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## Regular Articles

# Determining an unbounded potential for an elliptic equation with a power type nonlinearity 

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#### Abstract

In this article we focus on inverse problems for a semilinear elliptic equation. We show that a potential $q$ in $L^{n / 2+\varepsilon}, \varepsilon>0$, can be determined from the full and partial Dirichlet-to-Neumann map. This extends the results from [20] where this is shown for Hölder continuous potentials. Also we show that when the Dirichlet-toNeumann map is restricted to one point on the boundary, it is possible to determine a potential $q$ in $L^{n+\varepsilon}$. The authors of [25] proved this to be true for Hölder continuous potentials.


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## Contents

$\qquad$
2. Well-posedness ..... 4
3. Proof of Theorem 1.1 ..... 6
Proof of Theorem 1.2 ..... 7
5. Proof of Theorem 1.3 ..... 9
References ..... 10

## 1. Introduction

In this paper we consider an inverse problem of determining a potential in $L^{\frac{n}{2}+\varepsilon}$, for positive $\varepsilon$, from the Dirichlet-to-Neumann (DN) map related to the boundary value problem for a semilinear elliptic equation

$$
\begin{cases}\Delta u+q u^{m}=0, & \text { in } \Omega  \tag{1.1}\\ u=f, & \text { on } \partial \Omega\end{cases}
$$

[^0]where $m \geq 2, m \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^{n}$ open and bounded. This boundary value problem is well posed for $q \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ and a certain class of boundary values. In fact we show that there is $\delta>0$ such that for all (see [21] for Sobolev spaces)
$$
f \in U_{\delta}:=\left\{h \in W^{2-\frac{1}{p}, p}(\partial \Omega):\|h\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)}<\delta\right\}
$$
there exists a unique small solution $u \in W^{2, p}(\Omega)$ with sufficiently small norm. Here and in the rest of this article, we denote $p:=\frac{n}{2}+\varepsilon$. Thus the DN map can be defined as
$$
\Lambda_{q}: U_{\delta} \rightarrow W^{1-\frac{1}{p}, p}(\partial \Omega),\left.\quad f \mapsto \partial_{\nu} u_{f}\right|_{\partial \Omega}
$$

Our first main result shows that we can determine the potential from the knowledge of the DN map.
Theorem 1.1. Let $\Omega \subset \mathbb{R}^{n}, n \geq 2$, be a bounded open set with $C^{\infty}$ boundary, $\varepsilon>0$ and $q_{1}, q_{2} \in L^{\frac{n}{2}+\varepsilon}(\Omega)$. Let $\Lambda_{q_{j}}$ be the DN maps associated to the boundary value problems

$$
\begin{cases}\Delta u+q_{j} u^{m}=0, & \text { in } \Omega  \tag{1.2}\\ u=f, & \text { on } \partial \Omega,\end{cases}
$$

for $j=1,2$, and assume that $\Lambda_{q_{1}} f=\Lambda_{q_{2}}$ f for all $f \in U_{\delta}$ with $\delta>0$ sufficiently small. Then $q_{1}=q_{2}$ in $\Omega$.
This result is a special case of Theorem 1.2 but we give a proof because it is helpful for the other two main theorems of this paper. Also the proof of Theorem 1.1 gives a reconstruction formula for the potential $q$ via the Fourier transform (see Corollary 3.1).

The proof Theorem 1.1 is quite similar as in [19] and it uses the method of higher order linearization first introduced in [18] and further developed in the works [9], [19]. The key ingredient in this proof is the following integral identity which characterizes the $m$-th order linearization of the DN map $\left(D^{m} \Lambda_{q}\right)_{0}$ at 0 [19, Proposition 2.2]:

$$
\begin{equation*}
\int_{\partial \Omega}\left(D^{m} \Lambda_{q_{1}}-D^{m} \Lambda_{q_{2}}\right)_{0}\left(f_{1}, \ldots, f_{m}\right) f_{m+1} d S=-(m!) \int_{\Omega}\left(q_{1}-q_{2}\right) v_{f_{1}} \cdots v_{f_{m+1}} d x . \tag{1.3}
\end{equation*}
$$

Here $v_{f_{k}}$ are solutions to $\Delta v_{f_{k}}=0$ with boundary values $\left.v_{f_{k}}\right|_{\partial \Omega}=f_{k}$. Using this integral identity together with a result on density of products of solutions eventually gives $q_{1}=q_{2}$ in $\Omega$.

Theorem 1.1 has been proved for Hölder continuous potentials in [9] and [19] but in this article we give a first result for a less regular potential (at least to the best of our knowledge). The difference is in proving that (1.2) is well-posed when the potential is in $L^{p}(\Omega)$ and defining the DN map as a map from $U_{\delta}$ to $W^{1-\frac{1}{p}, p}(\partial \Omega)$.

In the linear case $(\Delta+q) u=0$, when $n \geq 3$, a similar result for $q \in L^{\frac{n}{2}}(\Omega)$ has been obtained in the works [23], [6] and in a more general Riemannian manifold setting in [8], where they used $L^{p}$ Carleman estimates in their proof. The case $q \in L^{\frac{n}{2}}(\Omega)$ is considered optimal in the sense of standard well-posedness theory and for the strong unique continuation principle [15]. There are also results when one assumes that $q \in W^{-1, n}(\Omega)$, see for example [11]. When $n=2$ the lowest regularity for the potential to have uniqueness in the inverse problem, at least to the best of our knowledge, is $L^{\frac{4}{3}}(\Omega)$ [3]. The same result is true on compact Riemannian surfaces with smooth boundary [22]. In dimension two the unique continuation principle holds for potentials in $L^{p}(\Omega)$ where $p>1$ (see for example [1], [2]).

In addition to the full data case, we consider some partial data results for the Schrödinger equation with unbounded potentials. In particular, let $\Gamma$ be an open subset of the boundary $\partial \Omega$. Define the partial Dirichlet-to-Neumann map for $f \in U_{\delta}, \operatorname{spt}(f) \subset \Gamma$, as

$$
\Lambda_{q}^{\Gamma} f=\left.\partial_{\nu} u\right|_{\Gamma} .
$$

Then from the knowledge of this partial DN map it is possible to determine the potential.
Theorem 1.2. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, be a connected open and bounded set with $C^{\infty}$ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial \Omega$. Let $\varepsilon>0, q_{1}, q_{2} \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ and $\Lambda_{q_{j}}^{\Gamma}$ be the partial DN maps associated to the boundary value problems

$$
\begin{cases}\Delta u+q_{j} u^{m}=0, & \text { in } \Omega \\ u=0, & \text { on } \partial \Omega \backslash \Gamma \\ u=f, & \text { on } \Gamma\end{cases}
$$

for $j=1,2$. Assume that

$$
\Lambda_{q_{1}}^{\Gamma} f=\Lambda_{q_{2}}^{\Gamma} f
$$

for all $f \in U_{\delta}$ with $\operatorname{spt}(f) \subset \Gamma$, where $\delta>0$ sufficiently small. Then $q_{1}=q_{2}$ in $\Omega$.
When the potentials are assumed to be Hölder continuous, then this theorem has been proved in [17] and [20] using the method of higher order linearization, which we will also use. Here again the key ingredients are the integral identity (1.3) and a density result for solutions of the Laplacian [25] (see also [5, Section 4]).

For the linear Schrödinger equation, partial data results with unbounded potentials have been proved only for special cases of partial data. When $n \geq 3$, it is proved in [7] that from the knowledge of the partial DN map in a specific situation it is possible to determine a potential in $L^{\frac{n}{2}}(\Omega)$. The authors use a method involving the construction of a Dirichlet Green's function for the conjugated Laplacian. In a similar situation on a manifold setting, [26] shows that a potential in $L^{\frac{n}{2}}$ can be determined from a particular case of partial data. When $n=2$ the best known result for the case of an arbitrary open subset of the boundary is for potentials in the Sobolev space $W^{1, p}(\Omega)$, for $p>2$ [14].

For partial data results, there is still the case when we are restricted to only one point on the boundary. In the situation of $\Delta u+q u^{m}$ with the potential $q$ in $C^{\alpha}(\bar{\Omega})$ this has been proved in [25] using the method of higher order linearization. Here we show that the same result holds even if we only assume that $q \in L^{n+\varepsilon}(\Omega)$ for a positive $\varepsilon$.

Theorem 1.3. Let $\Omega \subset \mathbb{R}^{n}, n \geq 2$, be a connected open and bounded set with $C^{\infty}$ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial \Omega$. Suppose that $\mu \not \equiv 0$ is a fixed measure on $\partial \Omega$ and let $\varepsilon>0$. Assume that $q_{1}, q_{2} \in L^{n+\varepsilon}(\Omega)$ satisfy

$$
\begin{equation*}
\int_{\partial \Omega} \Lambda_{q_{1}}(f) d \mu=\int_{\partial \Omega} \Lambda_{q_{2}}(f) d \mu \tag{1.4}
\end{equation*}
$$

for all $f \in U_{\delta}$ with $\operatorname{spt}(f) \subset \Gamma$, where $\delta>0$ sufficiently small. Then $q_{1}=q_{2}$ in $\Omega$. Thus when choosing $\mu=\delta_{x_{0}}$ for some fixed $x_{0} \in \partial \Omega$ the condition

$$
\Lambda_{q_{1}}(f)\left(x_{0}\right)=\Lambda_{q_{2}}(f)\left(x_{0}\right) \quad \text { for all } f \in U_{\delta} \text { with } \operatorname{spt}(f) \subset \Gamma
$$

gives $q_{1}=q_{2}$ in $\Omega$.
The proof of this theorem is very similar to the one in [25] and it uses heavily the identity (1.3) and a density result for solutions of the Laplacian [25].

It is an interesting question if in Theorems 1.1 and 1.2 it is enough to assume the potential $q$ to be in $L^{\frac{n}{2}}(\Omega)$ and if in Theorem 1.3 the potential $q$ could be in $L^{s}(\Omega)$ for $s=n$ or even $s<n$. The argument given for Theorems 1.1 and 1.2 fails when $q \in L^{\frac{n}{2}}(\Omega)$ since the well-posedness (Theorem 2.1) relies on Sobolev embedding theorems that fail for the exponent $\frac{n}{2}$. For Theorem 1.3 the restriction to $s>n$ comes from Lemma 5.1 and that we again use Sobolev embedding theorems that do not work for the exponent $n$ or exponents less than $n$.

The rest of this paper is organized as follows. In section 2 we prove the well-posedness of the boundary value problem (1.1). In sections 3 to 5 the proofs for Theorems 1.1, 1.2 and 1.3 are given.

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## 2. Well-posedness

A short reminder for the reader that we denote here and in the rest of this article $p:=\frac{n}{2}+\varepsilon$.
Theorem 2.1. (Well-posedness) Let $\Omega \subset \mathbb{R}^{n}, n \geq 2$, be a bounded open set with $C^{\infty}$ boundary, $\varepsilon>0$ and let $q \in L^{p}(\Omega)$. Then there exist $\delta, C>0$ such that for any

$$
f \in U_{\delta}:=\left\{h \in W^{2-\frac{1}{p}, p}(\partial \Omega):\|h\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)}<\delta\right\},
$$

there is a unique small solution $u_{f}$ in the class $\left\{v \in W^{2, p}(\Omega):\|w\|_{W^{2, p}(\Omega)} \leq C \delta\right\}$ of the boundary value problem

$$
\begin{cases}\Delta u+q u^{m}=0, & \text { in } \Omega  \tag{2.1}\\ u=f, & \text { on } \partial \Omega,\end{cases}
$$

where $m \in \mathbb{N}$ and $m \geq 2$. Moreover

$$
\|u\|_{W^{2, p}(\Omega)} \leq C\|f\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)},
$$

and there are $C^{\infty}$ maps

$$
\begin{aligned}
S: U_{\delta} & \rightarrow W^{2, p}(\Omega), \quad f \mapsto u_{f} \\
\Lambda_{q}: U_{\delta} & \rightarrow W^{1-\frac{1}{p}, p}(\partial \Omega),\left.\quad f \mapsto \partial_{\nu} u_{f}\right|_{\partial \Omega}
\end{aligned}
$$

The proof uses the implicit function theorem between Banach spaces [24, Theorem 10.6 and Remark 10.5] and is very similar to the one in [19, Proposition 2.1]. The difference here is that we replace Hölder spaces with Sobolev spaces and one needs to be careful with various embeddings for these spaces.

Proof. Let

$$
X=W^{2-\frac{1}{p}, p}(\partial \Omega), \quad Y=W^{2, p}(\Omega), \quad Z=L^{p}(\Omega) \times W^{2-\frac{1}{p}, p}(\partial \Omega)
$$

and $F: X \times Y \rightarrow Z$,

$$
F(f, u)=\left(Q(u),\left.u\right|_{\partial \Omega}-f\right),
$$

where $Q(u)=\Delta u+q u^{m}$. Let us now show that $F$ has the claimed mapping property. Since $u \in W^{2, p}(\Omega)$, this implies that $\left.u\right|_{\partial \Omega} \in W^{2-\frac{1}{p}, p}(\partial \Omega)$ (see [21]) and $\Delta u \in L^{p}(\Omega)$. Hence we need to show that the term $q u^{m} \in L^{p}(\Omega)$. Since $2\left(\frac{n}{2}+\varepsilon\right)>n$, then by the Sobolev embedding theorem [21] $u \in C^{0, \alpha}(\bar{\Omega})$, for $0<\alpha<1$, which is a subset of $L^{s}(\Omega)$ for every $1 \leq s \leq \infty$. Now this implies

$$
\left\|q u^{m}\right\|_{L^{p}(\Omega)} \leq\|q\|_{L^{p}(\Omega)}\left\|u^{m}\right\|_{L^{\infty}(\Omega)} \leq\|q\|_{L^{p}(\Omega)}\left(\|u\|_{L^{\infty}(\Omega)}\right)^{m}<\infty
$$

and thus $q u^{m} \in L^{p}(\Omega)$. Hence $F$ has the claimed mapping property.
Next we want to show that $F$ is a $C^{\infty}$ mapping. Since $u \mapsto \Delta u$ is a linear map $W^{2, p}(\Omega) \rightarrow L^{p}(\Omega)$, it is enough to show that $u \mapsto q u^{m}$ is a $C^{\infty} \operatorname{map} W^{2, p}(\Omega) \rightarrow L^{p}(\Omega)$. This follows since $u^{m}$ is a polynomial. More precisely, let $u, v \in W^{2, p}(\Omega)$ and use the Taylor formula:

$$
\begin{aligned}
q(u+v)^{m} & =\sum_{j=0}^{m} \frac{\partial_{u}^{j}\left(q u^{m}\right)}{j!} v^{j}+\int_{0}^{1} \frac{\partial_{u}^{m+1}\left(q(u+t v)^{m}\right)}{m!} v^{m+1}(1-t) d t \\
& =\sum_{j=0}^{m} \frac{\partial_{u}^{j}\left(q u^{m}\right)}{j!} v^{j}
\end{aligned}
$$

Now for $\|v\|_{W^{2, p}(\Omega)} \leq 1$ the above gives

$$
\left\|q(u+v)^{m}-\sum_{j=0}^{m} \frac{\partial_{u}^{j}\left(q u^{m}\right)}{j!} v^{j}\right\|_{L^{p}(\Omega)}=0 \leq\|v\|_{W^{2, p}(\Omega)}^{k+1}
$$

and thus the map $u \mapsto q(x) u^{m}$ is $C^{k}$ (in the sense of $\left[24\right.$, Definition 10.2]) for all $k \in \mathbb{N}$. Hence it is a $C^{\infty}$ map and $F$ is also $C^{\infty}$.

Our aim is to use the implicit function theorem for Banach spaces to get a unique solution for the boundary value problem (2.1). Firstly, the linearization of $F$ at $(0,0)$ in the second variable is

$$
\left.D_{u} F\right|_{(0,0)}(v)=\left(\Delta v,\left.v\right|_{\partial \Omega)}\right)
$$

which is linear and also $F(0,0)=0$. Secondly, $\left.D_{u} F\right|_{(0,0)}: Y \rightarrow Z$ is a homeomorphism. To see this, let $(\phi, g) \in Z$ and consider the boundary value problem

$$
\begin{cases}\Delta v=\phi, & \text { in } \Omega \\ v=g, & \text { on } \partial \Omega\end{cases}
$$

This problem has a unique solution for each pair $(\phi, g)$ (see for example [10, Theorem 9.15]), and thus $\left.D_{u} F\right|_{(0,0)}$ is bijective. We also have the estimate

$$
\left\|\left.D_{u} F\right|_{(0,0)}(v)\right\|_{Z}^{2}=\|\Delta v\|_{L^{p}(\Omega)}^{2}+\left\|\left.v\right|_{\partial \Omega}\right\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)}^{2} \leq M\|v\|_{W^{2, p}(\Omega)}^{2}
$$

because the trace operator from $W^{2, p}(\Omega)$ to $W^{2-\frac{1}{p}, p}(\partial \Omega)$ is bounded (see [21]). Hence $\left.D_{u} F\right|_{(0,0)}$ is also bounded and then the open mapping theorem (see e.g. [24, Theorem 8.33]) tells us that it is also a homeomorphism.

Now by the implicit function theorem [24, Theorem 10.6] there exists $\delta>0$, a neighborhood $U_{\delta}=$ $B(0, \delta) \subset X$ and a $C^{\infty} \operatorname{map} S: U_{\delta} \rightarrow Y$ such that $F(f, S(f))=0$ for $\|f\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)} \leq \delta$. Now $S$ is also Lipschitz continuous, $S(0)=0, S(f)=u$ and thus we have

$$
\|u\|_{W^{2, p}(\Omega)} \leq C\|f\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)}
$$

for $C>0$. By redefining $\delta$ if necessary we have the estimates $\|f\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)} \leq \delta,\|u\|_{W^{2, p}(\Omega)} \leq C \delta$ and the implicit function theorem gives that $u$ is the unique small solution of $F(f, u)=0$. Also the solution operator $S: U_{\delta} \rightarrow W^{2, p}(\Omega)$ is a $C^{\infty}$ map. Because $u \in W^{2, p}(\Omega)$, then $\nabla u \in W^{1, p}(\Omega)$. The trace operator is a bounded linear map from $W^{1, p}(\Omega)$ to $W^{1-\frac{1}{p}, p}(\partial \Omega)$ (see [21]) and thus $\partial_{\nu} u \in W^{1-\frac{1}{p}, p}(\partial \Omega)$ is defined almost everywhere on $\partial \Omega$. Hence $\Lambda_{q}$ is a well defined $C^{\infty}$ map between $U_{\delta}$ and $W^{1-\frac{1}{p}, p}(\partial \Omega)$.

Remark 2.2. In the previous proof, we showed that the mapping $\left.D_{u} F\right|_{(0,0)}$ is bijective and bounded and deduced that it is a homeomorphism. An alternative way to see this is to look at the inverse map $\left(\left.D_{u} F\right|_{(0,0)}\right)^{-1}: Z \rightarrow Y$ and show that it is bijective and bounded. In order to do this, one needs to prove the following estimate:

$$
\|v\|_{W^{2, p}(\Omega)} \leq C\left(\|\phi\|_{L^{p}(\Omega)}+\|g\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)}\right),
$$

where $C>0$ does not depend on $v, \phi$ and $g$. This can be done for example by combining the estimate

$$
\|v\|_{W^{2, p}(\Omega)} \leq C\left(\|\phi\|_{L^{p}(\Omega)}+\|g\|_{W^{2-\frac{1}{p}, p}(\partial \Omega)}+\|v\|_{L^{p}(\Omega)}\right)
$$

from [27, Theorem 9.1.3] with the assumption that 0 is not a Dirichlet eigenvalue and using a compactness argument.

## 3. Proof of Theorem 1.1

Using the method of higher order linearization we prove that it is possible to determine a potential in $L^{p}(\Omega)$ from the knowledge of full DN map.

Proof of Theorem 1.1. Let $\lambda_{1}, \ldots, \lambda_{m}$ be sufficiently small numbers, $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ and $f_{1}, \ldots, f_{m} \in$ $W^{2-\frac{1}{p}, p}(\partial \Omega)$. Let $u_{j}(x, \lambda) \in W^{2, p}(\Omega)$ be the unique small solution to

$$
\begin{cases}\Delta u_{j}+q_{j} u_{j}^{m}=0, & \text { in } \Omega  \tag{3.1}\\ u_{j}=\sum_{k=1}^{m} \lambda_{k} f_{k}, & \text { on } \partial \Omega\end{cases}
$$

Differentiating this with respect to $\lambda_{l}, l \in\{1, \ldots, m\}$ (possible by Theorem 2.1 which shows that $S$ is a $C^{\infty}$ map) and setting $\lambda=0$ gives that $v_{j}^{l}:=\left.\partial_{\lambda_{l}} u_{j}(x, \lambda)\right|_{\lambda=0}$ satisfies

$$
\begin{cases}\Delta v_{j}^{l}=0, & \text { in } \Omega  \tag{3.2}\\ v_{j}^{l}=f_{l}, & \text { on } \partial \Omega\end{cases}
$$

This has a unique solution in $W^{2, p}(\Omega)$ (see for example [10, Theorem 9.15]) and thus we can define $v^{l}:=$ $v_{1}^{l}=v_{2}^{l}$. Also the first linearizations of the DN maps $\Lambda_{q_{j}}$ are the DN maps of the Laplace equation.

Let $1<a \leq m-1$ be an integer and $l_{1}, \ldots, l_{a} \in\{1, \ldots, m\}$. Then the $a$-th order linearization of (3.1) is

$$
\begin{cases}\Delta\left(\left.\partial_{\lambda_{l_{1}}} \cdots \partial_{\lambda_{l_{a}}} u_{j}(x, \lambda)\right|_{\lambda=0}\right)=0, & \text { in } \Omega \\ \left.\partial_{\lambda_{l_{1}}} \cdots \partial_{\lambda_{l_{a}}} u_{j}(x, \lambda)\right|_{\lambda=0}=0, & \text { on } \partial \Omega,\end{cases}
$$

and uniqueness of solutions for the Laplace equation gives that 0 is the only solution. Thus the $a$-th order linearizations of the DN maps $\Lambda_{q_{j}}$ are equal to 0 .

Moving to the $m$-th order linearization, we apply $\left.\partial_{\lambda_{1}} \cdots \partial_{\lambda_{m}}\right|_{\lambda=0}$ to (3.1) which results in the boundary value problem

$$
\begin{cases}\Delta w_{j}=-m!q_{j} \prod_{k=1}^{m} v^{k}, & \text { in } \Omega  \tag{3.3}\\ w_{j}=0, & \text { on } \partial \Omega\end{cases}
$$

Here $w_{j}=\left.\partial_{\lambda_{1}} \cdots \partial_{\lambda_{m}} u_{j}(x, \lambda)\right|_{\lambda=0}$ and the functions $v^{k}, k \in\{1, \ldots, m\}$, are solutions to equation (3.2) with corresponding boundary values $f_{k}$. On the left hand side of (3.3) we are only left with a product of functions $v^{k}$, since after differentiating (3.1) $m$ times with respect to $\varepsilon$, all other terms involve a positive power of $u_{j}$. Proposition 2.1 says that the solution $u_{j}$ depends smoothly on $\varepsilon$ and thus when evaluating at $\varepsilon=0$, the function $u_{j}$ vanishes.

By our assumptions we have that $\Lambda_{q_{1}}\left(\sum_{k=1}^{m} \lambda_{k} f_{k}\right)=\Lambda_{q_{2}}\left(\sum_{k=1}^{m} \lambda_{k} f_{k}\right)$ and thus $\left.\partial_{\nu} u_{1}\right|_{\partial \Omega}=\left.\partial_{\nu} u_{2}\right|_{\partial \Omega}$. Applying $\left.\partial_{\lambda_{1}} \cdots \partial_{\lambda_{m}}\right|_{\lambda=0}$ to this gives $\left.\partial_{\nu} w_{1}\right|_{\partial \Omega}=\left.\partial_{\nu} w_{2}\right|_{\partial \Omega}$. Subtracting (3.3) for $j=1,2$ and integrating against $v \equiv 1$ (a solution of (3.2)) over $\Omega$ implies

$$
\begin{equation*}
\int_{\Omega} m!\left(q_{1}-q_{2}\right) \prod_{k=1}^{m} v^{k} d x=-\int_{\Omega} \Delta\left(w_{1}-w_{2}\right) d x=-\int_{\partial \Omega} \partial_{\nu}\left(w_{1}-w_{2}\right) d S=0 . \tag{3.4}
\end{equation*}
$$

Let us now choose $v^{1}, v^{2}$ to be the Calderón's exponential solutions [4]

$$
\begin{equation*}
v^{1}(x):=e^{(\eta+i \xi) \cdot x}, \quad v^{2}(x):=e^{(-\eta+i \xi) \cdot x} \tag{3.5}
\end{equation*}
$$

where $\eta, \xi \in \mathbb{R}^{n}, \eta \perp \xi$ and $|\eta|=|\xi|$, and $v^{k} \equiv 1$ for $k=3, \ldots, m$. Then we get that the Fourier transform of the difference $q_{1}-q_{2}$ at $-2 \xi$ vanishes. Thus $q_{1}=q_{2}$ since $\xi$ was arbitrary.

Notice that this proof gives a reconstruction formula for the potential. In particular, inspecting the last lines after equation (3.4) we have the following result which reconstructs the potential $q$ via its Fourier transform.

Corollary 3.1. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, be a bounded open set with $C^{\infty}$ boundary, $\varepsilon>0$ and $q \in L^{p}(\Omega)$. Let $\Lambda_{q}$ be the DN map associated to the boundary value problem

$$
\begin{cases}\Delta u+q u^{m}=0, & \text { in } \Omega \\ u=f, & \text { on } \partial \Omega .\end{cases}
$$

Then, denoting $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$,

$$
\hat{q}(-2 \xi)=-\left.\frac{1}{m!} \int_{\partial \Omega} \frac{\partial^{m}}{\partial \lambda_{1} \cdots \partial \lambda_{m}}\right|_{\lambda=0} \Lambda_{q}\left(\sum_{k=1}^{m} \lambda_{k} f_{k}\right) d S
$$

where $f_{1}, f_{2}$ are the boundary values of Calderón's exponential solutions (3.5), $f_{k} \equiv 1$ for $3 \leq k \leq m$ and $\hat{q}$ is the Fourier transform of $q$.

## 4. Proof of Theorem 1.2

We prove the partial data result for determining a potential in $L^{p}(\Omega)$ by using higher order linearization. The proof uses similar techniques as in [17] and [20].

Proof of Theorem 1.2. Let $\lambda_{1}, \ldots, \lambda_{m}$ be sufficiently small numbers, $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ and $f_{1}, \ldots, f_{m} \in$ $W^{2-\frac{1}{p}, p}(\partial \Omega)$ with $\operatorname{spt}(f) \subset \Gamma$. Let $u_{j}(x, \lambda) \in W^{2, p}(\Omega)$ be the unique small solution to

$$
\begin{cases}\Delta u_{j}+q_{j} u_{j}^{m}=0, & \text { in } \Omega \\ u_{j}=\sum_{k=1}^{m} \lambda_{k} f_{k}, & \text { on } \partial \Omega .\end{cases}
$$

The first and $m$-th order linearizations are the same as in the proof of Theorem 1.1, with corresponding boundary values. We also define $v^{l}:=v_{1}^{l}=v_{2}^{l}$ by uniqueness of solutions to (3.2). Let $v^{(0)}$ be the solution to

$$
\begin{cases}\Delta v^{(0)}=0, & \text { in } \Omega \\ v^{(0)}=0, & \text { on } \partial \Omega \backslash \Gamma \\ v^{(0)}=g, & \text { on } \Gamma,\end{cases}
$$

where $g \in C_{c}^{\infty}(\Gamma)$ with $g$ non-negative and not identically zero. By the maximum principle, $v^{(0)}>0$ in $\Omega$. Then subtracting (3.3) for $j=1,2$ and integrating against $v^{(0)}$ gives the following integral identity (compare to (3.4))

$$
\begin{align*}
-\int_{\Omega} m!\left(q_{1}-q_{2}\right) v^{(0)} \prod_{k=1}^{m} v^{k} d x & =\int_{\Omega} \Delta\left(w_{1}-w_{2}\right) v^{(0)} d x  \tag{4.1}\\
& =\int_{\Omega}\left(w_{1}-w_{2}\right) \Delta v^{(0)} d x \\
& +\int_{\partial \Omega} v^{(0)} \partial_{\nu}\left(w_{1}-w_{2}\right)-\left(w_{1}-w_{2}\right) \partial_{\nu} v^{(0)} d S \\
& =\int_{\partial \Omega} v^{(0)} \partial_{\nu}\left(w_{1}-w_{2}\right)-\left(w_{1}-w_{2}\right) \partial_{\nu} v^{(0)} d S
\end{align*}
$$

Here Green's formula and the fact that $\Delta v^{(0)}=0$ in $\Omega$ were used. Now our assumption on the DN maps coinciding gives $\left.\partial_{\nu} u_{1}\right|_{\Gamma}=\left.\partial_{\nu} u_{2}\right|_{\Gamma}$ and when applying $\left.\partial_{\lambda_{1}} \cdots \partial_{\lambda_{m}}\right|_{\lambda=0}$ to this, we have $\left.\partial_{\nu} w_{1}\right|_{\Gamma}=\left.\partial_{\nu} w_{2}\right|_{\Gamma}$. Also $w_{1}-w_{2}=0$ on $\partial \Omega$ by (3.3) and $v^{(0)}=0$ on $\partial \Omega \backslash \Gamma$. Using these (4.1) becomes

$$
\begin{align*}
-\int_{\Omega} m!\left(q_{1}-q_{2}\right) v^{(0)} \prod_{k=1}^{m} v^{k} d x & =\int_{\partial \Omega} v^{(0)} \partial_{\nu}\left(w_{1}-w_{2}\right)-\left(w_{1}-w_{2}\right) \partial_{\nu} v^{(0)} d S  \tag{4.2}\\
& =\int_{\partial \Omega \backslash \Gamma} v^{(0)} \partial_{\nu}\left(w_{1}-w_{2}\right) d S+\int_{\Gamma} v^{(0)} \partial_{\nu}\left(w_{1}-w_{2}\right) d S \\
& =0 .
\end{align*}
$$

Now we can apply Theorem 1.3 in [25] (see also [5, Section 4]) which says that the set of products of two harmonic functions that vanish on $\partial \Omega \backslash \Gamma$ is dense in $L^{1}(\Omega)$. Thus we can conclude from (4.2) that

$$
m!\left(q_{1}-q_{2}\right) v^{(0)} \prod_{k=3}^{m} v^{k}=0 \quad \text { in } \Omega
$$

Let $f_{k} \in C_{c}^{\infty}(\Gamma), f_{k}$ non-negative and $f_{k}>0$ somewhere for $k=3, \ldots, m$. Then again the maximum principle gives that $v^{k}>0$ in $\Omega$. Combining this with $v^{(0)}>0$ in $\Omega$ then implies $q_{1}=q_{2}$ in $\Omega$.

## 5. Proof of Theorem 1.3

As in [25], we need a lemma stating that the solution to the boundary value problem with a finite Borel measure $\mu$ as boundary value is in $L^{r}(\Omega)$ for $1 \leq r<\frac{n}{n-1}$. For the lemma, denote by $r^{\prime}$ the dual exponent of $1 \leq r \leq \infty$.

Lemma 5.1. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$ be a bounded open set with $C^{\infty}$ boundary and $\mu$ a finite complex Borel measure on $\partial \Omega$. Then for the function

$$
\begin{equation*}
\Psi(x)=\int_{\partial \Omega} P(x, y) d \mu(y), \quad x \in \Omega \tag{5.1}
\end{equation*}
$$

where $P(x, y)$ is the Poisson kernel for $\Delta$ in $\Omega$, we have $\Psi \in L^{r}(\Omega), 1 \leq r<\frac{n}{n-1}$. Additionally $\Psi$ solves the boundary value problem

$$
\begin{cases}\Delta \Psi=0, & \text { in } \Omega \\ \Psi=\mu, & \text { on } \partial \Omega,\end{cases}
$$

where $\Psi=\mu$ on $\partial \Omega$ means that for any $w \in W^{2, r^{\prime}}(\Omega)$ with $\left.w\right|_{\partial \Omega}=0$, in trace sense, one has

$$
\begin{equation*}
\int_{\partial \Omega} \partial_{\nu} w d \mu=\int_{\Omega}(\Delta w) \Psi d x \tag{5.2}
\end{equation*}
$$

Notice that the left hand side of relation (5.2) is well defined since $\partial_{\nu} w$ is continuous by the Sobolev embedding theorem (see for example [21]): The assumption $w \in W^{2, r^{\prime}}(\Omega)$ says that $\nabla w \in W^{1, r^{\prime}}(\Omega)$. This space embeds to $C^{0,1-\frac{n}{r^{\prime}}}(\bar{\Omega})$ if $r^{\prime}>n$. Notice that $r^{\prime}>n$ is equivalent with the assumption that $1 \leq r<\frac{n}{n-1}$. Also the right hand side of (5.2) is well defined by the fact that $\Delta w \in L^{r^{\prime}}(\Omega), \Psi \in L^{r}(\Omega)$ implies $(\Delta w) \Psi \in L^{1}(\Omega)$.

The proof of this lemma is the same as in [25, Lemma 2.1.]. The only difference when compared to the statement in [25], is that we assume $w \in W^{2, r^{\prime}}(\Omega)$ instead of $w \in C^{2}(\bar{\Omega})$.

Proof of Theorem 1.3. As before, we use the method of higher order linearization. Let $\lambda_{1}, \ldots, \lambda_{m}$ be sufficiently small numbers, $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ and $f_{1}, \ldots, f_{m} \in W^{2-\frac{1}{p}, p}(\partial \Omega)$ with $\operatorname{spt}(f) \subset \Gamma$. Let $u_{j}(x, \lambda) \in W^{2, p}(\Omega)$ be the unique small solution to

$$
\begin{cases}\Delta u_{j}+q_{j} u_{j}^{m}=0, & \text { in } \Omega \\ u_{j}=\sum_{k=1}^{m} \lambda_{k} f_{k}, & \text { on } \partial \Omega .\end{cases}
$$

The first and $m$-th order linearizations are the same as in the proof of Theorem 1.1, with corresponding boundary values. We also define $v^{l}:=v_{1}^{l}=v_{2}^{l}$ by uniqueness of solutions to (3.2).

Let $\varepsilon>0$ and $q_{1}, q_{2} \in L^{n+\varepsilon}(\Omega)$ be such that (1.4) holds for all $f \in U_{\delta}, \operatorname{spt}(f) \subset \Gamma$ with sufficiently small $\delta$. From $\partial_{\lambda_{1}} \cdots \partial_{\lambda_{m}} \Lambda_{q_{j}}(f)=\left.\partial_{\lambda_{1}} \cdots \partial_{\lambda_{m}} \partial_{\nu} u_{j}\right|_{\partial \Omega}=\left.\partial_{\nu} w_{j}\right|_{\partial \Omega}$, where $w_{j}$ is the solution to (3.3), and equation (1.4) we get that

$$
\int_{\partial \Omega}\left(\partial_{\nu} w_{1}-\partial_{\nu} w_{2}\right) d \mu=0 .
$$

Let $\Psi \in L^{(n+\varepsilon)^{\prime}}(\Omega)$ be the function given by (5.1) which is a solution to

$$
\begin{cases}\Delta \Psi=0, & \text { in } \Omega \\ \Psi=\mu, & \text { on } \partial \Omega\end{cases}
$$

in the sense of Lemma 5.1. Notice that $(n+\varepsilon)^{\prime}<\frac{n}{n-1}$ and $w_{j} \in W^{2, n+\varepsilon}(\Omega)$ because $-m!q_{j} \prod_{k=1}^{m} v^{k} \in$ $L^{n+\varepsilon}(\Omega)$ (see for example [10, Theorem 9.15]). Thus combining (5.2) and (3.3) gives

$$
0=\int_{\partial \Omega}\left(\partial_{\nu} w_{1}-\partial_{\nu} w_{2}\right) d \mu=\int_{\Omega} \Delta\left(w_{1}-w_{2}\right) \Psi d x=-\int_{\Omega} m!\left(q_{1}-q_{2}\right) \prod_{k=1}^{m} v^{k} \Psi d x
$$

where each $v^{k}$ is a solution to the Laplace equation with corresponding boundary value $f_{k}$. Let $f_{3}, \ldots, f_{m} \in$ $C^{\infty}(\partial \Omega)$ be such that $\operatorname{spt}\left(f_{k}\right) \subset \Gamma, f_{k} \geq 0$ and $f_{k}>0$ somewhere, then by the maximum principle $v^{k}>0$ in $\Omega$. Choosing the boundary values $f_{1}, f_{2} \in C^{\infty}(\partial \Omega), \operatorname{spt}\left(f_{1}\right), \operatorname{spt}\left(f_{2}\right) \subset \Gamma$, we get by elliptic regularity that $v^{1}, v^{2}$ are smooth and thus we may apply Theorem 1.3 from [25] (see also [5, Section 4]) to get

$$
m!\left(q_{1}-q_{2}\right) v_{3} \cdots v_{m} \Psi=0 \quad \text { a.e. in } \Omega .
$$

The positivity of $v_{3}, \ldots, v_{m}$ implies that $\left(q_{1}-q_{2}\right) \Psi=0$ a.e. in $\Omega$. Now we claim that $\Psi$ cannot vanish in any set $E \subset \Omega$ of positive measure. This can be seen as follows: We argue by contradiction and assume that $\Psi=0$ in $E \subset \Omega$ where $E$ has positive measure. Then by a unique continuation principle (see for example [12], $n>2$, and for $n=2$ [13]) $\Psi=0$ in $\Omega$. From [16] there is a constant $c>0$ such that for all $(x, y) \in \Omega \times \partial \Omega$

$$
c \cdot \frac{\operatorname{dist}(x, \partial \Omega)}{|x-y|^{n}} \leq P(x, y)
$$

In view of the definition of $\Psi$ in (5.1) this would imply that $\mu \equiv 0$ which is a contradiction. Hence we must have that $q_{1}=q_{2}$ a.e. in $\Omega$.

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