THE VOLUME OF THE BOUNDARY OF A SOBOLEV (p, q)-EXTENSION DOMAIN

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Abstract. Let $n \geq 2$ and $1 \leq q < p < \infty$. We prove that if $\Omega \subset \mathbb{R}^n$ is a Sobolev (p, q)-extension domain, with additional capacitary restrictions on the boundary in the case $q \leq n-1$, $n > 2$, then $|\partial\Omega| = 0$. In the case $1 \leq q < n-1$, we give an example of a Sobolev (p, q)-extension domain with $|\partial\Omega| > 0$.

1. Introduction

Let $1 \leq q \leq p \leq \infty$. Then a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is said to be a Sobolev (p, q)-extension domain if there exists a bounded extension operator

$$E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n).$$

Partial motivation for the study of Sobolev extensions comes from PDEs (see, for example, [27]). In [2, 36] it was proved that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, then there exists a bounded linear extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$, for each $k \geq 1$ and all $1 \leq p \leq \infty$. Here $W^{k,p}(\Omega)$ is the Banach space of L^p-integrable functions whose weak derivatives up to order k belong to $L^p(\Omega)$. More generally, the notion of (ε, δ)-domains was introduced in [16] and it was proved that, for every (ε, δ)-domain there exists a bounded linear extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$, for all $k \geq 1$ and $1 \leq p \leq \infty$.

A geometric characterization of simply connected planar Sobolev $(2, 2)$-extension domains was obtained in [43]. By later results in [18, 20, 21, 33], one now understands the geometry of simply connected planar Sobolev (p, p)-extension domains, for all $1 \leq p \leq \infty$. Geometric characterizations are also known in the case of homogeneous Sobolev spaces $L^{k,p}(\Omega)$, $2 < p < \infty$, defined on simply connected planar domains. Here $L^{k,p}(\Omega)$ is the seminormed space of locally integrable functions whose kth-order distributional partial derivatives belong to $L^p(\Omega)$. However, no characterizations are available in the general setting.

The boundary $\partial\Omega$ of a Sobolev (p, p)-extension domain is necessarily of volume zero when $1 \leq p < \infty$ by results in [9]. Actually, Ω has to be Ahlfors regular in the sense that

$$|B(x, r) \cap \Omega| \geq C|B(x, r)| \tag{1.1}$$

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for every $x \in \partial\Omega$ and all $0 < r < \min\{1, \frac{1}{4}\operatorname{diam}\Omega\}$ with a constant C independent of x, r. Even more is known if Ω is additionally a planar Jordan domain. In this case Ω has to be a so-called John domain when $1 \leq p \leq 2$ and the complementary domain needs to be a John domain when $2 \leq p < \infty$. Consequently, the Hausdorff dimension of $\partial\Omega$ is necessarily strictly less than two by results in [22]. For a sharp estimate see the very recent paper [26]. However, in general, the Hausdorff dimension of the boundary of a Sobolev (p,p)-extension domain $\Omega \subset \mathbb{R}^n$ can well be n.

Much less is known when $q < p$. Even though the case of Hölder-type cuspidal boundaries has been studied in detail [8, 28, 29, 30, 31], no geometric criteria are available even when Ω is planar and Jordan. The only existing result related to (1.1) is the generalized Ahlfors-type estimate [39] (also see [40])

$$\Phi(B(x,r))^{p-q}|B(x,r) \cap \Omega|^q \geq C|B(x,r)|^p \tag{1.2}$$

in the case $n < q < p < \infty$. Here Φ is a bounded and quasiadditive set function, see Section 3, defined on open sets $U \subset \mathbb{R}^n$. It is generated by the Sobolev (p,q)-extension property. By differentiating Φ with respect to the Lebesgue measure, one concludes that $|\partial\Omega| = 0$ if Ω is a Sobolev (p,q)-extension domain for $n < q < p < \infty$.

In this paper, we establish the optimal (capacitary) version of the generalized Ahlfors-type condition (1.2) for all $1 \leq q < p < \infty$, under additional capacitary restrictions on the boundary in the case $1 \leq q \leq n-1$, $n \geq 3$. With the help of the Lebesgue differentiation theorem it gives the following conclusion. The definition of q-fatness can be found in Section 2.

Theorem 1.1. *Let $1 \leq q < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a Sobolev (p,q)-extension domain which is q-fat at almost every $x \in \partial\Omega$. Then $|\partial\Omega| = 0$. In particular, if $\Omega \subset \mathbb{R}^2$ is a Sobolev (p,q)-extension domain with $1 \leq q < p < \infty$, then $|\partial\Omega| = 0$. Moreover, if $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ is a Sobolev (p,q)-extension domain with $n-1 < q < p < \infty$, then $|\partial\Omega| = 0$.*

Our second result shows that one indeed needs to pose some additional assumption besides the extension property in order to guarantee that $|\partial\Omega| = 0$ when $1 \leq q < n-1$.

Theorem 1.2. *Let $n \geq 3$ and $1 \leq q < n-1$. Then there exists $p > q$ and a Sobolev (p,q)-extension domain $\Omega \subset \mathbb{R}^n$ with a bounded linear extension operator such that $|\partial\Omega| > 0$.*

This paper is organized as follows. Section 2 contains definitions and preliminary results. Section 3 is devoted to set functions associated with extension operators. In Section 4, we establish the generalized Ahlfors density condition. In Section 5, we deduce Theorem 1.1 from our generalized Ahlfors density condition. Section 6 contains a discussion on Sobolev extension operators for outward cuspidal domains that give the sharpness of the generalized Ahlfors density condition. Section 7 is devoted to the construction behind Theorem 1.2. In the final section, Section 8, we pose open problems that arise from the results in this paper and discuss the locality of our estimates.

2. Preliminaries

2.1. **Definitions and notations.** Let Ω be a domain in the n-dimensional Euclidean space $\mathbb{R}^n$ with $n \geq 2$. By the symbol $\operatorname{Lip}(\Omega)$ we denote the class of all Lipschitz continuous functions defined on Ω. The Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, (see, for example, [27]) is defined as a Banach space of locally integrable and weakly differentiable functions $u : \Omega \to \mathbb{R}$ equipped with the norm:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)},$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$ is a weak gradient of u.

Let us give the definition of Sobolev extension domains.

Definition 2.1. *Let $1 \leq q \leq p < \infty$. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to be a Sobolev (p,q)-extension domain, if there exists a bounded operator*

$$E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$$

such that for every function $u \in W^{1,p}(\Omega)$, the function $E(u) \in W^{1,q}(\mathbb{R}^n)$ satisfies $E(u)\big|_{\Omega} \equiv u$ and

$$\|E\| := \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|E(u)\|_{W^{1,q}(\mathbb{R}^n)}}{\|u\|_{W^{1,p}(\Omega)}} < \infty.$$

Linear Sobolev extension operators form a subclass of homogeneous Sobolev extension operators. We prove that, for every Sobolev (p,q)-extension domain, there always exists a positively homogeneous Sobolev extension operator. When $q = p$ one in fact can find a linear extension operator [9] but it is not known if this could be the case when $q < p$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a Sobolev (p,q)-extension domain. Then every bounded Sobolev extension operator $E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$ promotes to a bounded, positively homogeneous Sobolev extension operator $E_h : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$ with the operator norm inequality $\|E_h\| \leq \|E\|$.*

Proof. Let $u \in W^{1,p}(\Omega)$. When $\|u\|_{W^{1,p}(\Omega)} \neq 0$, we define

$$E_h(u) := \|u\|_{W^{1,p}(\Omega)} E\left(\frac{u}{\|u\|_{W^{1,p}(\Omega)}}\right).$$

Then for $\lambda \geq 0$ we have

$$\begin{aligned} E_h(\lambda u) = \|\lambda u\|_{W^{1,p}(\Omega)} E\left(\frac{\lambda u}{\|\lambda u\|_{W^{1,p}(\Omega)}}\right) &= \lambda \|u\|_{W^{1,p}(\Omega)} E\left(\frac{\lambda u}{\lambda \|u\|_{W^{1,p}(\Omega)}}\right) \\ &= \lambda \|u\|_{W^{1,p}(\Omega)} E\left(\frac{u}{\|u\|_{W^{1,p}(\Omega)}}\right) = \lambda E_h(u). \end{aligned}$$

If $\|u\|_{W^{1,p}(\Omega)} = 0$, then necessarily $E(u) = 0$ and we set $E_h(u) = 0$. Finally,

$$\begin{aligned}\|E_h\| &= \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{1}{\|u\|_{W^{1,p}(\Omega)}} \left\| \|u\|_{W^{1,p}(\Omega)} E\left(\frac{u}{\|u\|_{W^{1,p}(\Omega)}}\right)\right\|_{W^{1,q}(\mathbb{R}^n)} \\ &= \sup_{\|u\|_{W^{1,p}(\Omega)}=1} \|E(u)\|_{W^{1,q}(\mathbb{R}^n)} \leq \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\|E(u)\|_{W^{1,q}(\mathbb{R}^n)}}{\|u\|_{W^{1,p}(\Omega)}} = \|E\|.\end{aligned}$$

□

By Lemma 2.1, from now on, we may always assume that $E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$ is a positively homogeneous, bounded Sobolev extension operator.

We continue with the definition of a strong bounded Sobolev extension operator.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a Sobolev (p, q)-extension domain with $1 \leq q \leq p < \infty$. A bounded Sobolev extension operator $E_s : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$ is said to be a strong bounded Sobolev extension operator if, for every function $u \in W^{1,p}(\Omega)$ with $u\big|_{B(x,r)\cap\Omega} \equiv c$ for some ball $B(x, r)$ with $B(x, r) \cap \Omega \neq \emptyset$ and some constant $c \in \mathbb{R}$, we have $E_s(u)(y) = c$ for $\mathcal{H}^n$-almost every $y \in B(x, r) \cap \partial\Omega$.*

2.2. **Fine Topology.** In this section, we recall some basic facts about the fine topology on $\mathbb{R}^n$. It is the coarsest topology on $\mathbb{R}^n$ in which all superharmonic functions on $\mathbb{R}^n$ are continuous, see [12, Chapter 12]. Let us recall the definition of the capacity [12]. Let Ω be an open set in $\mathbb{R}^n$ and $F \subset \Omega$ be a compact set. Fix $1 \leq p < \infty$. The class of admissible functions for the pair $(F; \Omega)$ is defined by setting

$$\mathsf{W}_p(F; \Omega) := \left\{u \in C_0(\Omega) \cap W^{1,p}(\Omega) : u \geq 1 \text{ on } F\right\}.$$

The p-capacity of F with respect to Ω is defined by

$$\mathsf{cap}_p(F; \Omega) = \inf_{u\in \mathsf{W}_p(F;\Omega)} \int_\Omega |\nabla u(x)|^p dx.$$

If $U \subset \Omega$ is an open set, we define

$$\mathsf{cap}_p(U; \Omega) = \sup\{\mathsf{cap}_p(F; \Omega) \,:\, F \subset U,\ F \text{ is compact}\}.$$

In the case of an arbitrary set $E \subset \Omega$ we define

$$\mathsf{cap}_p(E; \Omega) = \inf\{\mathsf{cap}_p(U; \Omega) \,:\, E \subset U \subset \Omega,\ U \text{ is open}\}. \tag{2.1}$$

The p-capacity is an outer measure on Ω [12].

Let us recall the notion of variational p-capacity [7, 12, 27].

Definition 2.3. *A condenser in a domain $\Omega \subset \mathbb{R}^n$ is a pair (E, F) of bounded compact subsets of $\overline{\Omega}$ with* $\operatorname{dist}(E, F) > 0$. *Fix $1 \leq p < \infty$. The set of admissible functions for the triple $(E, F; \Omega)$ is*

$$\mathcal{W}_p(E, F; \Omega) = \{u \in W^{1,p}(\Omega) \cap C(\Omega \cup E \cup F) : u \geq 1 \text{ on } E \text{ and } u \leq 0 \text{ on } F\}.$$

We define the p-capacity of the pair (E,F) with respect to Ω by setting.

$$Cap_p(E,F;\Omega) = \inf_{u\in\mathcal{W}_p(E,F;\Omega)} \int_\Omega |\nabla u(x)|^p dx.$$

For a pair (U,V) of arbitrary bounded open subsets of Ω with $\operatorname{dist}(U,V)>0$, we define the p-capacity by setting

$$Cap_p(U,V;\Omega) := \sup_{E\subset U, F\subset V} Cap_p(E,F;\Omega),$$

where supremum is taken over all compact subsets of Ω with $E\subset U$ and $F\subset V$.

To simplify notation, set $\mathcal{W}_p(E,F) := \mathcal{W}_p(E,F;\mathbb{R}^n)$ and $Cap_p(E,F) := Cap_p(E,F;\mathbb{R}^n)$ The following lemma gives the basic Loewner-type capacity estimate. The interested readers can find a proof in [13].

Lemma 2.2. *Let $B\subset\mathbb{R}^n$ be a ball with radius r and $n-1<p<\infty$. Suppose that $E,F\subset B$ are compact connected subsets with $\operatorname{dist}(E,F)>0$ and so that $\operatorname{diam} E\geq \delta r$ and $\operatorname{diam} F\geq\delta r$ for some $0<\delta<2$. Then we have*

$$Cap_p(E,F;B)\geq Cr^{n-p}, \tag{2.2}$$

where the constant C only depends on δ, n and p. The inequality also holds for $p=1$ when $n=2$.

We have the following capacity estimate for concentric balls. See, for example, [12, page 35].

$$\mathsf{cap}_p(\overline{B(x,r)};B(x,R)) = \begin{cases} \omega_{n-1}\left(\frac{|n-p|}{p-1}\right)^{p-1}\left|R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\right|^{1-p} & p\notin\{1,n\} \\ \omega_{n-1}\log^{1-n}\frac{R}{r} & p=n \end{cases} \tag{2.3}$$

where $0<r<R<\infty$ and ω_{n-1} is the $(n-1)$-dimensional mesure of the unit sphere $S^{n-1}(0,1)$.

Now, we are ready to define quasi-continuous functions, see [12].

Definition 2.4. *Let $1\leq p<\infty$. A function $u\in L^1_{\mathrm{loc}}(\Omega)$ is said to be p-quasi-continuous if, for every $\epsilon>0$, there exists an open set $E_\epsilon\subset\Omega$ with $\mathsf{cap}_p(E_\epsilon;\Omega)<\epsilon$ such that $u\big|_{\Omega\setminus E_\epsilon}$ is continuous.*

We record the fact that every Sobolev function can be redefined in a set of measure zero so as to become quasi-continuous. See [12, Chapter 4]. Actually, the capacity considered in [12] is different from the capacity we consider here. There capacity is defined to be the infimum of the Sobolev norms of all smooth admissible functions. Hence, that capacity is larger than or equal to the capacity we consider here.

Lemma 2.3. *Let $1\leq p<\infty$ and let $u\in W^{1,p}(\Omega)$. Then there exists a p-quasi-continuous function $\tilde{u}\in W^{1,p}(\Omega)$ with $\tilde{u}(z)=u(z)$ for almost every $z\in\Omega$. Furthermore, at every point of continuity of u, we have $\tilde{u}(z)=u(z)$.*

We continue with the definition of p-capacitary density.

Definition 2.5. *Let $1 \le p < \infty$. A set $E \subset \mathbb{R}^n$ is said to be p-capacitary dense at the point $z \in \mathbb{R}^n$, if*

$$\limsup_{r\to 0^+} \frac{\mathsf{cap}_p\left(E\cap B\left(z,\frac{r}{4}\right); B\left(z,\frac{r}{2}\right)\right)}{\mathsf{cap}_p\left(B\left(z,\frac{r}{4}\right); B\left(z,\frac{r}{2}\right)\right)} > 0.$$

The following definition of p-thin sets can be found in [12, Chapter 12] for $1 < p < \infty$ and in [25] for $p = 1$.

Definition 2.6. *Let $1 < p < \infty$. A set E is p-thin at z if*

$$\int_0^1 \left(\frac{\mathsf{cap}_p(E\cap B(z,t); B(z,2t))}{\mathsf{cap}_p(B(z,t); B(z,2t))}\right)^{\frac{1}{p-1}} \frac{dt}{t} < \infty.$$

A set E is 1-thin at z if

$$\lim_{t\to 0} t\frac{\mathsf{cap}_1(E\cap B(z,t); B(z,2t))}{\mathcal{H}^n(B(z,t))} = 0.$$

Furthermore, we say that E is p-fat at z if E is not p-thin at z.

The following proposition shows that capacitary density implies fatness. We will show in Section 5 that actually capacitary density can be strictly stronger than fatness.

Proposition 2.1. *Let $n \ge 3$. If a domain $\Omega \subset \mathbb{R}^n$ is q-capacitary dense at a point $z \in \mathbb{R}^n$ for some $1 \le p < \infty$, then Ω is also p-fat at z.*

Proof. Assuming that Ω is p-capacitary dense at the point z for $1 \le p < \infty$, there exists a positive constant $\delta_z > 0$ and a decreasing positive sequence $\{r_i\}_{i=1}^{\infty}$, which converges to 0, such that

$$\frac{\mathsf{cap}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right); B\left(z,\frac{r_i}{2}\right)\right)}{\mathsf{cap}_p\left(B\left(z,\frac{r_i}{4}\right); B\left(z,\frac{r_i}{2}\right)\right)} > \delta_z \tag{2.4}$$

for every r_i.

Let us first consider the case $p = 1$. We write $A \sim_c B$ if $\frac{1}{c}A \le B \le cA$ for a constant $c > 1$. By [7, Proposition 6.4] we have that

$$\mathsf{cap}_1\left(B\left(z,\frac{r_i}{4}\right); B\left(z,\frac{r_i}{2}\right)\right) \sim_c r_i^{n-1}$$

with an implicit constant independent of r_i. Hence we have

$$\frac{r_i\mathsf{cap}_1\left(\Omega\cap B\left(z,\frac{r_i}{4}\right); B\left(z,\frac{r_i}{2}\right)\right)}{\mathcal{H}^n(B(z,r_i))} > \tilde{\delta}_z > 0.$$

This implies that Ω is 1-fat at z.

Let now $1<p<\infty$. Without loss of generality, we may choose a sequence $\{r_i\}_{i=1}^{\infty}$ with $16r_{i+1}<r_i$ for every $i\in\mathbb{N}$ such that (2.4) holds. By (2.3), we have

$$\mathsf{cap}_p\left(B\left(z,\rho\right);B\left(z,2\rho\right)\right)\sim_c \mathsf{cap}_p\left(B\left(z,\frac{r_i}{4}\right);B\left(z,\frac{r_i}{2}\right)\right) \tag{2.5}$$

for every $\rho\in(\frac{r_i}{4},\frac{r_i}{2})$ with a constant c independent of ρ and r_i. Since $\rho\in(\frac{r_i}{4},\frac{r_i}{2})$,

$$\mathsf{W}_p\left(\Omega\cap B\left(z,\rho\right);B\left(z,2\rho\right)\right)\subset \mathsf{W}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right);B\left(z,2\rho\right)\right).$$

Hence, we have

$$\mathsf{cap}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right);B\left(z,2\rho\right)\right)\leq \mathsf{cap}_p\left(\Omega\cap B(z,\rho);B\left(z,2\rho\right)\right). \tag{2.6}$$

Let $u\in\mathsf{W}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right);B\left(z,2\rho\right)\right)$ be arbitrary. Then we define a function

$$\tilde{u}\in\mathsf{W}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right);B\left(z,\frac{r_i}{2}\right)\right)$$

by setting

$$\tilde{u}(x):=\begin{cases}u(x) & \text{if } x\in B(z,\frac{r_i}{4})\\ u\left((x-z)\frac{8\rho-r_i}{r_i}+\left(\frac{r_i}{2}-2\rho\right)\frac{x-z}{|x-z|}+z\right) & \text{if } x\in B\left(z,\frac{r_i}{2}\right)\setminus B\left(z,\frac{r_i}{4}\right)\end{cases}.$$

By the fact that $\frac{r_i}{4}\leq\rho\leq\frac{r_i}{2}$, we have

$$\int_{B\left(z,\frac{r_i}{2}\right)}|\nabla\tilde{u}(x)|^p dx\leq C\int_{B(z,2\rho)}|\nabla u(x)|^p dx$$

with a constant C independent of z, Ω and $\rho\in(\frac{r_i}{4},\frac{r_i}{2})$. Since the test function u was arbitrary, we have

$$\mathsf{cap}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right);B\left(z,\frac{r_i}{2}\right)\right)\leq C\mathsf{cap}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right);B\left(z,2\rho\right)\right) \tag{2.7}$$

with an absolute positive constant C independent of $\rho\in(\frac{r_i}{4},\frac{r_i}{2})$. By combining inequalities (2.6) and (2.7), we obtain

$$\mathsf{cap}_p\left(\Omega\cap B\left(z,\frac{r_i}{4}\right);B\left(z,\frac{r_i}{2}\right)\right)\leq C\mathsf{cap}_p\left(\Omega\cap B\left(z,\rho\right);B\left(z,2\rho\right)\right) \tag{2.8}$$

with a positive constant C independent of $\rho\in(\frac{r_i}{4},\frac{r_i}{2})$. Finally, by combining inequalities (2.4), (2.5) and (2.8), we obtain

$$\frac{\mathsf{cap}_p\left(\Omega\cap B(z,\rho);B\left(z,2\rho\right)\right)}{\mathsf{cap}_p\left(B(z,\rho);B\left(z,2\rho\right)\right)}>\tilde{\delta}_z$$

where $\tilde{\delta}_z > 0$ is a positive constant independent of $\rho \in (\frac{r_i}{4}, \frac{r_i}{2})$. Since $16r_{i+1} < r_i$ for every $i \in \mathbb{N}$, we have

$$\int_0^1 \left(\frac{\mathsf{cap}_p\left(\Omega \cap B(z,\rho); B\left(z,2\rho\right)\right)}{\mathsf{cap}_p\left(B(z,\rho); B\left(z,2\rho\right)\right)} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

$$\geq \sum_{i=1}^{\infty} \int_{\frac{r_i}{4}}^{\frac{r_i}{2}} \left(\frac{\mathsf{cap}_p\left(\Omega \cap B(z,\rho); B\left(z,2\rho\right)\right)}{\mathsf{cap}_p\left(B(z,\rho); B\left(z,2\rho\right)\right)} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

$$\geq \sum_{i=1}^{\infty} \frac{(\tilde{\delta}_z)^{\frac{1}{p-1}}}{2} = \infty.$$

Hence Ω is p-fat at the point z. $\square$

Let us consider the role of p in the validity of fatness and capacitary density. Given $x \in \mathbb{R}^n$ and $0 < s < t$, we set $A(x; s, t) := B(x, t) \setminus B(x, s)$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $n - 1 < p < \infty$. Then Ω is p-capacitary dense at every point of the boundary. A planar domain $\Omega \subset \mathbb{R}^2$ is also 1-capacitary dense at every point of the boundary.*

Proof. We may assume that Ω is not entire $\mathbb{R}^n$. Fix $x \in \partial\Omega$. Given $0 < t <$ diam $(\Omega)/3$, we may pick points $z \in B(x, t/2) \cap \Omega$ and $y \in \Omega \setminus B(x, 3t)$. Since Ω is open and connected, we find a curve γ that joins z to y in Ω. This curve gives us compact connected sets $E_t \subset \Omega \cap B(x, t)$ and $F_t \subset \Omega \cap A(x; 2t, 3t)$ with

$$\frac{t}{2} \leq \operatorname{diam} E_t$$

and

$$\operatorname{diam}(F_t) \geq t.$$

For every admissible function $u \in \mathsf{W}_p(\Omega \cap B(x, t); B(x, 2t))$, simply extend test functions to be zero outside the ball $B(x, 2t)$. We obtain an admissible function belongs to the class $\mathcal{W}_p(E_t, F_t; B(x, 4t))$. By Lemma 2.2, for every $n - 1 < p < \infty$, we have

$$\mathsf{cap}_p(\Omega \cap B(x,t); B(x,2t)) \geq Cap_p(E_t, F_t; B(x,4t)) \geq Ct^{n-p}$$

for some positive constant C independent of x and t. By (2.3) we conclude that

$$\limsup_{t \to 0^+} \frac{\mathsf{cap}_p\left(\Omega \cap B(x,t); B(x,2t)\right)}{\mathsf{cap}_p\left(B(x,t); B(x,2t)\right)} = \delta_x > 0.$$

Consequently, the domain Ω is p-capacitary dense at the point $x \in \partial\Omega$.

Finally, let us assume that $n = 2$ and $p = 1$. Similarly as above, by Lemma 2.2, we have

$$\mathsf{cap}_1\left(\Omega \cap B(x,t); B(x,2t)\right) \geq Cap_1\left(E_t, F_t; B(x,4t)\right) \geq Ct.$$

Since

$$\mathsf{cap}_1\left(B(x,t);B(x,2t)\right)\sim_c t$$

for some positive constant c independent of x and t, we have

$$\limsup_{t\to 0}\frac{\mathsf{cap}_1(\Omega\cap B(x,t);B(x,2t))}{\mathsf{cap}_1(B(x,t);B(x,2t))}=\delta_x>0.$$

Consequently, the domain $\Omega\subset\mathbb{R}^2$ is 1-capacitary dense at the point $x\in\partial\Omega$. □

Remark 2.9. We actually proved that

$$\mathsf{cap}_p(\Omega\cap B(x,t);B(x,2t))\geq Cap_p(E_t,F_t;B(x,4t))\geq C_p t^{n-p} \tag{2.10}$$

whenever $x\in\partial\Omega$, $0<t<\frac{1}{4}\operatorname{diam}(\Omega)$ and $p>n-1$ (also for $p=1$ in the plane).

Definition 2.7. *Let $1\leq p<\infty$. A set $U\subset\mathbb{R}^n$ is p-finely open if $\mathbb{R}^n\setminus U$ is p-thin at every $x\in U$, and*

$$\tau_p:=\{U\subset\mathbb{R}^n;U\text{ is }p-\text{finely open}\}$$

is the p-fine topology on $\mathbb{R}^n$.

The following lemma comes from [12, Corollary 12.18].

Lemma 2.4. *Suppose that a set $E\subset\mathbb{R}^n$ is p-fat at the point $x\in\mathbb{R}^n$. Then every p-finely open neighborhood of x intersects E. Consequently, x is a p-fine limit point of E.*

By a result due to Fuglede [4], we have the following lemma.

Lemma 2.5. *Let $1\leq p<\infty$. If a function u is p-quasi-continuous, then u is p-finely continuous except on a subset of p-capacity zero.*

By using the lemmata above, we can prove the following lemma. It is also a corollary of the result in [17].

Lemma 2.6. *Let $\Omega\subset\mathbb{R}^n$ be a domain such that Ω is p-fat at almost every point of the boundary $\partial\Omega$. If $u\in W^{1,p}(\mathbb{R}^n)$ is a Sobolev function such that $u\big|_{B(x,r)\cap\Omega}\equiv c$, where $x\in\partial\Omega$, $0<r<1$ and $c\in\mathbb{R}$, then $u(z)=c$ for almost every $z\in\partial\Omega\cap B(x,r)$.*

Proof. By Lemma 2.3, u has a p-quasi-continuous representative $\tilde{u}$ with $\tilde{u}(z)=c$ for every $z\in\Omega\cap B(x,r)$. By Lemma 2.5 and [3, Theorem 4.17], there exists a subset $E_1\subset\mathbb{R}^n$ with $|E_1|=0$ such that $\tilde{u}$ is p-finely continuous on $\mathbb{R}^n\setminus E_1$. Since Ω is p-fat at almost every $z\in\partial\Omega$, by Lemma 2.4, there exists a subset $E_2\subset\partial\Omega$ with $|E_2|=0$ such that, for every $z\in(\partial\Omega\cap B(x,r))\setminus(E_1\cup E_2)$, we have $\tilde{u}(z)=c$. Hence $u(z)=c$ for almost every $z\in\partial\Omega\cap B(x,r)$. □

2.3. **Gromov hyperbolicity.** For each $1 \leq q < n-1$, we will construct a Sobolev (p,q)-extension domain whose boundary is of positive volume. In order to establish the extension property of the domain, we will employ an approximation argument. Our domain Ω turns out to be δ-Gromov hyperbolic with respect to the quasihyperbolic metric, which implies that $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$.

Definition 2.8. *Let $\Omega \subsetneqq \mathbb{R}^n$ be a domain. Then the associated quasihyperbolic distance between a pair of points $x, y \in \Omega$ is defined as*

$$\operatorname{dist}_{qh}(x,y) = \inf_{\gamma} \int_{\gamma} \frac{dz}{\operatorname{dist}(z, \partial\Omega)},$$

where the infimum is taken over all the rectifiable curves $\gamma \subset \Omega$ connecting x and y. A curve attaining this infimum is called a quasihyperbolic geodesic between x and y. The distance between two sets is also defined in a similar manner.

The existence of quasihyperbolic geodesics comes from a result by Gehring and Osgood [5]. We continue with the definition of Gromov hyperbolicity with respect to the quasihyperbolic metric.

Definition 2.9. *Let $\delta > 0$. A domain is called δ-Gromov hyperbolic with respect to the quasihyperbolic metric, if for all $x, y, z \in \Omega$ and every corresponding quasihyperbolic geodesic $\gamma_{x,y}$, $\gamma_{y,z}$ and $\gamma_{x,z}$, we have*

$$\operatorname{dist}_{qh}(w, \gamma_{y,z} \cup \gamma_{x,z}) \leq \delta,$$

for arbitrary $w \in \gamma_{x,y}$.

Let us give the definition of quasiconformal mappings.

Definition 2.10. *Let Ω, Ω' be domains in $\mathbb{R}^n$ and let $1 \leq K < \infty$. A homeomorphism $f : \Omega \to \Omega'$ of the class $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to be a K-quasiconformal mapping, if*

$$|Df(x)|^n \leq K J_f(x), \text{ for almost every } x \in \Omega.$$

Here $|Df(x)|$ means the operator norm of the matrix $Df(x)$ and $J_f(x)$ is its determinant.

The following result was proved in [1].

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain which is quasiconformally equivalent to the unit ball. Then Ω is δ-Gromov hyperbolic with respect to the quasihyperbolic metric, where $\delta > 0$ depends only on the quasiconformality constant K and n.*

The following density result comes from [19].

Lemma 2.8. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain that is δ-Gromov hyperbolic with respect to the quasihyperbolic metric, then, for every $1 \leq p < \infty$, $W^{1,\infty}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

3. A quasiadditive set function

In this section, we will introduce a quasiadditive set function that is a minor modification to the one introduced by Ukhlov in [39, 40]. Also see [44, 45] for related set functions.

Let us recall that a function Φ defined on the class of open subsets of $\mathbb{R}^n$ and taking nonnegative values is called a quasiadditive set function (see, for example, [44]) if, for all open sets $U_1 \subset U_2 \subset \mathbb{R}^n$, we have

$$\Phi(U_1) \leq \Phi(U_2),$$

and there exists a positive constant C such that for every collection of pairwise disjoint open sets $\{U_i \subset \mathbb{R}^n\}_{i\in\mathbb{N}}$ we have

$$\sum_{i=1}^{\infty} \Phi(U_i) \leq C\Phi\left(\bigcup_{i=1}^{\infty} U_i\right). \tag{3.1}$$

The upper and lower derivatives of a quasiadditive set function are

$$\overline{D\Phi}(x) = \limsup_{r\to 0^+} \frac{\Phi(B_r)}{|B_r|} \text{ and } \underline{D\Phi}(x) = \liminf_{r\to 0^+} \frac{\Phi(B_r)}{|B_r|}.$$

Let us formulate a result from [38, 44] in a convenient form.

Lemma 3.1. *Let Φ be a quasiadditive set function defined on open subsets of $\mathbb{R}^n$. Then*

(1) *for every open set $U \subset \mathbb{R}^n$ we have*

$$\int_U \overline{D\Phi}(x)\, dx \leq C\Phi(U);$$

(2) *for almost all points $x \in \mathbb{R}^n$, the upper derivative is finite and*

$$\overline{D\Phi}(x) \leq C\underline{D\Phi}(x) < \infty.$$

The constants C above are the same as the one in (3.1).

We continue with a detailed construction of a set function associated with an extension operator, refining the constructions of [39, 40]. Let $\Omega \subset \mathbb{R}^n$ be a bounded Sobolev (p,q)-extension domain and $E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$ be the corresponding bounded extension operator. Suppose that $U \subset \mathbb{R}^n$ is an open set such that $U \cap \Omega \neq \emptyset$. We define

$$W_0^p(U,\Omega) := \left\{u \in C(\Omega) \cap W^{1,p}(\Omega) : u \equiv 0 \text{ on } \Omega \setminus U\right\}.$$

For every open set $U \subset \mathbb{R}^n$ with $U \cap \Omega \neq \emptyset$ and every $u \in W_0^p(U,\Omega)$, we define the q-Dirichlet energy $\Gamma_U^q(u)$ on U and with respect to the boundary value u by setting

$$\Gamma_U^q(u) := \inf\left\{\left(\int_U |\nabla v(z)|^q dz\right)^{\frac{1}{q}} : v \in W^{1,q}(U), v\big|_{U\cap\Omega} \equiv u\right\}. \tag{3.2}$$

Then the set function Φ is defined on U by setting

$$\Phi\left(U\right):=\sup\left\{\left(\frac{\Gamma_U^q(u)}{\|u\|_{W^{1,p}(U\cap\Omega)}}\right)^k : u\in W_0^p\left(U,\Omega\right)\right\} \tag{3.3}$$

with $\frac{1}{k}=\frac{1}{q}-\frac{1}{p}$, and by setting $\Phi(U)=0$ for those open sets that do not intersect Ω.

The Dirichlet energy defined in (3.2) has the following homogeneity.

Lemma 3.2. *Let $U\subset\mathbb{R}^n$ be an open set with $U\cap\partial\Omega\neq\emptyset$. Then for every $u\in W_0^p(U,\Omega)$ and every $\lambda\geq 0$, we have $\Gamma_U^q(\lambda u)=\lambda\Gamma_U^q(u)$.*

Proof. Let $\{v_i\}_{i=1}^\infty\subset W^{1,q}(U)$ be a sequence of functions with $v_i\big|_{U\cap\Omega}\equiv u$ and

$$\Gamma_U^q(u)=\lim_{i\to\infty}\left(\int_U|Dv_i(z)|^q dz\right)^{\frac{1}{q}}.$$

Then we claim that

$$\Gamma_U^q(\lambda u)=\lim_{i\to\infty}\left(\int_U|D\lambda v_i(z)|^q dz\right)^{\frac{1}{q}}=\lambda\Gamma_U^q(u).$$

If this fails, then there exists another sequence $\{\tilde{v}_i\}_{i=1}^\infty\subset W^{1,q}(U)$ with $\tilde{v}_i\big|_{U\cap\Omega}\equiv\lambda u$ and

$$\Gamma_U^q(\lambda u)=\left(\int_U|D\tilde{v}_i(z)|^q dz\right)^{\frac{1}{q}}<\lambda\Gamma_U^q(u).$$

Then we have $\{\frac{1}{\lambda}\tilde{v}_i\}_{i=1}^\infty\subset W^{1,q}(U)$ with $\frac{1}{\lambda}\tilde{v}_i\big|_{U\cap\Omega}\equiv u$ and

$$\frac{1}{\lambda}\Gamma_U^q(\lambda u)=\left(\int_U\left|D\frac{1}{\lambda}\tilde{v}_i(z)\right|^q dz\right)^{\frac{1}{q}}<\Gamma_U^q(u).$$

This is a contradiction. □

The following theorem gives the important properties of Φ.

Theorem 3.1. *Let $1\leq q<p<\infty$. Let $\Omega\subset\mathbb{R}^n$ be a bounded Sobolev (p,q)-extension domain. Then the set function Φ defined in (3.3) is a bounded quasiadditive set function defined on open subsets $U\subset\mathbb{R}^n$.*

Proof. The nonnegativity is immediate from the definition.

Let $E:W^{1,p}(\Omega)\to W^{1,q}(\mathbb{R}^n)$ be a bounded extension operator. For every open set $U\subset\mathbb{R}^n$ with $U\cap\Omega\neq\emptyset$ and every $u\in W_0^p(U,\Omega)$, we have

$$\Gamma_U^q(u)\leq\|\nabla E(u)\|_{L^q(U)}\leq C\|u\|_{W^{1,p}(U\cap\Omega)}.$$

This implies the boundedness of Φ.

Let $U_1 \subset U_2 \subset \mathbb{R}^n$ be two open sets. If $U_1 \cap \Omega = \emptyset$, we obviously have $0 = \Phi(U_1) \leq \Phi(U_2)$. Thus assume $U_1 \cap \Omega \neq \emptyset$. Let $u \in W_0^p(U_1, \Omega) \subset W_0^p(U_2, \Omega)$ be arbitrary. Then for each $v \in W^{1,q}(U_2)$ with $v\big|_{U_2 \cap \Omega} \equiv u$, we have

$$\left(\int_{U_2} |\nabla v(z)|^q dz\right)^{\frac{1}{q}} \geq \left(\int_{U_1} |\nabla v(z)|^q dz\right)^{\frac{1}{q}}.$$

This implies $\Gamma^q_{U_2}(u) \geq \Gamma^q_{U_1}(u)$. Since $u \in W_0^p(U_1, \Omega) \subset W_0^p(U_2, \Omega)$ is arbitrary, we obtain the monotonicity that is $\Phi(U_1) \leq \Phi(U_2)$.

Let $\{U_i\}_{i=1}^\infty$ be a pairwise disjoint collection of open sets. Fix $N \in \mathbb{N}$ and set $U_o := \bigcup_{i=1}^N U_i$. Let $0 < \epsilon < 1$. By (3.3), for every i, there exists a test function $u_i \in W_0^p(U_i, \Omega)$ such that

$$\Gamma^q_{U_i}(u_i) \geq \left(\Phi(U_i)\left(1 - \frac{\epsilon}{2^i}\right)\right)^{\frac{1}{k}} \|u_i\|_{W^{1,p}(U_i \cap \Omega)}. \tag{3.4}$$

Notice that also λu_i satisfies (3.4) by Lemma 3.2, when $\lambda > 0$. Hence, by replacing u_i with $\left(\Phi(U_i)\left(1 - \frac{\epsilon}{2^i}\right)\right)^{\frac{1}{p}} \frac{u_i}{\|u_i\|_{W^{1,p}(\Omega)}}$, we may also assume that

$$\|u_i\|^p_{W^{1,p}(U_i \cap \Omega)} = \Phi(U_i)\left(1 - \frac{\epsilon}{2^i}\right). \tag{3.5}$$

Set

$$u := \sum_{i=1}^N u_i.$$

Then $u \in W_0^p\left(\bigcup_{i=1}^N U_i, \Omega\right)$. By definition (3.2), there exists a function $v \in W^{1,q}\left(\bigcup_{i=1}^N U_i\right)$ with $v\big|_\Omega \equiv u$ and

$$2^k \Phi\left(\bigcup_{i=1}^N U_i\right) \geq \left(\frac{\|Dv\|_{L^q\left(\bigcup_{i=1}^N U_i\right)}}{\|u\|_{W^{1,p}\left(\bigcup_{i=1}^N U_i \cap \Omega\right)}}\right)^k \tag{3.6}$$

with $\frac{1}{k} = \frac{1}{q} - \frac{1}{p}$. Define $v_i := v\big|_{U_i}$ for $i = 1, \cdots, N$. Then, we have $v_i \in W^{1,q}(U_i)$ with $v_i\big|_{U_i} \equiv u_i$ and

$$\|Dv_i\|_{L^q(U_i)} \geq \Gamma^q_{U_i}(u_i).$$

Then, by making use of (3.4), we obtain

$$\|Dv\|_{L^q\left(\bigcup_{i=1}^N U_i\right)} = \left(\sum_{i=1}^N \|Dv_i\|^q_{L^q(U_i)}\right)^{\frac{1}{q}} \geq \left(\sum_{i=1}^N \left(\Phi(U_i)\left(1 - \frac{\epsilon}{2^i}\right)\right)^{\frac{q}{k}} \|u_i\|^q_{W^{1,p}(U_i \cap \Omega)}\right)^{\frac{1}{q}} \tag{3.7}$$

with $\frac{1}{k}=\frac{1}{q}-\frac{1}{p}$. The identity (3.5), together with the pairwise disjointness of U_i, gives (3.8)

$$\left(\sum_{i=1}^{N}\left(\Phi(U_i)\left(1-\frac{\epsilon}{2^i}\right)\right)^{\frac{q}{k}}\|u_i\|^q_{W^{1,p}(U_i\cap\Omega)}\right)^{\frac{1}{q}}=\left(\sum_{i=1}^{N}\Phi(U_i)\left(1-\frac{\epsilon}{2^i}\right)\right)^{\frac{1}{k}}\|u\|_{W^{1,p}\left(\bigcup_{i=1}^{N}U_i\cap\Omega\right)}.$$

First, suppose

$$\sum_{i=1}^{N}\Phi(U_i)>\Phi(U_o). \tag{3.9}$$

Since $0<\epsilon<1$, we also have

$$\sum_{i=1}^{N}\Phi(U_i)-\epsilon\Phi(U_o)>0.$$

By combining inequalities (3.7) and (3.8), we obtain

$$\begin{aligned}\|Dv\|_{L^q(\bigcup_{i=1}^{N}U_i)} &\geq \left(\sum_{i=1}^{N}\Phi(U_i)\left(1-\frac{\epsilon}{2^i}\right)\right)^{\frac{1}{k}}\|u\|_{W^{1,p}\left(\bigcup_{i=1}^{N}U_i\cap\Omega\right)} \\ &\geq \left(\sum_{i=1}^{N}\Phi(U_i)-\epsilon\Phi(U_o)\right)^{\frac{1}{k}}\|u\|_{W^{1,p}(\bigcup_{i=1}^{N}U_i\cap\Omega)}.\end{aligned} \tag{3.10}$$

Since $u\in W_0^p(\bigcup_{i=1}^{N}U_i,\Omega)$, we conclude from (3.6) and (3.10) that

$$C\Phi(U_0)^{\frac{1}{k}}\geq\frac{\|Dv\|_{L^q\left(\bigcup_{i=1}^{N}U_i\right)}}{\|u\|_{W^{1,p}(\bigcup_{i=1}^{N}U_i\cap\Omega)}}\geq\left(\sum_{i=1}^{N}\Phi(U_i)-\epsilon\Phi(U_o)\right)^{\frac{1}{k}}.$$

By letting ϵ tend to zero, we arrive at

$$\sum_{i=1}^{N}\Phi(U_i)\leq C\Phi\left(\bigcup_{j=1}^{N}U_i\right).$$

If (3.9) fails, then

$$\sum_{i=1}^{N}\Phi(U_i)\leq\Phi(U_o).$$

Hence, we always have

$$\sum_{i=1}^{N}\Phi(U_i)\leq(C+1)\Phi\left(\bigcup_{j=1}^{N}U_i\right).$$

Since $N \in \mathbb{N}$ is arbitrary and Φ is both nonnegative and monotone, we conclude that

$$\sum_{i=1}^{\infty} \Phi(U_i) \leq (C+1)\Phi\left(\bigcup_{j=1}^{\infty} U_i\right).$$

□

The following lemma is immediate from the definition (3.3) of the set function Φ.

Lemma 3.3. *Let $1 \leq q < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Sobolev (p,q)-extension domain and Φ be the set function from (3.3). Then, for a ball $B := B(x,r)$ with $x \in \partial\Omega$ and each function $u \in W_0^p(B,\Omega)$, there exists a function $v \in W^{1,q}(B)$ with $v\big|_{B\cap\Omega} \equiv u$ and*

$$\|Dv\|_{L^q(B)} \leq 2\Phi^{\frac{1}{k}}(B)\|u\|_{W^{1,p}(B\cap\Omega)}, \text{ where } 1/k = 1/q - 1/p. \tag{3.11}$$

4. The generalized Ahlfors condition

We begin with a general version of (1.2) for (p,q)-extension domains.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a Sobolev (p,q)-extension domain with $1 \leq q < p < \infty$. Then there exists a bounded quasiadditive set function Φ, defined on open sets, such that, for every $x \in \partial\Omega$ and each $0 < r < \min\{1, \frac{1}{4}\operatorname{diam}(\Omega)\}$, we have*

$$\Phi(B(x,r))^{p-q}|B(x,r)\cap\Omega|^q \geq r^{pq} Cap_q\left(\Omega\cap B\left(x,\frac{r}{4}\right), \Omega\cap A\left(x;\frac{r}{2},\frac{3r}{4}\right); B(x,r)\right)^p. \tag{4.1}$$

Proof. Let $\Omega \subset \mathbb{R}^n$ be a Sobolev (p,q)-extension domain with $1 \leq q < p < \infty$. Define the associated set function Φ by (3.3). Let $x \in \partial\Omega$ and $0 < r < \min\{1, \frac{1}{4}\operatorname{diam}(\Omega)\}$ be fixed. Then we define a function $u \in W^{1,p}(\Omega)\cap C(\Omega)$ by setting

$$u(y) = \begin{cases} 1 & \text{in } B(x,\frac{r}{4})\cap\Omega, \\ \frac{-4}{r}|y-x|+2 & \text{in } \big(B(x,\frac{r}{2})\setminus B(x,\frac{r}{4})\big)\cap\Omega, \\ 0 & \text{in } \Omega\setminus B(x,\frac{r}{2}). \end{cases} \tag{4.2}$$

We have

$$\left(\int_{B(x,r)\cap\Omega}|u(y)|^p dy + \int_{B(x,r)\cap\Omega}|\nabla u(y)|^p dy\right)^{\frac{1}{p}} \leq \frac{C}{r}|B(x,r)\cap\Omega|^{\frac{1}{p}}. \tag{4.3}$$

Because $u \in C(\Omega\cap B(x,r))$ with $u \equiv 0$ on $\Omega\cap\partial B(x,r)$, we conclude that $u \in W_0^p(B(x,r),\Omega)$. By Corollary 3.3, there exists a function $v \in W^{1,q}(\mathbb{R}^n)$ with $v\big|_{B(x,r)\cap\Omega} \equiv u$ and

$$\left(\int_{B(x,r)}|\nabla v(y)|^q dy\right)^{\frac{1}{q}} \leq 2(\Phi(B(x,r)))^{\frac{1}{k}}\left(\int_{B(x,r)\cap\Omega}|u(y)|^p + |\nabla u(y)|^p dy\right)^{\frac{1}{p}} \tag{4.4}$$

with $\frac{1}{k}=\frac{1}{q}-\frac{1}{p}$. For every small $\epsilon>0$, define v_ϵ to be the mollification of v as in [3, Section 4.2]. Then $v_\epsilon\in C(\mathbb{R}^n)\cap W^{1,q}(\mathbb{R}^n)$,

$$\lim_{\epsilon\to 0}\|\nabla v_\epsilon\|_{L^q(B(x,r))}=\|\nabla v\|_{L^q(B(x,r))}$$

and v_ϵ converges to v at every point of continuity of v. For every pair of compact sets (E,F) with $E\subset B(x,\frac{r}{4})\cap\Omega$ and $F\subset(B(x,\frac{r}{2})\setminus B(x,\frac{r}{4}))\cap\Omega$, there exists a sufficiently small $\epsilon_o>0$ such that for every $0<\epsilon<\epsilon_o$, we have $v_\epsilon\in\mathcal{W}_p(E,F;B(x,r))$. Hence, we have

$$\int_{B(x,r)}|\nabla v(x)|^q dx=\lim_{\epsilon\to 0}\int_{B(x,r)}|\nabla v_\epsilon(x)|^q dx\geq Cap_q\left(E,F;B(x,r)\right).$$

By taking the supremum over all pairs of compact sets (E,F) with $E\subset B(x,\frac{r}{4})\cap\Omega$ and $F\subset(B(x,\frac{r}{2})\setminus B(x,\frac{r}{4}))\cap\Omega$, we obtain

$$\int_{B(x,r)}|\nabla v(x)|^q dx\geq Cap_q\left(B\left(x,\frac{r}{4}\right)\cap\Omega,\Omega\cap A\left(x;\frac{r}{2},\frac{3r}{4}\right);B(x,r)\right).\tag{4.5}$$

By combining inequalities (4.3), (4.4) and (4.5), we obtain the inequality

$$C\Phi(B(x,r))^{p-q}|B(x,r)\cap\Omega|^q\geq r^{pq}Cap_q\left(\Omega\cap B\left(x,\frac{r}{4}\right),\Omega\cap A\left(x;\frac{r}{2},\frac{3r}{4}\right);B(x,r)\right)^p.$$

Our claim follows for the set function $\hat{\Phi}:=c\Phi$, where $c=C^{1/(p-q)}$. □

The following theorem gives the generalized Ahlfors condition in q-fat Sobolev (p,q)-extension domains for all $1\leq q<p<\infty$.

Theorem 4.2. *Let $\Omega\subset\mathbb{R}^n$ be a Sobolev (p,q)-extension domain, for $1\leq q<p<\infty$, which is q-fat at almost every $x\in\partial\Omega$. Then there exists a bounded quasiadditive set function Φ, defined on open sets, with the following property. For almost every $x\in\partial\Omega$, there exists $r_x>0$ such that, for every $0<r<r_x$, we have*

$$\Phi(B(x,r))^{p-q}|B(x,r)\cap\Omega|^q\geq|B(x,r)|^p.\tag{4.6}$$

Proof. Suppose that Ω is q-fat at almost every $x\in\partial\Omega$. By the Lebesgue density theorem and Lemma 3.1, there exists a subset $V\subset\overline{\Omega}$ with $|V|=|\overline{\Omega}|$ such that every $x\in V$ is a Lebesgue point of $\overline{\Omega}$ and Ω is q-fat at every $x\in V$. Fix $x\in V$. Let $\epsilon>0$ be sufficiently small such that $1-\epsilon\geq\frac{1}{2^{n-1}}$. Since $x\in V$ is a Lebesgue point of $\overline{\Omega}$, there exists $0<r_x<1$ such that for every $0<r<r_x$, we have

$$|B(x,r)\cap\overline{\Omega}|\geq(1-\epsilon)|B(x,r)|\geq\frac{1}{2^{n-1}}|B(x,r)|.\tag{4.7}$$

Let $r\in(0,r_x)$ be fixed. Since $|\partial B(x,s)|=0$ for every $0<s<r$, we have

$$\left|B\left(x,\frac{r}{4}\right)\cap\overline{\Omega}\right|\geq\frac{1}{2^{n-1}}\left|B\left(x,\frac{r}{4}\right)\right|\geq\frac{1}{2^{3n-1}}|B(x,r)|\tag{4.8}$$

and

$$\left|\left(B(x,r)\setminus B\left(x,\frac{r}{2}\right)\right)\cap\overline{\Omega}\right| \geq \left|B(x,r)\cap\overline{\Omega}\right| - \left|B\left(x,\frac{r}{2}\right)\right| \geq \frac{1}{2^n}|B(x,r)|. \tag{4.9}$$

Let u be defined by (4.2). Let Φ be the set function from (3.3). Then $u\in W_0^p(B(x,r),\Omega)$. By (4.4) and (4.3), we have $v\in W^{1,q}(B(x,r))$ with

$$\left(\int_{B(x,r)}|\nabla v(y)|^q dy\right)^{\frac{1}{q}} \leq C\left(\Phi(B(x,r))\right)^{\frac{1}{k}}\frac{C}{r}|B(x,r)\cap\Omega|^{\frac{1}{p}} \tag{4.10}$$

with $\frac{1}{k}=\frac{1}{q}-\frac{1}{p}$. Since Ω is q-fat at every $y\in V$, Lemma 2.6 implies that $v(y)=0$ for almost every $y\in(B(x,r)\setminus B(x,\frac{r}{2}))\cap V$ and $v(y)=1$ for almost every $y\in B(x,\frac{r}{4})\cap V$. Since $|V|=|\overline{\Omega}|$, $v(y)=1$ for almost every $y\in B(x,\frac{r}{4})\cap\overline{\Omega}$ and $E(u)(y)=0$ for almost every $y\in(B(x,r)\setminus B(x,\frac{r}{2}))\cap\overline{\Omega}$.

By the Poincaré inequality on balls, we have

$$Cr^q\int_{B(x,r)}|\nabla v(y)|^q dy \geq \int_{B(x,r)}|v(y)-v_{B(x,r)}|^q dy. \tag{4.11}$$

If $v_{B(x,r)}\geq\frac{1}{2}$, since $v(y)=0$ for almost every $y\in(B(x,r)\setminus B(x,\frac{r}{2}))\cap\overline{\Omega}$, we conclude from (4.9) that

$$\int_{B(x,r)}|v(y)-v_{B(x,r)}|^q dy \geq \left(\frac{1}{2}\right)^q\left|\left(B(x,r)\setminus B\left(x,\frac{r}{2}\right)\right)\cap\overline{\Omega}\right| \geq C|B(x,r)|.$$

In the case $v_{B(x,r)}<\frac{1}{2}$, since $v(y)=1$ for almost every $y\in B(x,\frac{r}{4})\cap\overline{\Omega}$, we conclude from (4.8) that

$$\int_{B(x,r)}|v(y)-v_{B(x,r)}|^q dy \geq \left(\frac{1}{2}\right)^q\left|B\left(x,\frac{r}{4}\right)\cap\overline{\Omega}\right| \geq C|B(x,r)|.$$

In conclusion, we always have

$$\int_{B(x,r)}|v(y)-v_{B(x,r)}|^q dy \geq C|B(x,r)|. \tag{4.12}$$

By combining inequalities (4.10), (4.11) and (4.12), we obtain the inequality

$$\Phi(B(x,r))^{p-q}|B(x,r)\cap\Omega|^q \geq C|B(x,r)|^p.$$

The desired inequality follows from this by replacing Φ with $c\Phi$ for a suitable c. $\square$

5. The volume of the boundary

We prove, relying on the generalized Ahlfors condition, that the boundary of a q-fat Sobolev (p,q)-extension domain is of volume zero.

Proof of Theorem 1.1. Let us assume that $|\partial\Omega| > 0$. Then, by the Lebesgue density theorem (see, for example [36]) and Theorem 4.2, there exists a subset $V \subset \partial\Omega$ with $|V| = |\partial\Omega| > 0$ such that every point $x \in V$ is a Lebesgue point of $\partial\Omega$, $\overline{D}\Phi(x) < \infty$ and

$$\Phi(B(x,r))^{p-q}|B(x,r) \cap \Omega|^q \geq C|B(x,r)|^p \tag{5.1}$$

holds for every $x \in V$ and each $0 < r < r_x$. Fix $x \in V$. Then by inequality (5.1), we have

$$|B(x,r) \cap \partial\Omega| \leq |B(x,r)| - |B(x,r) \cap \Omega| \leq |B(x,r)| - C\frac{|B(x,r)|^{\frac{p}{q}}}{\Phi(B(x,r))^{\frac{p-q}{q}}},$$

for every $0 < r < r_x$. Hence, by Lemma 3.1, we obtain

$$\begin{aligned}\limsup_{r\to 0^+} \frac{|B(x,r) \cap \partial\Omega|}{|B(x,r)|} &\leq \limsup_{r\to 0^+} \frac{|B(x,r)| - |B(x,r) \cap \Omega|}{|B(x,r)|} \\ &\leq \limsup_{r\to 0^+} \frac{|B(x,r)|}{|B(x,r)|} - C \liminf_{r\to 0^+} \frac{|B(x,r)|^{\frac{p-q}{q}}}{\Phi(B(x,r))^{\frac{p-q}{q}}} = 1 - C\overline{D}\Phi(x)^{\frac{q-p}{p}} < 1.\end{aligned}$$

This contradicts the assumption that $x \in V$ is a Lebesgue point of $\partial\Omega$. We conclude that $|\partial\Omega| = 0$. Thus we have verified the first claim of the theorem.

The final claims follow from Theorem 2.1 in combination with the first claim. □

We have not required our Sobolev extension operators to have any local properties. Let us consider such a requirement. Recall the definition of a strong extension operator from Definition 2.2. The following theorem shows that an extension operator can be promoted to a strong one precisely when the boundary of our extension domain is of volume zero.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Sobolev (p,q)-extension domain for $1 \leq q \leq p < \infty$. Then there exists a strong bounded extension operator $E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$ if and only if $|\partial\Omega| = 0$.*

Proof. Let $\Omega \subset \mathbb{R}^n$ be a Sobolev (p,q)-extension domain with $1 \leq q \leq p < \infty$. First, if $|\partial\Omega| = 0$, by Definition 2.2, every bounded extension operator $E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$ is a strong bounded extension operator.

Conversely, let us assume that there exists a strong bounded extension operator $E : W^{1,p}(\Omega) \to W^{1,q}(\mathbb{R}^n)$. Similarly to the set function Φ in (3.3), we define a new set function $\widetilde{\Phi}$ here. First, for every open set $U \subset \mathbb{R}^n$ with $U \cap \Omega \neq \emptyset$ and every $u \in W_0^p(U, \Omega)$, we define

$$\widetilde{\Gamma}_U^q(u) := \inf \left(\int_U |\nabla v(z)|^q dz\right)^{\frac{1}{q}},$$

where we take the infimum over all functions $v \in W^{1,q}(U)$ with $v\big|_{U\cap\Omega} \equiv u$ and which additionally satisfy the requirement that $v(y) = c$ for $\mathcal{H}^n$-almost every $y \in B(x,r) \cap U \cap \partial\Omega$ if $u\big|_{B(x,r)\cap\Omega} \equiv c$ for a ball $B(x,r)$ with $B(x,r) \cap U \neq \emptyset$. The existence of the strong extension operator E guarantees this class of desired functions is not empty. Then we define the set function $\widetilde{\Phi}$ by setting

$$\widetilde{\Phi}(U) := \sup\left\{\left(\frac{\widetilde{\Gamma}^q_U(u)}{\|u\|_{W^{1,p}(U\cap\Omega)}}\right)^k : u \in W^p_0(U,\Omega)\right\}$$

with $\frac{1}{k} = \frac{1}{q} - \frac{1}{p}$, and by setting $\widetilde{\Phi}(U) = 0$ for those open sets do not intersect Ω. Within a similar argument to the proof for Theorem 3.1, we obtain $\widetilde{\Phi}$ is a nonnegative, bounded and quasiadditive set function. Hence $\overline{D}\widetilde{\Phi}(x) < \infty$ for almost every $x \in \partial\Omega$. Fix a function $u \in W^{1,p}(\Omega)$ as in (4.2). By the definition of $\widetilde{\Phi}$, there exists a function $v \in W^{1,q}(B(x,r))$ with $v(y) = 1$ for almost every $y \in B(x, \frac{r}{4}) \cap \overline{\Omega}$ and $v(y) = 0$ for almost every $y \in (B(x,r) \setminus B(x, \frac{r}{2})) \cap \overline{\Omega}$ such that

$$\|\nabla v\|_{L^q(B(x,r))} \leq 2\widetilde{\Phi}^{\frac{1}{k}}(B(x,r))\|u\|_{W^{1,p}(B\cap\Omega)}.$$

Hence, similarly to the proof of Theorem 4.2, we obtain a similar point-wise density inequality with (4.6) for Φ replaced by $\widetilde{\Phi}$ for almost every $x \in \partial\Omega$. Finally, by making use of Lebesgue density theorem and repeating the proof of Theorem 1.1, we conclude that $|\partial\Omega| = 0$. $\square$

6. Cuspidal domains

6.1. **Outward cuspidal domains.** Since the lower bound in (4.1) comes with a term related to the capacitary size of a portion of Ω, let us analyze it carefully in the model case of an exterior spire of doubling order. More precisely, let $w : [0,\infty) \to [0,\infty)$ be continuous, increasing and differentiable with $w(0) = 0$, $w(1) = 1$ and so that $w(2t) \leq Cw(t)$ for all $t > 0$. We also require w' to be increasing on $(0,1)$ with $\lim_{t\to 0^+} w'(t) = 0$. We define

$$\Omega^n_w := \{z = (t,x) \in \mathbb{R}\times\mathbb{R}^{n-1} : 0 < t \leq 1, |x| < w(t)\} \cup B^n((2,0),\sqrt{2}). \tag{6.1}$$

See Figure 1. We call Ω^n_w an outward cuspidal domain with a doubling cuspidal function w. The boundary of Ω^n_w contains an exterior spire of order w at the origin.

The following theorem gives the sharp capacitary estimate at the origin for the outward cuspidal domain Ω^n_w.

In this section, we formulate and prove Theorem 6.1 that gives the sharp capacity estimate for outward cuspidal domains. After this, we use doubling order outward cuspidal domains to construct examples towards the sharpness of inequality (4.1) and show that capacitary fatness is a weaker condition than capacitary density.

The (p,q)-extendability properties for the domains Ω^n_w are known by [24, 28, 29, 30, 31]. We show that these domains give examples of settings where the exponents in (4.1) are optimal and where boundedness of Φ cannot be replaced, say, by an estimate of the type $\Phi(B(x,r)) \leq Cr^\alpha$. Besides of boundedness, the other crucial property of our set function Φ is quasiadditivity. It allows one to obtain better volume estimates when the center of $B(x,r)$

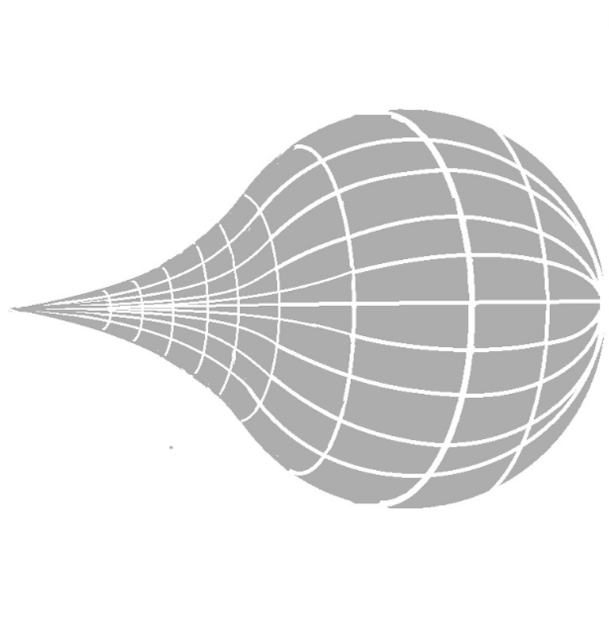

FIGURE 1. An outward cuspidal domain.

does not belong to a suitable exceptional set. These estimates will be shown to be sharp for wedges generated by Ω^n_w.

Theorem 6.1. *Let $n \geq 3$, and let $\Omega^n_w \subset \mathbb{R}^n$ be an outward cuspidal domain with a doubling cuspidal function w. Then, for every $0 < r < 1$, we have*

$$(6.2)\quad Cap_p\left(\Omega^n_w \cap B\left(0, \frac{r}{4}\right), \Omega^n_w \cap A\left(0; \frac{r}{2}, \frac{3r}{4}\right); B(0,r)\right) \sim_c \mathsf{cap}_p\left(\Omega^n_w \cap B\left(0, \frac{r}{4}\right); B\left(0, \frac{r}{2}\right)\right) \sim_c \begin{cases} r^{n-p} & \text{if } n-1 < p < \infty \\ \frac{r}{\log^{n-2}\frac{r}{w(r)}} & \text{if } p = n-1 \\ r(w(r))^{n-1-p} & \text{if } 1 \leq p < n-1 \end{cases}$$

where the constant c is independent of r.

Proof. Fix arbitrary $1 \leq p < \infty$ and a pair of compact sets (E, F) with $E \subset \Omega^n_w \cap B\left(0, \frac{r}{4}\right)$ and $F \subset \Omega^n_w \cap A\left(x; \frac{r}{2}\right)$. For every admissible function $u \in \mathsf{W}_p\left(\Omega^n_w \cap B\left(0, \frac{r}{4}\right); B\left(0, \frac{r}{2}\right)\right)$, by simply extending it to be zero on $A\left(0; \frac{r}{2}, r\right)$, we obtain an admissible function in $\mathcal{W}_p(E, F; B(0,r))$. Hence, by taking the supremum over all pairs of compact sets (E, F) with $E \subset \Omega^n_w \cap B\left(0, \frac{r}{4}\right)$

and $F \subset \Omega_w^n \cap A\left(x; \frac{r}{2}, \frac{3r}{4}\right)$, we obtain

$$\mathsf{cap}_p\left(\Omega_w^n \cap B\left(0, \frac{r}{4}\right); B\left(0, \frac{r}{2}\right)\right) \geq Cap_p\left(\Omega_w^n \cap B\left(0, \frac{r}{4}\right), \Omega_w^n \cap A\left(0; \frac{r}{2}, \frac{3r}{4}\right); B(0,r)\right).$$

We divide the argument for the remaining inequalities into three cases.

The case $n-1 < p < \infty$: By Lemma 2.2, we have

$$Cap_p\left(\Omega_w^n \cap B\left(0, \frac{r}{4}\right), \Omega_w^n \cap A\left(0; \frac{r}{2}, \frac{3r}{4}\right); B(0,r)\right) \geq Cr^{n-p}.$$

We define a test function v on $B(0,r)$ by setting

$$v(z) := \begin{cases} 1 & \text{if } \ |z| < \frac{r}{4} \\ \frac{-4}{r}|z| + 2 & \text{if } \ \frac{r}{4} \leq |z| \leq \frac{r}{2} \\ 0 & \text{if } \ \frac{r}{2} < |z| < r \end{cases}. \tag{6.3}$$

Since $v \in \mathsf{W}_p\left(\Omega_w^n \cap B\left(0, \frac{r}{4}\right); B\left(0, \frac{r}{2}\right)\right)$, we have

$$\mathsf{cap}_p\left(\Omega_w^n \cap B\left(0, \frac{r}{4}\right); B\left(0, \frac{r}{2}\right)\right) \leq \int_{B(0,r)} |\nabla v(z)|^p dz \leq Cr^{n-p}.$$

The case $1 \leq p < n-1$: Given $\frac{r}{5} < \rho < \frac{r}{4}$, we define an $(n-1)$-dimensional sphere S_ρ by

$$S_\rho := \left\{z \in \mathbb{R}^n : d\left(z, \left(\frac{3r}{8}, 0, \cdots, 0\right)\right) = \rho\right\}.$$

We set

$$S_\rho^+ := \{z = (t, x_1, x_2, \cdots, x_{n-1}) \in S_\rho : x_{n-1} > 0\}$$

and let $A_1^+(\rho) := S_\rho^+ \cap \left(B\left(0, \frac{r}{4}\right) \cap \Omega_w^n\right)$ and $A_0^+(\rho) := S_\rho^+ \cap \left(\Omega_w^n \setminus B\left(0, \frac{r}{2}\right)\right)$. Since w is doubling and

$$\lim_{r \to 0^+} w'(r) = 0,$$

we have

$$\mathcal{H}^{n-1}(A_0^+(\rho)) \sim_c (w(r))^{n-1} \text{ and } \mathcal{H}^{n-1}(A_1^+(\rho)) \sim_c (w(r))^{n-1}$$

for every $\rho \in (\frac{r}{5}, \frac{r}{4})$. The implicit constants are independent of r and ρ. There exists a bi-Lipschitz homeomorphism from S_ρ^+ to the $(n-1)$-dimensional disk $B^{n-1}(0,\rho)$ with a bi-Lipschitz constant independent of ρ, for example, see [14, Lemma 2.19]. Hence, for each $v \in \mathcal{W}_p\left(B\left(0, \frac{r}{4}\right) \cap \Omega_w^n, \Omega_w^n \cap A\left(0; \frac{r}{2}, \frac{3r}{4}\right); B(0,r)\right)$, by the Sobolev-Poincaré inequality on balls [3, Theorem 4.9], for almost every $\rho \in (\frac{r}{5}, \frac{r}{4})$, we have

$$\left(\fint_{S_\rho^+} |v(z) - v_{S_\rho^+}|^{p^\star} dz\right)^{\frac{1}{p^\star}} \leq Cr\left(\fint_{S_\rho^+} |\nabla v(z)|^p dz\right)^{\frac{1}{p}} \tag{6.4}$$

with $p^\star = \frac{(n-1)p}{n-1-p}$. Assuming $v_{S_\rho^+} \le \frac{1}{2}$, we have

$$\begin{aligned}(w(r))^{n-1-p} &\le C\left(\int_{A_1^+(\rho)} |v(z)-v_{S_\rho^+}|^{p^\star} dz\right)^{\frac{p}{p^\star}} \\ &\le C\left(\int_{S_\rho^+} |v(z)-v_{S_\rho^+}|^{p^\star} dz\right)^{\frac{p}{p^\star}} \le C\int_{S_\rho^+} |\nabla v(z)|^p dz.\end{aligned}$$

If $v_{S_\rho^+} > \frac{1}{2}$, we simply replace $A_1^+(\rho)$ by $A_0^+(\rho)$ in the inequality above. Hence, for almost every $\rho \in (\frac{r}{5}, \frac{r}{4})$, we have

$$(w(r))^{n-1-p} \le C\int_{S_\rho^+} |\nabla v(z)|^p dz.$$

By integrating over $\rho \in (\frac{r}{5}, \frac{r}{4})$, we obtain

$$(w(r))^{n-1-p} r \le C\int_{\frac{r}{5}}^{\frac{r}{4}} \int_{S_\rho^+} |\nabla v(z)|^p dz d\rho \le C\int_{\mathbb{R}^n} |\nabla v(z)|^p dz.$$

Since $v \in \mathcal{W}_p(E, F; B(0,r))$ for every pair of compact sets (E, F) with $E \subset B\left(0, \frac{r}{4}\right) \cap \Omega_w^n$ and $F \subset \Omega_w^n \cap A\left(0; \frac{r}{2}, \frac{3r}{4}\right))$, we conclude that

$$Cap_p\left(B\left(0, \frac{r}{4}\right) \cap \Omega_w^n, \Omega_w^n \cap A\left(0; \frac{r}{2}, \frac{3r}{4}\right); B(0,r)\right) \ge c_2 (w(r))^{n-1-p} r.$$

Towards the other direction of the inequality, we construct a suitable test function. We define a cut-off function F_1 by setting

$$F_1(z) = F_1(t,x) := \begin{cases} 1 & \text{if } \ |x| < w(\frac{r}{4}) \\ \frac{-|x|}{w(\frac{r}{2}) - w(\frac{r}{4})} + \frac{w(\frac{r}{2})}{w(\frac{r}{2}) - w(\frac{r}{4})} & \text{if } \ w(\frac{r}{4}) \le |x| \le w(\frac{r}{2}) \\ 0 & \text{if } |x| > w(\frac{r}{2}) \end{cases}.$$

Then we define our test function $v_1 \in \mathsf{W}_p\left(B\left(0, \frac{r}{4}\right) \cap \Omega_w^n; B\left(0, \frac{r}{2}\right)\right)$ by $v_1(z) := v(z) F_1(z)$, where v is defined in (6.3). Since w' is increasing on $(0, \infty)$ and w is doubling, we have

$$w\left(\frac{r}{4}\right) \le w\left(\frac{r}{2}\right) - w\left(\frac{r}{4}\right) \le w\left(\frac{r}{2}\right) \le w(r) \le Cw\left(\frac{r}{4}\right).$$

Hence, a simple computation shows that

$$|\nabla v_1(z)| \le \begin{cases} \frac{C}{w(r)} & \text{if } \ |t| < \frac{r}{2} \text{ and } |x| < w(r) \\ 0 & \text{otherwise} \end{cases}.$$

This implies

$$\mathsf{cap}_p\left(B\left(0, \frac{r}{4}\right) \cap \Omega_w^n; B\left(0, \frac{r}{2}\right)\right) \le \int_{B(0,r)} |\nabla v_1(z)|^p dz \le Cr(w(r))^{n-1-p}.$$

The case $p=n-1$: Let $z_1:=(-\rho,0,\cdots,0)$ and $z_2:=(\rho,0,\cdots,0)$ be a pair of antipodal points on the $(n-2)$-dimensional sphere $\partial B^{n-1}(0,\rho)$. Denote $\tilde{A}_1^+(\rho):=B^{n-1}(z_1,w(\rho))\cap B^{n-1}(0,\rho)$ and $\tilde{A}_0^+(\rho):=B^{n-1}(z_2,w(\rho))\cap B^{n-1}(0,\rho)$. For every $\rho\in(0,\frac{1}{4})$, there exists a bi-Lipschitz homeomorphism $H_\rho: S_\rho^+\to B^{n-1}(0,\rho)$ with $\tilde{A}_1^+(\rho)=H_\rho(A_1^+(\rho))$, $\tilde{A}_0^+(\rho)=H_\rho(A_0^+(\rho))$, with a bi-Lipschitz constant independent of ρ. Let

$$\{0\}\times\mathbb{R}^{n-2}:=\{x=(0,x_2,x_3,\cdots,x_{n-1}):x_i\in\mathbb{R}\text{ for }i=2,3,\cdots,n-1\}.$$

For $z\in\{0\}\times\mathbb{R}^{n-2}\cap B^{n-1}(0,\rho)$, we define $L_{z_1}^z$ to be the line segment with endpoints z_1,z and $L_{z_2}^z$ to be the line segment with endpoints z_2,z. We also define $S_{z_1}^z:=L_{z_1}^z\setminus B^{n-1}(z_1,w(\rho))$ and $S_{z_2}^z:=L_{z_2}^z\setminus B^{n-1}(z_2,w(\rho))$. For every pair of compact sets (E,F) with $E\subset B\left(0,\frac{r}{4}\right)\cap\Omega_w^n$ and $F\subset\Omega_w^n\cap A\left(0;\frac{r}{2},\frac{3r}{4}\right)$, fix a test function

$$\hat{v}\in\mathcal{W}_{n-1}\left(E,F;B(0,r)\right).$$

The function $\tilde{v}_\rho$ defined by $\tilde{v}_\rho:=\hat{v}\circ H_\rho^{-1}$, is continuous on $B^{n-1}(0,\rho)$ with $\tilde{v}_\rho\big|_{\tilde{A}_1^+(\rho)}\geq 1$ and $\tilde{v}_\rho\big|_{\tilde{A}_0^+(\rho)}\leq 0$. By the Fubini theorem, for almost every $\rho\in(\frac{r}{5},\frac{r}{4})$, $\tilde{v}_\rho\in W^{1,n-1}(B^{n-1}(0,\rho))\cap C(B^{n-1}(0,\rho))$. Let us fix such a $\rho\in(\frac{r}{5},\frac{r}{4})$. Then for $\mathcal{H}^{n-2}$-a.e. $z\in\{0\}\times\mathbb{R}^{n-2}\cap B^{n-1}(0,\rho)$, by the fundamental theorem of calculus, we have either

$$\frac{1}{2}\leq\int_{S_{z_1}^z}|\nabla\tilde{v}_\rho(x)|dx\quad\text{or}\quad\frac{1}{2}\leq\int_{S_{z_2}^z}|\nabla\tilde{v}_\rho(x)|dx.$$

Then the Hölder inequality implies either

$$\left(\int_{S_{z_1}^z}|x-z_1|^{-1}dx\right)^{2-n}\leq C\int_{S_{z_1}^z}|\nabla\tilde{v}_\rho(x)|^{n-1}|x-z_1|^{n-2}dx$$

or

$$\left(\int_{S_{z_2}^z}|x-z_2|^{-1}dx\right)^{2-n}\leq C\int_{S_{z_2}^z}|\nabla\tilde{v}_\rho(x)|^{n-1}|x-z_2|^{n-2}dx.$$

Hence, we have either

$$\frac{1}{\log^{n-2}\frac{r}{w(r)}}\leq C\int_{B^{n-1}(z_1,\sqrt{2}\rho)\cap B^{n-1}(0,\rho)}|\nabla\tilde{v}_\rho(x)|^{n-1}dx$$

or

$$\frac{1}{\log^{n-2}\frac{r}{w(r)}}\leq C\int_{B^{n-1}(z_2,\sqrt{2}\rho)\cap B^{n-1}(0,\rho)}|\nabla\tilde{v}_\rho(x)|^{n-1}dx.$$

In conclusion, for every $\rho\in\left(\frac{r}{5},\frac{r}{4}\right)$ with $\tilde{v}_\rho\in W^{1,n-1}(B^{n-1}(0,\rho))$, we have

$$\frac{1}{\log^{n-2}\frac{r}{w(r)}}\leq C\int_{B^{n-1}(0,\rho)}|\nabla\tilde{v}_\rho(x)|^{n-1}dx.$$

Since, for every $\rho \in \left(\frac{r}{5}, \frac{r}{4}\right)$, $H_\rho : S_\rho^+ \to B^{n-1}(0,\rho)$ is bi-Lipschitz with a bi-Lipschitz constant independent of ρ, we have

$$\frac{1}{\log^{n-2} \frac{r}{w(r)}} \leq C \int_{S_\rho^+} |\nabla \hat{v}(z)|^{n-1} dz.$$

By integrating over $\rho \in (\frac{r}{5}, \frac{r}{4})$, we obtain

$$\frac{r}{\log^{n-2} \frac{r}{w(r)}} \leq C \int_{B(0,r)} |\nabla \hat{v}(z)|^{n-1} dz.$$

Since the pair of compact sets (E,F) with $E \subset B\left(0, \frac{r}{4}\right) \cap \Omega_w^n$ and $F \subset \Omega_w^n \cap A\left(0; \frac{r}{2}, \frac{3r}{4}\right)$ is arbitrary and $\hat{v} \in \mathcal{W}_{n-1}(E,F;B(0,r))$ is arbitrary, we conclude that

$$Cap_{n-1}\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right), \Omega_w^n \cap A\left(0;\frac{r}{2},\frac{3r}{4}\right); B(0,r)\right) \geq C \frac{r}{\log^{n-2} \frac{r}{w(r)}}.$$

Towards the opposite direction of this inequality, we construct a suitable test function. We define a cut-off function F_2 by setting

$$F_2(z) = F_2(t,x) := \begin{cases} 1 & \text{if } \ |x| < w(\frac{r}{4}) \\ \frac{\log \frac{4|x|}{r}}{\log \frac{4w(\frac{r}{4})}{r}} & \text{if } \ w(\frac{r}{4}) \leq |x| \leq \frac{r}{4} \\ 0 & \text{if } |x| > \frac{r}{4} \end{cases}.$$

Then we define our test function $v_2 \in \mathsf{W}_{n-1}\left(B\left(0,\frac{r}{4}\right) \cap \Omega_u^n; B\left(0,\frac{r}{2}\right)\right)$ by

$$v_2(z) := \begin{cases} F_2(z) & \text{if } \ |z| < \frac{r}{4} \\ F_2(z)\frac{\log \frac{2|z|}{r}}{\log \frac{1}{2}} & \text{if } \ \frac{r}{4} \leq |z| \leq \frac{r}{2} \\ 0 & \text{if } |z| > \frac{r}{2} \end{cases}.$$

Since w is doubling, a simple computation shows that

$$|\nabla v_2(z)| \leq \begin{cases} \frac{C}{|x| \log \frac{r}{w(r)}} & \text{if } \ |t| < \frac{r}{2} \text{ and } w(\frac{r}{4}) < |x| < \frac{r}{4} \\ 0 & \text{elsewhere} \end{cases}.$$

Hence,

$$\mathsf{cap}_{n-1}\left(B\left(0,\frac{r}{4}\right) \cap \Omega_w^n; B\left(0,\frac{r}{2}\right)\right) \leq \int_{\mathbb{R}^n} |\nabla v_2(z)|^{n-1} dz \leq \frac{Cr}{\log^{n-2} \frac{r}{w(r)}}.$$

By combining the three cases above, we obtain the sought-for inequalities. $\square$

We proceed to show the sharpness of the inequality (4.1). We need the following lemma.

Lemma 6.1. *Let $0 \le \lambda \le n$ and Φ be a non-negative, bounded, monotone and quasiadditive set function defined on open sets. Define*

$$E = \left\{ x \in \overline{\Omega} : \limsup_{r \to 0} \frac{\Phi(B(x,r))}{r^\lambda} = \infty \right\}.$$

Then

$$\mathcal{H}^\lambda(E) = 0.$$

Proof. For each $x \in E$ and every $\delta > 0$, there exists $0 < r_x < \delta$ such that

$$\delta \Phi(B(x, r_x)) > r_x^\lambda.$$

Define

$$\mathcal{F} := \{B(x, r_x) : x \in E\}.$$

By the classical Vitali covering theorem, there exists an at most countable subclass of pairwise disjoint balls $\{B_i\}_{i=1}^\infty$ in $\mathcal{F}$ such that

$$E \subset \bigcup_{i=1}^\infty 5B_i.$$

Hence, writing r_i for the radius of B_i, we have

$$\begin{aligned} \mathcal{H}^\lambda_{10\delta}(E) &\le C\sum_{i=1}^\infty (5r_i)^\lambda \le C\delta \sum_{i=1}^\infty \Phi(B_i) \\ &\le C\delta\Phi\left(\bigcup_{i=1}^\infty B_i\right) \le C\delta\Phi(\mathbb{R}^n). \end{aligned}$$

The claim follows by letting δ tend to zero. $\square$

6.2. **Sharpness of the generalized capacitary Ahlfors type condition.**

Sharpness of (4.1). We use outward cuspidal domains to construct Sobolev extension domains that show the sharpness of (4.1). Given $s \in (1, \infty)$ and $\alpha > \frac{s-1}{s}$, let $\omega(t) = t^s \log^\alpha(\frac{e}{t})$, and consider the outward cuspidal domain $\Omega^n_{t^s \log^\alpha(\frac{e}{t})} := \Omega^n_\omega \subset \mathbb{R}^n$. By results due to Maz'ya and Poborchi in [28, 29, 30, 31], we have the following results. For $n \ge 3$, $\Omega^n_{t^s \log^\alpha(\frac{e}{t})}$ is a Sobolev (p,q)-extension domain for

$$\begin{cases} 1 \le q \le \frac{(1+(n-1)s)p}{1+(n-1)s+(s-1)p} & \text{if } \frac{1+(n-1)s}{2+(n-2)s} \le p \le \frac{(n-1)+(n-1)^2 s}{n}, \\ 1 \le q \le \frac{np}{1+(n-1)s} & \text{if } \frac{(n-1)+(n-1)^2 s}{n} \le p < \infty. \end{cases} \tag{6.5}$$

For $n = 2$, $\Omega^n_{t^s \log^\alpha(\frac{e}{t})}$ is a Sobolev (p,q)-extension domain for $\frac{1+s}{2} \le p < \infty$ and $1 \le q \le \frac{2p}{1+s}$.

Clearly, there exists a constant $C > 1$ such that for every $0 < r < 1$, we have

$$\frac{1}{C} r^{1+(n-1)s} \log^{\alpha(n-1)}\left(\frac{e}{r}\right) \le |B(0,r) \cap \Omega^n_{t^s \log^\alpha(\frac{e}{t})}| \le C r^{1+(n-1)s} \log^{\alpha(n-1)}\left(\frac{e}{r}\right). \tag{6.6}$$

Furthermore, (6.2) gives us a lower bound for the capacitary term in (4.1) in terms of r, q, s, n and $\log^{\alpha}\left(\frac{e}{r}\right)$.

By comparing the capacity estimate, (6.6) and (6.2) for the values of q, p given by (6.5) we see that (4.1) cannot hold for a bounded set function Φ for better exponents than the given ones.

Let us also analyze the additivity of Φ. Fix $n \geq 3$. Let $k \in \{1, 2, \cdots, n-2\}$, $s \in (1, \infty)$ and $\alpha > \frac{s-1}{k+1}$ be fixed. We define a domain $G_n^k(s,\alpha) \subset \mathbb{R}^n$ by setting

$$G_n^k(s,\alpha) := \Omega^{k+1}_{t^s \log^{\alpha}\left(\frac{e}{t}\right)} \times \mathbb{R}^{n-k-1}.$$

Since $G_n^k(s,\alpha)$ is the product of $\Omega^{k+1}_{t^s \log^{\alpha}\left(\frac{e}{t}\right)}$ and $\mathbb{R}^{n-k-1}$, by the extension results in [28, 29, 30, 31] and product results in [23, 46], we obtain the following conclusions. For $k \geq 2$, $G_n^k(s,\alpha)$ is a Sobolev (p,q)-extension domain for

$$\begin{cases} 1 \leq q \leq \frac{(1+ks)p}{1+ks+(s-1)p} & \text{if } \frac{1+ks}{2+(k-1)s} \leq p \leq \frac{k+k^2 s}{k+1}, \\ 1 \leq q \leq \frac{(k+1)p}{1+ks} & \text{if } \frac{k+k^2 s}{k+1} \leq p < \infty\,, \end{cases} \tag{6.7}$$

and $G_n^1(s,\alpha)$ is a Sobolev (p,q)-extension domain for $\frac{1+s}{2} \leq p < \infty$ and $1 \leq q \leq \frac{2p}{1+s}$.

Clearly, there exists a constant $C > 1$ such that, for every $x \in \{0\} \times \mathbb{R}^{n-k-1}$ and each $0 < r < 1$, we have

$$\frac{1}{C}(r)^{n+ks-k} \log^{\alpha k}\left(\frac{e}{r}\right) \leq |B(x,r) \cap G_n^k(s,\alpha)| \leq C r^{n+ks-k} \log^{\alpha k}\left(\frac{e}{r}\right). \tag{6.8}$$

Moreover, Fubini theorem, Theorem 6.1 and Lemma 2.2 give with some work the estimates

$$\begin{aligned} &Cap_q\left(G_n^k(s,\alpha) \cap B\left(x, \frac{r}{4}\right), A\left(x; \frac{r}{2}, \frac{3r}{4}\right); B(x,r)\right) \\ &\qquad \geq \begin{cases} c_1 (r)^{n-q} & \text{if } k < q < \infty \\ \frac{c_2 r^{n-k}}{\log^k \frac{e}{r}} & \text{if } q = k \\ c_3 r^{(k-q)s+n-k} \log^{\alpha(k-q)}\left(\frac{e}{r}\right) & \text{if } 1 \leq q < k \end{cases} \end{aligned} \tag{6.9}$$

and, for $k = 1$,

$$Cap_q\left(G_n^1(s,\alpha) \cap B\left(x, \frac{r}{4}\right), A\left(x; \frac{r}{2}, \frac{3r}{4}\right); B(x,r)\right) \geq c r^{n-q}. \tag{6.10}$$

By Lemma 6.1, for $\mathcal{H}^{n-k-1}$-almost every $x \in \{0\} \times \mathbb{R}^{n-k-1}$, there exists $M_x < \infty$ with

$$\Phi(B(x,r)) \leq M_x r^{n-k-1}. \tag{6.11}$$

If $k \geq 2$, by inserting (6.7), (6.9) and (6.11) into the inequality (4.1), we obtain the optimal bound in (6.8), modulo logarithmic terms. The case $k = 1$ is analogous.

In conclusion, there is no hope in improving on the boundedness of the set function Φ from (4.1) so as to obtain estimates that would hold at every boundary point. Moreover, the quasiadditivity of Φ gives rather optimal measure density properties for points outside exceptional sets. $\square$

6.3. **Fatness versus density.** Recall from Section 2 that p-capacitary density implies p-fatness and that every domain satisfies the density condition at each boundary point when $p > n-1$.

Given $1 < p \leq n-1$, we construct outward cuspidal domains $\Omega_w^n \subset \mathbb{R}^n$ with suitable functions w, such that Ω_w^n are p-fat but not p-capacitary dense at the tip 0.

Fix $1 < p < n-1$. We consider the function $w(t) = \frac{t}{\log^{\frac{p-1}{n-1-p}} \frac{e}{t}}$ and the corresponding outward cuspidal domain Ω_w^n. By Theorem 6.1, we have

$$\mathsf{cap}_p\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right) \sim_c Cap_p\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right), \Omega_w^n \cap A\left(0;\frac{r}{2},\frac{3r}{4}\right); B(0,r)\right) \sim_c \frac{r^{n-p}}{\log^{p-1}\frac{e}{r}}.$$

Hence, by (2.3), we have

$$\lim_{r\to 0^+} \frac{\mathsf{cap}_p\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)}{\mathsf{cap}_p\left(B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)} \sim_c \lim_{r\to 0^+} \frac{1}{\log^{p-1}\frac{e}{r}} = 0$$

and

$$\int_0^1 \left(\frac{\mathsf{cap}_p\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)}{\mathsf{cap}_p\left(B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)}\right)^{\frac{1}{p-1}} \sim_c \int_0^1 \frac{1}{r\log\frac{e}{r}} dr = \infty.$$

Hence, the outward cuspidal domain Ω_w^n is not p-capacitary dense but nevertheless p-fat at the tip 0. For $p = n-1$, we choose the function $w(t) = t^2$. By Theorem 6.1, for every $0 < r < \frac{1}{2}$, we have

$$\mathsf{cap}_{n-1}\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right) \sim_c Cap_{n-1}\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right), \Omega_w^n \cap A\left(0;\frac{r}{2},\frac{3r}{4}\right); B(0,r)\right) \sim_c \frac{r}{\log^{n-2}\frac{e}{r}}.$$

Hence, we have

$$\lim_{r\to 0^+} \frac{\mathsf{cap}_{n-1}\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)}{\mathsf{cap}_{n-1}\left(B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)} \sim_c \lim_{r\to 0^+} \frac{1}{\log^{n-2}\frac{e}{r}} = 0$$

and

$$\int_0^{\frac{1}{2}} \left(\frac{\mathsf{cap}_{n-1}\left(\Omega_w^n \cap B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)}{\mathsf{cap}_{n-1}\left(B\left(0,\frac{r}{4}\right); B\left(0,\frac{r}{2}\right)\right)}\right)^{\frac{1}{n-2}} \frac{dr}{r} \sim_c \int_0^{\frac{1}{2}} \frac{1}{r\log\frac{e}{r}} dr = \infty.$$

Consequently, the outward cuspidal domain Ω^n_w is not $(n-1)$-capacitary dense but nevertheless $(n-1)$-fat at the tip 0.

7. Sobolev (p,q)-extension domains with a positive boundary volume

In this section, for every $n \geq 3$ and $1 \leq q < n-1$, we construct a Sobolev (p,q)-extension domain $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| > 0$. We also use this construction to prove Theorem 7.2 that gives an even more striking example.

7.1. **The initial construction.** Let $\mathcal{Q}_o := (0,1)^n$ be the n-dimensional unit cube in $\mathbb{R}^n$, and $\mathcal{C}_0 := (0,1)^{n-1} \times (0,2)$ be an n-dimensional rectangle in $\mathbb{R}^n$. Let $\mathcal{S}_o := (0,1)^{n-1}$ be the $(n-1)$-dimensional unit cube in the $(n-1)$-dimensional Euclidean hyperplane $\mathbb{R}^{n-1}$. Let $E \subset [0,1]$ be a Cantor set with $0 < \mathcal{H}^1(E) < 1$. The Smith-Volterra-Cantor set guarantees the existence of such an E, see [35]. Define

$$E^{n-1} := \underbrace{E \times E \times \cdots \times E}_{n-1}.$$

Then $E^{n-1} \subset [0,1]^{n-1}$ is nowhere dense in $(0,1)^{n-1}$ with $0 < \mathcal{H}^{n-1}(E^{n-1}) < 1$. We let

$$W := \left\{Q \subset (0,1)^{n-1} \setminus E^{n-1} : Q \text{ is Whitney}\right\}$$

be the class of all Whitney cubes of the open set $(0,1)^{n-1} \setminus E^{n-1}$, see [37]. For every $k \in \mathbb{N}$, we define W_k to be the subclass of W with

$$W_k := \left\{Q \subset W : 2^{-k-1} \leq l(Q) < 2^{-k}\right\}$$

where $l(Q)$ is the edge-length of the cube Q. We number the elements in W_k by

$$W_k = \{Q^j_k : 1 \leq j \leq N_k\}.$$

Notice that $N_k \leq 2^{(n-1)(k+1)}$. For a Whitney cube Q^j_k, we refer to its center by x^j_k. Let $h : [0,1] \to [0,1]$ be an increasing and continuous function with $h(0) = 0$ and $h(t) > 0$ when $t > 0$. We define

$$r_k := \left(2^{-(n-1)(k+1)-k} h(8^{-k})\right)^{\frac{1}{n-1-q}}, \tag{7.1}$$

$\mathsf{D}^j_k := B^{n-1}(x^j_k, r_k)$ and $\widetilde{\mathsf{D}}^j_k := B^{n-1}(x^j_k, \frac{r_k}{2})$. Then $\widetilde{\mathsf{D}}^j_k \subset \mathsf{D}^j_k \subset Q^j_k$. Since E^{n-1} is nowhere dense in $[0,1]^{n-1}$, for an arbitrary $x \in E^{n-1}$ and each $\epsilon > 0$, there exists a large enough k and some $j \in \{1, 2, \cdots, N_k\}$ with $Q^j_k \subset (0,1)^{n-1} \cap B^{n-1}(x, \epsilon)$. Then $\widetilde{\mathsf{D}}^j_k \subset (0,1)^{n-1} \cap B^{n-1}(x, \epsilon)$. Hence, we have

$$E^{n-1} \subset \overline{\bigcup_{k=1}^{\infty} \bigcup_j \widetilde{\mathsf{D}}^j_k}.$$

We define

$$\mathcal{D}_h := \bigcup_{k=1}^{\infty} \bigcup_j \mathsf{D}^j_k \text{ and } \widetilde{\mathcal{D}}_h := \bigcup_{k=1}^{\infty} \bigcup_j \widetilde{\mathsf{D}}^j_k.$$

Then $E^{n-1} \subset \partial\widetilde{\mathcal{D}}_h$ and $\mathcal{H}^{n-1}(\partial\widetilde{\mathcal{D}}_h) \geq \mathcal{H}^{n-1}(E^{n-1}) > 0$.

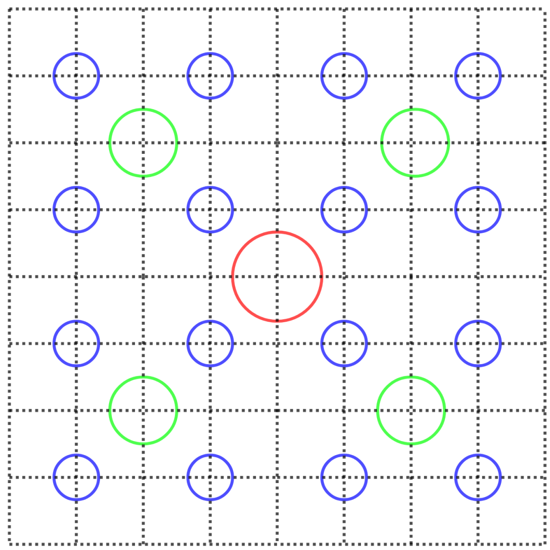

FIGURE 2. The set $\mathcal{D}_h$

We define $\mathsf{C}_k^j := \mathsf{D}_k^j \times [1, 2)$, $\widetilde{\mathsf{C}}_k^j := \widetilde{\mathsf{D}}_k^j \times [1, 2)$ and $A_k^j := C_k^j \setminus \widetilde{\mathsf{C}}_k^j$. We use the cylinders C_k^j and $\widetilde{\mathsf{C}}_k^j$ to define two domains:

$$\Omega_h := \mathcal{Q}_o \cup \bigcup_{k=1}^{\infty} \bigcup_j \mathsf{C}_k^j \text{ and } \widetilde{\Omega}_h := \mathcal{Q}_o \cup \bigcup_{k=1}^{\infty} \bigcup_j \widetilde{\mathsf{C}}_k^j.$$

Given $m \in \mathbb{N}$, we set

$$\Omega_h^m := \mathcal{Q}_o \cup \bigcup_{k=1}^{m} \bigcup_j \mathsf{C}_k^j \text{ and } \widetilde{\Omega}_h^m := \mathcal{Q}_o \cup \bigcup_{k=1}^{m} \bigcup_j \widetilde{\mathsf{C}}_k^j.$$

Figure 3 illustrates the construction of these domains.

The following lemma goes back to a result of Väisälä [41]. See [42, Pages 93-94] for a full proof. Also see [11].

Lemma 7.1. *The domain $\widetilde{\Omega}_h$ is quasiconformally equivalent to the unit ball: there is a quasiconformal mapping from the unit ball $B^n(0, 1)$ onto $\widetilde{\Omega}_h$.*

Hence, by Lemma 2.7 and Lemma 2.8, for arbitrary $1 \leq p < \infty$, $W^{1,\infty}(\widetilde{\Omega}_h)$ is dense in $W^{1,p}(\widetilde{\Omega}_h)$. Consequently, also $W^{1,\infty}(\widetilde{\Omega}_h) \cap C(\widetilde{\Omega}_h)$ is dense.

7.2. **Cut-off functions.** Let $\mathsf{C} := B^{n-1}(0,r) \times (0,1)$ and $\widetilde{\mathsf{C}} := B^{n-1}(0,\frac{r}{2}) \times (0,1)$. Then C is a cylinder and $\widetilde{\mathsf{C}}$ is a sub-cylinder of C. We define $A_{\mathsf{C}} := \mathsf{C} \setminus \overline{\widetilde{\mathsf{C}}}$. We employ the cylindrical coordinate system

$$\{x = (x_1, x_2, \cdots, x_n) = (s, \theta_1, \theta_2, \cdots, \theta_{n-2}, x_n) \in \mathbb{R}^n\}$$

where $\{(0,0,\cdots,0,x_n) : x_n \in \mathbb{R}\}$ is the rotation axis and $s = \sqrt{\sum_{i=1}^{n-1} x_i^2}$. For simplicity of notation, we write $\overrightarrow{\theta} = (\theta_1, \theta_2, \cdots, \theta_{n-2})$. Under this cylindrical coordinate system, we can write

$$\mathsf{C} = \left\{x = (s, \overrightarrow{\theta}, x_n) \in \mathbb{R}^n; x_n \in (0,1), s \in [0,r), \overrightarrow{\theta} \in [0,2\pi)^{n-2}\right\},$$

$$\widetilde{\mathsf{C}} = \left\{x = (s, \overrightarrow{\theta}, x_n) \in \mathbb{R}^n; x_n \in (0,1), s \in \left[0, \frac{r}{2}\right), \overrightarrow{\theta} \in [0,2\pi)^{n-2}\right\},$$

and

$$A_{\mathsf{C}} = \left\{x = (s, \overrightarrow{\theta}, x_n,) \in \mathbb{R}^n; x_n \in (0,1), s \in \left(\frac{r}{2}, r\right), \overrightarrow{\theta} \in [0,2\pi)^{n-2}\right\}.$$

We define a subset D_{C} of the cylinder C by setting

$$D_{\mathsf{C}} := \left\{x = (s, \overrightarrow{\theta}, x_n) \in \mathbb{R}^n; x_n \in \left(0, \frac{r}{2}\right), s \in \left(\frac{r}{2}, r - x_n\right), \overrightarrow{\theta} \in [0,2\pi)^{n-2}\right\}.$$

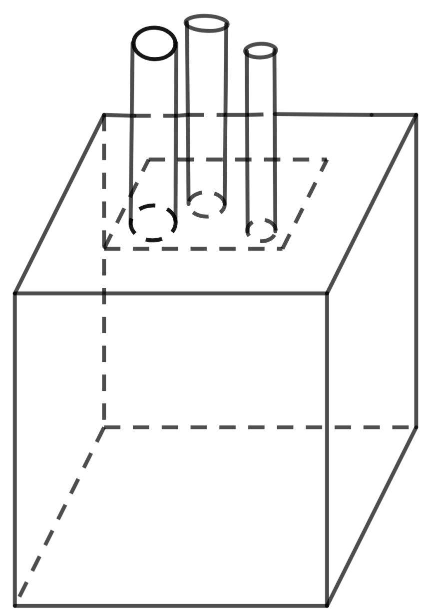

FIGURE 3. The domain Ω_3^λ

The next lemma gives two cut-off functions towards the construction of the desired extension operator.

Lemma 7.2. (1) : *There exists a function* $L^i_{\mathsf{C}} : \overline{A_{\mathsf{C}}} \to [0,1]$ *which is continuous on* $\overline{A_{\mathsf{C}}} \setminus \partial B^{n-1}(0,\frac{r}{2}) \times \{0\}$*, which equals zero both on* $\left(\overline{B}^{n-1}(0,r) \setminus B^{n-1}(0,\frac{r}{2})\right) \times \{0\}$ *and on the set* $\partial B^{n-1}(0,r) \times (0,1)$*, which equals* 1 *on* $\partial B^{n-1}(0,\frac{r}{2}) \times (0,1)$ *and which has the following additional properties. The function* L^i_{C} *is Lipschitz on* $A_{\mathsf{C}} \setminus \overline{D_{\mathsf{C}}}$ *with*

$$|\nabla L^i_{\mathsf{C}}(x)| \leq \frac{C}{r} \text{ for } x \in A_{\mathsf{C}} \setminus \overline{D_{\mathsf{C}}},$$

and L^i_{C} *is locally Lipschitz on* D_{C} *with*

$$|\nabla L^i_{\mathsf{C}}(x)| \leq \frac{C}{\sqrt{(s-\frac{r}{2})^2 + x_n^2}} \text{ for } x \in D_{\mathsf{C}}.$$

(2) : *There exists a function* $L^o_{\mathsf{C}} : \overline{A_{\mathsf{C}}} \to [0,1]$*, which is continuous on* $\overline{A_{\mathsf{C}}} \setminus \partial B^{n-1}(0,\frac{r}{2}) \times \{0\}$*, which equals zero on* $\partial B^{n-1}(0,\frac{r}{2}) \times [0,1)$*, and which equals* 1 *both on* $\partial B^{n-1}(0,r) \times (0,1)$ *and on* $\left(B^{n-1}(0,r) \setminus \overline{B}^{n-1}(0,\frac{r}{2})\right) \times \{0\}$*, and which has the additional following properties. The function* L^o_{C} *is Lipschitz on* $A_{\mathsf{C}} \setminus \overline{D_{\mathsf{C}}}$ *with*

$$|\nabla L^o_{\mathsf{C}}(x)| \leq \frac{C}{r} \text{ for } x \in A_{\mathsf{C}} \setminus \overline{D_{\mathsf{C}}},$$

and L^o_{C} *is locally Lipschitz on* D_{C} *with*

$$|\nabla L^o_{\mathsf{C}}(x)| \leq \frac{C}{\sqrt{(s-\frac{r}{2})^2 + x_n^2}} \text{ for } x \in D_{\mathsf{C}}.$$

Proof. (1) : We define the cut-off function L^i_{C} on $\overline{A_{\mathsf{C}}}$ with respect to the cylindrical coordinate system $\{x = (s, \overrightarrow{\theta}, x_n) \in \mathbb{R}^n\}$ by setting

$$L^i_{\mathsf{C}}(x) = \begin{cases} \frac{-2}{r}s + 2, & x \in \overline{A_{\mathsf{C}}} \setminus \overline{D_{\mathsf{C}}}, \\ \frac{x_n}{x_n + (s-\frac{r}{2})}, & x \in \overline{D_{\mathsf{C}}} \setminus \partial B^{n-1}(0,\frac{r}{2}) \times \{0\}, \\ 0, & x \in \partial B^{n-1}(0,\frac{r}{2}) \times \{0\}. \end{cases} \tag{7.2}$$

Then, if $x \in A_{\mathsf{C}} \setminus \overline{D_{\mathsf{C}}}$, we have

$$\frac{\partial L^i_{\mathsf{C}}(x)}{\partial \theta_1} = ... = \frac{\partial L^i_{\mathsf{C}}(x)}{\partial \theta_{n-2}} = \frac{\partial L^i_{\mathsf{C}}(x)}{\partial x_n} = 0 \text{ and } \left|\frac{\partial L^i_{\mathsf{C}}(x)}{\partial s}\right| = \frac{2}{r}.$$

If $x \in D_{\mathsf{C}}$, we have

$$\frac{\partial L^i_{\mathsf{C}}(x)}{\partial \theta_1} = ... = \frac{\partial L^i_{\mathsf{C}}(x)}{\partial \theta_{n-2}} = 0,$$

$$\left|\frac{\partial L^i_{\mathsf{C}}(x)}{\partial s}\right| = \left|\frac{x_n}{\left(x_n+(s-\frac{r}{2})\right)^2}\right| \leq \left|\frac{x_n+(s-\frac{r}{2})}{\left(x_n+(s-\frac{r}{2})\right)^2}\right| \leq \frac{1}{\sqrt{(s-\frac{r}{2})^2+x_n^2}}$$

and

$$\left|\frac{\partial L^i_{\mathsf{C}}(x)}{\partial x_n}\right| = \left|\frac{(s-\frac{r}{2})}{\left(x_n+(s-\frac{r}{2})\right)^2}\right| \leq \left|\frac{x_n+(s-\frac{r}{2})}{\left(x_n+(s-\frac{r}{2})\right)^2}\right| \leq \frac{1}{\sqrt{(s-\frac{r}{2})^2+x_n^2}}.$$

Hence, we obtain

$$|\nabla L^i_{\mathsf{C}}(x)| \leq \begin{cases} \frac{C}{r}, & x \in A_{\mathsf{C}} \setminus \overline{D_{\mathsf{C}}}, \\ \frac{C}{\sqrt{(s-\frac{r}{2})^2+x_n^2}}, & x \in D_{\mathsf{C}}. \end{cases} \tag{7.3}$$

(2) : We define the cut-off function L^o_{C} on $\overline{A_{\mathsf{C}}}$ with respect to the cylindrical coordinate system $\{x = (s, \overrightarrow{\theta}, x_n) \in \mathbb{R}^n\}$ by setting

$$L^o_{\mathsf{C}}(x) = \begin{cases} \frac{2}{r}s - 1, & x \in \overline{A_{\mathsf{C}}} \setminus \overline{D_{\mathsf{C}}}, \\ \frac{s-\frac{r}{2}}{x_n+(s-\frac{r}{2})}, & x \in \overline{D_{\mathsf{C}}} \setminus \partial B^{n-1}(0, \frac{r}{2}) \times \{0\}, \\ 0, & x \in \partial B^{n-1}(0, \frac{r}{2}) \times \{0\}. \end{cases} \tag{7.4}$$

By similar computations, we have

$$|\nabla L^o_{\mathsf{C}}(x)| \leq \begin{cases} \frac{C}{r}, & x \in A_{\mathsf{C}} \setminus \overline{D_{\mathsf{C}}}, \\ \frac{C}{\sqrt{x_n^2+(s-\frac{r}{2})^2}}, & x \in D_{\mathsf{C}}. \end{cases} \tag{7.5}$$

□

7.3. **The extension operator.** Towards the construction of our extension operator, we define piston-shaped domains P^j_k by setting

$$P^j_k := \mathsf{D}^j_k \times (0,1) \cup \widetilde{\mathsf{D}}^j_k \times [1,2).$$

The collection $\{P^j_k\}$ is pairwise disjoint. We set $U_1 := \mathcal{S}_o \times (1,2) \setminus \Omega_h$.

Given a cylinder C^j_k, in order to simplify our notation, we write $L^i_{k,j} = L^i_{\mathsf{C}^j_k}$, $L^o_{k,j} = L^o_{\mathsf{C}^j_k}$, $A^j_k = A_{\mathsf{C}^j_k}$ and $D^j_k = D_{\mathsf{C}^j_k}$. Then we define cut-off functions L^i and L^o by setting

$$L^i(x) := \sum_{k,j} L^i_{k,j}(x) \text{ for } x \in \bigcup_{k,j} \overline{A^j_k}, \tag{7.6}$$

and

$$L^o(x) := \sum_{k,j} L^o_{k,j}(x) \quad \text{for} \quad x \in \bigcup_{k,j} \overline{A^j_k}. \tag{7.7}$$

We define a reflection on $\mathcal{S}_o \times (1,2)$ by setting

$$\mathcal{R}_1(x) := (x_1, x_2, \cdots, x_{n-1}, 2 - x_n) \text{ for every } x = (x_1, x_2, \cdots, x_n) \in \mathcal{S}_o \times (1,2). \tag{7.8}$$

On the set $\bigcup_{k,j} A_k^j$, we define a mapping $\mathcal{R}_2$ which is a reflection on every A_k^j. With respect to the local cylindrical coordinate system on every A_k^j, we write

$$\mathcal{R}_2(x) := \mathcal{R}_2(s, \overrightarrow{\theta}, x_n) = \left(-\frac{s}{2} + \frac{3}{4}r_k, \overrightarrow{\theta}, x_n\right) \tag{7.9}$$

for $x = (s, \overrightarrow{\theta}, x_n) \in A_k^j$. Simple computations give the estimates

$$\frac{1}{C} \leq |J_{\mathcal{R}_1}(x)| \leq C \text{ and } |D\mathcal{R}_1(x)| \leq C, \tag{7.10}$$

for every $x \in \mathcal{S}_o \times (1,2)$, and

$$\frac{1}{C} \leq |J_{\mathcal{R}_2}(x)| \leq C \text{ and } |D\mathcal{R}_2(x)| \leq C, \tag{7.11}$$

for every $x \in \bigcup_{k,j} A_k^j$.

We begin by defining our linear extension operator on the dense subspace $W^{1,\infty}(\widetilde{\Omega}_h) \cap C(\widetilde{\Omega}_h)$ of $W^{1,p}(\widetilde{\Omega}_h)$. Given $u \in W^{1,\infty}(\widetilde{\Omega}_h) \cap C(\widetilde{\Omega}_h)$, we define the extension $E(u)$ on the rectangle $\mathcal{C}_0$ by setting

$$E(u)(x) := \begin{cases} u(x), & x \in \widetilde{\Omega}_h, \\ L^i(x)(u\circ\mathcal{R}_2)(x) + L^o(x)(u\circ\mathcal{R}_1)(x), & x \in \bigcup_{k,j} \overline{A_k^j}, \\ (u\circ\mathcal{R}_1)(x), & x \in U_1\,. \end{cases} \tag{7.12}$$

We continue with the local properties of our extension operator.

Lemma 7.3. *Let E be the extension operator defined in (7.12). Then, for every $u \in W^{1,\infty}(\widetilde{\Omega}_h) \cap C(\widetilde{\Omega}_h)$, we have:*

(1)***:*** *$E(u)$ is Lipschitz on U_1 with*

$$|\nabla E(u)(x)| \leq |\nabla(u\circ\mathcal{R}_1)(x)| \tag{7.13}$$

for almost every $x \in U_1$.

(2)***:*** *$E(u)$ is locally Lipschitz on $A_k^{n_k}$ with*

$$\begin{aligned} |\nabla E(u)(x)| \leq{}& |\nabla L^i_{k,n_k}(x)(u\circ\mathcal{R}_2)(x)| + |L^i_{k,n_k}(x)\nabla(u\circ\mathcal{R}_2)(x)| \\ &+ |\nabla L^o_{k,n_k}(x)(u\circ\mathcal{R}_1)(x)| + |L^o_{k,n_k}(x)\nabla(u\circ\mathcal{R}_1)(x)| \end{aligned} \tag{7.14}$$

for almost every $x \in A_k^j$.

Moreover, with respect to the local cylindrical system $x = (s, \overrightarrow{\theta}, x_n)$ on C_k^j, for every $1 \leq q < \infty$, we have

$$\int_{\mathsf{C}_k^j} |E(u)(x)|^q dx \leq C \int_{P_k^j} |u(x)|^q dx \tag{7.15}$$

and

$$\begin{aligned}\int_{\mathsf{C}_k^j} |\nabla E(u)(x)|^q dx \leq C \int_{P_k^j} |\nabla u(x)|^q dx \\ + C \int_{D_k^j} \left(\sqrt{\frac{1}{x_n^2 + \left(s - \frac{r_k}{2}\right)^2}} \right)^q (|u \circ \mathcal{R}_1(x)|^q + |u \circ \mathcal{R}_2(x)|^q) dx \\ + C \int_{A_k^j \setminus \overline{D_k^j}} \left(\frac{1}{r_k} \right)^q (|u \circ \mathcal{R}_1(x)|^q + |u \circ \mathcal{R}_2(x)|^q) dx,\end{aligned} \tag{7.16}$$

with some uniform positive constant C.

Proof. Since $u \in W^{1,\infty}(\widetilde{\Omega}_h) \cap C(\widetilde{\Omega}_h)$, definitions of cut-off functions L^i, L^o and reflections $\mathcal{R}_1, \mathcal{R}_2$ easily yield that $E(u)$ is Lipschitz on U_1 and that $E(u)$ is locally Lipschitz on A_k^j for every k and j. Inequalities (7.13) and (7.14) follow by the chain rule.

By the definition of $E(u)$ in (7.12), we have

$$\begin{aligned}\int_{\mathsf{C}_k^j} |E(u)(x)|^q dx \leq \int_{P_k^j} |u(x)|^q dx \\ + \int_{A_k^j} |L_{k,j}^i(x)(u \circ \mathcal{R}_2)(x) + L_{k,j}^o(x)(u \circ \mathcal{R}_1)(x)|^q dx.\end{aligned} \tag{7.17}$$

Since $0 \leq L_{k,j}^i(x) \leq 1$ and $0 \leq L_{k,j}^o(x) \leq 1$ for every $x \in A_k^j$, by (7.10), (7.11) and the change of variables formula, we have

$$\begin{aligned}\int_{A_k^j} |L_{k,j}^i(x)(u \circ \mathcal{R}_2)(x) + L_{k,j}^o(x)(u \circ \mathcal{R}_1)(x)|^q dx \\ \leq C \int_{A_k^j} |u \circ \mathcal{R}_1(x)|^q dx + C \int_{A_k^j} |u \circ \mathcal{R}_2(x)|^q dx \\ \leq C \int_{P_k^j} |u(x)|^q dx.\end{aligned} \tag{7.18}$$

By combining inequalities (7.17) and (7.18), we obtain inequality (7.15).

By inequality (7.14), we have

$$\int_{\mathsf{C}_k^j} |\nabla E(u)(x)|^q dx \leq \int_{P_k^j} |\nabla u(x)|^q dx + I_1^{k,j} + I_2^{k,j}, \tag{7.19}$$

where

$$I_1^{k,j} := \int_{A_k^j} |L_{k,j}^i(x) \nabla (u \circ \mathcal{R}_2)(x)|^q dx + \int_{A_k^j} |L_{k,j}^o(x) \nabla (u \circ \mathcal{R}_1)(x)|^q dx$$

and

$$I_2^{k,j} := \int_{A_k^j} |\nabla L_{k,j}^i(x)(u\circ\mathcal{R}_2)(x)|^q dx + \int_{A_k^j} |\nabla L_{k,j}^o(x)(u\circ\mathcal{R}_1)(x)|^q dx.$$

Arguing as for (7.18), we have

$$I_1^{k,j} \leq C\int_{P_k^j} |\nabla u(x)|^q dx. \tag{7.20}$$

By inequality (7.3), we have

$$\begin{aligned}\int_{A_k^j} |\nabla L_{k,j}^i(x)(u\circ\mathcal{R}_2)(x)|^q dx \\ \leq C\int_{A_k^j\setminus\overline{D_k^j}} \left(\frac{1}{r_k}\right)^q |(u\circ\mathcal{R}_2)(x)|^q dx \\ + C\int_{D_k^j}\left(\sqrt{\frac{1}{x_n^2+\left(s-\frac{r_k}{2}\right)^2}}\right)^q |(u\circ\mathcal{R}_2)(x)|^q dx.\end{aligned} \tag{7.21}$$

By (7.5), we have

$$\begin{aligned}\int_{A_k^j} |\nabla L_{k,j}^o(x)(u\circ\mathcal{R}_1)(x)|^q dx \\ \leq C\int_{A_k^j\setminus\overline{D_k^j}} \left(\frac{1}{r_k}\right)^q |(u\circ\mathcal{R}_1)(x)|^q dx \\ + C\int_{D_k^j}\left(\sqrt{\frac{1}{x_n^2+\left(s-\frac{r_k}{2}\right)^2}}\right)^q |(u\circ\mathcal{R}_1)(x)|^q dx.\end{aligned} \tag{7.22}$$

In conclusion, (7.21) and (7.22) give

$$\begin{aligned} I_2^{k,j} \leq C\int_{D_k^j}\left(\sqrt{\frac{1}{x_n^2+\left(s-\frac{r_k}{2}\right)^2}}\right)^q ((|u\circ\mathcal{R}_1)(x)|^q + |(u\circ\mathcal{R}_2)(x)|^q)dx \\ + C\int_{A_k^j\setminus\overline{D_k^j}}\left(\frac{1}{r_k}\right)^q (|(u\circ\mathcal{R}_1)(x)|^q + |(u\circ\mathcal{R}_2)(x)|^q)dx.\end{aligned} \tag{7.23}$$

Finally, by combining inequalities (7.19), (7.20) and (7.23), we obtain inequality (7.16). □

7.4. **An extension theorem.** The following theorem provides us with examples of irregular extension domains.

Theorem 7.1. *Let $1 \le q < n-1$ and $(n-1)q/(n-1-q) < p < \infty$ be fixed. Given $\lambda > 0$, define*

$$h_\lambda(t) := \left(\frac{1}{t}\right)^{((1-\lambda(n-1-q))(n-1)(k+1)+k)/3k}. \tag{7.24}$$

There exists $\lambda_o := \lambda_o(p,q) > 0$ such that $\widetilde{\Omega}_h \subset \mathbb{R}^n$ is a Sobolev (p,q)-extension domain with a bounded linear extension operator and with $|\partial\widetilde{\Omega}_h| > 0$ whenever $h(t) \le h_\lambda(t)$ for some $\lambda > \lambda_o$ and all $0 < t \le 1$.

Proof. By the definition of h_λ and (7.1), we have

$$r_k \le 2^{-\lambda(n-1)(k+1)}. \tag{7.25}$$

Set

$$\lambda_o(p,q) := \max\left\{\frac{n-1-p}{(n-1)^2}, \frac{p-q}{(n-1)(p-q)-pq}\right\}. \tag{7.26}$$

Then, for every $\lambda > \lambda_o$, we have $1 \le q < \frac{((n-1)\lambda-1)p}{\lambda p+(n-1)\lambda-1} < n-1$. Fix such a λ. To simplify our notation, we refer to h_λ by h in what follows. Since $E^{n-1} \subset \partial\widetilde{\mathcal{D}}_h$ and $\mathcal{H}^{n-1}(E^{n-1}) > 0$, we have $E^{n-1} \times [1,2] \subset \partial\widetilde{\Omega}_h$ and $\mathcal{H}^n(\partial\widetilde{\Omega}_h) \ge \mathcal{H}^n(E^{n-1} \times [1,2]) > 0$.

In order to prove that E defined in (7.12) is a bounded extension operator, we need an approximation argument; linearity is immediately from (7.12). Given $u \in W^{1,\infty}(\widetilde{\Omega}_h) \cap C(\widetilde{\Omega}_h)$ and $m \in \mathbb{N}$, we define $u_m := u\big|_{\widetilde{\Omega}_h^m}$. Since $\widetilde{\Omega}_h^m$ is clearly quasiconvex, it follows that u_m is Lipschitz and bounded. We define the extension $E^m(u_m)$ of u_m by setting

$$E^m(u_m)(x) := \begin{cases} u_m(x), & x \in \widetilde{\Omega}_h^m, \\ L^i(x)(u_m \circ \mathcal{R}_2)(x) + L^o(x)(u_m \circ \mathcal{R}_1)(x), & x \in \bigcup_{k=1}^m \bigcup_j \overline{A_k^j}, \\ (u_m \circ \mathcal{R}_1)(x), & x \in U_1^m, \end{cases}$$

where $U_1^m = \mathcal{S}_0 \times (0,1) \setminus \Omega_h^m$. Since u_m is Lipschitz, $E^m(u_m)$ is ACL on $\mathcal{C}_0$. By the definition of u_m and the Hölder inequality, we have

$$\int_{\widetilde{\Omega}_h^m} |u_m(x)|^q dx \le \int_{\widetilde{\Omega}_h} |u(x)|^q dx \le C\left(\int_{\widetilde{\Omega}_h} |u(x)|^p dx\right)^{\frac{q}{p}} \tag{7.27}$$

and

$$\int_{\widetilde{\Omega}_h^m} |\nabla u_m(x)|^q dx \le \int_{\widetilde{\Omega}_h} |\nabla u(x)|^q dx \le C\left(\int_{\widetilde{\Omega}_h} |\nabla u(x)|^p dx\right)^{\frac{q}{p}}. \tag{7.28}$$

Since the collection $\{P_k^j\}$ is pairwise disjoint, by summing over j and k, (7.15) and the Hölder inequality imply

$$\int_{\bigcup_{k=1}^m\bigcup_j \mathsf{C}_k^j}|E^m(u_m)(x)|^q dx \ \leq\ C\int_{\bigcup_{k=1}^m\bigcup_j P_k^j}|u_m(x)|^q dx \ \leq\ C\left(\int_{\widetilde{\Omega}_h}|u(x)|^p dx\right)^{\frac{q}{p}}. \tag{7.29}$$

By (7.10), the change of variables formula and the Hölder inequality, we have

$$\int_{U_1^m}|u_m\circ\mathcal{R}_1(x)|^q dx\leq\int_{\mathcal{R}_1\left(U_1^m\right)}|u_m(x)|^q dx\leq C\left(\int_{\widetilde{\Omega}_h}|u(x)|^p dx\right)^{\frac{q}{p}}. \tag{7.30}$$

Consequently, by combining (7.27), (7.29) and (7.30), we obtain

$$\left(\int_{\mathcal{C}_0}|E^m(u_m)(x)|^q dx\right)^{\frac{1}{q}}\leq C\left(\int_{\widetilde{\Omega}_h}|u(x)|^p dx\right)^{\frac{1}{p}}, \tag{7.31}$$

where the constant C is independent of m and u.

By (7.13), we have

$$\int_{U_1^m}|\nabla E^m(u_m)(x)|^q dx\leq\int_{U_1^m}|\nabla(u_m\circ\mathcal{R}_1)(x)|^q dx. \tag{7.32}$$

By (7.10), the change of variables formula and the Hölder inequality, we obtain

$$\begin{aligned}\int_{U_1^m}|\nabla(u_m\circ\mathcal{R}_1)(x)|^q dx&\leq\int_{U_1^m}|\nabla(u_m\circ\mathcal{R}_1)(x)|^q dx\\&\leq C\int_{\mathcal{R}_1\left(U_1^m\right)}|\nabla u_m(x)|^q dx\leq C\left(\int_{\widetilde{\Omega}_h}|\nabla u(x)|^p dx\right)^{\frac{q}{p}}.\end{aligned} \tag{7.33}$$

By combining (7.32) and (7.33), we obtain

$$\int_{U_1^m}|\nabla E^m(u_m)(x)|^q dx\leq C\left(\int_{\widetilde{\Omega}_h}|\nabla u(x)|^p dx\right)^{\frac{q}{p}}, \tag{7.34}$$

where the constant C is independent of m and u.

By (7.16) and the fact that the collection $\{P_k^j\}$ is pairwise disjoint, we have

$$\begin{aligned}\int_{\bigcup_{k=1}^m\bigcup_j \mathsf{C}_k^j}|\nabla E^m(u_m)(x)|^q dx&\leq C\int_{\bigcup_{k=1}^m\bigcup_j P_k^j}|\nabla u_m(x)|^q dx\\&+C\int_{\bigcup_{k=1}^m\bigcup_j D_k^j}\left(\sqrt{\frac{1}{x_n^2+\left(s-\frac{r_k}{2}\right)^2}}\right)^q(|(u_m\circ\mathcal{R}_1)(x)|^q+|(u_m\circ\mathcal{R}_2)(x)|^q)dx\\&+C\int_{\bigcup_{k=1}^m\bigcup_j A_k^j\setminus\overline{D_k^j}}\left(\frac{1}{r_k}\right)^q(|(u_m\circ\mathcal{R}_1)(x)|^q+|(u_m\circ\mathcal{R}_2)(x)|^q)dx.\end{aligned} \tag{7.35}$$

The Hölder inequality gives

$$\int_{\bigcup_{k=1}^{m}\bigcup_j P_k^j} |\nabla u_m(x)|^q dx \leq C\left(\int_{\widetilde{\Omega}_h} |\nabla u(x)|^p dx\right)^{\frac{q}{p}}, \tag{7.36}$$

$$\begin{aligned} &\int_{\bigcup_{k=1}^{m}\bigcup_j D_k^j} \left(\sqrt{\frac{1}{x_n^2+\left(s-\frac{r_k}{2}\right)^2}}\right)^q (|(u_m\circ\mathcal{R}_1)(x)|^q+|(u_m\circ\mathcal{R}_2)(x)|^q)dx \\ &\qquad \leq C\left(\int_{\bigcup_{k=1}^{m}\bigcup_j D_k^j} |(u_m\circ\mathcal{R}_1)(x)|^p+|(u_m\circ\mathcal{R}_2)(x)|^p dx\right)^{\frac{q}{p}} \\ &\qquad\qquad \times\left(\int_{\bigcup_{k=1}^{m}\bigcup_j D_k^j}\left(\sqrt{\frac{1}{x_n^2+\left(s-\frac{r_k}{2}\right)^2}}\right)^{\frac{pq}{p-q}} dx\right)^{\frac{p-q}{p}} \end{aligned} \tag{7.37}$$

and

$$\begin{aligned} &\int_{\bigcup_{k=1}^{m}\bigcup_j A_k^j\setminus\overline{D_k^j}} \left(\frac{1}{r_k}\right)^q (|(u_m\circ\mathcal{R}_1)(x)|^q+|(u_m\circ\mathcal{R}_2)(x)|^q)dx \\ &\qquad \leq \left(\int_{\bigcup_{k=1}^{m}\bigcup_j A_k^j\setminus\overline{D_k^j}} |(u_m\circ\mathcal{R}_1)(x)|^p+|(u_m\circ\mathcal{R}_2)(x)|^p dx\right)^{\frac{q}{p}} \\ &\qquad\qquad \times\left(\int_{\bigcup_{k=1}^{m}\bigcup_j A_k^j\setminus\overline{D_k^j}}\left(\frac{1}{r_k}\right)^{\frac{pq}{p-q}} dx\right)^{\frac{p-q}{p}}. \end{aligned} \tag{7.38}$$

By (7.10) and (7.11), the change of variables formula yields that

$$\int_{\bigcup_{k=1}^{m}\bigcup_j A_k^j\setminus\overline{D_k^j}} |(u_m\circ\mathcal{R}_1)(x)|^p+|(u_m\circ\mathcal{R}_2)(x)|^p dx \leq C\int_{\widetilde{\Omega}_h} |u(x)|^p dx \tag{7.39}$$

and

$$\int_{\bigcup_{k=1}^{m}\bigcup_j D_k^j} |(u_m\circ\mathcal{R}_1)(x)|^p+|(u_m\circ\mathcal{R}_2)(x)|^p dx \leq C\int_{\widetilde{\Omega}_h} |u(x)|^p dx. \tag{7.40}$$

With $l_k = \sqrt{x_n^2 + \left(s - \frac{r_k}{2}\right)^2}$, by (7.25) and (7.26), we have

$$\begin{aligned}(7.41)\quad \int_{\bigcup_{k=1}^m \bigcup_j D_k^j} \left(\frac{1}{\sqrt{x_n^2 + \left(s - \frac{r_k}{2}\right)^2}}\right)^{\frac{pq}{p-q}} dx &\le C \sum_{k=1}^m \sum_j r_k \int_0^{r_k} l_k^{n-2-\frac{pq}{p-q}} dl_k \\ &\le C \sum_{k=1}^m \sum_j r_k^{n-\frac{pq}{p-q}} \le C \sum_{k=1}^\infty 2^{(n-1)(k+1)\left(1-\lambda\left(n-\frac{pq}{p-q}\right)\right)} < \infty.\end{aligned}$$

Furthermore,

$$\begin{aligned}(7.42)\quad \int_{\bigcup_{k=1}^m \bigcup_j A_k^j \setminus \overline{D_k^j}} \left(\frac{1}{r_k}\right)^{\frac{pq}{p-q}} dx &\le C \sum_{k=1}^m \sum_{j=1}^{N_k} r_k^{n-1-\frac{pq}{p-q}} \\ &\le C \sum_{k=1}^\infty 2^{(n-1)(k+1)\left(1-\lambda\left(n-1-\frac{pq}{p-q}\right)\right)} < \infty.\end{aligned}$$

By combining inequalities (7.35)-(7.42), we deduce that

$$(7.43)\qquad \int_{\bigcup_{k=1}^m \bigcup_j \mathsf{C}_k^j} |\nabla E^m(u_m)(x)|^q dx \le C \left(\int_{\widetilde{\Omega}_h} |u(x)|^p + |\nabla u(x)|^p dx\right)^{\frac{q}{p}}.$$

Next, by combining (7.28), (7.34) and (7.43), we conclude that

$$(7.44)\qquad \int_{\mathcal{C}_0} |\nabla E^m(u_m)(x)|^q dx \le C \left(\int_{\widetilde{\Omega}_h} |u(x)|^p + |\nabla u(x)|^p dx\right)^{\frac{q}{p}}.$$

Hence, by combining (7.31) and (7.44), we infer that

$$(7.45)\qquad \|E^m(u_m)\|_{W^{1,q}(\mathcal{C}_0)} \le C \|u\|_{W^{1,p}(\widetilde{\Omega}_h)},$$

uniformly in m.

By the definitions of u_m and $E^m(u_m)$, for arbitrary $m, m' \in \mathbb{N}$ with $m < m'$, we have

$$\begin{aligned}(7.46)\quad \|E^m(u_m) - E^{m'}(u_{m'})\|^q_{W^{1,q}(\mathcal{C}_0)} &\le \int_{\bigcup_{k=m+1}^{m'} \bigcup_j \mathsf{C}_k^j} \left(|E^m(u_m)(x)|^q + |\nabla E^m(u_m)(x)|^q\right) dx \\ &+ \int_{\bigcup_{k=m+1}^{m'} \bigcup_j \mathsf{C}_k^j} \left(|E^{m'}(u_{m'})(x)|^q + |\nabla E^{m'}(u_{m'})(x)|^q\right) dx.\end{aligned}$$

By the definition of $E^m(u_m)$ and $E^{m'}(u_{m'})$, the Hölder inequality implies

$$\begin{aligned}(7.47)\quad \int_{\bigcup_{k=m+1}^{m'}\bigcup_j \mathsf{C}_k^j} & (|E^m(u_m)(x)|^q + |\nabla E^m(u_m)(x)|^q)\,dx \\ &\le C\int_{\mathcal{R}_1\left(\bigcup_{k=m+1}^{m'}\bigcup_j \mathsf{C}_k^j\right)} (|u_m(x)|^q + |\nabla u_m(x)|^q)\,dx \\ &\le C(p,q)\left(\int_{\mathcal{R}_1\left(\bigcup_{k=m+1}^{m'}\bigcup_j \mathsf{C}_k^j\right)} (|u(x)|^p + |\nabla u(x)|^p)\,dx\right)^{\frac{q}{p}},\end{aligned}$$

and

$$\begin{aligned}(7.48)\quad & \int_{\bigcup_{k=m+1}^{m'}\bigcup_j \mathsf{C}_k^j} \left(|E^{m'}(u_{m'})(x)|^q + |\nabla E^{m'}(u_{m'})(x)|^q\right)dx \le \\ & C\left(\int_{\bigcup_{k=m+1}^{m'}\bigcup_j \widetilde{\mathsf{C}}_k^j} (|u_{m'}|^p + |\nabla u_{m'}|^p)\,dx + \int_{\mathcal{R}_1\left(\bigcup_{k=m+1}^{m'}\bigcup_j A_k^j\right)} (|u_{m'}|^p + |\nabla u_{m'}|^p)\,dx\right)^{\frac{q}{p}} \\ &\le C\left(\int_{\bigcup_{k=m+1}^{m'}\bigcup_j \widetilde{\mathsf{C}}_k^j} (|u|^p + |\nabla u|^p)\,dx + \int_{\mathcal{R}_1\left(\bigcup_{k=m+1}^{m'}\bigcup_j A_k^j\right)} (|u|^p + |\nabla u|^p)\,dx\right)^{\frac{q}{p}}.\end{aligned}$$

Since the volumes of $\mathcal{R}_1\left(\bigcup_{k=m+1}^{m'}\bigcup_j A_k^j\right)$ and of $\bigcup_{k=m+1}^{m'}\bigcup_j \widetilde{\mathsf{C}}_k^j$ tend to zero as m, m' approach infinity, both terms in (7.47) and (7.48) converge to zero. Consequently, $\{E^m(u_m)\}$ is a Cauchy sequence in the Sobolev space $W^{1,q}(\mathcal{C}_0)$ and hence converges to some function $v \in W^{1,q}(\mathcal{C}_0)$ with respect to the $W^{1,q}$-norm. Furthermore, there exists a subsequence of $\{E^m(u_m)\}$ which converges to v almost everywhere in $\mathcal{C}_0$. On the other hand, by the definitions of $E^m(u_m)$ and $E(u)$, we have

$$\lim_{m\to\infty} E^m(u_m)(x) = E(u)(x)$$

for almost every $x \in \mathcal{C}_0$. Hence $v(x) = E(u)(x)$ almost everywhere. This implies that $E(u) \in W^{1,q}(\mathcal{C}_0)$ with

$$(7.49)\qquad \|E(u)\|_{W^{1,q}(\mathcal{C}_0)} = \|v\|_{W^{1,q}(\mathcal{C}_0)} = \lim_{m\to\infty}\|E^m(u_m)\|_{W^{1,q}(\mathcal{C}_0)} \le C\|u\|_{W^{1,p}(\widetilde{\Omega}_h)}.$$

We conclude that E defined in (7.12) is a linear extension operator from $W^{1,\infty}(\widetilde{\Omega}_h)\cap C(\widetilde{\Omega}_h)$ to $W^{1,q}(\mathcal{C}_0)$ with the norm inequality

$$\|E(u)\|_{W^{1,q}(\mathcal{C}_0)} \le C\|u\|_{W^{1,p}(\widetilde{\Omega}_h)},$$

where C is independent of u. Since $W^{1,\infty}(\widetilde{\Omega}_h)\cap C(\widetilde{\Omega}_h)$ is dense in $W^{1,p}(\widetilde{\Omega}_h)$, we can extend E to entire $W^{1,p}(\widetilde{\Omega}_h)$. It follows that $\widetilde{\Omega}_h$ is a Sobolev (p,q)-extension domain, since $\mathcal{C}_0$ is a (q,q)-extension domain. $\square$

7.5. **Positive boundary volume.**

Proof of Theorem 1.2. The claim is an immediate consequence of Theorem 7.1. $\square$

Theorem 1.1 relies on q-fatness which is a condition on relative capacity. This condition is automatically satisfied when $q > n-1$, but it may fail miserably even for Sobolev (p,q)-extension domains when $1 \le q < n-1$. This is the context of our next result.

Theorem 7.2. *Let $n \ge 3$ and $1 \le q < n-1$ be arbitrary, and let $h : [0,1] \to [0,1]$ be a strictly increasing and continuous function with $h(0) = 0$ and $h(1) = 1$. Then there exists $q < p < \infty$ and a Sobolev (p,q)-extension domain $\Omega_h \subset \mathbb{R}^n$ with a linear extension operator and with a subset $A \subset \partial\Omega_h$ of positive volume so that*

$$\lim_{r\to 0^+} \frac{Cap_q\left(\Omega_h \cap B\left(x, \frac{r}{4}\right), \Omega_h \cap A\left(x; \frac{r}{2}, \frac{3r}{4}\right); B(x,r)\right)}{h(r)} = 0$$

for every $x \in A$.

Proof. Let $n \ge 3$ and $1 \le q < n-1$. Fix $(n-1)q/(n-1-q) < p < \infty$ and a strictly increasing and continuous function $h : [0,1] \to [0,1]$. Fix $\lambda > \lambda_o$, where λ_o is from Theorem 7.1. Define

$$\tilde{h}(t) := \min\left\{h(t), \left(\frac{1}{t}\right)^{((1-\lambda(n-1-q))(n-1)(k+1)+k)/3k}\right\}.$$

Then $\widetilde{\Omega_{\tilde{h}}}$ is a Sobolev (p,q)-extension domain with a linear extension operator by Theorem 7.1. Let $A := E^{n-1} \times \left(\frac{3}{2}, 2\right)$. Then $A \subset \partial\widetilde{\Omega_{\tilde{h}}}$ and $|A| > 0$. Let $0 < r < \frac{1}{4}$ and $x \in A$ be arbitrary.

We define a cut-off function on the ball $B(x,r)$ by setting

$$F_h(y) = \begin{cases} 1 & \text{in } B(x, \frac{r}{4}), \\ \frac{-4}{r}|y-x| + 2 & \text{in } B(x, \frac{r}{2}) \setminus B(x, \frac{r}{4}), \\ 0 & \text{in } B(x,r) \setminus B(x, \frac{r}{2}). \end{cases} \tag{7.50}$$

Set $u(y) = \chi_{\widetilde{\Omega_{\tilde{h}}}}(y)$. Since $\widetilde{\Omega_{\tilde{h}}}$ is a Sobolev (p,q)-extension domain, $E(u) \in W^{1,q}(\mathcal{C}_0)$. The function v defined by $v(y) := F_h(y)E(u)(y)$ for $y \in B(x,r)$ satisfies

$$v \in \mathcal{W}_q\left(\widetilde{\Omega_{\tilde{h}}} \cap B\left(x, \frac{r}{4}\right), \widetilde{\Omega_{\tilde{h}}} \cap A\left(x; \frac{r}{2}, \frac{3r}{4}\right); B(x,r)\right).$$

Pick $k_r \in \mathbb{N}$ so that $2^{-k_r-2} < r \le 2^{-k_r-1}$. If $\mathsf{C}_k^j \cap B(x,r) \neq \emptyset$, then, by the definition of C_k^j, we have that $k > k_r$. Moreover, by the definition of $E(u)$ in (7.12), we have that

$$|\nabla v(y)| \le \begin{cases} \frac{C}{r_k}, & \text{for every } y \in \mathsf{C}_k^j \cap B(x,r), \\ 0, & \text{elsewhere}. \end{cases}$$

Hence, by the definition of v, we have

$$\int_{B(x,r)} |\nabla v(y)|^q dy \leq \sum_{k=k_r}^{\infty} \sum_{j} \int_{\mathsf{C}_k^j \cap B(x,r)} |\nabla v(y)|^q dy$$

$$\leq C \sum_{k=k_r}^{\infty} r 2^{(n-1)(k+1)} r_k^{n-1-q} \leq C \sum_{k=k_r}^{\infty} r 2^{-k} \tilde{h}(8^{-k}) \leq Crh(r).$$

Thus

$$Cap_q \left(\widetilde{\Omega_{\tilde{h}}} \cap B\left(x, \frac{r}{4}\right), \widetilde{\Omega_{\tilde{h}}} \cap A\left(x; \frac{r}{2}, \frac{3r}{4}\right); B(x,r) \right) \leq Crh(r).$$

This implies that

$$\limsup_{r \to 0^+} \frac{Cap_q \left(\widetilde{\Omega_{\tilde{h}}} \cap B\left(x, \frac{r}{4}\right), \widetilde{\Omega_{\tilde{h}}} \cap A\left(x; \frac{r}{2}, \frac{3r}{4}\right); B(x,r) \right)}{h(r)} \leq \lim_{r \to 0^+} Cr = 0,$$

as desired. □

Remark 7.1. *One can easily modify the construction of the domain from the previous proof so as to obtain a domain that fails to be $(n-1)$-fat at points of positive volume of the boundary. Let us sketch the necessary changes since we cannot use an extension operator as in the previous argument.*

First, define $r_k = 2^{-k-2} \exp(-\exp(2^k))$ and $R_k = 2^{-k-2}$. Instead of $E(u)$ in the above computation, we use a function u defined as follows. On each $Q_k^j \times [1,2)$, our function u as a function of (y', t) satisfies $u(y', t) = 1$ if $y \in \mathsf{D}_k^j$, $u(y', t) = 0$ if $y' \notin B^{n-1}(x_k^j, R_k)$ and

$$u(y', t) = \frac{\log\left(\frac{R_k}{|y' - x_k^j|}\right)}{\log\left(\frac{R_k}{r_k}\right)} \quad \text{otherwise.}$$

Define $v(y) = F_h(y) u(y)$. Then a simple computation gives what we want.

8. Final comments

In this section, we discuss in more detail some of the issues mentioned in the introduction and pose open problems that are motivated by the results in this paper.

First, let us comment on the locality of the estimate (4.6) from Theorem 4.2 that holds for almost every x for $0 < r < r_x$. When $q > n-1$, we actually have this estimate for all x and all $0 < r < \min\{1, \operatorname{diam}(\Omega)/4\}$. This also holds when $q = 1$ and $n = 2$.

Corollary 8.1. *Suppose that $1 \leq q < p$ when $n = 2$ or that $n - 1 < q < p$ when $n \geq 3$. If Ω is a Sobolev (p,q)-extension domain, then there is a nonnegative, bounded and quasiadditive* *set function Φ defined on open sets, with the following property. For each $x \in \partial\Omega$ and every $0 < r < \min\{1, \frac{1}{4}\operatorname{diam}(\Omega)\}$, we have*

$$\Phi(B(x,r))^{p-q}|B(x,r) \cap \Omega|^q \geq |B(x,r)|^q. \tag{8.1}$$

This conclusion follows by combining Theorem 4.1 with Remark 2.9, see inequality (2.10). Moreover, Theorem 4.1 shows that (8.1) holds also uniformly in x and r for $1 \leq q \leq n-1$ if we assume that (2.10) holds for these values.

One can view the uniform validity of (8.1) as the optimal analog of the Ahlfors-regularity condition (1.1). In [9, 10], it was shown, relying on (1.1), that a Sobolev (p,p)-extension domain can be equipped with a linear extension operator. We proved in Lemma 2.1 that a Sobolev (p,q)-extension domain can be equipped with a homogeneous extension operator but we do not know if one could promote this to linearity. This motivates the following problem.

Question 8.1. *Suppose that Ω is a bounded domain that satisfies the conclusion of Corollary 8.1. Find the additional assumptions that ensure the existence of a linear extension operator from $W^{1,p}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$.*

Given $1 \leq q < n-1$, we constructed a Sobolev (p,q)-extension domain whose boundary has positive volume. We do not know if such domains exist also when $q = n-1 > 1$.

Question 8.2. *Let $n \geq 3$. Does there exist a Sobolev $(p, n-1)$-extension domain $\Omega \subset \mathbb{R}^n$, for some $p > n-1$, so that $|\partial\Omega| > 0$?*

Furthermore, our constructions of examples of (p,q)-extension domains with positive boundary volume have restrictions on p in terms of q. Even though these restrictions are natural for our constructions, we do not know if some other constructions would allow p to be arbitrarily close to q.

Question 8.3. *Given $n \geq 3$, $1 \leq q < n-1$ and $p > q$, does there exist a Sobolev (p,q)-extension domain $\Omega \subset \mathbb{R}^n$ whose boundary has positive volume?*

Finally, the reader familiar with [9] may wonder why we do not employ the argument that was used there to prove (1.1) towards establishing (8.1) in the case $q < n$. We have indeed tried this but without success.

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References

[1] M. Bonk, J. Heinonen and P. Koskela, *Uniformizing Gromov hyperbolic spaces.* Astérisque No. 270 (2001), viii+99 pp.

[2] A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions,* Proc. Symp. Pure Math. 4 (1961), 33–49.
[3] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions.* Revised edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015.
[4] B. Fuglede, *Quasi topology and fine topology.* Séminaire Brelot-Choquet-Deny. Théorie du potentiel, tome 10, no.2 (1965-1966), exp. no.12, p. 1–14.
[5] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric.* J. Analyse Math. 36 (1979), 50–74 (1980).
[6] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
[7] V. M. Gol'dshtein, Yu. G. Reshetnyak, *Quasiconformal mappings and Sobolev spaces.* Dordrecht, Boston, London: Kluwer Academic Publishers, 1990.
[8] V. M. Gol'dshtein and V. N. Sitnikov, *Continuation of functions of the class W_p^1 across Hölder boundaries.* (Russian) Imbedding theorems and their applications, 31–43, Trudy Sem. S. L. Soboleva, No. 1, 1982, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1982.
[9] P. Hajłasz, P. Koskela and H. Tuominen, *Sobolev embeddings, extensions and measure density condition.* J. Funct. Anal. 254 (2008), 1217-1234.
[10] P. Hajłasz, P. Koskela and H. Tuominen, *Measure density and extendability of Sobolev functions.* Rev. Mat. Iberoam. 24 (2008), no. 2, 645–669.
[11] J. Heinonen, *The boundary absolute continuity of quasiconformal mappings II.* Rev. Mat. Iberoamericana 12 (1996), no. 3, 697–725.
[12] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.
[13] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry.* Acta Math. 181 (1998), no. 1, 1–61.
[14] S. Hencl and P. Koskela, *Lectures on mappings of finite distortion.* Lecture Notes on Mathematics 2096, Springer.
[15] D. Herron and P. Koskela, *Uniform, Sobolev extension and quasiconformal circle domains.* J. Anal. Math. 57 (1991), 172–202.
[16] P. W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces.* Acta Math. 147 (1981) 71–78.
[17] T. Kilpeläinen, *A remark on the uniqueness of quasi continuous functions.* Ann. Acad. Sci. Fenn. Math. 23 (1998), no. 1, 261–262.
[18] P. Koskela, *Extensions and imbeddings.* J. Funct. Anal. 159 (1998), no. 2, 369–383.
[19] P. Koskela, T. Rajala and Y. Zhang, *A density problem for Sobolev spaces on Gromov hyperbolic domains.* Nonlinear Anal. 154 (2017), 189–209.
[20] P. Koskela, T. Rajala and Y. Zhang, *A geometric characterization of planar Sobolev extension domains.* https://arxiv.org/abs/1502.04139.
[21] P. Koskela, T. Rajala and Y. Zhang, *Planar $W^{1,1}$-extension domains.* contained in JYU-dissertation.
[22] P.Koskela and S. Rohde, *Hausdorff dimension and mean porosity,* Math. Ann. 309 (1997), no. 4, 593–609.
[23] P. Koskela and Z. Zhu, *Product of extension domains is still an extension domain.* Indiana Univ. Math. J. 69 (2020), no. 1, 137–150.
[24] P. Koskela and Z. Zhu, *The extension property for domains with one singular point.* https://arxiv.org/abs/2110.07565

[25] P. Lahti, *A notion of fine continuity for BV functions on metric spaces.* Potential Anal. 46 (2017), no. 2, 279–294.

[26] D. Lučić, T. Rajala and J. Takanen, *Dimension extimates for the boundary of planar Sobolev extension domains*, https://arxiv.org/abs/2006.14213.

[27] V. Maz'ya, *Sobolev spaces: with applications to elliptic partial differential equations.* Springer, Berlin/Heidelberg, 2010.

[28] V. G. Maz'ya and S. V. Poborchi, *On extension of functions in Sobolev classes to the exterior of a domain with the vertex of a peak on the boundary*, Dokl. Akad. Nauk SSSR 275:5 (1984) 1066–1069 (Russian). English translation: Soviet Math. 29 (1984) 361–364.

[29] V. G. Maz'ya and S. V. Poborchi, *Extension of functions in Sobolev classes to the exterior of a domain with the vertex of a peak on the boundary I*, Czech. Math. Journ. 36:111 (1986) 634–661 (Russian).

[30] V. G. Maz'ya and S. V. Poborchi, *Differentiable functions on bad domains*, World Scientific Publishing Co., River Edge, NJ, 1997.

[31] V. G. Maz'ya and S. V. Poborchi, *Extension of functions in Sobolev classes to the exterior of a domain with the vertex of a peak on the boundary II*, Czech. Math. Journ. 37:112 (1987) 128–150 (Russian).

[32] P. Shvartsman, *Local approximations and intrinsic characterization of spaces of smooth functions on regular subsets of* $\mathbb{R}^n$. Math. Nachr. 279 (2006), no. 11, 1212–1241.

[33] P. Shvartsman, *On Sobolev extension domains in* $\mathbb{R}^n$. J. Funct. Anal. 258 (2010), no. 7, 2205–2245.

[34] P. Shvartsman, N. Zobin, *On planar Sobolev* L_p^m*-extension domains.* Adv. Math., 287 (2016), 237–346.

[35] H. J. S. Smith, *On the integration of discontinuous functions.* Proc. of the London Mathematical Society. 1874. First series. 6: 140–153.

[36] E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, 1971.

[37] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[38] T. Rado, P. V. Reichelderfer, *Continuous transformations in analysis.* Berlin: Springer-Verlag, 1955.

[39] A. D. Ukhlov, *Lower estimates for the norm of the extension operator of the weak differentiable functions on domains of Carnot groups.* Proc. of the Khabarovsk State Univ. Mathematics, 8 (1999), 33–44.

[40] A. D. Ukhlov, *Extension operators on Sobolev spaces with decreasing integrability.* Trans. Razmadze Math. Inst., 174 (2020), 381–388.

[41] J. Väisälä, *Quasiconformal maps and positive boundary measure.* Analysis 9 (1989), no. 1-2, 205–216.

[42] J. Väisälä, *The free quasiworld.* Freely quasiconformal and related maps in Banach spaces. Quasiconformal geometry and dynamics (Lublin, 1996), 55–118, Banach Center Publ., 48, Polish Acad. Sci. Inst. Math., Warsaw, 1999.

[43] S. K. Vodop'yanov, V. M. Gol'dstein, T. G. Latfullin, *A criterion for the extension of functions of the class* L_2^1 *from unbounded plane domains.* Siberian. Math. J., 20 (1979), 416–419.

[44] S. K. Vodop'yanov and A. D. Ukhlov, *Set functions and their applications in the theory of Lebesgue and Sobolev spaces. I* [Translation of Mat. Tr. 6 (2003), no. 2, 14–65; MR2033646]. Siberian Adv. Math. 14 (2004), no. 4, 78–125 (2005).

[45] S. K. Vodop'yanov and A. D. Ukhlov, *Set functions and their applications in the theory of Lebesgue and Sobolev spaces. II* [Translation of Mat. Tr. 7 (2004), no. 1, 13–49; MR2068275]. Siberian Adv. Math. 15 (2005), no. 1, 91–125.

[46] Z. Zhu, *Sobolev Functions and Mappings on cuspidal Domains*, JYU-Dissertation no. 253 (2020).

[47] W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation.* Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

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