mathcal{W}_Q$ with $\ell(Q') = \varepsilon 2^{-k}$ that is closest to $w_i$ satisfies

$$\begin{aligned}\operatorname{dist}(\mathbb{R}^2 \setminus A_k(x), Q') &\ge \operatorname{dist}(\mathbb{R}^2 \setminus A_k(x), B(w_i, 2^{-k-3})) - \sqrt{2}\operatorname{dist}(Q', \partial Q) - \operatorname{diam}(Q')\\ &\ge 2^{-k-6} - \sqrt{2}\ell(Q') - \sqrt{2}\ell(Q') \ge 2^{-k-6} - \varepsilon 2^{-k+2}.\end{aligned}$$

Therefore, by counting the consecutive squares of side-length $\varepsilon 2^{-k}$ in $\mathcal{W}_Q$ starting from this square, we obtain the estimate

$$\#\{Q' \in \mathcal{D} : Q' \in \mathcal{W}_Q, Q' \subset A_k(x) \text{ and } \ell(Q') = \varepsilon 2^{-k}\} \ge \frac{2^{-k-7}}{\varepsilon 2^{-k}} \ge \frac{2^{-7}}{\varepsilon}$$

and thus, $\chi_k^{\mathcal{D}}(x) = 1$.

Suppose then that for all $Q \in \mathcal{Q}$ we have $\ell(Q) \le 2^{-k-7}$. Then, by (3.6) for all $Q \in \mathcal{Q}$ we have $Q \subset A_k(x)$. Notice that by (2.1), $A$ is contained in the closure of the union of Whitney cubes $Q \in \mathcal{W}_{\mathbb{R}^2\setminus\overline{\Omega}}$ with $\ell(Q) \ge \varepsilon 2^{-k+2}$ and that $\mathcal{H}^1$-almost every point in $\mathbb{R}^2$ is contained in the closure of at most two $Q \in \mathcal{Q}$. Therefore, by using Lemma 3.3 and (3.5), we get

$$\sum_{Q\in\mathcal{Q}} \ell(Q) \ge \frac{1}{10}\sum_{Q\in\mathcal{Q}} \mathcal{H}^1(\gamma \cap \overline{Q}) \ge \frac{1}{20}\mathcal{H}^1(A \cap B(w_i, 2^{-k-3})) \ge 2^{-k-9}.$$

So, by (2.3)

$$\#\{Q' \in \mathcal{D} : Q' \subset A_k(x) \text{ and } \ell(Q') = \varepsilon 2^{-k}\} \ge \sum_{Q\in\mathcal{Q}} \frac{\ell(Q)}{\varepsilon 2^{-k+1}} \ge \varepsilon^{-1}2^{-10}.$$

Again, $\chi_k^{\mathcal{D}}(x) = 1$, concluding the proof. □

# 4 Examples

In this section we show the sharpness of our estimate between the constant in the curve condition and the dimension of the boundary. We also show that the dimension estimate via the John condition is necessarily less sharp. Let us write the conclusions from the two sets of examples we consider in the following theorem. The examples we consider are well known and the optimal John constants folklore. We still provide here full details for the convenience of the reader.

**Theorem 4.1.** *The following sets exist.*

(1) *For every $J \in (0, \frac{1}{2})$ there exists a Jordan $J$-John domain $\Omega \subset \mathbb{R}^2$ for which*

$$\dim_{\mathcal{H}}(\partial\Omega) \geq 2 - \frac{2}{\log(2)} J.$$

(2) *For every $p \in (1,2)$ and $C \in (\frac{72}{2-p}, \infty)$ there exists a Jordan domain $\Omega \subset \mathbb{R}^2$ satisfying the curve condition* (3.1) *with the constant $C$ and exponent $p$, for which*

$$\dim_{\mathcal{H}}(\partial\Omega) \geq 2 - \frac{24}{\log(2)(2-p)C}.$$

(3) *There exists a universal constant $c > 0$ so that for every $p \in (1,2)$ and $C \in (c, \infty)$ there exists a Jordan domain $\Omega \subset \mathbb{R}^2$ satisfying the curve condition* (3.1) *with the constant $C$, but failing to be a $J$-John domain for any*

$$J \geq c((2-p)C)^{\frac{1}{p-2}}.$$

Recall that the quasi-convexity of the complement of a domain $\Omega \subset \mathbb{R}^2$, and thus in particular the curve condition (3.1), implies that $\Omega$ is John, [9]. However, the curve condition (3.1) does not imply that the complementary open set $\mathbb{R}^2 \setminus \overline{\Omega}$ would be even connected. In particular, the complementary domain does not have to be a John domain in the Jordan domain case.

In the rest of the section we prove the existence of the sets mentioned in Theorem 4.1.

## 4.1 Cones

The first set of examples shows claim (3) in Theorem 4.1. We consider a fixed square and on top of it attach a cone whose width is the parameter $\varepsilon$ that we vary in order to change the constants in the curve condition (3.1) and the John condition.

**Example 4.2.** Let $\varepsilon \in (0, \frac{1}{2})$ and $1 < p < 2$. Let

$$\Omega := \{(x^1, x^2) : |x^1| < 1,\ |x^2 + 1| < 1\} \cup \{(x^1, x^2) : |x^1| < (1 - x^2)\varepsilon, x^2 \geq 0\} \subset \mathbb{R}^2.$$

Then the following hold:

(i) For $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ the curve condition (3.1) holds with constant $C = \frac{c}{2-p}\varepsilon^{p-2}$ with some constant $c > 0$ independent of $\varepsilon$.

(ii) The set $\Omega$ fails to be $J$-John for any $J > \varepsilon$.

*Proof of* (i). Notice first that for $\Omega' := \Omega \cup (0,1) \times (0,1)$ there exists a constant $C > 0$ independent of $\varepsilon$ so that $\Omega'$ satisfies (3.1) with this $C$. Write $z_i = (z_i^1, z_i^2)$. Thus, we may assume that $-1 \leq z_1^1 \leq 0 \leq z_2^1 \leq 1$ and $0 \leq z_1^2, z_2^2 \leq 1$.

Let us define $w_1 = (z_1^1 + z_1^2 - 1, 1)$ and $w_2 = (z_2^1 - z_2^2 + 1, 1)$. We claim that the concatenation $\gamma$ of the line-segments $[z_1, w_1]$, $[w_1, w_2]$ and $[w_2, z_2]$ satisfies the curve condition with the claimed constant. See Figure 2 for an illustration of the curve. For the lengths of the line-segments we have the estimates

$$\|w_i - z_i\| = \sqrt{2}|z_i^2 - 1| \leq \frac{\sqrt{2}}{\varepsilon}|z_i^1| \leq \frac{\sqrt{2}}{\varepsilon}\|z_1 - z_2\|$$

and

$$\|w_1 - w_2\| = |(z_1^1 + z_1^2 - 1) - (z_2^1 - z_2^2 + 1)| \leq |z_1^2 - 1| + |z_2^2 - 1| + |z_1^1 - z_2^1| \leq \frac{3}{\varepsilon}\|z_1 - z_2\|.$$

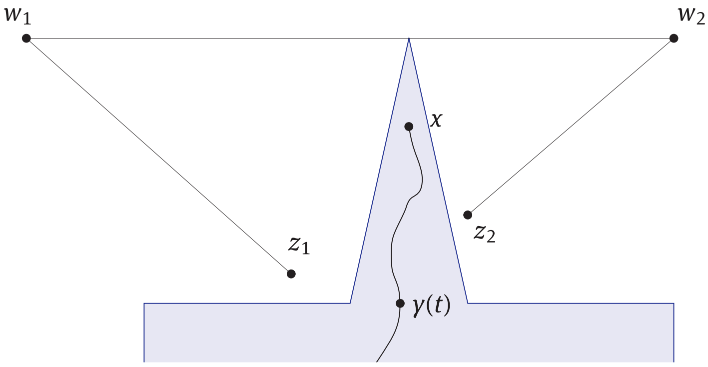


**Figure 2:** The failure of the John condition for $J > \varepsilon$ in Example 4.2 is seen by taking the point $x$ near the tip of the cone. Then every curve $\gamma$ connecting $x$ to a John center will fail the condition at a point $\gamma(t)$. The critical case for the curve condition (3.1) is the case where $z_1$ and $z_2$ are on the opposite sides of the cone. Up to a constant, an optimal way to connect them goes through the points $w_1$ and $w_2$.

Thus, we get

$$\int_{[z_i,w_i]} \operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}s(z) \le \int_0^{\frac{\sqrt{2}}{\varepsilon}\|z_1-z_2\|} \Big(\frac{t}{\sqrt{2}}\Big)^{1-p}\,\mathrm{d}t = \frac{2^{3/2-p}}{2-p}\varepsilon^{p-2}\|z_1-z_2\|^{2-p}$$

and

$$\int_{[w_1,w_2]} \operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}s(z) \le 2\int_0^{\frac{3}{\varepsilon}\|z_1-z_2\|} \Big(\frac{t}{\sqrt{2}}\Big)^{1-p}\,\mathrm{d}t = \frac{2^{(3-p)/2}3^{2-p}}{2-p}\varepsilon^{p-2}\|z_1-z_2\|^{2-p}.$$

Combining the above estimates, the claim is proven. □

*Proof of* (ii). Figure 2 shows the idea of the proof. Suppose $\Omega$ is a $J$-John domain with the John center $x_0 = (x_0^1, x_0^2) \in \Omega$. For $x_0^2 < x^2 < 1$, consider a John curve $\gamma\colon [0,\ell(\gamma)] \to \Omega$ from $(0,x^2)$ to $(x_0^1,x_0^2)$, and let $t \in [0,\ell(\gamma)]$ be such that $\gamma(t) \in \mathbb{R}\times\{\max(0,x_0^2)\}$. Then

$$Jt \le \operatorname{dist}(\gamma(t),\partial\Omega) \le \varepsilon\min(1-x_0^2, 1) \le \varepsilon\min\Big(\frac{1-x_0^2}{x^2-x_0^2}, \frac{1}{x^2-x_0^2}\Big)t \le \varepsilon\frac{1-x_0^2}{x^2-x_0^2}t.$$

Thus, by letting $x^2 \nearrow 1$, we see that $J \le \varepsilon$. □

## 4.2 Koch snowflakes

The second set of examples showing claims (1) and (2) in Theorem 4.1 is the von Koch snowflake with varying contraction constant $\lambda$ as the parameter.

**Example 4.3.** Let us first recall the construction of the von Koch curve $K$ with parameter $\lambda \in [\frac{1}{3}, \frac{1}{2})$. It is defined as the attractor of iterated function system $\{F_1, F_2, F_3, F_4\}$, where $F_1, \dots, F_4$ are the similitude mappings

$$F_1x = Sx, \quad F_2x = T_{(\lambda,0)}R_\theta Sx, \quad F_3x = T_{(\frac{1}{2},h)}R_{-\theta}Sx, \quad F_4x = T_{(1-\lambda,0)}Sx.$$

Here $Sx = \lambda x$ is the scaling by $\lambda$, $R_\tau$ is the rotation of the plane by the angle $\tau$, the used rotation angle $\theta$ here is defined by

$$\cos\theta = \frac{\frac{1}{2}-\lambda}{\lambda},$$

$T_a$ is the translation $T_a x = x + a$, and $h = \sqrt{\lambda - \frac{1}{4}}$. Recall that $K$ being the attractor means that it is the unique nonempty compact set satisfying

$$K = \bigcup_{i=1}^4 F_i(K).$$

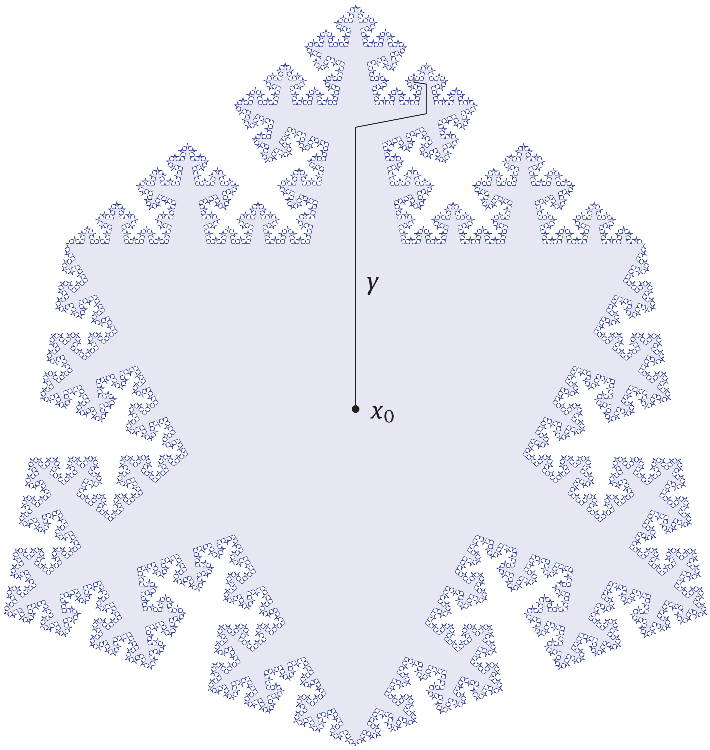


**Figure 3:** An illustration of the domain Ω bounded by three copies of a von Koch curve $K$ together with the John center $x_0$ and a John curve $\gamma$.

We define our domain Ω to be a snowflake domain whose boundary consists of three copies of $K$, see Figure 3. More precisely, Ω is the bounded component of the set

$$\mathbb{R}^2 \setminus \bigcup_{i=1}^{3} G_i(K),$$

where

$$G_1 x = x, \quad G_2 x = T_{(1,0)} R_{-\frac{2\pi}{3}} x, \quad G_3 x = T_{(\frac{1}{2}, -\frac{\sqrt{3}}{2})} R_{\frac{2\pi}{3}} x.$$

Notice that $K$ can also be obtained with an IFS consisting of just two functions. Consequently, the interior and exterior of the snowflake can be swapped in the arguments below. Since the iterated function system defining $K$ satisfies the open set condition, the Hausdorff dimension agrees with the similarity dimension, which gives

$$\dim_{\mathcal{H}}(\partial\Omega) = -\frac{\log(4)}{\log(\lambda)} \geq 2 - \frac{4}{\log(2)}\left(\frac{1}{2} - \lambda\right).$$

We claim that the following hold:
(i) The domain Ω is $\frac{\frac{1}{2}-\lambda}{\lambda}$-John, which is also optimal.
(ii) The domain Ω satisfies the curve condition (3.1) with $C = \frac{6\lambda^{2p-3}}{(2-p)(\frac{1}{2}-\lambda)}$.

Before proving the claims, let us introduce some additional notation for the Koch snowflake. For a nonnegative integer $k$, and a word $a_0 a_1 \dots a_k \in \{1, 2, 3\} \times \{1, 2, 3, 4\}^k$, we define the composed mapping

$$F_{a_0 \dots a_k} := G_{a_0} \circ F_{a_1} \circ \dots \circ F_{a_k}.$$

Now, we set $K_{a_0 \dots a_k} := F_{a_0 \dots a_k}(K)$. Similarly, by defining $L := [0, 1] \times \{0\}$, we set $L_{a_0 \dots a_k} := F_{a_0 \dots a_k}(L)$. We also fix the following notation:

$$\Delta_{a_0 \dots a_k} = \mathrm{ch}(L_{a_0 \dots a_k 2} \cup L_{a_0 \dots a_k 3}), \quad T_{a_0 \dots a_k} = L_{a_0 \dots a_k 2} \cap L_{a_0 \dots a_k 3},$$

where ch($A$) denotes the convex hull of set $A$.

*Proof of* (i). Let us first show that $\Omega$ cannot be John with a constant better than $\frac{\frac{1}{2}-\lambda}{\lambda}$. The proof is similar to the proof of (ii) in Example 4.2. Suppose that $\Omega$ is $J$-John with $x_0 \in \Omega$ the John center. Let $k \in \mathbb{N}$ be such that

$$x_0 \notin \operatorname{ch}(L_{12a_1\dots a_k} \cup L_{13b_1\dots b_k}) =: \Delta,$$

where $a_j = 4$ and $b_j = 1$ for all $1 \le j \le k$. Notice that the triangle $\Delta$ is similar to $\Delta_1$ with both having the same top vertex $T = T_1$. Let $\gamma$ be a unit speed curve connecting $T$ to $x_0$ in $\Omega \cup \{T\}$. Let $x \in \partial\Omega \cap (L_{12a_1\dots a_k 4} \cup L_{13b_1\dots b_k 1})$ and $t \in [0, \ell(\gamma)]$ be such that $\operatorname{dist}(\gamma(t), \partial\Omega) = \|\gamma(t) - x\| > 0$. Then

$$\frac{t}{\operatorname{dist}(\gamma(t), \partial\Omega)} \ge \frac{\|T - \gamma(t)\|}{\|\gamma(t) - x\|} \ge \frac{\lambda}{\frac{1}{2} - \lambda}.$$

Therefore, $J \le \frac{\frac{1}{2}-\lambda}{\lambda}$.

Let us then show that $\Omega$ is $\frac{\frac{1}{2}-\lambda}{\lambda}$-John. Let $x_0$ be the barycenter of $\Omega$, and let $x_1 \in \Omega$ be the point connected to $x_0$ with $\gamma$. Figure 3 shows the idea behind the following construction of the John curve $\gamma$. In the case $x_1 \in \Delta_0 := \operatorname{ch}(L_1 \cup L_2 \cup L_3)$ the claim is clear. Assume that $x_1 \in \Delta_{a_0\dots a_k}$, $k \ge 0$, $a_0 \in \{1, 2, 3\}$, $a_j \in \{1, 2, 3, 4\}$, $1 \le j \le k$. Let $P_{a_1\dots a_k} \in \Omega$ be the point on the line bisecting $\Delta_{a_0\dots a_k}$ through $T_{a_0\dots a_k}$ such that

$$\|T_{a_0\dots a_k} - P_{a_1\dots a_k}\| = \frac{\lambda^{k+1}}{2h}, \quad \text{where } h = \sqrt{\lambda - \frac{1}{4}}.$$

Now the line segment $[x_1, P_{a_0\dots a_k}]$ has length at most $\frac{\lambda^{k+1}}{2h}$ and (1.2) holds for all $x \in [x_1, P_{a_1\dots a_k}]$ with $J = \frac{\frac{1}{2}-\lambda}{\lambda}$.

By symmetry and self-similarity, the points $P_{a_0}, P_{a_0 a_1}, \dots, P_{a_0 a_1\dots a_k}$ have the following properties (where $a_0 \in \{1, 2, 3\}$ and $a_1, \dots, a_k \in \{1, 2, 3, 4\}$): for $x = tP_{a_0\dots a_m} + (1-t)P_{a_0\dots a_{m+1}}$, $t \in [0, 1]$ and $m \ge 0$,

$$\ell([P_{a_0\dots a_{m+1}}, x]) = t\frac{\lambda^{m+1}(1-\lambda)}{2h}$$

and

$$\operatorname{dist}(\partial\Omega, P_{a_0\dots a_m}) \ge \frac{(\frac{1}{2} - \lambda)\lambda^m}{2h},$$

which by the construction of $\Omega$ gives

$$\operatorname{dist}(\partial\Omega, x) \ge (1-t)\frac{(\frac{1}{2} - \lambda)\lambda^{m+1}}{2h} + t\frac{(\frac{1}{2} - \lambda)\lambda^m}{2h} = [(1-t)\lambda + t]\frac{(\frac{1}{2} - \lambda)\lambda^m}{2h}.$$

Therefore, for all $1 \le m \le k-1$ and $x \in [P_{a_0\dots a_{m+1}}, P_{a_0\dots a_m}]$,

$$\begin{aligned}
\ell(\gamma|_{x_1 \to x}) &= \ell([x_1, P_{a_0\dots a_k}]) + \sum_{j=m+2}^{k} \ell([P_{a_1\dots a_{j-1}}, P_{a_1\dots a_j}]) + \ell([P_{a_1\dots a_{m+1}}, x]) \\
&\le \frac{\lambda^{k+1}}{2h} + \sum_{j=m+2}^{k} \frac{\lambda^j(1-\lambda)}{2h} + t\frac{\lambda^{m+1}(1-\lambda)}{2h} \\
&= [\lambda(1-t) + t]\frac{\lambda^{m+1}}{2h} \\
&\le \frac{\lambda}{\frac{1}{2} - \lambda}\operatorname{dist}(\partial\Omega, x),
\end{aligned}$$

where $\gamma|_{x_1 \to x}$ denotes curve made of the line segments

$$[x_1, P_{a_0\dots a_k}], [P_{a_0\dots a_k}, P_{a_1\dots a_{k-1}}], \dots, [P_{a_0\dots a_{m+2}}, P_{a_0\dots a_{m+1}}], [P_{a_0\dots a_{m+1}}, x].$$

So (1.2) holds for all $x \in \gamma|_{x_1 \to P_{a_1}}$ with $J = \frac{\frac{1}{2}-\lambda}{\lambda}$ and (1.2) still holds (with the same constant) when $\gamma|_{x_1 \to P_{a_1}}$ is extended to $x_0$ with $[P_{a_1}, x_0]$. □

*Proof of* (ii). We will show that any two points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be connected by a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying (3.1) with

$$C = \frac{9\lambda^{3p-7}}{(2-p)(\frac{1}{2} - \lambda)}.$$

First of all, we may assume without loss of generality that $z_1, z_2 \in \partial\Omega$. Secondly, we may assume that $z_1, z_2 \in K_1$. We now divide the proof into three cases, them being Case 1: $z_1 \in K_{11}, z_2 \in K_{12}$, Case 2: $z_1 \in K_{12}$,

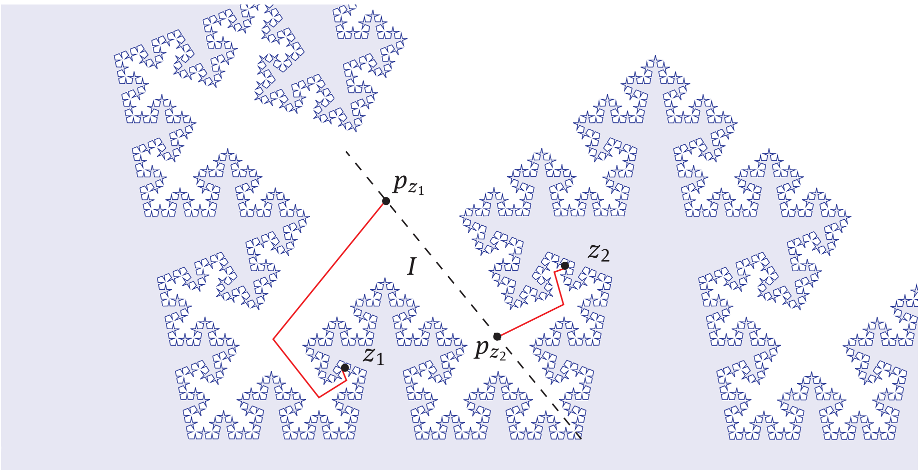


**Figure 4:** In the proof of the curve condition (3.1) we consider three critical cases. Here, in the zoomed in picture of Case 1, the points $z_1$ and $z_2$ are connected to points $p_{z_1}$ and $p_{z_2}$ on a line-segment $I$ using the curves constructed in the proof of claim (i).

$z_2 \in K_{13}$, and Case 3: $z_1 \in K_{11}$, $z_2 \in K_{13} \cup K_{14}$. Other cases follow then by symmetry, and from self-similarity by zooming in to the construction. We treat only the case 1 in detail, giving the ideas for the other two.

Case 1: $z_1 \in K_{11}$ and $z_2 \in K_{12}$. Let us call $z_1', z_2'$ the orthogonal projections of $z_1$ and $z_2$ on the line-segment

$$I := T_{(\lambda,0)}R_{(\pi-\theta)/2}L$$

(a line-segment in mirroring $K_{11}$ and $K_{12}$). We will define points $p_{z_1}$ and $p_{z_2}$ in $I$, that are connected to $z_1$ and $z_2$ by curves, which we will call $\gamma_1$ and $\gamma_2$. We then join the points $p_{z_1}$ and $p_{z_2}$ with a line-segment. See Figure 4 for an illustration. Let us write $\{o\} := K_{11} \cap K_{12}$. If $z_1 = o$, we take $p_{z_1} = z_1$. If not, then there exists $k \geq 1$ such that $z_1 \in K_{11a_1\ldots a_k}$ with $a_i = 4$ for all $i < k$ and $a_k \neq 4$. We can make a crude estimate

$$\|z_1 - z_1'\| \geq \Big(\frac{1}{2} - \lambda\Big)\lambda^{k+2}. \tag{4.1}$$

Now, by the proof of (i), $z_1$ can be connected to a point $p_{z_1} \in I$ by a John curve with John constant $\frac{\frac{1}{2}-\lambda}{\lambda}$ and length less than $\lambda^{k-1}$. Combining this with (4.1), we get

$$\begin{aligned}\int_{\gamma_1} \operatorname{dist}(z, \partial\Omega)^{1-p}\,dz &\leq 2\int_0^{\lambda^{k-1}} \Big(\frac{\frac{1}{2}-\lambda}{\lambda}t\Big)^{1-p}\,dt \\ &= \frac{2}{2-p}\Big(\frac{\frac{1}{2}-\lambda}{\lambda}\Big)^{1-p}(\lambda^{k-1})^{2-p} \\ &\leq \frac{2}{2-p}\frac{\lambda^{3p-7}}{\frac{1}{2}-\lambda}\|z_1 - z_1'\|^{2-p} \leq \frac{C}{3}\|z_1 - z_2\|^{2-p}.\end{aligned} \tag{4.2}$$

By symmetry, with the same arguments we also find $p_{z_2}$ and the curve $\gamma_2$ connecting $z_2$ to $p_{z_2}$, and get

$$\int_{\gamma_2} \operatorname{dist}(z, \partial\Omega)^{1-p}\,dz \leq \frac{2}{2-p}\frac{\lambda^{3p-7}}{\frac{1}{2}-\lambda}\|z_1 - z_1'\|^{2-p} \leq \frac{C}{3}\|z_1 - z_2\|^{2-p}. \tag{4.3}$$

For the line-segment $[p_{z_1}, p_{z_2}]$, notice that we have

$$\begin{aligned}\|p_{z_1} - p_{z_2}\| &\leq \|z_1' - z_2'\| + \|p_{z_1} - z_1'\| + \|p_{z_2} - z_2'\| \leq \|z_1' - z_2'\| + 2\lambda^{k-1} \\ &\leq \|z_1' - z_2'\| + \lambda^{-3}\Big(\frac{1}{2} - \lambda\Big)^{-1}(\|z_1 - z_1'\| + \|z_2 - z_2'\|) \\ &\leq 3\lambda^{-3}\Big(\frac{1}{2} - \lambda\Big)^{-1}\|z_1 - z_2\|\end{aligned}$$

and thus

$$\begin{aligned}\int\limits_{[p_{z_1},p_{z_2}]} \operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}z &\le \int\limits_0^{\|p_{z_1}-p_{z_2}\|} \Big(\frac{\frac{1}{2}-\lambda}{\lambda}t\Big)^{1-p}\,\mathrm{d}t \\ &\le \Big(\frac{\frac{1}{2}-\lambda}{\lambda}\Big)^{1-p}\frac{1}{2-p}\Big(3\lambda^{-3}\Big(\frac{1}{2}-\lambda\Big)^{-1}\|z_1-z_2\|\Big)^{2-p} \\ &\le \frac{3^{2-p}}{2-p}\frac{\lambda^{3p-7}}{\frac{1}{2}-\lambda}\|z_1-z_2\|^{2-p} \\ &\le \frac{C}{3}\|z_1-z_2\|^{2-p}.\end{aligned} \tag{4.4}$$

Combining (4.2), (4.3), and (4.4), we conclude the first case.

Case 2: $z_1 \in K_{12}$ and $z_2 \in K_{13}$. In this case, we connect $z_1$ and $z_2$ to the unique point $p \in K_{12} \cap K_{13}$ by curves $\gamma_1$ and $\gamma_2$. The estimate for $\gamma_1$ and $\gamma_2$ are exactly the same as in Case 1. We connect $z_1$ to $p_{z_1}$ with a John curve and then $p_{z_1}$ to $p$ (instead of $z_1'$) with a line-segment.

Case 3: $z_1 \in K_{11}$ and $z_2 \in K_{13} \cup K_{14}$. Similarly as in the second case, we can connect $z_1$ and $z_2$ to the unique point $p \in K_{12} \cap K_{13}$ obtaining the desired estimate also in this case. □

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# Dimension estimate for the two-sided points of planar Sobolev extension domains

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# DIMENSION ESTIMATE FOR THE TWO-SIDED POINTS OF PLANAR SOBOLEV EXTENSION DOMAINS

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ABSTRACT. In this paper we give an estimate for the Hausdorff dimension of the set of two-sided points of the boundary of bounded simply connected Sobolev $W^{1,p}$-extension domain for $1 < p < 2$. Sharpness of the estimate is shown by examples. We also prove the equivalence of different definitions of two-sided points.

## 1. INTRODUCTION

This paper is part of the study of the geometry of the boundary of Sobolev extension domains in Euclidean spaces. We investigate the size of the set of two-sided points of simply connected planar Sobolev extension domains. Recall that a domain $\Omega$ is a $W^{1,p}$-extension domain if there exists a bounded operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ with the property that $Eu|_\Omega = u$ for each $u \in W^{1,p}(\Omega)$. Here, for $p \in [1, \infty]$, we denote by $W^{1,p}(\Omega)$ the set of all functions in $L^p(\Omega)$ whose first distributional derivatives are in $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is normed by

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

When $p > 1$ operator $E$ can be assumed linear [9]. For $p = 1$ the linearity is known for the planar bounded simply connected case [14].

Several classes of domains are known to be $W^{1,p}$-extension domains. For instance, Lipschitz domains [3], [21]. Jones [11] introduced a wider class of $(\epsilon, \delta)$-domains and proved that every $(\epsilon, \delta)$-domain is a $W^{1,p}$-extension domain. Notice that the Hausdorff dimension of the boundary of a Lipschitz domain is $n - 1$ and the boundary is rectifiable. For an $(\epsilon, \delta)$-domain the Hausdorff dimension of the boundary may be strictly greater than $n - 1$ and it may not be locally rectifiable (for example the Koch snowflake). However, an easy argument shows that the boundary of an $(\epsilon, \delta)$-domain can not self-intersect.

The case we study in this paper is with $\Omega \subset \mathbb{R}^2$ bounded and simply connected. In this case, the $W^{1,p}$-extendability has been characterized. As we will see, from the characterizations it follows that the only relevant case for us is with $p < 2$. Firstly, for $p = 2$, from the results in [6], [7], [8], [11], we know that a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,2}$-extension domain if and only if $\Omega$ is a quasidisk, or equivalently a uniform domain.

For $2 < p < \infty$ and a finitely connected bounded planar domain $\Omega$, Shvartsman [20] proved that $\Omega$ is a Sobolev $W^{1,p}$-extension domain if and only if for some $C > 1$ the following condition is satisfied: for every $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining $x$ to $y$ such that

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-p}}\, \mathrm{d}s(z) \le C\|x - y\|^{\frac{p-2}{p-1}}.$$

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In particular, when $2 \leq p < \infty$, a finitely connected bounded $W^{1,p}$-extension domain $\Omega$ is quasiconvex, meaning that there exists a constant $C \geq 1$ such that any pair of points in $z_1, z_2 \in \Omega$ can be connected with a rectifiable curve $\gamma \subset \Omega$ whose length satisfies $\ell(\gamma) \leq C|z_1 - z_2|$. Let us point out that (uniformly locally) quasiconvex domains are exactly the $W^{1,\infty}$-extension domains [22], [9].

In paper [13] the case $1 < p < 2$ was characterized: a bounded simply connected $\Omega \subset \mathbb{R}^2$ is a Sobolev $W^{1,p}$-extension domain if and only if there exists a constant $C > 1$ such that for every $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting $z_1$ and $z_2$ and satisfying

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \leq C\|z_1 - z_2\|^{2-p}. \tag{1.1}$$

The above geometric characterizations give bounds for the size of the boundary of Sobolev extension domains. The following estimate for the Hausdorff dimension of the boundary for simply connected $W^{1,p}$-extension domain $\Omega$ in the case $p \in (1, 2)$ was given in [15] :

$$\dim_{\mathcal{H}}(\partial\Omega) \leq 2 - \frac{M}{C},$$

where $C$ is the constant in (1.1) and $M > 0$ is an universal constant. Recall, that for $s > 0$, the *s-dimensional Hausdorff measure* of a subset $A \subset \mathbb{R}^n$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A),$$

where $\mathcal{H}^s_\delta(A) = \inf \left\{\sum_i \operatorname{diam}(E_i)^s : A \subset \bigcup_i E_i, \operatorname{diam}(E_i) \leq \delta\right\}$, and $\mathcal{H}^0$ is the counting measure. The *Hausdorff dimension* of a set $\emptyset \neq A \subset \mathbb{R}^n$ is then given by

$$\dim_{\mathcal{H}}(A) = \inf\{t \geq 0 : \mathcal{H}^t(A) < \infty\}.$$

For notational convenience, we set $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

In this paper, we are interested in the case $1 < p < 2$, when the boundary of $\Omega$ may self-intersect, (for examples see [12, Example 2.5], [4], and Section 4). More accurately, we study the size of the set of two-sided points. Our motivation is to obtain more concrete measurements differentiating general simply-connected Sobolev extension domains from $(\epsilon, \delta)$-domains.

In the literature domains with self-intersecting boundary have been studied in relation to mixed boundary value problems (see [2], [10], [1]). Note that as an immediate consequence of the curve condition (1.1) we see that at most one of the boundary parts intersecting any given two-sided point can have a well-defined normal vector, allowing the Neumann boundary condition.

Before giving the definition of two-sided points let us briefly mention the cases where $p$ is not in the interval $(1, 2)$. In the case of $2 \leq p \leq \infty$, there are no two-sided points which can be seen from the quasiconvexity. The case $p = 1$ has been characterized in [14] as a variant of quasiconvexity of the complement. In this case the dimension of the set of two-sided points does not depend on the constant in quasiconvexity.

Let us now define what we mean by a two-sided point. Here we give a definition which generalizes to $\mathbb{R}^n$, but the proof of our main theorem will use an equivalent formulation based on conformal maps, see Section 2.

**Definition 1.1** (Two-sided points of the boundary of a domain)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. A point $x \in \partial\Omega$ is called* two-sided*, if there exists $R > 0$ such that for all $r \in (0, R)$ there exist* *connected components* $\Omega_r^1$ *and* $\Omega_r^2$ *of* $\Omega \cap B(x,r)$ *that are nested:* $\Omega_r^i \subset \Omega_s^i$ *for* $0 < r < s < R$ *and* $i \in \{1,2\}$.

We denote by $\mathcal{T}$ the two-sided points of $\Omega$. Note that the nestedness condition in Definition 1.1 for the connected components $\Omega_r^i$ implies that $x \in \partial\Omega_r^i$. We establish the following dimension estimate for $\mathcal{T}$ for simply connected planar $W^{1,p}$-extension domains.

**Theorem 1.2.** *Let* $1 < p < 2$ *and* $\Omega \subset \mathbb{R}^2$ *a simply connected, bounded Sobolev* $W^{1,p}$*-extension domain. Let* $\mathcal{T}$ *be the set of two-sided points of* $\Omega$*. Then*

$$\dim_{\mathcal{H}}(\mathcal{T}) \le 2 - p + \log_2\left(1 - \frac{2^{p-1}-1}{2^{5-2p}C}\right) \le 2 - p - \frac{M_1(p)}{C}, \tag{1.2}$$

*where* $M_1(p) = \frac{2^{p-1}-1}{2^{5-2p}\log 2}$ *and* $C \ge 1$ *is the constant in* (1.1).

*Remark* 1.3. If $p$ and $C$ are such that the right-hand side in (1.2) is strictly less than 0, then $\mathcal{T} = \emptyset$.

We divide the proof Theorem 1.2 in two parts. In Proposition 2.4 we show that $\mathcal{T}$ is covered by countably many curves satisfying (1.1) and in Lemma 3.1 we show that on each such curve we have the dimension bound (1.2).

In Section 4 we show the sharpness of Theorem 1.2 by proving the following existence of examples.

**Theorem 1.4.** *Let* $1 < p < 2$*. There exist constants* $M_2 > 0$ *and* $C(p) \ge 1$ *such that for each* $C > C(p)$ *there exists Sobolev* $W^{1,p}$*-extension domain* $\Omega_{p,C}$ *satisfying* (1.1) *with* $C$*, and*

$$\dim_{\mathcal{H}}(\mathcal{T}_{\Omega_{p,C}}) \ge 2 - p - \frac{M_2}{C}. \tag{1.3}$$

## 2. Equivalent definitions for two-sided points

In this section we give equivalent conditions for the set of two-sided points in the case that the domain is John. Although the equivalence stated in Theorem 2.1 is of independent interest, the main motivation for us is Proposition 2.4, where using one of the equivalent definitions for two-sided points we show the existence of a countable collection of curves covering $\mathcal{T}$ such that each of the curves fulfills a slightly refined version of (1.1).

We note that a bounded simply connected planar domain satisfying the condition (1.1) is John (this follows from [7, Chapter 6 Theorem 3.5] with [17, Theorem 4.5]). Recall, that $\Omega$ is a $J$-John domain, if there exists a constant $J > 0$ and a point $x_0 \in \Omega$ so that for every $x \in \Omega$ there exists a unit speed curve $\gamma\colon [0,\ell(\gamma)] \to \Omega$ such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \ge Jt \qquad \text{for all } t \in [0,\ell(\gamma)]. \tag{2.1}$$

We denote the open unit disk of the plane by $\mathbb{D}$. For a bounded simply connected John domain $\Omega \subset \mathbb{R}^2$, a conformal map $f\colon \mathbb{D} \to \Omega$ can always be extended continuously to a map $f\colon \overline{\mathbb{D}} \to \overline{\Omega}$. This is because a John domain is finitely connected along its boundary [17] and a conformal map from the unit disk to $\Omega$ can be extended continuously onto the closure $\overline{\Omega}$ if and only if the domain is finitely connected along its boundary [18].

**Theorem 2.1.** *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded simply connected John domain (especially, if* $\Omega$ *is a bounded simply connected* $W^{1,p}$*-extension domain for* $1 < p < 2$*). Let* $f\colon \mathbb{D} \to \Omega$ *be a conformal map extended continuously to a function* $\overline{\mathbb{D}} \to \overline{\Omega}$ *still denoted by* $f$*. Define*

$$E = \{x \in \partial\Omega : f^{-1}(\{x\}) \text{ disconnects } \partial\mathbb{D}\}$$

*and*

$$\tilde{E} = \{x \in \partial\Omega : \text{ card}\,(f^{-1}(\{x\})) > 1\}.$$

*Then*

$$\mathfrak{T} = E = \tilde{E},$$

*where* $\mathfrak{T}$ *is the set of two-sided points according to Definition 1.1.*

In the proof of Theorem 2.1 we need the following lemma.

**Lemma 2.2.** *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected John domain, let* $x \in \partial\Omega$*, and* $r \in (0, \operatorname{diam}(\Omega))$*. Suppose that there exist two disjoint open sets* $U_1, U_2 \subset \Omega \cap B(x,r)$ *such that* $x \in \partial U_1 \cap \partial U_2$ *and both of the sets* $U_1$ *and* $U_2$ *are unions of connected components of* $\Omega \cap B(x,r)$*. Then there exist connected components* $U_1'$ *and* $U_2'$ *of* $U_1$ *and* $U_2$ *respectively, such that* $x \in \partial U_1' \cap \partial U_2'$*.*

*Proof.* Let us first show that there exists $N \in \mathbb{N}$ independent of $x$ and $r$ such that

$$\text{card}\,\{\tilde{\Omega} : \tilde{\Omega} \text{ connected component of } \Omega \cap B(x,r) \text{ such that } \tilde{\Omega} \cap B(x, r/2) \neq \emptyset\} \leq N. \quad (2.2)$$

Take $M \in \mathbb{N}$ components $\tilde{\Omega}_i$ as in (2.2), and choose from each one a point $x_i \in \tilde{\Omega}_i \cap B(x, r/2)$. Let $\gamma_i$ be a John curve connecting $x_i$ to a fixed John center $x_0$ of $\Omega$. For each $i$ for which $x_0 \notin \tilde{\Omega}_i$, the curve $\gamma_i$ must exit $B(x, 2r/3)$. For these $i$ we consider points $y_i \in \gamma_i \cap S(x, 2r/3)$, which then exist for all but maybe one of the indexes $i$. By the John condition there exists balls $B_i = B(y_i, Jr/6) \subset \tilde{\Omega}_i$. As the balls $B_i$ are disjointed and $B_i$ covers an arc of $S(x, 2r/3)$ of length at least $Jr/3$, we have $(M-1)Jr/3 \leq \frac{4}{3}\pi r$, hence $M - 1 \leq (\frac{J}{4\pi})^{-1}$.

Next we show that (2.2) implies the claim of the lemma. Let us enumerate

$$\{A_j\}_{j=1}^k := \{\tilde{\Omega} \subset U_1 : \tilde{\Omega} \text{ connected component of } \Omega \cap B(x,r) \text{ such that } \tilde{\Omega} \cap B(x, r/2) \neq \emptyset\}.$$

By (2.2) we have $k \leq N$. Since $U_1$ consists of connected components of $\Omega \cap B(x,r)$, we have

$$U_1 \cap B(x, r/2) \subset \bigcup_{j=1}^{k} A_j.$$

Now, because $x \in \overline{\bigcup_{j=1}^k A_j} = \bigcup_{j=1}^k \overline{A_j}$ there exists $j$ such that $x \in \overline{A_j}$. We call this $A_j$ the set $U_1'$. Similarly we find $U_2'$ for $U_2$. □

Notice that Lemma 2.2 does not hold for general simply connected domain $\Omega$, for example, consider the topologist's comb.

*Proof of Theorem 2.1.* We divide the proof into several claims. Showing that

$$\tilde{E} \subset E \subset \mathfrak{T} \subset \tilde{E}.$$

Claim 1: $\tilde{E} \subset E$.

Let $z \in \partial\Omega$ such that $\text{card}\,(f^{-1}(\{z\})) > 1$, and $A = \partial\mathbb{D} \setminus f^{-1}(\{z\})$. Let $x_1, x_2 \in f^{-1}(\{z\})$. By [19, Theorem 10.9], the set $f^{-1}(\{z\})$ has Hausdorff dimension zero. Therefore, we find points of $A$ from both components of $\partial\mathbb{D} \setminus \{x_1, x_2\}$. Hence $A$ is disconnected in $\partial\mathbb{D}$, and thus $z \in E$.

Claim 2: $\mathfrak{T} \subset \tilde{E}$.

Let $z \in \mathfrak{T}$. By assumption there exists $R > 0$ such that for each $0 < r < R$ there exists disjoint connected components $\Omega_r^1, \Omega_r^2 \subset \Omega \cap B(z,r)$ with the property that $\Omega_r^i \subset \Omega_s^i$ when $0 < r < s$. Towards a contradiction, assume that $f^{-1}(\{z\})$ is a singleton ($w = f^{-1}(z)$). By continuity of $f$ (up to the boundary) there exists $\varepsilon > 0$ such that $f(B(w,\varepsilon)\cap\overline{\mathbb{D}}) \subset B(z,r)$. Being a continuous image of a connected set $f(B(w,\varepsilon)\cap\mathbb{D})$ is connected. We show that $f^{-1}(\Omega_r^j)\cap B(w,\varepsilon)\neq\emptyset$ for $j=1,2$ which gives a contradiction with $\Omega_r^j$ being the disjoint connected components of $B(z,r)\cap\Omega$. Let $(z_i^j)_{i=1}^\infty \subset \Omega_r^j$ be a sequence such that $z_i^j \to z$. By going to a subsequence, we may assume that $(f^{-1}(z_i^j))_{i=1}^\infty$ converges to a point $w^j \in \overline{f^{-1}(\Omega_r^j)}$. Since $f$ is continuous, $f(w^j) = z$. But then $w^j = w$ by the uniqueness of the preimage of $z$. Hence, $f^{-1}(z_i^j) \to w$, meaning that for some $i$ we have $f^{-1}(z_i^j) \in B(w,\varepsilon)$ showing $f^{-1}(\Omega_r^j)\cap B(w,\varepsilon)\neq\emptyset$. Therefore, $\Omega_r^j \cap f(B(w,\varepsilon)\cap\mathbb{D}) \neq \emptyset$, connecting sets $\Omega_r^j$. This completes the proof.

Claim 3: $E \subset \mathcal{T}$.

Let $z\in E$. We will show that $z\in\mathcal{T}$. We do this by first showing by induction that there exists $i_0\in\mathbb{N}$ so that for all $i\geq i_0$ there exist connected components $\Omega_{2^{-i}}^j$ of $\Omega\cap B(z,2^{-i})$, $j\in\{1,2\}$, that are nested for fixed $j\in\{1,2\}$. At each step of the induction we will have to make sure that $z\in\partial\Omega_{2^{-i}}^1\cap\partial\Omega_{2^{-i}}^2$.

Initial step: Let us show that there exists $r>0$ such that $B(z,r)\cap\Omega$ may be written as union of two disjointed open sets such that $z$ is contained in the boundary of both sets. First, since $f^{-1}(\{z\}) = \cap_{r>0} f^{-1}(B(z,r)\cap\partial\Omega)$, there exists $R>0$ such that $H = f^{-1}(B(z,R)\cap\partial\Omega)$ disconnects $\partial\mathbb{D}$. By the continuity of $f$, $K = f^{-1}(\bar{B}(z,R/2))$ is a closed set in the closed disk $\overline{\mathbb{D}}$. Let $y_1,y_2\in\partial\mathbb{D}\setminus H$ such that $y_1$ and $y_2$ are in different connected components of $\partial\mathbb{D}\setminus H$. Define $e = \min(\operatorname{dist}(y_1,K),\operatorname{dist}(y_2,K))/2$. Now $K\setminus B(0,1-e)$ is disconnected in $\overline{\mathbb{D}}$. Next we notice that $\operatorname{dist}(f(\bar{B}(0,1-e)),\partial\Omega) = R' > 0$. Thus the original claim holds with the radius $r=\min(R,R')/2$. Let us now define $i_0\in\mathbb{N}$ to be the smallest integer for which $2^{-i_0}\leq r$. Call $U_1$ and $U_2$ the two disjoint open sets for which $z\in\partial U_1\cap\partial U_2$ and $\Omega\cap B(x,2^{-i_0}) = U_1\cup U_2$. By Lemma 2.2 we have connected components $\Omega_{2^{-i_0}}^1\subset U_1$ and $\Omega_{2^{-i_0}}^2\subset U_2$ of $\Omega\cap B(z,2^{-i_0})$ such that $z\in\partial\Omega_{2^{-i_0}}^1\cap\partial\Omega_{2^{-i_0}}^2$.

Induction step: Assume that for some $i\in\mathbb{N}$ there exist disjoint connected components $\Omega_{2^{-i}}^1$ and $\Omega_{2^{-i}}^2$ of $\Omega\cap B(z,2^{-i})$ such that $z\in\partial\Omega_{2^{-i}}^1\cap\partial\Omega_{2^{-i}}^2$. Let $U_1 = \Omega_{2^{-i}}^1\cap B(z,2^{-i-1})$.

Let us show that $U_1$ is some union of connected components of $\Omega\cap B(z,2^{-i-1})$. Let $V$ be a connected component of $U_1$. It suffices to show that $V$ is a connected component of $\Omega\cap B(z,2^{-i-1})$. Take a connected component $V'\supset V$ of $\Omega\cap B(z,2^{-i-1})$. There exists connected component $W'$ of $\Omega\cap B(z,2^{-i})$ such that $W'\supset V'$. Since $\emptyset\neq V\subset W'\cap\Omega_{2^{-i}}^1$ we have $W'=\Omega_{2^{-i}}^1$. Furthermore $V'\subset\Omega_{2^{-i}}^1\cap B(z,2^{-i-1}) = U_1$. As $V'$ is connected we have $V'=V$.

Similarly for $U_2$. Now, by Lemma 2.2 we may choose connected components $U_1'\subset U_1$ and $U_2'\subset U_2$ (of $\Omega\cap B(z,2^{-i-1})$) such that $z\in\partial U_1'\cap\partial U_2'$.

General $r\in(0,2^{-i_0})$: Let $2^{-i-1}\leq r<2^{-i}$. Let $\Omega_r^1$ be the connected component of $\Omega\cap B(z,r)$ containing $\Omega_{2^{-i-1}}^1$. Since $\Omega_{2^{-i}}^1$ is connected component of $\Omega\cap B(z,2^{-i})$ containing $\Omega_{2^{-i-1}}^1$, we have $\Omega_r^1\subset\Omega_{2^{-i}}^1$. Let us show that $\Omega_r^1\subset\Omega_s^1$ for all $0<r<s$. Let $0<r<s$. We consider two cases: (1) If $2^{-i-1}\leq r<s<2^{-i}$ the sets $\Omega_r^1$ and $\Omega_s^1$ are connected components of $\Omega\cap B(z,r)$ and $\Omega\cap B(z,s)$, respectively, both containing $\Omega_{2^{-i-1}}^1$. Since $\Omega_r^1\subset\Omega\cap B(z,s)$ and $\Omega_r^1$ is connected we have $\Omega_r^1\subset\Omega_s^1$.

(2) If $2^{-i-1}\leq r\leq 2^{-i}\leq 2^{-j-1}\leq s\leq 2^{-j}$ sets $\Omega_r^1$ and $\Omega_s^1$ are connected components of $\Omega\cap B(z,r)$ and $\Omega\cap B(z,s)$ which contain $\Omega_{2^{-i-1}}^1$ and $\Omega_{2^{-j-1}}^1$, respectively. Similarly as in (1) we have $\Omega_r^1\subset\Omega_{2^{-i}}^1\subset\cdots\subset\Omega_{2^{-j-1}}^1\subset\Omega_s^1$. □

The proof of the following lemma follows closely the proof of [13, Lemma 2.3]. We present it here for the convenience of the reader and to point out that the condition (1.1) improves to subcurves without increasing the constant $C$ in (1.1).

**Lemma 2.3.** *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain for which the following holds: There exists $C > 0$ such that for each $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting $z_1, z_2$ for which (1.1) holds. Then the following stronger statement holds: For each pair of points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ there exists an injective curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting $z_1$ and $z_2$ such that for each subcurve $\gamma|_{[t_1,t_2]}$*

$$\int_{\gamma|_{[t_1,t_2]}} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \le C \|\gamma(t_1) - \gamma(t_2)\|^{2-p}, \tag{2.3}$$

*where $C$ is the constant in the assumption.*

*Proof.* Let $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$, and let $\gamma \subset \mathbb{R}^2 \setminus \Omega$ be a curve between $z_1$ and $z_2$ for which (1.1) holds. We then have the trivial estimate

$$\begin{aligned}\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) &\ge \int_{d(z_1,\partial\Omega)}^{d(z_1,\partial\Omega)+\ell(\gamma)} x^{1-p} \, \mathrm{d}x \\ &= \frac{1}{2-p}((\ell(\gamma) + d(z_1, \partial\Omega))^{2-p} - d(z_1, \partial\Omega)^{2-p}).\end{aligned}$$

This, in combination with (1.1), gives an upper bound for the length of $\gamma$:

$$\ell(\gamma) \le \left((2-p)C\|z_1 - z_2\|^{2-p} + d(z_1, \partial\Omega)^{2-p}\right)^{\frac{1}{2-p}} - d(z_1, \partial\Omega). \tag{2.4}$$

Let $\gamma_j \subset \mathbb{R}^2 \setminus \Omega$ be a sequence of curves joining $z_1$ and $z_2$ such that

$$\int_{\gamma_j} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \le c_j \|z_1 - z_2\|^{2-p},$$

where $c_j \le C$ converge to the infimum $c \le C$ of such constants $c_j$ for the pair $z_1$ and $z_2$. By the continuity of the distance function, and since $\sup_i \ell(\gamma_i) < \infty$ by (2.4), there exists (see for example [13, Lemma 2.1]) a sequence $j_i \to \infty$ and a limit curve $\gamma$ so that $\gamma_{j_i}(t) \to \gamma(t)$ for all $t$ as $i \to \infty$ and

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \le \liminf_{i\to\infty} \int_{\gamma_{j_i}} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \le c\|z_1 - z_2\|^{2-p}.$$

Thus there exists a curve minimizing the integral in (1.1). Now, fix $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ and let $\gamma \colon [0, T] \to \mathbb{R}^2 \setminus \Omega$ be a minimizer for the integral in (1.1) for $z_1$ and $z_2$. We claim that for any $0 \le t_1 < t_2 \le T$ the subcurve $\gamma|_{[t_1,t_2]}$ of $\gamma$ is also a minimizer between $\gamma(t_1)$ and $\gamma(t_2)$. Otherwise, let $\gamma'$ be a minimizer between $\gamma(t_1)$ and $\gamma(t_2)$. Then by the linearity of the integral we have that

$$\begin{aligned}\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) &= \left(\int_{\gamma|_{[0,t_1]}} + \int_{\gamma|_{[t_1,t_2]}} + \int_{\gamma|_{[t_2,T]}}\right) \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \\ &> \left(\int_{\gamma|_{[0,t_1]}} + \int_{\gamma'} + \int_{\gamma|_{[t_2,T]}}\right) \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \\ &= \int_{\gamma''} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z),\end{aligned}$$

where

$$\gamma'' = \gamma|_{[0,t_1]} * \gamma' * \gamma|_{[t_2,T]}$$

joins $z_1$ and $z_2$. This contradicts the minimality assumption on $\gamma$. Thus our claim follows. Lastly, the injectivity of the curve is given by [5, Lemma 3.1]. □

Following the ideas of [14, Lemma 4.6] we use the equivalent definition $E$ of two-sided points from Theorem 2.1 to show that the set of two-sided points can be covered by a countable union of injective curves fulfilling condition (1.1) for each subcurve.

**Proposition 2.4.** *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be bounded simply connected Sobolev $W^{1,p}$-extension domain. Then there exists a countable collection $\Gamma$ of injective curves $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying*

$$\int_{\gamma|_{[t_1,t_2]}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \le C\|\gamma(t_1) - \gamma(t_2)\|^{2-p}, \tag{2.5}$$

*for each subcurve $\gamma|_{[t_1,t_2]}$, where $C$ is the same constant as in (1.1), so that for the set $\mathcal{T}$ of two-sided points we have*

$$\mathcal{T} \subset \bigcup_{\gamma\in\Gamma} \gamma \cap \partial\Omega.$$

*Proof.* To prove the inclusion we use the equivalent definition $E$ of two-sided points given by Theorem 2.1. Let $f : \overline{\mathbb{D}} \to \overline{\Omega}$ be continuous, and conformal in $\mathbb{D}$. Let $\{x_j\}_{j\in\mathbb{N}} \subset \partial\mathbb{D}$ be dense. For each pair $(x_i, x_j)$, $i \ne j$ we define $\gamma_{i,j}$ as an injective curve connecting $f(x_i)$ and $f(x_j)$, with property (2.5) for each subcurve. The existence of such curves is given by Lemma 2.3.

Define $\Gamma = \{\gamma_{i,j} : i \ne j\}$, and let $z \in E$. By the definition of $E$ there exist $x_a, x_b \in f^{-1}(\{z\})$, $x_a \ne x_b$, which divide $\partial\mathbb{D}$ into two components $I_a$ and $I_b$, so that $f(I_a) \ne \{z\} \ne f(I_b)$. By the continuity of $f$ there exist $i, j$, $i \ne j$, such that $x_i \in I_a$ and $x_j \in I_b$ and $f(x_i) \ne z \ne f(x_j) \ne f(x_i)$. Let $\gamma_{i,j} \in \Gamma$ be the curve connecting $f(x_i) =: z_i$ and $f(x_j) =: z_j$. Let $\tilde\gamma := f([x_i, 0] \cup [0, x_j])$. The curve $[x_i, 0] \cup [0, x_j]$ divides $\mathbb{D}$ into two components $A$ and $B$. By interchanging $A$ and $B$ if necessary, we have $x_a \in \bar A$ and $x_b \in \bar B$, and by continuity $z \in \overline{f(A)} \cap \overline{f(B)}$.

Since the curve $\gamma_{i,j}$ is injective, and $z_i \ne z_j$, the curve $\tilde\gamma \cup \gamma_{i,j}$ is Jordan. Let $\tilde A$ and $\tilde B$ be the corresponding Jordan components. Since $f(A) \subset \tilde A$, $f(B) \subset \tilde B$ we have $z \in \overline{\tilde A} \cap \overline{\tilde B} = \gamma_{i,j} \cup \tilde\gamma$. Furthermore, since $\tilde\gamma \subset f(\mathbb{D}) \cup \{z_i, z_j\} = \Omega \cup \{z_i, z_j\}$, we have $z \in \gamma_{i,j}$. □

## 3. Proof of Theorem 1.2

By Proposition 2.4 the proof of Theorem 1.2 is now reduced to proving the following lemma.

**Lemma 3.1.** *Let $1 < p < 2$ and $\gamma \subset \mathbb{R}^2 \setminus \Omega$ an injective curve satisfying*

$$\int_{\gamma|_{[t_1,t_2]}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \le C\|\gamma(t_1) - \gamma(t_2)\|^{2-p},$$

*for each subcurve $\gamma|_{[t_1,t_2]}$. Then*

$$\dim_{\mathcal{H}}(\gamma \cap \partial\Omega) \le 2 - p + \log_2\left(1 - \frac{2^{p-1} - 1}{2^{5-2p}C}\right).$$

*In particular, if $p$ and $C$ are such that the right-hand side in* (1.2) *is strictly less than 0, then* $\gamma \cap \partial\Omega = \emptyset$.[1]

To prove Lemma 3.1 we need the following sufficient condition for an upper bound of the Hausdorff dimension.

**Lemma 3.2.** *Let $F \subset \mathbb{R}^d$, $s \in \mathbb{R}$, $0 < \lambda < 1$ and $i_0 \in \mathbb{N}$. Define for each $i \geq i_0$ a maximal $\lambda^i$-separated net*

$$\{x_k^i\}_{k \in I_i} \subset F.$$

*Assume that the following holds: For each $i \geq i_0$ and $k \in I_i$ there exists $j > i$, such that*

$$N_j < \lambda^{-(j-i)s},$$

*where $N_j = \operatorname{card}(\{l \in I_j : B(x_l^j, \lambda^j) \cap B(x_k^i, \lambda^i) \neq \emptyset\})$. Then $\dim_{\mathcal{H}}(F) \leq s$.*

*In particular, if $s < 0$, then $F = \emptyset$.*

*Proof.* Define $\mathcal{B}_{i_0} = \{B(x_k^{i_0}, \lambda^{i_0}) : k \in I_{i_0}\}$ and inductively for $n > i_0$ by

$$\mathcal{B}_n = \bigcup_{B(x_k^i, \lambda^i) \in \mathcal{B}_{n-1}} \{B(x_m^j, \lambda^j) : B(x_m^j, \lambda^j) \cap B(x_k^i, \lambda^i) \neq \emptyset\},$$

where $j = j(i,k) > i$ is given by the assumption. Clearly, $\mathcal{B}_n$ is a cover of $F$ for each $n \geq i_0$, and for all $B \in \mathcal{B}_n$

$$\operatorname{diam}(B) \leq 2\lambda^n.$$

By assumption, for each $B = B(x_k^i, \lambda^i) \in \mathcal{B}_{n-1}$ and with $j = j(i,k)$ again given by the assumption

$$\sum_{B(x_m^j, \lambda^j) \cap B \neq \emptyset} \operatorname{diam}(B(x_m^j, \lambda^j))^s = N_j (2\lambda^j)^s < (2\lambda^i)^s = \operatorname{diam}(B)^s,$$

and therefore

$$\sum_{B \in \mathcal{B}_n} \operatorname{diam}(B)^s \leq \sum_{B \in \mathcal{B}_{n-1}} \operatorname{diam}(B)^s.$$

Let $\delta > 0$ and choose $n \in \mathbb{N}$ such that $2\lambda^n < \delta$. Now

$$\begin{aligned} \mathcal{H}_\delta^s(F) &\leq \sum_{B \in \mathcal{B}_n} \operatorname{diam}(B)^s \leq \sum_{B \in \mathcal{B}_{n-1}} \operatorname{diam}(B)^s \leq \dots \\ &\leq \sum_{B \in \mathcal{B}_{i_0}} \operatorname{diam}(B)^s \leq \operatorname{card}(I_{i_0})(2\lambda^{i_0})^s < \infty. \end{aligned}$$

By letting $\delta \to 0$, we get $\mathcal{H}^s(F) \leq \operatorname{card}(I_{i_0})(2\lambda^{i_0})^s < \infty$, and consequently $\dim_{\mathcal{H}}(F) \leq s$. □

*Proof of Lemma 3.1.* Define the set

$$\{x_k^i\}_{k \in I_i} \subset \gamma \cap \partial\Omega$$

to be a maximal $2^{-i}$ separated net for all $i \in \mathbb{N}$. Let $s < \min(\dim_{\mathcal{H}}(\gamma \cap \partial\Omega), 2 - p)$. We make the extra assumption $s < 2 - p$ here to have convergence in (3.2). This has no consequence on the dimension argument as we will show that $s < 2 - p + \delta$ for some $\delta = \delta(C, p) < 0$.

[1] In order to make the estimate on the dimension formally correct we adopted the notational convention $\dim_{\mathcal{H}}(\emptyset) := -\infty$.

By Lemma 3.2, there exists $i \in \mathbb{N}$ and $k \in I_i$ such that $N_j \geq 2^{(j-i)s}$ for all $j > i$, where

$$N_j = \text{ card}\,(\{l \in I_j : B(x_l^j, 2^{-j}) \cap B(x_k^i, 2^{-i}) \neq \emptyset\}).$$

Note that, trivially also $N_i \geq 1$. Denote $B = B(x_k^i, 2^{-i+1})$. For all $j \geq i+1$ the ball $B$ contains at least $N_{j-1}$ pairwise disjoint balls $B(x_l^{j-1}, 2^{-j})$ centered at $\gamma \cap \partial\Omega$, and so we have

$$\mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < 2^{-j}\}) \geq N_{j-1} 2^{-j}. \tag{3.1}$$

Using (2.5), Cavalieri's principle, (3.1), and Lemma 3.2 we estimate

$$\begin{aligned}
C2^{-(i-2)(2-p)} &\geq \int_{\gamma \cap B} \text{dist}\,(z, \partial\Omega)^{1-p}\, \mathrm{d}s(z) \\
&= \int_0^\infty \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega)^{1-p} > t\})\, \mathrm{d}t \\
&= \int_0^\infty \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < t^{\frac{1}{1-p}}\})\, \mathrm{d}t \\
&= \sum_{j \in \mathbb{Z}} \int_{2^{-(j-1)(1-p)}}^{2^{-j(1-p)}} \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < t^{\frac{1}{1-p}}\})\, \mathrm{d}t \\
&\geq \sum_{j=i+1}^{\infty} \int_{2^{-(j-1)(1-p)}}^{2^{-j(1-p)}} \mathcal{H}^1(\{z \in \gamma \cap B : d(z, \partial\Omega) < 2^{-j}\})\, \mathrm{d}t \\
&\geq \sum_{j=i+1}^{\infty} 2^{-j(1-p)}(1 - 2^{1-p}) N_{j-1} 2^{-j} \\
&\geq \sum_{j=i+1}^{\infty} (2^{p-1} - 1) 2^{-(j-1)(1-p)} 2^{(j-1-i)s} 2^{-j},
\end{aligned}$$

which implies

$$\begin{aligned}
C &\geq (2^{p-1} - 1) 2^{2p-5} \sum_{j=i+1}^{\infty} 2^{(j-i-1)(s+p-2)} \\
&= (2^{p-1} - 1) 2^{2p-5} \frac{1}{1 - 2^{-(2-(p+s))}}.
\end{aligned} \tag{3.2}$$

A reordering of (3.2) gives

$$s \leq 2 - p + \log_2 \left(1 - \frac{2^{2p-5}(2^{p-1} - 1)}{C}\right).$$

Since $s < \min(\dim_{\mathcal{H}}(\gamma \cap \partial\Omega), 2 - p)$ was arbitrary, we have

$$\dim_{\mathcal{H}}(\gamma \cap \partial\Omega) \leq 2 - p + \log_2 \left(1 - \frac{2^{2p-5}(2^{p-1} - 1)}{C}\right).$$

□

## 4. Sharpness of the dimension estimate

In this section we show the sharpness of the estimate given in Theorem 1.2. We do this by constructing a domain whose set of two-sided points contains a Cantor type set.

Let $0 < \lambda < 1/2$. Let $\mathcal{C}_\lambda$ be the standard Cantor set obtained as the attractor of the iterated function system $\{f_1 = \lambda x, f_2 = \lambda x + 1 - \lambda\}$. For later use we fix some notation. Let $I_0^1 = [0,1]$, and $\tilde{I}_1^1 := (\lambda, 1-\lambda)$ be the first removed interval. We denote by $I_j^i$ the $2^j$ closed intervals left after $j$ iterations of the construction of the Cantor set, and similarly the $2^{j-1}$ removed open intervals by $\tilde{I}_j^i$. The lengths of the intervals are

$$|I_j^i| = \lambda^j, \quad i = 1, \dots, 2^j, j = 0, 1, 2, \dots$$

and

$$|\tilde{I}_j^i| = (1-2\lambda)\lambda^{j-1}, \quad i = 1, \dots, 2^{j-1}, j = 1, 2, 3, \dots.$$

Recall that, $\mathcal{C}_\lambda$ is of zero $\mathcal{H}^1$-measure, and $\dim_{\mathcal{H}}(\mathcal{C}_\lambda) = \frac{\log 2}{-\log \lambda}$ (see e.g. [16, p.60–62]).

Define

$$\Omega_\lambda = (-1,1)^2 \setminus \{(x,y) : x \geq 0, |y| \leq d(x, \mathcal{C}_\lambda)\}.$$

Set $\Omega_\lambda$ is clearly a domain and the set of two-sided points is $\mathcal{C}_\lambda \setminus \{(0,0)\}$.

**Lemma 4.1.** *The domain $\Omega_\lambda$ above satisfies the curve condition* (1.1) *for $1 < p < 2 + \frac{\log 2}{\log \lambda}$. That is, for each $x, y \in \Omega_\lambda^c$ there exists rectifiable curve $\gamma\colon [0, l(\gamma)] \to \Omega_\lambda^c$ connecting $x, y$ such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) \leq C(p,\lambda)\|x-y\|^{2-p}. \tag{4.1}$$

*Moreover, we have the estimate*

$$C(p,\lambda) \leq \frac{c}{(2-p)(1-2\lambda^{2-p})},$$

*where $c$ is an absolute constant.*

*Proof.* To prove the claim we construct a curve connecting $x$ and $y$ in $\mathbb{R}^2 \setminus \Omega_\lambda$ consisting of line segments either parallel to the coordinate axes or at an angle $\pm\frac{\pi}{4}$. To simplify the discussion, within the proof, we will call a *component of* $\Omega_\lambda^c$ the closure of an open connected component of $\operatorname{int}(\Omega_\lambda^c)$. Let us record the following observation: If $I \subset \mathbb{R}^2 \setminus \Omega_\lambda$ is a line segment which can be arclength parametrized by $t$ in such a manner that

$$\operatorname{dist}(z, \partial\Omega_\lambda) \geq \frac{t}{\sqrt{2}} \text{ for all } z = z(t) \in I, \tag{4.2}$$

then

$$\int_I \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) \leq 2^{\frac{p-1}{2}} \int_0^{|I|} t^{1-p} \, \mathrm{d}t = \frac{2^{\frac{p-1}{2}}}{2-p}|I|^{2-p}. \tag{4.3}$$

Note that any line segment $I \subset \mathbb{R}^2 \setminus (-1,1)$ with angle $\pm\frac{\pi}{4}$ can be decomposed into at most two subsegments on which (4.2) holds, and similarly for any $I$ parallel to coordinate axes contained in a bounded component of $\Omega_\lambda^c$. For such segments we have

$$\int_I \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p} \, \mathrm{d}s(z) \leq \frac{2^{\frac{p+1}{2}}}{2-p}|I|^{2-p}. \tag{4.4}$$

Let us assume first that $x$ and $y$ are in the same component of $\Omega^c$. If $x$ and $y$ are in the unbounded component $\mathbb{R}^2 \setminus (-1,1)^2$ of $\Omega^c$, $x$ and $y$ may be connected with at most 4 diagonal segments, two of which may have to be decomposed into two to fulfill (4.2). In case of $x, y$ being in the same bounded component of $\Omega^c$ $x$ and $y$ may be connected with two segments parallel to coordinate axes (both of which we may again have to decompose into two) for which (4.4) holds. Let us now consider the case where $x$ and $y$ are in different bounded components of $\Omega^c_\lambda$. By the above we may assume that $x$ and $y$ are on the real line. Let then $j \in \mathbb{N}$ be such that

$$\lambda^j < \|x-y\| \leq \lambda^{j-1}.$$

Now, $[x,y]$ intersects at most two of the intervals $I^i_j$ and one $\tilde{I}^i_j$, where $I^i_j$ and $\tilde{I}^i_j$ are the closed and open intervals, respectively, related to the $j$th step of the construction of the Cantor set. Interval $\tilde{I}^i_j$ is of the type considered above, so we have the estimate (4.4). Let us estimate the integral over $I^i_j$. By self-similarity we may consider the interval $[0,1]$ instead. Since the Cantor set in our construction has measure zero, the integral over $[0,1]$ is exactly the integral over all the removed intervals $\tilde{I}^i_j$. There are exactly $2^{j-1}$ of these with $|\tilde{I}^i_j| = (1-2\lambda)\lambda^{j-1}$ for $j \geq 1$, so

$$\begin{aligned}\int_{[0,1]} \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p}\, \mathrm{d}s(z) &\leq \frac{2^{\frac{p+1}{2}}}{2-p}(1-2\lambda)^{2-p}\sum_{j=1}^{\infty} 2^{j-1}\lambda^{(j-1)(2-p)} \\ &= \frac{2^{\frac{p+1}{2}}}{2-p}(1-2\lambda)^{2-p}\frac{1}{1-2\lambda^{2-p}}.\end{aligned} \tag{4.5}$$

To get to the integral over $I^i_j$ we multiply (4.5) by $|I^i_j|^{2-p}$.

Combining the above we get the following: Any two $x, y \in \Omega^c_\lambda$ can be joined using at most 6 line segments for which (4.4) holds and at most 2 segments to which (4.5), rescaled to the interval, applies. Calling the resulting path $\gamma$ and the segments $I_k$, we have

$$\begin{aligned}\int_\gamma \operatorname{dist}(z, \partial\Omega_\lambda)^{1-p}\, \mathrm{d}s(z) &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{6}|I_k|^{2-p} + \frac{(1-2\lambda)^{2-p}}{1-2\lambda^{2-p}}\sum_{k=7}^{8}|I_k|^{2-p}\right) \\ &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{8}|I_k|\right)^{2-p}\left(\sum_{k=1}^{6}1 + \sum_{k=7}^{8}\left(\frac{(1-2\lambda)^{2-p}}{1-2\lambda^{2-p}}\right)^{1/(p-1)}\right)^{p-1} \\ &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{8}|I_k|\right)^{2-p}\left(\sum_{k=1}^{6}\left(\frac{1}{1-2\lambda^{2-p}}\right)^{1/(p-1)} + \sum_{k=7}^{8}\left(\frac{1}{1-2\lambda^{2-p}}\right)^{1/(p-1)}\right)^{p-1} \\ &\leq \frac{2^{\frac{p+1}{2}}}{2-p}\left(\sum_{k=1}^{8}|I_k|\right)^{2-p}\frac{8^{p-1}}{1-2\lambda^{2-p}}\end{aligned}$$

by Hölder's inequality. By the definition of $\Omega_\lambda$, we may choose $I_k$'s so that $\sum_{k=1}^{8}|I_k| = |\gamma| \leq c\|x-y\|$ for an absolute constant $c$. □

*Proof of Theorem 1.4.* We show the existence of constants $M_2 > 0$ and $C(p) > 0$ so that (1.3) holds for $C \geq C(p)$. Fix $p \in (1,2)$, and let $M_2 = \frac{2c}{\log 2}$ where $c$ is the absolute constant from Lemma 4.1. In order to make estimates, we use the construction for $\lambda \in [\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}})$. In

Lemma 4.1 we established that domain $\Omega_\lambda$ satisfies the curve condition with the constant

$$\frac{c}{(2-p)(1-2\lambda^{2-p})}. \tag{4.6}$$

Setting $\lambda = \frac{1}{2}2^{\frac{1}{p-2}}$ in (4.6) we define

$$C(p) = \frac{c}{(2-p)(1-2^{p-2})}.$$

Now, for $C \geq C(p)$, by the continuity of the constant in (4.6) as a function of $\lambda$ and the fact that it tends to infinity as $\lambda \nearrow 2^{\frac{1}{p-2}}$, there exists $\lambda_C \in [\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}})$ such that

$$C = \frac{c}{(2-p)(1-2\lambda_C^{2-p})}.$$

We show that

$$\dim_{\mathcal{H}} \mathcal{C}_{\lambda_C} = -\frac{\log 2}{\log \lambda_C} \geq 2 - p - \frac{M_2}{C}. \tag{4.7}$$

In order to see that (4.7) holds, we show that

$$f_p(\lambda) = 2 - p - \frac{M_2}{c}(2-p)(1-2\lambda^{2-p}) + \frac{\log 2}{\log \lambda}$$

is non-positive on the interval $[\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}})$. This follows from

$$\begin{aligned}\min_{\lambda \in [\frac{1}{2}2^{\frac{1}{p-2}}, 2^{\frac{1}{p-2}}]} f_p'(\lambda) &\geq 2\frac{M_2}{c}(2-p)^2(2^{\frac{1}{p-2}})^{1-p} - \frac{\log 2}{2^{-1}2^{\frac{1}{p-2}}\log^2(2^{\frac{1}{p-2}})} \\ &= \frac{(2-p)^2}{2^{\frac{1}{p-2}}}\left(\frac{M_2}{c} - \frac{2}{\log 2}\right) \geq 0,\end{aligned}$$

and

$$f_p(\lambda) \leq f_p(2^{\frac{1}{p-2}}) = 0.$$

Hence, (4.7) holds.

□

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# Two-sided boundary points of Sobolev-extension domains on Euclidean spaces

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# TWO-SIDED BOUNDARY POINTS OF SOBOLEV EXTENSION DOMAINS ON EUCLIDEAN SPACES

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ABSTRACT. We prove an estimate on the Hausdorff dimension of the set of two-sided boundary points of general Sobolev extension domains on Euclidean spaces. We also present examples showing lower bounds on possible dimension estimates of this type.

## 1. Introduction

We continue the investigation of the geometric properties of Sobolev extension domains. In this paper, the space of Sobolev functions we use on a domain $\Omega \subset \mathbb{R}^n$ is the homogeneous Sobolev space $L^{1,p}(\Omega)$, which is the space of locally integrable functions whose weak derivatives belong to $L^p(\Omega)$. We endow this space with the homogeneous seminorm

$$\|f\|_{L^{1,p}(\Omega)} = \|\nabla f\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla f(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

The reason for working with the homogeneous Sobolev space is simply to make our dimension estimates scaling invariant. We will comment on the non-homogeneous spaces after stating our main result.

We say that $E\colon L^{1,p}(\Omega) \to L^{1,p}(\mathbb{R}^n)$ is an extension operator if $Eu(x) = u(x)$ for all $u \in L^{1,p}(\Omega)$ and $x \in \Omega$, and if there exists a constant $C \geq 1$ so that for every $u \in L^{1,p}(\Omega)$ we have $||Eu||_{L^{1,p}(\mathbb{R}^n)} \leq C||u||_{L^{1,p}(\Omega)}$. We name the infimum over such possible constants $C$ by $||E||$, and call it the norm of the extension operator. We say that a domain $\Omega \subset \mathbb{R}^n$ is an $L^{1,p}$-extension domain if such an operator exists. The same definition applies for the non-homogeneous spaces $W^{1,p}(\Omega)$.

Throughout this manuscript each time we refer to a Sobolev extension domain we mean it with respect to the homogeneous norm, unless otherwise stated.

Already from the work of Calderón and Stein [16] we know that Lipschitz domains are $W^{1,p}$-extension domains. However, much more complicated domains admit an extension operator. For instance, the Koch snowflake domains are extension domains and in some sense serve as sharp examples of extension domains in terms of the Hausdorff dimension of the boundary, see [13]. In [13], the question of the possible size of the boundary for simply connected planar Sobolev extension domains was studied. In particular, for these domains there is an upper bound on the Hausdorff dimension of the boundary in terms of the norm of the extension operator (although in [13] the bound was expressed in terms of a constant in a characterizing curve-condition property provided in [9]). Note that for the boundary of a general extension domain we cannot have a dimension estimate: take $\Omega = [0,1]^n \setminus C^n$ with $C \subset \mathbb{R}$ a Cantor set

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of $\dim_{\mathcal{H}}(C) = 1$ but Lebesgue measure zero. Then $\Omega$ is a Sobolev $L^{1,p}$-extension domain, but $\dim_{\mathcal{H}}(\partial\Omega) = n$.

With a bound on the dimension of the boundary, one might wonder what other geometric limitations does the existence of an extension operator imply. Let us approach this with a basic example of a domain that is not an $L^{1,p}$-extension domain for any $p$, the slit disc: $\Omega := \mathbb{D} \setminus [0,1] \times \{0\} \subset \mathbb{R}^2$. A continuous Sobolev function in $L^{1,p}(\Omega)$ which is one above $[1/2,1] \times \{0\}$ and zero below it serves as an example of a function that cannot be extended to a global Sobolev function since no extension would be absolutely continuous on almost every vertical line segment. Notice however, that the slit disc is an example of a $BV$-extension domain because its complement is quasiconvex (see [8]).

The slit disc example can be modified to a more delicate one by replacing the removed line segment $[0,1] \times \{0\}$ by a larger set where the two-sided points are at a Cantor set on the previously removed line segment, see Figure 1 in Section 4. This will give a domain where the extendability of Sobolev functions depends on the exponent $p$. Such constructions will also play a role in this paper, see Section 4.2 (also for the precise definitions of these domains). By removing small neighbourhoods of the two-sided points from the domain, one can actually make the example into a Jordan domain and still retain the property of being a Sobolev extension domain for some $p$'s, but not for other, see [11] for a similar construction (and also the earlier works [14, 15]).

The slit disc and its variations have boundary points that can be approached from two different sides in the domain. In this paper we study the question of how large this set of two-sided points can be for a Sobolev extension domain. This question was already investigated in [17] by the third named author in the case of planar simply connected domains. Before continuing, let us give the definition of two-sidedness that we will use in this paper. In the case of simply connected planar domains, the definition can also be reformulated in various ways using conformal mappings, see [17].

**Definition 1.1** (Two-sided points of the boundary of a domain)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. A point $x \in \partial\Omega$ is called* two-sided*, if there exists $R > 0$ such that for all $r \in (0, R)$ there exist disjoint connected components $\Omega_r^1$ and $\Omega_r^2$ of $\Omega \cap B(x, r)$ that are nested: $\Omega_s^i \subset \Omega_r^i$ for $0 < s < r < R$ and $i \in \{1, 2\}$.*

We denote the set of two-sided points of $\partial\Omega$ by $\mathcal{T}_\Omega$, or simply by $\mathcal{T}$, if there is no possibility for confusion. Notice that the set $\mathcal{T}$ need not be closed.

For $p \geq n$, we know that $L^{1,p}$ extension domains are quasiconvex (see [7, Theorem 3.1]). Therefore, for an $L^{1,p}$-extension domain with $p \geq n$, we have $\mathcal{T} = \emptyset$. The interesting case is thus $1 \leq p < n$. For this range we prove the following estimate on the size of $\mathcal{T}$:

**Theorem 1.2.** *Let $n \geq 2$ and $p \in [1, n)$ and let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^{1,p}$-extension domain. Then*

*(1) If $p = 1$, then $\mathcal{H}^{n-1}(\mathcal{T}_\Omega) = 0$.*
*(2) If $p > 1$ there exists a constant $C(n, p) > 0$ so that*

$$\dim_{\mathcal{H}}(\mathcal{T}_\Omega) \leq n - p - \frac{C(n,p)}{\|E\|^n \log(\|E\|)},$$

*where $\|E\|$ is the operator norm of the homogeneous Sobolev extension operator.*

*Here we use the convention that $\mathcal{T}_\Omega = \emptyset$ whenever the bound on the right-hand side of the estimate is strictly less than* 0.[1]

Let us now comment on the non-homogeneous Sobolev spaces. For bounded domains $\Omega$ it is known that Sobolev $L^{1,p}$-extension domains are the same as Sobolev $W^{1,p}$-extension domains (see [7]), so even though our main result is stated for homogeneous Sobolev extension domains, it can be applied to $W^{1,p}$-extension domains in the case that $\Omega$ is bounded. Let us note that there exist unbounded Sobolev $W^{1,p}$-extension domains which are not $L^{1,p}$-extension domains (see [7, Example 6.7]). However, one might expect that our result still applies for this unbounded case because having a dimension bound relies on local properties. Indeed, our method of proof will show that we can handle also with unbounded $W^{1,p}$-extension domains because the measure density condition (see Proposition 2.2) is still true for every $r \in (0,1)$ and the proof of Theorem 1.2 studies locally the set of two-sided points to estimate its dimension. We prefer to state our main theorem only for $L^{1,p}$-extension domains because of their homogeneous norm. If we stated it for $W^{1,p}$-extension domains, then a scaling of the domain $\Omega$ would perturb the norm of the operator $E$, and hence our estimate in the dimension of the two-sided points. Obviously, a scaling of a set will never change its dimension.

We will also give a size estimate on the two sided-points of $BV$-extension domains.

**Theorem 1.3.** *Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a $BV$-extension domain. Then $\mathcal{T}_\Omega$ has $\sigma$-finite $(n-1)$-dimensional Hausdorff measure.*

Observe that taking $\Omega$ to be a slit disc shows the sharpness of this result.

We will present the proof of Theorem 1.2 in Section 2 and the proof of Theorem 1.3 in Section 3. After that, in Section 4, we show that Theorem 1.2 (1) is sharp: there exist even a planar simply connected $L^{1,1}$-extension domain with $\dim_{\mathcal{H}}(\mathcal{T}) = 1$. We also give a class of domains $\Omega_\lambda$ for each $n \geq 2$ with the sets of two-sided points $\mathcal{T}_\lambda = C_\lambda$ being Cantor sets, so that for every $p \in (1,n)$ there exists a constant $C(n,p)$ for which, with the explicit extension operators $E_\lambda \colon L^{1,p}(\Omega_\lambda) \to L^{1,p}(\mathbb{R}^n)$ we construct, we have $\|E_\lambda\| \to \infty$ as $\dim_{\mathcal{H}}(C_\lambda) \to n-p$ and the estimate

$$\dim_{\mathcal{H}}(C_\lambda) \geq n - p - \frac{C(n,p)}{\|E_\lambda\|} \tag{1.1}$$

is satisfied.

This set of examples together with Theorem 1.2 shows that the possible optimal asymptotic behaviour for the dimension bound of the two-sided points in terms of the norm of the extension operator is between $n - p - C/\|E\|$ and $n - p - C/(\|E\|^n \log(\|E\|))$. We note that in [17] the exponents for the dimension bound and examples agreed, thus providing a possibly sharper estimate. However, as the study in [17] was done in terms of a constant in a characterizing curve condition, and since the dependence between this constant and the norm of the extension operator has not been clarified, the estimate in [17] does not yet translate to a sharp dimension estimate in terms of the norm of the extension operator in the planar simply connected case.

## 2. Dimension estimate for the set of two-sided points

In this section we will prove Theorem 1.2. Before doing so, we go through some notation and lemmata.

[1] To make this formally correct we adopt the notational convention $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

We often denote by $C(\cdot)$ a computable constant depending only on the parameters listed in the parenthesis. The constant may differ between appearances, even within a chain of (in)equalities. By $a \lesssim b$ we mean that $a \le Cb$ for some constant $C \ge 1$, that could depend on the dimension $n$. Similarly for $a \gtrsim b$. Then $a \sim b$ means that both $a \lesssim b$ and $a \gtrsim b$ hold. We denote by $\mathbf{m}_n$ the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. We will also denote by $Q(x, s)$ the cube of center $x$ and side length $s > 0$ and for a given cube $Q = Q(x, s)$ and some positive $K > 0$ we write $KQ = Q(x, Ks)$. We denote the side length of a cube by $\ell(Q)$.

We will use the following basic lemma, similar to [17, Lemma 3.2].

**Lemma 2.1.** *Let $F \subset \mathbb{R}^n$, $0 < \lambda < 1$, $s \ge 0$, and $i_0 \in \mathbb{N}$. For every $i \ge i_0$ let $\{x^i_k\}_{k\in I_i}$ be a maximal $\lambda^i$-separated net in $F$. Assume that for each $i \ge i_0$ and $k \in I_i$ there exists $j > i$ such that*

$$N_j < \lambda^{-(j-i)s},$$

*where $N_j = \#(\{l \in I_j : B(x^j_l, \lambda^j) \cap B(x^i_k, \lambda^i) \neq \emptyset\})$. Then $\dim_{\mathcal{H}}(F) \le s$.*

A measure density condition for Sobolev extension domains was proven in [4]. We will need to make the dependence of the parameters more explicit, so we modify slightly the proofs of [4, Lemma 11] and [4, Theorem 1] to obtain the following version of their measure density condition.

**Proposition 2.2** (Measure density condition). *Let $1 \le p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^{1,p}$-extension domain with an extension operator $E$.*

*(1) If $1 \le p < n$ then for all $x \in \overline{\Omega}$ and $r \in \left(0, \min\left\{1, \left(\frac{\mathbf{m}_n(\Omega)}{2\,\mathbf{m}_n(B(0,1))}\right)^{1/n}\right\}\right)$, denoting by $\Omega'$ a connected component of $\Omega \cap B(x, r)$ with $x \in \overline{\Omega'}$, we have*

$$\mathbf{m}_n(\Omega') \ge C(n, p)\|E\|^{-n} r^n.$$

*(2) If $p > n - 1$ then for all $x \in \overline{\Omega}$ and $r \in (0, \operatorname{diam}(\Omega))$, denoting by $\Omega'$ a connected component of $\Omega \cap B(x, r)$ with $x \in \overline{\Omega'}$, we have*

$$\mathbf{m}_n(\Omega') \ge C(n, p)\|E\|^{-p} r^n.$$

*Proof.* The case (2) follows by Theorem 2.2 and the proof of Theorem 4.1 from Koskela's dissertation [7], where he uses the concept of variational $p$-capacity. Notice that in the proof of Theorem 4.1 the support of the test function $u$ is contained in $\Omega'$.

We look now at the case $1 \le p < n$.

Let us denote $r_0 = r$. By induction, we define for every $i \in \mathbb{N}$ the radius $r_i \in (0, r_{i-1})$ by the equality

$$\mathbf{m}_n(\Omega' \cap B(x, r_i)) = \frac{1}{2}\mathbf{m}_n(\Omega' \cap B(x, r_{i-1})) = 2^{-i}\mathbf{m}_n(\Omega').$$

Since $x \in \overline{\Omega'}$, we have that $r_i \searrow 0$ as $i \to \infty$.

For each $i \in \mathbb{N}$, consider the function $f_i \colon \Omega \to \mathbb{R}$

$$f_i(y) = \begin{cases} 1, & \text{for } y \in B(x, r_i) \cap \Omega', \\ \frac{r_{i-1} - |x-y|}{r_{i-1} - r_i}, & \text{for } y \in (B(x, r_{i-1}) \setminus B(x, r_i)) \cap \Omega', \\ 0, & \text{otherwise.} \end{cases}$$

For the homogeneous Sobolev-norm of $f_i$ we can estimate

$$\begin{aligned}\|f_i\|^p_{L^{1,p}(\Omega)} &= \int_\Omega |\nabla f_i|^p \leq |r_i - r_{i-1}|^{-p}\mathbf{m}_n((B(x,r_{i-1}) \setminus B(x,r_i)) \cap \Omega') \\ &= |r_i - r_{i-1}|^{-p}2^{-i}\mathbf{m}_n(\Omega').\end{aligned} \tag{2.1}$$

Call $p^* = \frac{np}{n-p}$. For any $Ef_i \in L^{1,p}(\mathbb{R}^n)$, by the Sobolev-Poincaré-inequality (we will also prove a variant of this later, see (2.3)), we know the existence of a constant $c_i \in \mathbb{R}$ (that can be assumed to be between 0 and 1) so that

$$\|Ef_i - c_i\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p)\|Ef_i\|_{L^{1,p}(\mathbb{R}^n)}.$$

Hence we have the following chain of inequalities

$$\|f_i - c_i\|_{L^{p^*}(\Omega)} \leq \|Ef_i - c_i\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p)\|Ef_i\|_{L^{1,p}(\mathbb{R}^n)} \leq C(n,p)\|E\|\, \|f_i\|_{L^{1,p}(\Omega)}.$$

Recall that by our choice of $r = r_0$ we always have

$$\mathbf{m}_n(\Omega \setminus B(x,r_{i-1})) \geq \mathbf{m}_n(\Omega \setminus B(x,r_0)) \geq \mathbf{m}_n(\Omega) - \mathbf{m}_n(B(x,r_0)) \geq \frac{\mathbf{m}_n(\Omega)}{2}$$

and

$$\mathbf{m}_n(B(x,r_i) \cap \Omega') \leq \frac{\mathbf{m}_n(\Omega)}{2}$$

for every $i \geq 1$. Then

$$\begin{aligned}\int_\Omega |f_i(y) - c_i|^{p^*}\,dy &\geq \max\left\{\int_{\{y\,:\,f_i(y)=0\}} |c_i|^{p^*}\,dy, \int_{\{y\,:\,f_i(y)=1\}} |1-c_i|^{p^*}\,dy\right\} \\ &\geq \max\left\{|c_i|^{p^*}\mathbf{m}_n(\Omega \setminus B(x,r_{i-1})), |1-c_i|^{p^*}\mathbf{m}_n(B(x,r_i)\cap\Omega')\right\} \\ &\geq \mathbf{m}_n(B(x,r_i)\cap\Omega')\cdot\max\left\{|c_i|^{p^*}, |1-c_i|^{p^*}\right\} \geq \mathbf{m}_n(B(x,r_i)\cap\Omega')\cdot 2^{-p^*},\end{aligned}$$

so we write, using (2.1),

$$\begin{aligned}2^{-p^*-i}\mathbf{m}_n(\Omega') = 2^{-p^*}\mathbf{m}_n(B(x,r_i)\cap\Omega') &\leq \|f_i - c_i\|^{p^*}_{L^{p^*}(\Omega)} \leq C(n,p)\|E\|^{p^*}\|f_i\|^{p^*}_{L^{1,p}(\Omega)} \\ &= C(n,p)\|E\|^{p^*}\left(\int_\Omega |\nabla f_i(y)|^p\,dy\right)^{p^*/p} \\ &\leq C(n,p)\|E\|^{p^*}\left(|r_i - r_{i-1}|^{-p}2^{-i}\mathbf{m}_n(\Omega')\right)^{p^*/p} \\ &\leq C(n,p)\|E\|^{p^*}2^{-ip^*/p}\mathbf{m}_n(\Omega')^{p^*/p}|r_{i-1} - r_i|^{-p^*}.\end{aligned}$$

Consequently,

$$\begin{aligned}r_{i-1} - r_i &\leq C(n,p)\|E\|2^{i(1/p^*-1/p)}\mathbf{m}_n(\Omega')^{1/p-1/p^*} \\ &= C(n,p)\|E\|2^{-i/n}\mathbf{m}_n(\Omega')^{1/n}.\end{aligned}$$

By summing up all these quantities we conclude that

$$r = r_0 = \sum_{i=1}^\infty (r_{i-1} - r_i) \leq C(n,p)\|E\|\sum_{i=1}^\infty 2^{-i/n}\mathbf{m}_n(\Omega')^{1/n} = \frac{C(n,p)\|E\|}{2^{1/n}-1}\mathbf{m}_n(\Omega')^{1/n}.$$

This gives the claimed inequality. □

Observe that the measure density condition only holds for $1 \le p < \infty$. For $W^{1,\infty}$-extension domains this is not true. Take for instance $C \subset [0,1]$ a fat Cantor set with $\mathbf{m}_1(C) > 0$. Then almost every point of $C$ is of density 1 on $C$, so $[0,1] \setminus C$, whose closure is the whole interval $[0,1]$, cannot satisfy any measure density condition. Then take $\Omega = \mathbb{R}^n \setminus C^n$ which will be quasiconvex by [5, Theorem A], and consequently a $W^{1,\infty}$-extension domain by [4, Theorem 7], but does not satisfy any measure density condition either.

In the proof of Theorem 1.2 we will use the following consequence of a Sobolev-Poincaré type inequality (2.3). The proof of the lemma follows the proof for the classical Sobolev-Poincaré inequality that can be found in many text books. However, for our application of the lemma we need to include a set $F$ that is removed when integrating the gradient of the Sobolev function. This fact forces us to be more cautious. For the convenience of the reader, we provide here the proof with the needed modifications.

**Lemma 2.3.** *Let $1 \le p < n$, $Q \subset \mathbb{R}^n$ be a cube, $\delta \in (0,1)$ and $F \subset Q$ such that for any $i \in \{1, \dots, n\}$ we have*

$$\mathbf{m}_{n-1}(P_i(F)) \le \frac{\delta}{2n \cdot 2^n} \mathbf{m}_{n-1}(P_i(Q))$$

*with $P_i$ the projection $P_i \colon (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Then for any $f \in W^{1,p}(Q)$ so that $0 \le f \le 1$ and*

$$\min\left(\mathbf{m}_n(\{y \in \frac{1}{2}Q \,:\, f(y) = 0\}), \mathbf{m}_n(\{y \in \frac{1}{2}Q \,:\, f(y) = 1\})\right) > \delta \frac{\ell(Q)^n}{2^n},$$

*we have*

$$\int_{Q \setminus F} |\nabla f(y)|^p \,\mathrm{d}y \ge C(n,p) \delta^{\frac{n-p}{n}} \ell(Q)^{n-p}. \tag{2.2}$$

*Remark.* Observe that for the conclusion of Lemma 2.3 it is not enough to only require $\mathbf{m}_n(F)$ to be small. Consider for instance the cube minus a very thin central band which separates the cube in two connected components.

*Proof.* We will show that the following version of Sobolev-Poincaré inequality holds for our function $f$:

$$\left(\int_A |f(y) - f_A|^{\frac{pn}{n-p}} \,\mathrm{d}y\right)^{\frac{n-p}{pn}} \le C(n,p) \left(\int_{Q \setminus F} |\nabla f(y)|^p \,\mathrm{d}y\right)^{1/p}, \tag{2.3}$$

where $A = \left\{x \in \frac{1}{2}Q \,:\, P_i(x) \notin P_i(F) \text{ for every } i\right\}$ and

$$f_A = \frac{1}{\mathbf{m}_n(A)} \int_A f(y)\, dy.$$

Let us first observe that this implies

$$\begin{aligned}
\int_{Q \setminus F} |\nabla f(y)|^p \,\mathrm{d}y &\gtrsim \left(\int_A |f(y) - f_A|^{\frac{pn}{n-p}} \,\mathrm{d}y\right)^{\frac{n-p}{n}} \\
&\gtrsim \max\left(\mathbf{m}_n(\{y \in A \,:\, f(y) = 1\})^{\frac{n-p}{n}} |1 - f_A|^p, \mathbf{m}_n(\{y \in A \,:\, f(y) = 0\})^{\frac{n-p}{n}} |f_A|^p\right) \\
&\gtrsim \delta^{\frac{n-p}{n}} \ell(Q)^{n-p} \max\left(|1 - f_A|^p, |f_A|^p\right) \\
&\gtrsim \delta^{\frac{n-p}{n}} \ell(Q)^{n-p},
\end{aligned}$$

thus giving the inequality (2.2). Above we used the simple observation that

$$\mathbf{m}_n(\frac{1}{2}Q \setminus A) \le \frac{\delta}{4 \cdot 2^n}\mathbf{m}_n(Q). \tag{2.4}$$

To prove (2.3) we start by presenting the Sobolev embedding in the form

$$\left(\int_{A'} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y\right)^{\frac{n-p}{pn}} \le C(n,p,K)\left(\int_{Q\setminus F} |\nabla g(y)|^p\,\mathrm{d}y\right)^{1/p}, \tag{2.5}$$

for all $g \in W_0^{1,p}(Q)$ with $|g| \le 1$ and $\mathbf{m}_n(\{x \in A' : |g(x)| \ge 1/2\}) \ge K\delta\ell(Q)^n$ for some positive constant $K > 0$, and where $A' = \{x \in Q : P_i(x) \notin P_i(F)$ for every $i\}$. Following the proof of [2, Theorem 4.8] what we first get is

$$\left(\int_{A'} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y\right)^{\frac{n-1}{n}} \le C(n,p)\left(\int_{Q\setminus F} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y\right)^{\frac{p-1}{p}}\left(\int_{Q\setminus F} |\nabla g(y)|^p\,\mathrm{d}y\right)^{1/p}.$$

Note that by the properties of $g$ and by definition of $A'$

$$\int_{Q\setminus A'} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y \le \mathbf{m}_n(Q \setminus A') < \frac{n\delta}{2n \cdot 2^n}\ell(Q)^n$$

and

$$\int_{A'} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y \ge \left(\frac{1}{2}\right)^{\frac{pn}{n-p}} K\delta\ell(Q)^n.$$

Therefore,

$$\begin{aligned}\int_{Q\setminus F} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y \le \int_{Q} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y &= \int_{A'} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y + \int_{Q\setminus A'} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y \\ &\le (1 + C(n,p,K))\int_{A'} |g(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y,\end{aligned} \tag{2.6}$$

and finally we can get (2.5).

Secondly, we apply the inequality (2.5) to the function $g(y) = (f(y) - f_{A'})\phi(y)$, where $\phi \in C_0^\infty(\mathbb{R}^n)$ is supported in $Q$, is equal to 1 on $\frac{1}{2}Q$ and $|\nabla\phi| \lesssim \dfrac{1}{\ell(Q)}$. We get

$$\begin{aligned}\left(\int_{A} |f(y) - f_A|^{\frac{pn}{n-p}}\,\mathrm{d}y\right)^{\frac{n-p}{pn}} &\le \left(\int_{A'} |(f(y) - f_A)\phi(y)|^{\frac{pn}{n-p}}\,\mathrm{d}y\right)^{\frac{n-p}{pn}} \\ &\le C(n,p)\left(\int_{Q\setminus F} |\nabla f(y)|^p\,\mathrm{d}y\right)^{1/p} \\ &\quad + \frac{C(n,p)}{\ell(Q)}\left(\int_{Q\setminus F} |f(y) - f_A|^p\,\mathrm{d}y\right)^{1/p}.\end{aligned} \tag{2.7}$$

To handle the last term above, we first prove that

$$\begin{aligned}\left(\int_{Q\setminus F}|f(y)-f_A|^p\,dy\right)^{1/p} &\le C(n,p)\left(\int_{A'}|f(y)-f_A|^p\,dy\right)^{1/p}\\ &\le C(n,p)\left(\left(\int_{A'}|f(y)-f_{A'}|^p\,dy\right)^{1/p}+\left(\int_{A'}|f_{A'}-f_A|^p\,dy\right)^{1/p}\right)\\ &\le C(n,p)\left(\int_{A'}|f(y)-f_{A'}|^p\,dy\right)^{1/p}.\end{aligned}$$

In the first inequality we are using a similar trick like in (2.6) (that $0 \le f \le 1$ and that $f = 1$ and $f = 0$ in large enough sets). In the last inequality we use Hölder inequality and the fact that by (2.4) we have

$$\mathbf{m}_n(A) \ge \mathbf{m}_n(\frac{1}{2}Q) - \mathbf{m}_n(\frac{1}{2}Q \setminus A) \ge \frac{\mathbf{m}_n(Q)}{2^n} - \frac{\delta \mathbf{m}_n(Q)}{4\cdot 2^n} \ge 2^{-n-1}\mathbf{m}_n(Q) \ge 2^{-n-1}\mathbf{m}_n(A').$$

Finally, by modifying the standard proof for the Poincaré inequality (see [2, Section 4.5.2]) by first writing

$$|f(y)-f(x)| \le \sum_{i=1}^{n}|f(z_i)-f(z_{i-1})|,$$

with $z_i = (y_1, \dots, y_i, x_{i+1}, \dots x_n)$ so that $z_i$ and $z_{i-1}$ differ only in one coordinate, we are able to consider absolute continuity only along lines going in the coordinate directions. Thus, we obtain

$$\int_{A'}|f(y)-f_{A'}|^p\,dy \le C(n,p)\ell(Q)^p\int_{Q\setminus F}|\nabla f(y)|^p\,dy.$$

Combining the above with (2.7) concludes the proof. □

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let us first make some initial reductions. By the definition of two-sided points we can write $\mathcal{T}_\Omega = \bigcup_{i\in\mathbb{N}}\mathcal{T}_i$, where

$$\begin{aligned}\mathcal{T}_i = \{x \in \partial\Omega :&\text{ for every } r < 2^{-i}, \text{ there exist two different connected components } \Omega_r^1, \Omega_r^2\\ &\text{of } \Omega \cap B(x,r) \text{ that are nested, that is } \Omega_s^j \subset \Omega_r^j \text{ for } 0 < s < r,\ j = 1,2\}.\end{aligned}$$

Observe that if $x \in \mathcal{T}_i$ and $\Omega_r^1$, $\Omega_r^2$ are the associated nested connected components of $\Omega \cap B(x,r)$ for each $r \in (0, 2^{-i})$, then $x \in \overline{\Omega_r^1} \cap \overline{\Omega_r^2}$ for all $r \in (0, 2^{-i})$.

It is clear that it is enough to estimate $\dim_{\mathcal{H}}(\mathcal{T}_i)$ for a fixed $i \in \mathbb{N}$. We now cover $\mathcal{T}_i$ by countably many balls $B(z_k, 2^{-i}/6)$, where $z_k \in \mathcal{T}_i$. Then, for every $k \in \mathbb{N}$ we introduce the family of pairwise disjoint connected components of $B(z_k, 2^{-i}/2) \cap \Omega$, which we denote by $\{O_l^k\}_{l\in I}$. Let us check now that

$$\mathcal{T}_i \cap B(z_k, 2^{-i}/6) \subseteq \bigcup_{l\neq\tilde{l}} \partial O_l^k \cap \partial O_{\tilde{l}}^k. \tag{2.8}$$

Take $x \in \mathcal{T}_i \cap B(z_k, 2^{-i}/6)$. Since $x \in \mathcal{T}_i$ there exist two different connected components of $\Omega \cap B(x, 2^{-i})$, given by the definition of $\mathcal{T}_i$ which we call $U_1, U_2$, so that we in particular have $x \in \partial U_1 \cap \partial U_2$. Therefore, using that

$$B(z_k, 2^{-i}/2) \cap \Omega \subset B(x, 2^{-i}) \cap \Omega,$$

the sets $U_1 \cap B(z_k, 2^{-i}/2)$ and $U_2 \cap B(z_k, 2^{-i}/2)$ will have connected components, which we call $O_l^k, O_{\tilde{l}}^k$, so that $x \in \partial O_l^k \cap \partial O_{\tilde{l}}^k$. We have then proved (2.8). Observe that we can write

$$\mathcal{T}_\Omega = \bigcup_{i,k} \bigcup_{l \neq \tilde{l}} \mathcal{T}_i \cap B(z_k, 2^{-i}/6) \cap (\partial O_l^k \cap \partial O_{\tilde{l}}^k).$$

Therefore, it is enough to just estimate the Hausdorff dimension of $\mathcal{T}_i \cap B(z_k, 2^{-i}/6) \cap (\partial O_l^k \cap \partial O_{\tilde{l}}^k)$ for fixed $i, k, l, \tilde{l}$ with $l \neq \tilde{l}$. Each set of this type, that we call from now on $G$, has the following properties: there is some $x_0 \in \partial\Omega$ and some radius $r \in (0,1)$ so that

$$G \subset \partial\Omega \cap B(x_0, r),$$

and there exist connected components $\Omega_1, \Omega_2 \subset \Omega \cap B(x_0, 3r)$ for which

$$G \subset \partial\Omega_1 \cap \partial\Omega_2.$$

We will now proceed to estimate the Hausdorff dimension of such a set $G$.

(1) Let us first prove that $\mathcal{H}^{n-p}(G) = 0$ for all $1 \leq p < n$. In particular, this will handle the case $p = 1$ in the claim (1) of the theorem. We will use the well-known fact that for any given $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $0 \leq s < n$ we have

$$\mathcal{H}^s \left( \left\{ x \in \mathbb{R}^n \, : \, \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon^s} \int_{B(x,\varepsilon)} |h(y)| \, dy > 0 \right\} \right) = 0. \tag{2.9}$$

See for instance [2, Theorem 2.10] for a proof of this assertion.

We start by defining a function $u \in L^{1,p}(\Omega)$,

$$u(x) = \max\left(0, \min\left(1, 3 - r^{-1} \operatorname{dist}(x, x_0)\right) \chi_{\Omega_1}(x)\right),$$

where $\chi_{\Omega_1}$ denotes the indicator function of the set $\Omega_1$. Notice that $u$ is locally Lipschitz in $\Omega$ by the fact that $\Omega_1$ is a connected component of $\Omega \cap B(x_0, 3r)$. By Proposition 2.2, for every $x \in G$ and every $0 < \varepsilon \leq r$ sufficiently small,

$$\min\left(\mathbf{m}_n(\Omega_1 \cap B(x, \varepsilon/2\sqrt{n})), \mathbf{m}_n(\Omega_2 \cap B(x, \varepsilon/2\sqrt{n}))\right) \geq C(n,p)\|E\|^{-n}\varepsilon^n.$$

Now, we apply Lemma 2.3 with the removed set $F = \emptyset$ and the cube $Q(x, 2\varepsilon/\sqrt{n})$ centered at $x$ and with side length $2\varepsilon/\sqrt{n}$. Notice that $Q(x, 2\varepsilon/\sqrt{n})$ contains the ball $B(x, \varepsilon/\sqrt{n})$ and is contained in the ball $B(x, \varepsilon)$, and so the Lemma 2.3 gives

$$\int_{B(x,\varepsilon)} |\nabla Eu(y)|^p \, \mathrm{d}y \geq \int_{Q(x, 2\varepsilon/\sqrt{n})} |\nabla Eu(y)|^p \, \mathrm{d}y \geq C(n,p)\|E\|^{p-n}\varepsilon^{n-p}.$$

Therefore,

$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{n-p}} \int_{B(x,\varepsilon)} |\nabla Eu(y)|^p \, dy \geq C(n,p)\|E\|^{p-n} > 0$$

for every $x \in G$, and using (2.9) we conclude $\mathcal{H}^{n-p}(G) = 0$.

We are done with the case $p = 1$ of Theorem 1.2. For the case $p > 1$ we will be able to be more precise in the estimation of the Hausdorff dimension in terms of the norm of the extension operator $E \colon L^{1,p}(\Omega) \to L^{1,p}(\mathbb{R}^n)$. For this we will follow a different approach.

(2) Let us now focus on the case $p > 1$. First of all, call $C_1(n,p)$ and $C_2(n,p)$ the constants given by Proposition 2.2 and Lemma 2.3 respectively. Now we choose $i_0 \in \mathbb{N}$ and $0 < \lambda < 1$ such that $0 < \lambda^{i_0} < r$ small enough so that Proposition 2.2 is satisfied and

$$\frac{\lambda^{p-1}}{1-\lambda^{p-1}} = \frac{C_1(n,p)^{1+\frac{n-p}{n}} C_2(n,p)}{2^{2n+1}3^n n^{(2n+1-p)/2}\mathbf{m}_n(B(0,1))}\|E\|^{-2n}.$$

We can do this because the term on the left hand side tends to zero as $\lambda \to 0$.

For every $i \in \mathbb{N}$, let $\{x_k^i\}_{k\in I_i}$ be a maximal $\lambda^i$-separated net of points in $G$. For every $i \in \mathbb{N}$ and $k \in I_i$ define

$$\mathcal{B}_j^{i,k} = \{B(x_l^{i+j}, \lambda^{i+j}) : B(x_l^{i+j}, \lambda^{i+j}) \cap B(x_k^i, \lambda^i) \neq \emptyset\},$$

$N_j^{i,k} = \#\mathcal{B}_j^{i,k}$ for $j \geq 0$, and

$$A_k^i = B(x_k^i, \lambda^i) \setminus (\bigcup_{j=1}^{\infty} \bigcup_{l\in I_{i+j}} B(x_l^{i+j}, \lambda^{i+j})).$$

Now define $u_{i,k} = u \in L^{1,p}(\Omega)$ by

$$u(x) = \max\left(0, \min\left(1, 3 - \lambda^{-i}\operatorname{dist}(x, x_k^i)\right)\chi_{\Omega_1}(x)\right).$$

Without loss of generality we can assume that the extension operator applied to any function $0 \leq u \leq 1$ also satisfies $0 \leq Eu \leq 1$. We then have

$$\|u\|_{L^{1,p}(\Omega)}^p \leq \int_{B(x_k^i, 3\lambda^i)} |\nabla u(x)|^p \,\mathrm{d}x \leq (3^n \mathbf{m}_n(B(0,1)))\lambda^{i(n-p)}. \tag{2.10}$$

By Proposition 2.2 and because $\lambda^{i_0} < r$ $\lambda < r$, we have for $i \geq i_0$

$$\min\Big(\mathbf{m}_n(\Omega_1 \cap B(x_l^{i+j}, \lambda^{i+j}/2\sqrt{n})), \mathbf{m}_n(\Omega_2 \cap B(x_l^{i+j}, \lambda^{i+j}/2\sqrt{n}))\Big) \geq C_1(n,p)\|E\|^{-n}\frac{\lambda^{n(i+j)}}{2^n n^{n/2}}$$

for every $B(x_l^{i+j}, \lambda^{i+j}) \in \mathcal{B}_j^{i,k}$. (In the case $n-1 < p < n$ Proposition 2.2 will give a better estimate with $\|E\|^{-p}$ in the above estimate. We shall comment about this case in a remark at the end of the proof.) Applying Lemma 2.3 where again the removed set $F = \emptyset$, for the corresponding cube $Q(x_l^{i+j}, 2\lambda^{i+j}/\sqrt{n})$ centered at $x_l^{i+j}$ and side length $2\lambda^{i+j}/\sqrt{n}$ (thus containing the ball $B(x_l^{i+j}, \lambda^{i+j}/\sqrt{n})$ and contained in the ball $B(x_l^{i+j}, \lambda^{i+j})$), we have

$$\begin{aligned}\int_{B(x_l^{i+j}, \lambda^{i+j})} |\nabla Eu(y)|^p \,\mathrm{d}y &\geq \int_{Q(x_l^{i+j}, 2\lambda^{i+j}/\sqrt{n})} |\nabla Eu(y)|^p \,\mathrm{d}y \\ &\geq \frac{C_2(n,p)C_1(n,p)^{(n-p)/n}}{n^{(n-p)/2}}\|E\|^{p-n}\lambda^{(n-p)(i+j)}.\end{aligned} \tag{2.11}$$

Thus, since $\sum_{B\in\mathcal{B}_j^{i,k}}\chi_B(x)\le 3^n$ for all $x\in B(x_k^{i+j},\lambda^{i+j})$, and by using (2.11) and (2.10), we get the estimate

$$\begin{aligned}\frac{C_2(n,p)C_1(n,p)^{(n-p)/n}}{n^{(n-p)/2}}N_j^{i,k}\|E\|^{p-n}\lambda^{(n-p)(i+j)} &\le \sum_{B\in\mathcal{B}_j^{i,k}}\int_B|\nabla Eu(y)|^p\,\mathrm{d}y\\ &\le 3^n\int_{\mathbb{R}^n}|\nabla Eu(y)|^p\,\mathrm{d}y\\ &\le 3^n\|E\|^p\|u\|^p_{L^{1,p}(\Omega)}\\ &\le (\mathbf{m}_n(B(0,1))3^{2n})\|E\|^p\lambda^{i(n-p)}.\end{aligned}$$

This implies the bound

$$N_j^{i,k}\le\frac{\mathbf{m}_n(B(0,1))3^{2n}n^{(n-p)/2}}{C_2(n,p)C_1(n,p)^{(n-p)/n}}\|E\|^n\lambda^{-j(n-p)} \tag{2.12}$$

for every $i,j\in\mathbb{N}$ with $i\ge i_0$ and $k\in I_i$.

Let us next estimate the $\mathcal{H}^{n-1}$-measure of the $(n-1)$-projections of the sets

$$F_{i,k}=\bigcup_{j=1}^{\infty}\bigcup_{B\in\mathcal{B}_j^{i,k}}B$$

for all $i\ge i_0$ and $k\in I_i$. By applying the estimate (2.12) and by the choice of $\lambda$, for every $i\ge i_0$ and $m=1,\dots,n$,

$$\begin{aligned}\mathcal{H}^{n-1}(P_m(F_{i,k}))&\le\sum_{j=1}^{\infty}N_j^{i,k}(2\lambda^{i+j})^{n-1}\\ &\le 2^{n-1}\left(\frac{\mathbf{m}_n(B(0,1))3^{2n}n^{(n-p)/2}}{C_2(n,p)C_1(n,p)^{(n-p)/n}}\right)\|E\|^n\lambda^{i(n-1)}\sum_{j=1}^{\infty}\lambda^{j(p-1)}\\ &=2^{n-1}\left(\frac{\mathbf{m}_n(B(0,1))3^{2n}n^{(n-p)/2}}{C_2(n,p)C_1(n,p)^{(n-p)/n}}\right)\|E\|^n\frac{\lambda^{p-1}}{1-\lambda^{p-1}}\lambda^{i(n-1)}\\ &\le\frac{C_1(n,p)\|E\|^{-n}}{2n\cdot 4^n}\left(\frac{2\lambda^i}{\sqrt{n}}\right)^{n-1}.\end{aligned}$$

Note that in Proposition 2.2 one can always assume $C_1(n,p)\|E\|^{-n}<1$.

Suppose now that $s<\dim_{\mathcal{H}}(G)$. By Lemma 2.1 there exist $m_0\ge i_0$ and $k_0\in I_{m_0}$ such that $N_j^{m_0,k_0}\ge\lambda^{-js}$ for all $j\ge 0$.

For those fixed values $m_0,k_0$ and using the above estimate on the $\mathcal{H}^{n-1}$-measure of $P_m(F_{i,\ell})$, for the case $i=m_0+j$, $\ell\in I_i$, $j\ge 0$, we can apply Lemma 2.3 to the function $u_{m_0,k_0}=u$, that was defined before. That is,

$$\begin{aligned}\int_{A_l^{m_0+j}} |\nabla Eu(y)|^p \,\mathrm{d}y &\geq \int_{A_l^{m_0+j}\cap Q(x_l^{m_0+j},2\lambda^{m_0+j}/\sqrt{n})} |\nabla Eu(y)|^p \,\mathrm{d}y \\ &= \int_{Q(x_l^{m_0+j},2\lambda^{m_0+j}/\sqrt{n})\setminus F_{m_0+j,\ell}} |\nabla Eu(y)|^p \,\mathrm{d}y \\ &\geq C(n,p)\|E\|^{p-n}\lambda^{(n-p)(m_0+j)},\end{aligned}$$

where $A_l^{m_0+j} \subset B(x_l^{m_0+j}, \lambda^{m_0+j}) \in \mathcal{B}_j^{m_0,k_0}$. Now, by (2.10), and by summing over all the scales $j \geq 0$, we get

$$\begin{aligned}C(n)\|E\|^p\lambda^{m_0(n-p)} \geq C(n)\|E\|^p\|u\|^p_{L^{1,p}(\Omega)} &\geq C(n)\int_{\mathbb{R}^n} |\nabla Eu(y)|^p \,\mathrm{d}y \\ &\geq \sum_{j=0}^{\infty} \sum_{\{l\in I_{m_0+j}\,:\,B(x_l^{m_0+j},\lambda^{m_0+j})\in\mathcal{B}_j^{m_0,k_0}\}} \int_{A_l^{m_0+j}} |\nabla Eu(y)|^p \,\mathrm{d}y \\ &\geq \sum_{j=0}^{\infty} N_j^{m_0,k_0} C(n,p)\|E\|^{p-n}\lambda^{(n-p)(m_0+j)} \\ &\geq \sum_{j=0}^{\infty} \lambda^{-js} C(n,p)\|E\|^{p-n}\lambda^{(n-p)(m_0+j)} \\ &= C(n,p)\|E\|^{p-n}\frac{\lambda^{m_0(n-p)}}{1-\lambda^{n-p-s}}.\end{aligned}$$

This implies (observe that by the choice of $\lambda$ we have $C(n,p)\|E\|^{\frac{-2n}{p-1}} \leq \lambda$)

$$s \leq n-p-\frac{\log\left(1-C(n,p)\|E\|^{-n}\right)}{\log(\lambda)} \leq n-p-\frac{C(n,p)}{\|E\|^n\log(\|E\|)}.$$

Since $s < \dim_{\mathcal{H}}(G)$ was chosen arbitrarily, this concludes the proof of (2). □

*Remark* 2.4. Let us make a remark on the case $n-1 < p < n$. In this case, by applying Proposition 2.2 (2) we could slightly improve the estimates in the previous theorem. We would have that for the function $u$ defined above,

$$\int_{A_l^{i+j}} |\nabla Eu(y)|^p \,\mathrm{d}y \geq C(n,p)\|E\|^{-p\left(\frac{n-p}{n}\right)}\lambda^{(n-p)(i+j)},$$

and therefore

$$s \leq n-p-\frac{\log\left(1-C(n,p)\|E\|^{-p\left(\frac{n-p}{n}\right)-p}\right)}{\log(\lambda)} \leq n-p-\frac{C(n,p)}{\|E\|^{2p-\frac{p^2}{n}}\log(\|E\|)}.$$

## 3. Two-sided points of $BV$-extension domains

For a given domain $\Omega \subset \mathbb{R}^n$ the space of functions of bounded variation in $\Omega$ is

$$BV(\Omega) = \{u \in L^1(\Omega) \,:\, \|Du\|(\Omega) < \infty\},$$

where

$$\|Du\|(\Omega) = \sup\left\{\int_\Omega u\,\mathrm{div}(v)\,dx \,:\, v \in C_0^\infty(\Omega;\mathbb{R}^n),\ |v| \leq 1\right\}$$

denotes the total variation of $u$ on $\Omega$. We endow this space with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega)$. We say that $\Omega$ is a $BV$-extension domain if there exists a constant $C > 0$ and a (not necessarily linear) extension operator $T\colon BV(\Omega) \to BV(\mathbb{R}^n)$ so that $Tu|_\Omega = u$ and

$$\|Tu\|_{BV(\mathbb{R}^n)} \le C\|u\|_{BV(\Omega)}$$

for all $u \in BV(\Omega)$ and where $C > 0$ is an absolute constant, independent of $u$.

Let us point out that $\Omega$ being a $W^{1,1}$-extension domain always implies that it is also a $BV$-extension domain (see [8, Lemma 2.4]).

A Lebesgue measurable subset $E \subset \mathbb{R}^n$ has finite perimeter in $\Omega$ if $\chi_E \in BV(\Omega)$, where $\chi_E$ denotes the indicator function of the set $E$. We set $P(E,\Omega) = \|D\chi_E\|(\Omega)$ and call it the perimeter of $E$ in $\Omega$. Moreover, the measure theoretic boundary of a set $E \subset \mathbb{R}^n$ is defined as

$$\partial^M E = \left\{ x \in \mathbb{R}^n \,:\, \limsup_{r\searrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0 \text{ and } \limsup_{r\searrow 0} \frac{|(\mathbb{R}^n \setminus E) \cap B(x,r)|}{|B(x,r)|} > 0 \right\},$$

and for a set of finite perimeter in $\Omega$ one always has $P(E,\Omega) = \mathcal{H}^{n-1}(\partial^M E \cap \Omega)$. Finally, let us recall the useful coarea formula for $BV$ functions. Namely, for a given a function $u \in BV(\Omega)$, the superlevel sets $u_t = \{x \in \Omega : u(x) \ge t\}$ have finite perimeter in $\Omega$ for almost every $t \in \mathbb{R}$ and

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(u_t,\Omega)\,dt. \tag{3.1}$$

*Proof of Theorem 1.3.* We want to prove that $\mathcal{T}_\Omega$ has $\sigma$-finite $(n-1)$-dimensional Hausdorff measure. Similarly to the beginning part of the proof of Theorem 1.2 and reasoning by contradiction assume that there exists a set $G \subset \partial\Omega \cap B(x_0,r_0)$, with $r_0 \in (0,1)$, $x_0 \in G$, and two connected components $\Omega_1, \Omega_2 \subset B(x_0, 3r_0) \cap \Omega$ for which $G \subset \partial\Omega_1 \cap \partial\Omega_2$ such that $\mathcal{H}^{n-1}(G) = \infty$.

Consider the set $E = B(x_0,r_0) \cap \Omega_1$ for which we have $\chi_E \in BV(\Omega)$. Take any measurable function $v$ in $\mathbb{R}^n$ so that $v|_\Omega = \chi_E$. Note that $\widetilde{E}_t \cap \Omega = E$ for every $t \in (0,1)$ for the superlevel sets $\widetilde{E}_t = \{x \in \mathbb{R}^n : v(x) \ge t\}$. By using the measure density condition proved in [3, Proposition 2.3] applied to both connected components $\Omega_1$ and $\Omega_2$, we get that there exists $c > 0$ so that

$$\mathbf{m}_n(\Omega_i \cap B(x,r)) \ge cr^n$$

for $i = 1,2$ and all $x \in G$, $r \in (0, r_0)$. In particular, for every $x \in G$ we have

$$\limsup_{r\searrow 0} \frac{\mathbf{m}_n(B(x,r) \cap \widetilde{E}_t)}{\mathbf{m}_n(B(x,r))} \ge \limsup_{r\searrow 0} \frac{\mathbf{m}_n(B(x,r) \cap \Omega_1)}{\mathbf{m}_n(B(x,r))} > 0$$

and

$$\limsup_{r\searrow 0} \frac{\mathbf{m}_n(B(x,r) \cap (\mathbb{R}^n \setminus \widetilde{E}_t))}{\mathbf{m}_n(B(x,r))} \ge \limsup_{r\searrow 0} \frac{\mathbf{m}_n(B(x,r) \cap \Omega_2)}{\mathbf{m}_n(B(x,r))} > 0.$$

This means that $G \subset \partial^M \widetilde{E}_t$. Hence, $\mathcal{H}^{n-1}(\partial^M \widetilde{E}_t) \ge \mathcal{H}^{n-1}(G) = \infty$, so $\widetilde{E}_t$ does not have finite perimeter in $\mathbb{R}^n$ for any $t \in (0,1)$. Hence, by the coarea formula (3.1), $v \notin BV(\mathbb{R}^n)$. $\square$

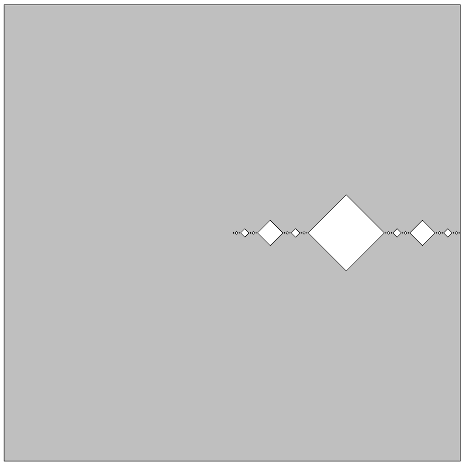

FIGURE 1. The domain showing the sharpness of Theorem 1.2 (1). The set $\mathcal{T}$ here is the fat Cantor set without its left-most point.

## 4. EXAMPLES

4.1. **Sharpness of the estimate for $p = 1$.** The following example shows the sharpness of Theorem 1.2 (1). In this case, where $p = 1$, we do not need to care about the norm of the extension operator and consequently, we can rely on previous non-quantitative characterizations of $W^{1,1}$-extension domains.

**Example 4.1.** *Let us define*

$$\Omega_2 = (-1, 1)^2 \setminus \{(x, y) : |y| \leq \operatorname{dist}(x, C), 0 \leq x \leq 1\}$$

*with $C \subset [0, 1]$ a Cantor set with $\dim_{\mathcal{H}}(C) = 1$ and $\mathcal{H}^1(C) = 0$. See Figure 1 for an illustration of the domain $\Omega_2$.*

*We claim that $\Omega_2$ is a $W^{1,1}$-extension domain and that*

$$\dim_{\mathcal{H}}(\mathcal{T}) = 1.$$

*It is easy to see that $\dim_{\mathcal{H}}(\mathcal{T}) = 1$, since $\mathcal{T} = (C \times \{0\}) \setminus \{(0, 0)\}$. In order to see that $\Omega_2$ is a $W^{1,1}$-extension domain, one can use the following characterization from* [10] *for bounded planar simply connected domains: $\Omega$ is a $W^{1,1}$-extension domain if and only if*

$$\begin{aligned}&\text{there exists a constant } K \text{ so that for every } x, y \in \Omega^c \text{ there exists a curve}\\ &\gamma \subset \Omega^c \text{ with } x, y \in \gamma, \ell(\gamma) \leq K|x - y|, \text{ and } \mathcal{H}^1(\gamma \cap \partial\Omega) = 0.\end{aligned} \tag{4.1}$$

*Now, the domain $\Omega_2$ clearly satisfies* (4.1) *and is thus a $W^{1,1}$-extension domain.*

Let us remark that Example 4.1 can also be generalized to higher dimensions $n > 2$ by defining $\Omega \subset \mathbb{R}^n$ as a product $\Omega_2 \times (-1, 1)^{n-2}$. It is then clear that

$$\dim_{\mathcal{H}}(\mathcal{T}) = n - 1.$$

The fact that $\Omega$ is a $W^{1,1}$-extension domain does not seem to immediately follow from known explicit results. One way to see that $\Omega$ is a $W^{1,1}$-extension domain is the following. Observe that the proof in [12] of the fact that a product of $W^{1,p}$-extension domains, with $p > 1$, is again a $W^{1,p}$-extension domain relies on the explicit form of the extension operators (which in that case can always be assumed to be a Whitney extension operators). In the case $p = 1$ it is unknown if the extension can always be done with a Whitney-type extension. However, the extension operator constructed in [10] for simply connected planar domains, and in particular for $\Omega_2$ is of Whitney-type. Thus, the argument in [12] goes through for our product domain $\Omega$.

4.2. **A bound for the estimates for $1 < p < n$.** The case $1 < p < n$ requires more work than the case $p = 1$, since the estimate in Theorem 1.2 depends on the norm of the extension operator. The assignment of reflected cubes in the construction of the extension operator, and the estimate of the norm of the extension operator follow roughly the proof of the sufficiency of the characterizing curve condition of planar simply connected $W^{1,p}$-extension domains [9].

Let us describe the family of domains $\Omega_\lambda$ we consider, where $\lambda \in (0, 1/2)$ refers to the contraction ratio of the Cantor set $C_\lambda \subset \mathbb{R}^{n-1}$. The Cantor sets $C_\lambda$ we use are the standard ones obtained as $C_\lambda = \prod_{i=1}^{n-1} K_\lambda$ with $K_\lambda$ being the Cantor set on the unit interval given as the attractor of the iterated function system $\{f_1(x) = \lambda x, f_2(x) = \lambda x + 1 - \lambda\}$.

We define first a set

$$D = (0,1)^{n-2} \times \big((-2,1) \times (-3/2, 3/2) \setminus [-1,0] \times [-1,1]\big)$$

and then the actual domain by carving out part of $D$:

$$\Omega_\lambda = D \setminus N_\lambda,$$

where

$$N_\lambda = \{(x_1, \ldots, x_n) \in [0,1]^n \,:\, |x_n| \leq \operatorname{dist}((x_1, \ldots, x_{n-1}), C_\lambda)\}.$$

Then, the set of two-sided points for $\Omega_\lambda$ is

$$\mathcal{T}_{\Omega_\lambda} = C_\lambda \times \{0\}$$

and so it has dimension

$$\dim_{\mathcal{H}}(\mathcal{T}_{\Omega_\lambda}) = \dim_{\mathcal{H}}(C_\lambda) = -\frac{(n-1)\log 2}{\log \lambda}. \tag{4.2}$$

Our aim is to build an extension operator $E_\lambda$ from $L^{1,p}(\Omega_\lambda)$ to $L^{1,p}(\mathbb{R}^n)$ for which we have

$$\dim_{\mathcal{H}}(C_\lambda) \geq n - p - \frac{C(n,p)}{||E_\lambda||}.$$

As will be explicit in (4.5), the constructed operator $E_\lambda$ will be bounded only for $p$ in the range

$$1 < p < n - \dim_{\mathcal{H}}(C_\lambda).$$

It is enough to construct an extension operator $E_\lambda \colon L^{1,p}(\Omega_\lambda) \to L^{1,p}(D)$, since the extension from $L^{1,p}(D)$ to $L^{1,p}(\mathbb{R}^n)$ is independent of $\lambda$, and exists since $D$ is a Lipschitz domain. Moreover, our definition of $E_\lambda$ will be independent of $p$ and will give a bounded operator between the Sobolev spaces $W^{1,p}(\Omega_\lambda)$ and $W^{1,p}(D)$. From now on we consider $\lambda \in (0, 1/2)$ fixed and we denote the extension operator by $E$ instead of $E_\lambda$ to simplify the notation.

Below by a dyadic cube we mean a set of the form $Q = [0, 2^{-k}]^n + \mathtt{j} \subset \mathbb{R}^n$ for some $k \in \mathbb{Z}$ and $\mathtt{j} \in 2^{-k}\mathbb{Z}^n$. Let $\mathcal{W} = \{Q_i\}_{i \in \mathbb{N}}$ be a Whitney decomposition of the interior of $N_\lambda$ and $\widetilde{\mathcal{W}} = \{\widetilde{Q}_i\}_{i \in \mathbb{N}}$ a Whitney decomposition of $\mathbb{R}^n \setminus N_\lambda$. This is

(W1) Each $Q_i$ is a closed dyadic cube inside $N_\lambda$.
(W2) $N_\lambda = \bigcup_i Q_i$ and for every $i \neq j$ we have $\operatorname{int}(Q_i) \cap \operatorname{int}(Q_j) = \emptyset$.
(W3) For every $i$ we have $\sqrt{n}\ell(Q_i) \leq \operatorname{dist}(Q_i, \partial N_\lambda) \leq 4\sqrt{n}\ell(Q_i)$.

(W4) If $Q_i \cap Q_j \neq \emptyset$, we have $\frac{1}{4}\ell(Q_i) \leq \ell(Q_j) \leq 4\ell(Q_i)$.

The definition of $\widetilde{\mathcal{W}}$ goes parallel. See [16, Chapter VI] for the existence of such Whitney decompositions. Consider also the subfamily of Whitney cubes

$$\mathcal{V} = \{Q \in \mathcal{W} : Q \cap ([0,1]^{n-1} \times \{0\}) \neq \emptyset\}.$$

Let us also distinguish an important subset of $\Omega_\lambda$, that we call

$$\widetilde{Q}_0 = (0,1)^{n-2} \times ((-2,1) \times (-3/2,3/2) \setminus [-1,1] \times [-1,1]) .$$

Note that $\widetilde{Q}$ are dyadic, so for every $\widetilde{Q} \in \widetilde{\mathcal{W}}$ we have $\partial((0,1)^{n-1} \times (-1,1)) \cap \mathrm{int}(\widetilde{Q}) = \emptyset$. Next we choose $\{\psi_i\}_{i\in\mathbb{N}}$ a partition of unity subordinate to the open cover $\{(9/8)\mathrm{int}(Q_i)\}_{i\in\mathbb{N}}$ and so that $|\nabla\psi_i(x)| \lesssim \ell(Q_i)^{-1}$.

Now, given $u \in L^{1,p}(\Omega_\lambda)$ we assign a value

$$a_i = \frac{1}{\mathbf{m}_n(\widetilde{Q}_{R(i)})} \int_{\widetilde{Q}_{R(i)}} u(x)\,dx$$

for every $i \in \mathbb{N}$, where the function $R\colon \mathbb{N} \to \mathbb{N}$ is defined as follows. If $Q_i \in \mathcal{V}$, then $R(i) = 0$. If $Q_i \notin \mathcal{V}$ we assign $R(i)$ to be the unique index so that $Q_i$ and $\widetilde{Q}_{R(i)}$ belong to the same half-space $\{x_n < 0\}$ or $\{x_n > 0\}$, $P_n(Q_i) \subset P_n(\widetilde{Q}_{R(i)})$, $\ell(\widetilde{Q}_{R(i)}) \leq 2\ell(Q_i)$ where $P_n\colon \mathbb{R}^n \to \mathbb{R}^{n-1}\colon (x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_{n-1})$, and $\widetilde{Q}_{R(i)}$ is the closest cube to $Q_i$ with the first three properties.

Now, we define the extension of the function $u$ by

$$Eu(x) = \begin{cases} u(x), & \text{if } x \in \Omega_\lambda \\ \sum_{i=1}^{\infty} a_i\psi_i(x), & \text{if } x \in \mathrm{int}(N_\lambda), \\ 0, & \text{if } x \in \partial N_\lambda \cap D, \,. \end{cases} \tag{4.3}$$

Let us explain first why $Eu \in L^{1,p}(D)$. On the one hand, $Eu \in L^{1,p}(\Omega_\lambda)$ and, on the other hand, we will see later that $Eu \in L^{1,p}(\mathrm{int}(N_\lambda))$. Since $C_\lambda$ has Hausdorff-dimension strictly less than $n-1$, we have that almost every line parallel to the coordinate axis does not meet the set $C_\lambda$. Consequently, any function in $L^{1,p}(D \setminus C_\lambda)$ is also ACL on $D$, implying that $L^{1,p}(D) = L^{1,p}(D \setminus C_\lambda)$ as sets. So, in order to then have $Eu \in L^{1,p}(D)$, it suffices to show that $Eu \in L^{1,p}(D \setminus C_\lambda)$. This follows by noticing that by the definition of the operator $E$, the trace of $u$

$$Tu(x) = \lim_{r\searrow 0} \frac{1}{\mathbf{m}_n(B(x,r) \cap \Omega_\lambda)} \int_{B(x,r)\cap\Omega_\lambda} u(y)\,dy$$

on $\partial N_\lambda \setminus C_\lambda$ coincides with that of $Eu|_{\mathrm{int}(N_\lambda)}$

$$TEu(x) = \lim_{r\searrow 0} \frac{1}{\mathbf{m}_n(B(x,r) \cap N_\lambda)} \int_{B(x,r)\cap\mathrm{int}(N_\lambda)} Eu(y)\,dy.$$

To conclude that $E$ is an extension operator it remains to control the $L^p$-norm of the gradient of the extension on $\mathrm{int}(N_\lambda)$ by the $L^p$-norm of the gradient of the initial function.

We know that $\mathrm{supp}(\psi_i) \subseteq \frac{9}{8}Q_i$ and that $|\nabla\psi_i(x)| \lesssim \ell(Q_i)^{-1}$ for every $x$ and $i \in \mathbb{N}$, so it is clear that for every $x \in Q_i$,

$$|\nabla Eu(x)| \leq \left| \sum_{Q_j\cap Q_i\neq\emptyset} \nabla\psi_j(x)(a_j - a_i) \right| \lesssim \sum_{Q_j\cap Q_i\neq\emptyset} \ell(Q_j)^{-1}|a_j - a_i|.$$

Now, if we take a cube $Q_i \in \mathcal{W}$, using that at most $C(n)$ other cubes of the Whitney decomposition are intersecting it and that $\ell(Q_i) \sim \ell(Q_j)$ if $Q_i \cap Q_j \neq \emptyset$ we write

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(Q_i)} = \int_{Q_i} |\nabla Eu(x)|^p \, dx &\lesssim \int_{Q_i} \sum_{Q_j \cap Q_i \neq \emptyset} \ell(Q_j)^{-p} |a_i - a_j|^p \, dx \\ &\lesssim \ell(Q_i)^{n-p} \sum_{Q_j \cap Q_i \neq \emptyset} |a_i - a_j|^p.\end{aligned} \tag{4.4}$$

It will be useful to work with chains of Whitney cubes that we next define. Given $i, j$ so that $Q_i \cap Q_j \neq \emptyset$ and $Q_i, Q_j \notin \mathcal{V}$ we define the chain of cubes joining $\widetilde{Q}_{R(i)}$ with $\widetilde{Q}_{R(j)}$, and denote it by $C(\widetilde{Q}_{R(i)}, \widetilde{Q}_{R(j)})$, to be a minimal family (in cardinality) of Whitney cubes whose union's interior is a connected set containing both the interiors of $\widetilde{Q}_{R(i)}$ and $\widetilde{Q}_{R(j)}$. Note that we always have $\#C(\widetilde{Q}_{R(i)}, \widetilde{Q}_{R(j)}) \leq C_0(n)$. Suppose $Q_i \notin \mathcal{V}$ is a cube so that there exists $Q_j \in \mathcal{V}$ with $Q_i \cap Q_j \neq \emptyset$. For the associated cube $\widetilde{Q}_{R(i)}$ we define $C(\widetilde{Q}_{R(i)}, \widetilde{Q}_0)$ as a minimal family of sets in $\widetilde{W} \cup \{\widetilde{Q}_0\}$ whose union's interior is a connected set containing both the interiors of $\widetilde{Q}_{R(i)}$ and $\widetilde{Q}_0$ and so that every $\widetilde{Q} \in C(\widetilde{Q}_{R(i)}, \widetilde{Q}_0)$ satisfies $P_n(\widetilde{Q}_{R(i)}) \subset P_n(\widetilde{Q})$.

We can assume there is an order in the chain when moving from $\widetilde{Q}_{R(i)}$ to $\widetilde{Q}_{R(j)}$ and call $\widetilde{Q}_{\text{next}}$ the next cube in the chain after $\widetilde{Q}$. We write

$$C(\widetilde{Q}_{R(i)}, \widetilde{Q}_{R(j)}) = \{\widetilde{Q}_{R(i)}, (\widetilde{Q}_{R(i)})_{\text{next}}, \ldots, \widetilde{Q}_{R(j)}\}.$$

To ease the notation in the following sums from now on we write

$$C_{i,j} = C(\widetilde{Q}_{R(i)}, \widetilde{Q}_{R(j)}) \setminus \{\widetilde{Q}_{R(j)}\} \quad \text{and} \quad C_{i,0} = C(\widetilde{Q}_{R(i)}, \widetilde{Q}_0) \setminus \{\widetilde{Q}_0\}.$$

Note that if $Q_i \notin \mathcal{V}$ and there does not exists $Q_j$ such that $Q_j \in \mathcal{V}$ and $Q_i \cap Q_j \neq \emptyset$ we define $C_{i,0} = \emptyset$.

Let also write

$$\mathcal{I} = \left\{\widetilde{Q} \in \widetilde{W} \,:\, \widetilde{Q} = \widetilde{Q}_{R(i)} \text{ for some } i \geq 1\right\}.$$

We assert that the following claim holds.

**Claim 4.2.** *With the above notation and for every $r > 0$ we have the following.*

*(i) For every $Q_i \notin \mathcal{V}$*

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(Q_i)} \lesssim &\sum_{\{\widetilde{Q} \in \mathcal{I} \,:\, \#C(\widetilde{Q}_{R(i)}, \widetilde{Q}) \leq C_0(n)\}} \int_{\widetilde{Q}} |\nabla u(x)|^p \, dx \\ &+ \ell(Q_i)^{n-p-rp} D(r,p) \sum_{\widetilde{Q} \in C_{i,0}} \ell(\widetilde{Q})^{p-n+rp} \int_{\widetilde{Q} \cup \widetilde{Q}_{next}} |\nabla u(x)|^p \, dx,\end{aligned}$$

*where $D(r,p) = (1 - 2^{\frac{-rp}{p-1}})^{1-p}$, and for every $Q_i \in \mathcal{V}$, we have*

$$\|\nabla Eu\|^p_{L^p(Q_i)} \lesssim \ell(Q_i)^{n-p-rp} \sum_{\substack{Q_j \cap Q_i \neq \emptyset \\ Q_j \notin \mathcal{V}}} D(r,p) \sum_{\widetilde{Q} \in C_{j,0}} \ell(\widetilde{Q})^{p-n+rp} \int_{\widetilde{Q} \cup \widetilde{Q}_{next}} |\nabla u(x)|^p \, dx.$$

*(ii) For a given $\widetilde{Q} \in \widetilde{\mathcal{W}}$ and $k \in \mathbb{Z}$ we have*

$$\#\left\{Q_i \in \mathcal{W} \setminus \mathcal{V} \,:\, Q_i \textit{ has a neighbouring cube in } \mathcal{V},\ \ell(\widetilde{Q}) = 2^k \ell(Q_i),\ \widetilde{Q} \in C_{i,0}\right\} \lesssim 2^{-(n-1)\frac{k \log 2}{\log \lambda}}.$$

Assuming for a moment that the claim is true let us show how one can estimate the full norm $\|\nabla Eu\|^p_{L^p(N_\lambda)}$. We first use Claim 4.2 (i) and change the order of summation to get

$$\begin{aligned}
\|\nabla Eu\|^p_{L^p(N_\lambda)} &= \sum_{Q_i\in\mathcal{W}} \|\nabla Eu\|^p_{L^p(Q_i)} = \sum_{Q_i\notin\mathcal{V}} \|\nabla Eu\|^p_{L^p(Q_i)} + \sum_{Q_i\in\mathcal{V}} \|\nabla Eu\|^p_{L^p(Q_i)} \\
&\lesssim \sum_{\widetilde{Q}\in\mathcal{I}} \sum_{\{i\,:\,\#C(\widetilde{Q}_{R(i)},\widetilde{Q})\le C_0(n)\}} \int_{\widetilde{Q}} |\nabla u(x)|^p\,dx \\
&\quad + 2\sum_{\widetilde{Q}\in\widetilde{\mathcal{W}}} \sum_{\widetilde{Q}\in C_{i,0}} \ell(Q_i)^{n-p-rp} D(r,p)\ell(\widetilde{Q})^{p-n+rp} \int_{\widetilde{Q}\cup\widetilde{Q}_{\mathrm{next}}} |\nabla u(x)|^p\,dx \\
&\lesssim \|\nabla u\|^p_{L^p(\Omega_\lambda)} + \sum_{\widetilde{Q}\in\widetilde{\mathcal{W}}} \sum_{\widetilde{Q}\in C_{i,0}} \ell(Q_i)^{n-p-rp} D(r,p)\ell(\widetilde{Q})^{p-n+rp} \int_{\widetilde{Q}\cup\widetilde{Q}_{\mathrm{next}}} |\nabla u(x)|^p\,dx.
\end{aligned}$$

Moreover, by Claim 4.2 (ii) it follows that

$$\sum_{\{i\,:\,\widetilde{Q}\in C_{i,0}\}} \ell(Q_i)^{n-p-rp} \lesssim \sum_{k=0}^{\infty} 2^{-(n-1)\frac{k\log 2}{\log\lambda}} (2^{-k}\ell(\widetilde{Q}))^{n-p-rp} = \frac{\ell(\widetilde{Q})^{n-p-rp}}{1-2^{-n+p-(n-1)\frac{\log 2}{\log\lambda}+rp}},$$

under the assumption

$$-n+p-(n-1)\frac{\log 2}{\log\lambda}+rp<0. \tag{4.5}$$

So, joining these facts together we get

$$\begin{aligned}
\|\nabla Eu\|^p_{L^p(N_\lambda)} &\lesssim \|\nabla u\|^p_{L^p(\Omega_\lambda)} + \sum_{\widetilde{Q}\in\widetilde{\mathcal{W}}_1} D(r,p)\left(\frac{1}{1-2^{-n+p-(n-1)\frac{\log 2}{\log\lambda}+rp}}\right) \int_{\widetilde{Q}\cup\widetilde{Q}_{\mathrm{next}}} |\nabla u(x)|^p\,dx \\
&\lesssim \left(\frac{1}{1-2^{\frac{-rp}{p-1}}}\right)^{p-1} \left(\frac{1}{1-2^{-n+p-(n-1)\frac{\log 2}{\log\lambda}+rp}}\right) \|\nabla u\|^p_{L^p(\Omega_\lambda)},
\end{aligned}$$

where $\widetilde{\mathcal{W}}_1 = \{\widetilde{Q}\in\widetilde{\mathcal{W}} : \#\{i : \widetilde{Q}\in C_{i,0}\} > 0\}$ and $\widetilde{Q}_{next}\in C_{i,0}$ for some $i\in\mathbb{N}$ such that $\widetilde{Q}\in C_{i,0}$. Choosing $r=\frac{p-1}{p^2}(n-p-\dim_{\mathcal{H}}(C_\lambda))$, we conclude that

$$||E|| \lesssim \frac{1}{1-2^{\frac{1}{p}(-n+p+\dim_{\mathcal{H}}(C_\lambda))}},$$

which yields

$$\dim_{\mathcal{H}}(C_\lambda) \ge n-p-\frac{C(n,p)}{||E||}.$$

Let us now prove the Claim 4.2.

*Proof of Claim 4.2.* To prove (i) we need to estimate $|a_i - a_j|^p$ in the expression (4.4). First note that from (4.4) one gets

$$\begin{aligned} |a_i - a_j|^p &\leq \left( \sum_{\widetilde{Q} \in C_{i,j}} \left| \frac{1}{\mathbf{m}_n(\widetilde{Q})} \int_{\widetilde{Q}} u(x)\,dx - \frac{1}{\mathbf{m}_n(\widetilde{Q}_{\text{next}})} \int_{\widetilde{Q}_{\text{next}}} u(x)\,dx \right| \right)^p \\ &\lesssim \left( \sum_{\widetilde{Q} \in C_{i,j}} \ell(\widetilde{Q})^{1-n} \int_{\widetilde{Q} \cup \widetilde{Q}_{\text{next}}} |\nabla u(x)|\,dx \right)^p, \end{aligned} \tag{4.6}$$

where we are using the Poincaré inequality in the last line (see [6, Lemma 2.2] and also [1]). Observe that if $Q_j, Q_i \in \mathcal{V}$ then $R(i) = R(j) = 0$ and $|a_j - a_i| = 0$. We now consider two cases.

(1) Suppose $i, j$ are so that $Q_i \cap Q_j \neq \emptyset$ and $Q_i, Q_j \notin \mathcal{V}$. Then using (4.6), that $\#C(\widetilde{Q}_{R(i)}, \widetilde{Q}_{(R(j)}) \leq C_0(n)$, that the sides of the cubes of the chain are comparable to that of $\widetilde{Q}_{R(i)}$, and hence that of $Q_i$, and applying Hölder inequality

$$\begin{aligned} |a_i - a_j|^p &\lesssim \sum_{\widetilde{Q} \in C_{i,j}} \ell(\widetilde{Q})^{(1-n)p} \left( \int_{\widetilde{Q} \cup \widetilde{Q}_{\text{next}}} |\nabla u(x)|\,dx \right)^p \\ &\lesssim \ell(Q_i)^{p-n} \sum_{\widetilde{Q} \in C_{i,j}} \int_{\widetilde{Q} \cup \widetilde{Q}_{\text{next}}} |\nabla u(x)|^p\,dx. \end{aligned} \tag{4.7}$$

(2) Suppose $i, j$ are such that $Q_i \cap Q_j \neq \emptyset$, $Q_j \in \mathcal{V}$ (then $R(j) = 0$) and $Q_i \notin \mathcal{V}$ then we fix $r > 0$ to be determined later and apply Hölder inequality to (4.6) to get

$$\begin{aligned} |a_i - a_j|^p &\lesssim \left( \sum_{\widetilde{Q} \in C_{i,0}} \ell(\widetilde{Q})^{-r} \ell(\widetilde{Q})^{1-n+r} \int_{\widetilde{Q} \cup \widetilde{Q}_{\text{next}}} |\nabla u(x)|\,dx \right)^p \\ &\leq \left( \sum_{\widetilde{Q} \in C_{i,0}} \ell(\widetilde{Q})^{-r\frac{p}{p-1}} \right)^{p-1} \left( \sum_{\widetilde{Q} \in C_{i,0}} \ell(\widetilde{Q})^{(1-n+r)p} \left( \int_{\widetilde{Q} \cup \widetilde{Q}_{\text{next}}} |\nabla u(x)|\,dx \right)^p \right) \\ &\lesssim \left( \sum_{k=0}^{\infty} (2^k \ell(\widetilde{Q}_{R(i)}))^{-r\frac{p}{p-1}} \right)^{p-1} \left( \sum_{\widetilde{Q} \in C_{i,0}} \ell(\widetilde{Q})^{p-n+rp} \int_{\widetilde{Q} \cup \widetilde{Q}_{\text{next}}} |\nabla u(x)|^p\,dx \right) \\ &\lesssim D(r,p) \ell(\widetilde{Q}_{R(i)})^{-rp} \left( \sum_{\widetilde{Q} \in C_{i,0}} \ell(\widetilde{Q})^{p-n+rp} \int_{\widetilde{Q} \cup \widetilde{Q}_{\text{next}}} |\nabla u(x)|^p\,dx \right). \end{aligned} \tag{4.8}$$

Going back to equation (4.4) for any $Q_i \notin \mathcal{V}$, using (4.7) and (4.8) we have

$$\begin{aligned}
\|\nabla Eu\|^p_{L^p(Q_i)} &\lesssim \ell(Q_i)^{n-p}\left(\sum_{Q_j\cap Q_i\neq\emptyset,\, Q_j\notin\mathcal{V}} |a_i-a_j|^p + \sum_{Q_j\cap Q_i\neq\emptyset,\, Q_j\in\mathcal{V}} |a_i-a_j|^p\right)\\
&\lesssim \ell(Q_i)^{n-p}\left(\sum_{Q_j\cap Q_i\neq\emptyset,\, Q_j\notin\mathcal{V}} \ell(Q_i)^{p-n}\sum_{\widetilde{Q}\in C_{i,j}}\int_{\widetilde{Q}\cup\widetilde{Q}_{\mathrm{next}}}|\nabla u(x)|^p\,dx\right.\\
&\quad\left. + \sum_{Q_j\cap Q_i\neq\emptyset,\, Q_j\in\mathcal{V}} D(r,p)\ell(Q_i)^{-rp}\sum_{\widetilde{Q}\in C_{i,0}}\ell(\widetilde{Q})^{p-n+rp}\int_{\widetilde{Q}\cup\widetilde{Q}_{\mathrm{next}}}|\nabla u(x)|^p\,dx\right)\\
&\lesssim \sum_{\{\widetilde{Q}\in\mathcal{I}\,:\,\#C(\widetilde{Q}_{R(i)},\widetilde{Q})\le C_0(n)\}}\int_{\widetilde{Q}}|\nabla u(x)|^p\,dx\\
&\quad + \ell(Q_i)^{n-p-rp}D(r,p)\sum_{\widetilde{Q}\in C_{i,0}}\ell(\widetilde{Q})^{p-n+rp}\int_{\widetilde{Q}\cup\widetilde{Q}_{\mathrm{next}}}|\nabla u(x)|^p\,dx,
\end{aligned}$$

and if $Q_i \in \mathcal{V}$, using only (4.8)

$$\begin{aligned}
\|\nabla Eu\|^p_{L^p(Q_i)} &\lesssim \ell(Q_i)^{n-p}\sum_{Q_j\cap Q_i\neq\emptyset,\, Q_j\notin\mathcal{V}}|a_i-a_j|^p\\
&\lesssim \ell(Q_i)^{n-p-rp}\sum_{Q_j\cap Q_i\neq\emptyset,\, Q_j\notin\mathcal{V}} D(r,p)\sum_{\widetilde{Q}\in C_{j,0}}\ell(\widetilde{Q})^{p-n+rp}\int_{\widetilde{Q}\cup\widetilde{Q}_{\mathrm{next}}}|\nabla u(x)|^p\,dx.
\end{aligned}$$

which proves (i).

Let us prove (ii). Let us write the Cantor set $C_\lambda \subset [0,1]^{n-1}$ as

$$C_\lambda = \bigcap_{i=0}^{\infty} C^i_\lambda = \bigcap_{i=0}^{\infty}\bigcup_{1\le j\le 2^{(n-1)i}} I_{i,j},$$

where $I_{i,j}$ is a translated copy of $[0,\lambda^i]^{n-1}$ for all $i=0,1,2,\ldots$ and $j=1,2,\ldots,2^{(n-1)i}$. It is clear that for $i<i'$, any cube $I_{i,j}$ contains $2^{(n-1)(i'-i)}$ cubes of side length $\lambda^{i'}$.

Fix $\widetilde{Q}\in\widetilde{\mathcal{W}}$ and $k\in\mathbb{N}$. Let $t\in\mathbb{N}$ such that $\ell(\widetilde{Q})=2^{-t}$. We count the cardinality of

$$A=\left\{Q_i\in\mathcal{W}\setminus\mathcal{V}\,:\, Q_i \text{ has a neighbouring cube in } \mathcal{V},\ \ell(\widetilde{Q})=2^k\ell(Q_i),\ \widetilde{Q}\in C_{i,0}\right\}.$$

Define $B=\{P_n(Q_i)\}_{Q_i\in A}$, where $P_n\colon\mathbb{R}^n\to\mathbb{R}^{n-1}\colon(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_{n-1})$.

Let $m$ be the least positive integer such that $\lambda^m<2^{-t}$ and let $l$ be the least positive integer so that $\lambda^l\le 2^{-t-k}$. By the properties of the Whitney decomposition, the construction of the Cantor set and the minimality of $m$ it is enough to count $\#\{Q\in B\,:\,\mathrm{dist}\,(Q,I_{m,j})\le C(n)\ell(Q)\}$ for a fixed $I_{m,j}$. Moreover, by the selection of $l$ none of the cubes $I_{l,j}$ contains any $Q\in B$.

Because $\lambda^l\le\ell(Q)$, we have

$$\#\left\{Q\in B\,:\,\mathrm{dist}\,(Q,I_{l,j'})\le C(n)\ell(Q)\right\}\le c(n)$$

for all $I_{l,j'} \subset I_{m,j}$. Finally since $I_{m,j} \cap C_\lambda^l$ is a disjoint union of $2^{(n-1)(l-m)}$ cubes $I_{l,j'}$ of side length $\lambda^l$ we conclude that

$$\#A \lesssim \#B \le c(n)2^{(n-1)(l-m)} \lesssim 2^{-k(n-1)\frac{\log 2}{\log \lambda}}.$$

□

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## A necessary condition for Sobolev extension domains in higher dimensions

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# A NECESSARY CONDITION FOR SOBOLEV EXTENSION DOMAINS IN HIGHER DIMENSIONS

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Abstract. We give a necessary condition for a domain to have a bounded extension operator from $L^{1,p}(\Omega)$ to $L^{1,p}(\mathbb{R}^n)$ for the range $1 < p < 2$. The condition is given in terms of a power of the distance to the boundary of $\Omega$ integrated along the measure theoretic boundary of a set of locally finite perimeter and its extension. This generalizes a characterizing curve condition for planar simply connected domains, and a condition for $W^{1,1}$-extensions. We use the necessary condition to give a quantitative version of the curve condition. We also construct an example of an extension domain in $\mathbb{R}^3$ that is homeomorphic to a ball and has 3-dimensional boundary.

## Contents



## 1. Introduction

A domain $\Omega \subset \mathbb{R}^n$ is called a $W^{k,p}$-extension domain, if we can extend each Sobolev function $u \in W^{k,p}(\Omega)$ to a global Sobolev function $u \in W^{k,p}(\mathbb{R}^n)$ so that the Sobolev norm of the extension is at most a constant times the norm of the original function. Sobolev extension domains are interesting in several fields of analysis because on those one can use many functional-analytic tools that are classically available for functions defined on the whole space. Examples of Sobolev extension domains include Lipschitz domains [5, 32] and more generally, $(\varepsilon, \delta)$-domains [18]. For our context, the Lipschitz and $(\varepsilon, \delta)$ results should be seen as sufficient conditions on the boundary of the domain for the extendability of Sobolev

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functions. In this paper, we continue investigating the converse direction by finding a new necessary condition for extendability.

Several necessary geometric conditions on the boundary of Sobolev extension domains are already known. For instance, all Sobolev extension domains have positive densities at all the points belonging to them (this is usually referred as to satisfy a measure density condition, [19, 15]). Then, by the Lebesgue differentiation theorem we must have that their boundaries are of zero Lebesgue measure. In general we cannot improve this to a non-trivial dimension upper bound on the boundary of a Sobolev extension domain: take for example $\Omega = [0,1]^n \setminus C^n$ with $C$ a Cantor set with zero Lebesgue measure and so that $\dim_{\mathcal{H}}(C) = 1$.

However, one can still meaningfully study the dimension of the boundary of extension domains. One approach is to limit the topology or other properties of the domain, and another one is to investigate only those points that are more relevant for the extendability. The second approach leads to the study of the size of the set of two-sided points of the boundaries of Sobolev extension domains (that is, points where the boundary might self-intersect and hence can be approached from two different sides in the domain). In the case $p \geq n$ we have that $W^{1,p}$-extension domains are quasiconvex (see [19, Theorem 3.1]) and then the set of two-sided points must be empty. The case $1 \leq p < n$ is more interesting and has been investigated in [33, 10], where bounds on the Hausdorff dimension of the set of two-sided points are found.

Non-trivial dimension upper bounds for the whole boundary have been obtained only in the special case of planar bounded simply connected extension domains [25]. These bounds are based on the porosity of the boundary that is implied by the geometric characterizations of bounded simply connected planar Sobolev extension domains, see (1.1) and (1.2) below. The first such characterizations established that a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,2}$-extension domain if and only if $\Omega$ is a quasidisk (see [11, 12, 13, 18]).

In the case $2 < p < \infty$, Shvartsman [30] proved that a bounded finitely connected domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,p}$-extension domain if and only if for some $C > 1$ the following condition is satisfied: for every $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining $x$ and $y$ so that

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-p}}\, ds(z) \leq C|x-y|^{\frac{p-2}{p-1}}. \tag{1.1}$$

Let us mention that in [31], the curve condition (1.1) was also shown to characterize $L^{k,p}$-extension domains for every $2 < p < \infty$ and $k \in \mathbb{N}$. Here we define the homogeneous Sobolev space $L^{k,p}(\Omega)$ to be the space of locally integrable functions whose distributional partial derivatives belong to $L^p(\Omega)$.

Finally, for the case $1 < p < 2$ the following result is proved in [22]: a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,p}$-extension domain if and only if there exists $C > 1$ such that for every $x, y \in \mathbb{R}^2 \setminus \Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting $x$ and $y$ such that

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \leq C|x-y|^{2-p}. \tag{1.2}$$

In Theorem 1.1 we generalize the condition (1.2) to higher dimensions; still for the range $1 < p < 2$ of exponents. Before stating our result, let us look at the limiting case $p = 1$ that partly motivates our formulation.

In the case of a bounded simply connected planar domain $\Omega$, by the results from [23], we know that $\Omega$ is a $W^{1,1}$-extension domain if and only if for every $x, y \in \Omega^c$ there exists a curve

$\gamma \subset \Omega^c$ connecting $x$ and $y$ with

$$\ell(\gamma) \le C|x-y|, \text{ and } \mathcal{H}^1(\gamma \cap \partial\Omega) = 0. \tag{1.3}$$

In other words, the correct limit of the term $\operatorname{dist}(z, \partial\Omega)^{1-p}$ in (1.2) is $1/\chi_{\mathbb{R}^2\setminus\Omega}(z)$ as $p \searrow 1$. The characterizing property (1.3) can also be seen as a combination of earlier results on $BV$-extension domains and the following more general planar result [9]: A bounded $BV$-extension domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,1}$-extension domain if and only if the 1-dimensional measure of the set

$$\partial\Omega \setminus \bigcup_{i\in I} \overline{\Omega_i}$$

intersected with any Lipschitz curve is zero, where $\{\Omega_i\}_{i\in I}$ are the connected components of $\mathbb{R}^2 \setminus \overline{\Omega}$. Recall that the space $BV(\Omega)$ consists of integrable functions $u \in L^1(\Omega)$ whose total variation

$$\|Du\|(\Omega) = \sup\left\{\int_\Omega u \operatorname{div}(v)\, dx \,:\, v \in C_0^\infty(\Omega; \mathbb{R}^n),\ |v| \le 1\right\}$$

is finite. As observed in [9], the above characterization of $W^{1,1}$-extension domains holds only in the plane. This is essentially because the planar topology allows one to write the measure theoretic boundary of a set of finite perimeter as the union of Jordan loops, see [1, Corollary 1] (recalled in Proposition 2.2 below).

In higher dimension where such decomposition result does not hold, the characterization is written in terms of sets of finite perimeter. Before going to this characterization, let us recall an earlier result on $BV_l$-extension domains, where

$$BV_l(\Omega) = \{u \in L^1_{loc}(\Omega) : \|Du\|(\Omega) < \infty\}.$$

In [4], Burago and Maz'ya proved the following characterization of $BV_l$-extension domains: $\Omega \subset \mathbb{R}^n$ is a $BV_l$-extension domain if and only if there exists some constant $C > 0$ so that any set $A \subset \Omega$ of finite perimeter in $\Omega$ admits an extension $\widetilde{A} \subset \mathbb{R}^n$ satisfying $\widetilde{A} \cap \Omega = A$ and

$$P(\widetilde{A}, \mathbb{R}^n) \le CP(A, \Omega).$$

Since $L^{1,1}$-extension domains are known to be $BV_l$-extension domains (the proof of this fact follows the same ideas as one may find in [21, Lemma 2.4]), the above property about extension of sets of finite perimeter is a necessary condition both for $BV_l$- and $L^{1,1}$-extension domains.

In order to turn this into a characterization of $L^{1,1}$- or $W^{1,1}$-extension domains, we have to account for the intersection of the boundary of the extended set with the boundary of the domain, analogously to (1.3). This leads to the following characterization in terms of strong extension of sets of finite perimeter [9]: A bounded domain $\Omega$ is a $W^{1,1}$-extension domain if and only if any set $A \subset \Omega$ of finite perimeter in $\Omega$ admits an extension $\widetilde{A} \subset \mathbb{R}^n$ satisfying $\widetilde{A} \cap \Omega = A$,

$$P(\widetilde{A}, \mathbb{R}^n) \le CP(A, \Omega) \text{ and also } \mathcal{H}^{n-1}(\partial^M \widetilde{A} \cap \partial\Omega) = 0,$$

where $\partial^M \widetilde{A}$ denotes the measure theoretic boundary of $\widetilde{A}$. In order to remind ourselves of the analogous condition in the planar simply connected case as the limit of (1.2), we can rewrite this in an integral form

$$\int_{\partial^M \widetilde{A}} \frac{1}{\chi_{\mathbb{R}^n\setminus\partial\Omega}(z)}\, d\mathcal{H}^{n-1}(z) \le C \int_{\Omega\cap\partial^M A} \frac{1}{\chi_{\mathbb{R}^n\setminus\partial\Omega}(z)}\, d\mathcal{H}^{n-1}(z).$$

This motivates the formulation of the following main theorem of this paper.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^n$ be an $L^{1,p}$-extension domain for some $1 < p < 2$. Then for any $\varepsilon > 0$ and any measurable set $A \subset \Omega$ there exists a set $\widetilde{A} \subset \mathbb{R}^n$ with $A = \widetilde{A} \cap \Omega$ and*

$$\int_{\partial^M \widetilde{A}} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) \le C(n,p,\varepsilon)\|E\|^{n+p+\varepsilon} \int_{\Omega\cap\partial^M A} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z), \tag{1.4}$$

*where $\|E\|$ denotes the norm of the $L^{1,p}$-extension operator, and the constant $C(n,p,\varepsilon)$ depends only on $n$, $p$ and $\varepsilon$.*

Let us immediately comment on the range $1 < p < 2$ of exponents and the use of the homogeneous Sobolev space $L^{1,p}$ in Theorem 1.1. The reason for the range of exponents is that if $p \ge 2$, then (1.4) is always satisfied by the choice $\widetilde{A} = A$ since the integral on the right-hand side is infinite in the nontrivial cases. Thus, for $p \ge 2$ the conclusion of Theorem 1.1 provides no information.

The use of the homogeneous Sobolev space is natural for scaling invariant results. In the case $\Omega$ is bounded, the result still applies for $W^{1,p}$-extension domains because these are known to be $L^{1,p}$-extension domains as well (see [19]). When thinking about moving between $W^{1,p}$- and $L^{1,p}$-extensions in bounded domains, one should observe that for a set $A$ occupying most of $\Omega$ (in our proof, for $A$ satisfying $|A| > \frac{1}{2}|\Omega|$) the extension $\widetilde{A}$ satisfying (1.4) has to contain all of the space $\mathbb{R}^n$ that is sufficiently far away from $\Omega$.

It is worth noticing also that, if $\Omega$ is bounded, any measurable set $A \subset \Omega$ for which the right hand side of the inequality (1.4) is finite must be of finite perimeter in $\Omega$, and also the set $\widetilde{A}$ that we construct will be of finite perimeter in $\mathbb{R}^n$. If $\Omega$ were unbounded we would only have that $A$ and $\widetilde{A}$ are locally of finite perimeter in $\Omega$ and in $\mathbb{R}^n$, respectively.

One might wonder if the condition in Theorem 1.1 is also sufficient for $\Omega$ to be an $L^{1,p}$-extension domain. It turns out that this is not the case: Suppose $\Omega' \subset \mathbb{R}^n$ is an arbitrary domain. We can modify $\Omega'$ to a new domain $\Omega = \Omega' \setminus \bigcup_{i=1}^{\infty} B(x_i, r_i)$, where the balls $B(x_i, r_i) \subset \Omega'$ are selected in such a way that $B(x_i, 2r_i) \setminus B(x_i, r_i) \subset \Omega$ (giving that we have an extension operator from $L^{1,p}(\Omega)$ to $L^{1,p}(\Omega')$), but so that they accumulate densely enough to the boundary of $\Omega'$ so that for any $A \subset \Omega$ we can take $\widetilde{A}$ to be zero outside $\Omega$ due to the right-hand side of (1.4) being infinite for any $A$ for which we would not be able to take $\widetilde{A} = A$ when considering (1.4) with respect to $\Omega'$. Again, the condition (1.4) gives us no real information on $\Omega$ as it holds regardless of $\Omega'$ and thus $\Omega$ being an extension domain or not.

In dimensions at least three, one can make the above idea into a construction of a topologically nice extension domain with large boundary. In the version of the construction that we use to prove the following theorem, the removed balls from the domain are replaced by removed tubes, and they accumulate only to a large portion of the boundary instead of the whole boundary.

**Theorem 1.2.** *There exists a domain $\Omega \subset \mathbb{R}^3$ such that $\Omega = h(B(0,1))$ for a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\dim_{\mathcal{H}}(\partial\Omega) = 3$ and $\Omega$ is a $W^{1,p}$-extension domain for all $p \in [1,\infty]$.*

Note that the domain in Theorem 1.2 cannot be an $(\varepsilon, \delta)$-domain, nor a John domain, since these domains have porous boundaries and hence their Hausdorff (and packing) dimensions would be strictly less than three. We also reiterate that the same type of example is not possible in $\mathbb{R}^2$ by the dimension bounds on the boundary of a simply connected planar Sobolev extension domain given in [25].

We wrote the dependence on the norm of the extension operator explicitly in Theorem 1.1 mainly in order to start the investigation of the dependence between the norm and the constant $C$ in (1.2). Using this explicit form, we obtain a more quantified version of the necessity of (1.2).

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^2$ a bounded simply connected $L^{1,p}$-extension domain for some $1 < p < 2$. Then for every $\varepsilon > 0$ there exists a constant $C(p,\varepsilon) > 0$ such that for all $z_1, z_2 \in \partial\Omega$ there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining $z_1$ and $z_2$ so that*

$$\int_\gamma \operatorname{dist}^{1-p}(z,\partial\Omega)\, ds(z) \leq C(p,\varepsilon)\|E\|^{\frac{4+4p-p^2}{2-p}+\varepsilon}|z_1 - z_2|^{2-p}. \tag{1.5}$$

We do not claim nor expect the dependence on $\|E\|$ in (1.5) to be sharp. However, our proof of Theorem 1.3 written in Section 4 gives the first explicit dependence. Since (1.2) is a characterization, one could also try to get the dependence of the operator norm $\|E\|$ on the curve condition constant $C$. This direction of the proof of the characterization in [22] is more technical. Consequently, we suspect the quantitative dependence in this direction to be more difficult to obtain.

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## 2. Preliminaries

In what follows, we use the notation $C(\cdot)$ to mean a strictly positive and finite function on the parameters listed in the parentheses, i.e. a constant once the listed parameters are fixed. The function (constant) may change between appearances even within a chain of inequalities.

For any point $x \in \mathbb{R}^n$ and radius $r > 0$ we denote the open ball by $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. More generally, for a set $A \subset \mathbb{R}^n$ we define the open $r$-neighbourhood as

$$B(A,r) = \bigcup_{x\in A} B(x,r).$$

We denote by $|A|$ the $n$-dimensional outer Lebesgue measure of a set $A \subset \mathbb{R}^n$. For any Lebesgue measurable subsets $A \subset \Omega \subset \mathbb{R}^n$ and any point $x \in \mathbb{R}^n$ we define the upper density of $A$ at $x$ over $\Omega$ as

$$\overline{D}(A,\Omega,x) = \limsup_{r\searrow 0} \frac{|A \cap B(x,r)|}{|B(x,r)\cap\Omega|},$$

and the lower density of $A$ at $x$ over $\Omega$ as

$$\underline{D}(A,\Omega,x) = \liminf_{r\searrow 0} \frac{|A \cap B(x,r)|}{|B(x,r)\cap\Omega|}.$$

If $\overline{D}(A,\Omega,x) = \underline{D}(A,\Omega,x)$, we call the common value the density of $A$ at $x$ over $\Omega$ and denote it by $D(A,\Omega,x)$. If $\Omega = \mathbb{R}^n$ we simply write $\underline{D}(A,x), \overline{D}(A,x)$, and $D(A,x)$. The measure theoretic interior of $A$ is then defined as

$$\mathring{A}^M = \{x \in \mathbb{R}^n : D(A,x) = 1\},$$

the measure theoretic closure of $A$ as

$$\overline{A}^M = \{x \in \mathbb{R}^n : \overline{D}(A,x) > 0\},$$

and the measure theoretic boundary of $A$ as

$$\partial^M A = \{x \in \mathbb{R}^n \,:\, \overline{D}(A,x) > 0 \text{ and } \overline{D}(\mathbb{R}^n \setminus A, x) > 0\}.$$

As usual, $\mathcal{H}^s(A)$ stands for the $s$-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ obtained as the limit

$$\mathcal{H}^s(A) = \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(A),$$

where $\mathcal{H}^s_\delta(A)$ is the $s$-dimensional Hausdorff $\delta$-content of $A$ defined as

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s \,:\, A \subset \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam}(U_i) \le \delta\right\}.$$

By a dyadic cube we refer to $Q = [0, 2^{-k}]^n + \mathtt{j} \subset \mathbb{R}^n$ for some $k \in \mathbb{Z}$ and $\mathtt{j} \in 2^{-k}\mathbb{Z}^n$. We denote the side-length of such dyadic cube $Q$ by $\ell(Q) := 2^{-k}$.

2.1. **Sets of finite perimeter.** A Lebesgue measurable subset $A \subset \mathbb{R}^n$ has finite perimeter in an open set $\Omega$ if $\chi_A \in BV(\Omega)$, where $\chi_A$ denotes the characteristic function of the set $A$. We set $P(A,\Omega) = \|D\chi_A\|(\Omega)$ and call it the perimeter of $A$ in $\Omega$. Here

$$\|D\chi_A\|(\Omega) = \sup\left\{\int_A \operatorname{div}(v)\,dx \,:\, v \in C_0^\infty(\Omega;\mathbb{R}^n),\ |v| \le 1\right\}$$

denotes the total variation of $\chi_A$ on $\Omega$.

It is well known that a set $E$ has finite perimeter in $\Omega$ if and only if $\mathcal{H}^{n-1}(\partial^M E \cap \Omega) < \infty$ (see [8, Section 4..5.11]). Let us recall as well the isoperimetric inequality, which follows from the $(1^*, 1)$-Poincaré inequality for $BV$ functions (see for instance [2, Theorem 3.44]).

**Proposition 2.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $A \subset \Omega$ a set of finite perimeter in $\Omega$. Let also $Q, Q' \subset \Omega$ be two dyadic cubes with $\frac{1}{4}\ell(Q') \le \ell(Q) \le 4\ell(Q')$ and so that $\operatorname{int}(Q \cup Q)$ is connected. Then we have*

$$P(A, \operatorname{int}(Q \cup Q')) \ge C(n) \min\{|A \cap (Q \cup Q')|^{1-1/n}, |(Q \cup Q') \setminus A|^{1-1/n}\}. \tag{2.1}$$

The study of the boundary of planar sets of finite perimeter can be reduced to the study of Jordan loops via the following decomposition result from [1, Corollary 1]. We will use this result in Section 4.

**Proposition 2.2.** *Let $E \subset \mathbb{R}^2$ have finite perimeter. Then, there exists a unique decomposition of $\partial^M E$ into rectifiable Jordan curves $\{C_i^+, C_k^- : i, k \in \mathbb{N}\}$, modulo $\mathcal{H}^1$-measure zero sets, such that*

1. *Given $\operatorname{int}(C_i^+)$, $\operatorname{int}(C_k^+)$, $i \neq k$ they are either disjoint or one is contained in the other; given $\operatorname{int}(C_i^-)$, $\operatorname{int}(C_k^-)$, $i \neq k$, they are either disjoint or one is contained in the other. Each $\operatorname{int}(C_i^-)$ is contained in one of the $\operatorname{int}(C_k^+)$.*
2. *$P(E, \mathbb{R}^2) = \sum_i \mathcal{H}^1(C_i^+) + \sum_k \mathcal{H}^1(C_k^-)$.*
3. *If $\operatorname{int}(C_i^+) \subset \operatorname{int}(C_j^+)$, $i \neq j$, then there is some rectifiable Jordan curve $C_k^-$ such that $\operatorname{int}(C_i^+) \subset \operatorname{int}(C_k^-) \subset \operatorname{int}(C_j^+)$. Similarly, if $\operatorname{int}(C_i^-) \subset \operatorname{int}(C_j^-)$, $i \neq j$, then there is some rectifiable Jordan curve $C_k^+$ such that $\operatorname{int}(C_i^-) \subset \operatorname{int}(C_k^+) \subset \operatorname{int}(C_j^-)$.*
4. *Setting $L_j = \{i : \operatorname{int}(C_i^-) \subset \operatorname{int}(C_j^+)\}$, the sets $Y_j = \operatorname{int}(C_j^+) \setminus \bigcup_{i \in L_j} \operatorname{int}(C_i^-)$ are pairwise disjoint, indecomposable and $E = \bigcup_j Y_j$.*

2.2. **Whitney decomposition.** If $\Omega \subset \mathbb{R}^n$ is an open set, not equal to the entire space $\mathbb{R}^n$, we let $\mathcal{W} = \{Q_i\}_{i=1}^{\infty}$ be the standard *Whitney decomposition* of $\Omega$, by which we mean that it satisfies the following properties:

(W1) Each $Q_i$ is a closed dyadic cube inside $\Omega$.
(W2) $\Omega = \bigcup_i Q_i$ and for every $i \neq j$ we have $\mathrm{int}(Q_i) \cap \mathrm{int}(Q_j) = \emptyset$.
(W3) For every $i$ we have $\ell(Q_i) \leq \mathrm{dist}\,(Q_i, \partial\Omega) \leq 4\sqrt{n}\ell(Q_i)$,
(W4) If $Q_i \cap Q_j \neq \emptyset$, we have $\frac{1}{4}\ell(Q_i) \leq \ell(Q_j) \leq 4\ell(Q_i)$.

The reader can find a proof of the existence of such a dyadic decomposition of the set $\Omega$ in [32, Chapter VI].

For such Whitney decomposition $\mathcal{W}$ we take a partition of unity $\{\psi_i\}_{i=1}^{\infty}$ so that for every $i$ we have $\psi_i \in C^{\infty}(\mathbb{R}^n)$, $\mathrm{spt}(\psi_i) = \{x \in \mathbb{R}^n : \psi_i(x) \neq 0\} \subset B(Q_i, \frac{1}{16}\ell(Q_i))$, $\psi_i \geq 0$, $|\nabla\psi_i| \leq C(n)\ell(Q_i)^{-1}$, and

$$\sum_{i=1}^{\infty} \psi_i = \chi_\Omega.$$

Notice that for each $Q_i \in \mathcal{W}$ the above together with the bound on the size of the supports and (W4) implies

$$\psi_i(x) = 1 - \sum_{j\neq i} \psi_j(x) = 1 \qquad \text{for all } x \in \frac{1}{2}Q_i. \tag{2.2}$$

In order to ease the notation, we denote for each $Q_i \in \mathcal{W}$ by $\mathcal{N}(Q_i)$ the collection of neighboring cubes that have a common face with $Q_i$:

$$\mathcal{N}(Q_i) = \{Q_j \in \mathcal{W} \setminus \{Q_i\} \,:\, \mathrm{int}(Q_i \cup Q_j) \text{ is connected}\}.$$

2.3. **Size estimates.** In this subsection we recall the remaining key auxiliary results that will be used in the paper.

The following lemma is a modification of [22, Lemma 3.2]. This version of the estimate was proven in [10, Lemma 2.3]. (Here we can simplify the presentation a bit since we do not need an exceptional set $F$.)

**Proposition 2.3.** *Let $Q$ be an $n$-dimensional cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Let $f \in C(Q) \cap W^{1,p}(\mathbb{R}^n)$ for some $1 \leq p < \infty$ and suppose there exists $\delta \in (0,1)$ so that*

$$\min\left(|\{y \in Q \,:\, f(y) \leq 0\}|, |\{y \in Q \,:\, f(y) \geq 1\}|\right) > \delta\ell(Q)^n.$$

*Then*

$$\int_Q |\nabla f(y)|^p \, dy \geq C(n,p)\delta^{\frac{n-p}{n}}\ell(Q)^{n-p}.$$

For $L^{1,p}$-extension domains $\Omega$ with $1 \leq p < \infty$ the following measure density condition holds for points $x \in \overline{\Omega}$. This version of the measure density condition was proven in [10, Proposition 2.2] following the results in [15], see also [19].

**Proposition 2.4.** *Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^{1,p}$-extension domain with an extension operator $E$. Then, for all $x \in \overline{\Omega}$ and*

$$r \in \left(0, \min\left\{1, \left(\frac{|\Omega|}{2\,|B(0,1)|}\right)^{1/n}\right\}\right),$$

*we have*

$$|\Omega \cap B(x,r)| \geq C(n,p)\|E\|^{-n}r^n.$$

## 3. Proof of the necessary condition

In this section we prove Theorem 1.1. In order to make the structure of the proof clearer, we first present the proof assuming the more technical parts proven. These technical parts are stated as separate lemmata. They are then proven after the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We start with a measurable set $A \subset \Omega$ so that

$$\int_{\Omega \cap \partial^M A} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) < \infty. \tag{3.1}$$

Notice that if (3.1) fails, we can simply take $\tilde{A} = A$ as the set satisfying the required inequality (1.4).

Following the definitions in Subsection 2.2, let $\mathcal{W} = \{Q_i\}$ and $\widetilde{\mathcal{W}} = \{\widetilde{Q}_i\}$ be the Whitney decompositions of $\Omega$ and $\mathbb{R}^n \setminus \overline{\Omega}$ respectively, and let $\{\psi_i\}_{i=1}^\infty$ be the partition of unity in $\Omega$ subordinate to $\mathcal{W} = \{Q_i\}$.

We first modify our set $A$ by means of selecting those Whitney cubes that intersect the set $A$ in a large enough measure set. Namely, we let

$$A' = \bigcup_{\substack{Q_i \in \mathcal{W} \\ |A \cap Q_i| > \frac{1}{2}|Q_i|}} Q_i.$$

It will be easier to handle this new set $A'$ rather than the original set $A$.

Next, for the constant $c = 20\sqrt{n}$ we define

$$A_0 = \bigcup_{\substack{\widetilde{Q} \in \widetilde{\mathcal{W}} \\ |c\widetilde{Q} \cap A'| > |c\widetilde{Q} \cap (\Omega \setminus A')|}} \widetilde{Q}.$$

Our extension of the set $A$ is then defined as

$$\widetilde{A} = A \cup A_0.$$

The task in proving Theorem 1.1 is now to show that the choice of $\widetilde{A}$ above works. We divide this task into several lemmata. The first lemma justifies the replacement of $A$ by $A'$.

**Lemma 3.1.** *For the sets $A$ and $A'$ above we have*

$$\int_{\Omega \cap \partial^M A'} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) \leq C(n) \int_{\Omega \cap \partial^M A} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z).$$

The next step is then to go from the set $A'$ to a Sobolev function to which we can apply our $L^{1,p}$-extension operator. This is done with a Whitney smoothing operator $S_{\mathcal{W}}$ defined via the partition of unity $\{\psi_i\}_{i=1}^\infty$ for $\Omega$. We define for any $v \in L^1_{loc}(\Omega)$ a smoothened version of $v$ as

$$(S_{\mathcal{W}}v)(x) = \sum_{i=1}^\infty \psi_i(x) \frac{1}{|Q_i|} \int_{Q_i} v(y)\, d(y). \tag{3.2}$$

Whitney smoothing operators similar to the one above have been used for instance in [14, 3, 24, 9].

In addition to smoothing the function, the operator $S_{\mathcal{W}}$ has the important property of leaving the trace of the function unmodified on the boundary of $\Omega$. Within our proof, this is the content of the last Lemma 3.4. The second lemma relates the integral in (1.4) to the $L^p$-norm of the gradient of the smoothened version of the indicator function. We write the lemma for a general set $F$, but here inside the proof of Theorem 1.1 use it only for the set $A'$.

**Lemma 3.2.** *Let $S_{\mathcal{W}}$ be the operator defined in* (3.2). *Then for any measurable $F \subset \Omega$ with*

$$\int_{\Omega \cap \partial^M F} \mathrm{dist}\,(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) < \infty$$

*we have $S_{\mathcal{W}}\chi_F \in C^\infty(\Omega)$ and*

$$\|\nabla S_{\mathcal{W}}\chi_F\|^p_{L^p(\Omega)} \le C(n,p) \int_{\Omega \cap \partial^M F} \mathrm{dist}\,(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z).$$

We now use $S_{\mathcal{W}}$ to pass from the characteristic function $\chi_{A'}$ to a Sobolev function

$$u = S_{\mathcal{W}}\chi_{A'} \in L^{1,p}(\Omega).$$

Lemma 3.2 together with Lemma 3.1 then gives us

$$\|\nabla u\|^p_{L^p(\Omega)} \le C(n,p) \int_{\Omega \cap \partial^M A} \mathrm{dist}\,(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z). \tag{3.3}$$

The third lemma shows that the extension $\widetilde{A}$ of the set $A$ has the correct property outside the closure of the domain $\Omega$. The fact that $\Omega$ is a Sobolev extension domain is used in the proof of this lemma. Recall that $\|E\|$ denotes the norm of the $L^{1,p}$-extension operator.

**Lemma 3.3.** *With the $A_0$ and $u$ defined above, for every $\varepsilon > 0$ we have*

$$\int_{\partial^M A_0 \setminus \overline{\Omega}} \mathrm{dist}\,(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) \le C(n,p,\varepsilon)\|E\|^{n+p+\varepsilon}\|\nabla u\|^p_{L^p(\Omega)}.$$

Now, the combination of Lemma 3.3 and the inequality (3.3) gives

$$\int_{\partial^M A_0 \setminus \overline{\Omega}} \mathrm{dist}\,(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) \le C(n,p,\varepsilon)\|E\|^{n+p+\varepsilon} \int_{\Omega \cap \partial^M A} \mathrm{dist}\,(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z). \tag{3.4}$$

The last lemma deals with the boundary of $\Omega$, where in principle some part of the measure theoretic boundary of $\widetilde{A}$ could live and cause the integral on the left-hand side of (1.4) to be infinite.

**Lemma 3.4.** *With our set $\widetilde{A}$ defined above, we have $\mathcal{H}^{n-p}(\partial^M \widetilde{A} \cap \partial\Omega) = 0$.*

Since we can write

$$\partial^M \widetilde{A} = (\partial^M A_0 \setminus \overline{\Omega}) \cup (\Omega \cap \partial^M A) \cup (\partial^M \widetilde{A} \cap \partial\Omega),$$

we can split the integral on the left-hand side of (1.4) and use the estimate (3.4) and Lemma 3.4 to obtain

$$\begin{aligned}\int_{\partial^M \widetilde{A}} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) &= \int_{\partial^M A_0\setminus\overline{\Omega}} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)\\ &\quad+\int_{\Omega\cap\partial^M A} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)\\ &\quad+\int_{\partial^M \widetilde{A}\cap\partial\Omega} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)\\ &\le C(n,p,\varepsilon)\|E\|^{n+p+\varepsilon}\int_{\Omega\cap\partial^M A} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)\\ &\quad+\int_{\Omega\cap\partial^M A} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)+0\\ &\le C(n,p,\varepsilon)\|E\|^{n+p+\varepsilon}\int_{\Omega\cap\partial^M A} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z).\end{aligned}$$

Thus, we conclude the proof of Theorem 1.1. □

Let us then focus on proving the lemmata we used in the proof of Theorem 1.1.

*Proof of Lemma 3.1.* Setting $a_i = \frac{|A'\cap Q_i|}{|Q_i|} \in \{0,1\}$ we start by writing

$$\begin{aligned}\int_{\Omega\cap\partial^M A'} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) &\le \sum_{Q_i}\int_{Q_i\cap\partial^M A'} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)\\ &\le \sum_{Q_i}\sum_{Q_j\in\mathcal{N}(Q_i)} \ell(Q_i)^{1-p}P(A',Q_i\cup Q_j)\\ &= \sum_{Q_i}\sum_{\substack{Q_j\in\mathcal{N}(Q_i)\\ a_i\neq a_j}} \ell(Q_i)^{1-p}P(A',Q_i\cup Q_j),\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega\cap\partial^M A} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) &\ge \frac{1}{2}\sum_{Q_i}\int_{Q_i\cap\partial^M A} \operatorname{dist}(z,\partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)\\ &\ge C(n,p)\sum_{Q_i}\sum_{Q_j\in\mathcal{N}(Q_i)} \ell(Q_i)^{1-p}P(A,Q_i\cup Q_j)\\ &\ge C(n,p)\sum_{Q_i}\sum_{\substack{Q_j\in\mathcal{N}(Q_i)\\ a_i\neq a_j}} \ell(Q_i)^{1-p}P(A,Q_i\cup Q_j).\end{aligned}$$

Hence, we only need to check that for $i,j\in\mathbb{N}$ with $Q_j\in\mathcal{N}(Q_i)$ and $a_i\neq a_j$ we have $P(A',Q_i\cup Q_j)\le C(n)P(A,Q_i\cup Q_j)$. Assuming without loss of generality that $a_i = 1$ and

$a_j = 0$, this is seen by using the isoperimetric inequality (2.1)

$$\begin{aligned} P(A, Q_i \cup Q_j) &\geq C(n) \min\{|A \cap (Q_i \cup Q_j)|^{1-1/n}, |(Q_i \cup Q_j) \setminus A|^{1-1/n}\} \\ &\geq C(n) \min\{|A \cap Q_i|^{1-1/n}, |Q_j \setminus A|^{1-1/n}\} \\ &\geq C(n) \min\{(1/2)^{1-1/n} \ell(Q_i)^{n-1}, (1/2)^{1-1/n} \ell(Q_j)^{n-1}\} \\ &\geq C(n) \ell(Q_i)^{n-1} \\ &\geq C(n) P(A', Q_i \cup Q_j). \end{aligned}$$

□

*Proof of Lemma 3.2.* From the definition of $S_{\mathcal{W}}$, for every $Q_k \in \mathcal{W}$ we get

$$\|\nabla S_{\mathcal{W}} \chi_F\|^p_{L^p(Q_k)} \leq C(n,p) \sum_{Q_k \cap Q_i \neq \emptyset} \sum_{Q_j \in \mathcal{N}(Q_i)} \ell(Q_i)^{n-p} |a_i - a_j|,$$

where

$$a_i = \frac{1}{|Q_i|} \int_{Q_i} \chi_F(x)\, dx = \frac{|F \cap Q_i|}{|Q_i|}.$$

Assume that we have $i, j \in \mathbb{N}$ with $Q_j \in \mathcal{N}(Q_i)$. We may further assume that $a_i \geq a_j$. Then, by using the isoperimetric inequality (2.1) we get

$$\begin{aligned} P(F, Q_i \cup Q_j) &\geq C(n) \min\{|F \cap (Q_i \cup Q_j)|^{1-1/n}, |(Q_i \cup Q_j) \setminus F|^{1-1/n}\} \\ &\geq C(n) \min\{|F \cap Q_i|^{1-1/n}, |Q_j \setminus F|^{1-1/n}\} \\ &\geq C(n) \min\{(a_i)^{1-1/n} \ell(Q_i)^{n-1}, (1-a_j)^{1-1/n} \ell(Q_j)^{n-1}\} \\ &\geq C(n) \ell(Q_i)^{n-1} |a_i - a_j|^{\frac{n-1}{n}} \\ &\geq C(n) \ell(Q_i)^{n-1} |a_i - a_j|. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|\nabla S_{\mathcal{W}} \chi_F\|^p_{L^p(Q_k)} &\leq C(n,p) \sum_{Q_k \cap Q_i \neq \emptyset} \sum_{Q_j \in \mathcal{N}(Q_i)} \ell(Q_j)^{n-p} |a_i - a_j| \\ &\leq C(n,p) \sum_{Q_k \cap Q_i \neq \emptyset} \sum_{Q_j \in \mathcal{N}(Q_i)} \ell(Q_i)^{1-p} P(F, Q_i \cup Q_j). \end{aligned}$$

Therefore, by using the finite overlapping between Whitney cubes, we have

$$\begin{aligned} \|\nabla S_{\mathcal{W}} \chi_F\|^p_{L^p(\Omega)} &\leq C(n,p) \sum_{Q_k} \sum_{Q_k \cap Q_i \neq \emptyset} \sum_{Q_j \in \mathcal{N}(Q_i)} \ell(Q_i)^{1-p} P(F, Q_i \cup Q_j) \\ &\leq C(n,p) \sum_{Q_k} \int_{Q_k \cap \partial^M F} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) \\ &\leq C(n,p) \int_{\Omega \cap \partial^M F} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z). \end{aligned}$$

□

*Proof of Lemma 3.3.* We introduce the following subfamily of Whitney cubes of $\widetilde{\mathcal{W}}$

$$\mathcal{V}_0 = \left\{ \widetilde{Q} \in \widetilde{\mathcal{W}} : \widetilde{Q} \subset A_0, \partial^M A_0 \cap \widetilde{Q} \neq \emptyset \right\}.$$

We then have

$$\partial^M A_0 \setminus \overline{\Omega} \subset \bigcup_{\widetilde{Q} \in \mathcal{V}_0} \partial(\widetilde{Q}).$$

Let us fix $\widetilde{Q} \in \mathcal{V}_0$ for the moment. Then there exists a neighbouring cube $\widetilde{Q}' \in \widetilde{\mathcal{W}}$, that is $\widetilde{Q}' \cap \widetilde{Q} \neq \emptyset$, so that $\widetilde{Q}' \not\subset A_0$. By the definition of $A_0$, we have

$$|c\widetilde{Q} \cap A'| > \frac{1}{2}|c\widetilde{Q} \cap \Omega| \tag{3.5}$$

and

$$|c\widetilde{Q}' \cap (\Omega \setminus A')| \geq \frac{1}{2}|c\widetilde{Q}' \cap \Omega|. \tag{3.6}$$

In particular, (3.5) and (3.6) imply that

$$\Omega \not\subset c\widetilde{Q} \qquad \text{or} \qquad \Omega \not\subset c\widetilde{Q}'.$$

Therefore,

$$\max\{\ell(\widetilde{Q}), \ell(\widetilde{Q}')\} \leq C(n) \operatorname{diam}(\Omega). \tag{3.7}$$

Combining (3.7),(3.6) and (3.5) with the measure density condition stated in Proposition 2.4, we get

$$\min\Big\{|c\widetilde{Q} \cap A'|, |c\widetilde{Q}' \cap (\Omega \setminus A')|\Big\} \geq C(n,p)\|E\|^{-n}\ell(\widetilde{Q})^n.$$

Recall that $u = S_{\mathcal{W}}\chi_{A'} = \sum_{i=1}^{\infty} a_i\psi_i$ where $a_i = \frac{|A' \cap Q_i|}{|Q_i|} \in \{0,1\}$. By (2.2) we have $\psi_i = 1$ on $\frac{1}{2}Q_i$ and so if $Q \subset A'$, then $u = 1$ on $\dfrac{1}{2}Q$ and if $Q \not\subset A'$, then $u = 0$ on $\dfrac{1}{2}Q$. Therefore,

$$\min\Big\{|\{y \in 9c\widetilde{Q} \cap \Omega \,:\, u(y) \leq 0\}|, |\{y \in 9c\widetilde{Q} \cap \Omega \,:\, u(y) \geq 1\}|\Big\} > C(n,p)\|E\|^{-n}\ell(9c\widetilde{Q})^n.$$

Let $s \in (1,p)$. Then by Proposition 2.3, we have

$$\left(\int_{9c\widetilde{Q}} |\nabla Eu(x)|^s\,dx\right)^{\frac{p}{s}} \geq \Big(C(n,p)\|E\|^{-n}\ell(\widetilde{Q})^{n-s}\Big)^{\frac{p}{s}} \geq C(n,p)\|E\|^{\frac{-np}{s}}\ell(\widetilde{Q})^{n-p}\ell(\widetilde{Q})^{(\frac{p}{s}-1)n}.$$

This concludes our estimate for the fixed $\widetilde{Q} \in \mathcal{V}_0$.

Now, since $p/s > 1$, we may use the boundedness of the Hardy-Littlewood maximal operator

$$M \colon L^{\frac{p}{s}}(\mathbb{R}^n) \to L^{\frac{p}{s}}(\mathbb{R}^n),$$

to get

$$\begin{aligned}
\int_{\partial^M A_0\setminus\overline{\Omega}} \operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}\mathcal{H}^{n-1}(z) &\le \sum_{\widetilde{Q}\in\mathcal{V}_0}\int_{\partial^M A_0\cap\widetilde{Q}} \operatorname{dist}(z,\partial\Omega)^{1-p}\,d\mathcal{H}^{n-1}(z)\\
&\le C(n)\sum_{\widetilde{Q}\in\mathcal{V}_0}\ell(\widetilde{Q})^{n-p}\\
&\le C(n,p)\|E\|^{\frac{np}{s}}\sum_{\widetilde{Q}\in\mathcal{V}_0}\ell(\widetilde{Q})^{(1-\frac{p}{s})n}\left(\int_{9c\widetilde{Q}}|\nabla Eu(x)|^s\,\mathrm{d}x\right)^{\frac{p}{s}}\\
&\le C(n,p)\|E\|^{\frac{np}{s}}\sum_{\widetilde{Q}\in\mathcal{V}_0}\ell(\widetilde{Q})^{n}\left(\fint_{9c\widetilde{Q}}|\nabla Eu(x)|^s\,\mathrm{d}x\right)^{\frac{p}{s}}\\
&\le C(n,p)\|E\|^{\frac{np}{s}}\sum_{\widetilde{Q}\in\mathcal{V}_0}\int_{\widetilde{Q}}|M(|\nabla Eu|^s)(x)|^{\frac{p}{s}}\,\mathrm{d}x\\
&\le C(n,p)\|E\|^{\frac{np}{s}}\int_{\mathbb{R}^n\setminus\overline{\Omega}}|M(|\nabla Eu|^s)(x)|^{\frac{p}{s}}\,\mathrm{d}x\\
&\le C(n,p)\|E\|^{\frac{np}{s}}\int_{\mathbb{R}^n}|M(|\nabla Eu|^s)(x)|^{\frac{p}{s}}\,\mathrm{d}x\\
&\le C(n,p,s)\|E\|^{\frac{np}{s}}\int_{\mathbb{R}^n}|\nabla Eu(x)|^p\,\mathrm{d}x\\
&\le C(n,p,s)\|E\|^{\frac{np}{s}}||E||^p\int_{\Omega}|\nabla u(x)|^p\,\mathrm{d}x.
\end{aligned}$$

Since we may choose $\frac{p}{s}>1$ to be arbitrarily close to 1 with the price of enlarging the constant $C(n,p,s)$, the lemma is proven. □

*Proof of Lemma 3.4.* We divide the proof into three parts. The parts 1 and 3 will imply the claim of the lemma, while part 2 is needed in the proof of part 3.

Part 1: For $\mathcal{H}^{n-p}$-a.e. $x\in\partial\Omega$ the limit $D(A',\Omega,x)=\lim_{r\to 0}\dfrac{|A'\cap B(x,r)|}{|B(x,r)\cap\Omega|}$ exists and is either 0 or 1.

*Proof of Part 1.* Let

$$F=\left\{x\in\partial\Omega:\ D(A',\Omega,x)\notin\{0,1\}\text{ or the limit does not exist}\right\}$$

and assume towards contradiction that $\mathcal{H}^{n-p}(F)>0$. Then, there exists $\delta>0$ so that $\mathcal{H}^{n-p}(F_\delta)>0$ for

$$F_\delta=\left\{x\in\partial\Omega:\ \exists r_i^x\searrow 0\ \text{ such that }\frac{|A'\cap B(x,r_i^x)|}{|B(x,r_i^x)\cap\Omega|}\in[\delta,1-\delta]\right\}.$$

Fix $\varepsilon\in(0,1)$ and for every $x\in F_\delta$ choose $i$ so that $r_i^x<\varepsilon$, then

$$F_\delta\subset\bigcup_{x\in F_\delta}B(x,r_i^x)$$

and hence by the Vitali covering theorem (see [6, Theorem 1.24]) there exists a countable disjointed collection $\{B(x_i, r_i)\}_{i\in\mathbb{N}}$ so that

$$\frac{|A' \cap B(x_i, r_i)|}{|B(x_i, r_i) \cap \Omega|} \in [\delta, 1-\delta] \tag{3.8}$$

and $F_\delta \subset \bigcup_{i\in\mathbb{N}} B(x_i, 5r_i)$. Recall that $u = S_{\mathcal{W}}\chi_{A'}$, and that by Lemma 3.2 and Lemma 3.1 we have

$$\begin{aligned}\|\nabla u\|^p_{L^p(\Omega)} &\le C(n,p)\int_{\Omega\cap\partial^M A'} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z)\\ &\le C(n,p)\int_{\Omega\cap\partial^M A} \operatorname{dist}(z, \partial\Omega)^{1-p}\, d\mathcal{H}^{n-1}(z) < \infty.\end{aligned}$$

So, we have $u \in L^{1,p}(\Omega)$. We extend $u$ to $Eu \in L^{1,p}(\mathbb{R}^n)$. Observe that for every $i \in \mathbb{N}$, by (3.8) and by the measure density condition (Proposition 2.4) we have

$$|A' \cap B(x_i, r_i)| \ge \delta |B(x_i, r_i)\cap\Omega| \ge C(n,p)\|E\|^{-n}\delta r_i^n$$

and

$$|B(x_i, r_i)\setminus A'| \ge |(B(x_i, r_i)\cap\Omega)\setminus A'| \ge \delta|B(x_i, r_i)\cap\Omega| \ge C(n,p)\|E\|^{-n}\delta r_i^n.$$

Therefore, by the definition of $u$ via $S_{\mathcal{W}}$, and the fact that $A'$ is the union of the same Whitney cubes used in the definition of $S_{\mathcal{W}}$, we have

$$|\{x \in B(x_i, r_i) :\, Eu \le 0\}| \ge C(n, p, \|E\|, \delta) r_i^n$$

and

$$|\{x \in B(x_i, r_i) :\, Eu \ge 1\}| \ge C(n, p, \|E\|, \delta) r_i^n.$$

Hence, we may apply Proposition 2.3 to get the estimate

$$\int_{B(x_i,r_i)} |\nabla u(y)|^p\, dy \ge C(n, p, \|E\|, \delta) r_i^{n-p}.$$

We can now conclude

$$\begin{aligned}\mathcal{H}^{n-p}_\varepsilon(F_\delta) &\le C(n,p)\sum_{i\in\mathbb{N}}(5r_i)^{n-p} \le C(n,p,\|E\|,\delta)5^{n-p}\sum_{i\in\mathbb{N}}\int_{B(x_i,r_i)}|\nabla u(y)|^p\, dy\\ &\le C(n,p,\|E\|,\delta)\int_{B(F_\delta,\varepsilon)}|\nabla u(y)|^p\, dy.\end{aligned}$$

Using that by the measure density condition $|F_\delta| \le |\partial\Omega| = 0$, the right hand side tends to zero as $\varepsilon \searrow 0$. So $\mathcal{H}^{n-p}(F_\delta) = 0$ which is a contradiction. We have thus proven Part 1. □

Part 2: The following two implications hold for $\mathcal{H}^{n-p}$-almost every $x \in \partial\Omega$:

$$\text{If } D(A', \Omega, x) = 1, \text{ then } D(A, \Omega, x) = 1, \tag{3.9}$$

and

$$\text{if } D(A', \Omega, x) = 0, \text{ then } D(A, \Omega, x) = 0. \tag{3.10}$$

*Proof of Part 2.* Let us first show that by going to complements, we only need to prove (3.9). Towards this, assume that (3.9) is true for every measurable set $A \subset \Omega$. Suppose then that $D(A', \Omega, x) = 0$. Call $B = \Omega \setminus A$ and consider the associated

$$B' = \bigcup_{\left\{Q_i \in \mathcal{W} \colon |B \cap Q_i| \geq \frac{1}{2}|Q_i|\right\}} Q_i.$$

We have $B' = \Omega \setminus A'$. Since $D(B', \Omega, x) = 1$, we have by assumption that $D(B, \Omega, x) = 1$. Thus, $D(A, \Omega, x) = 0$ and we have shown (3.10). (Notice that the form of the definitions of the sets $A'$ and $B'$ differ slightly in that one has a strict inequality while the other does not. However, it is easy to observe that this does not affect the proof below.)

Let us then prove (3.9). The argument is similar to the proof of Part 1. This time we write

$$G = \left\{x \in \partial\Omega : \, D(A', \Omega, x) = 1 \text{ and } D(A, \Omega, x) \neq 1\right\}$$

and assume towards contradiction that $\mathcal{H}^{n-p}(G) > 0$. Then, as in the previous proof, there exists $\delta > 0$ so that $\mathcal{H}^{n-p}(G_\delta) > 0$ for

$$G_\delta = \left\{ x \in \partial\Omega \, : \, \exists r_i^x \searrow 0 \text{ such that } \frac{|A \cap B(x, r_i^x)|}{|B(x, r_i^x) \cap \Omega|} < 1 - \delta \right.$$
$$\left. \text{and } \frac{|A' \cap B(x,r)|}{|B(x,r) \cap \Omega|} > \frac{1}{2} \text{ for all } 0 < r < \delta \right\}.$$

Now, at this stage it is enough to notice that by the definition of $A'$ we have

$$|A \cap B(x, Cr)| \geq \sum_{\substack{Q_i \in \mathcal{W} \\ Q_i \subset B(x,Cr)}} |Q_i \cap A| \geq \sum_{\substack{Q_i \in \mathcal{W} \\ Q_i \subset B(x,Cr)}} \frac{1}{2}|Q_i \cap A'| \geq \frac{1}{2}|A' \cap B(x,r)|$$

so that by the measure density, we have that for some $\delta' > 0$

$$G_\delta \subset \left\{ x \in \partial\Omega : \, \exists r_i^x \searrow 0 \text{ such that } \frac{|A \cap B(x, r_i^x)|}{|B(x, r_i^x) \cap \Omega|} \in [\delta', 1 - \delta'] \right\}. \tag{3.11}$$

Let us now consider $v = S_{\mathcal{W}}\chi_A$. By Part 1 of the proof the balls $B(x_i, r_i)$ for which we have

$$|\{x \in B(x_i, r_i) : \, Ev \leq \eta\}| \geq C(n, p, \|E\|, \delta')r_i^n \tag{3.12}$$

and

$$|\{x \in B(x_i, r_i) : \, Ev \geq 1 - \eta\}| \geq C(n, p, \|E\|, \delta')r_i^n \tag{3.13}$$

for some constant $\eta \in (0, \frac{1}{4})$ are well controlled. Therefore, we only need to control those balls for which either (3.12) or (3.13) fails. By taking the constant $\eta$ small enough, we have by the measure density condition and (3.11) that for such balls

$$|\{x \in B(x_i, r_i) : \, \eta < Ev < 1 - \eta\}| \geq C(n, p, \|E\|, \delta')r_i^n.$$

This in turn via (2.1) means that the combined Lebesgue measure of the Whitney cubes $Q \subset B(x_i, Cr_i) \cap \Omega$ for which we have

$$P(A, Q) \geq C(\eta, n)\ell(Q)^{n-1}$$

is at least $Cr_i^n$. Call the collection of these cubes $\mathcal{R}_i$. Now,

$$\begin{aligned}\int_{\partial^M A\cap\Omega\cap B(x_i,Cr_i)} \operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}\mathcal{H}^{n-1}(z) &\geq \sum_{Q\in\mathcal{R}_i}\int_{\partial^M A\cap Q} \operatorname{dist}(z,\partial\Omega)^{1-p}\,d\mathcal{H}^{n-1}(z)\\ &\geq C\sum_{Q\in\mathcal{R}_i}\ell(Q)^{n-p}\geq Cr_i^{n-p}.\end{aligned}\tag{3.14}$$

Since in the application of Vitali's covering theorem we may take $B(x_i, Cr_i)$ pairwise disjoint, the combination of the estimate from Part 1 for the balls satisfying (3.12) and (3.13), and the estimate (3.14) for the rest gives

$$\mathcal{H}^{n-p}_\varepsilon(F_\delta)\leq C\int_{B(F_\delta,C\varepsilon)}|\nabla v(y)|^p\,dy + C\int_{\partial^M A\cap\Omega\cap B(F_\delta,C\varepsilon)}\operatorname{dist}(z,\partial\Omega)^{1-p}\,\mathrm{d}\mathcal{H}^{n-1}(z).$$

This gives the needed contradiction and proves (3.9) and thus Part 2.

□

Part 3: The following two implications hold for $\mathcal{H}^{n-p}$-almost every $x\in\partial\Omega$:

$$\text{If } D(A',\Omega,x)=1, \text{ then } D(\widetilde{A},x)=1,\tag{3.15}$$

and

$$\text{if } D(A',\Omega,x)=0, \text{ then } D(\widetilde{A},x)=0.\tag{3.16}$$

*Proof of Part 3.* Since the definition of $A_0$ passes (up to the difference between a strict and non-strict inequality) to the complements, similarly to the Part 2 it is enough to prove the implication (3.15).

Let $x\in\partial\Omega$ with $D(A',\Omega,x)=D(A,\Omega,x)=1$ and $r>0$. (Notice that by Part 2 of the proof, $D(A,\Omega,x)=1$ holds for $\mathcal{H}^{n-p}$-almost every $x\in\partial\Omega$ with $D(A',\Omega,x)=1$.) Now, if $\widetilde{Q}\in\widetilde{\mathcal{W}}$ with $\widetilde{Q}\not\subseteq A_0$, by the definition of $A_0$ and the measure density condition (Proposition 2.4) we have

$$|c\widetilde{Q}\cap(\Omega\setminus A')|\geq\frac{1}{2}|c\widetilde{Q}\cap\Omega|\geq C(n,p,\|E\|)|\widetilde{Q}|.\tag{3.17}$$

Consider the collection

$$\mathcal{B}=\left\{\widetilde{Q}\in\widetilde{\mathcal{W}} : \widetilde{Q}\not\subseteq A_0, \widetilde{Q}\cap B(x,r)\neq\emptyset\right\}$$

and let $x_{\widetilde{Q}}$ be the center of each $\widetilde{Q}\in\widetilde{\mathcal{W}}$. By the Vitali covering theorem there exists a subcollection $\mathcal{B}'\subset\mathcal{B}$ so that

$$\bigcup_{\widetilde{Q}\in\mathcal{B}} B\left(x_{\widetilde{Q}},\sqrt{n}c\ell(\widetilde{Q})\right)\subset\bigcup_{\widetilde{Q}\in\mathcal{B}'} B\left(x_{\widetilde{Q}},5\sqrt{n}c\ell(\widetilde{Q})\right)$$

and

$$B\left(x_{\widetilde{Q}_1},\sqrt{n}c\ell(\widetilde{Q}_1)\right)\cap B\left(x_{\widetilde{Q}_2},\sqrt{n}c\ell(\widetilde{Q}_2)\right)=\emptyset\tag{3.18}$$

for any two $\widetilde{Q}_1,\widetilde{Q}_2\in\mathcal{B}'$ with $\widetilde{Q}_1\neq\widetilde{Q}_2$. Notice that (3.18) implies that also

$$c\widetilde{Q}_1\cap c\widetilde{Q}_2=\emptyset.$$

Hence, by (3.17)

$$\begin{aligned}|B(x,r)\setminus(A_0\cup\Omega)| &\le \sum_{\widetilde{Q}\in\mathcal{B}'}\left|B\left(x_{\widetilde{Q}},5\sqrt{n}c\ell(\widetilde{Q})\right)\right| \le C(n)\sum_{\widetilde{Q}\in\mathcal{B}'}|\widetilde{Q}|\\ &\le C(n,p,\|E\|)\sum_{\widetilde{Q}\in\mathcal{B}'}|c\widetilde{Q}\cap(\Omega\setminus A')| \\ &\le C(n,p,\|E\|)|B(x,Mr)\cap(\Omega\setminus A')|,\end{aligned} \tag{3.19}$$

where $M>0$ is a constant depending only on $n$ so that $c\widetilde{Q}\subset B(x,Mr)$ for any $\widetilde{Q}\in\widetilde{\mathcal{W}}$ with $\widetilde{Q}\cap B(x,r)\neq\emptyset$.

With (3.19) and the measure density condition we can estimate

$$\begin{aligned}\frac{|B(x,r)\cap\widetilde{A}|}{|B(x,r)|} &= 1-\frac{|B(x,r)\setminus(A_0\cup\Omega)|}{|B(x,r)|}-\frac{|B(x,r)\cap(\Omega\setminus A)|}{|B(x,r)|}\\ &\ge 1-C(n,p,\|E\|)\frac{|B(x,Mr)\cap(\Omega\setminus A')|}{|B(x,Mr)|}-\frac{|B(x,r)\cap(\Omega\setminus A)|}{|B(x,r)|}\\ &\ge 1-C(n,p,\|E\|)\frac{|B(x,Mr)\cap(\Omega\setminus A')|}{|B(x,Mr)\cap\Omega|}-C(n)\frac{|B(x,r)\cap(\Omega\setminus A)|}{|B(x,r)\cap\Omega|}\to 1,\end{aligned}$$

as $r\searrow 0$, since $D(A',\Omega,x)=D(A,\Omega,x)=1$. This proves (3.15). □

We can now conclude the proof of the lemma by taking $x\in\partial\Omega$ for which the conclusions of Part 1 and Part 3 above hold. Part 1 of the proof says that $D(A',\Omega,x)=\lim_{r\to 0}\dfrac{|A'\cap B(x,r)|}{|B(x,r)\cap\Omega|}$ exists and is either 0 or 1. Then by Part 3 of the proof

$$D(\widetilde{A},x)=D(A',\Omega,x)\in\{0,1\}$$

and hence $x\notin\partial^M\widetilde{A}$. □

## 4. A quantitative version of the curve condition

In the present section we use Theorem 1.1 to prove Theorem 1.3. This gives a quantitative version of a result proven in [22].

Theorem 1.3 states that if $\Omega\subset\mathbb{R}^2$ is a bounded simply connected $L^{1,p}$-extension domain for some $1<p<2$ with an extension operator $E$, then for every $\varepsilon>0$ there exists a constant $C(p,\varepsilon)>0$ such that for all $z_1,z_2\in\partial\Omega$ there exists a curve $\gamma\subset\mathbb{R}^2\setminus\Omega$ joining $z_1$ and $z_2$ so that

$$\int_\gamma \operatorname{dist}^{1-p}(z,\partial\Omega)\,\mathrm{d}s(z)\le C(p,\varepsilon)\|E\|^{\frac{4+4p-p^2}{2-p}+\varepsilon}|z_1-z_2|^{2-p}. \tag{4.1}$$

The curve condition (4.1) was proven in [22] to be a characterization of planar bounded simply connected $W^{1,p}$-extension domains for $1<p<2$ (a similar characterizing condition for the complement of a bounded finitely connected planar domain for $p>2$ was given in [30]). Here we only prove the necessity, but provide a more explicit estimate on the dependence of the operator norm $\|E\|$ in (4.1).

The proof in [22] of the necessity of (4.1) starts by observing that the domain $\Omega$ is $J$-John, by results in [19, Theorem 6.4], [12, Theorem 3.4], and [29, Theorem 4.5]. Recall that a bounded domain $\Omega\subset\mathbb{R}^2$ is called $J$-John for some constant $J\ge 1$ if there is a point $x_0\in\Omega$

and a constant $J \geq 1$ so that given $z \in \partial\Omega$ we can find a curve parameterized by arc length $\gamma \subset \Omega$ joining $z$ with $x_0$ so that

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq \frac{t}{J}. \tag{4.2}$$

The proof in [22] then continues by making a test function in $\Omega$ and by constructing the required curve using conformal maps. These steps make it difficult to track the constants.

The proof of (4.1) in our approach starts by examining two conditions similar to the John condition. We first prove a quantitative version of the so-called $\mathrm{cig}_d$ condition (4.3) (see [29] for this and similar conditions) for Sobolev extension domains. In the lemma below and elsewhere in this section, for an injective curve $\gamma \subset \mathbb{R}^2$ (possibly defined on an open or half-open interval) and two points $x, y \in \overline{\gamma}$ we denote by $\gamma_{x,y}$ a minimal subcurve of $\gamma$ so that $\gamma_{x,y} \cup \{x, y\}$ is connected.

**Lemma 4.1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected $L^{1,p}$-extension domain for some $1 < p < 2$. Then for every $x, y \in \overline{\Omega}$ there exists an injective curve $\gamma \subset \Omega \cup \{x, y\}$ connecting $x$ to $y$ and satisfying*

$$\min\{\operatorname{diam}(\gamma_{x,z}), \operatorname{diam}(\gamma_{y,z})\} \leq C_{cig\text{-}d} \operatorname{dist}(z, \partial\Omega) \tag{4.3}$$

*for all $z \in \gamma$, where $C_{cig\text{-}d} = C(p)\|E\|^{\frac{p}{2-p}}$.*

*Proof.* Let us first prove the claim for $x, y \in \partial\Omega$. By the Riemann mapping theorem there exists a conformal map $\varphi \colon \mathbb{D} \to \Omega$. Since we know that $\Omega$ is a John domain, by [29, Theorem 2.18] the domain $\Omega$ is finitely connected along its boundary and hence $\varphi$ extends as a continuous map to the boundary. We refer to this extension still by $\varphi$. Consider $a \in \varphi^{-1}(\{x\})$ and $b \in \varphi^{-1}(\{y\})$ so that one of the open arcs in $S^1$ connecting $a$ and $b$ does not intersect $\varphi^{-1}(\{x, y\})$. Call this arc $I_1$ and write $I_2 = S^1 \setminus (I_1 \cup \{a, b\})$.

Using the sets $I_1$ and $I_2$ we now define a set

$$G = \{z \in \mathbb{D} : \operatorname{dist}_{\Omega,\varphi}(z, I_1) = \operatorname{dist}_{\Omega,\varphi}(z, I_2)\},$$

where the distance $\operatorname{dist}_{\Omega,\varphi}(z, I)$ for a connected set $I \subset S^1$ and a point $z \in \mathbb{D}$ is defined by

$$\operatorname{dist}_{\Omega,\varphi}(z, I) = \inf\{\ell(\gamma) : \gamma \subset \Omega \text{ curve such that } \varphi^{-1}(\gamma) \cup I \cup \{z\} \text{ is connected}\}.$$

Notice that since $\Omega$ is a John domain we have for any non-empty arc $I$ and any $z \in \mathbb{D}$ that $\operatorname{dist}_{\Omega,\varphi}(z, I) < \infty$. This can be seen by taking $c \in I$, a sequence $c_i \in \mathbb{D}$ converging to $c$, the John curves $\gamma_i$ connecting $c_i$ to the John-center $x_0$, and finally a subsequence of $(\gamma_i)$ converging to the desired $\gamma$ giving $\operatorname{dist}_{\Omega,\varphi}(\varphi^{-1}(x_0), I) \leq \ell(\gamma) < \infty$. The passage to an arbitrary $z \in \mathbb{D}$ follows since any two points inside $\Omega$ can be connected by a curve in $\Omega$ of finite length. Notice moreover, that $\operatorname{dist}_{\Omega,\varphi}(\cdot, I)$ is a continuous function.

We claim that $G \subset \mathbb{D}$ is a closed set in $\mathbb{D}$ so that $a$ and $b$ are in the same connected component of $G \cup \{a, b\}$. Suppose this is not the case. Then there exists a path $\alpha$ from $I_1$ to $I_2$ that does not intersect $G$. However, the function

$$z \mapsto f(z) = \operatorname{dist}_{\Omega,\varphi}(z, I_1) - \operatorname{dist}_{\Omega,\varphi}(z, I_2)$$

is continuous in $\mathbb{D}$, and so in particular along the path $\alpha$. Since $f$ is negative near $I_1$ and positive near $I_2$ the function $f$ must be zero on some point of $\alpha$. This contradicts $G \cap \alpha = \emptyset$ and the claim is proven. Let us call $F$ the connected component of $G \cup \{a, b\}$ that contains the points $a$ and $b$.

Now, consider the following open neighbourhood of $G$

$$U = \left\{ z \in \mathbb{D} \, : \, \frac{1}{2} < \frac{\operatorname{dist}_{\Omega,\varphi}(z, I_1)}{\operatorname{dist}_{\Omega,\varphi}(z, I_2)} < 2 \right\}.$$

Since $G \cup \{a, b\} \subset U \cup \{a, b\}$ contains a connected component connecting $a$ to $b$, we can find an injective curve $\beta \colon (0, 1) \to U$ so that $\beta \cup \{a, b\}$ is connected. Notice that at this point we do not know if $\beta$ can be extended to 0 and 1 as a curve connecting $a$ and $b$, but after establishing (4.4) below, we have that the image curve $\varphi(\beta) \colon (0, 1) \to \Omega$ extends uniquely to a curve defined on $[0, 1]$ connecting $\varphi(a)$ to $\varphi(b)$.

Next we will show that for any $c \in \beta$ we have

$$\min \{ \operatorname{diam} (\varphi(\beta_{a,c})), \operatorname{diam} (\varphi(\beta_{b,c}))\} \le C(p)\|E\|^{\frac{p}{2-p}} \operatorname{dist} (\varphi(c), \partial\Omega). \tag{4.4}$$

Towards proving (4.4), let $c \in \beta$ and $C > 0$ be so that

$$\min \{ \operatorname{diam} (\varphi(\beta_{a,c})), \operatorname{diam} (\varphi(\beta_{b,c}))\} \ge C \operatorname{dist} (\varphi(c), \partial\Omega). \tag{4.5}$$

The estimate (4.4) is shown if we can prove that necessarily $C \le C(p)\|E\|^{\frac{p}{2-p}}$. We may assume that $C > 2$.

Let $\gamma^1$ be an injective curve in $\mathbb{D} \cup \{d_1\}$ joining $d_1 \in I_1$ to $c$ and let $\gamma^2$ be an injective curve in $\mathbb{D} \cup \{d_2\}$ joining $d_2 \in I_2$ to $c$ so that they satisfy

$$\ell(\varphi(\gamma^i)) < 2 \operatorname{dist}_{\Omega,\varphi}(c, I_i).$$

Let $c_i \in \gamma^i \cap \beta$ be such that $\gamma^i_{d_i,c_i} \cap \beta$ is a singleton. Now, the set $\mathbb{D} \setminus (\gamma^1_{d_1,c_1} \cup \gamma^2_{d_2,c_2} \cup \beta_{c_1,c_2})$ consists of two connected components, which we denote by $O_1$ and $O_2$. Then, by the injectivity of the curves $\gamma^i$ and $\beta$ and the selection of the points $c_1, c_2$, we have (by relabeling if necessary) $\beta_{a,c_1} \subset O_1 \cup \{a, c_1\}$ and $\beta_{b,c_2} \subset O_2 \cup \{b, c_2\}$.

Consequently, by (4.5) the sets $\Omega_i = \varphi(O_i)$ satisfy

$$\operatorname{diam} (\Omega^i) \ge C \operatorname{dist} (\varphi(c), \partial\Omega).$$

Denote $r = 4 \operatorname{dist} (\varphi(c), \partial\Omega)$ and notice that since $c \in U$, we have

$$r = 4 \min \{ \operatorname{dist}_{\Omega,\varphi}(c, I_1), \operatorname{dist}_{\Omega,\varphi}(c, I_2)\} \ge 2 \operatorname{dist}_{\Omega,\varphi}(c, I_i) > \ell(\varphi(\gamma^i)) \quad \text{for } i = 1, 2. \tag{4.6}$$

Define the test function

$$u(z) = \chi_{\Omega_1}(z) \max \left\{ \min \left\{ \frac{|\varphi(c) - z| - r}{r}, 1 \right\}, 0 \right\}.$$

Clearly $\operatorname{spt}(\nabla u) \subset \overline{B}(\varphi(c), 2r)$ and $|\nabla u| \le \frac{1}{r}$. Notice, that $u = 0$ on $B(\varphi(c), r)$ and by (4.6) we have $\varphi(\gamma^i) \subset B(\varphi(c), r)$. Hence, for each $z \in \varphi(\gamma^1 \cup \gamma^2)$ there exists $\varepsilon > 0$ such that $u \equiv 0$ in $B(z, \varepsilon)$. Thus, $u \in W^{1,p}(\Omega)$.

For the test function $u$ we have

$$\int_\Omega |\nabla u|^p \le 4\pi r^{2-p}.$$

Let $E \colon L^{1,p}(\Omega) \to L^{1,p}(\mathbb{R}^2)$ be the extension operator. Then in polar coordinates

$$\|E\|^p 4\pi r^{2-p} \ge \|E\|^p \int_\Omega |\nabla u|^p \ge \int_{\mathbb{R}^2} |\nabla E u|^p \ge \int_{2r}^{Cr} \int_0^{2\pi} |\nabla E u(\alpha, t)|^p t \, \mathrm{d}\alpha \, \mathrm{d}t. \tag{4.7}$$

By absolute continuity and Hölder's inequality for $2r < t < Cr$ we have

$$1 \le \int_0^{2\pi} |\nabla Eu(\alpha,t)| t \,\mathrm{d}\alpha \le \left(\int_0^{2\pi} |\nabla Eu(\alpha,t)|^p t \,\mathrm{d}\alpha\right)^{\frac{1}{p}} (2\pi t)^{1-\frac{1}{p}}.$$

Hence

$$\int_0^{2\pi} |\nabla Eu(\alpha,t)|^p t \,\mathrm{d}\alpha \ge (2\pi t)^{1-p}. \tag{4.8}$$

By combining (4.7) and (4.8) we get

$$\|E\|^p 4\pi r^{2-p} \ge (2\pi)^{1-p} \int_{2r}^{Cr} t^{1-p} = \frac{(2\pi)^{1-p}}{2-p}\left((Cr)^{2-p} - (2r)^{2-p}\right).$$

This gives the upper bound

$$C \le \left(\|E\|^p 2^{1+p}\pi^p(2-p) + 2^{2-p}\right)^{\frac{1}{2-p}}. \tag{4.9}$$

Thus we have established (4.4) and the lemma is proven in the special case $x, y \in \partial\Omega$.

Let us then consider the general case $x, y \in \overline{\Omega}$. In this case we repeat the previous construction but replace $\Omega$ by the simply connected domain $\Omega' = \Omega \setminus ([x,x'] \cup [y,y'])$ where $x', y' \in \partial\Omega$ satisfy

$$|x - x'| = \operatorname{dist}(x, \partial\Omega) \quad \text{and} \quad |y - y'| = \operatorname{dist}(y, \partial\Omega)$$

and $[x,x']$ and $[y,y']$ denote the line segments from $x$ to $x'$ and from $y$ to $y'$, respectively. Notice that $\Omega'$ is not necessarily a Sobolev extension domain. However, for points near $x$ and $y$ the condition (4.4) is satisfied trivially, and for points far from them, an enlarged ball meets the sets $\varphi(I_1) \setminus ([x,x'] \cup [y,y'])$ and $\varphi(I_2) \setminus ([x,x'] \cup [y,y'])$, so one can still use the argument from the special case. □

The next step is to go from the $\mathrm{cig}_d$ condition (4.3) to a $\mathrm{cig}_l$ condition (4.10). Before stating this as a lemma, let us recall the corresponding implication from [26, p. 385–386] from the so-called $\mathrm{car}_d$ condition to the so-called $\mathrm{car}_l$ condition. This latter condition is very close to the John condition (4.2), where one of the endpoints of all the curves is a fixed point $x_0$.

**Lemma 4.2.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $0 < \delta \le 1$. Suppose that there exists a curve $\gamma\colon [0,1] \to \Omega$ such that for every $t \in [0,1]$*

$$\gamma([0,t]) \subset B(\gamma(t), \frac{1}{\delta}\operatorname{dist}(\gamma(t), \partial\Omega)).$$

*Then there exists another arc length parametrized curve $\tilde{\gamma}\colon [0,d] \to \Omega$ with $\tilde{\gamma}(0) = \gamma(0)$, $\tilde{\gamma}(d) = \gamma(1)$ and*

$$\operatorname{dist}(\tilde{\gamma}(t), \partial\Omega) \ge 2^{-14}\delta^2 t \quad \textit{for} \quad t \in [0,d].$$

Following the proof of [29, Theorem 2.14] we now use Lemma 4.2 to obtain the passage from $\mathrm{cig}_d$ to $\mathrm{cig}_l$.

**Lemma 4.3.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain satisfying the condition (4.3) with some constant $C_{cig\text{-}d}$. Then for every $x, y \in \overline{\Omega}$ there exists an injective curve $\gamma \subset \Omega \cup \{x, y\}$ connecting $x$ to $y$ and satisfying*

$$\min\{\ell(\gamma_{x,z}), \ell(\gamma_{y,z})\} \le C_{cig\text{-}l} \operatorname{dist}(z, \partial\Omega) \tag{4.10}$$

*for all $z \in \gamma$, where $C_{cig\text{-}l} = 2^{14}C_{cig\text{-}d}^2$.*

*Proof.* Let us first consider the case where $x, y \in \Omega$. Let $\gamma \subset \Omega$ be a curve joining $x$ and $y$ and satisfying (4.3). Let $x_0 \in \gamma$ be a point such that $\operatorname{diam}(\gamma_{x,x_0}) = \operatorname{diam}(\gamma_{y,x_0})$. Then, by using Lemma 4.2 separately to the curves $\alpha^1 = \gamma_{x,x_0}$ and $\alpha^2 = \gamma_{y,x_0}$ there exist arc length parameterized curves $\tilde{\alpha}^i \colon [0, d_i] \to \Omega$, for $i = 1, 2$ so that $\tilde{\alpha}^1(0) = x$, $\tilde{\alpha}^2(0) = y$, $\tilde{\alpha}^1(d_1) = \tilde{\alpha}^2(d_2) = x_0$, and

$$\operatorname{dist}(\tilde{\alpha}^i(t), \partial\Omega) \geq 2^{-14} C_{\text{cig-d}}^{-2} t \quad \text{for} \quad t \in [0, d_i] \text{ and } i = 1, 2.$$

The concatenation of $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ now gives the curve satisfying (4.10).

Consider then the general case $x, y \in \overline{\Omega}$. Let $\{x_i\}, \{y_i\} \subset \Omega$ be sequences converging to the points $x$ and $y$, respectively, and let $\gamma^i \colon [0, 1] \to \Omega$ be a collection of constant speed parametrized curves connecting $x_i$ to $y_i$, and satisfying (4.10) with the same constant $C_{\text{cig-l}}$. Since $\Omega$ is bounded and the lengths of the curves are uniformly bounded, by Arzelá-Ascoli there exists a sequence $i_j \nearrow \infty$ and a curve $\gamma$ such that $\gamma_{i_j} \to \gamma$ uniformly. Moreover, by the lower semicontinuity of length

$$\begin{aligned}\min\{\ell(\gamma|_{[0,t]}), \ell(\gamma|_{[t,1]})\} &\leq \liminf_{i\to\infty} \min\{\ell(\gamma^i|_{[0,t]}), \ell(\gamma^i|_{[t,1]})\} \\ &\leq \liminf_{i\to\infty} C_{\text{cig-l}} \operatorname{dist}(\gamma^i(t), \partial\Omega) \leq C_{\text{cig-l}} \operatorname{dist}(\gamma(t), \partial\Omega),\end{aligned}$$

for all $t \in [0, 1]$. This concludes the general case.

Finally, note that the condition (4.10) still holds if we replace $\gamma$ with an injective subcurve [7, Lemma 3.1]. □

We are now ready to prove the main result of this section.

*Proof of Theorem 1.3.* Let $1 < p < 2$, $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^2$ a bounded and simply connected $L^{1,p}$-extension domain. Let $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$, $r = |z_1 - z_2|$, $x = \frac{z_1+z_2}{2}$, and denote by $\Omega_i$ the connected components of $B(x, 3r) \cap \Omega$ for which $\Omega_i \cap B(x, r) \neq \emptyset$. We divide the proof into two steps.

Step 1: Let us first show that we can connect any two points $z_1', z_2' \in \partial\Omega_i \cap \partial\Omega$ with a suitable curve. By Lemmata 4.1 and 4.3 we know that there exists a curve $\gamma \colon [0, d] \to \overline{\Omega}$ parametrized by the arc length between $z_1'$ and $z_2'$ so that

$$\min\left\{\ell(\gamma_{z_1',z}), \ell(\gamma_{z_2',z})\right\} \leq C_{\text{cig-l}} \operatorname{dist}(z, \partial\Omega) \tag{4.11}$$

for every $z \in \gamma$, where

$$C_{\text{cig-l}} = C(p)\|E\|^{\frac{2p}{2-p}}. \tag{4.12}$$

Let us denote by $\alpha^j$, $j = 1, 2$, the subcurves of $\gamma$ such that $\ell(\alpha^1) = \ell(\alpha^2)$ and $\gamma = \alpha^1 \cup \alpha^2$. We parametrize $\alpha^j \colon [0, \ell(\alpha^j)] \to \overline{\Omega}$ so that $\alpha^j(0) = z_j'$. We consider two cases. Assume first that $\gamma \subset B(x, 4r)$. By the condition (4.11) we have

$$\ell(\alpha^j|_{[0,t]}) = t \leq C_{\text{cig-l}} \operatorname{dist}(\alpha^j(t), \partial\Omega) \leq 7C_{\text{cig-l}} r \quad \text{for all } t \in [0, \ell(\alpha^j)].$$

Let us then consider the case $\gamma \not\subset B(x, 4r)$. Let $\Delta$ be the connected component of $\Omega_i \setminus \gamma$ with $z_1', z_2' \in \partial\Delta$ and let $C$ be a cross-cut in $\Delta$ connecting $z_1'$ to a point in $\gamma \setminus B(x, 4r)$. Let $w$ be the first point where $C$ intersects $S^1(x, 4r)$ when travelling from $z_1'$. Denote by $S \subset S^1(x, 4r)$ the maximal arc containing $w$ such that $S \cap (\alpha^1 \cup \alpha^2) = \emptyset$. Let $w_1, w_2$ be the endpoints of $S$. By reordering if necessary, there exist minimal times $t_1, t_2$ such that

$w_1 = \alpha^1(t_1)$ and $w_2 = \alpha^2(t_2)$. Let $\alpha$ be the curve parametrizing $\alpha^1|_{[0,t_1]} \cup S \cup \alpha^2|_{[0,t_2]}$ by arc length. By (4.11), we have

$$\ell(\alpha^j|_{[0,t]}) = t \leq C_{\text{cig-1}} \operatorname{dist}(\alpha^j(t), \partial\Omega) \leq C_{\text{cig-1}}|w_j - z_j| \leq 7C_{\text{cig-1}}r$$

for all $t \in [0, t_j]$. Suppose that $a \in S$ and $b \in \partial\Omega$ satisfy $|a - b| < \frac{1}{2}r$. Then, since $S$ is contained in the interior of $\Delta$ and $\Omega_i \subset B(x, 3r)$, we have that the line segment $[a, b]$ intersects one of the $\alpha^j$ at some point $\alpha^j(t)$. Since $\ell(\alpha^j|_{[0,t]}) > \frac{1}{2}r$, we have

$$|a - b| \geq |b - \alpha^j(t)| \geq \operatorname{dist}(\alpha^j(t), \partial\Omega) \geq \frac{r}{2C_{\text{cig-1}}}.$$

Consequently,

$$\operatorname{dist}(S, \partial\Omega) \geq \min\left\{r, \frac{r}{2C_{\text{cig-1}}}\right\} \geq \frac{r}{2C_{\text{cig-1}}}.$$

Therefore (setting $S = \emptyset$ in the first case),

$$\begin{aligned}\int_\alpha \operatorname{dist}(z, \partial\Omega)^{1-p} \, ds(z) &\leq \int_{\alpha^1} \operatorname{dist}(z, \partial\Omega)^{1-p} \, ds(z) + \int_{\alpha^2} \operatorname{dist}(z, \partial\Omega)^{1-p} \, ds(z) \\ &\quad + \int_S \operatorname{dist}(z, \partial\Omega)^{1-p} \, ds(z) \\ &\leq \frac{2}{2-p} \frac{1}{C_{\text{cig-1}}^{1-p}} (7C_{\text{cig-1}}r)^{2-p} + \left(\frac{r}{2C_{\text{cig-1}}}\right)^{1-p} 8\pi r \\ &\leq C(p) C_{\text{cig-1}} |z_1' - z_2'|^{2-p}.\end{aligned} \tag{4.13}$$

Finally, a connected component $A$ of $\Omega \setminus \alpha$ has finite perimeter in $\Omega$, and since the domain $\Omega$ is bounded, the extension $\widetilde{A}$ provided by Theorem 1.1 also has finite perimeter in $\mathbb{R}^2$. By Proposition 2.2, the boundary $\partial^M \widetilde{A}$ decomposes into Jordan loops $\{\Gamma_k\}$.

There exists one Jordan curve $\Gamma_k$ in the decomposition with $z_1, z_2 \in \Gamma_k$ $z_1', z_2' \in \Gamma_k$ because the points must be in the same connected component of $\partial^M \widetilde{A}$. We now write $\Gamma_k = \alpha * \tilde{\alpha}$ as a union of two curves, both having end points $z_1', z_2'$. Therefore $\tilde{\alpha} \subset \mathbb{R}^2 \setminus \Omega$ and by Theorem 1.1 and (4.13), we have

$$\begin{aligned}\int_{\tilde{\alpha}} \operatorname{dist}(z, \partial\Omega)^{1-p} \, ds(z) &\leq C(p, \varepsilon) \|E\|^{2+p+\varepsilon} \int_\alpha \operatorname{dist}(z, \partial\Omega)^{1-p} \, ds(z) \\ &\leq C(p, \varepsilon) \|E\|^{2+p+\varepsilon} C_{\text{cig-1}} |z_1' - z_2'|^{2-p}.\end{aligned} \tag{4.14}$$

Step 2: We now construct the curve $\gamma$ by connecting the sets $\Omega_i$ by suitable line-segments and by using Step 1 for each $\Omega_i$ to connect the entrance and exit points of $\Omega_i$. See Figure 1 for an illustration of the construction of $\gamma$.

Let us first check that by (4.11), we get an upper bound for the number $k \in \mathbb{N}$ of sets $\Omega_i$. By the definition of sets $\Omega_i$, $\Omega_i \cap B(x, r) \neq \emptyset$ for all $i = 1, \ldots, k$, so, there exists a curve in $\Omega_i$ satisfying (4.11) that starts in $B(x, r)$ and exits $B(x, 3r)$ at some point $z \in S^1(x, 3r)$ so that

$$\operatorname{dist}(z, \partial\Omega) \geq \frac{2r}{C_{\text{cig-1}}}.$$

Consequently, there exists an arc $S \subset S^1(x, 3r) \cap \overline{\Omega_i}$ such that $\mathcal{H}^1(S) > \frac{4r}{C_{\text{cig-1}}}$. Hence, $4k \frac{r}{C_{\text{cig-1}}} < 2\pi \cdot 3r$, and so

$$k < \frac{3}{2} \pi C_{\text{cig-1}}. \tag{4.15}$$

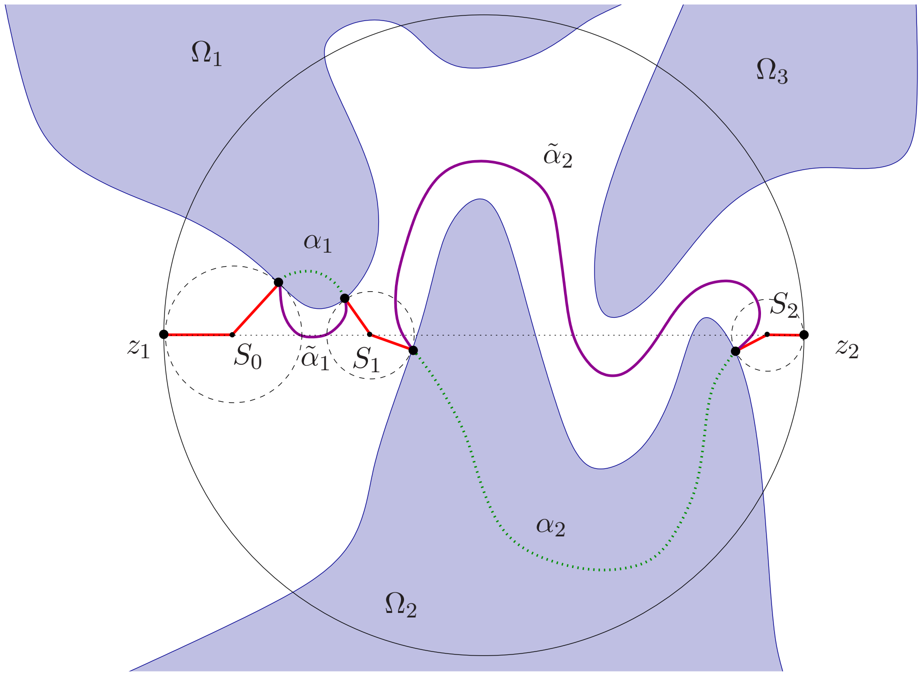


FIGURE 1. The curve $\gamma$ connecting $z_1$ and $z_2$ satisfying (4.1) is constructed by concatenating radial line segments (giving the curves $S_i$) inside disks completely contained in the complement of $\Omega$ and curves $\tilde{\alpha}_i$ that are obtained from Step 1 of the construction as (part of) the boundaries of extensions of sets whose boundary in $\Omega_i$ is $\alpha_i$.

Let us then construct a curve connecting $z_1$ and $z_2$ such that (4.1) holds. For notational convenience, write $\Omega_0 = \{z_1\}$ and $\Omega_{k+1} = \{z_2\}$.

Define $O_0 = \Omega_0$, $U_0 = \bigcup_{1\le i\le k+1} \Omega_i$ and a continuous function $f_0$ by

$$f_0 \colon t \mapsto \operatorname{dist}(O_0, tz_2 + (1-t)z_1) - \operatorname{dist}(U_0, tz_2 + (1-t)z_1), \quad 0 \le t \le 1.$$

Let $t_0 = \max\{t \in [0,1] : f_0(t) = 0\}$, and

$$R_0 := \operatorname{dist}(O_0, t_0 z_2 + (1-t_0)z_1) = \operatorname{dist}(U_0, t_0 z_2 + (1-t_0)z_1).$$

Denote by $P_0 = t_0 z_2 + (1-t_0)z_1$. By the selection of $P_0$ and $R_0$ we have $B(P_0, R_0) \subset \mathbb{R}^2 \setminus \Omega$, and there exists $i_0 \in \Big\{ i \in \{1, \ldots, k, k+1\} : \Omega_i \subset U_0 \Big\}$ such that $O_0 \cup \Omega_{i_0} \cup \overline{B}(P_0, R_0)$ is connected.

We continue by induction. Suppose we have found $P_0, \ldots, P_j$, $R_0, \ldots, R_j$, $O_0, \ldots, O_j$ and $U_0, \ldots, U_j$. Replacing $O_j$ with $O_{j+1} = O_j \cup \Omega_{i_0}$ and $U_j$ with $U_{j+1} = U_j \setminus \Omega_{i_0}$ we repeat the above process until $i_0 = k+1$.

The above process gives us a (relabeled) sequence $\Omega_0, \ldots, \Omega_{j+1}$ such that each adjacent pair $\Omega_m$, $\Omega_{m+1}$ may be connected with $S_m = [w_m^2, P_m] \cup [P_m, w_{m+1}^1]$ where $w_0^2 = z_1$, $w_{j+1}^1 = z_2$, and $w_m^2 \in \partial\Omega_m \cap \partial\Omega \cap \overline{B}(P_m, R_m)$ and $w_{m+1}^1 \in \partial\Omega_{m+1} \cap \partial\Omega \cap \overline{B}(P_m, R_m)$ for the other indices, so that

$$\int_{S_m} \operatorname{dist}(z, \partial\Omega)^{1-p} \, \mathrm{d}s(z) \le \frac{2}{2-p} \cdot R_m^{2-p} \le \frac{2}{2-p} |z_1 - z_2|^{2-p}. \tag{4.16}$$

For each $m \in \{1, \dots, j\}$ we can connect $w_m^1$ and $w_m^2$ with a curve $\tilde{\alpha}_m$ given by the special case that satisfies (4.14) with the obvious changes of notation.

The final curve $\gamma$ is obtained by the concatenation of the curves

$$S_0, \tilde{\alpha}_1, S_1, \dots, \tilde{\alpha}_{j-1}, S_j, \tilde{\alpha}_j, S_{j+1}.$$

By the bound (4.15) for the number of $\Omega_i$'s, combined with (4.14), (4.16), and (4.12), we see that the curve $\gamma$ satisfies

$$\begin{aligned}\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p}\,\mathrm{d}s(z) &\le \frac{3\pi C_{\text{cig-l}}}{2} \cdot \left(\frac{2}{2-p} + C(p,\varepsilon)\|E\|^{2+p+\varepsilon} C_{\text{cig-l}}\right) |z_1 - z_2|^{2-p} \\ &\le C(p,\varepsilon)\|E\|^{2+p+\varepsilon} C_{\text{cig-l}}^2 |z_1 - z_2|^{2-p} \\ &\le C(p,\varepsilon)\|E\|^{\frac{4p}{2-p}+2+p+\varepsilon} |z_1 - z_2|^{2-p} \\ &\le C(p,\varepsilon)\|E\|^{\frac{4+4p-p^2}{2-p}+\varepsilon} |z_1 - z_2|^{2-p}.\end{aligned}$$

This concludes the proof of the second step and the theorem. □

## 5. A Sobolev extension domain with large boundary

In this section we prove Theorem 1.2 which states the existence of a domain $\Omega \subset \mathbb{R}^3$ such that $\Omega = h(B(0,1))$ for a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\dim_{\mathcal{H}}(\partial\Omega) = 3$ and $\Omega$ is a $W^{1,p}$-extension domain for all $p \in [1, \infty]$.

We define first the following Cantor set $C \subset [0,1]^3$: Choose any strictly increasing sequence of positive numbers $\{\lambda_i\}$ satisfying

$$\lim_{i\to\infty} \lambda_i = 1/2 \quad ; \quad \prod_{i \ge 1} 2\lambda_i = 0.$$

For instance let $\lambda_i := (1/2)e^{-1/i}$. Define inductively a family of closed sets $C_n$, with $C_0 = [0,1]^3$, so that each $C_n = \bigcup_{i=1}^{8^n} C_{n,i}$ consists of $8^n$ disjoint cubes of sides $l_n = \lambda_1 \cdots \lambda_n$, in such a way that for each $i$, $C_{n+1} \cap C_{n,i}$ is formed by 8 cubes equally distributed inside $C_{n,i}$ (denote $l_0 = 1$). Namely, they are at least at a distance $e_{n+1} = l_n(1 - 2\lambda_{n+1})/3$ between themselves and also from the boundary of $C_{n,i}$.

Letting $C = \bigcap_{n \ge 0} C_n$, we get a Cantor set of Hausdorff dimension 3 but $\mathcal{L}^3(C) = 0$. The fact that $\dim_{\mathcal{H}}(C) = 3$ follows from $\lambda_i \nearrow \frac{1}{2}$, while $\mathcal{L}^3(C) = 0$ is implied by $\prod_{i\ge 1} 2\lambda_i = 0$. We refer to [28, Corollary 4] for further details.

Let us next define tubes that approach our Cantor set. The tubes will be removed from the open unit cube $(0,1)^3$ to form $\Omega$ so that $C \subset \partial\Omega$. Define first $x_{n,i}$ to be the middle point of the upper face of a cube $C_{n,i}$. Given a cube $C_{n,i}$ we denote by $C_{n-1,j(i)}$ the larger cube that contains it from the previous iteration. Define a decreasing sequence of positive constants $c_n$ by setting $c_0 = e_1/8 \in (0,1)$ and $c_n = c_{n-1}/64$ for all $n \ge 1$. In particular, we then have $c_n \le e_n/8$ and $c_n \le l_n$ for every $n \in \mathbb{N}$.

The tubes will be defined as tubular neighbourhoods of curves $L_{n,i}$ joining $x_{n,i}$ to a point $y_{n,i} \in \partial C_{n-1,j(i)}$ on the top face of $C_{n-1,j(i)}$. We require the curves $L_{n,i}$ and points $y_{n,i}$ to satisfy the following conditions:

- (L1) $L_{n,i} \subset C_{n-1,j(i)} \setminus \operatorname{int}(C_{n,i})$.
- (L2) $|y_{n,i} - x_{n-1,j(i)}| \le c_{n-1}/8$.
- (L3) $\operatorname{dist}(L_{n,i}, L_{n,j}) \ge c_n$ for $i \ne j$.

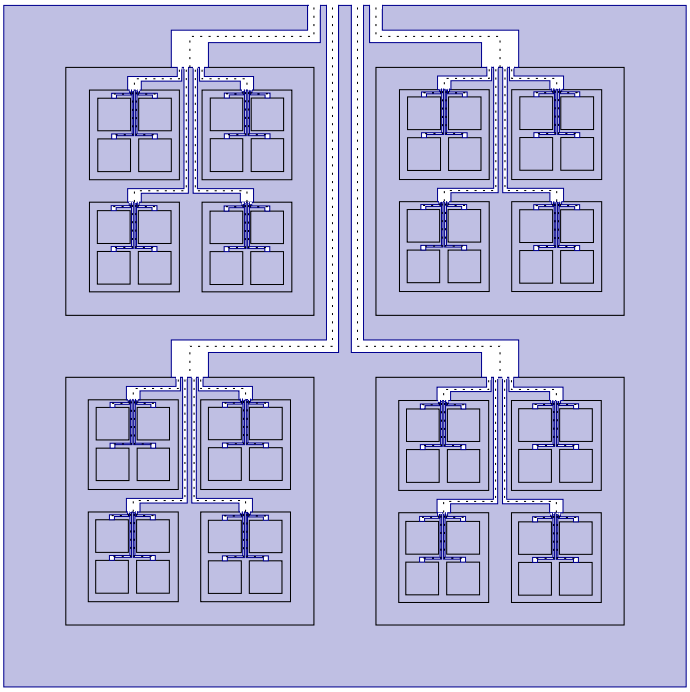

FIGURE 2. A two-dimensional illustration of the construction of $\Omega$. We start with a unit square and remove disjoint tubes that approach the top sides of Cantor set construction pieces. This two-dimensional version is homeomorphic to the unit ball, but contrary to $n \geq 3$ it is not a Sobolev extension domain.

(L4) The curves $L_{n,i}$ consist of segments that are parallel to the coordinate axes and have length at least $c_n$.

(L5) $L_{n,i}$ approaches $x_{n,i}$ and $y_{n,i}$ perpendicular to the faces of $C_{n,i}$ and $C_{n-1,j(i)}$, respectively.

Using the curves $L_{n,i}$ we then define the tubes as

$$T_{n,i} = \left\{ x \in C_{n-1,j(i)} \setminus \operatorname{int}(C_{n,i}) : \ \operatorname{dist}(x, L_{n,i}) \leq c_n/4 \right\}.$$

Next we define

$$T_n = \bigcup_{i=1}^{8^n} T_{n,i}$$

for every $n \geq 1$, and finally our domain as

$$\Omega = (0,1)^3 \setminus \overline{\bigcup_{n\geq 1} T_n}.$$

See Figure 2 for a two-dimensional illustration of the construction.

By construction, we have

$$C \subset \partial \bigcup_{n \geq 1} T_n \subset \partial\Omega \subset \partial(0,1)^3 \cup \bigcup_{n\geq 1} \bigcup_{i=1}^{8^n} \partial T_{n,i} \cup C.$$

Therefore, $\dim_{\mathcal{H}}(\partial\Omega) = 3$ and $\mathcal{L}^3(\partial\Omega) = 0$ as required. What remains to be checked is that $\Omega$ is homeomorphic to the open unit ball and that it is a Sobolev $W^{1,p}$-extension domain for all $1 \leq p \leq \infty$. We prove these separately in the following two lemmata.

**Lemma 5.1.** *There exists a homemorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ so that $h(B(0,1)) = \Omega$.*

*Proof.* It is enough to prove the existence of a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ so that $h(\Omega) = (0,1)^3$ since the unit cube and the unit ball are homeomorphic under a global homeomorphism.

Define the following decreasing sequence of open sets $\{U_n\}_{n\geq 1}$.

$$U_n = \left\{ x \in \mathbb{R}^3 : \operatorname{dist}\left( x, \overline{\bigcup_{i\geq n} T_i} \right) < \frac{c_n}{4} \right\}.$$

Observe that each $U_n$ has exactly $8^n$ disjoint connected components. Let us label these components as $U_{n,i} \supset T_{n,i}$. Notice also that

$$\bigcap_{n\geq 1} U_n = C.$$

We now define a sequence of homeomorphisms $h_n = h_{n,1} \circ h_{n,2} \circ \cdots \circ h_{n,8^n}$ so that each $h_{n,i}$ satisfies the following conditions:

(H1) For the supports we have

$$\operatorname{spt}(h_{n,i}) := \{x : h_{n,i}(x) \neq x\} \subset U_n.$$

In particular, for a given $n$ they are pairwise disjoint for different $i$.

(H2) For every two points $x, y \in U_{n+1} \cap U_{n,i}$ we have

$$|h_{n,i}(y) - h_{n,i}(x)| \leq |x - y|.$$

(H3) The map $h_{n,i}$ flattens the boundary of the tube $T_{n,i}$ to the top face of $C_{n-1,j(i)}$:

$$h_{n,i}\left( (\partial C_{n-1,j(i)} \setminus \partial T_{n,i}) \cup \overline{(\partial T_{n,i} \setminus \partial C_{n-1,j(i)})} \right) = \partial C_{n-1,j(i)}.$$

Using the maps $h_n$ we then define

$$h = \lim_{n\to\infty} h_1 \circ \cdots \circ h_n.$$

Let us next check that $h$ is well defined. On one hand, if $x \notin U_n$ for some $n$, then $h(x) = h_1 \circ \cdots \circ h_n(x)$, since by (H1) we have $h_k(x) = x$ when $k > n$. On the other hand, if $x \in \bigcap_n U_n$, by the pairwise disjointness of $U_{n,i}$ there exists a unique sequence $(i_n)$ so that $x \in U_{n,i_n}$. Since $\operatorname{diam}(U_{n,i_n}) \to 0$ as $n \to \infty$, by (H2) also

$$\operatorname{diam}(h(\overline{U_{n,i_n}})) = \operatorname{diam}(h_1 \circ \cdots \circ h_n(\overline{U_{n,i_n}})) \to 0$$

as $n \to \infty$. Hence $\{h(x)\} = \bigcap_n h(\overline{U_{n,i_n}})$.

Notice that $h$ maps the Cantor set $C$ bijectively to the Cantor set $\bigcap_n h(\overline{U_n})$. Hence, being a bijection outside the Cantor sets, $h$ is a bijection $\mathbb{R}^n \to \mathbb{R}^n$. Hence, in order to see that $h$ is a homeomorhism, by domain invariance it is enough to check that $h$ is continuous. This follows by the uniform continuity of the sequence $(h_1 \circ \cdots \circ h_n)_n$ of homeomorphisms given by (H1) and (H2). Thus, as the limit of uniformly continuous mappings, $h$ is continuous.

Let us finally observe that $h(\Omega) = (0,1)^3$. This is due to the condition (H3) implying $h(\partial T_n) \subset \partial(0,1)^3$ for all $n$ and hence, by continuity of $h$, we have $h(\partial\Omega) = \partial(0,1)^3$. □

**Lemma 5.2.** *$\Omega$ is a Sobolev $W^{1,p}$-extension domain for all $1 \leq p \leq \infty$.*

*Proof.* Since $\Omega$ is a bounded domain it is enough to prove that it is an $L^{1,p}$-extension domain for the homogeneous norm since for bounded domains these are the same (see [19, 17]). We will provide the extension in two steps. For this purpose we divide each of the tubes $T_{n,i}$ into an even number of shorter pieces of tubes to each of which we can extend the Sobolev function from its small neighbourhood.

Recall that our tubes are of the form

$$T_{n,i} = \left\{x \in C_{n-1,j(i)} \setminus \operatorname{int}(C_{n,i}) :\ \operatorname{dist}(x, L_{n,i}) \le c_n/2\right\}.$$

We now split each curve $L_{n,i}$ into finitely many parts $J(i,n)$

$$L_{n,i} = \bigcup_{j=1}^{J(i,n)} L_{n,i}^j$$

so that the following four properties hold:

(P1) $J(n,i)$ is an even number.
(P2) $\ell(L_{n,i}^j) \in [2c_n, 6c_n]$ for every $j = 1, \dots, J(n,i)$.
(P3) $L_{n,i}^j \cap L_{n,i}^{j+1}$ is just one point for every $j = 1, \dots, J(n,i) - 1$.
(P4) $L_{n,i}^1$ touches $\partial C_{n-1,j(i)}$ and $L_{n,i}^{J(n,i)}$ touches $C_{n,i}$.

The property (P1) together with (P2) can be satisfied because by the definition of $c_n$, we have $\ell(L_{n,i}) \ge 8c_n$. The condition (P3) and (P4) just say that the curves follow one after another in the desired direction. Using the shorter curves $L_{n,i}^j$ we then write $T_{n,i} = \bigcup_{j=1}^{J(n,i)} T_{n,i}^j$, where each shorter tube $T_{n,i}^j$ is the closure of the set of points of $T_{n,i}$ which are closer to $L_{n,i}^j$, for every $j$.

We define the following open sets from which we extend a given Sobolev function to the corresponding tube. If $j$ is odd, we set

$$U_{n,i}^j = \left\{x \in \operatorname{int}(C_{n-1,j(i)}) :\ \operatorname{dist}(x, L_{n,i}^j) < 2c_n, \text{and } x \text{ is closer to } L_{n,i}^j \text{ than to other } L_{n,i}^{j'}\right\},$$

and if $j$ is even, we set

$$U_{n,i}^j = \left\{x :\ \operatorname{dist}(x, T_{n,i}^j) < c_{n+1}\right\}.$$

By the assumptions (L4) and (P2), there exists a constant $L$ so that for every $n, i$ and $j$ there exists a map $f_{n,i}^j \colon \mathbb{R}^3 \to \mathbb{R}^3$ which is a composition of an $L$-biLipschitz map and a similitude so that

$$f_{n,i}^j(U_{n,i}^j) = U_{\text{odd}} \quad \text{and} \quad f_{n,i}^j(T_{n,i}^j) = T,$$

for $j$ odd, and

$$f_{n,i}^j(U_{n,i}^j) = U_{\text{even}} \quad \text{and} \quad f_{n,i}^j(T_{n,i}^j) = T,$$

for $j$ even, where

$$T = \left\{x = (x_1, x_2, x_3) \,:\, x_1 \in [0,1]\,,\ \sqrt{x_2^2 + x_3^2} \le 1\right\},$$

$$U_{\text{odd}} = \left\{x = (x_1, x_2, x_3) \,:\, x_1 \in (0,1)\,,\ \sqrt{x_2^2 + x_3^2} < 2\right\},$$

and

$$U_{\text{even}} = \{x \,:\, \operatorname{dist}(x, T) < 1\}.$$

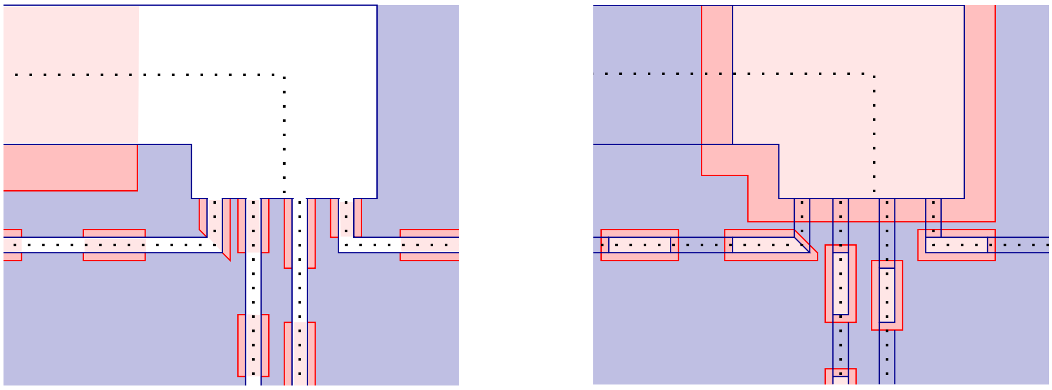

FIGURE 3. A two-dimensional illustration of the extension operator $E^1$ (on the left) that fills in every second piece of the tubes, and the extension operator $E^2$ (on the right) that fills in the rest of the pieces.

Now, for instance by Jones' theorem [18], there exists an extension operator

$$E_{\mathrm{odd}}\colon L^{1,p}(U_{\mathrm{odd}} \setminus T) \to L^{1,p}(U_{\mathrm{odd}}),$$

since $U_{\mathrm{odd}}$ is an $(\varepsilon, \delta)$-domain. Consequently, the norms of the operators for odd $j$

$$E^j_{n,i}\colon L^{1,p}(U^j_{n,i} \setminus T^j_{n,i}) \to L^{1,p}(U^j_{n,i})\colon u \mapsto (E_{\mathrm{odd}}(u \circ (f^j_{n,i})^{-1})) \circ f^j_{n,i}$$

are uniformly bounded. (Notice that the $L$-biLipschitz part of $f^j_{n,i}$ changes the norms only by a constant, whereas the similitude parts cancel out their effect on the norm since we use the homogenous norm.)

Similarly, for $j$ even, there exists an extension operator

$$E_{\mathrm{even}}\colon L^{1,p}(U_{\mathrm{even}} \setminus T) \to L^{1,p}(U_{\mathrm{even}}),$$

and so each of the operators for even $j$

$$E^j_{n,i}\colon L^{1,p}(U^j_{n,i} \setminus T^j_{n,i}) \to L^{1,p}(U^j_{n,i})\colon u \mapsto (E_{\mathrm{even}}(u \circ (f^j_{n,i})^{-1})) \circ f^j_{n,i}$$

are also uniformly bounded.

Next we see from the assumption (L3) that the collection $\{U^j_{n,i}\}_{j \text{ odd}}$ is pairwise disjoint. Hence, the extension operator

$$E^1\colon L^{1,p}(\Omega) \to L^{1,p}\left((0,1)^3 \setminus \overline{\bigcup_{\substack{i,n \\ j \text{ even}}} T^j_{n,i}}\right)$$

defined by

$$E^1u(x) = \begin{cases} E^j_{n,i}(u|_{U^j_{n,i} \setminus T^j_{n,i}})(x), & \text{if } x \in U^j_{n,i} \text{ with } j \text{ odd},\\ u(x), & \text{otherwise,} \end{cases}$$

is bounded. Then we use again the assumption (L3) to notice that also the collection $\{U^j_{n,i}\}_{j \text{ even}}$ is pairwise disjoint. Therefore, also the extension operator

$$E^2 \colon L^{1,p}\left((0,1)^3 \setminus \overline{\bigcup_{\substack{i,n \\ j \text{ even}}} T^j_{n,i}}\right) \to L^{1,p}((0,1)^3 \setminus C)$$

defined by

$$E^2 u(x) = \begin{cases} E^j_{n,i}(u|_{U^j_{n,i} \setminus T^j_{n,i}})(x), & \text{if } x \in U^j_{n,i} \text{ with } j \text{ even}, \\ u(x), & \text{otherwise}, \end{cases}$$

is bounded. See Figure 3 for an illustration of the extension operators $E^1$ and $E^2$.

Finally, we observe that $(0,1)^3$ is an extension domain (with some extension operator $E^3$) and the set $C$ is removable for Sobolev functions since its projection to any coordinate plane has zero two-dimensional measure. Thus, $E^3 \circ E^2 \circ E^1 \colon L^{1,p}(\Omega) \to L^{1,p}(\mathbb{R}^3)$ is a bounded extension operator. □

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