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Determining an unbounded potential for an elliptic equation with a power type nonlinearity

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A B S T R A C T

In this article we focus on inverse problems for a semilinear elliptic equation. We show that a potential q in $L^{n/2+\varepsilon}$, $\varepsilon > 0$, can be determined from the full and partial Dirichlet-to-Neumann map. This extends the results from [20] where this is shown for Hölder continuous potentials. Also we show that when the Dirichlet-to-Neumann map is restricted to one point on the boundary, it is possible to determine a potential q in $L^{n+\varepsilon}$. The authors of [25] proved this to be true for Hölder continuous potentials.

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Contents

1. Introduction

In this paper we consider an inverse problem of determining a potential in $L^{\frac{n}{2}+\varepsilon}$, for positive ε, from the Dirichlet-to-Neumann (DN) map related to the boundary value problem for a semilinear elliptic equation

$$\begin{cases} \Delta u + qu^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $m \geq 2$, $m \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^n$ open and bounded. This boundary value problem is well posed for $q \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ and a certain class of boundary values. In fact we show that there is $\delta > 0$ such that for all (see [21] for Sobolev spaces)

$$f \in U_\delta := \{h \in W^{2-\frac{1}{p},p}(\partial\Omega) \colon ||h||_{W^{2-\frac{1}{p},p}(\partial\Omega)} < \delta\}$$

there exists a unique small solution $u \in W^{2,p}(\Omega)$ with sufficiently small norm. Here and in the rest of this article, we denote $p := \frac{n}{2} + \varepsilon$. Thus the DN map can be defined as

$$\Lambda_q \colon U_\delta \to W^{1-\frac{1}{p},p}(\partial\Omega), \quad f \mapsto \partial_\nu u_f|_{\partial\Omega}.$$

Our first main result shows that we can determine the potential from the knowledge of the DN map.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, $\varepsilon > 0$ and $q_1, q_2 \in L^{\frac{n}{2}+\varepsilon}(\Omega)$. Let Λ_{q_j} be the DN maps associated to the boundary value problems*

$$\begin{cases} \Delta u + q_j u^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

for $j = 1, 2$, and assume that $\Lambda_{q_1} f = \Lambda_{q_2} f$ for all $f \in U_\delta$ with $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω.

This result is a special case of Theorem 1.2 but we give a proof because it is helpful for the other two main theorems of this paper. Also the proof of Theorem 1.1 gives a reconstruction formula for the potential q via the Fourier transform (see Corollary 3.1).

The proof Theorem 1.1 is quite similar as in [19] and it uses the method of higher order linearization first introduced in [18] and further developed in the works [9], [19]. The key ingredient in this proof is the following integral identity which characterizes the m-th order linearization of the DN map $(D^m \Lambda_q)_0$ at 0 [19, Proposition 2.2]:

$$\int_{\partial\Omega} (D^m \Lambda_{q_1} - D^m \Lambda_{q_2})_0 (f_1, \ldots, f_m) f_{m+1} \, dS = -(m!) \int_{\Omega} (q_1 - q_2) v_{f_1} \cdots v_{f_{m+1}} \, dx. \tag{1.3}$$

Here v_{f_k} are solutions to $\Delta v_{f_k} = 0$ with boundary values $v_{f_k}|_{\partial\Omega} = f_k$. Using this integral identity together with a result on density of products of solutions eventually gives $q_1 = q_2$ in Ω.

Theorem 1.1 has been proved for Hölder continuous potentials in [9] and [19] but in this article we give a first result for a less regular potential (at least to the best of our knowledge). The difference is in proving that (1.2) is well-posed when the potential is in $L^p(\Omega)$ and defining the DN map as a map from U_δ to $W^{1-\frac{1}{p},p}(\partial\Omega)$.

In the linear case $(\Delta + q)u = 0$, when $n \geq 3$, a similar result for $q \in L^{\frac{n}{2}}(\Omega)$ has been obtained in the works [23], [6] and in a more general Riemannian manifold setting in [8], where they used L^p Carleman estimates in their proof. The case $q \in L^{\frac{n}{2}}(\Omega)$ is considered optimal in the sense of standard well-posedness theory and for the strong unique continuation principle [15]. There are also results when one assumes that $q \in W^{-1,n}(\Omega)$, see for example [11]. When $n = 2$ the lowest regularity for the potential to have uniqueness in the inverse problem, at least to the best of our knowledge, is $L^{\frac{4}{3}}(\Omega)$ [3]. The same result is true on compact Riemannian surfaces with smooth boundary [22]. In dimension two the unique continuation principle holds for potentials in $L^p(\Omega)$ where $p > 1$ (see for example [1], [2]).

In addition to the full data case, we consider some partial data results for the Schrödinger equation with unbounded potentials. In particular, let Γ be an open subset of the boundary $\partial\Omega$. Define the partial Dirichlet-to-Neumann map for $f \in U_\delta$, $\text{spt}(f) \subset \Gamma$, as

$$\Lambda_q^\Gamma f = \partial_\nu u|_\Gamma.$$

Then from the knowledge of this partial DN map it is possible to determine the potential.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected open and bounded set with C^∞ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial\Omega$. Let $\varepsilon > 0$, $q_1, q_2 \in L^{\frac{n}{2}+\varepsilon}(\Omega)$ and $\Lambda_{q_j}^\Gamma$ be the partial DN maps associated to the boundary value problems*

$$\begin{cases} \Delta u + q_j u^m = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \setminus \Gamma \\ u = f, & \text{on } \Gamma \end{cases}$$

for $j = 1, 2$. Assume that

$$\Lambda_{q_1}^\Gamma f = \Lambda_{q_2}^\Gamma f$$

for all $f \in U_\delta$ with $\mathrm{spt}(f) \subset \Gamma$, where $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω.

When the potentials are assumed to be Hölder continuous, then this theorem has been proved in [17] and [20] using the method of higher order linearization, which we will also use. Here again the key ingredients are the integral identity (1.3) and a density result for solutions of the Laplacian [25] (see also [5, Section 4]).

For the linear Schrödinger equation, partial data results with unbounded potentials have been proved only for special cases of partial data. When $n \geq 3$, it is proved in [7] that from the knowledge of the partial DN map in a specific situation it is possible to determine a potential in $L^{\frac{n}{2}}(\Omega)$. The authors use a method involving the construction of a Dirichlet Green's function for the conjugated Laplacian. In a similar situation on a manifold setting, [26] shows that a potential in $L^{\frac{n}{2}}$ can be determined from a particular case of partial data. When $n = 2$ the best known result for the case of an arbitrary open subset of the boundary is for potentials in the Sobolev space $W^{1,p}(\Omega)$, for $p > 2$ [14].

For partial data results, there is still the case when we are restricted to only one point on the boundary. In the situation of $\Delta u + qu^m$ with the potential q in $C^\alpha(\bar{\Omega})$ this has been proved in [25] using the method of higher order linearization. Here we show that the same result holds even if we only assume that $q \in L^{n+\varepsilon}(\Omega)$ for a positive ε.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected open and bounded set with C^∞ boundary and let $\Gamma \neq \emptyset$ be an open subset of the boundary $\partial\Omega$. Suppose that $\mu \not\equiv 0$ is a fixed measure on $\partial\Omega$ and let $\varepsilon > 0$. Assume that $q_1, q_2 \in L^{n+\varepsilon}(\Omega)$ satisfy*

$$\int_{\partial\Omega} \Lambda_{q_1}(f)\, d\mu = \int_{\partial\Omega} \Lambda_{q_2}(f)\, d\mu \tag{1.4}$$

for all $f \in U_\delta$ with $\mathrm{spt}(f) \subset \Gamma$, where $\delta > 0$ sufficiently small. Then $q_1 = q_2$ in Ω. Thus when choosing $\mu = \delta_{x_0}$ for some fixed $x_0 \in \partial\Omega$ the condition

$$\Lambda_{q_1}(f)(x_0) = \Lambda_{q_2}(f)(x_0) \quad \text{for all } f \in U_\delta \text{ with } \mathrm{spt}(f) \subset \Gamma$$

gives $q_1 = q_2$ in Ω.

The proof of this theorem is very similar to the one in [25] and it uses heavily the identity (1.3) and a density result for solutions of the Laplacian [25].

It is an interesting question if in Theorems 1.1 and 1.2 it is enough to assume the potential q to be in $L^{\frac{n}{2}}(\Omega)$ and if in Theorem 1.3 the potential q could be in $L^s(\Omega)$ for $s = n$ or even $s < n$. The argument given for Theorems 1.1 and 1.2 fails when $q \in L^{\frac{n}{2}}(\Omega)$ since the well-posedness (Theorem 2.1) relies on Sobolev embedding theorems that fail for the exponent $\frac{n}{2}$. For Theorem 1.3 the restriction to $s > n$ comes from Lemma 5.1 and that we again use Sobolev embedding theorems that do not work for the exponent n or exponents less than n.

The rest of this paper is organized as follows. In section 2 we prove the well-posedness of the boundary value problem (1.1). In sections 3 to 5 the proofs for Theorems 1.1, 1.2 and 1.3 are given.

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2. Well-posedness

A short reminder for the reader that we denote here and in the rest of this article $p := \frac{n}{2} + \varepsilon$.

Theorem 2.1. *(Well-posedness) Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded open set with C^∞ boundary, $\varepsilon > 0$ and let $q \in L^p(\Omega)$. Then there exist $\delta, C > 0$ such that for any*

$$f \in U_\delta := \{h \in W^{2-\frac{1}{p},p}(\partial\Omega) \colon ||h||_{W^{2-\frac{1}{p},p}(\partial\Omega)} < \delta\},$$

there is a unique small solution u_f in the class $\{v \in W^{2,p}(\Omega) \colon ||w||_{W^{2,p}(\Omega)} \leq C\delta\}$ of the boundary value problem

$$\begin{cases} \Delta u + qu^m = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $m \in \mathbb{N}$ and $m \geq 2$. Moreover

$$||u||_{W^{2,p}(\Omega)} \leq C||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)},$$

and there are C^∞ maps

$$S \colon U_\delta \to W^{2,p}(\Omega), \quad f \mapsto u_f,$$
$$\Lambda_q \colon U_\delta \to W^{1-\frac{1}{p},p}(\partial\Omega), \quad f \mapsto \partial_\nu u_f|_{\partial\Omega}.$$

The proof uses the implicit function theorem between Banach spaces [24, Theorem 10.6 and Remark 10.5] and is very similar to the one in [19, Proposition 2.1]. The difference here is that we replace Hölder spaces with Sobolev spaces and one needs to be careful with various embeddings for these spaces.

Proof. Let

$$X = W^{2-\frac{1}{p},p}(\partial\Omega), \quad Y = W^{2,p}(\Omega), \quad Z = L^p(\Omega) \times W^{2-\frac{1}{p},p}(\partial\Omega)$$

and $F \colon X \times Y \to Z$,

$$F(f, u) = (Q(u), u|_{\partial\Omega} - f),$$

where $Q(u) = \Delta u + qu^m$. Let us now show that F has the claimed mapping property. Since $u \in W^{2,p}(\Omega)$, this implies that $u|_{\partial\Omega} \in W^{2-\frac{1}{p},p}(\partial\Omega)$ (see [21]) and $\Delta u \in L^p(\Omega)$. Hence we need to show that the term $qu^m \in L^p(\Omega)$. Since $2\left(\frac{n}{2}+\varepsilon\right) > n$, then by the Sobolev embedding theorem [21] $u \in C^{0,\alpha}(\bar{\Omega})$, for $0 < \alpha < 1$, which is a subset of $L^s(\Omega)$ for every $1 \le s \le \infty$. Now this implies

$$||qu^m||_{L^p(\Omega)} \le ||q||_{L^p(\Omega)}||u^m||_{L^\infty(\Omega)} \le ||q||_{L^p(\Omega)}\left(||u||_{L^\infty(\Omega)}\right)^m < \infty$$

and thus $qu^m \in L^p(\Omega)$. Hence F has the claimed mapping property.

Next we want to show that F is a C^∞ mapping. Since $u \mapsto \Delta u$ is a linear map $W^{2,p}(\Omega) \to L^p(\Omega)$, it is enough to show that $u \mapsto qu^m$ is a C^∞ map $W^{2,p}(\Omega) \to L^p(\Omega)$. This follows since u^m is a polynomial. More precisely, let $u, v \in W^{2,p}(\Omega)$ and use the Taylor formula:

$$\begin{aligned} q(u+v)^m &= \sum_{j=0}^{m} \frac{\partial_u^j(qu^m)}{j!}v^j + \int_0^1 \frac{\partial_u^{m+1}\left(q(u+tv)^m\right)}{m!}v^{m+1}(1-t)\,dt \\ &= \sum_{j=0}^{m} \frac{\partial_u^j(qu^m)}{j!}v^j. \end{aligned}$$

Now for $||v||_{W^{2,p}(\Omega)} \le 1$ the above gives

$$\left|\left|q(u+v)^m - \sum_{j=0}^{m} \frac{\partial_u^j(qu^m)}{j!}v^j\right|\right|_{L^p(\Omega)} = 0 \le ||v||_{W^{2,p}(\Omega)}^{k+1}$$

and thus the map $u \mapsto q(x)u^m$ is C^k (in the sense of [24, Definition 10.2]) for all $k \in \mathbb{N}$. Hence it is a C^∞ map and F is also C^∞.

Our aim is to use the implicit function theorem for Banach spaces to get a unique solution for the boundary value problem (2.1). Firstly, the linearization of F at $(0,0)$ in the second variable is

$$D_uF|_{(0,0)}(v) = (\Delta v, v|_{\partial\Omega}),$$

which is linear and also $F(0,0) = 0$. Secondly, $D_uF|_{(0,0)}\colon Y \to Z$ is a homeomorphism. To see this, let $(\phi, g) \in Z$ and consider the boundary value problem

$$\begin{cases} \Delta v = \phi, & \text{in } \Omega \\ v = g, & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique solution for each pair (ϕ, g) (see for example [10, Theorem 9.15]), and thus $D_uF|_{(0,0)}$ is bijective. We also have the estimate

$$||D_uF|_{(0,0)}(v)||_Z^2 = ||\Delta v||_{L^p(\Omega)}^2 + ||v|_{\partial\Omega}||_{W^{2-\frac{1}{p},p}(\partial\Omega)}^2 \le M||v||_{W^{2,p}(\Omega)}^2,$$

because the trace operator from $W^{2,p}(\Omega)$ to $W^{2-\frac{1}{p},p}(\partial\Omega)$ is bounded (see [21]). Hence $D_uF|_{(0,0)}$ is also bounded and then the open mapping theorem (see e.g. [24, Theorem 8.33]) tells us that it is also a homeomorphism.

Now by the implicit function theorem [24, Theorem 10.6] there exists $\delta > 0$, a neighborhood $U_\delta = B(0,\delta) \subset X$ and a C^∞ map $S\colon U_\delta \to Y$ such that $F(f, S(f)) = 0$ for $||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)} \le \delta$. Now S is also Lipschitz continuous, $S(0) = 0, S(f) = u$ and thus we have

$$||u||_{W^{2,p}(\Omega)} \leq C||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)}$$

for $C > 0$. By redefining δ if necessary we have the estimates $||f||_{W^{2-\frac{1}{p},p}(\partial\Omega)} \leq \delta$, $||u||_{W^{2,p}(\Omega)} \leq C\delta$ and the implicit function theorem gives that u is the unique small solution of $F(f,u) = 0$. Also the solution operator $S\colon U_\delta \to W^{2,p}(\Omega)$ is a C^∞ map. Because $u \in W^{2,p}(\Omega)$, then $\nabla u \in W^{1,p}(\Omega)$. The trace operator is a bounded linear map from $W^{1,p}(\Omega)$ to $W^{1-\frac{1}{p},p}(\partial\Omega)$ (see [21]) and thus $\partial_\nu u \in W^{1-\frac{1}{p},p}(\partial\Omega)$ is defined almost everywhere on $\partial\Omega$. Hence Λ_q is a well defined C^∞ map between U_δ and $W^{1-\frac{1}{p},p}(\partial\Omega)$. □

Remark 2.2. In the previous proof, we showed that the mapping $D_uF|_{(0,0)}$ is bijective and bounded and deduced that it is a homeomorphism. An alternative way to see this is to look at the inverse map $(D_uF|_{(0,0)})^{-1}\colon Z \to Y$ and show that it is bijective and bounded. In order to do this, one needs to prove the following estimate:

$$||v||_{W^{2,p}(\Omega)} \leq C\left(||\phi||_{L^p(\Omega)} + ||g||_{W^{2-\frac{1}{p},p}(\partial\Omega)}\right),$$

where $C > 0$ does not depend on v, ϕ and g. This can be done for example by combining the estimate

$$||v||_{W^{2,p}(\Omega)} \leq C\left(||\phi||_{L^p(\Omega)} + ||g||_{W^{2-\frac{1}{p},p}(\partial\Omega)} + ||v||_{L^p(\Omega)}\right)$$

from [27, Theorem 9.1.3] with the assumption that 0 is not a Dirichlet eigenvalue and using a compactness argument.

3. Proof of Theorem 1.1

Using the method of higher order linearization we prove that it is possible to determine a potential in $L^p(\Omega)$ from the knowledge of full DN map.

Proof of Theorem 1.1. Let $\lambda_1, \ldots, \lambda_m$ be sufficiently small numbers, $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $f_1, \ldots, f_m \in W^{2-\frac{1}{p},p}(\partial\Omega)$. Let $u_j(x,\lambda) \in W^{2,p}(\Omega)$ be the unique small solution to

$$\begin{cases} \Delta u_j + q_j u_j^m = 0, & \text{in } \Omega \\ u_j = \sum_{k=1}^m \lambda_k f_k, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Differentiating this with respect to $\lambda_l, l \in \{1, \ldots, m\}$ (possible by Theorem 2.1 which shows that S is a C^∞ map) and setting $\lambda = 0$ gives that $v_j^l := \partial_{\lambda_l} u_j(x,\lambda)|_{\lambda=0}$ satisfies

$$\begin{cases} \Delta v_j^l = 0, & \text{in } \Omega \\ v_j^l = f_l, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

This has a unique solution in $W^{2,p}(\Omega)$ (see for example [10, Theorem 9.15]) and thus we can define $v^l := v_1^l = v_2^l$. Also the first linearizations of the DN maps Λ_{q_j} are the DN maps of the Laplace equation.

Let $1 < a \leq m - 1$ be an integer and $l_1, \ldots, l_a \in \{1, \ldots, m\}$. Then the a-th order linearization of (3.1) is

$$\begin{cases} \Delta(\partial_{\lambda_{l_1}} \cdots \partial_{\lambda_{l_a}} u_j(x,\lambda)|_{\lambda=0}) = 0, & \text{in } \Omega \\ \partial_{\lambda_{l_1}} \cdots \partial_{\lambda_{l_a}} u_j(x,\lambda)|_{\lambda=0} = 0, & \text{on } \partial\Omega, \end{cases}$$

and uniqueness of solutions for the Laplace equation gives that 0 is the only solution. Thus the a-th order linearizations of the DN maps Λ_{q_j} are equal to 0.

Moving to the m-th order linearization, we apply $\partial_{\lambda_1} \cdots \partial_{\lambda_m}|_{\lambda=0}$ to (3.1) which results in the boundary value problem

$$\begin{cases} \Delta w_j = -m! q_j \prod_{k=1}^m v^k, & \text{in } \Omega \\ w_j = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Here $w_j = \partial_{\lambda_1} \cdots \partial_{\lambda_m} u_j(x,\lambda)|_{\lambda=0}$ and the functions $v^k, k \in \{1,\ldots,m\}$, are solutions to equation (3.2) with corresponding boundary values f_k. On the left hand side of (3.3) we are only left with a product of functions v^k, since after differentiating (3.1) m times with respect to ε, all other terms involve a positive power of u_j. Proposition 2.1 says that the solution u_j depends smoothly on ε and thus when evaluating at $\varepsilon = 0$, the function u_j vanishes.

By our assumptions we have that $\Lambda_{q_1}\left(\sum_{k=1}^m \lambda_k f_k\right) = \Lambda_{q_2}\left(\sum_{k=1}^m \lambda_k f_k\right)$ and thus $\partial_\nu u_1|_{\partial\Omega} = \partial_\nu u_2|_{\partial\Omega}$. Applying $\partial_{\lambda_1} \cdots \partial_{\lambda_m}|_{\lambda=0}$ to this gives $\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}$. Subtracting (3.3) for $j = 1,2$ and integrating against $v \equiv 1$ (a solution of (3.2)) over Ω implies

$$\int_\Omega m!(q_1 - q_2) \prod_{k=1}^m v^k \, dx = -\int_\Omega \Delta(w_1 - w_2)\, dx = -\int_{\partial\Omega} \partial_\nu (w_1 - w_2)\, dS = 0. \tag{3.4}$$

Let us now choose v^1, v^2 to be the Calderón's exponential solutions [4]

$$v^1(x) := e^{(\eta + i\xi)\cdot x}, \quad v^2(x) := e^{(-\eta + i\xi)\cdot x}, \tag{3.5}$$

where $\eta, \xi \in \mathbb{R}^n$, $\eta \perp \xi$ and $|\eta| = |\xi|$, and $v^k \equiv 1$ for $k = 3, \ldots, m$. Then we get that the Fourier transform of the difference $q_1 - q_2$ at -2ξ vanishes. Thus $q_1 = q_2$ since ξ was arbitrary. □

Notice that this proof gives a reconstruction formula for the potential. In particular, inspecting the last lines after equation (3.4) we have the following result which reconstructs the potential q via its Fourier transform.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, $\varepsilon > 0$ and $q \in L^p(\Omega)$. Let Λ_q be the DN map associated to the boundary value problem*

$$\begin{cases} \Delta u + q u^m = 0, & \textit{in } \Omega \\ u = f, & \textit{on } \partial\Omega. \end{cases}$$

Then, denoting $\lambda = (\lambda_1, \ldots, \lambda_m)$,

$$\hat{q}(-2\xi) = -\frac{1}{m!} \int_{\partial\Omega} \frac{\partial^m}{\partial\lambda_1 \cdots \partial\lambda_m}\Big|_{\lambda=0} \Lambda_q \left(\sum_{k=1}^m \lambda_k f_k \right) dS,$$

where f_1, f_2 are the boundary values of Calderón's exponential solutions (3.5), $f_k \equiv 1$ for $3 \leq k \leq m$ and $\hat{q}$ is the Fourier transform of q.

4. Proof of Theorem 1.2

We prove the partial data result for determining a potential in $L^p(\Omega)$ by using higher order linearization. The proof uses similar techniques as in [17] and [20].

Proof of Theorem 1.2. Let $\lambda_1, \dots, \lambda_m$ be sufficiently small numbers, $\lambda = (\lambda_1, \dots, \lambda_m)$ and $f_1, \dots, f_m \in W^{2-\frac{1}{p},p}(\partial\Omega)$ with $\mathrm{spt}(f) \subset \Gamma$. Let $u_j(x,\lambda) \in W^{2,p}(\Omega)$ be the unique small solution to

$$\begin{cases} \Delta u_j + q_j u_j^m = 0, & \text{in } \Omega \\ u_j = \sum_{k=1}^m \lambda_k f_k, & \text{on } \partial\Omega. \end{cases}$$

The first and m-th order linearizations are the same as in the proof of Theorem 1.1, with corresponding boundary values. We also define $v^l := v_1^l = v_2^l$ by uniqueness of solutions to (3.2). Let $v^{(0)}$ be the solution to

$$\begin{cases} \Delta v^{(0)} = 0, & \text{in } \Omega \\ v^{(0)} = 0, & \text{on } \partial\Omega \setminus \Gamma \\ v^{(0)} = g, & \text{on } \Gamma, \end{cases}$$

where $g \in C_c^\infty(\Gamma)$ with g non-negative and not identically zero. By the maximum principle, $v^{(0)} > 0$ in Ω. Then subtracting (3.3) for $j = 1, 2$ and integrating against $v^{(0)}$ gives the following integral identity (compare to (3.4))

$$\begin{aligned} -\int_\Omega m!(q_1 - q_2) v^{(0)} \prod_{k=1}^m v^k \, dx &= \int_\Omega \Delta(w_1 - w_2) v^{(0)} \, dx \\ &= \int_\Omega (w_1 - w_2) \Delta v^{(0)} \, dx \\ &\quad + \int_{\partial\Omega} v^{(0)} \partial_\nu (w_1 - w_2) - (w_1 - w_2) \partial_\nu v^{(0)} \, dS \\ &= \int_{\partial\Omega} v^{(0)} \partial_\nu (w_1 - w_2) - (w_1 - w_2) \partial_\nu v^{(0)} \, dS \end{aligned} \tag{4.1}$$

Here Green's formula and the fact that $\Delta v^{(0)} = 0$ in Ω were used. Now our assumption on the DN maps coinciding gives $\partial_\nu u_1|_\Gamma = \partial_\nu u_2|_\Gamma$ and when applying $\partial_{\lambda_1} \cdots \partial_{\lambda_m}|_{\lambda=0}$ to this, we have $\partial_\nu w_1|_\Gamma = \partial_\nu w_2|_\Gamma$. Also $w_1 - w_2 = 0$ on $\partial\Omega$ by (3.3) and $v^{(0)} = 0$ on $\partial\Omega \setminus \Gamma$. Using these (4.1) becomes

$$\begin{aligned} -\int_\Omega m!(q_1 - q_2) v^{(0)} \prod_{k=1}^m v^k \, dx &= \int_{\partial\Omega} v^{(0)} \partial_\nu (w_1 - w_2) - (w_1 - w_2) \partial_\nu v^{(0)} \, dS \\ &= \int_{\partial\Omega\setminus\Gamma} v^{(0)} \partial_\nu (w_1 - w_2) \, dS + \int_\Gamma v^{(0)} \partial_\nu (w_1 - w_2) \, dS \\ &= 0. \end{aligned} \tag{4.2}$$

Now we can apply Theorem 1.3 in [25] (see also [5, Section 4]) which says that the set of products of two harmonic functions that vanish on $\partial\Omega \setminus \Gamma$ is dense in $L^1(\Omega)$. Thus we can conclude from (4.2) that

$$m!(q_1 - q_2) v^{(0)} \prod_{k=3}^m v^k = 0 \quad \text{in } \Omega.$$

Let $f_k \in C_c^\infty(\Gamma)$, f_k non-negative and $f_k > 0$ somewhere for $k = 3, \dots, m$. Then again the maximum principle gives that $v^k > 0$ in Ω. Combining this with $v^{(0)} > 0$ in Ω then implies $q_1 = q_2 \quad \text{in } \Omega$. □

5. Proof of Theorem 1.3

As in [25], we need a lemma stating that the solution to the boundary value problem with a finite Borel measure μ as boundary value is in $L^r(\Omega)$ for $1 \le r < \frac{n}{n-1}$. For the lemma, denote by r' the dual exponent of $1 \le r \le \infty$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be a bounded open set with C^∞ boundary and μ a finite complex Borel measure on $\partial\Omega$. Then for the function*

$$\Psi(x) = \int_{\partial\Omega} P(x,y)\, d\mu(y), \quad x \in \Omega, \tag{5.1}$$

where $P(x,y)$ is the Poisson kernel for Δ in Ω, we have $\Psi \in L^r(\Omega)$, $1 \le r < \frac{n}{n-1}$. Additionally Ψ solves the boundary value problem

$$\begin{cases} \Delta\Psi = 0, & \text{in } \Omega \\ \Psi = \mu, & \text{on } \partial\Omega, \end{cases}$$

where $\Psi = \mu$ on $\partial\Omega$ means that for any $w \in W^{2,r'}(\Omega)$ with $w|_{\partial\Omega} = 0$, in trace sense, one has

$$\int_{\partial\Omega} \partial_\nu w\, d\mu = \int_\Omega (\Delta w)\Psi\, dx. \tag{5.2}$$

Notice that the left hand side of relation (5.2) is well defined since $\partial_\nu w$ is continuous by the Sobolev embedding theorem (see for example [21]): The assumption $w \in W^{2,r'}(\Omega)$ says that $\nabla w \in W^{1,r'}(\Omega)$. This space embeds to $C^{0,1-\frac{n}{r'}}(\bar{\Omega})$ if $r' > n$. Notice that $r' > n$ is equivalent with the assumption that $1 \le r < \frac{n}{n-1}$. Also the right hand side of (5.2) is well defined by the fact that $\Delta w \in L^{r'}(\Omega), \Psi \in L^r(\Omega)$ implies $(\Delta w)\Psi \in L^1(\Omega)$.

The proof of this lemma is the same as in [25, Lemma 2.1.]. The only difference when compared to the statement in [25], is that we assume $w \in W^{2,r'}(\Omega)$ instead of $w \in C^2(\bar{\Omega})$.

Proof of Theorem 1.3. As before, we use the method of higher order linearization. Let $\lambda_1, \dots, \lambda_m$ be sufficiently small numbers, $\lambda = (\lambda_1, \dots, \lambda_m)$ and $f_1, \dots, f_m \in W^{2-\frac{1}{p},p}(\partial\Omega)$ with $\operatorname{spt}(f) \subset \Gamma$. Let $u_j(x,\lambda) \in W^{2,p}(\Omega)$ be the unique small solution to

$$\begin{cases} \Delta u_j + q_j u_j^m = 0, & \text{in } \Omega \\ u_j = \sum_{k=1}^m \lambda_k f_k, & \text{on } \partial\Omega. \end{cases}$$

The first and m-th order linearizations are the same as in the proof of Theorem 1.1, with corresponding boundary values. We also define $v^l := v_1^l = v_2^l$ by uniqueness of solutions to (3.2).

Let $\varepsilon > 0$ and $q_1, q_2 \in L^{n+\varepsilon}(\Omega)$ be such that (1.4) holds for all $f \in U_\delta$, $\operatorname{spt}(f) \subset \Gamma$ with sufficiently small δ. From $\partial_{\lambda_1} \cdots \partial_{\lambda_m} \Lambda_{q_j}(f) = \partial_{\lambda_1} \cdots \partial_{\lambda_m} \partial_\nu u_j|_{\partial\Omega} = \partial_\nu w_j|_{\partial\Omega}$, where w_j is the solution to (3.3), and equation (1.4) we get that

$$\int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, d\mu = 0.$$

Let $\Psi \in L^{(n+\varepsilon)'}(\Omega)$ be the function given by (5.1) which is a solution to

$$\begin{cases} \Delta\Psi = 0, & \text{in } \Omega \\ \Psi = \mu, & \text{on } \partial\Omega \end{cases}$$

in the sense of Lemma 5.1. Notice that $(n+\varepsilon)' < \frac{n}{n-1}$ and $w_j \in W^{2,n+\varepsilon}(\Omega)$ because $-m!q_j \prod_{k=1}^{m} v^k \in L^{n+\varepsilon}(\Omega)$ (see for example [10, Theorem 9.15]). Thus combining (5.2) and (3.3) gives

$$0 = \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2)\, d\mu = \int_{\Omega} \Delta(w_1 - w_2)\Psi\, dx = -\int_{\Omega} m!(q_1 - q_2) \prod_{k=1}^{m} v^k \Psi\, dx,$$

where each v^k is a solution to the Laplace equation with corresponding boundary value f_k. Let $f_3, \ldots, f_m \in C^\infty(\partial\Omega)$ be such that $\operatorname{spt}(f_k) \subset \Gamma$, $f_k \geq 0$ and $f_k > 0$ somewhere, then by the maximum principle $v^k > 0$ in Ω. Choosing the boundary values $f_1, f_2 \in C^\infty(\partial\Omega)$, $\operatorname{spt}(f_1), \operatorname{spt}(f_2) \subset \Gamma$, we get by elliptic regularity that v^1, v^2 are smooth and thus we may apply Theorem 1.3 from [25] (see also [5, Section 4]) to get

$$m!(q_1 - q_2)v_3 \cdots v_m \Psi = 0 \quad \text{a.e. in } \Omega.$$

The positivity of $v_3, \ldots, v_m$ implies that $(q_1 - q_2)\Psi = 0$ a.e. in Ω. Now we claim that Ψ cannot vanish in any set $E \subset \Omega$ of positive measure. This can be seen as follows: We argue by contradiction and assume that $\Psi = 0$ in $E \subset \Omega$ where E has positive measure. Then by a unique continuation principle (see for example [12], $n > 2$, and for $n = 2$ [13]) $\Psi = 0$ in Ω. From [16] there is a constant $c > 0$ such that for all $(x, y) \in \Omega \times \partial\Omega$

$$c \cdot \frac{\operatorname{dist}(x, \partial\Omega)}{|x - y|^n} \leq P(x, y).$$

In view of the definition of Ψ in (5.1) this would imply that $\mu \equiv 0$ which is a contradiction. Hence we must have that $q_1 = q_2$ a.e. in Ω. □

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