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The interplay between the guidance from the digital learning environment and the teacher in supporting folding back

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Abstract
Previous studies have proposed that students' mathematical understanding develops dynamically through the process known as folding back, in which learners revisit earlier forms of understanding and use them to build even deeper levels of mathematical understanding. Digital learning environments, where students can manipulate representations, are often used to enable students to notice properties, patterns, or rules. When working in such an environment, students usually receive support from the environment and the teacher. The interplay between these different sources of support is important according to previous studies. In this study, we examine this interplay in the case of folding back. The study aims to understand how the teacher, together with the learning environment, can support the process of folding back. We collected data from second, fourth, and sixth grade students as they worked in groups to develop a rule for balancing a balance beam in a digital learning environment designed to support folding back. One pre-service teacher guided each three-student group. Data were analyzed by identifying occasions for folding back and characterizing different ways in which the interplay between the teacher and the environment supported students' folding back. We found different kinds of synergy between the two sources of support. The teachers followed up on and augmented the support from the environment, initiated supplementary folding back, and reinforced the support from the environment. We also found non-synergy between the two sources of support, when the teachers' support was not aligned with support from the environment.

Keywords Folding back · Distributed support · Digital learning environment · Synergistic support · Teacher support · Technology

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1 Introduction

The use of digital learning environments is common practice in schools. In mathematics, digital environments help students notice connections, properties, patterns, or rules as a result of manipulating representations (e.g., Erbas & Yenmez, 2011; Olsson & Granberg, 2019). Despite working in digital environments, students still need teachers to support their mathematical reasoning (Drijvers et al., 2010; Hähkiöniemi et al., 2013).

A key process in mathematical understanding is the concept of *folding back* (Martin, 2008; Pirie & Kieren, 1994). When students face a challenge while working on a mathematical task, they may need to fold back to revisit earlier understandings in order to then build even deeper understandings and use them to overcome the challenge. Except for some research initiatives (e.g., Martin, 2008; Martin & Towers, 2016a; Yao & Manouchehri, 2022), few studies have examined the role of the teacher or technology in supporting folding back.

Research on distributed scaffolding has shown that student learning may be supported by multiple sources, such as a teacher and learning materials (Brown et al., 1993; Kolodner et al., 2003). This line of research has pointed out that considering the *interplay between different sources of support* is important (Martin et al., 2019; Tabak, 2004). According to Tabak (2004), synergy between sources can contribute to productive support. Teachers or other social supports play an important role in making sure that the support is responsive to students' needs by complementing the support from the environments with their own support (Puntambekar, 2022). Consistent with this position, Martin et al. (2019) call for more research on understanding the interplay between support from a teacher and material sources.

Drawing on insights from the research on distributed scaffolding, more attention could be given to interplay between different sources of support in the case of folding back. So far, no studies have investigated the interplay between sources of support in guiding students' folding back. Yet, this kind of research is needed to advance our understanding of how to support mathematical understanding through folding back and to improve the design of learning environments. This study addresses this research gap by examining the interplay between support from the teacher and from a digital learning environment in promoting primary school students' folding back when the students explore a rule for balancing a balance beam. The following research question guided the study: What relations exist between guidance from the teacher and the digital learning environment in cases of supporting folding back?

2 Theoretical background

2.1 Promoting mathematical understanding and folding back

The Pirie-Kieren theory of growth of mathematical understanding is a well-established theoretical perspective on the nature of mathematical understanding (see, e.g., Kieren et al., 1999; Martin & Pirie, 2003; Pirie & Kieren, 1994). According to this theory, growth of mathematical understanding is neither linear nor monodirectional. Instead, it is the result of a dynamical and active process involving a continuous movement back and forth between eight layers of understanding (see Fig. 1). The four layers that apply to the present study are summarized in Fig. 1.

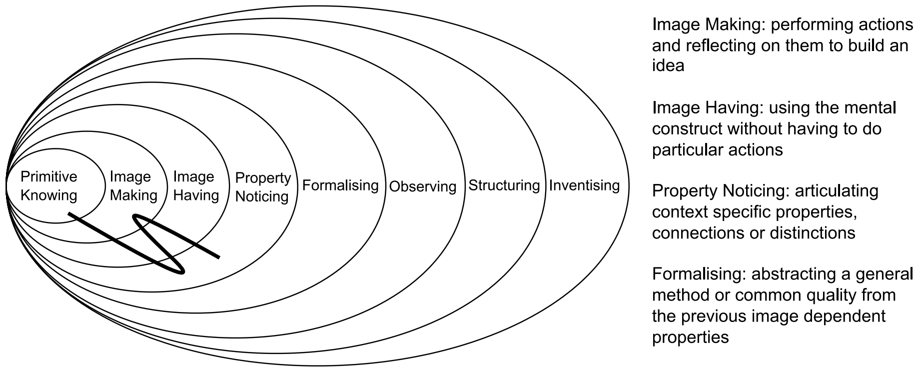

Fig. 1 Layers of understanding and an instance of folding back in a hypothetical path of growth of understanding (Pirie & Kieren, 1994)

In the Pirie-Kieren theory, when faced with a problem that is not immediately solvable, a person functioning at an outer layer of understanding may need to revisit an inner layer of understanding to examine and modify their current ideas and thinking about a concept. This process is known as "folding back" to imply that when learners revisit an earlier layer of understanding, they carry with them understandings from the outer layer, a phenomenon called "thickening" (Fig. 1). Two key features define folding back and differentiate it from a simple act of going back: The return to an inner layer is "stimulated and guided by outer level knowing" and the"folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding" (Pirie & Kieren, 1991, p. 172). Folding back promotes understandings because when learners revisit an earlier understanding in response to a challenge, they can modify, collect, or build anew conceptions that will allow the difficulty to be overcome through an extended understanding of the topic (Martin, 2008).

2.2 Supporting folding back

Martin (2008) developed a framework for folding back intended as an observational analytical tool. The framework identifies three higher-level categories that describe key aspects of folding back. *Source* refers to the stimulus that prompts the learner to fold back and four sources are identified: the teacher (teacher intervention), another student (peer intervention), curriculum material (material intervention) or the student who decides to fold back (self-invoked). *Form* refers to the kinds of actions engaged at the inner layer (e.g., collecting an existing understanding). *Outcome* refers to the effect of folding back on growth of students' understanding (e.g., effective or not in enabling continued growth). Each category is subdivided into subcategories to provide further descriptions of folding back. Based on Martin's framework, support for folding back consists of creating occasions for and stimulating folding back, helping learners engage in appropriate inner layer activity, and making sure that learners return to the outer layer and use the extended understanding to solve the challenge that motivated the folding back. In this study, we focus on two support sources: the teacher and the material.

Research on teacher support for folding back shows that teachers can support folding back by using moves such as rug-pulling, which shifts the focus of students' attention to something that confuses them and requires them to reassess their mathematical process (Towers, 1998; Towers & Proulx, 2013), and focusing students' attention on contradictions in their current understanding (Hähkiöniemi & Hirvonen, 2013). Martin and Towers (2016a) reported a case of a high school teacher who encouraged folding back by getting students to revisit and build on ideas from previous studies. Teachers can also use folding back as a pedagogical design tool for planning their teaching to create occasions for folding back (Martin & Towers, 2016a, b).

Research on support for folding back from material sources are rare. However, Gulkilik et al. (2020) showed that virtual manipulatives can support a student's understanding as defined in Pirie-Kieren theory. Poon and Wong (2017) designed dynamic geometry materials that encouraged students to fold back. They found that materials provided learning opportunities. Yao and Manouchehri (2022) found that technology may mediate folding back initiated by the teacher or by the students themselves. However, studies have not elaborated on the interplay between teachers and material in supporting folding back.

2.3 Synergy between different sources of support

Research on distributed scaffolding has contributed to understanding the interplay between sources of support. Tabak (2004) described two patterns of distributed scaffolds that existed in the research literature. In the differentiated scaffolds pattern, support is provided through different means to address diverse learning needs. In the redundant scaffolds pattern, different sources support the same learning need at different points in time. Tabak introduced a third pattern, the *synergistic scaffolds*, in which multiple sources of support interact to target the same learning need at the same time. Tabak argues that it is an "important conceptual tool in understanding how different constituents interact to produce support that is greater than the sum of the constituents" and "the central question is not whether interaction between supports can occur, but how this interaction can come into play and what functions it can serve" (Tabak, 2004, p. 308). While Tabak emphasized the importance of synergistic scaffolds, he noted that it has not received much attention in research. In mathematics education, the synergy between sources of support has been examined only in some studies (Tropper et al., 2015; van Zoest & Stockero, 2008).

When sources of support include the teacher, the teacher may be particularly important in interpreting the situation with respect to students' thinking and use of the other sources of support. Puntambekar et al. (2007) highlighted the teacher's role in building connections between different materials and activities. As an example of the teacher's role, Lehtinen and Hähkiöniemi (2016) found that the use of technology may create occasions for productive student explanation, but how this opportunity is exploited depends on teachers' complementary guidance. Lehtinen and Viiri (2017) provide an example of synergistic guidance in which pre-service teachers advised students on selecting an appropriate difficulty level in a game. Furthermore, findings by Martin et al. (2019) point to the importance of complementarity between support from the teacher and material sources. Teacher support that only replicated guidance provided by the software was not as productive as teacher support that provided augmenting guidance (Martin et al., 2019).

3 Methods

3.1 Design of the learning environment

We developed a digital learning environment by using the Graasp authoring platform (Graasp, 2021) and GeoGebra (GeoGebra, 2021). The environment consists of seven tabs (Table 1) in which weights and locations of two birds on a balance beam can be changed while the beam is supported to stay in balance. When the supports are removed, the beam stays in balance or tilts. In Lab (Fig. 2), students can experiment freely and start building an image about the functioning of the balance beam, by noticing properties in the balance cases and formulating rules. It was expected that at first, students would develop lower-level rules such as weights and distances being equal.

In the tasks, students are asked to apply their rule (shown above the task in noneditable form). If they solve a task correctly so that the beam stays in balance, the environment prompts them to move on to the next task. If they do not succeed, the screen turns into gray, the birds cannot be moved, and they receive a prompt to return to Lab to develop their rule before returning to the task. To avoid using trial and error,

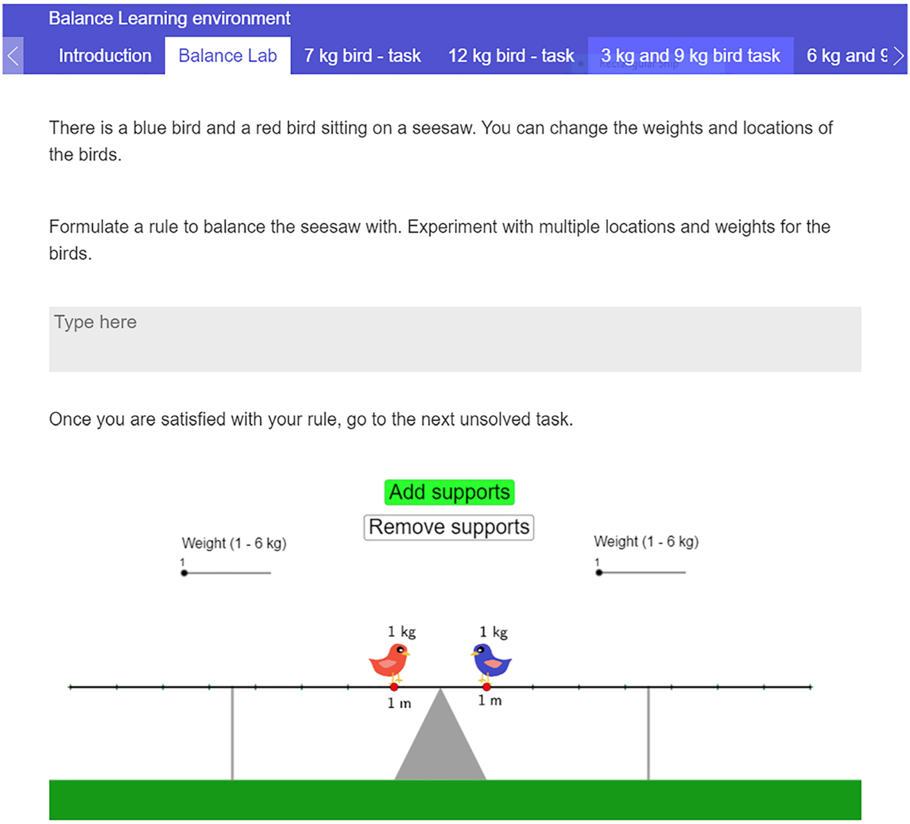

Fig. 2 Lab

Table 1 Weights and distances of the birds on a balance beam in different tabs

	Bird on the left		Bird on the right		Assignment
	Weight	Distance	Weight	Distance	
Intro	Fixed, not shown	Not shown	Fixed, not shown	Not shown	Try how to move the birds and how to remove the supports
Lab	1–6 kg	1–8 m	1–6 kg	1–8 m	Formulate a rule to balance the seesaw with. Experiment with multiple weights and locations for the birds
Task 1	7 kg	2 m	1–20 kg	1–8 m	Use your rule to balance the beam. You have only one try
Task 2	12 kg	1 m	1–7 kg	1–8 m	Use your rule to balance the beam. You have only one try
Task 3	3 kg	2–8 m	9 kg	2–8 m	Use your rule to balance the beam. You have only one try
Task 4	6 kg	1–8 m	9 kg	4 m	Use your rule to balance the beam. You have only one try
Additional tasks	Balancing Act PhET simulation (PhET Interactive Simulations, 2021)				E.g., balance the beam, decide whether the beam tilts, find the weight of an object

the number of tries in the tasks was limited by locking the screen after an answer. To enable opening the task again after visiting in Lab, a start-button appears on the corner of the screen after 7 s.

The tasks were designed to require progressively more advanced rules. For example, task 2 can be solved using a rule that involves the proportion of weights being 1:2, while the proportion of weights is 1:3 in task 3 and 2:3 in task 4. Thus, it was expected that at some point students would solve a task incorrectly, which would point to a limitation in their current rule at Property Noticing (rule is tied to particular cases) or at Formalising (general rule). It was assumed that students would fold back to Image Making by experimenting in Lab and finally improve their rule so that the task could be solved. Thus, the sequence of the tasks, the possibility to experiment in Lab and the prompt to return to Lab were expected to support folding back.

3.2 Data collection

The students were randomly divided into groups of three students per laptop. Two groups included second grade (8-year-old) students, five groups included fourth grade (10-year-old) students, and five groups included sixth grade (12-year-old) students. They worked in these groups for the full duration of the lesson (approximately 40 min). Each group had one pre-service primary school teacher guiding their work. Thus, 12 pre-service teachers participated in the study.

The 12 pre-service teachers were participants in a course that focused on guiding students based on their thinking when using dynamic representation for learning mathematics and science. One session (2.5 h) was devoted to familiarizing with the balance beam activity and preparing for guiding students. The pre-service teachers used the same environment as students. Ideas about different rules and what would prompt someone to modify a rule were collected. The participants also discussed how to interpret certain hypothetical rules and what kind of support could be provided. The concept of folding back was not discussed, but, the idea of building rules based on empirical experimenting in Lab and trying to apply these rules in the tasks, then returning to experiment in Lab when the rule did not help to solve a task, were discussed.

The screen of each laptop was recorded using screen capture software. The software also captured audio from the laptop microphone and video from the laptop webcam in sync with screen capture. In addition, a small action video camera recorded the group from the side to enable the recognition of gestures and who was talking. All these data sources from each group were synchronized in one video file when preparing the data for analysis.

3.3 Data analysis

We used the video analysis model developed by Powell et al. (2003) as the model suits the purpose of developing insights and building an understanding of complex phenomena. The heart of this method is selecting critical events that help build insights related to the research question—in this case, the interplay between the teacher and the learning environment in supporting folding back.

In this study, the critical events were episodes in which the teacher or the environment pointed to a limitation in students' current rules and prompted the students to experiment more. In these events there was an opportunity for folding back to Image Making, although it did not always happen. Image Making was interpreted when students experimented with the balance beam (whether in Lab or in tasks). When working in the environment, students can fold back to various layers, but the analysis focused particularly on folding back to Image Making because the environment was designed to support folding back primarily to that layer.

The analysis consisted of six steps. First, data were transcribed and imported into video analysis software together with the video for each student group. Second, we familiarized ourselves with the data by viewing videos in parallel with transcripts. Third, we described the students' development of ideas and teacher guidance to achieve an overall picture of each group. Fourth, we identified critical events in which there was an opportunity for folding back to Image Making. Fifth, we analyzed the critical events for how the teacher and the environment supported (or not) folding back. In connection with this, we examined how the teacher complemented support from the environment. Sixth, we recognized and characterized different ways in which the interplay between the teacher and the environment supported the students' folding back to Image Making.

4 Results

We found that the interplay between the teacher and the learning environment in supporting folding back appeared in three synergistic and two non-synergistic ways.

4.1 Synergistic guidance

In synergistic guidance, the teachers' support complemented the support from the environment so that the two sources of support worked in concert. Both sources shared the same aim of sending students from tasks back to Lab to continue image making when needed.

4.1.1 The teacher follows up on and augments support from the environment

Often, students' progress in the learning environment created an occasion where students needed to revise their rule for balance and the environment prompted them to fold back, but, the students did not follow the prompt. However, the teachers followed up on the prompt from the environment, added support that was adjusted to students' work, and guided the students to fold back as intended by the environment.

For example, Evan, Lucy, and Landon (sixth grade) built their image by experimenting in Lab and noticed the following property: "the heavier bird is closer and the lighter bird is further away from the center." Related to this, Lucy added that "If it is 2 kg heavier, it has to be two (inaudible: toward there)." They solved task 2 incorrectly (12 kg 1 m $\wedge$ 7 kg 5 m). Despite the environment prompting the students to return to Lab, they continued to think about task 2:

1 Lucy: Or if it has to be 5 between them, because 12 – 7 = 5. Maybe there has to be 5 between them, like 1, 2, 3, 4, 5 [points to 5 jumps from the bird on the left and ends at 4 m].

2 Mr. Perez: So 12…
3 Evan: Like, how much is the difference between them [weights], that is, how far should it be from that [distance between the birds].

In the above instance, the students did not fold back to Image Making. Instead, they continued to think based on their current image and adjusted the property they had noticed before (turns 1 and 3). The teacher intervened and guided the students to fold back to Image Making by asking the students to test their new rule in Lab:

10 Mr. Perez: We can try the rule here [opens Lab]. […] Let's put, for example, six kilos and three kilos [changes the weights to 6 kg and 3 kg]. Now try your rule. How did it go, six minus?
11 Evan: Three.
12 Mr. Perez: And?
13 Evan: So, it should.
14 Lucy: Let's put it.
15 Evan: One more toward there.
16 Mr. Perez: 6 – 3 equals what?
17 Lucy: Three.
18 Evan: Three.
19 Mr. Perez: And?
20 Evan: And now it has three.
21 [Lucy removes the supports, balance case 6 kg 1 m ∧ 3 kg 2 m] […]
26 Mr. Perez: Okay. So now 6 – 3, and then, it gives three to here [points at the difference between the birds].
27 Lucy: Yes.
28 Mr. Perez: Okay, good. Would you write your rule? Or would you like to test it again with some other weights?
29 Evan: No need to.

In the above instance, when the students folded back (turns 11–21), their image making was guided by the properties they had found. The teacher intervention was important in guiding the students to develop understanding through image making, instead of just adjusting their rule as in turns 1–3. In this case, the students' image making strengthened their incorrect property that just happened to work. However, the teacher continued to support further image making by insisting on testing the property in another case:

30 Mr. Perez: I would like to [test the rule] because I have not invented a rule like that. I had a bit different rule when I solved this. [Changes the weights to 5 kg and 2 kg.] How about now?
31 Evan: This has to be three.
32 Lucy: It has to be one more forward.
33 Evan: The difference has to be three.
34 Mr. Perez: So?
35 Lucy: One, two.
36 Evan: Yes, one more toward there.
37 Lucy: One more there.
38 [Lucy removes the supports, non-balance case 5 kg 2 m ∧ 2 kg 1 m.]
39 Lucy: Help. […]
49 Lucy: Then, our rule does not work.

Finally, the students noticed that their rule was not working (turns 38–49). After this, the students continued image making by experimenting in Lab. Their understanding at Image Making was thicker as they knew that they had to search for another kind of balance rule. They built balance cases 5 kg 2 m ∧ 2 kg 5 m, and 3 kg 2 m ∧ 2 kg 3 m and noticed that the weight of one bird was equal to the distance of the other bird ($w_1 = d_2$ and $w_2 = d_1$). Thus, the students noticed another property.

The teacher's support was important in guiding the students to fold back to Image Making instead of just relying on their existing image. After an incorrect solution, the environment gave a prompt to return to developing the rule, but the environment could not interpret whether the students really folded back to Image Making. However, the teacher was able to interpret the situation and provide support that was adapted to students' way of working and to the rule they were thinking about. The characteristic feature of this type of synergistic guidance was that the teacher's guidance augmented the support from the environment by adapting to the situation.

4.1.2 The teacher initiates supplementary instances of folding back

The teachers also initiated instances of folding back that were additional to those prompted by the environment. These instances were created by the teachers on the fly when they saw the need.

For example, Hailey, Nicholas, and Kylie (sixth grade) had noticed the following property while in Lab: "When the other bird is half lighter, it has to be double further." Nicholas proposed the correct solution to task 1 and task 2 by using this rule. Then, in task 3, he immediately adapted the rule to the new situation by saying that "it has to be three times further." Thus, Nicholas quickly adapted the rule to a different proportion of weights (3:1). However, it seems that Nicholas proceeded so quickly that Hailey and Kylie were not able to follow him in generalizing the rule. Indeed, Kylie suggested a different answer than Nicholas in task 3. Nicholas continued to proceed quickly and generalized the property for task 4 straight away:

1 Nicholas: This is one-third lighter (6 kg is 1/3 lighter than 9 kg). Then, this must be 6 meters (4 m is 1/3 closer than 6 m).
2 Kylie: Yeah.
3 Nicholas: Like this. Maybe.
4 Ms. Hall: Does. Wait. Does…
5 Kylie: Wait.
6 Ms. Hall: …everyone agree?
7 Nicholas: Yeah. 6 × 6 equals 36 and 9 × 4 equals.
8 Ms. Hall: Wait, wait.
9 Nicholas: It went already. [Removes the supports, balance case 6 kg 6 m ∧ 9 kg 4 m.]
10 Ms. Hall: What did you say?
11 Nicholas: 9 × 4 = 36 and 6 × 6 = 36.

At first, Nicholas generalized the property to include an instance where the proportion of weights is 2:3 (turn 1). Then, he modified the property to include multiplying the weight and distance on both sides (turns 7 and 11). All of this happened in 34 s after opening task 4. He had already solved the task before the teacher had time to stop for a moment. However, the teacher continued to ask Nicholas to explain his thinking:

19 Ms. Hall: Is this now according to your rule?
20 Nicholas: No.
21 Ms. Hall: Okay. How should you change the rule then?
22 Nicholas: Those multiplications have to be equal. Maybe. I don't know.
23 Ms. Hall: […] Do you girls agree with Nicholas?
24 Kylie: I suppose so. I did not quite understand what he said. Or, I did not hear well.
25 Ms. Hall: Nicholas, would you like to explain what you were thinking? You are on the right track, but the girls did not catch your thinking.
26 Nicholas: So, like $9 \times 4 = 36$ and $6 \times 6 = 36$.
27 Kylie: Oh.
28 Hailey: Yeah.
29 Kylie: Should we then change it [the rule].
30 Nicholas: I don't know if it works. Whether it is always. At least here.
31 Ms. Hall: Well, would you like to test it in Lab?
32 Nicholas: No.
33 Ms. Hall: You can…
34 Hailey: Let's try.
35 Ms. Hall: …test it there.
36 Hailey: Let's test. It would be good.
37 Ms. Hall: Go and test. Whether it works.
38 [Students go to Lab where they have a non-balance case 2 kg 6 m $\wedge$ 4 kg 4 m.]
39 Nicholas: Well, 2×6 equals 12 and 4×4, well it does not equal 12.
40 Hailey: Let's put three.
41 Nicholas: Now it is in balance. [Removes the supports, balance case 2 kg 6 m $\wedge$ 3 kg 4 m]

Because Nicholas answered tasks correctly, the environment did not prompt them to go to Lab. However, the teacher affected the students' work in three important ways. First, Nicholas noticed that he was not totally sure whether the property always works (turns 22 and 30). Second, Nicholas repeated his idea so that the others had time to start making sense of it (turns 7, 11, 22, and 26). Third, the students folded back to Image Making by exploring Nicholas's property (turns 38–41). After this, the students wrote the rule as, "When both birds have the weight times distance from the middle of the beam the same, the beam stays in balance." Afterward, the students used the rule several times in the additional tasks. Thus, folding back helped Kylie and Hailey to generalize the initial property where the proportion of weights was 2:1. Also, Nicholas' understanding was thicker after folding back as he now knew that the property works in several cases. Because the students finally expressed the rule in a general form and used it to answer several additional tasks with first try, they were at Formalising.

In contrast with the first type of synergistic guidance, now the teacher made the initiative for folding back. The synergy between the teacher and the environment was important as the students could see their current written rule above the task and the teacher pointed their attention to the fact that their solution was not according to the rule. The environment provided a platform for experimenting in more than the only case in the task, but the role of the teacher was important in guiding the students to use this functionality of the environment. The teacher adapted her support to the situation to compensate for the fact that the environment could not recognize differences between students or the need for folding back when students answered a task correctly.

4.1.3 The teacher reinforces guidance from the environment

In some cases, the teachers only emphasized the prompt given by the learning environment. Unlike in the first type of synergistic guidance, the teachers only insisted that the students follow the prompt without having to intervene in other ways to achieve folding back to Image Making. For example, when Avery, Lucas, and Samuel (second grade) experimented in Lab, they noticed a property, which they wrote as “if birds have same weight, (they are at) same spot (distance).” They solved task 1 correctly (7 kg 2 m ∧ 7 kg 2 m). In task 2, the students tried to use the same rule and Avery said “put twelve”, but since this was not possible, they decided to “put the biggest it can be” (12 kg 1 m ∧ 7 kg 1 m). Thus, the environment prompted them to return to Lab and the teacher gave some additional advice:

1 Mr. Lewis: No worries. Let’s go back to Lab and try the same thing there. […]
7 Mr. Lewis: Let’s try, for example one and two kilograms [1 kg ∧ 2 kg].
8–42 [Students make several experiments that lead to an unbalanced beam. Finally, they build a balance case 1 kg 2 m ∧ 2 kg 1 m.]
43 Samuel: Now it stayed.
44 Lucas: I said.
45 Mr. Lewis: Yeah, what is it? They are in different places.
46 Samuel: You just have to look. No, there is one kilogram…
47 Mr. Lewis: Yeah.
48 Samuel:…and there is two meters [points to the numbers]. Yeah [flaps his forehead].
49 Mr. Lewis: Would it work? How could it be said with words, the rule? When you think about the numbers. What did you understand?
50 Samuel: Yes, we understood that there is one kilogram and two meters, and then two kilograms and one meter…
51 Mr. Lewis: Yeah.
52 Samuel:…like here are kilograms and there are (inaudible: other kilograms), but they have been kind of switched.

Because of the prompt by the environment and by the teacher, the students folded back to Image Making (turns 8–42). Their image making was now thicker as they knew that their previous property did not cover all the cases and that they should explore cases where the birds did not have equal weights. This was emphasized by the teacher’s advice in selecting the weights (turn 7). Finally, the students noticed a new property ($w_1 = d_2$ and $w_2 = d_1$, turns 50–52).

In this type of interplay, the teachers reinforced the prompt by the environment. In several groups, the students did not stay in Lab and experiment when the teacher did not insist that they do so. In addition, sometimes the teachers emphasized something when returning to Lab (e.g., Mr. Lewis selected the weights).

4.2 Non-synergistic guidance

In non-synergistic guidance, the teachers’ support was not aligned with the support from the environment. The environment prompted the students to return to Lab to experiment there, but the teachers guided the students in different directions in two ways. First, the teachers guided the students to solve the task through experimenting. Second, the teachers gave the students hints to guess the rule.

4.2.1 The teacher guides to solve a task through experimenting

In some cases, the teachers contradicted the prompt from the environment to return to Lab and guided students to re-start the task and solve it with a new try. For example, Aubrey, Elliot and Ryan (fourth grade) had noticed the property of switched numbers ($w_1=d_2$ and $w_2=d_1$) which they wrote as "opposite numbers." They used it in task 1 (7 kg 2 m ∧ 2 kg 7 m). However, as it could not be used in task 2, the students solved task 2 incorrectly. When the environment prompted them to return to Lab, the teacher contradicted the prompt:

1 Mr. Martin: You can click Start from there below. […]
3 Mr. Martin: Like that. You will get a new try from there. Think for a moment.
4–5 [Ryan and Aubrey make disappointed sounds and faces.]
6 Mr. Martin: It is not far. You have worked really well…
7 Ryan: Look. If we had. No, nothing.
8 Mr. Martin: … (inaudible) there are always some small obstacles.
9 Elliot: Put four (inaudible).
10 Aubrey: Choose one kilo and put it to twelve. Then, put five. Let's test five. […]
14 [Non-balance case 12 kg 1 m ∧ 5 kg 8 m. Students re-start the task.]
15 Aubrey: No.
16 Elliot: Four. Four.
17 Aubrey: Our technique does not work, our rule. […]
20 [Non-balance case 12 kg 1 m ∧ 4 kg 6 m. Students re-start the task.]
21 Elliot: Well, four. Then, there.
22 [Balance case 12 kg 1 m ∧ 4 kg 3 m.]
23 Ryan: Yes.
24 Mr. Martin: Oh! It was a bit random shooting now, but, but …
25 Ryan: Well yeah, but
26 Mr. Martin: …it's not always, it's not always.
27 Elliot: I figured it out now, because it is three and then it goes four, three, twelve [points to 4 kg, 3 m and 12 kg in the screen].
28 Mr. Martin: Good. Good, Elliot.
29 Aubrey: So, four times three equals 12. So, you can think like that.

The teacher instructed the students to re-start the task even though the environment prompted to return to Lab. This led the students to fold back to Image Making and search for balance in the task as if they were in Lab (turns 7–22). Their understanding at Image Making was thicker as they knew that the previously noticed property did not work in that case and that they must look for other connections between the variables. Finally, the students also noticed another property, i.e., the mass of the bird sitting at 1 m equals the product of the mass and distance of the other bird (turns 27 and 29). Thus, although the teacher contradicted the guidance from the environment, it helped the students to improve their understanding through folding back to Image Making. However, now the students had built their rule in a more constrained situation based on only one balance case, as opposed to using Lab to experiment with freely chosen variables to construct a rule and then try to apply the rule to different weights in the task.

Contrary to the above episode, re-starting a task did not always lead to improved understanding. In some cases, the focus of the activity changed from experimenting to building a rule to experimenting just to find a balance. This happened to Gabriel, Ella and Mia (second grade) when they solved task 2 incorrectly and ignored the prompts by the environment and by Mr. Clark to return to Lab. They re-started the task several times and tried to search for balance. They spent most of their time trying out different weights and locations and did not develop new rules, although the teacher tried to orient their image making towards experimenting to build rules: "the purpose is not to just get them in balance but to construct a rule that is a principle of how to get a balance."

In this category, the teacher support contradicted the support from the environment by guiding the students to experiment within the task. This could lead to improved understanding of the balance rule through folding back, or to abandoning the development of the rule and just focus on finding a balance.

4.2.2 The teacher gives hints to guess the rule

Some teachers also guided the students to propose different rules based on the teacher's hints instead of experimentation in Lab. For example, Benjamin, Dylan, and Layla (fourth grade) solved task 4 incorrectly several times. Instead of guiding the students to experiment in Lab, the teacher started to hint towards multiplication:

1 Mr. Lopez: If you forget plus and minus. Then division, and what would be the friend of division. What kinds of calculations can you make, if you divide? So, then, if we reverse. What is the opposite of division?
2 Benjamin: Multiplication.
3 Mr. Lopez: Um. How could you multiply these numbers? So that you could through that.
4 Benjamin: Well, 4 × 6. […]
5 [Dylan re-builds a balance case of 6 kg 6 m ∧ 9 kg 4 m that they had previously.]
6 Mr. Lopez: Yeah, just right. Now it's in balance. Now, what do we have there? What did you Benjamin just [inaudible]? That we multiply.
7 Benjamin: We multiply four by six.
8 Mr. Lopez: Here?
9 Benjamin: Yes.
10 Mr. Lopez: What do we do then? What is it?
11 Benjamin: 24.
12 Mr. Lopez: Yeah. And what do we do then?
13 Benjamin: Um, 24.
14 Mr. Lopez: Do you Dylan have anything?
15 Dylan: Well, if all the numbers are added together and if it would give.
16 Mr. Lopez: Um. Do these two [points at 6 kg and 6 m] and these two [points at 9 kg and 4 m] numbers have something common?
17 Dylan: No.
18 Benjamin: They all are less than 10.
19 Mr. Lopez: Yeah, but if we think about a calculation. [Pause.] Let's use multiplication. What is 6 × 6? What is 6 × 6?
20 Benjamin: 36.
21 Mr. Lopez: Yeah. [Points to 9 kg and 4 m on the screen.]
22 Benjamin: 9 × 4.

23 Mr. Lopez: What is 9×4?
24 Dylan: 36.
25 Mr. Lopez: Yeah.
26 Dylan: Um, they are the same.

Based on their image and the teacher's hint (turn 1), Benjamin suggested a particular multiplication to have something to do with the balance (turn 4). The students were at Image Having as they had some idea about how the balance beam works and stated only a partial relation between the variables. The teacher asked the students to continue this idea (turns 6–14). When Dylan (turn 15) and Benjamin (turn 18) gave some suggestions, the teacher's feedback suggested to them that they think about something else (turns 16 and 19). Finally, the teacher suggested calculating particular multiplications (turns 19 and 21) and the students noticed the products to be equal. Thus, the teacher guided the students to use the teacher as the source for finding the rule. It may seem as if the students' folded back to Image Making when they suggested something and the teacher gave feedback. However, the students did not work on their images by experimenting. Rather, they gave unconnected suggestions based on the teacher's hints until they reached the point where the teacher was aiming at. Thus, students did not fold back to Image Making.

5 Discussion

We set out to explore the interplay between guidance provided by the teacher and guidance provided by a digital learning environment in supporting folding back. The students' interactions with the digital environment created instances of folding back as intended in the design of the environment. Thus, our results support suggestions that material sources can be designed to intentionally launch folding back (Martin, 2008; Poon & Wong, 2017; Yao & Manouchehri, 2022). In our study, material support made it possible that the need for folding back emerged from the students' work in the environment. However, teachers had to complement material support in several ways, which was crucial for folding back. The teachers were able to adapt their support to how the students received the prompts from the environment and how the students worked. Thus, this study supports the importance of teachers and digital learning environments supporting students in a synergistic way (Martin et al., 2019; Tabak, 2004; Tropper et al., 2015). Furthermore, in the case of synergistic support for complex and challenging processes, such as folding back, our results highlight the importance of teachers adapting support for students. In this study, material support alone seemed insufficient for creating folding back. The reason may be that initiating folding back involves pointing to limitations in students' current understanding and prompting them to take a step back. Productivity of this kind of move, that makes students' work more problematic, may depend on further guidance (Reiser, 2004). Indeed, our results show several ways in which further adaptive guidance from the teacher was important for supporting folding back.

We identified three ways in which the teacher and the environment can synergistically support folding back. First, the environment prompted folding back, and the teacher followed up with adaptive guidance. The prompt from the environment was based on the students' work, and the students saw that their work had some drawbacks. However, for the folding back to actually happen, it was important that the teachers provide further support by taking into account the students' thinking and how they reacted to the prompt. This synergy helped the students react to the prompt from the environment in a productive manner.

This is similar to how Tabak (2004) described that teacher modeling augmented software tools so that culturally appropriate uses of the tools were made visible to the students.

Second, the teachers initiated supplementary instances of folding back on the fly. Here, the teachers augmented the support for folding back, as the support preplanned in the tasks and the prompts from the environment were not enough. Unlike the environment, the teachers could interpret students' thinking behind a correct answer and create an occasion for folding back when needed.

Third, the teachers reinforced the guidance from the environment by repeating the prompts from the environment. Even with this kind of simple guidance, the teachers augmented the support of the environment by insisting that students actually engage in image making and not just quickly visit Lab. This augmentation does not require a thorough analysis of students' ideas, but it still shows teachers adapting their support to students' general ways of receiving the prompt. Although Martin et al. (2019) found that replicating the guidance provided by the software was not as productive as complementing the guidance, we found it important that the teachers insist that students follow the guidance from the environment. This kind of reinforcing may be particularly important in supporting folding back, as the purpose is to challenge the students as opposed to offering help.

We also found instances of non-synergistic guidance where the teachers' support was not aligned with support from the environment. First, the teachers contradicted the prompt from the environment intended to support folding back to Image Making in Lab and guided the students to solve tasks through experimenting. This led the students to fold back to Image Making within the task. Instead of open exploration in Lab, the teachers supported searching for one balance case within the constraints of the task. This is similar to Tower's (1998, 2002) blocking move, where the teacher blocked the potential folding back that was about to happen. Towers questioned the productivity of this move. However, in our study the move seemed to be productive when the teacher blocked a particular kind of folding back prompted by the environment and instead supported another one. Yet, we also found that experimenting within a task can change the nature of the activity so that students only solve the tasks through trial and error without the aim of developing a rule. This is similar to what Martin (2008) calls "going back," as the students go back to Image Making, but they do not connect their new image making with their previous understanding at the outer layer, and thus, their understanding is not thickened. Second, instead of supporting folding back to Image Making by experimenting, the teachers started to give hints for the rule. This led the students to focus on guessing what the teacher was thinking instead of exploring the balance themselves. It seemed as if the students were using the teacher as an experimenting device. Through this process the students expressed a sophisticated rule, but this was not based on image making. Instead, the students just adopted the new rule that was not connected to their understanding at the inner layers. Pirie and Kieren (1994) warned that, if a teacher offers information in a ready-made form, it may lead to a disjointed piece of understanding that may be difficult to use in building further understanding.

While this study shows that material support can be designed to intentionally support folding back, it is important that teachers share this intention. Non-synergistic guidance shows that teachers may change the nature of the activity. Thus, our results support Tabak's (2004) suggestion that productive synergy requires teachers' conceptions to be consistent with material support. However, our results also show that even non-synergistic guidance where a teacher contradicts the guidance from the environment can productively support folding back.

As an implication for practice, the results suggest that folding back can be used as a design principle for creating digital learning environments in a similar way to how Martin and Towers (2016a) described the use of folding back as a pedagogical tool in planning lessons. Environments can be planned not only to help students, but also to purposefully make things more problematic for students at an appropriate point in the growth of their understanding. However, in the case of folding back, the teacher plays an important role in complementing support from the environment. Thus, learning environments can be enriched by planning synergistic guidance from the environment and the teacher. Teachers could be prepared to follow up on and augment the guidance from the environment, launch supplementary folding back, and reinforce the prompts from the environment.

In addition, suggestions for improving the design of this and similar digital learning environments can be drawn from the study. To emphasize the image making activity, the environment could automatically transfer the students to Lab after an incorrect answer to a task. To emphasize solving tasks by applying the rule, students could be prompted to explain how the rule is used and to return to Lab even before answering the task if they cannot use their current rule. However, despite any improvements that can be made, teachers will still play an important role in supporting understanding in digital environments, as they can establish synergistic interactions with the environment that ensure that support is responsive to students' needs.

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