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Calculus of Variations

Optimal C^∞-approximation of functions with exponentially or sub-exponentially integrable derivative

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Abstract

We discuss Meyers-Serrin's type results for smooth approximations of functions $b = b(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$, with convergence of an energy of the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} w(t, x)\varphi\left(|Db(t, x)|\right) \mathrm{d}x\mathrm{d}t\,,$$

where $w > 0$ is a suitable weight function, and $\varphi : [0, \infty) \to [0, \infty)$ is a convex function with $\varphi(0) = 0$ having exponential or subexponential growth.

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Contents

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1 Introduction and results

In this note we deal with the approximation of functions $b = b(t, x) : I \times \Omega \to \mathbb{R}^m$ by smooth ones, where $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ are an open interval and an open set, respectively. Our main motivation comes from [4], where $m = n$ and b is a possibly nonautonomous vector field. In that paper we are dealing with *a priori* upper bounds for Sobolev norms of the flow of b by means of quantities $N_{\varphi,w}(|Db|)$ that depend on the spatial derivative Db of b. The quantities $N_{\varphi,w}(|Db|)$ are *energies* of the form

$$N_{\varphi,w}(|Db|) := \int_I \int_\Omega w(t,x)\varphi\left(|Db(t,x)|\right)\, \mathrm{d}x\, \mathrm{d}t,$$

where $w > 0$ is a suitable weight function (related to $\mathrm{dist}(x, \partial\Omega)$, or to the length of the maximal interval of the ODE associated to b) and $\varphi : [0,\infty) \to [0,\infty)$ is a convex function with $\varphi(0) = 0$ having exponential or sub-exponential growth, the model case being $\exp_*(t) := \exp(t) - 1$. When one tries to extend the *a priori* estimates from the case of smooth vector fields b to those having only a Sobolev spatial regularity, one faces the difficulty of passing the quantity $N_{\varphi,w}$ to the limit.

In this context, if φ had polynomial growth, a weighted and t-dependent version of the celebrated Meyers-Serrin Theorem [9] would be applicable, providing even a smooth approximation $(b_h)_h$ with $N_{\varphi,w}(|D(b_h - b)|) \to 0$. In general, as the discussion below shows, this kind of approximation fails when φ does not satisfy a doubling condition. However, we realized that the exponential (or subexponential) case is a borderline one. Indeed, thanks to the weak subadditivity condition

$$\varphi(a+b) \le k\left[(1+\varphi(b))\,\varphi(a) + \varphi(b)\right]$$

we are able to prove convergence of the energy $N_{\varphi,w}$ and density of smooth functions with respect to modular convergence when φ is strictly convex. This kind of convergence in energy, though weaker than convergence with respect to Luxenburg norm (or modular convergence, when φ is not strictly convex), should be compared with the theory of BV functions, where smooth functions are not dense in BV norm, but dense in energy. Moreover, this convergence will be sufficient to pass our *a priori* estimates to the limit in the paper [4].

In the classical setting of Orlicz spaces (see Sect. 2) the weighted φ-energy $N_{\varphi,w}$ is called a *modular* and we put some of our results in this context. The main approximation result of the note reads as follows.

Theorem 1 *Let $I \subset \mathbb{R}$ be an open interval and $\Omega \subset \mathbb{R}^n$ an open set. Let $w : I \times \Omega \to (0,\infty)$ be a Borel function uniformly bounded from above and from below on compact subsets of $I \times \Omega$. Let $\varphi : [0,\infty) \to [0,\infty)$ be a convex function satisfying $\varphi(0) = 0$ and and for* *which there exists a positive constant* k_φ *such that*

$$\varphi(a+b) \le k_\varphi[\varphi(a)\,\varphi(b) + \varphi(a) + \varphi(b)] \quad \text{for all } a,\ b \in [0,\infty). \tag{1}$$

Let $b \in L^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)) \cap C^0(I \times \Omega;\mathbb{R}^m)$ *satisfy*

$$N_{\varphi,w}(|Db|) < \infty. \tag{2}$$

Then there exist $b_h \in C^\infty(I \times \Omega;\mathbb{R}^m)$ *satisfying*

$$b_h \to b \text{ in } L^1_{\mathrm{loc}}(I \times \Omega;\mathbb{R}^m), \qquad Db_h \to Db \text{ in } L^1_{\mathrm{loc}}(I \times \Omega;\mathbb{R}^{nm}), \tag{3}$$

and

$$w\varphi\,(|Db_h|) \to w\varphi\,(|Db|) \text{ in } L^1(I \times \Omega). \tag{4}$$

In particular,

$$\lim_{h\to\infty} N_{\varphi,w}(|Db_h|) = N_{\varphi,w}(|Db|). \tag{5}$$

Besides the model case of $\varphi = \exp_*$, we are able to consider the functions

$$\exp^*_{\gamma,\tau}(t) := \exp_{\gamma,\tau}(t) - 1, \tag{6}$$

where $0 \le \gamma \le 1$, $\tau > 0$ and

$$\exp_{\gamma,\tau}(t) := \exp\left(\frac{t}{(\log(t+\tau))^\gamma}\right). \tag{7}$$

It is easy to see that a convex function φ with polynomial growth satisfies (1), see Remark 6. We will show in Lemma 21 that $\exp^*_{\gamma,\tau}$ satisfies the conditions in Theorem 1 for φ with $k_\varphi = 1$, if τ is sufficiently large. The functions $\exp_{\gamma,\tau}$, though convex, do not have null derivative at 0, and therefore do not fit exactly in the theory of N-functions. Therefore, in order to provide a bridge with the theory of N-functions of Orlicz spaces, we will also consider the modified functions

$$\widetilde{\exp}_{\gamma,\tau}(t) := \exp_{\gamma,\tau}(t) - 1 - \frac{t}{(\log\tau)^\gamma} = \exp^*_{\gamma,\tau}(t) - \frac{t}{(\log\tau)^\gamma}, \tag{8}$$

which are indeed N-functions and can be treated it by comparison with $\exp^*_{\gamma,\tau}$.

Corollary 2 *Let* $w : I \times \Omega \to (0,\infty)$ *be a Borel function uniformly bounded from above and from below on compact subsets of* $I \times \Omega$. *Let* $b \in L^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)) \cap C^0(I \times \Omega;\mathbb{R}^m)$ *satisfy* (2) *with*

$$\varphi = \widetilde{\exp}_{\gamma,\tau} \text{ with } \tau \text{ sufficiently large} \tag{9}$$

and

$$\text{either } w|Db| \in L^1(I \times \Omega), \text{ or } w \in L^1(I \times \Omega). \tag{10}$$

Then there exist $b_h \in C^\infty(I \times \Omega;\mathbb{R}^m)$ *satisfying* (3), (4) *and* (5).

Notice that, when the functions φ we are dealing with have a more than polynomial growth at infinity and the weight function w is uniformly bounded from 0 on compact subsets, Corollary 2, the Sobolev Embedding Theorem grants continuity of b with respect to the spatial variable. But, in the proof of Theorem 1, it seems that the continuity of b with respect to t is also needed (cf. the estimate of the term z_δ). However, if we assume the weight w to be time-independent, we can adapt the proof of Theorem 1 to drop the continuity assumption on b, and we obtain the following extension of Corollary 2.

Theorem 3 *Let $w : \Omega \to (0,\infty)$ be a Borel function uniformly bounded from above and from below on compact subsets of Ω. Assume that either $\varphi = \exp^*_{\gamma,\tau}$ or $\varphi = \widetilde{\exp}_{\gamma,\tau}$ and (10) holds, with τ sufficiently large and let $b \in L^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^m))$ satisfy (2). Then there exist $b_h \in C^\infty(I \times \Omega;\mathbb{R}^m)$ satisfying (3), (4) and (5).*

When the function φ is strictly convex, from the φ-energy convergence of the Jacobian matrices of autonomous vector fields, one can obtain the *modular convergence* (see Definition 10).

Theorem 4 *Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly convex function, and assume that $b_h \to b$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ with*

$$\int_\Omega \varphi(|Db_h|)\,\mathrm{d}x \to \int_\Omega \varphi(|Db|)\,\mathrm{d}x < \infty. \tag{11}$$

Then

$$\int_\Omega \varphi\left(\frac{|Db_h - Db|}{2}\right)\mathrm{d}x \to 0. \tag{12}$$

Finally, if $\varphi = \widetilde{\exp}_{\gamma,\tau}$, being $\widetilde{\exp}_{\gamma,\tau}$ a N-function (see Lemma 21.(ii)), we can set our result in the classical setting of Orlicz-Sobolev spaces (see Sect. 2). Therefore, an immediate consequence of Theorem 3 is the following approximation result in the Orlicz-Sobolev class $W^1K_{\widetilde{\exp}_{\gamma,\tau}}(\Omega)$.

Theorem 5 *Let $\varphi = \widetilde{\exp}_{\gamma,\tau}$ be the N-function in (8) with τ given by Lemma 21 and $u \in W^1K_{\widetilde{\exp}_{\gamma,\tau}}(\Omega)$. Suppose that*

$$\text{either } |Du| \in L^1(\Omega), \text{ or } \Omega \text{ has finite measure.} \tag{13}$$

Then there exists $(u_h)_h \subset C^\infty(\Omega) \cap W^1K_{\widetilde{\exp}_{\gamma,\tau}}(\Omega)$ such that $(u_h)_h$ is mean convergent to u (with respect to the modular $N_{\widetilde{\exp}_{\gamma,\tau}}$) and $(|Du_h|)_h$ is $\widetilde{\exp}_{\gamma,\tau}$-energy convergent to $|Du|$, that is,

$$\lim_{h\to\infty} N_{\widetilde{\exp}_{\gamma,\tau}}(u_h - u) = 0 \text{ and } \lim_{h\to\infty} N_{\widetilde{\exp}_{\gamma,\tau}}(|Du_h|) = N_{\widetilde{\exp}_{\gamma,\tau}}(|Du|)\,. \tag{14}$$

To our knowledge, the previous result does not seem be a consequence of the well-known results about approximation by smooth functions in Orlicz-Sobolev spaces (see Sect. 2.6), even in the classical case with $\gamma = 0$.

2 Recalls of some density results of smooth functions in Orlicz and Orlicz-Sobolev spaces

We will quickly recall here the notions of Orlicz and Orlicz-Sobolev spaces and some their main properties. In particular, we will focus on the main density results of smooth functions in Orlicz and Orlicz-Sobolev spaces. We will mainly use the notation from [1, Ch. VIII].

2.1 N-functions

A function $\varphi : [0,\infty) \to [0,\infty)$ is called a *N-function*, if

$$\varphi(t) := \int_0^t a(s)\,\mathrm{d}s \text{ if } t \geq 0,$$

with $a : [0, \infty) \to [0, \infty)$ satisfying:

- $a(0) = 0$, $a(t) > 0$ if $t > 0$, and $\lim_{t\to\infty} a(t) = \infty$;
- a is nondecreasing, that is, if $t \geq s \geq 0$, then $a(t) \geq a(s)$;
- a is right continuous, that is, if $t \geq 0$, then $\lim_{s\to t^+} a(s) = a(t)$.

Given a N-function φ and $\lambda > 0$, we denote by $\varphi_\lambda : [0, \infty) \to [0, \infty)$ the function

$$\varphi_\lambda(t) := \varphi\left(\frac{t}{\lambda}\right) \text{ if } t \geq 0,$$

which is still a N-function.

A function φ is said to satisfy a *global* Δ_2*-condition* if there exists $k > 0$ such that

$$\varphi(2t) \leq k\,\varphi(t) \text{ for each } t \geq 0.$$

A function φ is said to satisfy a Δ_2*-condition near infinity* if there exist $k, t_0 > 0$ such that

$$\varphi(2t) \leq k\,\varphi(t) \text{ for each } t \geq t_0.$$

Remark 6 Observe that a convex function φ satisfying a global Δ_2-condition trivially fulfills condition (1). Indeed, by the convexity and Δ_2-condition, we can get the following estimate

$$\varphi(a+b) \leq \frac{1}{2}\varphi(2a) + \frac{1}{2}\varphi(2b) \leq \frac{k}{2}(\varphi(a)+\varphi(b)) \text{ for each } a,\ b \in \mathbb{R}.$$

Given $\Omega \subset \mathbb{R}^n$ and a N-function φ, a pair (φ, Ω) is said to be Δ-*regular* if

- φ satisfies a global Δ_2-condition, or
- φ satisfies a Δ_2-condition near infinity and Ω has finite measure.

2.2 The Orlicz class $K_\varphi(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be an open set and let φ be a N-function. The *Orlicz class* $K_\varphi(\Omega)$ is the set of all (equivalence classes modulo equality a.e. on Ω of) measurable functions $u : \Omega \to \mathbb{R}$ such that

$$N_\varphi(u) := \int_\Omega \varphi(|u(x)|)\, \mathrm{d}x < \infty.$$

In the theory of modular spaces, the map $u \mapsto N_\varphi(u)$ is called a *modular* ([10, pg. 82]). A comprehensive account of modular function spaces can be found in [7]. We treat the case of real-valued functions for simplicity, but all results have an obvious extension to the case of $\mathbb{R}^m$-valued maps.

Let us recall some properties of the Orlicz class $K_\varphi(\Omega)$.

Proposition 7 *Given an open set $\Omega \subset \mathbb{R}^n$ and a N-function φ, the following statements hold:*

(i) *$K_\varphi(\Omega)$ is a convex set of measurable functions.*
(ii) *$K_{\varphi_\lambda}(\Omega) \supseteq K_\varphi(\Omega)$ if $\lambda \geq 1$ and $K_{\varphi_\lambda}(\Omega) \subseteq K_\varphi(\Omega)$ if $\lambda \leq 1$, where $\varphi_\lambda(t) := \varphi(t/\lambda)$ is a N-function for all $\lambda > 0$.*
(iii) *If $f,\ g \in K_\varphi(\Omega)$, then $f + g \in K_{\varphi_2}(\Omega)$ and*

$$N_{\varphi_2}(f+g) \leq \frac{1}{2}N_\varphi(f) + \frac{1}{2}N_\varphi(g).$$

(iv) *If $f \in K_\varphi(\Omega)$ and $\lambda > 0$, then $\lambda f \in K_{\varphi_\lambda}(\Omega)$.*

(v) If Ω has finite measure, then

$$L^\infty(\Omega) \subset K_\varphi(\Omega) \subsetneq L^1(\Omega).$$

(vi) If Ω has finite measure, then for every $u \in L^1(\Omega)$ there is a N-function φ such that $u \in K_\varphi(\Omega)$.

Proof Properties (i), (ii), (iii) and (iv) are immediate consequences of the definition of $K_\varphi(\Omega)$ and the convexity of φ. For the proof of properties (v) and (vi) see, for instance, [8]. □

Lemma 8 *([1, Lem. 8.8] or [8, Ch. III, Th. 8.2]) $K_\varphi(\Omega)$ is a vector space if and only if (φ, Ω) is Δ-regular.*

2.3 The Orlicz space $L_\varphi(\Omega)$

The *Orlicz space* $L_\varphi(\Omega)$ is defined to be the linear hull of the Orlicz class $K_\varphi(\Omega)$, that is the smallest vector subspace of $L^1_{\text{loc}}(\Omega)$ containing $K_\varphi(\Omega)$. It is easy to see that, since $K_\varphi(\Omega)$ is convex, one has

$$L_\varphi(\Omega) := \left\{\lambda u : \lambda \in \mathbb{R},\ u \in K_\varphi(\Omega)\right\}.$$

Moreover, from Lemma 8, $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if (φ, Ω) is Δ-*regular*.

We can endow $L_\varphi(\Omega)$ with the following norm, called *Luxemburg norm*,

$$\|u\|_\varphi = \|u\|_{\varphi,\Omega} := \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|u(x)|}{\lambda}\right)\,\mathrm{d}x \le 1\right\}.$$

Theorem 9 *([1, Thm. 8.10]) $(L_\varphi(\Omega), \|\cdot\|_\varphi)$ is a Banach space.*

2.4 Convergences in $L_\varphi(\Omega)$

The typical convergences that apply in Orlicz spaces are the following.

Definition 10 A sequence of functions $(u_h)_h \subset L_\varphi(\Omega)$ is said to be *norm convergent* to $u \in L_\varphi(\Omega)$ if

$$\|u_h - u\|_\varphi \to 0 \text{ as } h \to \infty.$$

A sequence of functions $(u_h)_h \subset L_\varphi(\Omega)$ is said to be *modular convergent* to $u \in L_\varphi(\Omega)$ if there exists $\lambda > 0$ such that

$$N_\varphi\left(\frac{u_h - u}{\lambda}\right) \to 0 \text{ as } h \to \infty. \tag{15}$$

If $\lambda = 1$ in (15), $(u_h)_h$ is said to be *mean convergent* to $u \in L_\varphi(\Omega)$. A sequence of functions $(u_h)_h \subset K_\varphi(\Omega)$ is said to be φ-*energy convergent* to $u \in K_\varphi(\Omega)$ if

$$N_\varphi(u_h) \to N_\varphi(u) \text{ as } h \to \infty. \tag{16}$$

Norm and modular convergences are classical in the theory of Orlicz spaces (see, for instance, [1, 8]). We do not know whether the φ-energy convergence has been already named in the literature.

The following implications between norm, mean, modular and φ-energy convergence hold.

Proposition 11 *Let $(u_h)_h$ and u be in $L_\varphi(\Omega)$.*

(i) Suppose that $(u_h)_h$ is norm convergent to u. Then it is also mean convergent. The converse implication in general does not hold. It holds if (φ, Ω) is Δ-regular.
(ii) Suppose that (φ, Ω) is Δ-regular, φ is strictly convex, $u_h \to u$ a.e. in Ω and $(u_h)_h$ is φ-energy convergent to u. Then $(u_h)_h$ is norm convergent to u.
(iii) $(u_h)_h$ is norm convergent to u if and only if, for each $\lambda > 0$,

$$N_\varphi\left(\frac{u_h - u}{\lambda}\right) \to 0 \text{ as } h \to \infty.$$

(iv) Suppose that (φ, Ω) is Δ-regular and $(u_h)_h \subset K_\varphi(\Omega)$ is mean convergent to $u \in K_\varphi(\Omega)$. Then $(u_h)_h$ is φ-energy convergent to u.
(v) Suppose that $(2u_h)_h \subset K_\varphi(\Omega)$ is mean convergent to $2u \in K_\varphi(\Omega)$ (with respect to the modular N_φ). Then $(u_h)_h \subset K_\varphi(\Omega)$, $u \in K_\varphi(\Omega)$ and $(u_h)_h$ is φ-energy convergent to u.
(vi) Suppose that $2u \in K_\varphi(\Omega)$ and $(u_h)_h$ is norm convergent to u. Then $u_h \in K_\varphi(\Omega)$ for h large and (u_h) is also φ-energy convergent to u.

Proof (i) and (ii) are proven in [10, Chap. III, Sect. 3.4, Thm. 12]. The proof of (iii) is somehow elementary, see for instance [2, Lem. 2.7] and [7, pg. 4].

We prove (iv) when φ satisfies a global Δ_2-condition: in the other case, when Ω has finite measure and the φ satisfies a Δ_2-condition near infinity, has a similar proof. From the assumptions, the sequence $(\varphi(|u_h - u|))_h \subset L^1(\Omega)$ converges in $L^1(\Omega)$ to 0. Thus, up to a subsequence, we can assume that

$$\varphi(|u_h - u|) \to 0 \text{ a.e. in } \Omega, \text{ as } h \to \infty.$$

Since φ is a N-function, then $\varphi : [0, \infty) \to [0, \infty)$ is bijective and $\varphi^{-1} : [0, \infty) \to [0, \infty)$ is still continuous. Thus, we also get that

$$|u_h - u| = \varphi^{-1}(\varphi(|u_h - u|)) \to \varphi^{-1}(\varphi(0)) = 0 \text{ a.e. in } \Omega, \text{ as } h \to \infty. \tag{17}$$

From the convexity of φ and the global Δ_2-condition of φ, it follows that

$$\begin{aligned} \varphi(|u_h|) &\le \frac{1}{2}\varphi(2|u_h - u|) + \frac{1}{2}\varphi(2|u|) \\ &\le \frac{k}{2}\left(\varphi(|u_h - u|) + \varphi(|u|)\right). \end{aligned} \tag{18}$$

By (17) and (18), we can apply Vitali's convergence theorem and then

$$\varphi(|u_h|) \to \varphi(|u|) \text{ in } L^1(\Omega), \text{ as } h \to \infty.$$

Thus (16) follows.

For (v), we get at once that $(u_h)_h \subset K_\varphi(\Omega)$ and $u \in K_\varphi(\Omega)$, because φ is increasing. We can show (17) as in the proof of claim (iv), and the convexity of φ implies

$$\varphi(|u_h|) \le \frac{1}{2}\varphi(2|u_h - u|) + \frac{1}{2}\varphi(2|u|)$$

Thus, applying Vitali's convergence theorem, we still get (16).

Finally, we prove (vi). From the norm convergence and (iii), we can infer that, up to a subsequence,

$$\varphi(|u_h - u|) \to 0 \text{ a.e. in } \Omega, \text{ as } h \to \infty,$$

and

$$\varphi(2|u_h - u|) \to 0 \text{ in } L^1(\Omega), \text{ as } h \to \infty.$$

We can show again (17) as in claim (iv) and $\varphi(|u_h|) \le \frac{1}{2}\varphi(2|u_h - u|) + \frac{1}{2}\varphi(2|u|)$ from the convexity of φ. Thus, applying Vitali's convergence theorem, we get (16). □

Example 12 From items (iv) and (v) of Proposition 11, one could get the wrong impression that mean convergence implies φ-energy convergence. We show that this is not the case if (φ, Ω) is not Δ-regular: for $\Omega = (0, 1)$ and $\varphi = \widetilde{\exp}_0$ (cf. (8)), we give a sequence of functions $u_h \in K_\varphi((0, 1))$ that is mean convergent to $u \in K_\varphi((0, 1))$, but that is not φ-energy convergent.

Let $f_h, f : (0, 1) \to \mathbb{R}$ be the functions

$$f_h(x) := \begin{cases} \frac{2\sqrt{h}}{\log h} & \text{if } 0 < x < \frac{1}{h} \\ \frac{1}{\log h}\frac{1}{\sqrt{x}} & \text{if } \frac{1}{h} \le x < 1 \end{cases}, \quad f(x) := \frac{1}{\sqrt{x}}.$$

Direct computations show that

$$\int_0^1 f(x)\,\mathrm{d}x = 2, \qquad \int_0^1 \log(f(x))\,\mathrm{d}x = \frac{1}{2},$$

$$\int_0^1 f_h(x)\,\mathrm{d}x \overset{h\to\infty}{\longrightarrow} 0, \qquad \int_0^1 \log(1 + f_h)\,\mathrm{d}x \overset{h\to\infty}{\longrightarrow} 0,$$

$$\int_0^1 f(x)\, f_h(x)\,\mathrm{d}x = \frac{4}{\log(h)} + 1 \overset{h\to\infty}{\longrightarrow} 1.$$

Define

$$u := \log(f) \quad \text{and} \quad u_h := \log(f) + \log(1 + f_h).$$

Then, for $\varphi(s) = \exp(s) - 1 - s$, we have

$$\begin{aligned} N_\varphi(u) &= \int_0^1 f(x)\,\mathrm{d}x - 1 - \int_0^1 \log(f(x))\,\mathrm{d}x, \\ N_\varphi(u_h) &= \int_0^1 f(x)\,\mathrm{d}x - 1 - \int_0^1 \log(f(x))\,\mathrm{d}x \\ &\quad + \int_0^1 f(x)\, f_h(x)\,\mathrm{d}x - \int_0^1 \log(1 + f_h)\,\mathrm{d}x, \\ N_\varphi(u_h - u) &= \int_0^1 f_h(x)\,\mathrm{d}x - \int_0^1 \log(1 + f_h(x))\,\mathrm{d}x. \end{aligned}$$

We conclude that $u, u_h \in K_\varphi((0, 1))$, $N_\varphi(|u - u_h|) \to 0$ but

$$N_\varphi(|u_h|) - N_\varphi(|u|) = \int_0^1 f(x)\, f_h(x)\,\mathrm{d}x - \int_0^1 \log(1 + f_h)\,\mathrm{d}x \overset{h\to\infty}{\longrightarrow} 1,$$

that is, u_h is not φ-energy convergent to u. Notice that the key fact is that $f_h \to 0$ in $L^1((0, 1))$ but $f \cdot f_h \not\to 0$. Let us also observe that $(2u_h)_h \subset K_\varphi(\Omega)$, but neither $2u \in K_\varphi(\Omega)$ nor $(2u_h)_h$ is mean convergent to $2u$ with respect to N_φ.

2.5 The vector space $E_\varphi(\Omega)$

Let $E_\varphi(\Omega)$ denote the closure in $(L_\varphi(\Omega), \|\cdot\|_\varphi)$ of the space of functions u which are bounded in Ω with bounded support in Ω.

One can see ([1, Sect. 8.14]) that $E_\varphi(\Omega) \subset K_\varphi(\Omega)$ and that, if (φ, Ω) is Δ-*regular*, then

$$E_\varphi(\Omega) = K_\varphi(\Omega) = L_\varphi(\Omega).$$

Moreover the following characterization of $E_\varphi(\Omega)$ holds ([A, Lemma 8.15]).

Lemma 13 *$E_\varphi(\Omega)$ is the maximal linear subspace of $K_\varphi(\Omega)$.*

Corollary 14 *If (φ, Ω) is not Δ-regular, it holds that*

$$E_\varphi(\Omega) \subsetneq K_\varphi(\Omega) \subsetneq L_\varphi(\Omega).$$

Proof By Lemma 8, $K_\varphi(\Omega)$ cannot be a vector space. Thus, by Lemma 13, we get the desired conclusions. □

Let us now recall some density results in $(E_\varphi(\Omega), \|\cdot\|_A)$.

Theorem 15 *([1, Thm. 8.20]) Let $\Omega \subset \mathbb{R}^n$ be an open set and let φ be a N-function.*

(i) $C_c^\infty(\Omega)$ are dense in $(E_\varphi(\Omega), \|\cdot\|_\varphi)$.
(ii) $(E_\varphi(\Omega), \|\cdot\|_\varphi)$ is separable.
(iii) Let us extend $u \in E_\varphi(\Omega)$ to the whole $\mathbb{R}^n$ so as to vanish outside Ω and let $(\rho_\varepsilon)_\varepsilon$ be a family of mollifiers on $\mathbb{R}^n$. Then

$$\rho_\varepsilon * u \to u \text{ in } (E_\varphi(\Omega), \|\cdot\|_\varphi), \text{ as } \varepsilon \to 0.$$

An immediate consequence of Theorem 15 is that, if (φ, Ω) is not Δ-regular, then $C_c^0(\Omega)$ is not dense in $(L_\varphi(\Omega), \|\cdot\|_\varphi)$. In fact, one can prove the following stronger result:

Theorem 16 *([8, Chap. II, Thm. 10.2]) If the pair (φ, Ω) is not Δ-regular, then $(L_\varphi(\Omega), \|\cdot\|_\varphi)$ is not separable.*

Let us also point out some density results in $K_\varphi(\Omega)$ with respect to the modular convergence.

Theorem 17 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let φ be a N-function.*

(i) The set of bounded functions on Ω contained in $K_\varphi(\Omega)$ with bounded support is dense in $K_\varphi(\Omega)$ with respect to the mean convergence, that is, for each $u \in K_\varphi(\Omega)$ there exists a sequence of bounded functions $(u_h)_h \subset K_\varphi(\Omega)$ such that

$$N_\varphi(u_h - u) \to 0, \text{ as } h \to \infty.$$

(ii) $C_c^0(\Omega)$ is dense in $K_\varphi(\Omega)$ with respect to the modular convergence with $\lambda = 4$. More precisely, for each $u \in K_\varphi(\Omega)$, there is a sequence $(u_h)_h \subset C_c^0(\Omega)$ such that

$$N_\varphi\left(\frac{u_h - u}{4}\right) \to 0, \text{ as } h \to \infty.$$

Proof The proof of part (i) can be found in [8, Chap. II, pg. 77] or [1, Sect. 8.14].

To prove part (ii), given $u \in K_\varphi(\Omega)$ and $\epsilon > 0$, we first notice that, by a standard truncation argument in Ω, there is a function $\tilde{u} \in K_\varphi(\Omega)$ with support compactly contained in Ω and with $N_\varphi(u - \tilde{u}) < \epsilon$.

Next, let $\tilde{u}_k := \max\{-k, \min\{\tilde{u}, k\}\}$ be the standard truncation of u, $F_k := \{|\tilde{u}| > k\}$ and $f \in C^0_c(\Omega)$ with $\sup|f| \le k$. We estimate

$$\begin{aligned} N_{\varphi_2}(\tilde{u} - f) &= \int_{\Omega\setminus F_k} \varphi\left(\frac{|\tilde{u}_k - f|}{2}\right) \mathrm{d}x + \int_{F_k} \varphi\left(\frac{|\tilde{u} - f|}{2}\right) \mathrm{d}x \\ &\le \int_{\Omega\setminus F_k} \varphi\left(\frac{|\tilde{u}_k - f|}{2}\right) \mathrm{d}x + \frac{1}{2}\int_{F_k} \varphi(|\tilde{u}|)\, \mathrm{d}x + \frac{1}{2}\int_{F_k} \varphi(|f|)\, \mathrm{d}x \\ &\le \int_{\Omega\setminus F_k} \varphi\left(\frac{|\tilde{u}_k - f|}{2}\right) \mathrm{d}x + \int_{F_k} \varphi(|\tilde{u}|)\, \mathrm{d}x. \end{aligned}$$

Now, since $\int_\Omega \varphi(|\tilde{u}|)\, \mathrm{d}x < \infty$, we can choose k so large that the second integral is smaller than $\epsilon/2$. Since $\tilde{u}_k$ has compact support and thanks to Lusin's theorem, we can find $f \in C^0_c(\Omega)$ with $|f| \le k$ and the Lebesgue measure of $\{x \in \Omega : \ \tilde{u}_k(x) \ne f(x)\}$ sufficiently small, in such a way that also the first integral gives a contribution smaller than $\epsilon/2$.

In conclusion, for every $\epsilon > 0$ we have $f \in C^0_c(\Omega)$ such that

$$N_{\varphi_4}(u - f) \le \frac{1}{2}(N_{\varphi_2}(u - \tilde{u}) + N_{\varphi_2}(\tilde{u} - f)) \le \frac{1}{2}(N_\varphi(u - \tilde{u})/2 + \epsilon) \le \epsilon.$$

□

2.6 Orlicz-Sobolev spaces and density results of smooth functions.

Given a N-function φ, the *Orlicz-Sobolev vector space* $W^1L_\varphi(\Omega)$ consists of those (equivalence classes of) functions $u \in L_\varphi(\Omega) \cap W^{1,1}_{loc}(\Omega)$ whose weak derivatives $D_i u \in L_\varphi(\Omega)$ for each $i = 1, \ldots, n$. The vector space $W^1E_\varphi(\Omega)$ and the convex set $W^1K_\varphi(\Omega)$ are defined in analogous fashion. Obviously

$$W^1E_\varphi(\Omega) \subset W^1K_\varphi(\Omega) \subset W^1L_\varphi(\Omega).$$

It is easy to see (see, for instance, [1, §8.27]) that $W^1L_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,\varphi} := \max\{\|u\|_\varphi, \|D_1 u\|_\varphi, \ldots, \|D_n u\|_\varphi\}.$$

Notice also that, since

$$\max_i |D_i u| \le |Du| \le \sum_{i=1}^n |D_i u| \text{ a.e. in } \Omega,$$

and $L_\varphi(\Omega)$ is a linear space, an equivalent norm on $W^1L_\varphi(\Omega)$ is given by.

$$\|u\|_\varphi + \||Du|\|_\varphi.$$

Observe that $W^1E_\varphi(\Omega)$ turns out to be a closed subspace of $W^1L_\varphi(\Omega)$. Moreover $W^1E_\varphi(\Omega)$ coincides with $W^1L_\varphi(\Omega)$ if and only if (φ, Ω) is Δ-*regular*. Notice also that, for the applications we have in mind, what is more relevant is the φ-integrability of the derivative, rather than the integrability of the function which, also in view of Sobolev embeddings, could be qualified in a different way, see also Remark 24.

Celebrated Meyers-Serrin's result was extended from the classical Sobolev spaces to the Orlicz-Sobolev space $W^1E_\varphi(\Omega)$ in [5] (see also [2]).

Theorem 18 *([5]) $C^\infty(\Omega) \cap W^1E_\varphi(\Omega)$ is dense in $(W^1E_\varphi(\Omega), \|\cdot\|_{1,\varphi})$.*

It is easy to see that the previous result also fails for functions in the Orlicz-Sobolev class $W^1K_\varphi(\Omega)$, and so also in the Orlicz-Sobolev space $W^1L_\varphi(\Omega)$, provided that (φ, Ω) is not Δ-regular, as the following example shows.

Example 19 Assume that $n = 1$, $\Omega = (-1, 1)$, let $\varphi = \widetilde{\exp}_0$ be N-function in (8) with $\gamma = 0$ and let

$$u(x) := \begin{cases} \frac{x}{2}\log\frac{1}{e|x|} & \text{if } |x| \le \frac{1}{e} \\ 0 & \text{if } \frac{1}{e} < |x| < 1 \end{cases}.$$

Then it is easy to see that $u \in W^1K_\varphi(\Omega)\backslash W^1E_\varphi(\Omega)$, since the weak derivative

$$u'(x) := \begin{cases} \log\dfrac{1}{e\sqrt{|x|}} & \text{if } |x| < \frac{1}{e} \\ 0 & \text{if } \frac{1}{e} < |x| < 1 \end{cases} \quad \text{a.e. } x \in \Omega$$

belongs to $K_\varphi(\Omega) \setminus E_\varphi(\Omega)$. Indeed

$$\int_{-1}^{1} \varphi(|u'|)\,\mathrm{d}x = \int_{-1}^{1}\left(\exp(|u'|) - |u'| - 1\right)\,\mathrm{d}x < \infty,$$

but $2|u'| \notin K_\varphi(\Omega)$, since

$$\int_{-1}^{1} \varphi(2|u'|)\,\mathrm{d}x = \int_{-1}^{1}\left(\exp(2|u'|) - 2|u'| - 1\right)\,\mathrm{d}x = \infty.$$

Thus $|u'| \notin E_\varphi(\Omega)$, since $E_\varphi(\Omega)$ is a linear subspace. By contradiction, assume there exists a sequence $(u_h)_h \subset C^\infty(\Omega) \cap W^1L_\varphi(\Omega)$ such that $u_h \to u$ in $W^1L_\varphi(\Omega)$, as $h \to \infty$. In particular, it also follows that

$$u_h' \to u' \text{ in } L_\varphi(\Omega) \text{ as } h \to \infty. \tag{19}$$

Let $\psi \in C_c^0(\Omega)$ such that $0 \le \psi \le 1$ and $\psi \equiv 1$ in $(-1/e, 1/e)$ and let

$$v_h := \psi\, u_h'.$$

By Proposition 11(iii) and (19), it still holds that

$$E_\varphi \ni \psi\, u_h' \to \psi\, u' = u' \text{ in } L_\varphi(\Omega), \text{ as } h \to \infty.$$

Then a contradiction since $u' \notin E_\varphi$.

A weaker density result of regular functions in $W^1L_\varphi(\Omega)$ holds by using the modular convergence, as shown in [6].

Theorem 20 *([6]) Let $u \in W^1L_\varphi(\Omega)$. Then there exist $\lambda > 0$ and a sequence of functions $(u_h)_h \subset C^\infty(\Omega) \cap W^1L_\varphi(\Omega)$ such that*

$$N_\varphi\left(\frac{u_h - u}{\lambda}\right) \to 0 \text{ and } N_\varphi\left(\frac{D_i u_h - D_i u}{\lambda}\right) \to 0 \text{ as } h \to \infty,$$

for each $i = 1, \ldots, n$. In particular it suffices to choose λ such that $\frac{16}{\lambda} D_i u \in K_\varphi(\Omega)$.

3 Exponential and sub-exponential N-functions

It is easy to see that a convex function φ with polynomial growth satisfies (1), see Remark 6. We will show in Lemma 21 that $\exp^*_{\gamma,\tau}$ satisfies the conditions in Theorem 1 for φ with $k_\varphi = 1$, if τ is sufficiently large.

Recall from (6) and (7) that we have set

$$\exp_{\gamma,\tau}(t) := \exp\left(\frac{t}{(\log(t+\tau))^\gamma}\right), \quad \text{and} \quad \exp^*_{\gamma,\tau}(t) := \exp_{\gamma,\tau}(t) - 1.$$

The functions $\exp^*_{\gamma,\tau}$, though convex, do not have null derivative at 0, and therefore do not fit exactly in the theory of N-functions. Therefore, in order to provide a bridge with the theory of N-functions of Orlicz spaces, we will also consider the modified functions

$$\widetilde{\exp}_{\gamma,\tau}(t) := \exp_{\gamma,\tau}(t) - 1 - \frac{t}{(\log\tau)^\gamma} = \exp^*_{\gamma,\tau}(t) - \frac{t}{(\log\tau)^\gamma},$$

which are indeed N-functions and can be treated by comparison with $\exp^*_{\gamma,\tau}$.

Lemma 21 *There exists $\tau_0 > 0$ such that, for all $\tau \geq \tau_0$ and $0 \leq \gamma \leq 1$, one has*

(i) $\exp_{\gamma,\tau}$ *is a smooth strictly convex increasing function. Moreover, for all $t, s \in [0,\infty)$,*

$$\exp_{\gamma,\tau}(t+s) \leq \exp_{\gamma,\tau}(t)\,\exp_{\gamma,\tau}(s), \tag{20}$$

and $\exp^*_{\gamma,\tau}$ *satisfies* (1) *with $k_\varphi = 1$, that is,*

$$\exp^*_{\gamma,\tau}(t+s) \leq \exp^*_{\gamma,\tau}(t)\,\exp^*_{\gamma,\tau}(s) + \exp^*_{\gamma,\tau}(t) + \exp^*_{\gamma,\tau}(s). \tag{21}$$

(ii) $\widetilde{\exp}_{\gamma,\tau}$ *is a N-function satisfying*

$$\widetilde{\exp}_{\gamma,\tau}(t) \leq \exp^*_{\gamma,\tau}(t) \text{ if } t \geq 0 \text{ and } \lim_{t\to\infty} \frac{\widetilde{\exp}_{\gamma,\tau}(t)}{\exp^*_{\gamma,\tau}(t)} = 1. \tag{22}$$

Proof (i) By a simple calculation, it is easy to see that, if $\tau > 1$, $\exp_{\gamma,\tau}$ is well-defined, $\exp_{\gamma,\tau} \in C^\infty([0,\infty))$ and

$$\exp'_{\gamma,\tau}(t) = \exp\left(\frac{t}{(\log(t+\tau))^\gamma}\right) \frac{\log(t+\tau) - \dfrac{\gamma t}{t+\tau}}{(\log(t+\tau))^{\gamma+1}},$$

$$\exp''_{\gamma,\tau}(t) = \frac{\exp\left(\dfrac{t}{(\log(t+\tau))^\gamma}\right)}{(\log(t+\tau))^{2\gamma+2}} \Bigg[\left(\log(t+\tau) - \frac{\gamma t}{t+\tau}\right)^2 - (\log(t+\tau))^{\gamma+1}\frac{\gamma\tau + \gamma(t+\tau)}{(t+\tau)^2} + \frac{\gamma(\gamma+1)t(\log(t+\tau))^\gamma}{(t+\tau)^2}\Bigg]$$

for all $t \geq 0$. Now, observe that, if $\tau > e$ and $t \geq 0$, then

$$\frac{\gamma t}{t+\tau} \leq \gamma, \qquad \frac{\gamma\tau + \gamma(t+\tau)}{(t+\tau)^2} \leq \frac{2\gamma}{\tau}, \text{ and } \log(t+\tau) > 1 \geq \gamma. \tag{23}$$

Combining these inequalities, it follows that, if $\tau > e$,

$$\exp'_{\gamma,\tau}(t) > 0 \text{ for each } t \geq 0, \tag{24}$$

so that $\exp_{\gamma,\tau}$ is strictly increasing on $[0,\infty)$. Let us now show that, for sufficiently large τ, one has

$$\exp''_{\gamma,\tau}(t) > 0 \text{ for each } t \geq 0. \tag{25}$$

By (23), for each $t \geq 0$ and $\tau > e$, we obtain that

$$\begin{aligned}
\left(\log(t+\tau) - \frac{\gamma t}{t+\tau}\right)^2 &- (\log(t+\tau))^{\gamma+1}\frac{\gamma\tau + \gamma(t+\tau)}{(t+\tau)^2} + \frac{\gamma(\gamma+1)t\log(t+\tau))^\gamma}{(t+\tau)^2} \\
&\geq \left(\log(t+\tau) - \frac{\gamma t}{t+\tau}\right)^2 - (\log(t+\tau))^{\gamma+1}\frac{\gamma\tau + \gamma(t+\tau)}{(t+\tau)^2} \\
&\geq (\log(t+\tau) - \gamma)^2 - \frac{2\gamma}{\tau}(\log(t+\tau))^2 \\
&= \log(t+\tau)\left(\log(\tau) - 2\gamma\left(\frac{\log(\tau)}{\tau} + 1\right)\right) + \gamma^2 \\
&\geq \log(t+\tau)\left(\log(\tau) - 2\left(\frac{\log(\tau)}{\tau} + 1\right)\right).
\end{aligned}$$

It is clear that, there is $\tau_0 > 0$ (independent on γ), so that the latter quantity is positive for all $\tau \geq \tau_0$ and $t \geq 0$. Hence, (25) follows and $\exp_{\gamma,\tau}$ is strictly convex on $[0,\infty)$.

Let us show (20), that is, for every $t, s \in [0,\infty)$,

$$\begin{aligned}
\exp_{\gamma,\tau}(t+s) &= \exp\left(\frac{t+s}{(\log(t+s+\tau))^\gamma}\right) \\
&\leq \exp_{\gamma,\tau}(t)\,\exp_{\gamma,\tau}(s) \\
&= \exp\left(\frac{t}{(\log(t+\tau))^\gamma} + \frac{s}{(\log(s+\tau))^\gamma}\right).
\end{aligned} \tag{26}$$

Observe that

$$\begin{aligned}
\frac{t+s}{(\log(t+s+\tau))^\gamma} &= \frac{t}{(\log(t+s+\tau))^\gamma} + \frac{s}{(\log(t+s+\tau))^\gamma} \\
&\leq \frac{t}{(\log(t+\tau))^\gamma} + \frac{s}{(\log(s+\tau))^\gamma},
\end{aligned}$$

whence (26) follows, being the exponential function nondecreasing. Inequality (21) follows by using (20) and the fact that $\exp_{\gamma,\tau}(t) = \exp^*_{\gamma,\tau}(t) + 1$.
(ii) Notice that, if

$$a(t) := \exp'_\gamma(t) - \exp'_\gamma(0) \text{ if } t \geq 0,$$

by (24) and (25), a is continuous, (strictly) increasing, $a(0) = 0$, $a(t) > 0$ if $t > 0$ and $\lim_{t\to\infty} a(t) = \infty$. Moreover, being a increasing, we have for $t \geq 0$,

$$\widetilde{\exp}_{\gamma,\tau}(t) = \exp_{\gamma,\tau}(t) - 1 - \frac{t}{(\log\tau)^\gamma} = \int_0^t \left(\exp'_\gamma(s) - \exp'_\gamma(0)\right) ds = \int_0^t a(s)\,ds.$$

Thus, $\widetilde{\exp}_{\gamma,\tau}$ is (strictly) convex.

Finally, since the functions have a more than linear growth, it is clear that $\widetilde{\exp}_{\gamma,\tau}(t)/\exp_{\gamma,\tau}(t)$ tends to 1 as $t \to \infty$. □

Remark 22 Notice that, by (22), the pair $(\widetilde{\exp}_{\gamma,\tau}, \Omega)$ is never Δ-regular for any $\Omega \subset \mathbb{R}^n$. In particular, by Lemma 8, $K_\varphi(\Omega)$ is never a vector space if $\varphi = \widetilde{\exp}_{\gamma,\tau}$.

Remark 23 Exponential growth functions fall also in the class treated by Theorem 1. More precisely, the functions

$$e^*_\alpha(t) := \exp(\alpha t) - 1,$$

for $\alpha > 0$ satisfy the conditions on φ given in Theorem 1.

4 Proof of the approximation results

In this section we are going to show our results.

Proof of Theorem 1 Uniform positivity of w on compact subsets gives

$$\int_R \varphi\left(|Db(s,x)|\right)\,\mathrm{d}s\,\mathrm{d}x < \infty \qquad \text{whenever } R \Subset I \times \Omega. \tag{27}$$

We are going to exploit an adaptation of the technique of the proof of Meyers-Serrin's theorem (see, for instance, [3, Thm. 3.9]). Let $Q := I \times \Omega$ and let U_j, $j = 0, 1, \ldots$, be the nondecreasing sequence of open subsets

$$U_0 := \emptyset, \quad U_j := \left\{(s,x) \in Q : \operatorname{dist}((s,x), \partial Q) > \frac{1}{j},\ |s| + |x| < j\right\} \quad (j = 1, 2, \ldots),$$

and let

$$Q_j := U_{j+1} \setminus \overline{U}_{j-1} \quad j = 1, 2, \ldots.$$

Then $\cup_j Q_j = Q$, each Q_j has compact closure in Q and any point of Q belongs to at most four sets Q_j. More specifically, if $j \geq 3$ and $x \in Q_j$, then x may belong at most to Q_{j-1} and Q_{j+1}.

Let $(\zeta_j)_j$ be a partition of unity relative to the covering (Q_j), that is, nonnegative functions $\zeta_j \in C^\infty_c(Q_j)$ such that $\sum_{j=1}^\infty \zeta_j \equiv 1$ in Q. Moreover, let $\psi_j \in C^\infty_c(Q)$ be cut-off functions such that $0 \leq \psi_j \leq 1$ in Q and $\psi_j \equiv 1$ in Q_j.

For each $j = 1, 2, \ldots$, let $b_j : \mathbb{R}^{n+1} = \mathbb{R}_s \times \mathbb{R}^n_x \to \mathbb{R}^m$ denote

$$b_j(s,x) := \begin{cases} \psi_j(s,x)\, b(s,x) & \text{if } (s,x) \in I \times \Omega \\ 0 & \text{if } (s,x) \in \mathbb{R}^{n+1} \setminus I \times \Omega, \end{cases}$$

so that it is clear that $\operatorname{spt}(b_j) \Subset Q$, $b_j \in L^1(\mathbb{R}_s; W^{1,1}(\mathbb{R}^n_x; \mathbb{R}^m)) \cap C^0_c(\mathbb{R}^{n+1}, \mathbb{R}^m)$, and

$$Db_j = D(\psi_j b) = \psi_j\, Db + \nabla\psi_j \otimes b \text{ a.e. in } Q. \tag{28}$$

In particular,

$$b_j = b \text{ and } Db_j = Db \text{ a.e. in } Q_j\,, \tag{29}$$

$$Db_j \in L^1(\mathbb{R}^{n+1}; \mathbb{R}^{nm})\,. \tag{30}$$

The monotonicity of φ and the weak subadditivity condition (1) give

$$\begin{aligned} \varphi\left(|Db_j|\right) &\leq \varphi\left(\psi_j\,|Db| + |\nabla\psi_j \otimes b|\right) \\ &\leq k_\varphi\left[\varphi\left(\psi_j\,|Db|\right)\varphi\left(|\nabla\psi_j \otimes b|\right) + \varphi\left(\psi_j\,|Db|\right) + \varphi\left(|\nabla\psi_j \otimes b|\right)\right]. \end{aligned} \tag{31}$$

Since, by (27),

$$\varphi\left(\psi_j\,|Db|\right) \in L^1(\mathbb{R}^{n+1}) \quad \text{and} \quad \varphi\left(|\nabla\psi_j \otimes b|\right) \in L^1(\mathbb{R}^{n+1}) \cap L^\infty(\mathbb{R}^{n+1}),$$

we obtain from (31) that

$$\varphi\left(|Db_j|\right) \in L^1(\mathbb{R}^{n+1}). \tag{32}$$

Let $(\rho_\varepsilon(s,x))_\varepsilon$ be space-time mollifiers on $\mathbb{R}^{n+1} = \mathbb{R}_s \times \mathbb{R}^n_x$. For each $\delta \in (0,1)$ we will make a suitable choice of $0 < \varepsilon_j < \delta$ and define $b_\delta : Q \to \mathbb{R}^m$ as

$$b_\delta(s,x) := \sum_{j=1}^{\infty} \zeta_j(s,x)\,(\rho_{\epsilon_j} * b_j)(s,x).$$

Since the sum is locally finite, b_δ is well defined. Moreover, by construction, $b_\delta \in C^\infty(I \times \Omega; \mathbb{R}^m)$ and one has

$$b_\delta \to b \text{ in } L^1_{\text{loc}}(Q;\mathbb{R}^m), \text{ as } \delta \to 0. \tag{33}$$

Notice now that

$$\begin{aligned} Db_\delta &= \sum_{j=1}^{\infty} D(\zeta_j(\rho_{\varepsilon_j} * b_j)) = \sum_{j=1}^{\infty} \zeta_j(\rho_{\varepsilon_j} * Db_j) + \sum_{j=1}^{\infty} \nabla\zeta_j \otimes (\rho_{\varepsilon_j} * b) \\ &= v_\delta + z_\delta \text{ in } Q, \end{aligned} \tag{34}$$

where

$$v_\delta := \sum_{j=1}^{\infty} \zeta_j(\rho_{\varepsilon_j} * Db_j),$$

and

$$z_\delta := \sum_{j=1}^{\infty} \left(\nabla\zeta_j \otimes (\rho_{\varepsilon_j} * b_j) - \nabla\zeta_j \otimes b_j\right),$$

where we used the fact that, since $\sum_{j=1}^{\infty} \nabla\zeta_j \equiv 0$, we have $\sum_{j=1}^{\infty} \nabla\zeta_j \otimes b \equiv 0$.

For each $\delta > 0$ and $j = 1, 2, \ldots$, we can find $0 < \varepsilon_j < \delta$ such that

$$\int_Q |\zeta_j\,(\rho_{\varepsilon_j} * b_j) - \zeta_j\, b|\,\mathrm{d}x\,\mathrm{d}s < \frac{\delta}{2^j} \tag{35}$$

and

$$\|\zeta_j(\rho_{\varepsilon_j} * Db_j) - \zeta_j Db_j\|_{L^1(\mathbb{R}^{n+1};\mathbb{R}^m)} < \frac{\delta}{2^{j+1}}. \tag{36}$$

In addition, setting

$$M_j := \max\{1, \sup_{Q_j} w\},$$

we can also ensure that

$$\begin{aligned} &\|\nabla\zeta_j \otimes (\rho_{\varepsilon_j} * b_j) - \nabla\zeta_j \otimes b\|_{L^p(Q_j;\mathbb{R}^{nm})} \\ &= \|\nabla\zeta_j \otimes (\rho_{\varepsilon_j} * b_j) - \nabla\zeta_j \otimes b\|_{L^p(\mathbb{R}^{n+1};\mathbb{R}^{nm})} < \frac{\delta}{2^{j+1}M_j} \text{ if } p = 1, \infty \end{aligned} \tag{37}$$

and

$$\|\rho_{\varepsilon_j} * \varphi(|Db_j|) - \varphi(|Db_j|)\|_{L^1(\mathbb{R}^{n+1})} < \frac{\delta}{2^j M_j}. \tag{38}$$

Notice that, by assumption on w, $M_j < \infty$. Notice also that the choices in (35) and (37) are possible thanks to the continuity of b (in particular, for (37) with $p = \infty$, we are using also continuity of b with respect to the time variable), while the choices in (36) and (38) are possible thanks to the classical properties of convolution together with (30) and (32), respectively. From (37), it follows that

$$\|z_\delta\|_{L^p(Q;\mathbb{R}^{nm})} < \frac{\delta}{2} \text{ if } p = 1, \infty, \text{ and } \int_Q |z_\delta|\, w \,\mathrm{d}x\,\mathrm{d}s < \delta. \tag{39}$$

Moreover, by (39) with $p = \infty$ and the convexity of φ, if we set $L := \varphi(1)$, then $\varphi(0) = 0$ and the monotonicity of difference quotients give

$$\sigma_\delta := \varphi(|z_\delta|) \leq L|z_\delta| \text{ in } Q. \tag{40}$$

Notice now that, since $Db = \sum_{j=1}^\infty \zeta_j Db = \sum_{j=1}^\infty \zeta_j Db_j$, by (36) and (39) with $p = 1$, we have

$$\begin{aligned}
\|Db_\delta - Db\|_{L^1(Q;\mathbb{R}^m)} &= \|v_\delta + z_\delta - Db\|_{L^1(Q;\mathbb{R}^m)} \\
&\leq \|v_\delta - Db\|_{L^1(Q;\mathbb{R}^m)} + \|z_\delta\|_{L^1(Q;\mathbb{R}^m)} \\
&\leq \sum_{j=1}^\infty \|\zeta_j(\rho_{\varepsilon_j} * Db_j) - \zeta_j Db_j\|_{L^1(\mathbb{R}^{n+1};\mathbb{R}^m)} + \frac{\delta}{2} \\
&\leq \sum_{j=1}^\infty \frac{\delta}{2^{j+1}} + \frac{\delta}{2} = \delta.
\end{aligned} \tag{41}$$

By (41), it follows that

$$\lim_{\delta\to 0} \|Db_\delta - Db\|_{L^1(Q;\mathbb{R}^m)} = 0. \tag{42}$$

In particular, by the continuity of φ and (42), there exists an infinitesimal sequence $(\delta_h)_h$ such that, if $b_h := b_{\delta_h}$,

$$\varphi(|Db_h|) \to \varphi(|Db|) \text{ a.e. in } Q, \text{ as } h \to \infty. \tag{43}$$

Let us now show (4). One has, a.e. in Q,

$$\begin{aligned}
\varphi(|Db_\delta|) &\leq \varphi(|z_\delta| + |v_\delta|) \leq k_\varphi(\varphi(|z_\delta|)\,\varphi(|v_\delta|) + \varphi(|z_\delta|) + \varphi(|v_\delta|)) \\
&\leq k_\varphi((1 + \sigma_\delta)\,\varphi(|v_\delta|) + \sigma_\delta),
\end{aligned}$$

where $\sigma_\delta = \varphi(|z_\delta|)$. Set $\sigma_\delta^\infty := \|\sigma_\delta\|_{L^\infty(Q)}$, so that, a.e. in Q,

$$\varphi(|Db_\delta|) \leq k_\varphi((1 + \sigma_\delta^\infty)\,\varphi(|v_\delta|) + \sigma_\delta). \tag{44}$$

By the monotonicity and convexity of φ, by Jensen's inequality and taking into account that $(\zeta_j)_j$ is a partition of unity, we get, a.e. in Q,

$$\begin{aligned}\varphi\left(|v_\delta|\right) = \varphi\left(\left|\sum_{j=1}^{\infty}\zeta_j(\rho_{\varepsilon_j} * Db_j)\right|\right) &\le \varphi\left(\sum_{j=1}^{\infty}\zeta_j(\rho_{\varepsilon_j} * |Db_j|)\right)\\ &\le \sum_{j=1}^{\infty}\zeta_j(\rho_{\varepsilon_j} * \varphi\left(|Db_j|\right)) =: G_\delta.\end{aligned} \tag{45}$$

Hence, by (44) and (45), if follows that

$$w\varphi\left(|Db_\delta|\right) \le k_\varphi((1+\sigma_\delta^\infty)\, wG_\delta + w\sigma_\delta) \quad \text{a.e. in } Q. \tag{46}$$

It is clear that, by (39) and (40),

$$w\,\sigma_\delta \text{ converges to 0 in } L^1(Q) \text{ and } \sigma_\delta^\infty \to 0, \text{ as } \delta \to 0. \tag{47}$$

Let us now prove that

$$w\,G_\delta \text{ converges to } w\varphi(|Db|) \text{ in } L^1(Q), \text{ as } \delta \to 0. \tag{48}$$

Observe that, by (29),

$$\varphi(|Db|) = \sum_{j=1}^{\infty}\zeta_j\varphi(|Db|) = \sum_{j=1}^{\infty}\zeta_j\varphi(|Db_j|) \text{ a.e. in } Q,$$

so that (38) gives (48).

The combination of (48) and (46) gives the equi-integrability of $w\varphi\left(|Db_\delta|\right)$. By using (43), Vitali's form of the dominated convergence theorem (see, for instance, [3, Exercise 1.18]), gives (4) and the proof is complete. □

Remark 24 Let $\Psi : [0,\infty) \to [0,\infty)$ be any continuous function with $\Psi(0) = 0$ and linear growth at the origin. Notice that all the terms $\zeta_j(\rho_{\epsilon_j} * b_j) - \zeta_j b$ can be made arbitrarily small not only in $L_t^\infty(L_x^\infty)$, but also in $L_t^1(L_x^1)$, choosing $\epsilon_j \ll 1$. Then, using the representation

$$(b_{\delta_h} - b) = \sum_{j=1}^{\infty}\zeta_j(\rho_{\epsilon_j} * b_j) - \zeta_j b$$

we can improve the construction to get also

$$\lim_{h\to\infty}\int_I\int_\Omega \Psi(|b_{\delta_h} - b|)\,\mathrm{d}x\,\mathrm{d}s = 0.$$

More precisely, choosing $\epsilon_j \ll 1$ properly, we can make arbitrarily small all terms

$$\int_{Q_j}\Psi(|\sum_{j=1}^{\infty}\zeta_j(\rho_{\epsilon_j} * b_j) - \zeta_j b|)\,\mathrm{d}x\,\mathrm{d}s$$

since the sum is locally finite and $Q_j \Subset I \times \Omega$.

Remark 25 Since the proof of Theorem 1 is based on a convolution argument, we have more control on the convergence depending on the properties of b. We give two cases that can be of interest.

First, if $b \in L^1(I; C(\Omega; \mathbb{R}^m))$ is continuous in the spatial variable, then the approximating sequence $b_h \in C^\infty(I \times \Omega; \mathbb{R}^m)$ in Theorem 1 can be taken so that $b_h \to b$ in $b \in L^1(I; C(\Omega; \mathbb{R}^m))$.

Second, if there exists a bounded open set $\Omega' \Subset \Omega$ such that

$$\operatorname{spt}(b(t,\cdot)) \subset \Omega' \text{ for every } t \in I, \tag{49}$$

then the approximating sequence $b_h \in C^\infty(I \times \Omega; \mathbb{R}^m)$ can be taken with

$$\operatorname{spt}(b_h(t,\cdot)) \subset \Omega' \text{ for each } t \in I \text{ and } h \in \mathbb{N}. \tag{50}$$

Proof of Corollary 2 Let us prove that

$$\int_I \int_\Omega w(s,x) \exp^*_{\gamma,\tau}(|Db(s,x)|)\, \mathrm{d}x\, \mathrm{d}s < \infty. \tag{51}$$

Notice that, since b satisfies (2) with (9), then $w\, \widetilde{\exp}_{\gamma,\tau}(|Db|) \in L^1(I \times \Omega)$. On the one hand, if $w|Db| \in L^1(I \times \Omega)$, since

$$w \exp^*_{\gamma,\tau}(|Db|) = w \widetilde{\exp}_{\gamma,\tau}(|Db|) + \frac{w|Db|}{(\log\tau)^\gamma}$$

and $w\, \widetilde{\exp}_{\gamma,\tau}(|Db|) \in L^1(I \times \Omega)$ we immediately obtain (51). On the other hand, if $w \in L^1(I \times \Omega)$, by (22), there exists $\bar{t} > 0$ such that

$$\frac{1}{2} \exp^*_{\gamma,\tau}(t) \le \widetilde{\exp}_{\gamma,\tau}(t) \text{ for each } t \ge \bar{t}. \tag{52}$$

Thus

$$\begin{aligned}
\int_{I\times\Omega} w \exp^*_{\gamma,\tau}(|Db|)\; \mathrm{d}s\, \mathrm{d}x &= \int_{\{|Db|<\bar{t}\}} w \exp^*_{\gamma,\tau}(|Db|)\; \mathrm{d}s\, \mathrm{d}x \\
&\quad + \int_{\{|Db|\ge\bar{t}\}} w \exp^*_{\gamma,\tau}(|Db|)\; \mathrm{d}s\, \mathrm{d}x \\
&\le \exp^*_{\gamma,\tau}\left(\bar{t}\right) \int_{I\times\Omega} w\; \mathrm{d}s\, \mathrm{d}x \\
&\quad + 2\int_{I\times\Omega} w\, \widetilde{\exp}_{\gamma,\tau}(|Db|)\; \mathrm{d}s\, \mathrm{d}x < \infty
\end{aligned}$$

and (51) follows once more.

By (51), we can apply Theorem 1 to get the existence of $b_h \in C^\infty(I \times \Omega; \mathbb{R}^m)$ satisfying (3) and (4) with $\varphi = \exp^*_{\gamma,\tau}$. Since $w\, \widetilde{\exp}_{\gamma,\tau}(|Db_h|) \le w \exp^*_{\gamma,\tau}(|Db_h|)$, by applying Vitali's convergence theorem, we obtain again the desired conclusion. □

In the proof of Theorem 3 we will need the following lemma.

Lemma 26 *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_s \times \Omega)$ and $(\rho_\varepsilon(s))_\varepsilon$ be a family of time mollifiers in $\mathbb{R}_s$. Then, for a.e. $x \in \Omega$, the time convolution product $f^\varepsilon(\cdot,x) : \mathbb{R}_s \to \mathbb{R}$*

$$\begin{aligned}
f^\varepsilon(s,x) &= (\rho_\varepsilon * f(\cdot,x))(s) \\
&:= \int_{\mathbb{R}} \rho_\varepsilon(s-v)\, f(v,x)\, dv \textit{ for each } s \in \mathbb{R}
\end{aligned}$$

and

$$f^\varepsilon(\cdot,x) \in C^0(\mathbb{R}_s) \textit{ for each } \varepsilon > 0, \textit{ for a.e. } x \in \Omega. \tag{53}$$

In addition, for any open set $\omega \subset \Omega$ one has

$$\|f^{\varepsilon}\|_{L^1(\mathbb{R}_s\times\omega)} \le \|f\|_{L^1(\mathbb{R}_s\times\omega)} \text{ for each } \varepsilon > 0 \tag{54}$$

and

$$f^{\varepsilon} \to f \text{ in } L^1(\mathbb{R}_s \times \omega) \text{ as } \varepsilon \to 0, \tag{55}$$

provided that $f \in L^1(\mathbb{R}_s \times \omega)$.

Finally, if we assume that, for each ball $B(x_0, r) \Subset \Omega$ one has

$$f \in L^1\big(\mathbb{R}_s; C^0(B(x_0, r))\big), \tag{56}$$

then

$$f^{\varepsilon} \in C^0(\mathbb{R}_s \times \Omega) \text{ for each } \varepsilon > 0. \tag{57}$$

Proof Properties (53), (54) and (55) can be proved as in the case of the global (s, x)-convolution by mollifiers (see, for instance, [3, Section 2.1]). Let us prove (57). Let $(s_0, x_0) \in \mathbb{R}_s \times \Omega$ and let $((s_h, x_h))_h \subset \mathbb{R}_s \times \Omega$ a sequence converging to (s_0, x_0). From (56),

$$\int_{\mathbb{R}} F(s)\, ds < \infty \text{ if } F(s) := \sup_{B(x_0,r)} |f(s,\cdot)|,$$

and, without loss of generality, we can assume that $(x_h)_h \subset B(x_0, r)$ for a fixed $r > 0$. Then, since

$$|\rho_\varepsilon(s_h - v)\, f(v, x_h)| \le \sup_{\mathbb{R}} \rho_\varepsilon\, F(v) \quad \text{for a.e.} v \in \mathbb{R},$$

by Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}\lim_{h\to\infty} f^{\varepsilon}(s_h, x_h) &= \lim_{h\to\infty} \int_{\mathbb{R}} \rho_\varepsilon(s_h - v)\, f(v, x_h)\, dv \\ &= \int_{\mathbb{R}} \lim_{h\to\infty} (\rho_\varepsilon(s_h - v)\, f(v, x_h))\, dv \\ &= \int_{\mathbb{R}} \rho_\varepsilon(s_0 - v)\, f(v, x_0)\, dv = f^{\varepsilon}(s_0, x_0).\end{aligned}$$

□

Proof of Theorem 3 We extend b to $\mathbb{R} \times \Omega$ setting $b(t, x) = 0$ whenever $t \notin I$. Denoting by b^{ϵ} the mollified functions with respect to the time variable, one has

$$Db^{\epsilon}(t, x) = \int_{\mathbb{R}} \rho_\epsilon(t - s) Db(s)\, \mathrm{d}s$$

and therefore Jensen's inequality gives

$$\begin{aligned}\int_{\Omega} w(x)\varphi(Db^{\epsilon}(t, x))\, \mathrm{d}x &\le \int_{\Omega} w(x) \int_{\mathbb{R}} \rho_\epsilon(t - s)\varphi(|Db(s, x)|)\, \mathrm{d}s\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \rho_\epsilon(t - s) \int_{\Omega} w(x)\varphi(Db(s, x)|)\, \mathrm{d}x\, \mathrm{d}s.\end{aligned}$$

By integration on I, it follows that $N_{\varphi,w}(|Db^{\epsilon}|) \le N_{\varphi,w}(|Db|)$. Notice now that b^{ϵ} satisfy the assumptions of Corollary 2, thanks to (57). Thus for all $\epsilon > 0$ we get the existence of a sequence $(b^{\epsilon}_h)_h \subset C^{\infty}(I \times \Omega; \mathbb{R}^m)$ satisfying (3) and (4). Finally, by taking a diagonal sequence, we get the desired conclusion. □

Proof of Theorem 4 Notice that for any open set $A \subset \Omega$ the weak $L^1(A; \mathbb{R}^{mn})$ convergence of derivatives grants

$$\liminf_{h\to\infty} \int_A \varphi\left(|Db_h|\right)\,\mathrm{d}x \geq \int_A \varphi\left(|Db|\right)\,\mathrm{d}x\,.$$

Therefore, by applying an elementary lemma (see, for instance, the proof of Proposition 1.80 in [3]) the convergence of the integrals on Ω can be localized, getting

$$\lim_{h\to\infty} \int_A \varphi\left(|Db_h|\right)\,\mathrm{d}x = \int_A \varphi\left(|Db|\right)\,\mathrm{d}x$$

whenever $A \subset \Omega$ is open with Lebesgue negligible boundary. In particular, choosing $A \Subset \Omega$ with this property, since A has finite measure we can use the strict convexity of φ and [11] to get that $Db_h \to Db$ in $L^1(A; \mathbb{R}^{mn})$. It follows that $Db_h \to Db$ in $L^1_{\mathrm{loc}}(I \times \Omega; \mathbb{R}^{nm})$ and therefore, modulo the extraction of a subsequence, we can assume that $Db_h \to Db$ a.e. in Ω.

Combining the pointwise convergence

$$\lim_{h\to\infty} \varphi(|Db_h|) = \varphi(|Db|) \quad \text{a.e. in } \Omega$$

with the convergence of the integrals, Scheffé's lemma gives that $\varphi(|Db_h|)$ converges in $L^1(\Omega)$ to $\varphi(|Db|)$. Now, the inequality

$$\varphi\left(\frac{|Db_h - Db|}{2}\right) \leq \frac{1}{2}\varphi\left(|Db_h|\right) + \frac{1}{2}\varphi\left(|Db|\right)$$

grants the equi-integrability of $\varphi(|Db_h - Db|/2)$. Vitali's convergence theorem can finally be applied to get the result. □

Proof of Theorem 5 Notice that, being $\widetilde{\exp}$ a N-function, it has a linear growth at the origin. Thus by applying Remark 24 with $\Psi = \widetilde{\exp}$ and Corollary 2 with $b(t, x) = u(x)$ and $w \equiv 1$, we get the desired conclusion. □

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