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Systematic Derivation of Partial Differential Equations for Second Order Boundary Value Problems

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Summary

Software systems designed to solve second order boundary value problems are typically restricted to hardwired lists of partial differential equations. In order to come up with more flexible systems, we introduce a systematic approach to find partial differential equations that result in eligible boundary value problems. This enables one to construct and combine one's own partial differential equations instead of choosing those them from a pre-given list expanding significantly end users possibilities to employ boundary value problems in modelling. To introduce the main ideas we employ differential geometry to examine the mathematical structure involved in second order boundary value problems and exploit electromagnetism as a working example. This provides us with an organized view on the key building blocks behind boundary value problems. Thereafter the approach is naturally generalized to a class of second order boundary value problems that covers field theories from statics to wave problems. As a result, we obtain a systematic framework to construct partial differential equations and to test whether they form eligible boundary value problems.

KEYWORDS:
partial differential equations, action principle, differential forms, product, coproduct, category

1 | INTRODUCTION

Flexibility is a topical issue in computing as the modern needs in engineering design call for solving multi-physical problems. The existing software systems, however, typically do not allow the users to choose their own partial differential equations (PDEs) to be solved within boundary value problems (BVP). Instead, the available PDEs are typically restricted to a pre-given list. While some very nice exceptions, such as GetDP [1], exist, still, there is no underlying framework that guided which pairs of PDEs form well-posed boundary value problems at the first place.

Historically, the finite element (FEM) kind of systems haven arisen from specific needs such as those of elasticity or electromagnetism. As the resources of the early computers were minimal, an apparent approach was to design FEM systems for specific needs. By hardwiring the partial differential equations into the software, it became also easier to prevent the users from establishing inadvertently ill-posed problems.

A modern view on programming, however, does not seek for developing distinct software for every specific problem, but rather, to cover as many demands as possible with a single system as long as efficiency, clarity, and plainness can be maintained. In this paper we are after a framework that enabled to choose "all" pairs of PDEs that compose well-posed second order boundary value problems –"all" should be understood here with respect to some set or class that is larger than a finite set.

It is plain all the mathematical knowledge required to meet our goal must already exist. The issue is rather to reorganize the required pieces of knowledge. Field theories are studied in depth and the literature is vast, but they are also presented in many ways. The mixture of interpretations make it challenging to find a view that condensed into formal terms what the seemingly different theories have actually in common.

[0] **Abbreviations:** BVP, boundary value problem; PDE, partial differential equation

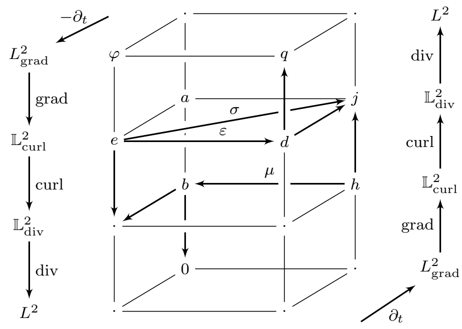

FIGURE 1 Maxwell's house, a functional framework for Maxwell's equations.[10]

The endeavor to compress information is also the key to design apt software. In designing multifunctional software one should also take into account there is always a trade-off between flexibility and power[2]. Hence, it is also justified to ask whether the pursued flexibility would result in an inefficient software system. The example of GetDP, however, shows such concerns can be overcame; It is the solution process and not the setup of system equations that requires computing resources. It is plain, if one tried to come up with an all-purpose solver for systems of equations the trade-off between flexibility and power would pop up.

2 | HODGE DECOMPOSITIONS AND MAXWELL'S HOUSE

Let us start from the literature and the structural properties of electromagnetism. Some of the pioneering works in this field are Kotiuga's PhD-thesis[3], Tonti diagrams[4], and Bossavit's study on the curl operator[5]. The mathematical background of these studies lies in the generalization of classical Helmholtz decompositions[6] to Hilbert and Sobolev spaces on bounded domains. Such generalizations are known as Hodge-Kodaira decompositions[7,8,9]. Kotiuga, Tonti and Bossavit exploited Hodge decompositions to the needs of electromagnetic modelling, and Bossavit conveyed the outcome graphically aptly to what he calls by the name Maxwell's house[10], see Fig. 1.

The Hilbert space $\mathbb{L}^2(\Omega)$ consists of square integrable fields supported in domain Ω. When it comes to electromagnetism the underlying inner product has to do with energy or power such as with the energy stored in the electric field, i.e., with $\frac{1}{2}\int_\Omega \varepsilon\, e \cdot e$. Accordingly, the Hodge decomposition can be employed to find mutually orthogonal components of energy or power. The decompositions provide one also with the de Rham complex[11,12]. Hodge decompositions can also be employed to construct systems of equations for field problems as demonstrated in ref.[13].

While the Maxwell house condenses nicely the information of Maxwell's equations –the value at a node is the sum of its incoming arrows– the arrows in the Maxwell house are not meant to be composed: If the map $e \xrightarrow{\varepsilon} d$ is joint from the right with $d \xrightarrow{\partial_t} j$, the composition did not coincide with map $e \xrightarrow{\sigma} j$. This implies the Maxwell house is a flow graph. To express the framework of Maxwell's theory with arrow compositions we need to shift to categories. For, by definition, the arrows, i.e., the so-called morphisms of a category[14,15] have to fulfil the composition law. Consequently, if the arrows represent functions, then their compositions are also granted.

The Maxwell house is not related to any particular electromagnetic boundary value problem, but rather, it conveys information that should hold for all electromagnetic problems. To maintain such an idea we need to consider the collection of all the function spaces associated with electromagnetic fields supported in all possible domains. Functions spaces are sets, but such a collection of functions spaces is hardly a set. Rather, it is a class. For this reason we aim for categories whose objects are classes of function spaces. This makes it possible to make statements that hold for all electromagnetic problems.

To summarise, our goal is to come up with a diagram that aptly condensed the information of electromagnetic boundary value problems, but which is simultaneously a category. The Hodge decompositions yield a firm background for this. Especially so, as the Hodge decompositions are closely related to the action principle[16,17], which in turn is the key tool to construct PDEs for various needs. As Baez and Munian writes[17]: "In modern physics one rarely starts with differential equations for fields; rather, one derives them from a Lagrangian", that is, from a weak form with the action principle. In addition, we also notice that the Maxwell house consists of a dual pair of de Rham complexes, where the grad, curl, and div operators and their adjoints are proxies of the exterior operator and its adjoint, respectively. The exterior derivative and its adjoint are mappings between differential forms of degree $p = 0, \ldots, 3$. However, to derive PDEs neither the dimension of the domain (i.e., of the manifold) nor the degrees of forms are in our primal interest. We are rather interested in all (meaningful) dimensions and in forms of all degrees. This suggests to take a step backwards from the Maxwell house and to rebuild the desired framework starting from the Hodge operator, Hodge decompositions, and the action principle.

3 | HODGE OPERATOR AND THE CONSTITUTIVE LAWS

Let n stand for the dimension. The Hodge operator is a map between p- and $(n-p)$-vector spaces. In electromagnetism it pops up in the constitutive laws. We would like to stress out the constitutive laws are strictly local. That is, they are maps between cotangent spaces [18,19], that is, between fibers over a point of a manifold, and the Hodge map over a point has no effect on the fibers over its neighbouring points.

The $\star$-operator is defined by Riesz representation theorem [20,18]: Let V_p denote the space of p-vectors. The unit n-vector –the so called "canonical volume"– is $\omega_n \in V_n$, and $\langle \cdot, \cdot \rangle$ is the inner product of p-vectors. The Hodge operator is the map $\star : V_p \to V_{n-p}, v \mapsto \star v$ such that for all $u \in V_p$ condition

$$u \wedge \star v = \omega_n \langle u, v \rangle \tag{1}$$

holds. For an elementary example, see fig. 2.

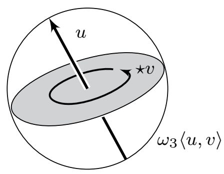

FIGURE 2 An elementary geometric example of the $\star$ In 3d euclidean space.

To express constitutive laws as a commutative diagram we employ notation $(u, \cdot)_{u \in V}$ to say argument u goes through all elements of V and then introduce maps[1]

$$h_p : \{V_p\} \times V_p \to \{V_n\},\ (v', v)_{v' \in V_p} \mapsto \omega_n \langle v', v \rangle \quad \text{and} \quad h_{n-p} : \{V_{n-p}\} \times V_{n-p} \to \{V_n\},\ (w', w)_{w' \in V_{n-p}} \mapsto \omega_n \langle w', w \rangle.$$

Let the disjoint union be denoted by $\sqcup$. If we introduce inclusions

$$i_p : \{V_p\} \times V_p \ \to\ \{V_p\} \times V_p \sqcup \{V_{n-p}\} \times V_{n-p} \quad \text{and} \quad i_{n-p} : \{V_{n-p}\} \times V_{n-p} \ \to\ \{V_p\} \times V_p \sqcup \{V_{n-p}\} \times V_{n-p},$$

then map

$$h : \{V_p\} \times V_p \sqcup \{V_{n-p}\} \times V_{n-p} \to \{V_n\},\ \begin{cases} i_p(v', v)_{v' \in V_p} & \mapsto v' \wedge \star v \\ i_{n-p}(w', w)_{w' \in V_{n-p}} & \mapsto w' \wedge \star w\,, \end{cases}$$

factorizes out the common part of maps h_p and h_{n-p}. This is to say that

$$h_p = h \circ i_p \quad \text{and} \quad h_{n-p} = h \circ i_{n-p}$$

hold. In terms of function values such a commutation property corresponds with equations

$$v' \wedge \star v = \omega_n \langle v', v \rangle \quad \forall v' \in V_p \quad \text{and} \quad w' \wedge \star w = \omega_n \langle w', w \rangle \quad \forall w' \in V_{n-p},$$

see fig. 3. This construction is a coproduct, see ref. [21,22].

$$\begin{tikzcd}
& \{V_n\} & \\
\{V_p\} \times V_p \arrow[ur, "h_p : (v', v)_{v' \in V_p} \mapsto \omega_n \langle v', v \rangle"] \arrow[r, "i_p"] & \{V_p\} \times V_p \sqcup \{V_{n-p}\} \times V_{n-p} \arrow[u, "h : (v', v)_{v' \in V_p} \mapsto v' \wedge \star v,\ (w', w)_{w' \in V_{n-p}} \mapsto w' \wedge \star w"'] & \{V_{n-p}\} \times V_{n-p} \arrow[ul, "h_{n-p} : (w', w)_{w' \in V_{n-p}} \mapsto \omega_n \langle w', w \rangle"'] \arrow[l, "i_{n-p}"']
\end{tikzcd}$$

FIGURE 3 Structure of the constitutive law expressed as a coproduct.

[1] Symbol $\{V_p\}$ stands for a set whose only element is the space of p-vectors. The reason we need V_p as an element of a set lies in the Riesz representation theorem: eq. (1) should hold for all elements u in V_p.

Remark: The constitutive laws involve also the structural layer of orientation, for details see[23,24,25]. For brevity, in this paper we will bypass this.

Example: The dielectric constitutive law $d = \varepsilon \star e$. Locally, on ordinary points x of the domain, the 1-covector electric field e is an element of cotangent space T_x^1. Correspondingly, the electric flux density $d \in T_x^{n-1}$. (If orientation is taken into account, d is recognized as a so-called twisted 2-covector[26,27,23].) The inner product for the 1-covectors and for the $(n-1)$-covectors can be given by $\langle v', \varepsilon v\rangle$ and $\langle w', \frac{1}{\varepsilon} w\rangle$, respectively. The constitutive laws $d = \varepsilon \star e$ and $e = \frac{1}{\varepsilon} \star d$ correspond with the commutative diagram of fig. 4.

$$\begin{tikzcd}
 & \{T_x^n\} & \\
\{T_x^1\} \times T_x^1 \arrow[ur, "(e',e)_{e' \in T_x^1} \mapsto \omega^n \langle e', \varepsilon e\rangle"] \arrow[r, "\imath_p"] & \{T_x^1\} \times T_x^1 \sqcup \{T_x^{n-1}\} \times T_x^{n-1} \arrow[u, "(e',e)_{e' \in T_x^1} \mapsto e' \wedge \varepsilon \star e,\ (d',d)_{d' \in T_x^{n-1}} \mapsto d' \wedge \frac{1}{\varepsilon} \star d"'] & \{T_x^{n-1}\} \times T_x^{n-1} \arrow[l, "\imath_{n-p}"'] \arrow[ul, "(d',d)_{d' \in T_x^{n-1}} \mapsto \omega^n \langle d', \frac{1}{\varepsilon} d\rangle"']
\end{tikzcd}$$

FIGURE 4 A coproduct that corresponds with the equations of dielectric constitutive law.

4 | HODGE-KODAIRA DECOMPOSITIONS

Let us now move to Hodge-Kodaira decompositions. For this, we first assume PDEs that are mappings between p and $(p+1)$-forms, $F^p(\Omega) \xrightarrow{\mathrm{d}} F^{p+1}(\Omega)$. We assume a sufficiently smooth $f \in F^p(\Omega)$ equipped with boundary condition $\mathrm{t} f = \mathrm{t} f_f$ on the component $\partial_f \Omega$ of boundary $\partial\Omega$.

To maintain the linearity of the exterior derivative, we restrict d to act only on those functions whose trace t vanish on $\partial_f\Omega$. We denote such a restriction of d by d_f:

$$\mathrm{dom}(\mathrm{d}_f) = \{f \in \mathrm{dom}(\mathrm{d}) \mid \ \mathrm{t} f = 0 \text{ on } \partial_f \Omega\}.$$

Symmetrically, let $\partial_g\Omega$, which is the complement of $\partial_f\Omega$ on the boundary, $\partial\Omega = \partial_f\Omega + \partial_g\Omega$. Then

$$\mathrm{dom}(\mathrm{d}_g) = \{g \in \mathrm{dom}(\mathrm{d}) \mid \ \mathrm{t} g = 0 \text{ on } \partial_g \Omega\}$$

is the restriction of d on $\partial_g\Omega$.

The Stokes' theorem implies for any $a' \in F^{p-1}(\Omega)$ and $f \in F^p(\Omega)$

$$\int_\Omega \mathrm{d}(a' \wedge \star f) = \int_{\partial\Omega} \mathrm{t}(a' \wedge \star f)\,.$$

Integrating this by parts yields

$$\int_\Omega \mathrm{d}a' \wedge \star f - (-1)^p \int_\Omega a' \wedge \mathrm{d}\star f = \int_{\partial\Omega} \mathrm{t}a' \wedge \mathrm{t}\star f.$$

As $\star\star = (-1)^{p(n-p)}$, this is equivalent to

$$\int_\Omega \mathrm{d}a' \wedge \star f - (-1)^k \int_\Omega a' \wedge \star(\star \mathrm{d} \star f) = \int_{\partial\Omega} \mathrm{t}a' \wedge \mathrm{t}\star f. \tag{2}$$

where $k = p + (p-1)(n-p+1)$. This is Green's formula.

Now, notice, if $(p-1)$-form $a' \in \mathrm{dom}(\mathrm{d}_f)$ and $\star f \in \mathrm{dom}(\mathrm{d}_g)$, then we have

$$\langle \mathrm{d}_f a', f\rangle = (-1)^k \langle a', \star \mathrm{d}_g \star f\rangle,$$

which is to say, $\delta_g = (-1)^k \star \mathrm{d}_g \star$ is the adjoint[20] of d_f.

This provides us with the Hodge-Kodaira decompositions[28] of the Hilbert spaces $F^{p-1}(\Omega)$ and $F^p(\Omega)$: Firstly, the kernel of d_f is $\ker(\mathrm{d}_f) = \{a \in F^{p-1}(\Omega), \mathrm{d}_f a = 0\}$, and its orthogonal complement is $\ker(\mathrm{d}_f)^\perp = \{a \in F^{p-1}(\Omega),\ \langle a, a'\rangle = 0\,\forall a' \in \ker(\mathrm{d}_f)\}$. Secondly, as δ_g is the adjoint of d_f, the complement $\ker(\mathrm{d}_f)^\perp$, coincides with $\mathrm{cod}(\delta_g)$.[20] Symmetrically, the orthogonal complement of $\ker(\delta_g) \subset F^p(\Omega)$ is $\mathrm{cod}(\mathrm{d}_f)$, and hence, we may write

$$\begin{aligned} F^{p-1}(\Omega) \ &= \ \ker(\mathrm{d}_f) \ \oplus \ \mathrm{cod}(\delta_g), \\ F^p(\Omega) \ &= \ \ker(\delta_g) \ \oplus \ \mathrm{cod}(\mathrm{d}_f). \end{aligned}$$

Next, to express the decomposition $F^p(\Omega) = \ker(\mathrm{d}_f) \oplus \mathrm{cod}(\delta_g)$ with a commutative diagram we introduce projections

$$\begin{aligned} \pi_f &: \ker(\mathrm{d}_f) \ \oplus \ \mathrm{cod}(\delta_g) \ \to \ \ker(\mathrm{d}_f),\ (f,g) \mapsto f, \\ \pi_g &: \ker(\mathrm{d}_f) \ \oplus \ \mathrm{cod}(\delta_g) \ \to \ \mathrm{cod}(\delta_g),\ (f,g) \mapsto g. \end{aligned}$$

Then, for every $F^p(\Omega)$ and maps

$$d_f : F^p(\Omega) \to \ker(\mathrm{d}_f), \begin{cases} f \mapsto f, & \text{if } \mathrm{d}_f f = 0 \\ f \mapsto 0, & \text{if } \mathrm{d}_f f \neq 0 \end{cases}$$

and

$$d_g : F^p(\Omega) \to \mathrm{cod}(\delta_g), \begin{cases} f \mapsto f, & \text{if } \langle f, f' \rangle = 0 \ \forall f' \in \ker(\mathrm{d}_f) \\ f \mapsto 0, & \text{if } \mathrm{d} f = 0\,, \end{cases}$$

there exist a map $d : F^p(\Omega) \to \ker(\mathrm{d}_f) \oplus \mathrm{cod}(\delta_g)$ such that

$$d_f = \pi_f \circ d \qquad \text{and} \qquad d_g = \pi_g \circ d$$

hold. This is a product [21,29], see fig. 5.

$$\begin{tikzcd}
 & F^p(\Omega) \arrow[dl, "d_f"'] \arrow[d, "d"] \arrow[dr, "d_g"] & \\
\ker(\mathrm{d}_f) & \ker(\mathrm{d}_f) \oplus \mathrm{cod}(\delta_g) \arrow[l, "\pi_f"'] \arrow[r, "\pi_g"] & \mathrm{cod}(\delta_g)
\end{tikzcd}$$

FIGURE 5 The orthogonal decomposition expressed as a product.

5 | ACTION PRINCIPLE

Next, let us move to boundary value problems and derive PDEs from the action principle [16,17]. At first, we need to introduce an action such as

$$\mathcal{A} = \frac{1}{2} \int_{\Omega} f \wedge \star f \; - \int_{\Omega} a \wedge \star(\star h) \; - \int_{\partial\Omega} \mathrm{t}a \wedge \mathrm{t}g_g \,, \tag{3}$$

where the pair $a \in F^{p-1}(\Omega)$, $f \in F^p(\Omega)$ fulfils $\mathrm{d}a = f$, $\star h \in F^{p-1}(\Omega)$ is a source term, which is assumed to be known a priori. On the boundary we set $\mathrm{t}a = \mathrm{t}a_f$ on $\partial_f\Omega$ and $\mathrm{t}\star f = \mathrm{t}g_g$ on $\partial_g\Omega$.

As $\mathrm{dd} \equiv 0$, the map $\mathrm{dom}(\mathrm{d}) \xrightarrow{\mathrm{d}} F^p(\Omega)$ provides us with the first PDE[2]

$$\begin{cases} \mathrm{d}f &= 0\,, \\ \mathrm{t}f &= \mathrm{t}(\mathrm{d}a_f) \text{ on } \partial_f\Omega\,. \end{cases} \tag{4}$$

Thereafter, by insisting the variation

$$\delta\mathcal{A} \;=\; \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{A}(a + \alpha a')\Big|_{\alpha=0}$$

to vanish for all $a' \in \mathrm{dom}(\mathrm{d}_f)$, one gets

$$\delta\mathcal{A} \;=\; \int_{\Omega} \mathrm{d}a' \wedge \star f \;-\; \int_{\Omega} a' \wedge \star(\star h) \;-\; \int_{\partial\Omega} \mathrm{t}a' \wedge \mathrm{t}g_g = 0 \quad \forall a' \in \mathrm{dom}(\mathrm{d}_f). \tag{5}$$

To derive the second PDE, we employ Green's formula (2) to write (5) in the form

$$\begin{cases} (-1)^k \int_{\Omega} a' \wedge \star(\star\mathrm{d}\star f) \;=\; \int_{\Omega} a' \wedge \star(\star h) \;\; \forall a' \in \mathrm{dom}(\mathrm{d}_f) \\ \qquad\qquad \mathrm{t}\star f \;=\; \mathrm{t}g_g \qquad\qquad \text{on } \partial_g\Omega\,. \end{cases} \tag{6}$$

This is equivalent to

$$(-1)^k \langle a', \star\mathrm{d}\star f \rangle \;=\; \langle a', \star h \rangle \quad \forall a' \in \mathrm{dom}(\mathrm{d}_f) \quad \text{and} \quad \mathrm{t} \star f \;=\mathrm{t}g_g \text{ on } \partial_g\Omega\,.$$

Consequently, as space $\mathrm{dom}(\mathrm{d}_f)$ is dense in $F^p(\Omega)$, eq. (5) is the weak form [10] of PDE

$$\begin{cases} (-1)^k \star\mathrm{d}g \;=\; \star h\,, \\ \qquad \mathrm{t}g \;=\; \mathrm{t}g_g \text{ on } \partial_g\Omega \end{cases} \quad \Leftrightarrow \quad \begin{cases} (-1)^k \mathrm{d}g \;=\; h\,, \\ \qquad \mathrm{t}g \;=\; \mathrm{t}g_g \text{ on } \partial_g\Omega \end{cases} \tag{7}$$

where $g = \star f$.

Summing up, from the action $\mathcal{A}$ given in (3) the action principle yields (4) in the strong form and (7) in the weak form.

Remark: Notice, as $f \in F^p(\Omega) = \ker(\mathrm{d}_f) \oplus \mathrm{cod}(\delta_g)$, any $f \in \ker(\mathrm{d})$ with a boundary condition $\mathrm{t}f = \mathrm{td}a$ on $\partial_f\Omega$ can be decomposed into $f = f_f + \mathrm{d}a_f$, where $f_f \in \ker(\mathrm{d}_f)$ and $\mathrm{d}a_f \in \mathrm{cod}(\delta_g)$. This is to say there exists a $g \in F^{p+1}(\Omega)$ that fulfils $\delta_g g = \mathrm{d}a_c$.

[2] This, however, assumes the homology of Ω is trivial. If the homology is non-trivial one has to introduce "cuts" [30,31] that make all the p-cycles mod $\partial_f\Omega$ to bound $(p+1)$-cycles mod $\partial_f\Omega$. Such "cuts" will imply additional terms, but to avoid unnecessary technical details we will not deal with them here.

5.1 | Diagram of boundary value problems

Our next aim is to find a diagram of boundary value problems, and for this we express first the action principle as a product. We start by specifying spaces

$$
\begin{aligned}
D_{a,f} &= \{(a,f) \in F^{p-1}(\Omega) \times F^p(\Omega) \ \mid f = \mathrm{d}a,\ \mathrm{t}a = \mathrm{t}a_f \text{ on } \partial_f\Omega\}\,,\\
D_f &= \{f \in F^p(\Omega) \qquad\qquad\qquad\quad \mid \mathrm{d}f = 0,\ \mathrm{t}f = \mathrm{t}(\mathrm{d}a_f) \text{ on } \partial_f\Omega\}\,,\\
D_g(g) &= \{g \in F^{n-p}(\Omega) \qquad\qquad\qquad \mid \mathrm{d}g = h,\ \mathrm{t}g = \mathrm{t}g_g \text{ on } \partial_g\Omega\}\,,
\end{aligned}
$$

and

$$
A_0 = \{g \in F^{n-p}(\Omega) \,|\, \int_\Omega \mathrm{d}a' \wedge g - \int_\Omega a' \wedge \star(\star h) - \int_{\partial\Omega} \mathrm{t}a' \wedge \mathrm{t}g_g = 0 \quad \forall a' \in \mathrm{dom}(\mathrm{d}_f)\}\,.
$$

The action principle expressed as a product is object $D_f \times D_g$ together with projections

$$
\begin{aligned}
\pi_s &: D_f \times D_g \to D_f\,,\\
\pi_w &: D_f \times D_g \to D_g\,,
\end{aligned}
$$

with the condition that for every $D_a \times A_0$ and pairs of maps

$$
\begin{aligned}
s &: D_{a,f} \times A_0 \to D_f\,,\ ((a,f),g) \mapsto f\,,\\
w &: D_{a,f} \times A_0 \to D_g\,,\ ((a,f),g) \mapsto g\,,
\end{aligned}
$$

there exists map b from $D_a \wedge A_0$ to $D_f \wedge D_g$ such that

$$
s = \pi_s \circ b \quad \text{and} \quad w = \pi_w \circ b
$$

hold, see fig. 6.

$$
\begin{tikzcd}
 & D_{a,f} \times A_0 \arrow[dl, "s"'] \arrow[d, "b"] \arrow[dr, "w"] & \\
D_f & D_f \times D_g \arrow[l, "\pi_s"'] \arrow[r, "\pi_w"] & D_g
\end{tikzcd}
$$

FIGURE 6 The action principle expressed as a product.

A boundary value problem consists of a pair of PDEs equipped with the boundary conditions and of the constitutive law. Correspondingly, a combination of the product of fig. 6 with the constitutive law coproduct of fig. 3 should represent boundary value problems.

For this, recall that p-forms can be interpreted as sections of cotangent bundle $T^p\Omega = \bigcup_{x\in\Omega}\{x\} \times T^p_x$, and the set $\Gamma(T^p\Omega)$ of all sections on Ω coincides with space $F^p(\Omega)$. We call by π^p_x the map that projects $f \in F^p(\Omega)$ at point $x \in \Omega$ to $\{T^p_x\} \times T^p_x$,

$$
\pi^p_x : F^p(\Omega) \to \{T^p_x\} \times T^p_x\,,\ f \mapsto (T^p_x, f_x)\,.
$$

Saying, "find pair $(f,g) \in F^p(\Omega) \times F^{n-p}(\Omega)$ such that PDEs (4) and (7) and the constitutive law hold", corresponds now to the diagram of fig. 7.

$$
\begin{tikzcd}
 & D_{a,f} \times A_0 \arrow[dl, "s"'] \arrow[d, "b"] \arrow[dr, "w"] & \\
D_f \arrow[d, "\pi^p_x"'] & D_f \times D_g \arrow[l, "\pi_s"'] \arrow[r, "\pi_w"] & D_g \arrow[d, "\pi^{n-p}_x"] \\
\{T^p_x\} \times T^p_x \arrow[r, "i_p"] \arrow[dr, "h_p"'] & \{T^p_x\} \times T^p_x \sqcup \{T^{n-p}_x\} \times T^{n-p}_x \arrow[d, "h"] & \{T^{n-p}_x\} \times T^{n-p}_x \arrow[l, "i_{n-p}"'] \arrow[dl, "h_{n-p}"] \\
 & \{T^n_x\} &
\end{tikzcd}
$$

FIGURE 7 The diagram representing boundary value problems: BVPs consists of pairs of PDEs and of the constitutive law that connects the underlying fields over each point $x \in \Omega$. The BVPs themselves are derived from the action principle

5.2 | Finite dimensional counterpart

Before going on, let us briefly notice that the finite element method is a finite dimensional counterpart of (5). To verify this, we first choose a $a_c \in \mathrm{cod}(\delta_g)$ that fulfils the boundary condition of a, i.e., $\mathrm{t}a_c = \mathrm{t}a = \mathrm{t}a_f$ on $\partial_f\Omega$. Any a_c satisfying the condition is feasible. Thereafter any $f = \mathrm{d}a \in F^p(\Omega)$ can be given in the form $f = f_f + \mathrm{d}a_c$ where $f_f \in \mathrm{dom}(\mathrm{d}_f)$. Consequently, instead of (5) we may write

$$\delta\mathcal{A} = \int_\Omega \mathrm{d}a' \wedge (f_f + \mathrm{d}a_c) - \int_\Omega a' \wedge \star(\star h) - \int_{\partial\Omega} \mathrm{t}a' \wedge \mathrm{t}g_g = 0 \quad \forall a' \in \mathrm{dom}(\mathrm{d}_f),$$

$$\Leftrightarrow \quad \int_\Omega \mathrm{d}a' \wedge \star f_f = \int_\Omega a' \wedge \star(\star h) - \int_\Omega \mathrm{d}a' \wedge \star \mathrm{d}a_c - \int_{\partial\Omega} \mathrm{t}a' \wedge \mathrm{t}g_g \quad \forall a' \in \mathrm{dom}(\mathrm{d}_f)\,. \tag{8}$$

Now, if we set $a = \sum_i a_i w_i^{p-1}$, where the w_i^{p-1} are the basis functions of Whitney space W^{p-1}, and the a_i's are the degrees of freedom. In addition, if we choose W^{p-1} in place of $\mathrm{dom}(\mathrm{d}_f)$, then (8) corresponds with a finite element kind of system of equations.

For an example, let us consider electrostatics and say the dimension $n = 3$. In this case the counterpart of $f \in F^p(\Omega)$ is the electric field $e \in F^1(\Omega)$. Hence, we have $p = 1$. We denote the spaces of Whitney p-forms by $W^p, p = 0, \ldots n$, and in (8) in place of $\mathrm{dom}(\mathrm{d})$ we chose W^0 and instead of $a' \in \mathrm{dom}(\mathrm{d}_f)$ write $\varphi' \in \mathrm{dom}(\mathrm{d}_f) \subset W^0$. The counterpart of $f_f \in \mathrm{dom}(\mathrm{d}_f)$ is $e_f = \sum_i -\mathrm{d}(\varphi_i w_i^0)$, $w_i^0 \in \ker(\mathrm{d}_f)$, and the counterpart of a_c is $-\varphi_c$. To express $-\varphi_c$ in the Whitney basis, we denote by X_f the set of nodes on $\partial_f\Omega$ and by $\varphi(x_j)$ the value of $\mathrm{t}\varphi$ at the node $x \in X_f$ indexed by j. Function φ_c is then given by $\varphi_c = \sum_j \varphi(x_j) w_j^0$, $x_j \in X_f$, and now, the electric field e fulfils boundary condition $\mathrm{t}e = \mathrm{t}(e_f - \mathrm{d}\varphi_c) = -\mathrm{t}\mathrm{d}\varphi_c$ on $\partial_F\Omega$. On the complement $\partial_g\Omega$ the boundary condition to the electric flux is imposed by writing $\mathrm{t}(\varepsilon \star e) = \mathrm{t}d_g$. Finally, in place of $\star$ we write $\varepsilon\star$, where ε is permittivity.

The finite dimensional counterpart of (8) is now

$$-\int_\Omega \mathrm{d}w_i^0 \wedge \varepsilon \star \sum_j \mathrm{d}(\varphi_j w_j^0) = \int_\Omega w_i^0 \wedge \rho + \int_\Omega w_i^0 \wedge \varepsilon \star \sum_j \mathrm{d}(\varphi_c(x_j) w_j^0) - \int_{\partial\Omega} \mathrm{t}w_i^0 \wedge \mathrm{t}d_g \quad \forall w_i^0 \in \mathrm{dom}(\mathrm{d}_f)\,,$$

which we recognize as the finite element formulation of the electrostatic field.

This example exemplifies representation of boundary value problems in finite dimensional spaces and the significance of Whitney forms.

6 | "ALL" BOUNDARY VALUE PROBLEMS

The aim is to find a systematic approach to construct "all" pairs of PDEs that compose well-posed second order boundary value problems. To move in this direction, next we will derive at once the PDEs for all differential forms $p = 0, \ldots n$ on a four dimensional Minkowski manifold. The manifold is equipped with a metric tensor and signature $(-, +, +, +)$. The metric tensor makes it possible to introduce a Hodge operator and Hodge decompositions also to a Minkowski manifold. To distinct between the Hodge operator on Minkowski and Riemannian manifolds, we denote the Hodge operator on the Minkowski manifold by $*$. Under the assumptions made we have $** = (-1)^{p(n-p)+1}$.

As the degree p of the differential forms is not in our primal interest, we bury all the degrees into a formal sum of all p-form spaces

$$F(\Omega) = \bigoplus_{p=0}^{n} F^p(\Omega)\,.$$

This is to say, all degrees are of equal significance. This also implies the elements of $F(\Omega)$ are formal sums of differential form degree $p = 0, \ldots, n$ such as $f = f^0 + f^1 + f^2 + f^3 + f^4 \in F(\Omega)$.

Our aim is now to extend differential operator d and its adjoint δ_g to the elements of $F(\Omega)$. This will then enable us to write PDEs for any $f \in F(\Omega)$. For this we pick a $h \in F(\Omega)$ and decompose the $h^p \in F^p(\Omega), p = 0, \ldots 4$ into components in $\ker(\mathrm{d}_f)$ and $\mathrm{cod}(\delta_g)$ by writing

$$\mathrm{D}f = h \quad \Leftrightarrow \quad \begin{bmatrix} & -\delta_g & & & \\ \mathrm{d}_f & & \delta_g & & \\ & \mathrm{d}_f & & -\delta_g & \\ & & \mathrm{d}_f & & \delta_g \\ & & & \mathrm{d}_f & \end{bmatrix} \begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \\ f^4 \end{bmatrix} = \begin{bmatrix} h^0 \\ h^1 \\ h^2 \\ h^3 \\ h^4 \end{bmatrix}\,. \tag{9}$$

These decompositions are in space-time, while boundary value problems are solved in space and time. Accordingly, we assume Ω is decomposed into space and time. To replace (9) with space and time-like equations we also decompose $f^p \in F^p(\Omega)$ into $f^p = f_t^p + f_s^p$, where subscripts s and t refer to space and time-like components, respectively. In the same manner the exterior derivative is decomposed into $\mathrm{d} = \mathrm{d}t \wedge \partial_t + \mathrm{d}^s$. The derivatives of f^p can then be given in space and time by

$$\begin{aligned} \mathrm{d}f^0 &= \mathrm{d}t \wedge \partial_t f^0 + \mathrm{d}^s f^0\,, \\ \mathrm{d}f^p &= \mathrm{d}t \wedge \partial_t f^p_s + \mathrm{d}^s f^p_t + \mathrm{d}^s f^p_s, \quad \forall p > 0\,, \\ \delta f^p = *\mathrm{d}*f^p &= \star\mathrm{d}t \wedge \partial_t \star f^p_t + \star\mathrm{d}^s\star f^p_t + \star\mathrm{d}^s\star f^p_s\,, \ \forall p < n\,, \\ \delta f^n = *\mathrm{d}*f^n &= \star\mathrm{d}t \wedge \partial_t \star f^n_t + \star\mathrm{d}^s\star f^n_t\,. \end{aligned} \tag{10}$$

Here, as our aim is only to work out PDEs that hold within $\Omega = \Omega_t \times \Omega_s$. For this reason we may neglect the boundary conditions, and accordingly, there is no need to write d_f and δ_g in place of d and δ.

With (10) it can now be shown that the counterpart of (9) in space and time is (here f^p and h^p are space-like p-forms, and F^p and H^p are the space-like components of $(p+1)$-forms $dt \wedge F^p$ and $dt \wedge H^p$, for details, see [32]):

$$\begin{bmatrix} \partial_t & & & & & -\mathrm{d} & & \\ & \star\partial_t\star & & & \mathrm{d} & & & \\ & & \partial_t & & & \star\mathrm{d}\star & & -\mathrm{d} \\ & & & \star\partial_t\star & \star\mathrm{d}\star & & \mathrm{d} & \\ & \star\mathrm{d}\star & & -\mathrm{d} & \partial_t & & & \\ \star\mathrm{d}\star & & \mathrm{d} & & & -\star\partial_t\star & & \\ & & & \star\mathrm{d}\star & & & \partial_t & \\ & & \star\mathrm{d}\star & & & & & -\star\partial_t\star \end{bmatrix} \begin{bmatrix} f^3 \\ F^3 \\ f^1 \\ F^1 \\ f^2 \\ F^2 \\ f^0 \\ F^0 \end{bmatrix} = \begin{bmatrix} H^3 \\ h^3 \\ H^1 \\ h^1 \\ H^2 \\ h^2 \\ H^0 \\ h^0 \end{bmatrix}, \tag{11}$$

Eq. (11) provides us with the desired class of all PDEs expressible with ordinary differential forms. This class covers wave problems, quasistatic problems as well as static ones. Particular PDEs are obtained with appropriate choices of F and H.

6.1 | Examples

For an example let us demonstrate that Maxwell's equations is a particular instance of this class; Let us denote the electric field strength and magnetic flux density by e and b, respectively, and imbed permittivity ε, permeability μ, and reluctivity $\nu = 1/\mu$ into the inner product involved in the definition of the Hodge operator $\star$. Now,

$$\text{the choice} \begin{cases} F^1 &= -e \\ f^2 &= b \\ h^1 &= \star j \\ H^0 &= -\star q \end{cases} \text{results in} \begin{cases} \mathrm{d}b &= 0 \\ \mathrm{d}e + \partial_t b &= 0 \\ -\star\partial_t\star_\varepsilon e + \star\mathrm{d}\star_\nu b &= \star j \\ -\star\mathrm{d}\star_\varepsilon e &= -\star q \end{cases} \iff \begin{cases} \mathrm{d}b &= 0 \\ \mathrm{d}e + \partial_t b &= 0 \\ -\partial_t\star_\varepsilon e + \mathrm{d}\star_\nu b &= j \\ \mathrm{d}\star_\varepsilon e &= q \end{cases}.$$

For another example, the choice $f^2 = g, h^1 = -\star 4\pi G\rho$ –where g is the gravitational field, G the universal gravity constant, and ρ is mass density– yields the Newtonian gravity field $\mathrm{d}g = 0$ and $\mathrm{d}\star g = -4\pi G\rho$. For further details, see ref. [32].

Summing up, the system of equations (11) answers the question, how to find systematically PDEs for ordinary differential forms, and (7) answers to the question, when a boundary value problem is well posed, and Whitney forms make it possible to express such BVPs systematically in finite dimensional spaces.

6.2 | Extension of the class: E-valued forms

Not all boundary value problems are expressible in ordinary forms. Elasticity, which calls for so-called E and E^*-valued forms –also called by the name vector and covector -valued forms– is a good example of this. Vector and covector valued forms are differentiated with the covariant exterior derivative d_∇, where ∇ is a connection. For a very good exposition, see ref. [17]. In addition, the standard definitions of the wedge product and the Hodge operator need also to be generalized to E and E^*-valued forms. Despite of this, PDEs can still derived and boundary value problems established in the same way as in case of ordinary differential forms.

For an example, boundary value problems of small-strain elasticity can also be formulated exploiting the structure shown in fig. 7. To find the basic equations, notice, displacement is an E-valued 0-form ν, and the time derivative of this is velocity $u = \partial_t \nu$. The body force f_V is an E-valued 3-form, stress σ is a E^*-valued 2-form, and the linearized strain ε is an E-valued 1-form obtained by differentiating displacement, $\varepsilon = \mathrm{d}_\nabla \nu$. [33,34,35] The stress-strain relation is given by $\sigma = \star^C \varepsilon$, where C is the so-called tensor of elasticity [16,36], which we imbed into the Hodge operator. The basic equations of elasticity are now obtained from system (11), where d is replaced with d_∇, and the wedge product and the Hodge operator are extended to their counterparts for E and E^*-valued forms, as follows:

$$\text{the choice} \quad \begin{cases} F^0 &= u \\ f^1 &= \varepsilon \\ g^0 &= -\star f_V \end{cases} \quad \text{results in} \quad \begin{cases} -\partial_t \varepsilon + \mathrm{d}_\nabla u &= 0 \\ \mathrm{d}_\nabla \varepsilon &= 0 \\ \star \mathrm{d}_\nabla \star^C \varepsilon - \star \partial_t \star_\rho u &= -\star f_V \end{cases} \iff \begin{cases} -\partial_t \varepsilon + \mathrm{d}_\nabla u &= 0 \\ \star_\rho \partial_t u - \mathrm{d}_\nabla \sigma &= f_V \\ \sigma = \star^C \varepsilon,\ u &= \partial_t \nu \end{cases} .$$

This suggests a recipe to find a very large class of boundary value problems: Extend system (11) and the structure shown in fig. (7) to E and $\mathrm{End}(E)$-valued forms [17] in all meaningful dimensions. Such an approach seems to provide one with solid foundations to software systems like GetDP to enable users to establish and solve their own problems.

7 | CONCLUSION

We have expressed the structure of second order boundary value problems to come up with a system from which a class of PDEs can be derived. In addition, we have formalized the use of action principle to answer the question, when a boundary value problem is well posed. These results can be exploited in system designed to solve numerically second boundary value problems laying some foundations for machines to check and find end users unintended mistakes in feeding in boundary value problems.

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