

JYX



This is a self-archived version of an original article. This version may differ from the original in pagination and typographic details.

Author(s): Fässler, Katrin; Orponen, Tuomas

Title: A note on Keakeya sets of horizontal and $SL(2)$ lines

Year: 2023

Version: Published version

Copyright: © 2023 The Author(s). The Bulletin of the London Mathematical Society is copyright

Rights: CC BY 4.0

Rights url: <https://creativecommons.org/licenses/by/4.0/>

Please cite the original version:

Fässler, K., & Orponen, T. (2023). A note on Keakeya sets of horizontal and $SL(2)$ lines. Bulletin of the London Mathematical Society, 55(5), 2195-2204. <https://doi.org/10.1112/blms.12844>

RESEARCH ARTICLE

A note on Keakeya sets of horizontal and $SL(2)$ lines

Katrin Fässler | **Tuomas Orponen**Department of Mathematics and
Statistics, University of Jyväskylä,
Jyväskylä, Finland**Correspondence**Tuomas Orponen, Department of
Mathematics and Statistics, University of
Jyväskylä, P.O. Box 35 (MaD), FI-40014
Jyväskylä, Finland.
Email: tuomas.t.orponen@jyu.fi**Funding information**Academy of Finland, Grant/Award
Numbers: 321696, 321896**Abstract**We consider unions of $SL(2)$ lines in \mathbb{R}^3 . These are lines
of the form

$$L = (a, b, 0) + \text{span}(c, d, 1),$$

where $ad - bc = 1$. We show that if \mathcal{L} is a Keakeya set of
 $SL(2)$ lines, then the union $\cup \mathcal{L}$ has Hausdorff dimension
3. This answers a question of Wang and Zahl. The $SL(2)$
lines can be identified with *horizontal lines* in the first
Heisenberg group, and we obtain the main result as a
corollary of a more general statement concerning unions
of horizontal lines. This statement is established via a
point-line duality principle between horizontal and *con-*
ical lines in \mathbb{R}^3 , combined with recent work on *restricted*
families of projections to planes, due to Gan, Guo, Guth,
Harris, Maldague and Wang. Our result also has a corol-
lary for Nikodym sets associated with horizontal lines,
which answers a special case of a question of Kim.**MSC 2020**

28A78 (primary), 28A80 (secondary)

Contents

1. INTRODUCTION	2
1.1. Nikodym sets associated with horizontal lines	3
1.2. Ingredients of the proof.	5

2. PROOFS CONCERNING $SL(2)$ LINES 5
 3. PROOFS CONCERNING HORIZONTAL LINES 7
 ACKNOWLEDGEMENTS. 9
 REFERENCES. 9

1 | INTRODUCTION

The purpose of this note is to study the Hausdorff dimension of unions of $SL(2)$ lines in \mathbb{R}^3 . Here is the definition of $SL(2)$ lines, following [10].

Definition 1.1 ($\mathcal{L}_{SL(2)}$). The family $\mathcal{L}_{SL(2)}$ consists of the following lines $L \subset \mathbb{R}^3$. Either L is a line contained in the xy -plane, and $0 \in L$, or then

$$L := L_{\alpha,\beta,\gamma,\delta} := (\alpha, \beta, 0) + \text{span}(\gamma, \delta, 1),$$

where $\alpha\delta - \beta\gamma = 1$.

We also use the following notation. If \mathcal{L} is any family of lines in \mathbb{R}^3 , we write $\text{dir}(\mathcal{L}) := \{e \in S^2 : \ell \parallel \text{span}(e) \text{ for some } \ell \in \mathcal{L}\}$. Here is the main result of the note.

Theorem 1.2. *Let $\mathcal{L} \subset \mathcal{L}_{SL(2)}$ be a set with $\mathcal{H}^2(\text{dir}(\mathcal{L})) > 0$. Then*

$$\dim_{\mathbb{H}}(\cup \mathcal{L}) = 3.$$

Here ‘ $\dim_{\mathbb{H}}(\cup \mathcal{L})$ ’ is the Euclidean Hausdorff dimension of the union $\cup \mathcal{L} := \bigcup_{\ell \in \mathcal{L}} \ell$.

Remark 1.3. Theorem 1.2 answers a question posed by Wang and Zahl in [10, Section 1.2]. This question was motivated by earlier work of Katz and Zahl [5]. Theorem 1.2 continues to hold if the full lines in \mathcal{L} are replaced by line segments of positive length. We will discuss this briefly below (3.2).

Katz, Wu and Zahl [4] also proved Theorem 1.2 independently, using a different method.

The $SL(2)$ lines are essentially (up to a change in coordinates) the same as *horizontal lines in the first Heisenberg group* $\mathbb{H} = (\mathbb{R}^3, *)$, viewed as subsets of \mathbb{R}^3 (see Proposition 2.1). We will infer Theorem 1.2 from a more general statement concerning unions of these horizontal lines, Theorem 1.5 below. We first need to define the concepts properly.

The family of all horizontal lines is denoted by $\mathcal{L}(\mathbb{H})$. The ‘Heisenberg’ definition of these lines is the following. Let $\Pi_0 := \{(x, y, 0) : x, y \in \mathbb{R}\}$ be the xy -plane, and for $p \in \mathbb{R}^3$, let $\Pi_p := p * H_0$ be the *left translate* of Π_0 by the Heisenberg group product

$$(x, y, t) * (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y) \right).$$

Then, $\mathcal{L}(\mathbb{H})$ consists of all the lines in Π_p (for every $p \in \mathbb{R}^3$) which contain the point p .

The family $\mathcal{L}(\mathbb{H})$ is a three-dimensional submanifold of the full (four-dimensional) family of lines in \mathbb{R}^3 . In fact, the definition above of horizontal lines will not be used in the note: rather, we

focus attention on the following parameterised subset of $\mathcal{L}(\mathbb{H})$:

$$\mathcal{L}'(\mathbb{H}) = \{\ell_{(a,b,c)} : (a, b, c) \in \mathbb{R}^3\},$$

where

$$\ell_{(a,b,c)} = \left\{ (as + b, s, \frac{b}{2}s + c) : s \in \mathbb{R} \right\}.$$

The subset $\mathcal{L}'(\mathbb{H})$ consists of all elements of $\mathcal{L}(\mathbb{H})$, except for those contained in some translate of the plane $\mathbb{W}_0 := \{(x, 0, t) : x, t \in \mathbb{R}\}$. By definition, every set $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$ can be written as

$$\mathcal{L} = \ell(P) := \{\ell_{(a,b,c)} : (a, b, c) \in P\}$$

for some set $P \subset \mathbb{R}^3$. This identification of $\mathcal{L}'(\mathbb{H})$ with \mathbb{R}^3 allows us to transport notions like ‘Borel set’ and ‘dimension’ from \mathbb{R}^3 to corresponding notions for subsets of $\mathcal{L}'(\mathbb{H})$.

Definition 1.4. Let $\mathcal{L} = \ell(P) \subset \mathcal{L}'(\mathbb{H})$. We say that \mathcal{L} is a Borel set if $P \subset \mathbb{R}^3$ is a Borel set. We define $\dim_{\mathbb{H}} \mathcal{L} := \dim_{\mathbb{H}} P$, where ‘ $\dim_{\mathbb{H}} P$ ’ refers to the Euclidean Hausdorff dimension of $P \subset \mathbb{R}^3$.

Now we can state our main result about unions of horizontal lines.

Theorem 1.5. Let $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$. Then,

$$\dim_{\mathbb{H}}(\cup \mathcal{L}) = \min\{\dim_{\mathbb{H}} \mathcal{L} + 1, 3\}.$$

The following corollary implies Theorem 1.2, as we will verify in Section 2.

Corollary 1.6. Let $\mathcal{L} \subset \mathcal{L}(\mathbb{H})$ with $\mathcal{H}^2(\text{dir}(\mathcal{L})) > 0$. Then,

$$\dim_{\mathbb{H}}(\cup \mathcal{L}) = 3.$$

Remark 1.7. Theorem 1.5 and Corollary 1.6 continue to hold if full lines are replaced by line segments of positive length, see the discussion below (3.2). Thus, if $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$, and every line $\ell \in \mathcal{L}$ contains a segment $I(\ell) \subset \ell$ of positive length, then

$$\dim_{\mathbb{H}} \left(\bigcup_{\ell \in \mathcal{L}} I(\ell) \right) = \min\{\dim_{\mathbb{H}} \mathcal{L} + 1, 3\}. \tag{1.8}$$

1.1 | Nikodym sets associated with horizontal lines

Theorem 1.5 easily yields information about the dimension of *Nikodym sets* associated with horizontal lines. A set $N \subset \mathbb{R}^3$ is called an $\mathcal{L}(\mathbb{H})$ -*Nikodym set* if for every $p \in \mathbb{R}^3$ (or more generally every $p \in \mathbb{R}^3$ in a measurable set of positive measure $\Omega \subset \mathbb{R}^3$), there exists a line $\ell_p \in \mathcal{L}(\mathbb{H})$ containing p such that N contains a line segment $I_p \subset \ell_p$ of positive length.

Corollary 1.9. Every $\mathcal{L}(\mathbb{H})$ -*Nikodym set* $N \subset \mathbb{R}^3$ has $\dim_{\mathbb{H}} N = 3$.

It is well known that bounds for Kakeya sets yield bounds for Nikodym sets: we only repeat the standard details below for the reader's convenience. For a similar argument in the case of classical Kakeya and Nikodym sets, see [9, Section 11.3].

Proof of Corollary 1.9. We may assume without loss of generality that all the lines $\ell_p \in \mathcal{L}(\mathbb{H})$ appearing in the definition of 'N' lie in $\mathcal{L}'(\mathbb{H})$. Namely, if this is true for a positive measure subset of the points $p \in \Omega$, we simply replace Ω by that subset. If this fails for Lebesgue almost every point $p \in \Omega$, then we apply a rotation R of, say, 10° around the t -axis to the objects Ω , N , and the lines ℓ_p , $p \in \Omega$. Rotations around the t -axis preserve $\mathcal{L}(\mathbb{H})$, and the measure and dimension of Ω and N . After this procedure, we moreover have $\ell_p \in \mathcal{L}'(\mathbb{H})$ for a.e. $p \in R(\Omega)$.

Using Fubini's theorem, start by picking $y_0 \in \mathbb{R}$ such that $\mathcal{H}^2(\Omega \cap \mathbb{W}_{y_0}) > 0$. Here, $\mathbb{W}_y = \{(x, y, t) : x, t \in \mathbb{R}^3\}$ for $y \in \mathbb{R}$. By assumption, for every $p = (x, y_0, t) \in \Omega \cap \mathbb{W}_{y_0}$, there exists a line

$$\ell_p := \ell_{(a(p), b(p), c(p))} \in \mathcal{L}'(\mathbb{H})$$

containing p such that N contains a line segment $I_p \subset \ell_p$ of positive length.

Now, note that the map $(a, b, c) \mapsto \Psi(a, b, c) = (ay_0 + b, y_0, \frac{b}{2}y_0 + c)$ is Lipschitz, and

$$\Omega \cap \mathbb{W}_{y_0} \subset \Psi(\{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{y_0}\}).$$

(This is because the lines ℓ_p contain the points $p \in \Omega \cap \mathbb{W}_{y_0}$.) Therefore,

$$\dim_{\mathbb{H}}\{(a(p), b(p), c(p)) : p \in \Omega \cap \mathbb{W}_{y_0}\} \geq \dim_{\mathbb{H}}(\Omega \cap \mathbb{W}_{y_0}) = 2.$$

In particular, the set of lines $\mathcal{L} := \{\ell_p : p \in \Omega\} \subset \mathcal{L}'(\mathbb{H})$ has $\dim_{\mathbb{H}} \mathcal{L} \geq 2$ by definition. Therefore, it follows from Theorem 1.5, or to be precise (1.8), that

$$\dim_{\mathbb{H}} N \geq \dim_{\mathbb{H}} \left(\bigcup_{p \in \Omega} I_p \right) = 3.$$

This completes the proof. □

Remark 1.10. Nikodym set for 'restricted' families of lines was earlier considered by Kim [6]. Corollary 1.9 answers (a special case of) a question raised on [6, p. 478]. We elaborate on this a little further. The paper [6] considered general families of 2-planes $p \mapsto \Pi_a(p) \subset \mathbb{R}^3$, where $p \mapsto \mathbf{a}(p)$ is a non-vanishing measurable vector field, and

$$p \in \Pi_a(p) \quad \text{and} \quad \text{span}(\mathbf{a}(p)) = \Pi_a(p)^\perp.$$

One can associate Nikodym sets $N \subset \mathbb{R}^3$ to such a plane family, as follows: for every $p \in \mathbb{R}^3$, the requirement is that there exists a line $\ell_p \subset \mathbb{R}^3$ satisfying

$$p \in \ell_p \subset \Pi_a(p),$$

and a non-trivial segment $I_p \subset N \cap \ell_p$. How small can such a Nikodym set $N \subset \mathbb{R}^3$ be? In [6], Kim approached the question via maximal function estimates, and his results depend on the properties of the vector field \mathbf{a} . Kim considered vector fields \mathbf{a} of the form

$$\mathbf{a}(p) = (a_{11}p_1 + a_{21}p_2, a_{12}p_1 + a_{22}p_2, -1), \quad p = (p_1, p_2, p_3) \in \mathbb{R}^3,$$

and defined the ‘discriminant’ $D_{\mathbf{a}} = (a_{12} + a_{21})^2 - 4a_{11}a_{22}$. In [6, Corollary 1, p. 478], it was shown that the dimension of N equals 3 if $D_{\mathbf{a}} \neq 0$. Right after the corollary, the question is raised, what happens in the situation $D_{\mathbf{a}} = 0$.

Now, recall the definition of horizontal lines $\mathcal{L}(\mathbb{H})$: these were the lines contained in the planes $\Pi_p = p * \Pi_0$, and passing through p . The planes Π_p fit in the framework of [6], choosing $\mathbf{a}(p) = (-p_2/2, p_1/2, -1)$, or $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$. In particular, $D_{\mathbf{a}} = 0$. Also, the $\mathcal{L}(\mathbb{H})$ -Nikodym sets defined above Corollary 1.9 are the same as the Nikodym sets of [6] associated with the planes $\Pi_p = p * \Pi_0$. Thus, Corollary 1.9 covers the special case $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, \frac{1}{2}, -\frac{1}{2}, 0)$ of the problem raised on [6, p. 478].

1.2 | Ingredients of the proof

The proof of Theorem 1.5 is based on two ingredients. The first one is a *point-line duality* between horizontal lines and *conical lines* in \mathbb{R}^3 , namely translates of lines contained in the light cone $\{(x, y, t) : t^2 = x^2 + y^2\}$. This duality was formalised in our paper [2], although it was already implicit in the work [7] of Liu. Using this point-line duality, Kakeya-type problems for horizontal lines can be transformed into projection problems in \mathbb{R}^3 . These projection problems concern ‘restricted’ families of projections to planes in \mathbb{R}^3 . Sharp results for such families were recently established by Gan, Guo, Guth, Harris, Maldague and Wang [3]. This is the second key component in the proof of Theorem 1.5.

2 | PROOFS CONCERNING $SL(2)$ LINES

In this section, we formalise the connection between $SL(2)$ lines and horizontal lines. We also deduce our main result, Theorem 1.2, from Corollary 1.6.

Recall the $SL(2)$ lines from Definition 1.1. We write $\mathcal{L}'_{SL(2)}$ for all the lines in $\mathcal{L}_{SL(2)}$, except for the x -axis, and lines of the form $L_{\alpha,\beta,\gamma,\delta}$ with $\delta = 0$. The difference between $\mathcal{L}_{SL(2)}$ and $\mathcal{L}'_{SL(2)}$ is the same as the difference between $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}'(\mathbb{H})$. Consider the map

$$\Xi(x, y, t) := (x, y, t/2).$$

We claim that Ξ maps the $SL(2)$ lines to horizontal lines. More precisely:

Proposition 2.1. *If $L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}'_{SL(2)}$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$, then*

$$\Xi(L_{\alpha,\beta,\gamma,\delta}) = \ell_{(a,b,c)} \in \mathcal{L}'(\mathbb{H}), \tag{2.2}$$

where

$$\begin{cases} a = \gamma/\delta, \\ b = 1/\delta, \\ c = -\beta/(2\delta). \end{cases}$$

Proof. Fix $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$. Write $L_{\alpha, \beta, \gamma, \delta}(s) = (\alpha, \beta, 0) + (s\gamma, s\delta, s)$. It is a straightforward computation to check that

$$\Xi(L_{\alpha, \beta, \gamma, \delta}(s)) = \ell_{(a, b, c)}(\beta + s\delta), \quad s \in \mathbb{R}.$$

Since $\delta \neq 0$ by assumption, this completes the proof. \square

We are then prepared to prove Theorem 1.2.

Proof of Theorem 1.2. We may assume that $\mathcal{L} \subset \mathcal{L}'_{SL(2)}$, since the directions of the lines in $\mathcal{L}_{SL(2)} \setminus \mathcal{L}'_{SL(2)}$ are contained in the \mathcal{H}^2 null set $S^2 \cap \{(x, 0, t) : x, t \in \mathbb{R}\}$. Similarly, we may assume that \mathcal{L} contains no lines in the xy -plane; thus, every $L \in \mathcal{L}$ has the form $L = L_{\alpha, \beta, \gamma, \delta}$ for some $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$ and $\alpha\delta - \beta\gamma = 1$.

Since $\mathcal{L} \subset \mathcal{L}'_{SL(2)}$, we infer from Proposition 2.1 that $\Xi(\mathcal{L}) := \{\Xi(\ell) : \ell \in \mathcal{L}\} \subset \mathcal{L}'(\mathbb{H})$. We claim that

$$\mathcal{H}^2(\text{dir}(\Xi(\mathcal{L}))) > 0. \quad (2.3)$$

According to Corollary 1.6, this will imply that

$$\dim_{\mathbb{H}}(\cup \mathcal{L}) = \dim_{\mathbb{H}} \Xi(\cup \mathcal{L}) = \dim_{\mathbb{H}}(\cup \Xi(\mathcal{L})) = 3,$$

and complete the proof.

To verify (2.3), fix $L = L_{\alpha, \beta, \gamma, \delta} \in \mathcal{L}$. Then, by (2.2), we have $\Xi(L) = \ell_{(a, b, c)}$ with

$$\begin{cases} a = \gamma/\delta, \\ b = 1/\delta \\ c = -\beta/(2\delta). \end{cases}$$

We will use this information in the form of the following inclusion: writing $F(\gamma, \delta) := (\gamma/\delta, 1/\delta)$, we have

$$\{(a, b) \in \mathbb{R}^2 : \ell_{(a, b, c)} \in \Xi(\mathcal{L})\} \supset \{F(\gamma, \delta) : L_{\alpha, \beta, \gamma, \delta} \in \mathcal{L}\}. \quad (2.4)$$

Since $\mathcal{H}^2(\text{dir}(\mathcal{L})) > 0$, and the direction of $L_{\alpha, \beta, \gamma, \delta} = (\alpha, \beta, 0) + \text{span}(\gamma, \delta, 1)$ is determined by γ and δ , we know that

$$\mathcal{H}^2(\{(\gamma, \delta) \in \mathbb{R}^2 : L_{\alpha, \beta, \gamma, \delta} \in \mathcal{L}\}) > 0.$$

It now follows from (2.4), and the fact that F is locally bilipschitz in the set $\mathbb{R}^2 \setminus \{(\gamma, \delta) : \delta = 0\}$ (since $|\det DF(\gamma, \delta)| = 1/\delta^3$), that also

$$\mathcal{H}^2(\{(a, b) \in \mathbb{R}^2 : \ell_{(a,b,c)} \in \Xi(\mathcal{L})\}) \geq \mathcal{H}^2(\{F(\gamma, \delta) : L_{\alpha,\beta,\gamma,\delta} \in \mathcal{L}\}) > 0.$$

Since the direction of $\ell_{(a,b,c)} = (b, 0, c) + \text{span}(a, 1, b/2)$ is determined by (a, b) , we may now infer that $\mathcal{H}^2(\text{dir}(\Xi(\mathcal{L}))) > 0$, as claimed in (2.3). □

3 | PROOFS CONCERNING HORIZONTAL LINES

We start by proving Theorem 1.5.

Proof of Theorem 1.5. Without loss of generality, we may assume that $\mathcal{L} = \ell(P)$ is a Borel set of lines, that is, $P \subset \mathbb{R}^3$ is a Borel set. For the full details of this reduction, see [7, Section 3] or [1, Theorem 7.9]. The idea is that we can first replace $\cup \mathcal{L}$ by a G_δ -set $G \supset \cup \mathcal{L}$ without affecting $\dim_{\mathbb{H}}(\cup \mathcal{L})$. Then, it is easy to check that the set of parameters $P' := \{p \in \mathbb{R}^3 : \ell(p) \subset G\}$ is a Borel set with $P' \supset P$, in particular $\dim_{\mathbb{H}} P' \geq \dim_{\mathbb{H}} P$. Finally, writing $\mathcal{L}' := \ell(P')$, we have

$$\dim_{\mathbb{H}}(\cup \mathcal{L}) = \dim_{\mathbb{H}} G \geq \dim_{\mathbb{H}}(\cup \mathcal{L}').$$

So, if the result is known for Borel sets of lines, it follows for \mathcal{L} .

Write $\mathcal{L} := \ell(P)$, where $P \subset \mathbb{R}^3$ is Borel. Write also

$$K_y := \left\{ \left(ay + b, \frac{b}{2}y + c \right) : (a, b, c) \in P \right\}, \quad y \in \mathbb{R},$$

and note that K_y is a ‘slice’ of $\cup \mathcal{L}$ with the plane $\mathbb{W}_y := \{(x, y, t) : x, t \in \mathbb{R}\}$:

$$(\cup \mathcal{L}) \cap \mathbb{W}_y \cong K_y,$$

where ‘ \cong ’ refers to the isometry $\iota_y : \mathbb{R}^2 \rightarrow \mathbb{W}_y$, defined by $\iota_y(x, t) = (x, y, t)$. In order to prove that

$$\dim_{\mathbb{H}}(\cup \mathcal{L}) \geq \min\{\dim_{\mathbb{H}} \mathcal{L} + 1, 3\}, \tag{3.1}$$

we now claim that

$$\dim_{\mathbb{H}} K_y = \min\{\dim_{\mathbb{H}} P, 2\} \quad \text{for a.e. } y \in \mathbb{R}. \tag{3.2}$$

If \mathcal{L} consisted of line segments of positive length, and not full lines, then we would have to modify (3.2) as follows: for every $\epsilon > 0$, there exists an interval $I \subset \mathbb{R}$ of positive length such that $\dim_{\mathbb{H}} K_y \geq \min\{\dim_{\mathbb{H}} P - \epsilon, 2\}$ for a.e. $y \in I$. This interval would (be chosen to) consist of points $y \in \mathbb{R}$ with the property that the plane \mathbb{W}_y intersects a family of segments corresponding to a $(\dim_{\mathbb{H}} P - \epsilon)$ -dimensional Borel subset $P' \subset P$. We refer the reader to [7, Section 3] for a very similar argument.

Clearly, (3.1) follows from (3.2) by the ‘Fubini inequality’ for Hausdorff measures (hence dimension), see [1, Theorem 5.8] or [8, Theorem 7.7]. To prove (3.2), we define

$$v(y) := (y, 1, 0) \quad \text{and} \quad w(y) := (0, y/2, 1), \quad y \in \mathbb{R}.$$

Then, we note that for $y \in \mathbb{R}$ fixed, K_y can be expressed as

$$\begin{aligned} K_y &= \{(\langle p, v(y) \rangle, \langle p, w(y) \rangle) : p \in P\} \\ &= \{(\langle \pi_{V_y}(p), v(y) \rangle, \langle \pi_{V_y}(p), w(y) \rangle) : p \in P\}, \end{aligned} \quad (3.3)$$

where ‘ $\langle \cdot, \cdot \rangle$ ’ is the Euclidean dot product and π_{V_y} the Euclidean orthogonal projection from \mathbb{R}^3 onto the plane

$$V_y := \text{span}(\{v(y), w(y)\}).$$

It is then easy to see that

$$\dim_{\mathbb{H}} K_y = \dim_{\mathbb{H}} \pi_{V_y}(P), \quad y \in \mathbb{R}. \quad (3.4)$$

Indeed, expression (3.3) shows that K_y can be written as the image of $\pi_{V_y}(P)$ under the linear map

$$M_y : V_y \rightarrow \mathbb{R}^2, \quad M_y(q) = (\langle q, v(y) \rangle, \langle q, w(y) \rangle),$$

and thus, $\dim_{\mathbb{H}} K_y = \dim_{\mathbb{H}} M_y(\pi_{V_y}(P))$. Moreover, $\dim_{\mathbb{H}} M_y(\pi_{V_y}(P)) = \dim_{\mathbb{H}} \pi_{V_y}(P)$ holds as the linear map M_y is invertible by the linear independence of $v(y)$ and $w(y)$. Hence, (3.4) holds as desired.

To complete the proof, we claim that

$$\dim_{\mathbb{H}} \pi_{V_y}(P) = \min\{\dim_{\mathbb{H}} P, 2\} \quad \text{for a.e. } y \in \mathbb{R}. \quad (3.5)$$

The idea is that $\{\pi_{V_y}\}_{y \in \mathbb{R}}$ is a one-parameter family of orthogonal projections to planes in \mathbb{R}^3 which satisfies the hypotheses of [3, Corollary 1].

Which planes are the planes V_y ? Note that

$$v(y) \times w(y) = (1, -y, y^2/2) =: e_y.$$

Thus, $V_y = e_y^\perp$. Moreover, the lines $\ell_y := \text{span}(e_y)$ are all contained in a 45° rotated copy of the light cone

$$C := \{(x, y, t) \in \mathbb{R}^3 : t^2 = x^2 + y^2\},$$

see [2, Section 2.2] for the details. This implies that the projections $\{\pi_{V_y}\}_{y \in \mathbb{R}}$ satisfy the curvature condition [3, (1)]. In fact, up to the rotation by 45° , this family of projections is precisely the ‘model

example' mentioned just below [3, (1)]. Therefore, (3.5) follows from [3, Corollary 1], and the proof is complete. \square

We conclude the paper by proving Corollary 1.6.

Proof of Corollary 1.6. Firstly, note that $\mathcal{H}^2(\text{dir}(\mathcal{L} \cap \mathcal{L}'(\mathbb{H}))) > 0$. This is because $\text{dir}(\mathcal{L}'(\mathbb{H}))$ contains all the directions on S^2 , except for those contained in the null set $\{(x, 0, t) : x, t \in \mathbb{R}\}$. Therefore, we may assume that $\mathcal{L} \subset \mathcal{L}'(\mathbb{H})$.

Write $\mathcal{L} = \ell(P)$, where $P \subset \mathbb{R}^3$. Recall that

$$\begin{aligned} \mathcal{L} = \ell(P) &= \left\{ \left(as + b, s, \frac{b}{2}s + c \right) : s \in \mathbb{R}, (a, b, c) \in P \right\} \\ &= \left\{ (b, 0, c) + \text{span} \left(a, 1, \frac{b}{2} \right) : (a, b, c) \in P \right\}. \end{aligned}$$

Since $\mathcal{H}^2(\text{dir}(\mathcal{L})) > 0$ by assumption, we see that

$$\mathcal{H}^2 \left(\left\{ \left(a, \frac{b}{2} \right) : (a, b, c) \in P \right\} \right) > 0,$$

and consequently, $\dim_{\mathbb{H}} P \geq 2$. The claim now follows from Theorem 1.5. \square

ACKNOWLEDGEMENTS

We are grateful to the anonymous reviewer for reading the paper carefully, and providing us with many helpful comments.

K.F. is supported by the Academy of Finland via the project *Singular integrals, harmonic functions, and boundary regularity in Heisenberg groups*, grant No. 321696. T.O. is supported by the Academy of Finland via the project *Incidences on Fractals*, grant No. 321896.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. K. J. Falconer, *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1986.
2. K. Fässler and T. Orponen, *Vertical projections in the Heisenberg group via point-plate incidences*, arXiv e-prints, arXiv:2210.00458, October 2022.
3. S. Gan, S. Guo, L. Guth, T. L. J. Harris, D. Maldague, and H. Wang, *On restricted projections to planes in \mathbb{R}^3* , arXiv e-prints, arXiv:2207.13844, July 2022.
4. N. Katz, S. Wu, and J. Zahl, *Kakeya sets from lines in SL_2* , arXiv e-prints, arXiv:2211.05194, November 2022.
5. N. H. Katz and J. Zahl, *An improved bound on the Hausdorff dimension of Besicovitch sets in \mathbb{R}^3* , *J. Amer. Math. Soc.* **32** (2019), no. 1, 195–259.
6. J. Kim, *Nikodym maximal functions associated with variable planes in \mathbb{R}^3* , *Integral Equations Operator Theory* **73** (2012), no. 4, 455–480.

7. J. Liu, *On the dimension of Kakeya sets in the first Heisenberg group*, Proc. Amer. Math. Soc. **150** (2022), no. 8, 3445–3455.
8. P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, 1st paperback ed., Cambridge University Press, Cambridge, 1999.
9. P. Mattila, *Fourier analysis and Hausdorff dimension*, vol. 150 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2015.
10. H. Wang and J. Zahl, *Sticky Kakeya sets and the sticky Kakeya conjecture*, arXiv e-prints, arXiv:2210.09581, October 2022.