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On the Modulus Duality in Arbitrary Codimension

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We study the modulus of dual families of k- and $(n-k)$-dimensional Lipschitz chains of Euclidean n-cubes and establish half of the modulus duality identity.

1 Introduction

Suppose $D \subset \mathbb{R}^2$ is a Jordan domain, whose boundary is divided into four segments $\zeta_1, \ldots, \zeta_4$, in cyclic order. Let $\Gamma(\zeta_1, \zeta_3; D)$ be the family of all paths of D that connect ζ_1 and ζ_3. Then for every $1 < p < \infty$

$$(\mathrm{mod}_p \Gamma(\zeta_1, \zeta_3; D))^{1/p} (\mathrm{mod}_q \Gamma(\zeta_2, \zeta_4; D))^{1/q} = 1. \tag{1}$$

Here $q = \frac{p}{p-1}$ and the p-modulus of a path family Γ is defined by

$$\mathrm{mod}_p \Gamma = \inf_\rho \int_D \rho^p \, \mathrm{d}\mathcal{H}^2,$$

where the infimum is taken over all positive Borel-functions ρ with

$$\int_\gamma \rho \, \mathrm{d}s \geq 1$$

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for every locally rectifiable path $\gamma \in \Gamma$. The path modulus is a fundamental tool in geometric function theory and nonsmooth analysis [10, 18, 22]. To prevent confusion, we use the plural form of the word modulus, moduli, sparingly. We emphasize that the concepts and results of this paper are not closely related to moduli spaces.

For conformal moduli, that is, $p = 2 = q$, the duality relation (1) was already known to Beurling and Ahlfors, see for example, [1, Lemma 4] and [2, Ch. 14], although instead of modulus they considered its reciprocal, called extremal length. For general p the identity (1) follows from the results of [24]. It has found applications in connection with uniformization theorems [12, 17] and Sobolev extension domains [23].

The duality property of the modulus is also present in Euclidean spaces of higher dimension [6, 8, 24] and sufficiently regular metric spaces [13–15]. Moreover, discrete analogues can be found in the context of graphs and networks, see [3] and the references therein. For example, in [24], it is shown that for $1 < p < \infty$

$$(\mathrm{mod}_p \Gamma(E, F; G))^{1/p} (\mathrm{mod}_q \Gamma^*(E, F; G))^{1/q} = 1, \tag{2}$$

where $G \subset \mathbb{R}^n$ is open and connected, E and F are disjoint, compact and connected subsets of G and $\Gamma^*(E, F; G)$ is the set of all relatively closed sets of G that separate E from F. The modulus of separating sets is a natural generalization of the path modulus. See Section 2 for definitions of moduli and other concepts appearing in the introduction.

Separating sets are generally of codimension 1, so (1) and (2) deal with objects of either dimension or codimension 1. In fact, this is a common theme in all of the results cited above. Indeed, not much is known about the relationship of the modulus and objects of higher (co)dimension. However, the modulus has been applied to study such objects in [11, 16], where the nonexistence of quasisymmetric parametrizations of certain spaces was established. Studying more general moduli could therefore lead to finding tools to approach such parametrization questions in higher dimensions.

An observation by Freedman and He (see the discussion after Theorem 2.5 in [6]) hints that a duality result could hold for objects of higher (co)dimension as well. In this paper we explore this question in the setting of cubes of $\mathbb{R}^n$.

Our first problem is defining suitable classes of k- and $(n-k)$-dimensional objects, since simple descriptions such as “connecting paths” or “separating surfaces” do not seem to exist. We follow [6] and define the objects as representatives of certain relative homology classes. For example, in the context of (1), we can think of the paths of $\Gamma(\zeta_1, \zeta_3; D)$ as singular relative cycles, which are representatives of either generator of $H_1(D, \zeta_1 \cup \zeta_3) \simeq \mathbb{Z}$. Since we also want to integrate over the chains, we need to assume some regularity. For this reason we will consider Lipschitz chains instead of singular chains.

Let $Q \subset \mathbb{R}^n$ be a compact set homeomorphic to the closed unit n-cube I^n. Fix a homeomorphism $h : Q \to I^n$ and an integer $0 < k < n$, and let

$$A = h^{-1}(\partial I^k \times I^{n-k}) \text{ and } B = h^{-1}(I^k \times \partial I^{n-k}).$$

Then A and B are $(n-1)$-dimensional submanifolds of ∂Q with $\partial Q = A \cup B$ and $\partial A = A \cap B = \partial B$. We assume that A, B, and Q are locally Lipschitz neighborhood retracts. This includes triples (Q, A, B) that are smooth or polygonal and cubes that are images of the standard cube under bi-Lipschitz automorphisms of $\mathbb{R}^n$.

We denote the Lipschitz homology groups by H^L_*. We consider only groups with integer coefficients. This notation should not be confused with the Hausdorff measures, which are denoted by $\mathcal{H}^*$. Note that

$$H^L_k(Q, A) \simeq \mathbb{Z} \simeq H^L_{n-k}(Q, B),$$

since the same is true for singular homology, and the two homology theories are equivalent for pairs of locally Lipschitz retracts (see Lemma 2.1).

Let Γ_A (resp. Γ_B) be the collection of the images of relative Lipschitz k-cycles of $Q - B$ that generate $H^L_k(Q, A)$ ($(n-k)$-cycles of $Q - A$ that generate $H^L_{n-k}(Q, B)$). Define

$$\text{mod}_p \Gamma_A := \inf_\rho \int_Q \rho^p \, \mathrm{d}\mathcal{H}^n,$$

where the infimum is taken over positive Borel-functions ρ, for which

$$\int_S \rho \, \mathrm{d}\mathcal{H}^k \geq 1$$

for every $S \in \Gamma_A$. The modulus $\text{mod}_p \Gamma_B$ is defined analogously. In this paper we will prove the following upper bound.

Theorem 1.1. For every $1 < p < \infty$

$$(\text{mod}_p \Gamma_A)^{1/p} (\text{mod}_q \Gamma_B)^{1/q} \leq 1,$$

where $q = \frac{p}{p-1}$.

It is unknown whether Theorem 1.1 holds with an equality. We will prove Theorem 1.1 in Section 3. A similar result for de Rham cohomology classes, with an equality, is proved in the setting of Riemannian manifolds in pages 212–213 of [6].

In light of the results of [5, Ch. 4], it would be interesting to know whether analogues of Theorem 1.1 hold for homology classes of integral currents.

Remark 1.2. The assumption on Ω, A, and B being locally Lipschitz neighborhood retracts can be relaxed. The proof of Theorem 1.1 only requires that there exists a pair of Lipschitz chains that generate $H_k(\Omega, A)$ and $H_{n-k}(\Omega, B)$. The assumption on retracts was chosen for its simplicity and its use in [5]. It is also likely that such minimal assumptions for the upper bound of Theorem 1.1 are not sufficient for the corresponding lower bound. We will discuss the lower bound in Section 4.

2 Definitions

2.1 Lipschitz homology

Let us recall the definition and basic properties of the integral homology groups. See for example [4, 9] or other texts on basic algebraic topology for a more comprehensive treatment.

For an integer $k \geq 0$ the standard k-simplex Δ_k is the convex hull of the standard unit vectors $e_0, \ldots, e_k$ of $\mathbb{R}^{k+1}$. Given a metric space (X, d), a singular k-simplex is a continuous map from Δ_k to X. Finite formal linear combinations

$$\sigma = \sum_i k_i \sigma_i$$

of singular k-simplices σ_i with integer coefficients k_i are called singular k-chains. Singular k-chains of X form a free abelian group denoted by $C_k(X)$. The boundary $\partial\sigma$ of a singular k-simplex σ is the singular $(k-1)$-chain

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i \sigma \circ F_k^i,$$

where $F_k^i : \Delta_{k-1} \to \Delta_k$ is the unique linear map that maps each e_j to e_j for $j < i$ and to e_{j+1} for $j \geq i$. For singular 0-simplices we set $\partial\sigma = 0$. The boundary defines a collection of homomorphisms $\partial : C_k(X) \to C_{k-1}(X)$, all denoted by the same symbol ∂. Then $\partial\partial = 0$.

The image of a singular k-simplex σ is the compact set $|\sigma| = \sigma(\Delta_k)$. The image of a k-chain $\sigma = \sum_i k_i \sigma_i$ is the compact set $|\sigma| = \bigcup_i |\sigma_i|$.

Given a subspace $Y \subset X$, we identify each singular simplex σ of Y with the singular simplex $i_Y \circ \sigma$ of X, where $i_Y : Y \hookrightarrow X$ is the inclusion map. We define the groups of relative chains by

$$C_k(X, Y) := \frac{C_k(X)}{C_k(Y)},$$

with the convention $C_k(X, \emptyset) = C_k(X)$. The boundary map induces homomorphisms $\partial : C_k(X, Y) \to C_{k-1}(X, Y)$, which are again denoted by the same symbol. A chain $\sigma \in C_k(X)$ is called a cycle relative to Y, if $\partial\sigma \in C_{k-1}(Y)$, or simply a relative cycle if the choice of Y is clear from the context. Similarly, σ is called a relative boundary if $\sigma = \partial\sigma' + \sigma''$, where $\sigma' \in C_{k+1}(X)$ and $\sigma'' \in C_k(Y)$.

The singular relative homology groups of the pair (X, Y) are the quotient groups

$$H_k(X, Y) := \frac{\ker(\partial : C_k(X, Y) \to C_{k-1}(X, Y))}{\operatorname{im}(\partial : C_{k+1}(X, Y) \to C_k(X, Y))}.$$

The homology groups of X are the groups $H_k(X) := H_k(X, \emptyset)$. The homology class of a (relative) chain σ is denoted by $[\sigma]$. The homology classes of $H_k(X, Y)$ are represented by relative k-cycles, and two relative k-cycles define the same class if and only if their difference is a relative boundary.

If X' is another metric space with a subset Y', and $f : X \to X'$ is a continuous map with $f(Y) \subset Y'$, we denote by f_* the induced homomorphisms $f_* : C_k(X, Y) \to C_k(X', Y')$, and also the homomorphisms $f_* : H_k(X, Y) \to H_k(X', Y')$. These are given by $f_*\sigma = f \circ \sigma$ for singular simplices, $f_* \sum_i k_i\sigma_i = \sum_i k_i f_*\sigma_i$ for chains and $f_*[\sigma] = [f_*\sigma]$ for homology classes.

Given a continuous homotopy $H : X \times I \to X'$ with $H(Y \times I) \subset Y'$, there exists a sequence of homomorphisms

$$P : C_k(X, Y) \to C_{k+1}(X', Y'),$$

such that

$$H_{1*} - H_{0*} = P\partial + \partial P. \tag{3}$$

Here $H_t(x) = H(x, t)$. Formula (3) is called the homotopy formula.

A continuous $f : X \to Y$ is called a retraction if $f \circ i_Y = \mathrm{id}_Y$. The set Y is then called a retract of X. If Y is a retract of one if its neighborhoods in X, it is called a neighborhood retract.

The corresponding objects in the Lipschitz category are obtained by replacing each occurrence of "singular" or "continuous" with "Lipschitz." The homotopies involved in these definitions are then required to be Lipschitz with respect to the metric $d((x,t),(x',t')) = d(x,x') + |t-t'|$. We denote the groups of Lipschitz chains by $C_*^L(X,Y)$ and the Lipschitz homology groups by $H_*^L(X,Y)$. We define locally Lipshitz objects similarly. However, due to compactness, there is often no difference between the corresponding objects of Lipschitz and locally Lipschitz categories.

Lemma 2.1. Let $Y \subset X \subset \mathbb{R}^n$ be locally Lipschitz neighborhood retracts of $\mathbb{R}^n$. Then the inclusions

$$i : C_*^L(X,Y) \hookrightarrow C_*(X,Y)$$

induce isomorphisms on homology.

Lemma 2.1 follows from a more general result [19,Cor. 11.1.2], which holds for pairs of locally Lipschitz contractible metric spaces (see [19] for the definition). It is straightforward to show that the existence of locally Lipschitz neighborhood retractions implies locally Lipschitz contractibility.

2.2 Modulus

Given a $1 < p < \infty$ and a family $\mathcal{M}$ of Borel measures of $\mathbb{R}^n$, the p-modulus of $\mathcal{M}$ is the number

$$\mathrm{mod}_p\mathcal{M} := \inf_\rho \int_{\mathbb{R}^n} \rho^p \, \mathrm{d}\mathcal{H}^n, \tag{4}$$

where the infimum is taken over all Borel functions $\rho : \mathbb{R}^n \to [0,\infty]$ with

$$\int_{\mathbb{R}^n} \rho \, \mathrm{d}\nu \geq 1 \tag{5}$$

for every $\nu \in \mathcal{M}$. Such functions are called admissible for $\mathcal{M}$. If there exists a subfamily $\mathcal{N} \subset \mathcal{M}$ such that $\mathrm{mod}_p\mathcal{N} = 0$ and (5) holds for all $\nu \in \mathcal{M} - \mathcal{N}$, we say that ρ is p-weakly admissible or simply weakly admissible if the choice of p is clear from the context. It follows that the infimum in (4) does not change if we take it over p-weakly admissible functions instead. Let us list some useful properties of the modulus.

Lemma 2.2. Let $\mathcal{M}$ be a collection of Borel measures of $\mathbb{R}^n$. Let $1 < p < \infty$.

i) If ρ_i are p-integrable Borel functions that converge to a function ρ in L^p, there exists a subsequence $(\rho_{i_j})_j$ for which

$$\int_{\mathbb{R}^n} \rho_{i_j} \,\mathrm{d}\nu \xrightarrow{j\to\infty} \int_{\mathbb{R}^n} \rho \,\mathrm{d}\nu$$

for almost every $\nu \in \mathcal{M}$. In particular, Borel representatives of L^p-limits of admissible functions are weakly admissible.

ii) If $\mathrm{mod}_p\mathcal{M} < \infty$, then

$$\mathrm{mod}_p\mathcal{M} = \int_{\mathbb{R}^n} \rho^p \,\mathrm{d}\mathcal{H}^n$$

for a weakly admissible minimizer ρ, unique up to sets of $\mathcal{H}^n$-measure zero. Moreover,

$$\mathrm{mod}_p\mathcal{M} \le \int_{\mathbb{R}^n} \phi\rho^{p-1} \,\mathrm{d}\mathcal{H}^n$$

for any other p-integrable weakly admissible ϕ.

iii) If $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}_i$ with $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ for all i, then

$$\mathrm{mod}_p\mathcal{M} = \lim_{i\to\infty} \mathrm{mod}_p\mathcal{M}_i.$$

Claim *i*) is often referred to as Fuglede's lemma. Proofs for *i*) and the first part of *ii*) can be found in [7, Thm. 3]. The second part of *ii*) and *iii*) are generalizations of [15, Lemma 5.2] and [25, Lemma 2.3], respectively. The same proofs apply.

In this paper we abbreviate

$$\mathrm{mod}_p\Gamma_A = \mathrm{mod}_p\{\mathcal{H}^k \llcorner S \mid S \in \Gamma_A\},$$

and

$$\mathrm{mod}_q\Gamma_B = \mathrm{mod}_q\{\mathcal{H}^{n-k} \llcorner S^* \mid S^* \in \Gamma_B\}.$$

2.3 Rectifiable sets

A subset of $\mathbb{R}^n$ is k-rectifiable if it is covered by the image of a subset of $\mathbb{R}^k$ under a Lipschitz map. A subset of $\mathbb{R}^n$ is countably k-rectifiable if $\mathcal{H}^k$-almost all of it is contained in a countable union of k-rectifiable sets.

See for example [5, 21] for basic theory on rectifiable sets. Note that the definition of countable rectifiability in [5, 3.2.14] is slightly different from ours.

Let us record some useful facts on rectifiable sets. The following Fubini-type lemma is an application of [5, 3.2.23] and [5, 2.6.2].

Lemma 2.3. Suppose S^* is a countably k-rectifiable subset of $\mathbb{R}^n$ and S is a countable union of l-rectifiable subsets of $\mathbb{R}^m$. Then $S^* \times S$ is a countably $(k+l)$-rectifiable subset of $\mathbb{R}^n \times \mathbb{R}^m$, and

$$\int_{S^* \times S} g(x,y)\,\mathrm{d}\mathcal{H}^{k+l}(x,y) = \int_{S^*}\int_S g(x,y)\,\mathrm{d}\mathcal{H}^l(y)\,\mathrm{d}\mathcal{H}^k(x)$$

for any positive Borel function g on $\mathbb{R}^n \times \mathbb{R}^m$.

Lemma 2.3 does not hold for general countably l-rectifiable sets S, see [5, 3.2.24]. The second tool we need is the coarea formula, see for example, [21,12.7].

Lemma 2.4. Suppose $m \leq k$. Let S be a countably k-rectifiable subset of $\mathbb{R}^n$ and let $u : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz. Then

$$\int_{\mathbb{R}^m}\int_{u^{-1}(z)\cap S} g\, d\mathcal{H}^{k-m}\,\mathrm{d}\mathcal{H}^m(z) = \int_S g J_u^S\,\mathrm{d}\mathcal{H}^k \tag{6}$$

for every positive Borel function g on S.

Let us define the Jacobian J_u^S appearing in (6). Details can be found in [21, §12]. Suppose first that S is an embedded C^1 k-submanifold (without boundary) of $\mathbb{R}^n$. Then u is differentiable at $\mathcal{H}^k$-almost every $x \in S$. Fix such an x, and let $\{E_1, \dots, E_k\}$ be an orthonormal basis for the tangent space of S at x. Let $Du(x)$ be the Jacobian matrix of u at x with respect to standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. We set

$$J_u^S(x) := \sqrt{\det(d^S u(x) d^S u(x)^t)},$$

where $d^S u(x)$ is the matrix with columns $Du(x)E_i$. It can be shown that $J_u^S(x)$ does not depend on the choice of the basis $\{E_i\}$.

More generally, every countably k-rectifiable set S can be expressed as a disjoint union $S = \bigcup_{i=0}^\infty M_i$, where $\mathcal{H}^k(M_0) = 0$ and each M_i for $i \geq 1$ is contained in an embedded

C^1 k-submanifold N_i of $\mathbb{R}^n$. Given an $x \in M_i$ with $i \geq 1$, we set

$$J_u^S(x) := J_u^{N_i}(x).$$

Then J_u^S is well defined $\mathcal{H}^k$-almost everywhere on S. It can be shown that J_u^S does not depend on the decomposition $S = \bigcup_{i=0}^{\infty} M_i$, up to sets of $\mathcal{H}^k$-measure zero.

3 Proof of Theorem 1.1

Given any set $S \subset \mathbb{R}^n$ and a vector $y \in \mathbb{R}^n$ we denote

$$S_y = \{x + y \mid x \in S\}$$

and

$$N_\varepsilon(S) = \{x \mid d(x, S) < \varepsilon\}.$$

Denote by Γ_A^* the collection of $(n-k)$-rectifiable subsets S^* of $\Omega - A$, such that the homomorphism

$$i_* : H_k^L(\Omega - S^*, A) \to H_k^L(\Omega, A)$$

induced by inclusion is trivial. Lemma 3.5 below implies that $\Gamma_B \subset \Gamma_A^*$. Every set $S^* \in \Gamma_A^*$ intersects with every $S \in \Gamma_A$ in a nonempty set. To see this, note that if $|\sigma| \cap S^*$ is empty for some Lipschitz cycle $\sigma \in C_k(\Omega)$ relative to A, then $[\sigma] = i_*[\sigma] = 0$ in $H_k^L(\Omega, A)$ by the definition of Γ_A^*.

We abbreviate

$$\operatorname{mod}_q \Gamma_A^* := \operatorname{mod}_q \{\mathcal{H}^{n-k} \llcorner S^* \mid S^* \in \Gamma_A^*\}.$$

Theorem 1.1 is then implied by the following more general result.

Theorem 3.1. For every $1 < p < \infty$

$$(\operatorname{mod}_p \Gamma_A)^{1/p} (\operatorname{mod}_q \Gamma_A^*)^{1/q} \leq 1,$$

where $q = \frac{p}{p-1}$.

The rest of this section is focused on the proof of Theorem 3.1.

For each $\delta > 0$ let Γ_A^δ be the subcollection of Γ_A consisting of those sets whose distance to B is at least 100δ. Analogously, the subcollection $\Gamma_A^{*\delta}$ consists of the elements of Γ_A^* whose distance to A is at least 100δ. In light of *iii*) of Lemma 2.2, it suffices to show that

$$(\mathrm{mod}_p \Gamma_A^\delta)^{1/p} (\mathrm{mod}_q \Gamma_A^{*\delta})^{1/q} \leq 1 \tag{7}$$

for all δ. Fix a δ for the rest of the proof. We may assume without loss of generality that the moduli in question are nonzero and the collections Γ_A^δ and $\Gamma_A^{*\delta}$ are nonempty.

The following intersection property of the elements of Γ_A and Γ_A^* forms the topological core of Theorem 3.1.

Proposition 3.2. The intersection $S_z \cap S^*$ is nonempty for every $S \in \Gamma_A^\delta$, $S^* \in \Gamma_A^{*\delta}$ and $|z| < 10\delta$.

We postpone the proof to Subsection 3.1.

Let $S \in \Gamma_A^\delta$. Observe that the map

$$g \mapsto \int_S g \, \mathrm{d}\mathcal{H}^k \tag{8}$$

is a distribution in $\mathbb{R}^n$. Thus we have by [5, 4.1.2] that

$$\int_\Omega \phi_\varepsilon^S g \, \mathrm{d}\mathcal{H}^n \overset{\varepsilon \to 0}{\longrightarrow} \int_S g \, \mathrm{d}\mathcal{H}^k \tag{9}$$

for every smooth compactly supported function g, where

$$\phi_\varepsilon^S(x) := \int_S \phi_\varepsilon(x-y) \, \mathrm{d}\mathcal{H}^k(y)$$

is the convolution of the distribution (8) with respect to a smooth kernel ϕ. That is, $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$ and ϕ is a nonnegative smooth function on $\mathbb{R}^n$ that vanishes outside the unit ball $\mathbb{B}^n$ and satisfies $\int_{\mathbb{B}^n} \phi \, \mathrm{d}\mathcal{H}^n = 1$.

Smoothness is convenient for avoiding tedious technicalities, but to see the geometry behind the arguments that follow, the reader is encouraged to repeat the proof with the nonsmooth kernel $\phi = |\mathbb{B}^n|^{-1}\chi_{\mathbb{B}^n}$.

Theorem 3.1 follows via (7) from the following proposition.

Proposition 3.3. The convolution $\phi_\varepsilon^{S_z}$ is admissible for $\Gamma_A^{*\delta}$ for all $\varepsilon < \delta$ and all $|z| < \delta$.

Proof. Fix an $\varepsilon < \delta$ and a set $S^* \in \Gamma_A^{*\delta}$. Let $z = 0$ for now. By Lemma 2.3

$$\begin{aligned}\int_{S^*} \phi_\varepsilon^S(x)\, d\mathcal{H}^{n-k}(x) &= \int_{S^*}\int_S \phi_\varepsilon(x-y)\, \mathrm{d}\mathcal{H}^k(y)\mathrm{d}\mathcal{H}^{n-k}(x)\\ &= \int_{S^*}\int_{S\cap N_\varepsilon(S^*)} \phi_\varepsilon(x-y)\, \mathrm{d}\mathcal{H}^k(y)\mathrm{d}\mathcal{H}^{n-k}(x)\\ &= \int_{(S^*\times S)\cap\{|x-y|<\varepsilon\}} \phi_\varepsilon(x-y)\, \mathrm{d}\mathcal{H}^n(x,y).\end{aligned}$$

Now we can apply the coarea formula, Lemma 2.4, on the map $u(x,y) = x - y$ to obtain

$$\int_{S^*} \phi_\varepsilon^S(x)\, \mathrm{d}\mathcal{H}^{n-k}(x) \geq \int_{\varepsilon\mathbb{B}^n}\int_{(S^*\times S)\cap\{x-y=w\}} \phi_\varepsilon(w)\, \mathrm{d}\mathcal{H}^0 d\mathcal{H}^n(w) \tag{10}$$

since $J_u^{S^*\times S} \leq 1$. To see this, note that for any $(n-k)$- and k-dimensional embedded C^1 submanifolds N^* and N of $\mathbb{R}^n$ the matrix $d^{N^*\times N}u$ consists of unit column vectors. Thus $J_u^{N^*\times N} \leq 1$. It follows that $J_u^{S^*\times S} \leq 1$ as well, since it can be computed via $J_u^{M_i^*\times M_j}$ with $i,j \geq 1$, where $S^* = \bigcup_{i=0}^\infty M_i^*$ and $S = \bigcup_{i=0}^\infty M_j$ are decompositions of S^* and S as in the discussion following Lemma 2.4. Note that the sets $M_0^* \times S$ and $S^* \times M_0$ have zero $\mathcal{H}^n$-measure by Lemma 2.3.

Finally, we apply Proposition 3.2 on (10) and obtain

$$\int_{S^*} \phi_\varepsilon^S(x)\, \mathrm{d}\mathcal{H}^{n-k}(x) \geq \int_{\varepsilon\mathbb{B}^n} \phi_\varepsilon(w)\, \mathrm{d}\mathcal{H}^n(w) = 1.$$

The proof in the case of general z reduces to the case $z = 0$ via

$$\phi_\varepsilon^{S_z}(x) = \phi_\varepsilon^S(x-z), \tag{11}$$

since Proposition 3.2 can still be applied. ■

Proof of Theorem 3.1. The q-modulus of $\Gamma_A^{*\delta}$ is finite by Proposition 3.3. Let ρ be the unique weak minimizer of $\mathrm{mod}_q\Gamma_A^{*\delta}$ given by *ii*) of Lemma 2.2. We may assume that ρ vanishes in $N_{10\delta}(A)$ and is defined as zero outside Q. Let g_r be the smooth convolution

$$g_r(x) := \int_{r\mathbb{B}^n} \rho^{q-1}(x+y)\phi_r(y)\, \mathrm{d}\mathcal{H}^n(y).$$

Let $S \in \Gamma_A^\delta$ and let $\varepsilon < \delta$. Proposition 3.3 and *ii*) of Lemma 2.2 imply

$$\operatorname{mod}_q \Gamma_A^{*\delta} \le \int_\Omega \phi_\varepsilon^{S_z} \rho^{q-1} \,\mathrm{d}\mathcal{H}^n$$

for all $|z| < \delta$ and $S \in \Gamma_A^\delta$. Note that the product $\phi_\varepsilon^{S_z} \rho^{q-1}$ vanishes in $N_{10\delta}(\partial\Omega)$, so by (11) and a change of variables

$$\operatorname{mod}_q \Gamma_A^{*\delta} \le \int_\Omega \phi_\varepsilon^{S}(x) \rho^{q-1}(x+z) \,\mathrm{d}\mathcal{H}^n(x)$$

for all $|z| < \delta$. Multiplying both sides by $\phi_r(z)$ and integrating over z yields

$$\operatorname{mod}_q \Gamma_A^{*\delta} \le \int_\Omega \phi_\varepsilon^{S} g_r \,\mathrm{d}\mathcal{H}^n$$

by Fubini's theorem. Letting $\varepsilon \to 0$ and then $r \to 0$ yields

$$\operatorname{mod}_q \Gamma_A^{*\delta} \le \int_S \rho^{q-1} \,\mathrm{d}\mathcal{H}^k$$

for mod_p-almost every $S \in \Gamma_A^\delta$ by (9) and *i*) of Lemma 2.2. Thus

$$\frac{1}{\operatorname{mod}_q \Gamma_A^{*\delta}} \rho^{q-1}$$

is weakly admissible for Γ_A^δ, so

$$\operatorname{mod}_p \Gamma_A^\delta \le (\operatorname{mod}_q \Gamma_A^{*\delta})^{1-p},$$

which is a rearrangement of (7). ■

3.1 Topological lemmas

In this subsection we complete the proof of Theorem 1.1 by proving Proposition 3.2 and showing that $\Gamma_B \subset \Gamma_A^*$. These are implied by the following two lemmas.

Lemma 3.4. Suppose $S \in \Gamma_A^\delta$ and $|y| < 10\delta$. Then there exists a singular relative cycle σ_y, such that it generates $H_k(\Omega, A)$ and its image coincides with S_y outside $N_{100\delta}(A)$.

Lemma 3.5. Suppose σ_A and σ_B are relative singular chains that generate nontrivial elements of $H_k(\Omega, A)$ and $H_{n-k}(\Omega, B)$, respectively. Then $|\sigma_A| \cap |\sigma_B|$ is nonempty.

Proof of Lemma 3.4. The lemma follows from the homotopy formula (3). By the definition of Γ_A there is a relative cycle σ that generates $H_k(Q, A)$ and has S as its image. By applying barycentric subdivision multiple times, if necessary, we may assume that σ splits into $\sigma = \sigma_1 + \sigma_2$, where $|\sigma_1| \subset N_{30\delta}(A)$ and $|\sigma_2| \subset Q - N_{20\delta}(\partial Q)$. Let H_t be the homotopy $H_t(x) = x + ty$ for some fixed y with $|y| < 10\delta$. Then by (3) there exist homomorphisms $P : C_l(U) \to C_{l+1}(U_y)$ for all l and all open sets $U \subset \mathbb{R}^n$, such that

$$H_{1*} - H_{0*} = \partial P + P\partial. \tag{12}$$

By applying this with $U = Q - N_{20\delta}(\partial Q)$, we see that $P(\partial\sigma_2)$ and $H_{1*}\sigma_2$ are chains in $Q - N_{10\delta}(\partial Q)$. We let $\sigma_y = \sigma_1 - P(\partial\sigma_2) + H_{1*}\sigma_2$. Then $\sigma_y - \sigma = \partial P\sigma_2$ by (12), so σ_y belongs to the same relative homology class as σ. To prove the final part of the lemma, note that $|\partial\sigma_2| \subset N_{30\delta}(A)$, since $|\partial\sigma_2| = |\partial\sigma_1| \cap \operatorname{int}(Q)$. Thus $|P(\partial\sigma_2)| \subset N_{40\delta}(A)$ and $|\sigma_y|$, $|H_{1*}\sigma_2| = |\sigma_2|_y$ and S_y all coincide outside $N_{100\delta}(A)$. ■

Proof of Lemma 3.5. The lemma follows from the theory of intersection numbers developed in [4]. We may assume that $Q = J^n$, where $J = [-1, 1]$, and respectively $A = \partial J^k \times J^{n-k}$ and $B = J^k \times \partial J^{n-k}$. Let σ_A and σ_B be representatives of some nontrivial classes of $H_k(Q, A)$ and $H_{n-k}(Q, B)$, respectively. Suppose $|\sigma_A| \cap |\sigma_B| = \emptyset$. Then we can deform σ_A and σ_B slightly, if necessary, and assume that $|\sigma_A| \cap B = \emptyset = |\sigma_B| \cap A$. This allows us to define the intersection number $[\sigma_A] \circ [\sigma_B] \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \simeq \mathbb{Z}$ of the classes $[\sigma_A]$ and $[\sigma_B]$, as in [4, VII.4].

The intersection number of the two classes is defined (up to sign) by pushing the outer product

$$[\sigma_A] \times [\sigma_B] \in H_n(Q \times Q, A \times Q \cup Q \times B)$$

forward with the map $u(x, y) = x - y$. Notice the analogy with the proof of Proposition 3.3. We do not describe the definition of the outer product here, as it is rather complicated and would take us too far away from the main topic.

Let us compute the intersection number by using two different pairs of representatives for $[\sigma_A]$ and $[\sigma_B]$. On one hand, since the images of the representatives σ_A and σ_B do not intersect, Propositions 4.5 and 4.6 of [4,VII] imply that $[\sigma_A] \circ [\sigma_B] = 0$. On the other hand, $[\sigma_A]$ and $[\sigma_B]$ admit representatives that are integer multiples of triangulations of the subspaces $J^k \times \{0\}$ and $\{0\} \times J^{n-k}$, so combining Proposition 4.5 and Example 4.10 of [4, VII] shows that $[\sigma_A] \circ [\sigma_B]$ is nontrivial. ■

4 Lower Bound and Related Open Problems

Theorems 1.1 and 3.1 raise the question:

Question 4.1. Do the lower bounds

$$1 \leq (\mathrm{mod}_p \Gamma_A)^{1/p} (\mathrm{mod}_q \Gamma_B)^{1/q} \tag{13}$$

or

$$1 \leq (\mathrm{mod}_p \Gamma_A)^{1/p} (\mathrm{mod}_q \Gamma_A^*)^{1/q} \tag{14}$$

hold whenever Ω, A, and B are as in Theorem 1.1?

Since $\Gamma_B \subset \Gamma_A^*$, (13) implies (14). Most existing proofs of such lower bounds rely on some variation of the coarea formula, Lemma 2.4. However, the proof in [6] is different. Alternative approaches can also be found in the discrete setting [3].

In [6] a lower bound is proved for de Rham cohomology classes. Hence it may be possible to answer Question 4.1 by finding a connection between the modulus of Γ_A (or Γ_B), which can be thought of as the modulus of a homology class, and the modulus of a suitable cohomology class. This is of course easier said than done. For instance, it is not very clear what "suitable cohomology" should mean, when Ω is nonsmooth. It seems these kinds of questions are still largely unexplored.

Let us sketch a proof of (14) in the special case $k = 1$. Then A consists of two opposite faces A_0 and A_1 of Ω and, recalling the notation from the introduction,

$$\mathrm{mod}_p \Gamma_A = \mathrm{mod}_p \Gamma(A_0, A_1; \Omega).$$

Moreover, by [20]

$$\mathrm{mod}_p \Gamma(A_0, A_1; \Omega) = \mathrm{cap}_p \Gamma(A_0, A_1; \Omega), \tag{15}$$

where the (Lipschitz) *capacity* is defined by

$$\mathrm{cap}_p \Gamma(A_0, A_1; \Omega) := \inf_u \int_\Omega |\nabla u|^p \, \mathrm{d}\mathcal{H}^n,$$

and the infimum is taken over Lipschitz functions $u : \Omega \to I$ with $u|_{A_0} = 0$ and $u|_{A_1} = 1$. Then by the coarea formula

$$1 \leq \int_I \int_{u^{-1}(t)} \rho \, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t = \int_\Omega \rho |\nabla u| \, \mathrm{d}\mathcal{H}^n$$

for any integrable ρ admissible for Γ_A^*, since by [5, 3.2.15] almost every level set $u^{-1}(t)$ is an element of Γ_A^*. Now the lower bound (14) follows from Hölder's inequality and (15).

Similar ideas can be used to prove that Theorems 1.1 and 3.1 are sharp for any n and k. Let us show that (13) holds whenever $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subset \mathbb{R}^k$ and $\Omega_2 \subset \mathbb{R}^{n-k}$ are k- and $(n-k)$-dimensional topological cubes as in Theorem 1.1, $A = \partial\Omega_1 \times \Omega_2$ and $B = \Omega_1 \times \partial\Omega_2$. Then it suffices to show that

$$\operatorname{mod}_p \Gamma_A = \frac{\mathcal{H}^{n-k}(\Omega_2)}{\mathcal{H}^k(\Omega_1)^{p-1}} \text{ and } \operatorname{mod}_q \Gamma_B = \frac{\mathcal{H}^k(\Omega_1)}{\mathcal{H}^{n-k}(\Omega_2)^{q-1}}.$$

The proofs of the two formulas are identical, so we only consider Γ_A. For every $y \in \Omega_2$ and ρ admissible for Γ_A

$$1 \leq \int_{\Omega_1 \times \{y\}} \rho \, \mathrm{d}\mathcal{H}^k,$$

so by Hölder's inequality

$$1 \leq \left(\int_{\Omega_1 \times \{y\}} \rho^p \, \mathrm{d}\mathcal{H}^k \right)^{1/p} \mathcal{H}^k(\Omega_1)^{1/q},$$

from which we obtain the inequality "$\geq$" by integrating over y and applying Fubini's theorem (or the coarea formula applied on the projection $\pi_2(x, y) = y$). The reverse inequality follows from the observation that $\mathcal{H}^k(\Omega_1)^{-1}\chi_\Omega$ is admissible for Γ_A.

It is also noteworthy that in this case $\operatorname{mod}_q \Gamma_B = \operatorname{mod}_q \Gamma_A^*$, and both are equal to the q-modulus of the slices $\{x\} \times \Omega_2$.

Observe that if we let $\lambda = \mathcal{H}^k(\Omega_1)^{-1/k}$ and use a scaled projection map $\lambda\pi_1(x, y) = \lambda x$ instead, we find that $\mathcal{H}^k(\lambda\pi_1(\Omega_1 \times \Omega_2)) = 1$ and $J_{\lambda\pi_1} = \mathcal{H}^k(\Omega_1)^{-1}\chi_\Omega$. That is, the minimizer of $\operatorname{mod}_p \Gamma_A$ is the Jacobian of $\lambda\pi_1$. Moreover, the level sets of $\lambda\pi_1$ are elements of Γ_B.

Inspired by this example we extend the definition of the capacity to general Ω and A by

$$\mathrm{cap}_p \Gamma_A := \inf_u \int_\Omega J_u^p \, \mathrm{d}\mathcal{H}^n,$$

where the infimum is taken over all such Lipschitz maps $u : (\Omega, A) \to (\bar{U}, \partial U)$, that U is a domain in $\mathbb{R}^k$ normalized with $\mathcal{H}^k(U) = 1$, $(\bar{U}, \partial U)$ is homeomorphic to $(\bar{\mathbb{B}}^k, \partial \mathbb{B}^k)$, and the induced homomorphism

$$u_* : H_k(\Omega, A) \to H_k(\bar{U}, \partial U) \simeq \mathbb{Z} \tag{16}$$

is an isomorphism. We observe that $U \subset u(S)$ for any $S \in \Gamma_A$, so almost every level set of u is in Γ_A^*, since $H_k(\bar{U} - \{x\}, \partial U)$ is trivial for all $x \in U$. Moreover, the Cauchy–Binet formula implies that $J_u \geq J_u^S$, so

$$\int_S J_u \, \mathrm{d}\mathcal{H}^k \geq \int_S J_u^S \, \mathrm{d}\mathcal{H}^k \geq \int_U \mathrm{d}\mathcal{H}^k = 1$$

by Lemma 2.4. Thus J_u is admissible for Γ_A and

$$\mathrm{mod}_p \Gamma_A \leq \mathrm{cap}_p \Gamma_A.$$

It is unknown whether the reverse inequality is true, but it would imply (14). To prove the reverse inequality one would have to be able to construct the required Lipschitz maps u. This seems to be very difficult when $k > 1$, especially with a given J_u. If $k = 1$, the situation is considerably simpler, since then $J_u = |\nabla u|$ and the unit interval I is practically the only choice of U.

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References

[1] Ahlfors, L. and A. Beurling. "Conformal invariants and function-theoretic null-sets." *Acta Math.* 83 (1950): 101–29. https://doi.org/10.1007/BF02392634.

[2] Ahlfors, L. and L. Sario. *Riemann Surfaces*. Princeton Mathematical Series, No. 26. Princeton University Press, Princeton, NJ, 1960.

[3] Albin, N., J. Clemens, N. Fernando, and P. Poggi-Corradini. "Blocking duality for p-modulus on networks and applications." *Ann. Mat. Pura Appl., (4)* 198, no. 3 (2019): 973–99, https://doi.org/10.1007/s10231-018-0806-0.

[4] Dold, A. *Lectures on Algebraic Topology*, 2nd ed. Grundlehren der Mathematischen Wissenschaften, vol. 200. Springer, Berlin–New York, 1980.

[5] Federer, H. *Geometric Measure Theory*. Die Grundlehren der Mathematischen Wissenschaften, Band 153. New York: Springer New York Inc., 1969.

[6] Freedman, M. and Z.-X. He. "Divergence-free fields: energy and asymptotic crossing number." *Ann. Math. (2)* 134, no. 1 (1991): 189–229.

[7] Fuglede, B. "Extremal length and functional completion." *Acta Math.* 98 (1957): 171–219. https://doi.org/10.1007/BF02404474.

[8] Gehring, F. "Extremal length definitions for the conformal capacity of rings in space." *Michigan Math. J.* 9 (1962): 137–50. https://10.1307/mmj/1028998672.

[9] Hatcher, A. *Algebraic Topology*. Cambridge: Cambridge University Press, 2002.

[10] Heinonen, J., P. Koskela, N. Shanmugalingam, and J. Tyson. *Sobolev Spaces on Metric Measure Spaces, an Approach Based on Upper Gradients*. New Mathematical Monographs, vol. 26. Cambridge University Press, Cambridge, 2015, https://doi.org/10.1017/CBO9781316135914.

[11] Heinonen, J. and J.-M. Wu. "Quasisymmetric nonparametrization and spaces associated with the Whitehead continuum." *Geom. Topol.* 14, no. 2 (2010): 773–98. https://doi.org/10.2140/gt.2010.14.773.

[12] Ikonen, T. "Uniformization of metric surfaces using isothermal coordinates." Preprint, arXiv:1909.09113.

[13] Jones, R. and P. Lahti. "Duality of moduli and quasiconformal mappings in metric spaces." *Anal. Geom. Metr. Spaces* 8, no. 1 (2020): 166–81. https://doi.org/10.1515/agms-2020-0112.

[14] Lohvansuu, A. "Duality of moduli in regular toroidal metric spaces." *Ann. Acad. Sci. Fenn. Math.*, in press.

[15] Lohvansuu, A. and K. Rajala. "Duality of moduli in regular metric spaces." *Indiana Univ. Math. J.*, in press.

[16] Pankka, P. and J.-M. Wu. "Geometry and quasisymmetric parametrization of Semmes spaces." *Rev. Mat. Iberoam.* 30, no. 3 (2014): 893–960. https://doi.org/10.4171/RMI/802.

[17] Rajala, K. "Uniformization of two-dimensional metric surfaces." *Invent. Math.* 207, no. 3 (2017): 1301–75. https://doi.org/10.1007/s00222-016-0686-0.

[18] Rickman, S. *Quasiregular Mappings*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 26. Springer, Berlin, 1993, https://doi.org/10.1007/978-3-642-78201-5.

[19] Riedweg, C. "Virtual flat chains and homologies in metric spaces." PhD thesis, Zürich, Switzerland: ETH Zürich, 2011.

[20] Shlyk, V. "On the equality between p-capacity and p-modulus." *Sibirsk. Mat. Zh.* 34, no. 6 (1993): 216–21, v, x.

[21] Simon, L. *Lectures on Geometric Measure Theory*. Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[22] Väisälä, J. *Lectures on n-Dimensional Quasiconformal Mappings*. Lecture Notes in Mathematics, vol. 229. Springer, Berlin–New York, 1971, https://doi.org/10.1007/BFb0061216.

[23] Zhang, Y. "Duality of capacities and Sobolev extendability in the plane." *Complex Anal. Synerg.*, in press.

[24] Ziemer, W. "Extremal length and conformal capacity." *Trans. Amer. Math. Soc.* 126 (1967): 460–73. https://doi.org/10.1090/S0002-9947-1967-0210891-0.

[25] Ziemer, W. "Extremal length and p-capacity." *Michigan Math. J.* 16 (1969): 43–51. https://doi.org/10.1307/mmj/1029000164.