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Hardy spaces and quasiconformal maps in the Heisenberg group ☆

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ABSTRACT

We define Hardy spaces H^p, $0 < p < \infty$, for quasiconformal mappings on the Korányi unit ball B in the first Heisenberg group $\mathbb{H}^1$. Our definition is stated in terms of the Heisenberg polar coordinates introduced by Korányi and Reimann, and Balogh and Tyson. First, we prove the existence of $p_0(K) > 0$ such that every K-quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$ belongs to H^p for all $0 < p < p_0(K)$. Second, we give two equivalent conditions for the H^p membership of a quasiconformal map f, one in terms of the radial limits of f, and one using a nontangential maximal function of f. As an application, we characterize Carleson measures on B via integral inequalities for quasiconformal mappings on B and their radial limits. Our paper thus extends results by Astala and Koskela, Jerison and Weitsman, Nolder, and Zinsmeister, from $\mathbb{R}^n$ to $\mathbb{H}^1$. A crucial difference between the proofs in $\mathbb{R}^n$

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and $\mathbb{H}^1$ is caused by the nonisotropic nature of the Korányi unit sphere with its two characteristic points.

Contents

1. Introduction

A holomorphic function f on the unit disk $\mathbb{D}$ in $\mathbb{C}$ belongs to the *Hardy class* H^p, for some $0 < p < \infty$, if

$$\sup_{0<s<1} \left(\int_0^{2\pi} |f(se^{\mathrm{i}\varphi})|^p \, d\varphi \right)^{1/p} < \infty. \tag{1.1}$$

According to a result by Hardy and Littlewood [36], condition (1.1) holds for a holomorphic function f on $\mathbb{D}$ if and only if the nontangential maximal function of f belongs to $L^p(S^1)$. Here and in the rest of this paper, we consider Lebesgue spaces L^p of p-integrable functions for all exponents $0 < p < \infty$, not only in the normed case $p \geq 1$.

While Hardy spaces play an important role in complex analysis, they have also spurred the development of real-variable methods used to study their analogs in $\mathbb{R}^n$, or even on homogeneous groups [22,25,26,29,69]. Another line of research, close in spirit to the condition (1.1), is the H^p theory for quasiconformal or quasiregular mappings on the unit ball in $\mathbb{R}^n$, where "quasiconformal" and "quasiregular" serve as substitutes for "conformal" and "holomorphic", respectively, see [5,44,62–64,71,2].

The purpose of the present paper is to develop an H^p theory for quasiconformal mappings on the unit ball in the first Heisenberg group $\mathbb{H}^1$, thus extending some of the previously mentioned results from $\mathbb{R}^n$ to $\mathbb{H}^1$. Quasiconformal maps in $\mathbb{H}^1$ have been studied extensively, both as prototypes for quasiconformal maps in non-Euclidean metric measures spaces [38] and due to their connection with complex hyperbolic geometry [66,52].

We consider mappings defined on the unit ball $B := B(0,1) \subset \mathbb{H}^1$ with respect to the *Korányi norm* $\|\cdot\|$. This gauge function plays a distinguished role on $\mathbb{H}^1$ as it is connected to the fundamental solution of the sub-Laplacian, and it gives rise to a system of *polar coordinates* $(s,\omega) \in (0,\infty)\times[\partial B \setminus \{z=0\}]$, see [51,8]. For the precise definitions, we refer the reader to Section 2; for now we simply mention that (1.1) motivates the following definition.

Definition 1.2. Let $0 < p < \infty$. A quasiconformal map $f : B \subset \mathbb{H}^1 \to f(B) \subset \mathbb{H}^1$ belongs to the *Hardy class* H^p if

$$\|f\|_{H^p} := \sup_{0<s<1} \left(\int\limits_{\partial B} \|f(\gamma(s,\omega))\|^p \, d\mathcal{S}^3(\omega) \right)^{1/p} < \infty.$$

Here, $\mathcal{S}^3$ denotes the 3-dimensional spherical Hausdorff measure with respect to a metric that is bi-Lipschitz equivalent to the Korányi distance d. This is a natural measure to work with since, with respect to the Korányi metric, ∂B has Hausdorff dimension 3, while the entire space $\mathbb{H}^1$ is 4-dimensional. However, unlike in $\mathbb{R}^n$, we will employ several different canonical measures on ∂B, depending on the context. In addition to the Hausdorff measure $\mathcal{S}^3|_{\partial B}$, these are the measure σ_0 appearing in the polar coordinates formula, and the measure σ from the formula for the modulus of a ring domain, see Section 2. These measures qualitatively differ from each other in a neighborhood of the north and south pole of the Korányi sphere, which are the two *characteristic points* of ∂B. The nonisotropic nature of ∂B is the main reason for the challenges one faces when extending the H^p theory to $\mathbb{H}^1$.

To motivate our definition of Hardy spaces, we first prove that every quasiconformal map on the Korányi unit ball belongs to some H^p space:

Theorem 1.3. *For every $K \geq 1$, there exists a constant $p_0 = p_0(K) > 0$ such that every K-quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$ belongs to H^p for all $0 < p < p_0$.*

This extends a result by Jerison and Weitsman [44] from $\mathbb{R}^n$ to $\mathbb{H}^1$. In $\mathbb{R}^n$, more precise information on the admissible exponents is due to Nolder [64], Astala and Koskela [5]. In the second part of our paper, we give necessary and sufficient conditions for the H^p membership of a quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$ for a fixed exponent p.

Theorem 1.4. *Let $0 < p < \infty$. The following conditions are equivalent for a quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$:*

(1) $f \in H^p$,
(2) *the nontangential maximal function Mf of f belongs to $L^p(\mathcal{S}^3|_{\partial B})$,*
(3) *the Korányi norm of the radial limit f^* of f belongs to $L^p(\mathcal{S}^3|_{\partial B})$.*

This extends earlier work by Zinsmeister [71] and by Astala and Koskela [5]. The implication "(3) ⇒ (2)" in Theorem 1.4 is the most challenging, and analogously as in $\mathbb{R}^n$, we also obtain quantitative information, see Proposition 5.4. Our proof combines elements from [5] (the use Hardy-Littlewood maximal functions on ∂B) and [71] (the use of specific Carleson measures on B). However, an important tool in [71,5], the subgroup of Möbius transformations that keep B invariant, is not flexible enough in our setting. We use as a substitute a specific class of 1-quasiconformal maps that do not necessarily preserve the unit ball, but nonetheless have useful metric properties.

In order to put Proposition 5.4 in a slightly wider perspective, let us mention that similar inequalities, for a more restrictive range of integrability exponents, have been studied for harmonic functions e.g., on Lipschitz domains in $\mathbb{R}^n$ by Dahlberg, see [17, (2.1)]. Integral inequalities for nontangential maximal functions appear also in connection with the Dirichlet problem for the sub-Laplacian on certain domains in $\mathbb{H}^1$ or more general H-type groups, with rough boundary values, see [65, Theorem 1.8], [12, Theorem 1.1].

Our main application of Theorem 1.4 is a characterization of Carleson measures on B in terms of radial limits of quasiconformal maps on B.

Theorem 1.5. *Assume that μ is a Carleson measure on B. If $f : B \to f(B) \subset \mathbb{H}^1$ is K-quasiconformal, then*

$$\int_B \|f(q)\|^p \, d\mu(q) \leq C \int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega), \quad \text{for all } 0 < p < \infty \tag{1.6}$$

where C depends only on p, K, and the Carleson measure constant of μ. Conversely, for every $K \geq 1$, there exists $p(K) < 3$ such that if $p > p(K)$ is fixed and μ is a Borel measure for which (1.6) holds for all K-quasiconformal mappings, then μ is a Carleson measure.

In fact we prove a more general result for the α-Carleson measures, Theorem 6.13, which can be applied to relate H^p and Bergman-type spaces A^p, analogously as in [5, Theorem 9.1], see Theorem 6.18. Theorems 1.5 and 6.13 are Heisenberg versions of results by Nolder [62,63], and Astala and Koskela [5] which in turn were motivated by Carleson's embedding theorem [13]. We prove the first part of Theorem 6.13 in two steps: (i) a result for Carleson measures and nontangential maximal functions in rather general metric spaces (Proposition 6.3) and (ii) the relation between nontangential maximal functions and radial limits of quasiconformal maps given by Theorem 1.4, or more specifically, Proposition 5.4. The second part of Theorem 6.13 follows by applying (1.6) to specific maps f that are constructed using the $\mathbb{H}^1$ radial stretch map from [6].

In the process of proving the stated theorems, we establish results of independent interest. Extending a theorem by Jones [45], we show that for every quasiconformal map f on B omitting the origin, $|\nabla_H \log \|f(q)\|| \, dq$ defines a Carleson measure on B. This is a consequence of Proposition 4.25. Related to the Heisenberg polar coordinates, we prove that there exists a parameter κ such that every radial curve segment connecting a point $\omega \in \partial B$ to the origin stays inside the nontangential approach region $\Gamma_\kappa(\omega)$ in B (Proposition 2.15). This is a nontrivial statement in $\mathbb{H}^1$, due to the non-geodesic feature of the radial curves, and the non-isotropic nature of ∂B. The Heisenberg geometry enters the picture in other ways, too. For instance, rotations do not act transitively on ∂B. In Section 4.1 we introduce a family of canonical maps on B that serve as the mentioned substitutes for the Möbius self-maps of the unit ball. Since these maps do not necessarily keep B invariant, we have to formulate several of the auxiliary results for a more general class of domains in $\mathbb{H}^1$. This can be done by replacing explicit computations for the Euclidean unit ball with abstract arguments using concepts from metric geometry such as *corkscrew* and *QED* domains.

Structure of the paper. In Section 2 we introduce preliminaries about the Heisenberg group and quasiconformal maps, and state the definitions of radial limits and nontangential maximal functions. In Section 3 we prove our first main result: quasiconformal maps on the Korányi ball belong to Hardy spaces. Section 4 and Appendix A contain auxiliary results on Carleson measures and radial curves, respectively. In Section 5 we prove our second main result, a characterization of the H^p-membership of a quasiconformal map, for a fixed exponent p. We apply this result in Section 6 to characterize Carleson measures on B using radial limits of quasiconformal maps on B.

Notation. If $f, g \geq 0$, the notation $f \lesssim g$ signifies the existence of a positive absolute constant C such that $f \leq Cg$. The notation $f \lesssim_p g$ means that C is allowed to depend on a parameter "p". Finally, $f \sim g$ is an abbreviation of $f \lesssim g \lesssim f$.

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2. Preliminaries

2.1. The Heisenberg group

The *Heisenberg group* $\mathbb{H}^1$ is the set $\mathbb{R}^3$ endowed with the group product

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx')$$

for $(x, y, t), (x', y', t') \in \mathbb{R}^3$. Sometimes we will identify $(x, y) \in \mathbb{R}^2$ with $z = x + \mathrm{i}y \in \mathbb{C}$. Similarly, if $g = (g_1, g_2, g_3)$ is a $\mathbb{H}^1$-valued map, we will occasionally denote $g_I = g_1 + \mathrm{i}g_2$, or $g_I = (g_1, g_2)$. We collect here the properties of $\mathbb{H}^1$ that are most relevant for this paper and we refer the reader to [10] for more details.

2.1.1. Metric structure

We denote the *Korányi norm* by

$$\|(z,t)\| = (|z|^4 + t^2)^{1/4}, \quad (z,t) \in \mathbb{H}^1$$

and the Korányi unit ball by $B := \{p \in \mathbb{H}^1 : \|p\| < 1\}$. The *Korányi distance* is the left-invariant metric given by

$$d(p,q) := \|q^{-1} \cdot p\|, \quad p, q \in \mathbb{H}^1.$$

For $q \in \mathbb{H}^1$ and $A \subset \mathbb{H}^1$, we write $d(q, A) := \inf_{a \in A} d(q,a)$. A curve $\gamma : [a,b] \to \mathbb{H}^1$ is rectifiable with respect to d if and only if it is absolutely continuous as a curve in $\mathbb{R}^3$ and for almost every $s \in [a,b]$ the tangent vector $\dot{\gamma}(s)$ is contained in the *horizontal plane* $H_{\gamma(s)}\mathbb{H}^1$ given by

$$H_{(x,y,t)}\mathbb{H}^1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2y \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2x \end{pmatrix} \right\}.$$

We denote the left-invariant vector fields

$$(x,y,t) \mapsto (1,0,2y) \quad \text{and} \quad (x,y,t) \mapsto (0,1,-2x)$$

by X and Y, respectively, and identify them with the differential operators $\partial_x + 2y\,\partial_t$ and $\partial_y - 2x\,\partial_t$, respectively. The *horizontal gradient* of a function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subset \mathbb{H}^1$ is

$$\nabla_H u = (Xu)X + (Yu)Y.$$

We also equip $H_q\mathbb{H}^1$ with the norm $|\cdot|$ defined by $|aX_q + bY_q| := \sqrt{a^2 + b^2}$.

2.1.2. Polar coordinates and radial curves

The following polar coordinates formula was first proved by Korányi and Reimann [51], and later in greater generality by Balogh and Tyson [8, Example 3.11]. Here and in the following, integration on $\mathbb{H}^1$ is performed with respect to the Lebesgue measure on $\mathbb{R}^3$, which is a Haar measure of the group $\mathbb{H}^1$. We will use the symbol $|A|$ to denote the Lebesgue measure of a set A.

Theorem 2.1. *There exists a unique Radon measure σ_0 on $\partial B \setminus \{z = 0\}$ such that for all $u \in L^1(\mathbb{H}^1)$,*

$$\int_{\mathbb{H}^1} u(q)\,dq = \int_{\partial B \setminus \{z=0\}} \int_0^\infty u(\gamma(s,\omega)) s^3\,ds\,d\sigma_0(\omega), \tag{2.2}$$

where the radial curves *are given by the horizontal curves*

$$\gamma(s,(z,t)) = \left(sze^{-\mathrm{i}\frac{t}{|z|^2}\log s}, s^2 t\right), \quad (z,t) \in \partial B \setminus \{z=0\}.$$

Moreover, using the parametrization

$$\partial B \setminus \{z=0\} = \left\{(z,t) = (\sqrt{\cos\alpha}\,e^{\mathrm{i}\varphi}, \sin\alpha) : -\tfrac{\pi}{2} < \alpha < \tfrac{\pi}{2}, 0 \le \varphi < 2\pi\right\}, \tag{2.3}$$

the measure σ_0 takes the form $d\sigma_0 = d\alpha\,d\varphi$.

Abusing notation, we will often identify a set $E \subset \partial B \setminus \{z=0\}$ with the corresponding set in the parameter space $(-\pi/2, \pi/2) \times [0, 2\pi)$.

2.1.3. Modulus of curve families

The modulus of curve families is a crucial tool in this paper. We refer to [57] for a detailed discussion, and only recall the relevant definitions and properties. Given a family Γ of curves in $\mathbb{H}^1$, a Borel function $\rho : \mathbb{H}^1 \to [0, +\infty]$ is said to be *admissible* for Γ, denoted $\rho \in \mathrm{adm}(\Gamma)$, if $\int_\gamma \rho\,\mathrm{d}s \ge 1$ for all locally rectifiable $\gamma \in \Gamma$. The 4-*modulus of* Γ is then defined as

$$\mathrm{mod}_4(\Gamma) := \inf_{\rho \in \mathrm{adm}(\Gamma)} \int_{\mathbb{H}^1} \rho^4(q)\,dq.$$

Given sets $U \subset \mathbb{H}^1$ and $E, F \subset U$, we denote by $\Gamma(E,F,U)$ the family of all curves contained in U that connect E and F. Korányi and Reimann proved in [51] that the family

$$\Gamma_{a,b} := \Gamma(\partial B(0,a), \partial B(0,b), \mathbb{H}^1)$$

of all curves joining the Korányi spheres $\partial B(0,a)$ and $\partial B(0,b)$ for $0 < a < b < \infty$ has modulus

$$\mathrm{mod}_4(\Gamma_{a,b}) = \pi^2 \left(\log \frac{b}{a}\right)^{-3}. \tag{2.4}$$

A slight modification of the argument in [51] yields the following useful formula concerning the radial curves introduced in Theorem 2.1.

Proposition 2.5. *Fix $0 < r < 1$ and a Borel set $E \subset \partial B \setminus \{z=0\}$. If Γ denotes the family of radial curves joining $\partial B(0,r)$ to E, then*

$$\mathrm{mod}_4(\Gamma) = \sigma(E)\left(\ln \frac{1}{r}\right)^{-3}, \tag{2.6}$$

where $d\sigma = \cos^2\alpha\, d\alpha\, d\varphi$ *in the coordinates given by* (2.3).

Proof. The proof is almost verbatim the same as for (2.4) with $a = r$, $b = 1$, observing that $\sigma(\partial B \setminus \{z = 0\}) = \pi^2$. The inequality "$\geq$" in (2.6) is proven like [51, (4.5)], with the only difference that instead of integrating over $(\alpha, \varphi) \in (-\pi/2, \pi, 2) \times [0, 2\pi]$, the domain of integration now corresponds to the set E. To prove the converse inequality, we follow the proof of [51, (4.6)], but observe that it suffices to consider radial curves, instead of arbitrary rectifiable ones. Moreover, the domain of integration is now restricted to the part of the annulus $\{q \in \mathbb{H}^1 : r \leq \|q\| \leq 1\}$ foliated by segments of the radial curves passing through the set E. □

2.1.4. Measures on the Korányi unit sphere

The two points $q_\pm := (0, 0, \pm 1) \in \partial B$ are *characteristic points* of the Korányi sphere: the tangent planes to the surface ∂B agree with the horizontal planes $H_{\pm q}\mathbb{H}^1$ at the respective points. The distinguished role of these points is reflected in the behavior of the measures that we consider on the Korányi unit sphere outside $q_\pm$: the measure σ_0 from the polar coordinates formula (2.2), the measure σ from the modulus formula (2.6), and the Hausdorff measure $\mathcal{S}^3|_{\partial B \setminus \{z=0\}}$. The latter is a restriction of the 3-dimensional spherical Hausdorff measure computed with respect to the metric d_∞ induced by the gauge function $\|(z, t)\|_\infty = c \max\{|z|, \sqrt{|t|}\}$ for a suitable constant $c > 0$. Since d_∞ is bi-Lipschitz equivalent to the Korányi distance d, we might as well use the standard 3-dimensional Hausdorff measure with respect to d for our purposes, but $\mathcal{S}^3|_{\partial B}$ is related to the horizontal perimeter measure of B and we can thus use some results from [27].

Lemma 2.7. *If we parametrize* $\partial B \setminus \{z = 0\}$ *as in* (2.3), *then*

$$d\mathcal{S}^3|_{\partial B \setminus \{z=0\}} = \sqrt{\cos\alpha}\, d\alpha\, d\varphi.$$

Proof. Since the boundary ∂B of the Korányi unit ball is of class C^1, we know by [27, Corollary 7.7] that, if the constant c in the definition of $\|\cdot\|_\infty$ is chosen suitably, we have

$$d\mathcal{S}^3|_{\partial B} = |C n_B|\, d\mathcal{H}^2|_{\partial B}, \tag{2.8}$$

where n_B is the Euclidean outward unit normal to the Korányi unit sphere, $\mathcal{H}^2$ denotes the 2-dimensional Euclidean Hausdorff measure and

$$C(x, y, t) := \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}.$$

To prove the lemma, it suffices to express the right-hand side of (2.8) using the coordinates (α, φ) from (2.3). First, since $n_B = \nabla v / |\nabla v|$ with $v(x, y, t) := \|(x, y, t)\|^4 =$

$(x^2+y^2)^2+t^2$, we compute for $(x,y,t)=(\sqrt{\cos\alpha}\cos\varphi,\sqrt{\cos\alpha}\sin\varphi,\sin\alpha)\in\partial B\backslash\{z=0\}$ that

$$\nabla v(x,y,t)=\begin{pmatrix}4(x^2+y^2)x\\4(x^2+y^2)y\\2t\end{pmatrix}=\begin{pmatrix}4\cos^{3/2}\alpha\cos\varphi\\4\cos^{3/2}\alpha\sin\varphi\\2\sin\alpha\end{pmatrix}$$

and

$$C(x,y,t)\nabla v(x,y,t)=\begin{pmatrix}4(x^2+y^2)x+4yt\\4(x^2+y^2)y-4xt\end{pmatrix}=4\begin{pmatrix}\cos^{3/2}\alpha\cos\varphi+\cos^{1/2}\alpha\sin\alpha\sin\varphi\\\cos^{3/2}\alpha\sin\varphi-\cos^{1/2}\alpha\sin\alpha\cos\varphi\end{pmatrix}.$$

Hence

$$|Cn_B|=\frac{|C\nabla v|}{|\nabla v|}=\frac{|\nabla_{\mathbb{H}}v|}{|\nabla v|}=\frac{\sqrt{\cos\alpha}}{\sqrt{\cos^3\alpha+\frac{\sin^2\alpha}{4}}}. \tag{2.9}$$

On the other hand, using the parametrization $\Phi(\alpha,\varphi)=(\sqrt{\cos\alpha}\cos\varphi,\sqrt{\cos\alpha}\sin\varphi,\sin\alpha)$, we find that the surface measure is given by

$$d\mathcal{H}^2|_{\partial B\backslash\{z=0\}}=|\Phi_\alpha\times\Phi_\varphi|\,d\alpha d\varphi=\sqrt{\cos^3\alpha+\frac{\sin^2\alpha}{4}}\,d\alpha d\varphi. \tag{2.10}$$

Inserting (2.9) and (2.10) in formula (2.8) then yields the claim. □

In addition to its simple expression in the coordinates (α,φ), the measure $\mathcal{S}^3|_{\partial B}$ has another useful feature:

Lemma 2.11. *The measure $\mathcal{S}^3|_{\partial B}$ is 3-regular, that is, there exists a constant $C\geq 1$ such that*

$$C^{-1}r^3\leq\mathcal{S}^3(B(p,r)\cap\partial B)\leq Cr^3,\quad \textit{for all } p\in\partial B,\ 0<r<\operatorname{diam}(\partial B).$$

The lemma follows e.g., from [20, Theorem 6.2]. Alternatively, one can also obtain it by combining [19, Theorem 5.1], [11, Propositions 9 and 10] and [24, Proposition 4.1].

The formulae for σ_0, σ, and $\mathcal{S}^3|_{\partial B\backslash\{z=0\}}$ given in Theorem 2.1, Proposition 2.5, and Lemma 2.7, respectively, show that

$$\sigma(E)\leq\mathcal{S}^3(E)\leq\sigma_0(E),\quad \text{for all Borel sets } E\subseteq\partial B\setminus\{z=0\}. \tag{2.12}$$

The difference between the measures becomes more pronounced for sets E concentrated near the characteristic points of ∂B as such sets are harder to reach by short radial curves from inside the ball. Indeed, the curves $\gamma(\cdot,(z,t))$ defined in Theorem 2.1 begin to spiral more as z tends to 0, approaching in the limit the t-axis, a curve that fails to be locally rectifiable with respect to the Korányi metric. However, since $\cos\alpha > 0$ for $\alpha \in (-\pi/2, \pi/2)$, the three measures are still mutually absolutely continuous:

$$\sigma(E) = 0 \quad \Leftrightarrow \quad \sigma_0(E) = 0 \quad \Leftrightarrow \quad \mathcal{S}^3(E) = 0, \quad \text{for all } E \subset \partial B \setminus \{z = 0\} \text{ Borel.} \tag{2.13}$$

2.1.5. Nontangential regions and nontangential maximal functions

To study how a mapping on the Korányi unit ball B behaves close to the boundary ∂B, we use two different tools: *nontangential maximal functions* and *radial limits*. While the former are purely metric concepts, our definition of radial limit is tailored specifically to the Korányi unit ball, as it makes use of the *radial curves* in Theorem 2.1. The geometry of B is also reflected in the choice of the parameter κ for which we apply the definition of nontangential region and associated maximal function, see Proposition 2.15. However, the nontangential maximal functions will not appear in our final result, Theorem 6.13, where they are only used as tools along the way. The first step in the proof of Theorem 6.13, namely Proposition 6.3, works in abstract proper metric spaces, which is why we state the following definitions in this generality.

Definition 2.14. Let (X, d) be a metric space, and Ω a fixed nonempty domain in X. For a point $\omega \in \partial\Omega$ and a parameter $\kappa > 0$, we define the *nontangential region in Ω with parameter κ centered at ω* as follows:

$$\Gamma_{\Omega,\kappa}(\omega) := \{x \in \Omega : \ d(x, \omega) < (1 + \kappa)d(x, \partial\Omega)\}.$$

If $(X, d) = (\mathbb{H}^1, d)$, $\Omega = B$, and κ is as in Proposition 2.15, we often abbreviate

$$\Gamma(\omega) := \Gamma_\kappa(\omega) = \Gamma_{B,\kappa}(\omega).$$

If (X, d) is the Euclidean plane, and Ω the open unit disk, then $\Gamma_{\kappa,\Omega}(\omega)$ is a *Stolz region* (or *nontangential approach region*) in the usual sense. In general, we do not require the domain Ω in Definition 2.14 to have the interior corkscrew property or satisfy other geometric conditions, but in our main application, the nontangential regions carry relevant information thanks to the following result.

Proposition 2.15. *Let B be the Korányi unit ball in $\mathbb{H}^1$. There exists $\kappa > 0$ such that for every $\omega \in \partial B \setminus \{z = 0\}$,*

$$\gamma(s, \omega) \in \Gamma_{B,\kappa}(\omega), \quad \textit{for all } s \in (0, 1).$$

We postpone the proof to the Appendix.

Remark 2.16. By [11, Corollary 1] and by the discussion in the beginning of Section 6 in [31], we have that B is a *John domain*. This already shows that there is a constant

κ, and for every $\omega \in \partial B$, a rectifiable curve in B emanating from ω and contained in $\Gamma_{B,\kappa}(\omega)$ until it hits the John center of B. The purpose of Proposition 2.15 is to provide specific information about the *radial* curves $\gamma(\cdot, \omega)$.

Definition 2.17. Let (X, d) be a metric space, and Ω a fixed domain with nonempty boundary in X. For a point $x \in \Omega$ and a parameter $\kappa > 0$, we define the *shadow* associated to x and $\kappa > 0$ as

$$S_{\Omega,\kappa}(x) := \partial\Omega \cap B(x, (1+\kappa)d(x, \partial\Omega)).$$

If $(X, d) = (\mathbb{H}^1, d)$, $\Omega = B$, and κ is as in Proposition 2.15, we often abbreviate

$$S(q) := S_\kappa(q) := S_{B,\kappa}(q),$$

and we also call the shadow a *spherical cap* in this case.

The definition is tailored so that nontangential regions and shadows are related in the following way:

$$x \in \Gamma_{\Omega,\kappa}(\omega) \quad \Leftrightarrow \quad \omega \in S_{\Omega,\kappa}(x).$$

Definition 2.18. Let (X, d) be a metric space, and Ω a fixed nonempty domain in X with nonempty boundary $\partial\Omega$. If $\kappa > 0$ is such that $\Gamma_{\Omega,\kappa}(\omega) \neq \emptyset$ for all $\omega \in \partial\Omega$, we define the *$\kappa$-nontangential maximal function*

$$N_{\Omega,\kappa}h(\omega) := \sup_{x \in \Gamma_{\Omega,\kappa}(\omega)} h(x), \quad \omega \in \partial\Omega$$

of a function $h : \Omega \to [0, +\infty)$.

If $(X, d) = (\mathbb{H}^1, d)$, $\Omega = B$, and κ is as in Proposition 2.15, we define the *nontangential maximal function of $f : B \to \mathbb{H}^1$* as

$$M(f)(\omega) := M_\kappa(f)(\omega) := N_{B,\kappa}\|f\|(\omega) = \sup_{q \in \Gamma_{B,\kappa}(\omega)} \|f(q)\|.$$

Remark 2.19. For (X, d), $\Omega \subset X$, $\kappa > 0$ and $h : \Omega \to [0, +\infty)$ as in Definition 2.18, the nontangential maximal function $N_{\Omega,\kappa}h : \partial\Omega \to [0, +\infty]$ is lower semicontinuous. Indeed, if $\lambda \in \mathbb{R}$ and $\omega \in \partial\Omega$ are such that $N_{\Omega,\kappa}h(\omega) > \lambda$, then there exists $x \in \Gamma_{\Omega,\kappa}(\omega)$ such that $h(x) > \lambda$. Since

$$x \in \Gamma_{\Omega,\kappa}(\omega') \quad \text{for all } \omega' \in \partial\Omega \text{ with } d(\omega, \omega') < [(1+\kappa)d(x, \partial\Omega) - d(x, \omega)],$$

we see that $N_{\Omega,\kappa}h(\omega') > \lambda$ for all ω' in a relatively open neighborhood of ω in $\partial\Omega$.

2.2. Quasiconformal mappings

A homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{H}^1$ is *quasiconformal* if

$$H_f(q) := \limsup_{r\to 0} \frac{\max_{d(q,q')=r} d(f(q), f(q'))}{\min_{d(q,q')=r} d(f(q), f(q'))}$$

is uniformly bounded on Ω. We say that f is K-*quasiconformal* for a constant $K \geq 1$, if

$$\|H_f\|_\infty := \mathrm{esssup}_{q\in\Omega} H_f(q) \leq K,$$

where the essential supremum is computed with respect to the Lebesgue measure on $\mathbb{R}^3$.

We refer to the literature for equivalent characterizations of quasiconformal mappings and simply recall that a K-quasiconformal map $f : \Omega \to \Omega'$ between domains in $\mathbb{H}^1$ has the following properties:

(1) The map f is absolutely continuous along mod_4 a.e. curve in Ω, and there exists a constant K', depending only on K, such that for each curve family Γ in Ω,

$$\frac{1}{K'}\mathrm{mod}_4(\Gamma) \leq \mathrm{mod}_4(f(\Gamma)) \leq K'\mathrm{mod}_4(\Gamma). \tag{2.20}$$

(2) The components f_1, f_2, f_3 all belong the *horizontal Sobolev space* $HW^{1,4}_{loc}(\Omega)$ of $L^4_{loc}(\Omega)$-functions with weak X and Y derivatives in $L^4_{loc}(\Omega)$, they satisfy the *contact conditions* $Xf(q), Yf(q) \in H_q\mathbb{H}^1$ for almost every $q \in \Omega$, and there exists a constant K'', depending only on K, such that

$$|D_H f(q)|^4 \leq K'' J_f(q), \quad \text{for almost every } q \in \Omega, \tag{2.21}$$

where the *operator norm* is $|D_H f(q)| := \sup_{\xi\in H_q\mathbb{H}^1, |\xi|=1} |D_H f(q)\xi|$, and the *formal horizontal derivative* is given with respect to the frame $\{X, Y\}$ by

$$D_H f(q) := \begin{pmatrix} Xf_1(q) & Yf_1(q) \\ Xf_2(q) & Yf_2(q) \end{pmatrix},$$

and $J_f(q) = (\det D_H f(q))^2$.

These properties follow from [66,52,38], see also [18], and [40, Section 9] for a discussion in abstract metric measure spaces of locally Q-bounded geometry. If the quasiconformal map f is also a diffeomorphism, then J_f agrees with its standard Jacobian, see [52, Section 2.3].

2.2.1. Radial limits

Definition 2.22. The *radial limit* of a map $f : B \subset \mathbb{H}^1 \to \mathbb{H}^1$ is defined as

$$f^*(\omega) := \lim_{s \to 1} f(\gamma(s, \omega))$$

for all $\omega \in \partial B$, where the limit exists.

Lemma 2.23. *If $f : B \to f(B) \subseteq \mathbb{H}^1$ is quasiconformal, then the radial limit f^* exists almost everywhere on $\partial B \setminus \{z = 0\}$ with respect to any of the measures σ_0, σ, and $\mathcal{S}^3|_{\partial B \setminus \{z=0\}}$. Moreover, f^* is Borel measurable.*

By (2.6), the proof is a straightforward adaptation of [5, p. 21]. Polar coordinates were also used in connection with radial limits at ∞ for homogeneous Sobolev functions [54].

Proof. If $\omega \in \partial B \setminus \{z = 0\}$ is such that $f \circ \gamma(\cdot, \omega) : [1/2, 1) \to \mathbb{H}^1$ is rectifiable, then it extends to a rectifiable curve on the closed interval $[1/2, 1]$ (see e.g. [3, Theorem 2.1]), and consequently, $f^*(\omega)$ is defined. In other words, the set A_0 of points $\omega \in \partial B \setminus \{z = 0\}$ for which the radial limit $f^*(\omega)$ does *not* exist is a subset of the set E of points ω in $\partial B \setminus \{z = 0\}$ for which $f \circ \gamma(\cdot, \omega) : [1/2, 1) \to \mathbb{H}^1$ fails to be rectifiable.

Now let $A \subset E$ be an arbitrary Borel set and denote by Γ_A the family of radial curves connecting $\partial B(0, 1/2)$ to A. Then, since the modulus of non-rectifiable curves is zero, see e.g. [41, Proposition 5.3.3.], we have $\mathrm{mod}_4(f(\Gamma_A)) = 0$ and hence $\mathrm{mod}_4(\Gamma_A) = 0$ by the quasiconformality of f and (2.20). Then formula (2.6) implies that $\sigma(A) = 0$. This holds for arbitrary Borel sets $A \subset E$, in particular for the set A_0 defined above, which is indeed a Borel set by standard arguments. Thus we conclude that f^* is defined on the Borel set $\partial B \setminus [\{z = 0\} \cup A_0]$ with $\sigma(A_0) = 0$, and hence also $\sigma_0(A_0) = \mathcal{S}^3(A_0) = 0$ by (2.13). Moreover, since each component of f^* is the limit of a convergent sequence of Borel functions, f^* itself is Borel measurable. □

3. Quasiconformal maps belong to Hardy spaces

In this section we prove Theorem 1.3, which states that every K-quasiconformal map on the Korányi unit ball is of class H^p, for all p less than a threshold that depends only on the distortion K. We show this by establishing the following counterpart of a result by Jerison and Weitsman [44, Theorem 1]. Obviously Theorem 3.1 implies Theorem 1.3.

Theorem 3.1. *For every $K \geq 1$, there exists a constant $p_0 = p_0(K) > 0$ such that every K-quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$ satisfies*

$$\limsup_{r \to 1} \int_{\partial B} \|f(\gamma(r, \omega))\|^p \, d\mathcal{S}^3(\omega) < \infty \quad \textit{for all } p < p_0,$$

where $\gamma(\cdot,\omega)$, $\omega \in \partial B \setminus \{z=0\}$, are the radial curves given by Theorem 2.1.

3.1. Consequences of modulus estimates

Readers interested in the main argument used to prove Theorem 3.1 may want to proceed directly to Section 3.2. The purpose of the present section is to establish growth estimates for quasiconformal maps on B that will be used as a tool in the proof of Theorem 3.1. Analogous estimates for balls in $\mathbb{R}^n$ have been proven using Euclidean techniques, such as spherical symmetrization and Grötzsch rings [53,5,59]. Our proof relies on standard estimates for abstract Ahlfors regular Loewner spaces. This allows us to formulate the statement not only for the unit ball, but also for bounded 1-quasiconformal images thereof that contain the origin. This generalization is possible since the Korányi ball and its conformal images are *quasiextremal distance domains (QED)* in the terminology of [32], [57, (13.33)], with a universal constant. Our use for this property, and a challenge in proving it, is that the Korányi inversion with center at 0 does not keep the sphere ∂B pointwise fixed. Hence simple reflection arguments as in $\mathbb{R}^n$ are not available. On the other hand, it is then natural to discuss the proof for more general domains than B, following the reasoning used to prove [57, Lemma 3.6] in $\mathbb{R}^n$.

Adapting the Euclidean terminology [46], we say that a domain $D \subset \mathbb{H}^1$ is an *extension domain for the Dirichlet energy space (EDE)* if there exists a bounded extension operator

$$\mathrm{ext} : \mathcal{L}^1_4(D) \to \mathcal{L}^1_4(\mathbb{H}^1) \quad \text{with} \quad \|\mathrm{ext}(u)\|_{\mathcal{L}^1_4(\mathbb{H}^1)} \leq C\|u\|_{\mathcal{L}^1_4(D)}, \quad \mathrm{ext}(u)|_D = u, \tag{3.2}$$

for the *homogeneous horizontal Sobolev space* $\mathcal{L}^1_4$, i.e., for the semi-normed space of locally integrable functions with weak X and Y derivatives in L^4.

Lemma 3.3. *If $D \subset \mathbb{H}^1$ is an extension domain for the Dirichlet energy space, then it is a quasiextremal distance domain, quantitatively, that is, for all disjoint nonempty continua E and F in D, it holds*

$$\mathrm{mod}_4(\Gamma(E,F,\mathbb{H}^1)) \leq C\mathrm{mod}_4(\Gamma(E,F,D)), \tag{3.4}$$

where C is the constant in (3.2).

In particular, all domains that arise as images of the Korányi ball B under 1-quasiconformal maps $T : B \to T(B) \subset \mathbb{H}^1$ are QED with a constant C that does not depend on T.

The first part is based on results about 4-capacities. The QED property of B follows from work by Lu [55], see also Greshnov [33], which extends a result by Jones [46, Theorem 2] to Carnot groups. The QED property of $T(B)$ is then a consequence of a Liouville-type theorem in $\mathbb{H}^1$ and conformal invariance of the 4-modulus.

Proof. Let D be an EDE as in the first part of the Lemma. To establish (3.4) for D, we follow the same reasoning as used to prove [57, Lemma 3.6] in $\mathbb{R}^n$. The existence of the extension operator (3.2) has immediate consequences for capacities. For a domain $U \subset \mathbb{H}^1$, and nonempty disjoint compact sets $C_0, C_1 \subset U$, we define the 4-*capacity*

$$\mathrm{cap}_4(C_0, C_1; U) := \inf_{u \in W} \int_U |\nabla_H u(q)|^4 \, dq, \tag{3.5}$$

where $W := W(C_0, C_1; U) = \{u \in C(U) \cap \mathcal{L}_4^1(U) : u|_{C_0} \leq 0 \text{ and } u|_{C_1} \geq 1\}$. We apply this definition first for $U = D$, $C_0 = E$ and $C_1 = F$ as in the statement of the lemma. Thus, for every $\varepsilon > 0$, there exists $u \in W(E, F; D)$ such that

$$\int_D |\nabla_H u(q)| \, dq \leq \mathrm{cap}_4(E, F; D) + \frac{\varepsilon}{2}.$$

Our goal is to control the left-hand side of the inequality from below by $\mathrm{cap}_4(E, F; \mathbb{H}^1)$. To achieve this, we will apply the extension operator (3.2) to a modification v of u, so as to obtain a competitor in $W(E, F; \mathbb{H}^1)$. Namely, we choose $r > 0$ small enough such that $v := \frac{1+r}{1-r}(u - r)$ satisfies

$$\int_D |\nabla_H v(q)|^4 \, dq \leq \mathrm{cap}_4(E, F; D) + \varepsilon, \quad v|_E \leq -r, \quad v|_F \geq 1 + r. \tag{3.6}$$

Clearly, $v \in \mathcal{L}_4^1(D)$ with $\nabla_H v = \frac{1+r}{1-r} \nabla_H u$. By (3.2), there exists $\mathrm{ext}(v) \in \mathcal{L}_4^1(\mathbb{H}^1)$ with

$$\mathrm{ext}(v)|_D = v \quad \text{and} \quad \int_{\mathbb{H}^1} |\nabla_H \mathrm{ext}(v)(q)|^4 \, dq \leq C \int_D |\nabla_H v(q)|^4 \, dq, \tag{3.7}$$

for the constant C given by (3.2). Even if v is continuous on D, a priori, $\mathrm{ext}(v)$ is only an element in $\mathcal{L}_4^1(\mathbb{H}^1)$ and does not necessarily have a continuous representative on $\mathbb{H}^1$. However, there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset C^\infty(\mathbb{R}^3)$ such that

$$v_n \to \mathrm{ext}(v) \text{ locally in } L^1(\mathbb{R}^3) \quad \text{and} \quad \nabla_H v_n \to \nabla_H \mathrm{ext}(v) \text{ in } L^4(\mathbb{R}^3),$$

see [52, Section 3.1], and also [26, Chapter 1.B]. Recalling that $\mathrm{ext}(v)|_E \leq -r$ and $\mathrm{ext}(v)|_F \geq 1 + r$ almost everywhere, this allows us to choose a (smooth) function $\varphi \in W(E, F; \mathbb{H}^1)$ with

$$\int_{\mathbb{H}^1} |\nabla_H \varphi(q)|^4 \, dq \leq \int_{\mathbb{H}^1} |\nabla_H \mathrm{ext}(v)(q)|^4 \, dq + \varepsilon.$$

Combining this with (3.6) and (3.7), and then letting ε tend to 0, we deduce that

$$\mathrm{cap}_4(E,F;\mathbb{H}^1) \le C\,\mathrm{cap}_4(E,F;D).$$

To conclude the proof of the QED property of D, it suffices to replace the 4-capacities in the above estimate by the 4-modulus of the curve families $\Gamma(E,F;\mathbb{H}^1)$ and $\Gamma(E,F;D)$, respectively. In $\mathbb{R}^n$, the corresponding modulus-capacity equality is due to Hesse [42]. There are several related statements for $\mathbb{H}^1$, but the assumptions for instance in [38] and Eichman's result [23] cited in [52, Section 3.2] are a bit different from ours. Instead, we apply [47, Theorem 1.1], which holds for p-modulus and all disjoint compact non-empty sets C_0 and C_1 in arbitrary domains U of proper φ-convex metric measure spaces with a doubling measure supporting a $(1,p)$-Poincaré inequality with $1<p<\infty$. This class of metric measure spaces includes $(\mathbb{H}^1,d)$ with the Lebesgue measure for $p=4$. The caveat is that the capacities appearing in [47] are defined using upper gradients, specifically,

$$\mathrm{Cont}-\mathrm{Cap}_4(C_0,C_1;U) = \inf_g \int_U g(q)^4\,dq, \tag{3.8}$$

where the infimum is taken over all non-negative Borel functions g that are upper gradients – or weak upper gradients – of functions $u \in C(U)$ with the property $u|_{C_0} \le 0$ and $u|_{C_0} \ge 1$. By the proof of [48, Proposition C.12], the definitions in (3.5) and (3.8) yield the same result. Thus we have shown that every EDE $D \subset \mathbb{H}^1$ is indeed a QED domain.

The second part of the lemma can be deduced by applying the previous statement to the Korányi unit ball $D=B$. The EDE property of B follows from [55, Theorem C] since B is uniform and hence an (ϵ,∞)-domain in the sense of [55, Definition 1.1]; see also [33, Theorem 5], [60, Theorem 3.2.], and the comment below [61, Theorem 1.1]. Then every domain $T(B)$ as in the lemma is also a QED domain, with the same constant, by conformal invariance of mod_4. Indeed, by a version of Liouville's theorem [9, Corollary 1.4], T is the restriction of a conformal self-map of the one-point compactification $\widehat{\mathbb{H}}^1$. The claim (3.4) for $D=T(B)$ follows from the QED property of B by applying T to the curve families $\Gamma(T^{-1}(E),T^{-1}(F),B)$ and $\Gamma(T^{-1}(E),T^{-1}(F),\mathbb{H}^1)$, recalling that the family of all nonconstant curves passing through one point in $\mathbb{H}^1$ has vanishing 4-modulus by [37, Corollary 7.20], so we may ignore the points $T(\infty)$ and $T^{-1}(\infty)$. $\square$

Lemma 3.3 allows us to prove the following proposition with constants C and α that do not depend on the particular domain Ω in the statement. The class of sets Ω covered by Proposition 3.9 is strictly larger than the class of Korányi balls, since the 1-quasiconformal maps on B include suitable compositions with *the Korányi inversion*, see Section 4.1.

Proposition 3.9. *For every $K \ge 1$ and $0<m<M<\infty$, there exist constants $C,\alpha>1$ such that whenever $\Omega \subset \mathbb{H}^1$ is a 1-quasiconformal image of B with $B(0,m) \subset \Omega \subset B(0,M)$ and $g:\Omega \to g(\Omega) \subset \mathbb{H}^1 \setminus \{0\}$ is a K-quasiconformal map, then*

$$C^{-1}\, d(q,\partial\Omega)^{\alpha} \leq \frac{\|g(q)\|}{\|g(0)\|} \leq C d(q,\partial\Omega)^{-\alpha}, \quad \textit{for all } q\in\Omega. \tag{3.10}$$

Proof of Proposition 3.9. If Ω is as assumed in the proposition, then

$$d(q,\partial\Omega) \leq M, \quad \text{for all } q\in\Omega. \tag{3.11}$$

Indeed, since $q\in\Omega$ belongs to $B(0,M)$, there exists a horizontal line segment ℓ of d-length at most M connecting q to $\mathbb{H}^1\setminus B(0,M)\subseteq\mathbb{H}^1\setminus\Omega$. As ℓ is a connected set, it must intersect $\partial\Omega$ in at least one point, which proves that $d(q,\partial\Omega)\leq M$ as claimed.

Fix now an arbitrary point $q\in\Omega$. We first show the upper bound in (3.10). There are two cases to consider. If $\|g(0)\| > \frac{1}{2}\|g(q)\|$, then we find by (3.11) that

$$\frac{\|g(q)\|}{\|g(0)\|} < 2 \leq 2M^{\alpha} d(q,\partial\Omega)^{-\alpha} \leq C d(q,\partial\Omega)^{-\alpha}$$

for every $\alpha>0$ and $C\geq 2M^{\alpha}$, establishing the upper bound in (3.10) in that case.

If instead $\|g(0)\|\leq\frac{1}{2}\|g(q)\|$, we will use the modulus of suitable curve families Γ_n to prove the corresponding estimate. To define Γ_n, let first E' be the line segment connecting $g(0)$ to 0, and let F' be the half ray on the line through 0 and $g(q)$ that emanates from $g(q)$ and does not contain 0. By convexity of Korányi balls,

$$E'\subset\overline{B}(0,\|g(0)\|) \quad \text{and} \quad F'\subset\mathbb{H}^1\setminus B(0,\|g(q)\|). \tag{3.12}$$

Moreover, E' and F' are disjoint since $\|g(0)\|<\|g(q)\|$ by assumption. Essentially, we would like to work with the family of curves connecting $E'\cap g(\Omega)$ and $F'\cap g(\Omega)$ inside $g(\Omega)$, but this would lead to technical problems since the two sets are neither compact nor necessarily connected. To address this, we consider suitable sequences of continua $E'_n\subset E'$, $F'_n\subset F'$. By assumption, Ω is the image of B under a 1-quasiconformal map T, and we define $\Omega_n := T(B(0,1-\frac{1}{n}))$. Then $g(0)\in g(\Omega_n)$, and for n large enough, also $g(q)\in g(\Omega_n)$. On the other hand $0\notin\overline{g(\Omega_n)}$ by our assumption on g, and also the unbounded ray F' must contain points outside the compact set $\overline{g(\Omega_n)}$. It follows that E' and F' must intersect $\partial g(\Omega_n)$. These considerations imply that, for n large enough depending on q, the sets

$$\begin{aligned} E'_n &= \text{connected component of } E'\cap\overline{g(\Omega_n)} \text{ containing } g(0),\\ F'_n &= \text{connected component of } F'\cap\overline{g(\Omega_n)} \text{ containing } g(q) \end{aligned}$$

are disjoint continua as desired. If

$$\Gamma'_n := \Gamma(E'_n, F'_n, g(\Omega))$$

denotes the family of curves in $g(\Omega)$ that connect E'_n and F'_n, then every element in Γ'_n must have a subcurve connecting $\partial B(0,\|g(0)\|)$ and $\partial B(0,\|g(q)\|)$ by (3.12). This yields

$$\mathrm{mod}_4(\Gamma'_n) \leq \mathrm{mod}_4(\Gamma(\partial B(0, \|g(0)\|), \partial B(0, \|g(q)\|), \mathbb{H}^1)),$$

and formula (2.4) for the modulus of Korányi annuli implies

$$\mathrm{mod}_4(\Gamma'_n) \leq \pi^2 \left(\ln \frac{\|g(q)\|}{\|g(0)\|} \right)^{-3}. \tag{3.13}$$

On the other hand, if Γ_n denotes the family $g^{-1}(\Gamma'_n)$, we know by (2.20) that

$$\mathrm{mod}_4(\Gamma_n) \leq K' \mathrm{mod}_4(\Gamma'_n), \tag{3.14}$$

with K' depending only on K. To find a lower bound for $\mathrm{mod}_4(\Gamma_n)$, we observe that Γ_n consists of all curves in Ω that connect the two continua $E_n := g^{-1}(E'_n)$ and $F_n := g^{-1}(F'_n)$. By (3.14) and the QED property stated in Lemma 3.3, there exists a universal constant $0 < c \leq 1$, independent of the choice of Ω in Proposition 3.9, and independent of E_n, F_n, such that

$$\mathrm{mod}_4(\Gamma'_n) \geq \frac{c}{K'} \mathrm{mod}_4(\Gamma(E_n, F_n, \mathbb{H}^1)). \tag{3.15}$$

The right-hand side can be bounded from below using the fact that $(\mathbb{H}, d)$ equipped with the Haar measure is a 4-regular 4-Loewner space in the sense of [39], see for instance [37,34] and references therein. This means that

$$\mathrm{mod}_4(\Gamma(E_n, F_n, \mathbb{H}^1)) \geq \psi(\Delta(E_n, F_n)), \tag{3.16}$$

where ψ denotes a decreasing homeomorphism as in [39, (3.9)] and

$$\Delta(E_n, F_n) := \frac{\mathrm{dist}(E_n, F_n)}{\min\{\mathrm{diam} E_n, \mathrm{diam} F_n\}}$$

is the relative distance of E_n and F_n. Hence, by (3.13), (3.15), and (3.16), we obtain for large enough n,

$$\pi^2 \left(\ln \frac{\|g(q)\|}{\|g(0)\|} \right)^{-3} \geq \frac{c}{K'} \psi(\Delta(E_n, F_n)). \tag{3.17}$$

Since $0 \in E_n \subset \Omega_n$ and $q \in F_n \subset \Omega_n$, and the endpoints of E_n and F_n lie in $\partial\Omega_n$, we find

$$\mathrm{diam} E_n \geq d(0, \partial\Omega_n), \quad \mathrm{diam} F_n \geq d(q, \partial\Omega_n) \quad \text{and} \quad \mathrm{dist}(E_n, F_n) \leq \|q\| \leq M. \tag{3.18}$$

Recall that Ω is the image of $B = B(0,1)$ under a 1-quasiconformal map T, and $\Omega_n = T(B(0, 1 - \frac{1}{n}))$, where T is the restriction of a conformal self-map of $\widehat{\mathbb{H}}^1$, which we continue to denote by T. Since the image $T(B) \subset B(0, M)$ is bounded, T is uniformly continuous on $\overline{B}$. This can be used to prove for every $q \in \Omega$ that

$$2d(q, \partial\Omega_n) \geq d(q, \partial\Omega) \tag{3.19}$$

for all n large enough, depending on q and T. Since $B(0, m) \subset \Omega$ by assumption, we have $d(0, \partial\Omega) \geq m$, and hence we may assume that

$$d(0, \partial\Omega_n) \overset{(3.19)}{\geq} \frac{m}{2} \overset{(3.11)}{\geq} \frac{m\, d(q, \partial\Omega)}{2M}$$

for n large enough. If we apply (3.19) also to the lower bound for $\mathrm{diam} F_n$ in (3.18), then the monotone decrease of the Loewner function yields

$$\psi(\Delta(E_n, F_n)) \geq \psi\left(\frac{2M^2}{m\, d(q, \partial\Omega)}\right). \tag{3.20}$$

By [39, Theorem 3.6], we may assume that $\psi(t) \sim (\ln t)^{-3}$ for large enough t, i.e.,

$$\psi(\Delta(E_n, F_n)) \gtrsim \left(\ln\left(\frac{2M^2}{m\, d(q, \partial\Omega))}\right)\right)^{-3} \tag{3.21}$$

if $d(q, \partial\Omega)$ is small enough, where the notation "$\gtrsim$" means that the inequality holds up to a positive multiplicative constant that does not depend on q, E_n or F_n. Hence, if $d(q, \partial\Omega)$ is, say, such that

$$\frac{md(q, \partial\Omega)}{2M^2} \leq \frac{1}{t_0}, \tag{3.22}$$

for some t_0 that depends only on the geometry of $\mathbb{H}^1$, then (3.21) holds for all large enough n. On the other hand, if (3.22) fails, then we will simply use the estimate

$$\psi(\Delta(E_n, F_n)) \geq \psi(t_0) > 0. \tag{3.23}$$

This suffices to treat that case since we always have that $d(q, \partial\Omega) \leq M$ by (3.11), so if $d(q, \partial\Omega)$ is also bounded from below by a positive constant in terms of m and M (and the absolute constant t_0), then actually $d(q, \partial\Omega)$ is comparable to a constant depending on m and M. Hence in that case

$$\pi^2 \left(\ln \frac{\|g(q)\|}{\|g(0)\|}\right)^{-3} \geq \frac{c}{K'}\psi(t_0)$$

implies that q satisfies the second inequality in (3.10) for any α with large enough constant C, depending only on K, m, and M. Hence it remains to discuss the case of q as in (3.22). By (3.17) and (3.21) applied to large enough n, we obtain

$$\pi^2 \left(\ln \frac{\|g(q)\|}{\|g(0)\|}\right)^{-3} \gtrsim \left(\ln \left(\frac{2M^2}{m\, d(q, \partial\Omega)}\right)\right)^{-3}. \tag{3.24}$$

It follows that

$$\frac{\|g(q)\|}{\|g(0)\|} \lesssim \left(\frac{2M^2}{md(q, \partial\Omega)}\right)^{C(K)^3},$$

which concludes the proof of the upper bound in (3.10).

In order to show the lower estimate in the assertion (3.10), we follow a similar approach as above, and hence we only sketch the proof. First assume that $\|g(q)\| > \frac{1}{2}\|g(0)\|$. By (3.11) we immediately obtain that

$$\|g(0)\| < 2\|g(q)\| \leq 2M\|g(q)\|d(q, \partial\Omega)^{-1}.$$

So in that case the lower estimate in (3.10) holds for every $\alpha > 0$ and with sufficiently large C, depending on M. If $\|g(q)\| \leq \frac{1}{2}\|g(0)\|$, then we define sets E' and F' as above with $g(q)$ instead of $g(0)$ in the definition of E' and with the opposite change in the definition of F'. Then the counterpart for (3.13) reads:

$$\mathrm{mod}_4(\Gamma') \leq \mathrm{mod}_4(\Gamma(\partial B(0, \|g(q)\|), \partial B(0, \|g(0)\|), \mathbb{H}^1)) = \pi^2 \left(\ln \frac{\|g(0)\|}{\|g(q)\|}\right)^{-3},$$

and the rest of the proof follows as before. □

Remark 3.25. By applying Proposition 3.9 to the case $\Omega = B$ (with $m = M = 1$), we get the growth estimates (3.10) for quasiconformal maps on B omitting the origin, hence generalizing Lemma 2.2 in [5].

Having proven Proposition 3.9 for quasiconformal mappings that omit the origin, we next deduce information for maps with $f(0) = 0$.

Proposition 3.26. *For every $K \geq 1$, there is $\alpha > 0$ such that whenever $f : B \subset \mathbb{H}^1 \to f(B) \subset \mathbb{H}^1$ is a K-quasiconformal map with $f(0) = 0$, then*

$$\|f(q)\| \leq C_1(f) + C_2(f)\, d(q, \partial B)^{-\alpha}, \quad \textit{for all } q \in B,$$

for positive and finite constants $C_1(f)$ and $C_2(f)$, which do not depend on q.

The proof below will in fact yield

$$\frac{d(f(0), f(q))}{d(f(0), \partial f(B))} \leq 1 + Cd(q, \partial B)^{-\alpha}, \quad \text{for all } q \in B.$$

The crucial feature is that this holds for *all* points $q \in B$, arbitrarily close to the boundary, with a constant C depending only on K thus controlling the rate at which $\|f(q)\|$ can grow as q approaches ∂B.

Proof. If f satisfies the assumptions of the proposition, then $f(B)$ is a *strict* subset of $\mathbb{H}^1$. Indeed, suppose towards a contradiction that $f(B) = \mathbb{H}^1$. This yields a quasiconformal map $f^{-1} : \mathbb{H}^1 \to B$, which is in fact *quasisymmetric*, see for instance [67, Lemma 5.2]. Since the quasisymmetric image of a complete space is complete, this would imply that the open ball B is complete, a contradiction. Hence we can pick a point $\tau \in \mathbb{H}^1 \setminus \{0\}$ that is omitted by f. The constants $C_1(f)$ and $C_2(f)$ will depend on the choice of τ. Now the map $g : B \to g(B)$, defined by $g(q) := \tau^{-1} \cdot f(q)$, is quasiconformal with the same constant K as f, and it fulfills the assumptions of Proposition 3.9 for $\Omega = B$. Then,

$$\begin{aligned}\|f(q)\| = d(\tau^{-1}, g(q)) \leq \|\tau^{-1}\| + \|g(q)\| &\leq \|\tau^{-1}\| + C\,\|g(0)\|\,d(q, \partial B)^{-\alpha} \\ &= \|\tau\| + C\,\|\tau\|\,d(q, \partial B)^{-\alpha}\end{aligned}$$

for all $q \in B$, and constants C and α depending only on the distortion K of f. $\square$

3.2. Proof that quasiconformal maps belong to Hardy classes

To prove Theorem 3.1, we have to consider the radial curves $\gamma(\cdot, \omega)$ from Theorem 2.1 more closely. We will use in particular that they are horizontal curves, that is

$$\dot{\gamma}(s, \omega) \in H_{\gamma(s,\omega)}\mathbb{H}^1 = \operatorname{span}\{X_{\gamma(s,\omega)}, Y_{\gamma(s,\omega)}\},$$

and that

$$\left|\frac{\partial \gamma}{\partial s}(s, (z, t))\right| = \frac{1}{|z|}, \tag{3.27}$$

where $|\cdot|$ on the left-hand side of (3.27) denotes the norm on $H_{\gamma(s,(z,t))}\mathbb{H}^1$ that makes $X_{\gamma(s,(z,t))}, Y_{\gamma(s,(z,t))}$ orthonormal, see the formula below (4.4) in [51], or [8, Lemma 3.3 and Example 3.6]. This allows us to prove the following:

Lemma 3.28. *If $f : B \to f(B) \subset \mathbb{H}^1$ is quasiconformal, then for almost every $\omega = (z, t) \in \partial B \setminus \{z = 0\}$ (with respect to any of the measures σ_0, σ, or $\mathcal{S}^3$) and for almost every $s \in (0, 1)$, we have*

$$\left|\frac{\partial}{\partial s}\|f(\gamma(s, \omega))\|\right| \leq \frac{|f_I(\gamma(s, \omega))|}{\|f(\gamma(s, \omega))\|}|D_H f(\gamma(s, \omega))|\frac{1}{|z|},$$

where $|D_H f|$ is defined as below (2.21).

Proof. Taking into account Proposition 2.5, it follows from well-known properties of quasiconformal mappings [52], and of rectifiable curves in the Heisenberg group [35], that $f \circ \gamma(\cdot,\omega)$ is a horizontal curve for σ almost every $\omega \in \partial B \setminus \{z = 0\}$. On the other hand, being quasiconformal, the map f is differentiable in the sense of Pansu [66] at Lebesgue almost every point in B. By the polar coordinates formula stated in Theorem 2.1, this means that for σ almost every ω in the Korányi sphere, for almost every $s \in (0,1)$, the point $\gamma(s,\omega)$ is a Pansu differentiability point of f. We fix now $\omega \in \partial B \setminus \{z = 0\}$ such that $\lambda := f \circ \gamma(\cdot,\omega)$ is horizontal, and we further fix $s \in (0,1)$ such that the tangent vectors $\partial_s\gamma(\omega,s)$ and $\dot\lambda(s)$ exist and are horizontal, and such that f is Pansu differentiable at $\gamma(s,\omega)$ with Pansu differential given by its *formal Pansu differential* as in [18, Theorem 5.1]. The horizontality of λ means that

$$\dot\lambda_3 = 2(\dot\lambda_1\lambda_2 - \dot\lambda_2\lambda_1), \quad \text{almost everywhere on } (0,1). \tag{3.29}$$

Then, almost everywhere,

$$\begin{aligned}\frac{\partial}{\partial s}\sqrt[4]{(\lambda_1^2+\lambda_2^2)^2+\lambda_3^2} &= \frac{1}{4}\frac{4(\lambda_1^2+\lambda_2^2)(\lambda_1\dot\lambda_1+\lambda_2\dot\lambda_2)+2\dot\lambda_3\lambda_3}{\|\lambda\|^3}\\ &\overset{(3.29)}{=} \frac{(\lambda_1^2+\lambda_2^2)(\lambda_1\dot\lambda_1+\lambda_2\dot\lambda_2)+\dot\lambda_1\lambda_2\lambda_3-\dot\lambda_2\lambda_1\lambda_3}{\|\lambda\|^3}\\ &= \frac{\dot\lambda_1((\lambda_1^2+\lambda_2^2)\lambda_1+\lambda_2\lambda_3)+\dot\lambda_2((\lambda_1^2+\lambda_2^2)\lambda_2-\lambda_1\lambda_3)}{\|\lambda\|^3}.\end{aligned}$$

Hence

$$\left|\frac{\partial}{\partial s}\|\lambda\|\right| \le \frac{|\dot\lambda_I|}{\|\lambda\|^3}|\lambda_I|\|\lambda\|^2 = |\dot\lambda_I|\frac{|\lambda_I|}{\|\lambda\|}, \tag{3.30}$$

where we have denoted $\lambda_I = \lambda_1 + \mathrm{i}\lambda_2$. The lemma follows upon observing that

$$\dot\lambda(s) = D_H f(\gamma(s,\omega))\partial_s\gamma(s,\omega) \in H_{\lambda(s)}\mathbb{H}^1,$$

which is based on the fact that the restriction of the Pansu derivative of f to $H_{\gamma(s,\omega)}\mathbb{H}^1$ coincides with the formal horizontal derivative $D_H f(\gamma(s,\omega))$ defined below (2.21), and a chain rule holds for Pansu derivatives, see [56]. Hence we obtain

$$|\dot\lambda_I(s)| \le |D_H f(\gamma(s,\omega))|\left|\frac{\partial}{\partial s}\gamma(s,\omega)\right|,$$

which yields the claim in Lemma 3.28 by (3.30) and (3.27). □

We now prove the main result of this section. With Proposition 3.26 and Lemma 3.28 in place, our argument follows the proof by Jerison and Weitsman of [44, Theorem 1].

Proof of Theorem 3.1. Using left translations and Heisenberg dilations, we may assume without loss of generality that $f(0) = 0$ and that there exists $\varepsilon > 0$ so that

$$\|f|_{B\setminus B(0,1/2)}\| > \varepsilon. \tag{3.31}$$

Points $\omega \in \partial B$ will be denoted by $\omega = (z,t)$. Since f is quasiconformal, it is absolutely continuous along mod_4 almost every curve. By the modulus formula (2.6), this implies that f is absolutely continuous along the radial curve segment $\gamma(\cdot,\omega) : [1/2, 1] \to \mathbb{H}^1$ for $\mathcal{S}^3$ almost every ω, recalling that $\mathcal{S}^3|_{\partial B\setminus\{z=0\}}$ is absolutely continuous with respect to σ.

Let us now fix an exponent $p > 0$, to be determined later, and $\frac{1}{2} < r < 1$. We apply Lemma 3.28 to obtain

$$\begin{aligned}
&\int\limits_{\partial B} \|f(\gamma(r,\omega))\|^p \, d\mathcal{S}^3(\omega) - \int\limits_{\partial B} \|f(\gamma(\tfrac{1}{2},\omega))\|^p \, d\mathcal{S}^3(\omega) \\
&= \int\limits_{\partial B} \int\limits_{\frac{1}{2}}^{r} \frac{\partial}{\partial s} \|f(\gamma(s,\omega))\|^p \, ds \, d\mathcal{S}^3(\omega) \\
&\le \int\limits_{\partial B} \int\limits_{\frac{1}{2}}^{r} \left| \frac{\partial}{\partial s} \|f(\gamma(s,\omega))\|^p \right| \, ds \, d\mathcal{S}^3(\omega) \\
&\le p \int\limits_{1/2}^{r} \int\limits_{\partial B} \|f(\gamma(s,\omega))\|^{p-2} |f_I(\gamma(s,\omega)| \, |D_H f(\gamma(s,\omega))| \frac{1}{|z|} d\mathcal{S}^3(\omega) \, ds.
\end{aligned}$$

The second integral on the left-hand side is a finite positive number $C = C(f,p)$ that does not depend on r. Thus, by the distortion inequality (2.21) for quasiconformal maps,

$$\begin{aligned}
&\int\limits_{\partial B} \|f(\gamma(r,\omega)\|^p d\mathcal{S}^3(\omega) \\
&\le (K'')^{1/4} p \int\limits_{1/2}^{r} \int\limits_{\partial B} \|f(\gamma(s,\omega))\|^{p-2} |f_I(\gamma(s,\omega)| \, J_f(\gamma(s,\omega))^{1/4} \frac{1}{|z|} d\mathcal{S}^3(\omega) \, ds + C \\
&= (K'')^{1/4} p \int\limits_{1/2}^{r} \int\limits_{\partial B} g(s,\omega) \cdot h(s,\omega) d\mathcal{S}^3(\omega) \, ds + C =: I(r) + C,
\end{aligned}$$

where, for $\omega = (z,t)$, we have

$$g(s,\omega) := \|f(\gamma(s,\omega))\|^{-(p+1)} J_f(\gamma(s,\omega))^{1/4} \frac{1}{|z|^{1/4}} s^{3/4}$$

$$h(s,\omega) := \|f(\gamma(s,\omega))\|^{2p-1} |f_I(\gamma(s,\omega)| \frac{1}{|z|^{3/4}} s^{-3/4}.$$

We estimate I by applying Hölder's inequality with exponents 4 and 4/3. This yields $I(r) \leq (K'')^{1/4} p I_1(r)^{1/4} I_2(r)^{3/4}$, where by the area formula for quasiconformal maps [18, Theorem 5.4], and (3.31),

$$\begin{aligned}
I_1(r) &:= \int_{1/2}^{r} \int_{\partial B} \|f(\gamma(s,\omega))\|^{-4(p+1)} J_f(\gamma(s,\omega)) s^3 \frac{1}{|z|} \, d\mathcal{S}^3(\omega)\, ds \\
&= \int_{1/2}^{r} \int_{\partial B} \|f(\gamma(s,\omega))\|^{-4(p+1)} J_f(\gamma(s,\omega)) s^3 \, d\sigma_0(\omega)\, ds \\
&= \int_{B(0,r)\setminus B(0,1/2)} \|f(q)\|^{-4(p+1)} J_f(q)\, dq \\
&\leq \int_{\mathbb{H}^1 \setminus B(0,\varepsilon)} \|q\|^{-4(p+1)}\, dq < \infty,
\end{aligned}$$

and

$$\begin{aligned}
I_2(r) &:= \int_{1/2}^{r} \int_{\partial B} \|f(\gamma(s,\omega))\|^{(2p-1)4/3} |f_I(\gamma(s,\omega))|^{4/3} s^{-1}\, d\sigma_0(\omega)\, ds \\
&\leq \int_{1/2}^{r} \int_{\partial B} \|f(\gamma(s,\omega))\|^{2p4/3} s^{-1}\, d\sigma_0(\omega)\, ds \\
&\leq C \int_{1/2}^{r} (1-s)^{-2p\alpha 4/3}\, ds + C.
\end{aligned}$$

Here we used that $d\mathcal{S}^3 = |z| d\sigma_0$ by Lemma 2.7 for σ_0 as in the polar coordinates formula of Theorem 2.1. The estimate in the last line is a consequence of Proposition 3.26, which yields that

$$\|f(q)\| \leq C_1(f) + C_2(f) d(q,\partial B)^{-\alpha} \leq C_1(f) + C_2(f) \left(1 - \|q\|\right)^{-\alpha}, \quad \text{for all } q \in B,$$

for some positive and finite constants $C_1(f)$ and $C_2(f)$, and $\alpha = \alpha(K) > 0$ that only depends on the distortion K of f. If $2p\alpha 4/3 < 1$, then $\sup_{1/2<r<1} I_2(r)$ is clearly finite. This proves the theorem with $p_0(K) := \frac{3}{8\alpha(K)}$. $\square$

4. Carleson measures

The aim of this section is to prove Proposition 4.25, which states that every quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1 \setminus \{0\}$ gives rise to certain Carleson measures on B. This will be an important tool in the proof of Theorem 1.4. A straightforward corollary of Proposition 4.25 says that $|\nabla_H \log \|f(q)\|| \, dq$ defines a Carleson measure for every quasiconformal map f as above. This extends to $\mathbb{H}^1$ a result that was proven in $\mathbb{R}^n$ first by Jones [45, Lemma 4.2 and p. 65] in order to show that $\log |f|$ belongs to $BMO(S^{n-1})$, quantitatively, for each K-quasiconformal map f on the unit ball in $\mathbb{R}^n$ that omits 0, where f on S^{n-1} is understood as the radial limit. Jones' result about Carleson measures was also obtained, with a different method, by Astala and Koskela [5, Lemma 5.6], and we follow roughly their approach. However, the class of Möbius self-maps of B is not rich enough to perform the standard normalization arguments done in [45,5]. The generality in which we stated Proposition 3.9 allows us to work with 1-quasiconformal maps that do not necessarily preserve B.

4.1. The canonical Möbius transformation

Möbius transformations are a commonly used tool to simplify proofs concerning Hardy spaces and Carleson measures on the unit ball B^n in $\mathbb{R}^n$. Möbius self-maps of B^n are discussed in great detail in Ahlfors' monograph [4, p. 24 ff.]. The computations there use specific properties of the Euclidean metric and Möbius transformations in $\mathbb{R}^n$. The relevant maps in $\mathbb{H}^1$ arise as restrictions of conformal maps of the compactified Heisenberg group, they are compositions of *left translations*, *Heisenberg dilations*, *rotations about the vertical t-axis*, and the *Korányi inversion*. While these share many properties with Möbius transformations in $\mathbb{R}^n$, see e.g. [14,28], there are important differences. To give some examples, the conformal group in our case is not transitive on the set of triples of distinct points [50], rotations are not transitive on ∂B, and unlike $x \mapsto \frac{x}{|x|^2}$, the Korányi inversion does not keep ∂B pointwise fixed. Nonetheless, 1-quasiconformal maps of, but not necessarily onto, B play an important role in our proof of Theorem 1.4. Here we discuss the preliminaries.

The *Korányi inversion* in the Korányi unit sphere centered at the origin is defined as follows: $I(y) = -\frac{1}{\|y\|^4}\left(y_z(|y_z|^2 + iy_t), y_t\right)$, where $y = (y_z, y_t) \in \mathbb{H}^1 \setminus \{0\}$. It is the restriction of a conformal self-map of the compactification $\widehat{\mathbb{H}}^1$, with $I(0) = \infty$ and $I(\infty) = 0$, see [52]. This inversion was introduced by Korányi [49, (1.8)] to define a Kelvin transform for functions on the Heisenberg group. The inversion has the crucial property that

$$d(I(y), I(y')) = \frac{d(y, y')}{\|y\| \, \|y'\|}, \quad y, y' \in \mathbb{H}^1 \setminus \{0\}, \tag{4.1}$$

see, e.g., [10, p. 19]. The Jacobian of I at a point $y \in B$, $y \neq 0$, can be expressed as follows: $J_I(y) = (Xf_1Yf_2 - Xf_2Yf_1)^2(y) = \frac{1}{\|y\|^8}$, see [16, (3.5)]. We will use the inversion I to define certain *canonical* 1-quasiconformal mappings. We start with the most general definition, and later add more requirements on the parameters, as we prove finer properties.

Proposition 4.2. *For $x \in \mathbb{H}^1$, $a \in \mathbb{H}^1 \setminus \{x\}$, and $\rho > 0$, the map*

$$T := T_{x,a,\rho} : \hat{\mathbb{H}}^1 \to \hat{\mathbb{H}}^1, \quad T(y) := \delta_\rho \left(\left[I(a^{-1} \cdot x) \right]^{-1} \cdot \left[I(a^{-1} \cdot y) \right] \right)$$

has the following properties:

$$T(x) = 0, \quad T(a) = \infty, \quad T(\infty) = \delta_\rho \left([I(a^{-1} \cdot x)]^{-1} \right), \tag{4.3}$$

$$T|_{\mathbb{H}^1 \setminus \{a\}} : \mathbb{H}^1 \setminus \{a\} \to \mathbb{H}^1 \setminus \{\delta_\rho \left([I(a^{-1} \cdot x)]^{-1} \right)\} \quad \text{is 1-quasiconformal,} \tag{4.4}$$

for all $y, y' \in \mathbb{H}^1 \setminus \{a\}$, it holds that

$$d(T(y), T(y')) = \rho \frac{d(y, y')}{d(a, y) d(a, y')}, \tag{4.5}$$

$$\|T(y)\| = \rho \frac{d(x, y)}{d(a, y) d(a, x)}, \tag{4.6}$$

$$J_T(y) = \frac{\rho^4}{d(a, y)^8}, \tag{4.7}$$

and for all $r > 0$, one has

$$T(\partial B(a, r)) = \partial B \left(\delta_\rho \left([I(a^{-1} \cdot x)]^{-1} \right), \frac{\rho}{r} \right). \tag{4.8}$$

Proof. Property (4.3) is immediate from the definition of T, recalling that $I(0) = \infty$ and $I(\infty) = 0$. Then T maps $\mathbb{H}^1 \setminus \{a\}$ homeomorphically onto $\mathbb{H}^1 \setminus \{\delta_\rho \left([I(a^{-1} \cdot x)]^{-1} \right)\}$ and it is 1-quasiconformal on $\mathbb{H}^1 \setminus \{a\}$ as a composition of left-translations, dilations, and the 1-quasiconformal inversion I. This proves (4.4). Since the metric d is invariant under left translations, scales by factor ρ under the dilation δ_ρ, and interacts with the inversion I as stated in (4.1), we can deduce (4.5) immediately from the definition of T. Next, (4.6) follows from (4.5) applied to $y' = x$ (recalling that $T(x) = 0$ as stated in (4.3)). The Jacobian of T can be computed by the chain rule. Denoting left translations by $L_{q_0}(q) := q_0 \cdot q$, we find

$$\begin{aligned} J_T(y) &= J_{\delta_\rho} \left([I(a^{-1} \cdot x)]^{-1} \cdot [I(a^{-1} \cdot y)] \right) \, JL_{[I(a^{-1} \cdot x)]^{-1}}(I(a^{-1} \cdot y)) \, JI(a^{-1} \cdot y) \, JL_{a^{-1}}(y) \\ &= \rho^4 \cdot 1 \cdot \frac{1}{d(a, y)^8} \cdot 1. \end{aligned}$$

This yields (4.7). Finally, (4.8) follows since $I(\partial B(0, r)) = \partial B(0, 1/r)$ for all $r > 0$. $\square$

Corollary 4.9. *There exists a constant $C \geq 1$ such that for all $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$ and $\rho > 0$, the map $T = T_{x,a,\rho}$ from Proposition 4.2 satisfies*

$$T(B) \subset B\left(0, \frac{C\rho}{d(a,\partial B)} + \frac{\rho}{d(a,x)}\right).$$

In particular, if

$$\rho \lesssim d(a,\partial B) \quad \textit{and} \quad \rho \lesssim d(a,x), \tag{4.10}$$

then $T(B) \subset B(0,M)$ for a radius M that depends on x, a, and ρ only through the implicit multiplicative constants in the inequalities in (4.10).

Proof. Formula (4.6) in Proposition 4.2 yields by the triangle inequality that

$$\|T(y)\| = \rho \frac{d(x,y)}{d(a,y)d(a,x)} \leq \frac{\rho}{d(a,y)} + \frac{\rho}{d(a,x)}, \quad y \in \mathbb{H}^1 \setminus \{0\}.$$

Now if $y \in B$, then $Cd(a,y) \geq d(a,\partial B)$ for a universal constant C. Indeed, the inequality holds true with constant 1 if d is replaced by the sub-Riemannian distance, and then the constant C can be found by comparing the two distances. $\square$

Corollary 4.9 concerned the size of balls $B(0,M)$ that *contain* $T_{x,a,\rho}(B)$. Similarly, we next study the size of a ball $B(0,m)$ that can be *included* in $T_{x,a,\rho}(B)$.

Corollary 4.11. *Assume that $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$ and $\rho > 0$ are such that*

$$d(x,\partial B) \gtrsim \rho, \quad d(a,x) \lesssim \rho. \tag{4.12}$$

Then $B(0,m) \subset T_{x,a,\rho}(B)$ for a constant $m > 0$ that depends on x, a, and ρ only through the implicit multiplicative constants in the inequalities in (4.12).

Proof. By Proposition 4.2, the map $T = T_{x,a,\rho}$ satisfies for all $w' \in \partial B$ that

$$\begin{aligned}\|T(w')\| = d(T(x),T(w')) = \frac{\rho d(x,w')}{d(a,x)\,d(a,w')} &\overset{(4.12)}{\gtrsim} \frac{d(x,w')}{d(a,w')} \gtrsim \frac{d(x,w')}{d(a,x)+d(x,w')} \\ &\overset{(4.12)}{\gtrsim} \frac{d(x,w')}{\rho + d(x,w')} \\ &\overset{(4.12)}{\gtrsim} \frac{d(x,w')}{d(x,w')+d(x,w')} \gtrsim 1. \quad \square\end{aligned}$$

We now discuss the behavior of $T_{x,a,\rho}$ on $B(\omega,r) \cap B$ for $\omega \in \partial B$ and $r > 0$, under certain conditions on these parameters.

Proposition 4.13. *Let $\omega \in \partial B$, $x \in B$, and $\rho > 0$. Assume that $a \in \mathbb{H}^1 \setminus \overline{B}$ and $r > 0$ are such that*

$$d(a,\omega) \lesssim r, \quad d(a,B) \geq Cr \tag{4.14}$$

for a constant $C > 1$. Then, the map $T = T_{x,a,\rho}$ from Proposition 4.2 satisfies

$$\frac{d(T(y),\partial T(B))}{d(y,\partial B)} \sim_C \frac{\rho}{d(y,a)^2}, \quad \text{for all } y \in B(\omega,r)\cap B. \tag{4.15}$$

Proof. By (4.5) in Proposition 4.2, we know that

$$\frac{\rho}{d(y,a)^2} = \frac{d(T(y),T(y'))\, d(a,y')}{d(a,y)\, d(y,y')}, \quad y,y' \in \mathbb{H}^1 \setminus \{a\}. \tag{4.16}$$

First, we apply (4.16) for $y \in B$ and $y' \in \partial B$ with the property that

$$d(T(y),T(y')) = d\left(T(y),T(\partial B)\right).$$

For this pair of points, (4.16) yields

$$\begin{aligned}\frac{\rho}{d(y,a)^2} = \frac{d(T(y),T(\partial B))\, d(a,y')}{d(a,y)\, d(y,y')} &\leq d(T(y),T(\partial B))\left[\frac{1}{d(y,y')} + \frac{1}{d(a,y)}\right]\\ &\lesssim \frac{d(T(y),\partial TB)}{d(y,\partial B)}.\end{aligned}$$

Here the last inequality holds since $y \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$, and $y' \in \partial B$, hence $d(y,\partial B) \leq d(y,y')$ and, as in the proof of Corollary 4.9, $d(y,\partial B) \lesssim d(y,a)$. Thus,

$$\frac{\rho}{d(y,a)^2} \lesssim \frac{d(T(y),\partial TB)}{d(y,\partial B)}, \quad y \in B.$$

Second, to prove the converse inequality in (4.15), we apply (4.16) to $y \in B(\omega,r)\cap B$ and $y' \in \partial B$ with the property that

$$d(y,y') = d\left(y,\partial B\right).$$

For this pair of points, (4.16) yields

$$\frac{\rho}{d(y,a)^2} = \frac{d(T(y),T(y'))\, d(a,y')}{d(a,y)\, d(y,\partial B)} \geq \frac{d(T(y),T(\partial TB))\, d(a,y')}{d(a,y)\, d(y,\partial B)}.$$

Thus it suffices to show that $d(a,y') \gtrsim_C d(a,y)$. To this end, we use the assumptions stated in (4.14), which yield, since $C > 1$ and $d(y,\partial B) \leq d(y,\omega) < r$, that

$$d(a,y') \geq d(a,y) - d(y,y') = d(a,y) - d(y,\partial B) \overset{(4.14)}{\geq} (C-1)r,$$

and

$$d(a,y) \leq d(a,\omega) + d(\omega,y) \overset{(4.14)}{\lesssim} r.$$

Combining these estimates, we deduce that $d(a,y') \gtrsim_C d(a,y)$ and conclude the proof. $\square$

Lemma 4.17. *Assume that $\omega \in \partial B$, $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$, $\rho > 0$ and $r > 0$ satisfy the conditions in Proposition 4.13, and additionally, $\rho \sim r$. Let $f : B \to f(B) \subset \mathbb{H}^1 \setminus \{0\}$ be a K-quasiconformal map, and let $T = T_{x,a,\rho}$ be the 1-quasiconformal map defined in Proposition 4.2. Then*

$$g|_{T(B(\omega,r)\cap B)} := f \circ T^{-1}|_{T(B(\omega,r)\cap B)}$$

satisfies

$$\frac{|D_H f(y)|^p}{\|f(y)\|^p}\, d(y,\partial B)^{p-1} \sim_{p,K} r^3 \frac{|D_H g(T(y))|^p}{\|g(T(y))\|^p}\, d(T(y),\partial T(B))^{p-1}\, J_T(y) \tag{4.18}$$

for $0 < p < 4$ and almost every $y \in B(\omega,r) \cap B(0,1)$.

Proof. The maps f and g are K-quasiconformal on $B(\omega,r)\cap B$ and $T(B(\omega,r)\cap B)$, respectively. Hence their Pansu derivatives exist and agree with the formal Pansu derivatives almost everywhere. Since T is conformal and T-preimages of null sets are null sets, the chain rule [56, Proposition 3.2.5] and elementary linear algebra imply that the operator norms satisfy

$$|D_H f| \sim_K |D_H g(T(\cdot))|\, J_T^{1/4}, \quad \text{almost everywhere.} \tag{4.19}$$

By Proposition 4.2, the assumptions $\rho \sim r \sim d(a,y)$ for $y \in B(\omega,r)\cap B$, and by Proposition 4.13, we have

$$J_T(y)^{1/4} \overset{(4.7)}{\sim} \frac{\rho}{d(y,a)^2} \overset{(4.15)}{\sim} \frac{d(T(y),\partial T(B))}{d(y,\partial B)} \sim \frac{1}{r}, \quad \text{for all } y \in B(\omega,r)\cap B. \tag{4.20}$$

Therefore we can proceed as follows:

$$\begin{aligned} |D_H f|^p &\overset{(4.19)}{\sim}_{p,K} |D_H g(T(\cdot))|^p\, J_T^{p/4} \sim |D_H g(T(\cdot))|^p\, J_T^{(p-1)/4} J_T^{-3/4} J_T \\ &\overset{(4.20)}{\sim} |D_H g(T(\cdot))|^p \left(\frac{d(T(\cdot),\partial T(B))}{d(\cdot,\partial B)}\right)^{p-1} r^3\, J_T \end{aligned}$$

almost everywhere on $B(\omega,r)\cap B$. This yields (4.18). $\square$

Remark 4.21. Given $\omega \in \partial B$ and $0 < r \lesssim 1$, it is possible to choose $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$, and $\rho > 0$ such that all the assumptions of the results stated in this section are satisfied. To see this, we use the fact that B satisfies interior and exterior corkscrew conditions [11] (or apply Proposition 2.15 to find interior corkscrew points). This means that there exist constants $M_0 \geq 1$ and $r_0 > 0$ such that the following holds:

- For every $\omega \in \partial B$ and $0 < r < r_0$, there exists $A_{i,r}(\omega) \in B$ such that
$$\frac{r}{M_0} \leq d(A_{i,r}(\omega), \partial B) \leq d(A_{i,r}(\omega), \omega) \leq r.$$
- For every $\omega \in \partial B$ and $0 < r < r_0$, there exists $A_{o,r}(\omega) \in \mathbb{H}^1 \setminus \overline{B}$ such that
$$\frac{r}{M_0} \leq d(A_{o,r}(\omega), \partial B) \leq d(A_{o,r}(\omega), \omega) \leq r.$$

We consider $r < r_0/(M_0 N) =: r_*$ for a fixed constant $N > 1$ to be determined. We claim that N can be chosen such that

$$\rho := r, \quad x := A_{i,r}(\omega) \in B, \quad a := A_{o,M_0 N r}(\omega) \in \mathbb{H}^1 \setminus \overline{B}$$

have the desired properties. Indeed, we have

(1) $d(x, \partial B) \geq r/M_0$,
(2) $r/M_0 \leq d(x, \omega) \leq r$,
(3) $Nr \leq d(a, \partial B) \leq d(a, \omega) \leq M_0 N r$,
(4) $(N-1)r \leq d(a, \omega) - d(\omega, x) \leq d(x, a) \leq d(x, \omega) + d(\omega, a) \leq (1 + M_0 N)r$.

Finally, arguing as in the proof of Corollary 4.9, we find a universal constant $\eta < 1$ such that

$$d(a, q) \geq \eta d(\partial B, a) \geq \eta N r =: Cr, \quad q \in B.$$

Thus we choose $N > 1$ large enough, depending on η, such that $C := \eta N > 1$.

Remark 4.22. The maps $T_{x,a,\rho}$ are in general *not* self-maps of B. For a simple example, consider $x = (x_1, 0, 0)$ and $a = (a_1, 0, 0)$ with $0 < x_1 < 1 < a_1$. Then $T = T_{x,a,\rho}$ does not map B onto itself for any choice of $\rho > 0$. This can be seen by computing $\|T(y)\|$ for $y \in \{(1,0,0), (-1,0,0), (0, 2^{-\frac{1}{4}}, 2^{-\frac{1}{2}})\}$. This is in contrast to the situation in $\mathbb{R}^n$, where the relevant maps preserve the unit ball thanks to the identity [4, (33)].

4.2. Carleson measures related to quasiconformal maps

Carleson characterized in [13, Theorem 1] the measures μ on the unit disk $\mathbb{D}$ in $\mathbb{C}$ for which $\|f\|_{L^p(\mu)} \leq C(\mu)\|f\|_{H^p}$ holds for all holomorphic functions on $\mathbb{D}$ that belong to the

Hardy space H^p, $p \geq 1$. These are measures μ with the property that $\mu(\mathbb{D} \cap B(\omega, r)) \lesssim r$ for $\omega \in \partial\mathbb{D}$ and $r > 0$. Such measures, and various generalizations thereof, are now known as *Carleson measures*. Carleson's result was reproved by Hörmander [43] and, in extended form, by Duren [21]. Later Nolder [63], and Astala and Koskela [5, 4.5. Corollary], generalized Carleson's and Duren's result to Hardy spaces of quasiconformal mappings on the unit ball in $\mathbb{R}^n$. We will prove an analogous result for $\mathbb{H}^1$ in Section 6. This motivates the following definition (cf. Definition 6.1 for metric spaces).

Definition 4.23. Let $1 \leq \alpha < \infty$. We say that a (positive) Borel measure μ on B is an *α-Carleson measure on the Korányi unit ball B*, if there exists a constant $C > 0$ such that

$$\mu(B \cap B(\omega, r)) \leq Cr^{3\alpha}, \quad \text{for all } \omega \in \partial B \text{ and } r > 0. \tag{4.24}$$

The *α-Carleson measure constant* of μ is defined by

$$\gamma_\alpha(\mu) := \inf\{C > 0 \text{ such that (4.24) holds for all } \omega \in \partial B \text{ and } r > 0\}$$

We also call 1-Carleson measures simply *Carleson measures.*

The following is an extension to $\mathbb{H}^1$ of a result originally due to Jones [45]. Our argument is inspired by a proof of Jones' result that was given later by Astala and Koskela in [5, Lemma 5.6], but we avoid the use of Möbius self-maps of B.

Proposition 4.25. *Fix $0 < p < 4$ and let f be a quasiconformal map on $B \subset \mathbb{H}^1$ with $f(q) \neq 0$ for all $q \in B$. Then the following measure μ is a Carleson measure in B, with an upper bound for the Carleson measure constant that depends only on K and p:*

$$d\mu = \frac{|D_H f(q)|^p}{\|f(q)\|^p} d(q, \partial B)^{p-1} dq. \tag{4.26}$$

Proof. Let f be an arbitrary map satisfying the assumptions of the proposition, and fix $0 < p < 4$. We aim to prove that the associated measure μ in (4.26) is a Carleson measure. In the course of the proof below, we will show that $\mu(B(0, 1))$ can be bounded by a finite constant depending only on p and K, see (4.35). Taking this for granted, it suffices to verify the Carleson condition for small $r > 0$, say $r < r_*$, where r_* is the absolute constant in Remark 4.21 that only depends on the geometry of $\mathbb{H}^1$. Indeed, for $r > r_*$ and $\omega \in \partial B$, we trivially have

$$\mu\left(B \cap B(\omega, r)\right) \leq \mu(B) \lesssim_{K,p} (1/r_*)^3 r^3.$$

Thus let us fix $\omega \in \partial B$ and $0 < r < r_*$. Then choose $x \in B$, $a \in \mathbb{H}^1 \setminus \overline{B}$ and $\rho > 0$, depending on ω and r, as in Remark 4.21, and consider the associated 1-quasiconformal map $T = T_{x,a,\rho}$ defined in Proposition 4.2. Writing $g := f \circ T^{-1}|_{TB}$, Lemma 4.17 yields:

$$\int_{B\cap B(\omega,r)} \frac{|D_H f(q)|^p}{\|f(q)\|^p}\, d(q,\partial B)^{p-1} dq$$
$$\sim_K r^3 \int_{B\cap B(\omega,r)} \frac{|D_H g(T(q))|^p}{\|g(T(q))\|^p}\, d(T(q),\partial T(B))^{p-1} J_T(q) dq. \tag{4.27}$$

By the change of variables $q \mapsto T(q)$ and Hölder's inequality with exponents $4/p$ and $4/(4-p)$, the last line in (4.27) becomes:

$$r^3 \int_{T(B\cap B(\omega,r))} \frac{|D_H g(q))|^p}{\|g(q)\|^p}\, d(q,\partial T(B))^{p-1} dq \tag{4.28}$$
$$\leq r^3 \left(\int_{T(B\cap B(\omega,r))} \frac{|D_H g(q)|^4}{\|g(q)\|^4} d(q,\partial T(B))^{\varepsilon \frac{4}{p}} dq \right)^{\frac{p}{4}}$$
$$\times \left(\int_{T(B\cap B(\omega,r))} d(q,\partial T(B))^{\frac{4(p-1-\varepsilon)}{4-p}} dq \right)^{\frac{4-p}{4}}.$$

Here, $\varepsilon > 0$ is a suitably chosen constant (depending only on p) that makes the last integral converge. To see that such a constant ε exists, recall from Proposition 4.13, formula (4.7) and the choice of parameters according to Remark 4.21, that

$$d(T(q),\partial TB) \sim \frac{1}{r} d(q,\partial B), \quad J_T(q) \sim \frac{1}{r^4}, \quad q \in B(\omega,r)\cap B, \tag{4.29}$$

with implicit constants independent of ω and r. Hence, for any $\eta < 1$, we find

$$\int_{T(B\cap B(\omega,r))} d(q,\partial T(B))^{-\eta}\, dq = \int_{B\cap B(\omega,r)} d(T(q),\partial T(B))^{-\eta} J_T(q)\, dq$$
$$\overset{(4.29)}{\sim} \frac{r^\eta}{r^4} \int_{B\cap B(\omega,r)} d(q,\partial B)^{-\eta}\, dq$$
$$\sim \frac{r^\eta}{r^4} \sum_{j=0}^{\infty} r^{-\eta} 2^{\eta j} |\{q \in B \cap B(\omega,r) :\, r2^{-j-1} \leq d(q,\partial B) < r2^{-j}\}|$$
$$\lesssim \frac{1}{r^4} \sum_{j=0}^{\infty} 2^{\eta j} |\{q \in B\cap B(\omega,r) :\, d(q,\partial B) < r2^{-j}\}|. \tag{4.30}$$

It remains to find a good upper bound for the Lebesgue measure of the sets

$$A_j(\omega,r) := \{q \in B\cap B(\omega,r) :\, d(q,\partial B) < r2^{-j}\}, \quad j \in \mathbb{N}_0.$$

First, we observe that

$$A_j(\omega, r) \subset \bigcup_{\widetilde{\omega}\in B(\omega,2r)\cap\partial B} B(\widetilde{\omega}, r2^{-j}). \tag{4.31}$$

Indeed, if $q \in A_j(\omega, r)$, then $d(q, \omega_q) < r2^{-j}$ for some $\omega_q \in \partial B$ with $d(q, \omega_q) = d(q, \partial B)$. Since

$$d(\omega_q, \omega) \leq d(\omega_q, q) + d(q, \omega) < r2^{-j} + r,$$

we see that $\omega_q \in B(\omega, 2r) \cap \partial B$, so (4.31) holds. Then we apply the $5r$-covering lemma to find a disjoint subfamily

$$\{B(\widetilde{\omega}_i, r2^{-j}) : i = 1, 2, \dots, I_j(\omega, r)\}$$

such that the 5-times enlarged balls still cover $A_j(\omega, r)$. To conclude the argument, we control the cardinality $I_j(\omega, r)$ from above by observing that

$$I_j(\omega, r)\, r^3 2^{-3j} \lesssim \mathcal{S}^3 \left(\bigcup_{i=1}^{I_j(\omega,r)} B(\widetilde{\omega}_i, r2^{-j}) \cap \partial B \right) \leq \mathcal{S}^3(B(\omega, 3r) \cap \partial B) \lesssim r^3$$

by the 3-regularity of $\mathcal{S}^3|_{\partial B}$, recall Lemma 2.11. Thus $I_j(\omega, r) \lesssim 2^{3j}$, and hence

$$|A_j(\omega, r)| \lesssim I_j(\omega, r)\, r^4 2^{-4j} \lesssim r^4 2^{-j}. \tag{4.32}$$

Inserting this estimate in (4.30), we conclude that for every $\eta < 1$ (positive or negative), it holds

$$\int_{T(B\cap B(\omega,r))} d(q, \partial T(B))^{-\eta}\, dq \lesssim \sum_{j=0}^{\infty} 2^{(\eta-1)j} \lesssim_\eta 1.$$

Applying these considerations for $\eta = 4(\varepsilon + 1 - p)/(4 - p)$, we see that if we choose $\varepsilon = \varepsilon(p) < \frac{3}{4}p$, the second integral in (4.28) can be bounded from above by a finite constant depending only on p.

Combining the above estimates, we have so far found that

$$\begin{aligned} \int_{B\cap B(\omega,r)} \frac{|D_H f(q)|^p}{\|f(q)\|^p} d(q, \partial B)^{p-1} dq &\lesssim_{K,p} r^3 \left(\int_{T(B\cap B(\omega,r))} \frac{|D_H g(q)|^4}{\|g(q)\|^4} d(q, \partial T(B))^{\varepsilon \frac{4}{p}} dq \right)^{\frac{p}{4}} \\ &\lesssim_{K,p} r^3 \left(\int_{T(B)} \frac{|D_H g(q)|^4}{\|g(q)\|^4} d(q, \partial T(B))^{\varepsilon \frac{4}{p}} dq \right)^{\frac{p}{4}}. \end{aligned}$$

In order to estimate the last integral, we proceed as follows, using the fact that g is quasiconformal, with constant K, and the change-of-variables formula holds,

$$\begin{aligned}\int\limits_{T(B)} \frac{|D_H g(q)|^4}{\|g(q)\|^4} d(q,\partial T(B))^{\varepsilon\frac{4}{p}} dq &\lesssim_K \int\limits_{T(B)} \frac{d(q,\partial T(B))^{\varepsilon\frac{4}{p}}}{\|g(q)\|^4} J_g(q) dq \\ &\sim_K \int\limits_{g(T(B))} \frac{d(g^{-1}(q),\partial T(B))^{\varepsilon\frac{4}{p}}}{\|q\|^4} dq \\ &\sim_K I_1 + I_2, \end{aligned} \tag{4.33}$$

where

$$I_1 := \int\limits_{g(T(B))\cap\{q:\|q\|<\|g(0)\|\}} \frac{d(g^{-1}(q),\partial T(B))^{\varepsilon\frac{4}{p}}}{\|q\|^4} dq$$

and

$$I_2 := \int\limits_{g(T(B)\cap\{q:\|q\|\geq\|g(0)\|\}} \frac{d(g^{-1}(q),\partial T(B))^{\varepsilon\frac{4}{p}}}{\|q\|^4} dq.$$

The integrals I_1 and I_2 can be bounded from above using the first and second inequality in Proposition 3.9, respectively:

$$I_1 \lesssim_{p,K} \int\limits_{\|q\|<\|g(0)\|} \|q\|^{-4} \frac{\|q\|^{\frac{\varepsilon 4}{p\alpha}}}{\|g(0)\|^{\frac{\varepsilon 4}{p\alpha}}} dq \lesssim_{p,K} \frac{1}{\|g(0)\|^{\frac{\varepsilon 4}{p\alpha}}} \int\limits_0^{\|g(0)\|} s^{\frac{\varepsilon 4}{p\alpha}-1} ds \sim_{p,K} 1$$

and

$$I_2 \lesssim_{p,K} \int\limits_{\|q\|\geq\|g(0)\|} \|q\|^{-4} \frac{\|g(0)\|^{\frac{\varepsilon 4}{p\alpha}}}{\|q\|^{\frac{\varepsilon 4}{p\alpha}}} dq \lesssim_{p,K} \|g(0)\|^{\frac{\varepsilon 4}{p\alpha}} \int\limits_{\|g(0)\|}^{\infty} s^{-\frac{\varepsilon 4}{p\alpha}-1} ds \sim_{p,K} 1.$$

In conclusion, we have shown that

$$\int\limits_{B\cap B(\omega,r)} \frac{|D_H f(q)|^p}{\|f(q)\|^p} d(q,\partial B)^{p-1} dq \lesssim_{K,p} r^3, \quad \text{for all } \omega\in\partial B,\ 0<r<r_*. \tag{4.34}$$

Finally, the argument by Hölder's inequality and change-of-variables $q \mapsto g(q)$ that we applied above to "g" and "$T(B)$" works also for "f" and "B" to show that

$$\int_B \frac{|D_H f(q)|^p}{\|f(q)\|^p} d(q, \partial B)^{p-1} dq \lesssim_{K,p} 1. \tag{4.35}$$

More precisely, we have

$$\begin{aligned}&\int_B \frac{|D_H f(q))|^p}{\|f(q)\|^p} d(q, \partial B))^{p-1} dq \\ &\leq \left(\int_B \frac{|D_H f(q)|^4}{\|f(q)\|^4} d(q, \partial B)^{\varepsilon \frac{4}{p}} dq\right)^{\frac{p}{4}} \left(\int_B d(q, \partial B)^{\frac{4(p-1-\varepsilon)}{4-p}} dq\right)^{\frac{4-p}{4}},\end{aligned}$$

where ε is as before. The first integral can be bounded by a finite constant depending only on K and p by the same argument as we used for (4.33), with "g" and "$T(B)$" replaced by "f" and "B" The second integral is finite constant depending only on p, thanks to (4.32) (for $r \sim 1$). As remarked at the beginning of the proof, (4.34) and (4.35) together suffice to show the Carleson measure condition for all scales $r > 0$. □

The following corollary of Proposition 4.25 extends [45, Lemma 4.2] to $\mathbb{H}^1$.

Corollary 4.36. *If f is a quasiconformal map on $B \subset \mathbb{H}^1$ with $f(q) \neq 0$ for all $q \in B$, then*

$$|\nabla_H \log \|f(q)\|| \; dq$$

defines a Carleson measure on B, with an upper bound for the Carleson measure constant depending only on K.

Proof. Since for Lebesgue almost every $q \in B$, we have

$$|\nabla_H \log \|f\|(q)| = \frac{|\nabla_H(\|\cdot\| \circ f)(q)|}{\|f(q)\|} \lesssim \frac{|D_H f(q)|}{\|f(q)\|}$$

this is an immediate consequence of Proposition 4.25 for $p = 1$. The above inequality can be verified by a direct computation using the contact equations for f, or observing that $|\nabla_H \|\cdot\|| \leq 1$ and using the chain rule for Pansu derivatives. □

5. Characterization of the H^p property for quasiconformal mappings

The purpose of this section is to prove Theorem 1.4, which characterizes membership in H^p for a quasiconformal mapping in terms of its radial limit and nontangential maximal function, respectively. This is motivated by a result for quasiconformal mappings on the Euclidean unit ball in $\mathbb{R}^n$, $n \geq 2$, originally due to Zinsmeister [71], and later obtained with different methods by Astala and Koskela in [5, Theorem 4.1]. Our proof combines elements from both approaches with arguments tailored to the Heisenberg geometry.

5.1. Conditions implying p-integrability of the radial limit

By Lemma 2.23 and (2.13), the radial limit f^* of a quasiconformal map f on the Korányi unit ball B exists for $\mathcal{S}^3$-almost every boundary point, and it is a Borel function. We will use this fact throughout the section to prove the straightforward implications (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (3), and (2) $\Rightarrow$ (1) in Theorem 1.4.

Lemma 5.1. *Fix $0 < p < \infty$ and $f : B \subset \mathbb{H}^1 \to f(B) \subset \mathbb{H}^1$ be a quasiconformal map. Then*

$$\int_{\partial B} \|f^*\|^p \, d\mathcal{S}^3 \leq \|f\|_{H^p}^p.$$

In particular, $f \in H^p$ implies $\|f^*\| \in L^p(\mathcal{S}^3|_{\partial B})$.

Proof. Since f is quasiconformal and f^* exists $\mathcal{S}^3|_{\partial B}$ almost everywhere on ∂B, the claim follows by Fatou's lemma:

$$\begin{aligned}\int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega) &= \int_{\partial B} \|\lim_{r\nearrow 1} f(\gamma(r,\omega))\|^p \, d\mathcal{S}^3(\omega) \\ &\leq \liminf_{r\nearrow 1} \int_{\partial B} \|f(\gamma(r,\omega))\|^p \, d\mathcal{S}^3(\omega) \overset{\text{Def. 1.2}}{\leq} \|f\|_{H^p}^p. \quad \square\end{aligned}$$

In what follows we will frequently refer to Proposition 2.15. It asserts that there exists $\kappa > 0$ such that for every $\omega \in \partial B \setminus \{z = 0\}$, we have

$$\gamma(s,\omega) \in \Gamma(\omega) := \Gamma_\kappa(\omega), \quad \text{for all } s \in (0,1),$$

see the Appendix for the proof. The nontangential maximal function $M(f)$ is defined with respect to that parameter κ. We also recall from Remark 2.19 that $M(f)$ is Borel measurable. These observations immediately yield the next two propositions.

Proposition 5.2. *Let $f : B \subset \mathbb{H}^1 \to f(B) \subset \mathbb{H}^1$ be a quasiconformal map. Then*

$$\|f^*(\omega)\| \leq M(f)(\omega), \quad \mathcal{S}^3 \text{ a.e. } \omega \in \partial B.$$

In particular, for $0 < p < \infty$, the condition $M(f) \in L^p(\mathcal{S}^3|_{\partial B})$ implies that $\|f^*\| \in L^p(\mathcal{S}^3|_{\partial B})$.

Proposition 5.3. *Let $0 < p < \infty$ and let $f : B \to f(B) \subseteq \mathbb{H}^1$ be a quasiconformal map. Then*

$$\|f\|_{H^p}^p \leq \int_{\partial B} M(f)^p \, d\mathcal{S}^3.$$

In particular, $M(f) \in L^p\left(\mathcal{S}^3|_{\partial B}\right)$ implies that $f \in H^p$.

5.2. Conditions implied by the p-integrability of the radial limit

In this section, we prove the main implication in Theorem 1.4, namely (3) ⇒ (2):

Proposition 5.4. *Assume that $K \geq 1$, $0 < p < \infty$, and that $f : B \to f(B) \subset \mathbb{H}^1$ is K-quasiconformal. Then*

$$\|M(f)\|_{L^p(\mathcal{S}^3|_{\partial B})} \leq C(K,p)\big\|\|f^*\|\big\|_{L^p(\mathcal{S}^3|_{\partial B})}$$

for a constant $C(K,p)$ that depends only on K and p.

Proposition 5.4 is a counterpart for Zinsmeister's result [71, Proposition 1] in Euclidean spaces, which was re-proven with a different argument by Astala and Koskela in [5, Corollary 4.3 and Theorem 4.1]. We establish our result by combining the use of a specific Carleson measure inspired by [71] with a Hardy-Littlewood maximal function argument as in [5]. The core of the proof is then to relate the nontangential maximal function of f and the Hardy-Littlewood maximal function of its radial limit. This is achieved with Lemma 5.6, where the geometry of the Heisenberg group enters the picture due to the nonisotropic nature of the Korányi ball. In contrast to [71], we do not involve $\log\|f\|$ in our discussion, although Corollary 4.36 shows that for a quasiconformal f on $B \subset \mathbb{H}^1$, omitting the origin, $|\nabla_H \log\|f\||$ always defines a Carleson measure on B. In [71], an analogous result is used to observe that if a Sobolev mapping f is such that $f \neq 0$ in B and $|\nabla \log\|f\||$ defines a Carleson measure, then the radial limit f^* exists (see [71, pg. 128]), whereas for us f is quasiconformal and f^* exists by Lemma 2.23. The Carleson measure induced by $|\nabla \log\|f\||$ appears in [71] also in a more subtle way, through condition [71, (10)]. There is a similar element in our argument, where we apply the Carleson measure defined by $|D_H f|/\|f\|$ to establish Lemma 5.11, but our proof necessarily looks different due to the presence of characteristic points in the Korányi sphere and the rigidity of Möbius self-maps of the Korányi ball. We now turn to the details.

For a Borel function $h : \partial B \to [0,+\infty]$, we define the *non-centered Hardy-Littlewood maximal function*

$$\mathcal{M}_{\partial B} h(\omega) := \sup_{B(\omega',r)\ni\omega} \frac{1}{\mathcal{S}^3(B(\omega',r)\cap\partial B)} \int_{B(\omega',r)\cap\partial B} h\, d\mathcal{S}^3, \quad \text{for all } \omega \in \partial B.$$

Proposition 5.4 will be established through a series of intermediate results, but the core of the argument is the chain of inequalities

$$\int_{\partial B} M(f)^p\, d\mathcal{S}^3 = \int_{\partial B} \left(\sup_{\Gamma(\cdot)} \|f\|^q\right)^{\frac{p}{q}} d\mathcal{S}^3 \overset{\text{Lem. 5.6}}{\lesssim} \int_{\partial B} (\mathcal{M}_{\partial B}(\|f^*\|^q))^{\frac{p}{q}}\, d\mathcal{S}^3 \lesssim \int_{\partial B} \|f^*\|^p\, d\mathcal{S}^3, \tag{5.5}$$

for $0 < q < p$ and every quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$, with implicit constants depending on p, q, and K. Exactly as in the proof of [5, Theorem 4.1], the last inequality holds since the operator $\mathcal{M}_{\partial B}$ is of strong type (s, s) for all $s > 1$, see e.g., [15,37] and recall that $(\partial B, d|_{\partial B}, \mathcal{S}^3|_{\partial B})$ is a doubling metric measure space. By fixing a suitable constant q, depending on p, Proposition 5.4 follows from (5.5), so we concentrate on Lemma 5.6.

Lemma 5.6. *For $K \geq 1$ and $0 < q < \infty$, there exists a constant C such that for every quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$, its radial limit f^* satisfies*

$$\sup_{\Gamma(\omega)} \|f\|^q \leq C \mathcal{M}_{\partial B}(\|f^*\|^q)(\omega), \quad \textit{for all } \omega \in \partial B.$$

Lemma 5.6 follows from the subsequent result:

Lemma 5.7. *Let $0 < q < \infty$ and $K \geq 1$. Then there exists a constant C, depending on q and K, such that for every quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$, we have*

$$\|f(x)\|^q \leq C \frac{1}{\mathcal{S}^3(S(x))} \int\limits_{S(x)} \|f^*(\omega)\|^q \, d\mathcal{S}^3(\omega), \quad \textit{for all } x \in B. \tag{5.8}$$

Here $S(x) = B\,(x, (1+\kappa) d(x, \partial B)) \cap \partial B$ with κ given by Proposition 2.15.

Proof of Lemma 5.6 using Lemma 5.7. Under the assumptions of Lemma 5.7, we have for every $\omega_0 \in \partial B$ that

$$\sup_{x \in \Gamma(\omega_0)} \|f(x)\|^q \leq C \sup_{x \in \Gamma(\omega_0)} \frac{1}{\mathcal{S}^3(S(x))} \int\limits_{S(x)} \|f^*(\omega)\|^q \, d\mathcal{S}^3(\omega). \tag{5.9}$$

Now for every $x \in B$, there exists $\omega_x \in \partial B$ such that $d(x, \omega_x) = d(x, \partial B)$ and it is easy to see that then

$$B\,(\omega_x, \kappa\, d(x, \partial B)) \cap \partial B \subset S(x) \subset B\,(\omega_x, (2+\kappa)\, d(x, \partial B)) \cap \partial B. \tag{5.10}$$

Using the 3-regularity of $\mathcal{S}^3|_{\partial B}$, we can deduce from (5.9) that

$$\begin{aligned} &\sup_{x \in \Gamma(\omega_0)} \|f(x)\|^q \\ &\leq C \sup_{x \in \Gamma(\omega_0)} \frac{1}{\mathcal{S}^3(B\,(\omega_x, (2+\kappa)\, d(x, \partial B)) \cap \partial B)} \int\limits_{B(\omega_x, (2+\kappa)\, d(x, \partial B)) \cap \partial B} \|f^*(\omega)\|^q \, d\mathcal{S}^3(\omega), \end{aligned}$$

where C now also depends on κ and the 3-regularity constant of $\mathcal{S}^3|_{\partial B}$, which we consider as universal constants. Since $\omega_0 \in B(\omega_x, (2+\kappa) d(x, \partial B) \cap \partial B)$ if $x \in \Gamma(\omega_0)$, the right-hand side of the above inequality can be bounded from above by $\mathcal{M}_{\partial B}\|f^*\|^q(\omega_0)$ on ∂B as claimed. $\square$

Thus Proposition 5.4 will follow if we manage to prove Lemma 5.7.

5.2.1. Points and shadows: proof of Lemma 5.7

The main ingredient for Lemma 5.7 is a statement for a quasiconformal map f on B, saying that if $0 \notin f(B)$, then $\|f^*(\omega)\|$ cannot be too small compared to $\|f(x)\|$ for too many $\omega \in S(x)$.

Lemma 5.11. *For every $K \geq 1$, there exists $N(K)$ and a function Ψ_K with $\lim_{N\to\infty} \Psi_K(N) = 0$ such that the following holds. If $f : B \to f(B) \subset \mathbb{H}^1 \setminus \{0\}$ is K-quasiconformal, then, for all $x \in B$ and all $N \geq N(K)$, we have*

$$\mathcal{S}^3\left(\left\{\omega \in S(x) : \|f^*(\omega)\| \leq \frac{\|f(x)\|}{N}\right\}\right) \leq \Psi_K(N)\mathcal{S}^3(S(x)).$$

This lemma is easy to prove by standard modulus techniques if x is deep inside B and the shadow $S(x)$ is large, see Lemma 5.16 for the case $x = 0$. In Euclidean spaces, the general case can be reduced to this one by a suitable Möbius self-map of the unit ball that sends a small spherical cap to a large one in a canonical, metrically controlled way, see the proof of [5, Lemma 4.2]. Möbius automorphisms of the Korányi unit ball B are not flexible enough for this approach. In our setting, we therefore give a separate proof of the statement in Lemma 5.11 for x close to ∂B with the help of a Carleson measure provided by Proposition 4.25. To do so, we again make use of the growth estimate in Proposition 3.9 and the specific Möbius transformations introduced in Section 4.1. Yet this time we start from a point $x \in B$ (close to ∂B), and assign to it the data

$$\omega_x \in \partial B, \quad a_x \in \mathbb{H}^1 \setminus \overline{B}, \quad \rho_x > 0$$

and associated Möbius transformation

$$T_x := T_{x,a_x,\rho_x} \tag{5.12}$$

as described in the following. Without loss of generality, we may assume that the constant r_* in Remark 4.21 satisfies $r_* < 1$. Then there exists $R_* \in [1 - r_*, 1)$ such that

$$d(y, \partial B) < r_* \quad \text{for all } \|y\| > R_*.$$

Now, for $x \in B$ with $\|x\| > R_*$, we define:

(d1) $\rho_x := d(x, \partial B)$,
(d2) ω_x a point in ∂B such that $d(x, \omega_x) = d(x, \partial B)$,

(d3) $a_x := A_{0,M_0 N\rho_x}(\omega_x)$ (outer corkscrew point with "N" as in Remark 4.21).

To prove Lemma 5.11 for $\|x\| > R_*$, we will study the behavior of $f \circ T_x^{-1}$ on

$$T_x\left(\left[\bigcup_{\omega\in S(x)} B(\omega,\rho_x)\right]\cap B\right),$$

recalling that $T_x(x) = 0$. The precise statement, given in Lemma 5.15, therefore requires us to control the time for which the radial segment $\gamma(\cdot,\omega)$ stays inside $B(\omega,\rho_x)$.

Lemma and Definition 5.13. *Let κ be as in Proposition 2.15. For all $\omega \in \partial B \setminus \{z=0\}$ and $\rho \in (0,1)$, there exists $s_{\omega,\rho} \in (0,1)$ such that*

(1) $\gamma(s,\omega) \in B(\omega,\rho) \cap \Gamma_\kappa(\omega)$ *for all* $s \in [s_{\omega,\rho}, 1)$,
(2) $\gamma(s_{\omega,\rho},\omega) \in \partial B(\omega,\rho)$,
(3) $s_{\omega,\rho} \geq 1-\rho$.

Moreover, the choice $\omega \mapsto s_{\omega,\rho}$ can be made Borel measurable on $\partial B \setminus \{z=0\}$. If $x \in B \setminus \{0\}$ and $\rho = \rho_x := d(x,\partial B)$, we denote $s_{\omega,x} := s_{\omega,\rho_x}$.

Proof. Let $\omega \in \partial B \setminus \{0\}$. By Proposition 2.15 we know that $\gamma(s,\omega) \in \Gamma_\kappa(\omega)$ for all $s \in (0,1)$. Since the radial curves are continuous with

$$d(\gamma(0,\omega),\omega) = 1 \quad \text{and} \quad d(\gamma(1,\omega),\omega) = 0,$$

we can take $s_{\omega,\rho}$ to be the largest number in $(0,1)$ such that

$$\gamma(s_{\omega,\rho},\omega) \in \partial B(\omega,\rho).$$

By maximality, this satisfies also

$$\gamma(s,\omega) \in B(\omega,\rho), \quad \text{for all } s \in [s_{\omega,\rho},1).$$

Moreover, it is clear that

$$s_{\omega,\rho} = \|\gamma(s_{\omega,\rho},\omega)\| \geq \|\omega\| - d(\gamma(s_{\omega,\rho},\omega),\omega) \geq 1-\rho.$$

To prove the Borel measurability of $\omega \mapsto s_{\omega,\rho}$, we use that the function $(s,\omega) \mapsto d(\gamma(s,\omega),\omega)$ is continuous on $(0,1) \times (\partial B \setminus \{z=0\})$ and it extends to a continuous function $h : [0,1] \times \partial B \to [0,1]$, cf. (A.3). Then, for a given $0 < \rho < 1$, the function

$$\omega \mapsto s(\omega) := \max\{s \in [0,1] : h(s,\omega) = \rho\}$$

is upper semicontinuous. Indeed, for any $\lambda \leq 1$, if $s(\omega) < \lambda$, then $h(s,\omega) < \rho$ for all $s \in [\lambda, 1]$ while $h(s(\omega),\omega) = \rho$, and thus there exists a small relatively open neighborhood U of ω in ∂B such that for all $\omega' \in U$ and $s \geq \lambda$, we also have $h(s,\omega') < \rho$, and hence $s(\omega') < \lambda$. This shows that $\omega \mapsto s(\omega)$ is upper semicontinuous as claimed, and in particular,

$$\omega \mapsto s_{\omega,\rho} := \max\{s \in (0,1) : \, d(\gamma(s,\omega),\omega) = \rho\}$$

is a Borel function. $\square$

If a point $x \in B$ is close to ∂B, then so is $\gamma(s_{\omega,x},\omega)$. The map T_x allows us to normalize the situation in such a way that we obtain a point at a uniformly bounded distance away from the boundary of the new domain, independently of the choice of x and $\omega \in S(x)$.

Lemma 5.14 *(Normalization). There exists a constant $c > 0$ such that, for all $x \in B$ with $\|x\| > R_*$ and all $\omega \in S(x) \setminus \{z = 0\}$, we have*

$$d\left(T_x(\gamma(s_{\omega,x},\omega)), \partial T_x B\right) \geq c.$$

Proof. Let $x \in B$ and $\omega \in S(x)$ be as assumed in the lemma. By definition of T_x, it holds for every $\widetilde{\omega} \in \partial B$ that

$$d\left(T_x(\gamma(s_{\omega,x},\omega)), T_x(\widetilde{\omega})\right) = \rho_x \frac{d(\gamma(s_{\omega,x},\omega),\widetilde{\omega})}{d(\gamma(s_{\omega,x},\omega),a_x)\, d(\widetilde{\omega},a_x)}$$

Since $\omega \in S(x)$, we can apply Lemma 5.13 (2) and Remark 4.21 to deduce that

$$d\left(T_x(\gamma(s_{\omega,x},\omega)), T_x(\widetilde{\omega})\right) \gtrsim \rho_x \frac{d(\gamma(s_{\omega,x},\omega),\widetilde{\omega})}{\rho_x \left[d(\gamma(s_{\omega,x},\omega),\widetilde{\omega}) + d(\gamma(s_{\omega,x},\omega),a_x)\right]} \gtrsim \frac{\rho_x}{\rho_x} = 1.$$

In the last step we also used that

$$d(\gamma(s_{\omega,x},\omega),a_x) \lesssim \rho_x \simeq d(\gamma(s_{\omega,x},\omega),\partial B) \lesssim d(\gamma(s_{\omega,x},\omega),\widetilde{\omega}). \quad \square$$

Lemma 5.13 and Lemma 5.14 are useful to deduce consequences of a small radial limit:

Lemma 5.15. *There exists a constant $C > 0$ and, for every $K \geq 1$, a number $N(K) > 1$ such that the following holds. Whenever $x \in B$ is such that $\|x\| > R_*$ and $f : B \to f(B) \subset \mathbb{H}^1 \setminus \{0\}$ is K-quasiconformal, then for all $N \geq N(K)$ and $\mathcal{S}^3$ almost every $\omega \in S(x)$, we have*

$$\|f^*(\omega)\| \leq \frac{\|f(x)\|}{N} \quad \Rightarrow \quad \int\limits_{s_{\omega,x}}^{1} \frac{|D_H(f \circ T_x^{-1})(T_x(\gamma(s,\omega)))|}{\|f(\gamma(s,\omega))\|} s^3 \, ds \geq C\rho_x |(\omega_1,\omega_2)| \log N$$

where T_x is the Möbius transformation defined in (5.13), and $s_{\omega,x}$ is given by Lemma 5.13.

We postpone the proof to Section 5.2.2. The next result serves as a substitute for Lemma 5.15 in case $x \in B$ is far away from the boundary. This is a Heisenberg version of [5, Lemma 4.2] in the special case when the distinguished point is the origin. This case is particularly simple since

$$S_\kappa(0) = \partial B \cap B(0, (1+\kappa)d(0, \partial B)) = \partial B.$$

Lemma 5.15 and Lemma 5.16 look different at first, but their similarity will become clear latest in Remark 5.22. The integral in Lemma 5.15 is a new element in our proof, but the inspiration for using it came from the proof of Zinsmeister's [71, Lemma 4], where Möbius self-maps of B appear through the definition of $\|u\|_*$ at the bottom of [71, p. 127].

Lemma 5.16. *If $f : B \to f(B) \subset \mathbb{H}^1 \setminus \{0\}$ is K-quasiconformal, then*

$$\mathcal{S}^3 \left(\left\{ \omega \in \partial B : \ \|f^*(\omega)\| \leq \tfrac{\|f(0)\|}{N} \right\} \right) \leq C(K) \, (\ln N)^{-\frac{3}{4}} ,$$

where the constant $C(K)$ depends only on K.

Proof. We first prove a similar inequality for the measure σ on $\partial B \setminus \{0\}$. Namely, for every $K \geq 1$, we show that there exists a constant $C(K)$ such that if $f : B \subset \mathbb{H}^1 \to f(B) \subset \mathbb{H}^1 \setminus \{0\}$ is a K-quasiconformal map, then for all $N > 1$, one has

$$\sigma \left(\{\omega \in \partial B \setminus \{z = 0\} : \ \|f^*(\omega)\| < \|f(0)\|/N\} \right) \leq C(K) \sigma(\partial B \setminus \{z = 0\}) (\ln N)^{-3}. \tag{5.17}$$

As in the proof of [5, Lemma 4.2], we set

$$d(f(0), \partial f(B)) := c.$$

We apply Corollary 3.5 in [1] with $g = f$, $U = B$, $U' = f(B)$, $\beta := 5$ and a ball $B = B(0, r_0)$, where $r_0 = r_0(K)$ is such that $3kB \subset B$ (k depends on K only) to obtain that for all $q \in B(0, r_0)$ we have

$$d(f(0), f(q)) \leq \operatorname{diam} f(B(0, r_0)) \leq d(f(0), \partial f(B(0, r_0))) \leq \frac{c}{2}.$$

Since $c := d(f(0), \partial f(B)) \leq \|f(0)\|$, it holds for any $q' \in f(B(0, r_0))$ that

$$d(q', 0) = \|q'\| \geq \|f(0)\| - d(q', f(0)) \geq \|f(0)\| - \frac{c}{2} \geq \frac{1}{2}\|f(0)\|.$$

Thus $f(B(0, r_0)) \cap B(0, \frac{1}{2}\|f(0)\|) = \emptyset$. Then, for any given $N > 1$ we denote

$$E := E_N := \{\omega \in \partial B \setminus \{z=0\} : \|f^*(\omega)\| < \|f(0)\|/N\},$$

and we define a family of radial curves $\Gamma_E = \Gamma(\partial B(0,r_0), E, B)$, see the discussion following Theorem 2.1. By (2.6) we obtain that $\mathrm{mod}_4(\Gamma_E) = \pi^2 \left(\ln \frac{1}{r_0}\right)^{-3} \sigma(E)$. If $N > 3$, then (radial) curves in $f(\Gamma_E)$ have one endpoint in $\partial f(B(0,r_0))$ and another in $B(0, \|f(0)\|/N)$ and so, in particular, they connect the complement of $B(0,\|f(0)\|/2)$ to $B(0,\|f(0)\|/N)$. Hence, by (2.4),

$$\mathrm{mod}_4(f(\Gamma_E)) \lesssim_K \left(\ln \frac{N}{2}\right)^{-3} \leq C(\ln N)^{-3}.$$

Therefore, $\sigma(E) \lesssim_K \frac{(\ln \frac{1}{r_0})^3}{\pi^2}(\ln N)^{-3}$. If $1 < N \leq 3$, then the estimate is trivial:

$$\sigma(E) \leq \left(\frac{\ln 3}{\ln N}\right)^3 \sigma(\partial B \setminus \{z=0\}).$$

Thus $\sigma(E) \lesssim_K (\ln N)^{-3}$, and the proof of (5.17) is complete.

The statement of Lemma 5.16 can be reduced to this estimate. Since

$$d\sigma = \cos^2\alpha \, d\sigma_0 \quad \text{and} \quad d\mathcal{S}^3 = \sqrt{\cos\alpha}\, d\sigma_0,$$

Hölder's inequality with exponents $p = 4$ and $q = 4/3$ yields

$$\mathcal{S}^3(E) = \int_E \sqrt{\cos\alpha}\, d\sigma_0 \leq \left[\int_E \cos^2\alpha \, d\sigma_0\right]^{\frac{1}{4}} \left[\int_E d\sigma_0\right]^{\frac{3}{4}} = \sigma(E)^{\frac{1}{4}} \sigma_0(E)^{\frac{3}{4}}.$$

Now we bound $\sigma(E)$ from above with the help of (5.17), and thus conclude that

$$\mathcal{S}^3(E) \leq C(K)^{\frac{1}{4}} \sigma(\partial B \setminus \{z=0\})^{\frac{1}{4}} \, (\ln N)^{-\frac{3}{4}} \, \sigma_0(\partial B \setminus \{z=0\})^{\frac{3}{4}}. \quad \square$$

Lemmas 5.15 and 5.16 allow to control the set of points ω in $S(x)$ where $\|f^*(\omega)\|$ is small compared to $\|f(x)\|$, as made precise by Lemma 5.11.

Proof of Lemma 5.11. The proof is split in two cases: $\|x\| \leq R_*$ and $\|x\| > R_*$, where the first case is handled with Lemma 5.16 and Proposition 3.9, while Lemma 5.15 and the Carleson measure $|D_H f(q)|/\|f(q)\|\, dq$ are used to treat the second case.

Let us assume first that $\|x\| \leq R_*$. Since $0 \notin f(B)$, Proposition 3.9 for $\Omega = B$ and $g = f$ implies that there exist constants $C_K, \alpha_K > 1$ such that

$$\frac{\|f(x)\|}{\|f(0)\|} \leq C_K\, d(x, \partial B)^{-\alpha_K} \leq C_K\, (1 - R_*)^{-\alpha_K}.$$

Then Lemma 5.16 with $\Psi(N) = (\ln N)^{-\frac{3}{4}}$ implies that

$$\begin{aligned}
&\mathcal{S}^3\left(\left\{\omega \in S(x) : \|f^*(\omega)\| \leq \frac{\|f(x)\|}{N}\right\}\right) \\
&\quad \leq \mathcal{S}^3\left(\left\{\omega \in \partial B : \|f^*(\omega)\| \leq C_K\,(1-R_*)^{-\alpha_K}\,\frac{\|f(0)\|}{N}\right\}\right) \\
&\quad \leq C(K)\Psi\left(\frac{N}{C_K\,(1-R_*)^{-\alpha_K}}\right) \\
&\quad \leq C(R^*)\,C(K)\,\Psi\left(\frac{N}{C_K\,(1-R_*)^{-\alpha_K}}\right)\mathcal{S}^3(S(x)),
\end{aligned}$$

where we have used in the last inequality that $\mathcal{S}^3(S(x)) \gtrsim_{R_*} 1$ for $\|x\| \leq R_*$. This is the case by the inclusion (5.10), the inequality $d(x, \partial B) \geq 1 - R_*$, and the 3-regularity of $\mathcal{S}^3|_{\partial B}$ stated in Lemma 2.11. Thus the estimate in Lemma 5.11 holds for all $x \in \overline{B(0, R_*)}$ with any function Ψ_K satisfying $\lim_{N\to\infty} \Psi_K(N) = 0$ and

$$\Psi_K(N) \geq C(R^*)\,C(K)\,\Psi\left(\frac{N}{C_K\,(1-R_*)^{-\alpha_K}}\right).$$

In the second part of the proof, we assume that $\|x\| > R_*$. Choosing $N(K)$ as in Lemma 5.15, we find for $N \geq N(K)$ that

$$\begin{aligned}
&\mathcal{S}^3\left(\left\{\omega \in S(x) : \|f^*(\omega)\| < \frac{\|f(x)\|}{N}\right\}\right) \\
&\leq \mathcal{S}^3\left(\left\{\omega \in S(x) : \frac{1}{\rho_x\,C\,\log N\,|(\omega_1,\omega_2)|}\int_{s_{\omega,x}}^{1}\frac{|D_H(f\circ T_x^{-1})(T_x(\gamma(s,\omega)))|}{\|f(\gamma(s,\omega))\|}\,s^3\,ds \geq 1\right\}\right) \\
&\leq \frac{1}{\rho_x\,C\,\log N}\int_{S(x)}\frac{1}{|(\omega_1,\omega_2)|}\int_{s_{\omega,x}}^{1}\frac{|D_H(f\circ T_x^{-1})(T_x(\gamma(s,\omega)))|}{\|f(\gamma(s,\omega))\|}\,s^3\,ds\,d\mathcal{S}^3(\omega) \\
&= \frac{1}{\rho_x\,C\,\log N}\int_0^1\int_{S(x)}\chi_{[s_{\omega,x},1]}(s)\frac{|D_H(f\circ T_x^{-1})(T_x(\gamma(s,\omega)))|}{\|f(\gamma(s,\omega))\|}\,d\sigma_0(\omega)\,s^3 ds,
\end{aligned}$$

where we have used $d\mathcal{S}^3|_{\partial B}(\omega) = |(\omega_1,\omega_2)|d\sigma_0(\omega)$ and the Borel measurability of $\omega \mapsto s_{\omega,x}$, recall Lemma 5.13.

Let us take a closer look at the domain of the double integration. By the choice of $s_{\omega,x}$ in Lemma 5.13, we know that for all $\omega \in \partial B$, we have

$$d(\gamma(s,\omega),\omega) \leq \rho_x, \quad \text{for all } s \in [s_{\omega,x}, 1]. \tag{5.18}$$

If we assume that ω is contained in the spherical cap $S(x)$, then by definition we also have

$$d(\omega, x) \leq (1+\kappa)\rho_x. \tag{5.19}$$

Finally, by definitions (d1)-(d2) below (5.12), we have

$$d(x, \omega_x) = \rho_x. \tag{5.20}$$

Recalling that $S(x) := S_\kappa(x)$, a combination of (5.18), (5.19), and (5.20) shows that

$$\{\gamma(s,\omega) : s \in [s_{\omega,x}, 1], \omega \in S(x)\} \subset B \cap B(\omega_x, (3+\kappa)\rho_x).$$

Thus we can continue with the previous estimate as follows

$$\begin{aligned} &\mathcal{S}^3\left(\left\{\omega \in S(x) : \|f^*(\omega)\| < \frac{\|f(x)\|}{N}\right\}\right) \\ &\quad \leq \frac{1}{\rho_x\, C\, \log N} \int\limits_{B\cap B(\omega_x,(3+\kappa)\rho_x)} \frac{|D_H(f\circ T_x^{-1})(T_x(q))|}{\|f(q)\|}\, dq. \end{aligned}$$

We now derive an upper bound for the operator norm of the horizontal derivative appearing in that integral. The chain rule for Pansu derivatives, the contact and the Lusin property of quasiconformal maps yield for Lebesgue almost every $q \in B$ that

$$D_H(f\circ T_x^{-1})(T_x(q)) = D_H f(q)\, D_H T_x^{-1}(T_x(q)).$$

Then, for almost every $q \in B \cap B(\omega_x, (3+\kappa)\rho_x)$, we find by similar computations as in the proof of Lemma 4.17 that

$$\begin{aligned} |D_H(f\circ T_x^{-1})(T_x(q))| \leq |D_H f(q)|\, |D_H T_x^{-1}(T_x(q))| &= |D_H f(q)|\, J_{T_x}(q)^{-\frac{1}{4}} \\ &= |D_H f(q)|\, \frac{d(a_x, q)^2}{\rho_x} \\ &\lesssim_\kappa |D_H f(q)|\, \rho_x. \end{aligned}$$

Here the first equation holds since T_x^{-1} is 1-quasiconformal and the next equation is due to (4.7), the formula for the Jacobian of T_x. Finally, the last inequality holds because $q \in B \cap B(\omega_x, (3+\kappa)\rho_x)$ and $d(a_x, \omega_x) \lesssim \rho_x$ by the choices we made below (5.12) in (d1)-(d3). Inserting the obtained estimate for $|D_H(f\circ T_x^{-1})(T_x(q))|$ in our chain of inequalities, we find that

$$\mathcal{S}^3\left(\left\{\omega \in S(x) : \|f^*(\omega)\| < \frac{\|f(x)\|}{N}\right\}\right) \leq \frac{1}{C\,\log N} \int\limits_{B\cap B(\omega_x,(3+\kappa)\rho_x)} \frac{|D_H f(q)|}{\|f(q)\|}\, dq.$$

Now we apply Proposition 4.25 for $p = 1$ to deduce that

$$d\mu(q) = \frac{|D_H f(q)|}{\|f(q)\|}\, dq$$

is a Carleson measure with Carleson measure constant depending only on K. Hence,

$$\mathcal{S}^3\left(\left\{\omega \in S(x) : \|f^*(\omega)\| < \frac{\|f(x)\|}{N}\right\}\right) \lesssim_K \frac{1}{\log N}\rho_x^3 \lesssim \frac{1}{\log N}\mathcal{S}^3(S(x)),$$

where in the last step we used the 3-regularity of $\mathcal{S}^3|_{\partial B}$, recall Lemma 2.11, and (5.10). This concludes the proof of Lemma 5.11 in the second case, that is, if $\|x\| > R_*$. $\square$

With these preparations in hand, we are now ready to prove Lemma 5.7, following the proof of [5, Corollary 4.3] by Astala and Koskela.

Proof of Lemma 5.7. If $0 \notin f(B)$, then we can directly apply Lemma 5.11 by choosing N large enough, depending only on K, such that

$$\mathcal{S}^3\left(\left\{\omega \in S(x) : \|f^*(\omega)\| \le \frac{\|f(x)\|}{N}\right\}\right) < \frac{1}{2}\mathcal{S}^3(S(x)), \quad \text{for all } x \in B.$$

This yields (5.8) in that case, cf., the proof of Corollary 4.3 in [5]. If, on the other hand, $0 \in f(B)$, then we can choose a point $y \in \mathbb{H}^1$ such that

$$\|y\| \le \|f^*(\omega)\|, \quad \text{for almost all } \omega \in \partial B.$$

Then the map $g := L_{y^{-1}} \circ f$ is K-quasiconformal on B with $0 \notin g(B)$. Applying the inequality from the previous case yields

$$\begin{aligned}
\|f(x)\|^q &\le 2^q\|g(x)\|^q + 2^q\|y\|^q \\
&\le 2^q\|g(x)\|^q + \frac{2^q}{\mathcal{S}^3(S(x))}\int_{S(x)} \|f^*(\omega)\|^q \, d\mathcal{S}^3(\omega) \\
&\le \frac{2^q C}{\mathcal{S}^3(S(x))}\int_{S(x)} \|g^*(\omega)\|^q \, d\mathcal{S}^3(\omega) + \frac{2^q}{\mathcal{S}^3(S(x))}\int_{S(x)} \|f^*(\omega)\|^q \, d\mathcal{S}^3(\omega) \\
&\le \frac{2^q C}{\mathcal{S}^3(S(x))}\int_{S(x)} (2\|f^*(\omega)\|)^q \, d\mathcal{S}^3(\omega) + \frac{2^q}{\mathcal{S}^3(S(x))}\int_{S(x)} \|f^*(\omega)\|^q \, d\mathcal{S}^3(\omega),
\end{aligned}$$

which concludes the proof also in that case. $\square$

5.2.2. Proof of Lemma 5.15

We will deduce Lemma 5.15 for quasiconformal maps f on B from a related statement for quasiconformal maps on certain conformal images of B, similarly to the reasoning in the proof of Proposition 4.25.

Lemma 5.21. *There exists a constant $C > 0$, and for every $K \geq 1$ a number $N(K) > 1$, such that for all $x \in B$ with $\|x\| > R_*$ with associated Möbius transformation T_x as in (5.12), and for all K-quasiconformal $g : T_x(B) \to g(T_x(B)) \subset \mathbb{H}^1 \setminus \{0\}$, we have for all $N \geq N(K)$ and $\mathcal{S}^3$ almost all $\omega = (\omega_1, \omega_2, \omega_3) \in S(x)$ that*

$$\|\lim_{s\to 1} g(T_x(\gamma(s,\omega)))\| \leq \frac{\|g(T_x(x))\|}{N} \quad \Rightarrow \quad G_x(\omega) \geq C\rho_x|(\omega_1,\omega_2)|\log N,$$

where

$$G_x(\omega) = \int_{s_{\omega,x}}^{1} \frac{|D_H g(T_x(\gamma(s,\omega)))|}{\|g\|(T_x(\gamma(s,\omega)))} s^3\, ds.$$

Remark 5.22. Lemma 5.21 implies

$$\mathcal{S}^3\left(\omega \in S(x):\ \|\lim_{s\to 1} g(T_x(\gamma(s,\omega)))\| < \frac{\|g(0)\|}{N}\right) \leq \frac{1}{C\,\rho_x\,|(\omega_1,\omega_2)|\,\log N}\|G_x\|_{L^1(\mathcal{S}^3|_{\partial B})}$$

Lemma 5.15 follows immediately by applying Lemma 5.21 to $g := f \circ T_x^{-1}$. So Lemma 5.21 is the last missing piece. We prove it by applying Proposition 3.9, which is possible thanks to the normalization provided by T_x. This is inspired by ideas from [71], but at the same time geometric properties of radial curves and Möbius transformations in $\mathbb{H}^1$ play an important role in our argument via the following auxiliary result.

Lemma 5.23. *For a point $x \in B$ with $\|x\| > R_*$, let T_x be the associated Möbius transformation defined in (5.12), and let $g : T_x(B) \to g(T_x(B)) \subset \mathbb{H}^1 \setminus \{0\}$ be quasiconformal. Then, for $\mathcal{S}^3$ almost every $\omega \in S(x)$ and almost every $s \in (0,1)$, we have*

$$|D_H g(T_x(\gamma(s,\omega)))| \geq \left|\frac{\partial}{\partial s}\|g(T_x(\gamma(s,\omega)))\|\right| \frac{|(\omega_1,\omega_2)|}{J_{T_x}(\gamma(s,\omega))^{1/4}}.$$

Proof. Since $g \circ T_x$ is quasiconformal, the curve $s \mapsto g(T_x(\gamma(s,\omega)))$ is horizontal for $\mathcal{S}^3$ almost every $\omega \in S(x)$. Fix such $\omega = (\omega_1, \omega_2, \omega_3)$. Applying Lemma 3.28 to $f = g \circ T_x$ and using the chain rule we find

$$\left|\frac{\partial}{\partial s}\|g(T_x(\gamma(s,\omega)))\|\right| \leq \frac{|(g\circ T_x)_I(\gamma(s,\omega)))|}{\|g(T_x(\gamma(s,\omega)))\|}|D_H g(T_x(\gamma(s,\omega)))||D_H T_x(\gamma(s,\omega))|\frac{1}{|(\omega_1,\omega_2)|}.$$

We conclude that

$$|D_H g(T_x(\gamma(s,\omega)))| \geq \left|\frac{\partial}{\partial s}\|g(T_x(\gamma(s,\omega)))\|\right| \frac{\|g(T_x(\gamma(s,\omega)))\|}{|(g\circ T_x)_I(\gamma(s,\omega)))|} \frac{|(\omega_1,\omega_2)|}{|D_H T_x(\gamma(s,\omega))|}$$
$$\geq \left|\frac{\partial}{\partial s}\|g(T_x(\gamma(s,\omega)))\|\right| \frac{|(\omega_1,\omega_2)|}{J_{T_x}(\gamma(s,\omega))^{1/4}}. \quad \square$$

With these preparations in hand, we can conclude the proof of Lemma 5.21, and thus, in particular the whole proof of Proposition 5.4.

Proof of Lemma 5.21. Fix $x \in B$ with $\|x\| > R_*$, and a constant N to be determined (not to be confused with the universal constant N from Remark 4.21). To simplify notation, we denote

$$F_{x,N} := \left\{\omega \in S(x) \text{ where } (g\circ T_x)^*(\omega) \text{ exists and } \|\lim_{s\to 1} g(T_x(\gamma(s,\omega)))\| < \frac{\|g(0)\|}{N}\right\}.$$

Our goal is to prove for $\mathcal{S}^3$ almost every $\omega \in F_{x,N}$ that

$$G_x(\omega) \geq C\,\rho_x\,|(\omega_1,\omega_2)|\,\log N \tag{5.24}$$

for some universal constant $C > 0$. Indeed, for such ω, we find by the definition of $s_{\omega,x}$ (recall Lemma 5.13 (3)) and by the choice of R_* below (5.12) that

$$G_x(\omega) \geq (1-r_*)^3 \int_{s_{\omega,x}}^{1} \frac{|D_H g(T_x(\gamma(s,\omega)))|}{\|g\|(T_x(\gamma(s,\omega))} \, ds.$$

Applying the bound for the horizontal derivative given in Lemma 5.23, we observe that

$$\int_{s_{\omega,x}}^{1} \frac{|D_H g(T_x(\gamma(s,\omega)))|}{\|g\|(T_x(\gamma(s,\omega))} \, ds \geq \int_{s_{\omega,x}}^{1} \left|\frac{\partial_s\|g(T_x(\gamma(s,\omega)))\|}{\|g(T_x(\gamma(s,\omega)))\|}\right| \frac{|(\omega_1,\omega_2)|}{J_{T_x}(\gamma(s,\omega))^{1/4}}\, ds$$
$$\geq \int_{s_{\omega,x}}^{1} \left|\frac{\partial}{\partial s}\log\|g(T_x(\gamma(s,\omega)))\|\right| \frac{|(\omega_1,\omega_2)|}{J_{T_x}(\gamma(s,\omega))^{1/4}}\, ds.$$

By the formula for the Jacobian J_{T_x} stated in (4.7), the last expression equals

$$\int_{s_{\omega,x}}^{1} \left|\frac{\partial}{\partial s}\log\|g(T_x(\gamma(s,\omega)))\|\right| \frac{|(\omega_1,\omega_2)|\, d(a_x,\gamma(s,\omega))^2}{\rho_x}\, ds.$$

By Remark 4.21, we have

$$d(a_x,\gamma(s,\omega))^2 \geq C\rho_x,$$

and hence we obtain from the above chain of inequalities that

$$G_x(\omega) \geq C^2 (1-r_*)^3 \rho_x |(\omega_1,\omega_2)| \left| \int_{s_{\omega,x}}^{1} \frac{\partial}{\partial s} \log \|g(T_x(\gamma(s,\omega)))\| \, ds \right|$$
$$\geq C^2 (1-r_*)^3 \rho_x |(\omega_1,\omega_2)| \left| \log \frac{\|(g\circ T_x)(\omega)\|}{\|g(T_x(\gamma(s_{\omega,x},\omega)))\|} \right|.$$

To control the logarithm term, we first apply Proposition 3.9 to obtain

$$\|g(0)\| \leq \frac{C_K \|g(T_x(\gamma(s_{\omega,x},\omega)))\|}{d(T_x(\gamma(s_{\omega,x},\omega)),\partial T_x B)^{\alpha_K}} \tag{5.25}$$

for some constants C_K and α_K, which depend only on the distortion K. To justify this application of Proposition 3.9, we observe that there exist universal constants $0<m<M<\infty$ such that

$$B(0,m) \subset T_x(B) \subset B(0,M) \tag{5.26}$$

holds for all $x \in B$ with $\|x\| \geq R_*$ and $T_x = T_{x,a_x,\rho_x}$ defined in Proposition 4.2 for $a = a_x$ and $\rho = \rho_x$ as in (d1)-(d3) below (5.12). Indeed, the second inclusion in (5.26) can be arranged by Corollary 4.9 since the choice of parameters ρ_x, ω_x, and a_x in (d1)-(d3) implies that conditions (4.10) are satisfied for $\rho=\rho_x$ and $a = a_x$:

$$d(a_x,\partial B) \overset{(d3)}{=} d(A_{o,M_0 N\rho_x}(\omega_x),\partial B) \geq \rho_x \quad \text{and} \quad d(a_x,x) \overset{(d1)-(d3)}{>} d(x,\partial B) \overset{(d1)}{=} \rho_x.$$

On the other hand, the first inclusion in (5.26) can be arranged by Corollary 4.11 since (d1)-(d3) ensure that conditions (4.12) are satisfied for our choice of x, $\rho=\rho_x$, and $a=a_x$:

$$d(x,\partial B) \overset{(d1)}{=} \rho_x, \quad d(a_x,x) \leq d(a_x,\omega_x) + d(\omega_x,x) \overset{(d2),(d3)}{\lesssim} \rho_x.$$

Then we use the normalization provided by Lemma 5.14 to continue the estimate in (5.25):

$$\|g(0)\| \lesssim_K \|g(T_x(\gamma(s_{\omega,x},\omega)))\|. \tag{5.27}$$

On the other hand, since $\omega \in F_{x,N}$, we know that

$$\|(g\circ T_x)^*(\omega)\| < \frac{\|g(0)\|}{N}. \tag{5.28}$$

Combining (5.27) and (5.28), we find for some constant $C(K)$, which depends only on the distortion K, that

$$\frac{\|(g\circ T_x)^*(\omega)\|}{\|g(T_x(\gamma(s_{\omega,x},\omega)))\|} < \frac{\|g(0)\|}{N}\frac{C(K)}{\|g(0)\|}$$

and hence

$$\log\frac{\|(g\circ T_x)^*(\omega)\|}{\|g(T_x(\gamma(s_{\omega,x},\omega)))\|} < \log\frac{C(K)}{N} < \log\frac{1}{N^{1/2}} = -\frac{1}{2}\log N(<0)$$

if we have initially chosen

$$N > C(K)^2 =: N(K),$$

so that $C(K)/N < 1/N^{1/2}$. Thus for this choice of N, the previous estimates yield

$$\begin{aligned} G_x(\omega) &\geq C^2\,(1-r_*)^3\,\rho_x\,|(\omega_1,\omega_2)|\left|\log\frac{\|(g\circ T_x)(\omega)\|}{\|g(T_x(\gamma(s_{\omega,x},\omega)))\|}\right| \\ &\geq C^2\,(1-r_*)^3\,\rho_x\,|(\omega_1,\omega_2)|\frac{1}{2}\log N \end{aligned}$$

for almost every $\omega = (\omega_1,\omega_2,\omega_3) \in F_{x,N}$. $\square$

6. Carleson measures and radial limits of quasiconformal maps on B

We apply the results from the previous section, notably Proposition 5.4, to characterize Carleson measures on B. While this proposition concerns the nontangential maximal function, the inequalities concerning this maximal function will only be used as intermediate results and not appear in the main result. Below we take the strategy to first discuss a general type result in metric spaces and then apply it in the Heisenberg setting.

6.1. Carleson measures and nontangential maximal functions in metric spaces

It is well known that several variants of Carleson's embedding theorem on the Euclidean unit ball and half-space can be proven using nontangential maximal functions as an intermediate tool, see [68, VII. 4.4], [30, I. Exercise 19] and [5, Corollary 4.5]. The first step in these arguments works in rather general metric spaces, as we now show. Later we will apply this abstract result in the context of quasiconformal maps on the Korányi unit ball.

Recall that a non-empty domain $\Omega \subset X$ of a metric space (X,d) has *s-regular boundary* for some $s>0$, if its boundary is Ahlfors s-regular with respect to the Hausdorff measure on X restricted to $\partial\Omega$, i.e., there exists a constant $C\geq 1$ such that

$$C^{-1}\,r^s \leq \mathcal{H}^s(B(x,r)\cap\partial\Omega) \leq C\,r^s, \quad \text{for all } x\in\partial\Omega \text{ and } 0<r<\operatorname{diam}(\partial\Omega).$$

Definition 6.1. Fix $1 \leq \alpha < \infty$ and $s > 0$. Let (X, d) be a metric space and $\Omega \subset X$ a domain with nonempty s-regular boundary. We say that a (positive) Borel measure μ on Ω is an *α-Carleson measure on* Ω if there exists a constant $C > 0$ such that

$$\mu(\Omega \cap B(\omega, r)) \leq C r^{\alpha\, s}, \quad \text{for all } \omega \in \partial\Omega \text{ and } r > 0. \tag{6.2}$$

The *α-Carleson measure constant* of μ is defined by

$$\gamma_\alpha(\mu) := \inf\{C > 0 \text{ such that (6.2) holds for all } \omega \in \partial\Omega \text{ and } r > 0\}$$

We also call 1-Carleson measures simply *Carleson measures.*

Recall Definition 2.18 and Remark 2.19.

Proposition 6.3. *Fix $s > 0$ and $1 \leq \alpha < \infty$. Let (X, d) be a proper metric space and let $\Omega \subset X$ be a bounded domain with nonempty s-regular boundary and let $\kappa > 0$ be such that the nontangential region $\Gamma_\kappa(\omega)$ is nonempty for all $\omega \in \partial\Omega$. Assume that μ is an α-Carleson measure on Ω. Then the κ-nontangential maximal function $N_\kappa h$ of an arbitrary Borel function $h : \Omega \to [0, +\infty)$ satisfies*

$$\int\limits_{\Omega} h^{\alpha p}\, d\mu \leq C \left(\int\limits_{\partial\Omega} (N_{\Omega,\kappa} h)^p \, d\mathcal{H}^s \right)^{\alpha}, \quad \textit{for all } 0 < p < \infty \tag{6.4}$$

where C depends on p, α, s, κ, $\gamma_\alpha(\mu)$, and the s-regularity constant of $\partial\Omega$. If $\alpha = 1$, then C can be chosen independently of p.

Our proof of Proposition 6.3 is inspired by the first part of the proof of [5, Corollary 4.5] in the context of the Euclidean unit ball. We generalize this approach using a Whitney decomposition in abstract doubling metric spaces, in the spirit of [15, Theorem 3.2]. For this purpose, it will be more natural to work directly with covering balls centered at points in $\partial\Omega$, rather than analogs of the spherical caps $S(x_k)$, $k = 1, 2, \ldots$, in [5].

Proof. Let p, α, s, κ, $\gamma_\alpha(\mu)$, and h be as in the statement of the proposition. For simplicity, we abbreviate throughout the proof $N_\kappa h := N_{\Omega,\kappa} h$ and $S(x) := S_\kappa(x)$. To prove (6.4), we define the superlevel sets

$$E(\lambda) := \{x \in \Omega : \, h(x) > \lambda\} \quad \text{and} \quad U(\lambda) := \{\omega \in \partial\Omega : \, N_\kappa h(\omega) > \lambda\}, \quad \lambda > 0.$$

It suffices to show that there exists a constant C, depending only on α, s, κ, $\gamma_\alpha(\mu)$, and the s-regularity constant of $\partial\Omega$, such that

$$\mu(E(\lambda)) \leq C \mathcal{H}^s(U(\lambda))^\alpha \quad \text{for all } \lambda > 0. \tag{6.5}$$

If $\alpha = 1$, then (6.4), with C independent of p, follows immediately from (6.5) by a standard application of Cavalieri's principle. If $\alpha > 1$, then by a similar reasoning, one concludes as follows:

$$\int_{\partial\Omega} h^{\alpha p}\, d\mu = \alpha p \int_0^\infty \lambda^{\alpha p-1}\mu(E(\lambda))\, d\lambda \le C \int_0^\infty \lambda^{\alpha p-1}\mathcal{H}^s(U(\lambda))^\alpha\, d\lambda$$
$$\lesssim \left(\sum_{j=-\infty}^{\infty} \mathcal{H}^s(U(2^j))2^{jp}\right)^\alpha \lesssim \left(\int_{\partial\Omega} (N_\kappa h)^p\, d\mathcal{H}^s\right)^\alpha,$$

where the implicit constants now depend additionally also on α and p.

It remains to prove the superlevel set estimate (6.5), which we will do by a Whitney-type decomposition of $U(\lambda)$ if the latter is a strict subset of $\partial\Omega$ (otherwise the claim is trivial). Recall that $U(\lambda)$ is a relatively open subset in $\partial\Omega$, and $(\partial\Omega, d|_{\partial\Omega})$ is metrically doubling since it is s-regular. Thus we can for instance apply the general result [41, Proposition 4.1.15] to the metric space $(\partial\Omega, d|_{\partial\Omega})$ and the open set $U(\lambda)$ to find a countable collection $\mathcal{W}_\lambda = \{B(\omega_i, r_i) : i = 1, 2, \ldots\}$ of balls with $\omega_i \in U(\lambda)$ such that

$$U(\lambda) = \bigcup_{i=1,2,\ldots} B(\omega_i, r_i) \cap \partial\Omega, \tag{6.6}$$

$$\sum_i \chi_{B(\omega_i,2r_i)\cap\partial\Omega} \le 2N^5, \tag{6.7}$$

where $r_i = (1/8)d(\omega_i, \partial\Omega \setminus U(\lambda))$ and N depends only on s and the s-regularity constant of $\partial\Omega$. (For our purposes, it would in fact suffice to obtain (6.7) with "$B(\omega_i, r_i)$" instead of "$B(\omega_i, 2r_i)$".) To prove the superlevel set estimate, we want to show that $E(\lambda)$ is included in the union of the balls $B(\omega_i, Cr_i)$, for a suitable geometric constant $C = C(\kappa)$. If x is an arbitrary point in $E(\lambda)$, then

$$N_\kappa h(\omega) > \lambda, \quad \text{for all } \omega \in S(x) = B\left(x, (1+\kappa)d(x, \partial\Omega)\right) \cap \partial\Omega,$$

and hence

$$S(x) \subset U(\lambda) \overset{(6.6)}{=} \bigcup_i B(\omega_i, r_i) \cap \partial\Omega, \quad \text{for all } x \in E(\lambda). \tag{6.8}$$

Next, for $x \in E(\lambda)$, let $\omega_x \in \partial B$ be such that

$$d(x, \omega_x) = d(x, \partial\Omega). \tag{6.9}$$

Such a point may not be unique, but there exists at least one since $\partial\Omega$ is compact. By definition, $\omega_x \in S(x)$, and therefore (6.8) implies that there exists $i_x \in \{1, 2, \ldots\}$ such that $\omega_x \in B(\omega_{i_x}, r_{i_x})$. Since $S(x) \subset U(\lambda)$, we moreover know that

$$d(\omega_x, \partial\Omega \setminus U(\lambda)) \geq d(\omega_x, \partial\Omega \setminus S(x)). \tag{6.10}$$

Combining this information, we find that

$$\begin{aligned} r_{i_x} = \frac{1}{8} d(\omega_{i_x}, \partial\Omega \setminus U(\lambda)) &\geq \frac{1}{8}\left[d(\omega_x, \partial\Omega \setminus U(\lambda)) - d(\omega_x, \omega_{i_x})\right] \\ &\overset{(6.10)}{\geq} \frac{1}{8} d(\omega_x, \partial\Omega \setminus S(x)) - \frac{1}{8} r_{i_x}. \end{aligned} \tag{6.11}$$

Since $d(\omega_x, x) = d(x, \partial\Omega)$, it is easy to see that

$$B\left(\omega_x, \kappa d(x, \partial\Omega)\right) \cap \partial\Omega \subset S(x).$$

Hence the above chain of inequalities implies that

$$9\, r_{i_x} \overset{(6.11)}{\geq} d(\omega_x, \partial\Omega \setminus S(x)) \geq d(\omega_x, \partial\Omega \setminus B\left(\omega_x, \kappa d(x, \partial\Omega)\right)).$$

As the right-hand side of the above inequality is bounded from below by $\kappa d(x, \partial\Omega) = \kappa d(x, \omega_x)$, we obtain that

$$x \in B\left(\omega_{i_x}, \left(\tfrac{9}{\kappa} + 1\right) r_{i_x}\right). \tag{6.12}$$

Since x was chosen arbitrarily from $E(\lambda)$, we have thus shown that $E(\lambda)$ is covered by the countable family of balls $B(\omega_i, C r_i)$, $i = 1, 2, \ldots$, where $C = C(\kappa) = \frac{9}{\kappa} + 1$. Using that μ is an α-Carleson measure by assumption, the fact that $\mathcal{H}^s|_{\partial\Omega}$ is s-regular, and the multiplicity of the Whitney balls is controlled by (6.7), we deduce that

$$\begin{aligned} \mu(E(\lambda)) \leq \mu\left(\bigcup_i B(\omega_i, C r_i) \cap \Omega\right) &\leq \sum_i \mu(B(\omega_i, C r_i) \cap \Omega) \\ &\leq \gamma_\alpha(\mu)\ \left(\tfrac{9}{\kappa} + 1\right)^{s\alpha} \sum_i r_i^{s\alpha} \\ &\leq \gamma_\alpha(\mu)\ \left(\tfrac{9}{\kappa} + 1\right)^{s\alpha} \left(\sum_i r_i^s\right)^{\alpha} \\ &\lesssim \left(\sum_i \mathcal{H}^s(B(\omega_i, r_i) \cap \partial\Omega)\right)^{\alpha} \overset{(6.7)}{\lesssim} \mathcal{H}^s(U(\lambda))^{\alpha}, \end{aligned}$$

as desired. This concludes the proof of Proposition 6.3, as explained above. □

6.2. Carleson measures and nontangential maximal functions in $\mathbb{H}^1$

We now apply Proposition 6.3 in $\mathbb{H}^1$ and observe that in this setting it can be strengthened to give an integral inequality involving the radial limit of a quasiconformal map.

The fact that such inequalities actually *characterize* Carleson measures on the Korányi ball B follows by an application of the Heisenberg radial stretch map [6]. Moreover, we provide consequences of this characterization in Section 6.2.1, where we present a relation between Hardy spaces and Bergman-type spaces for quasiconformal maps on B, see Theorem 6.18.

Theorem 6.13. *Let $1 \leq \alpha < \infty$ and assume that μ is an α-Carleson measure on B. If $f : B \to f(B) \subset \mathbb{H}^1$ is K-quasiconformal, then*

$$\int_B \|f(q)\|^{\alpha p} \, d\mu(q) \leq C_p \left(\int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega) \right)^{\alpha}, \quad \text{for all } 0 < p < \infty, \tag{6.14}$$

where C_p depends only on p, α, K, and a Carleson measure constant $\gamma_\alpha(\mu)$.

Conversely, for every $K \geq 1$, there is $p(K) < 3$ such that if $p > p(K)$ is fixed and μ is a Borel measure for which (6.14) holds for all K-quasiconformal maps, then μ is an α-Carleson measure.

By Lemma 5.1, the first part of Theorem 6.13 immediately yields a necessary condition for α-Carleson measures μ on B in terms of $L^{\alpha p}(\mu)$ integral inequalities for quasiconformal maps in H^p and $\|\cdot\|_{H^p}$.

Corollary 6.15. *Let $1 \leq \alpha < \infty$ and assume that μ is an α-Carleson measure on B. If $f : B \to f(B) \subset \mathbb{H}^1$ is K-quasiconformal, then, for all $0 < p < \infty$,*

$$\left(\int_B \|f(q)\|^{\alpha p} \, d\mu(q) \right)^{\frac{1}{\alpha p}} \leq C_p \|f\|_{H^p},$$

where C_p depends only on p, α, K, and $\gamma_\alpha(\mu)$.

Proof of Theorem 6.13. In order to show the sufficiency part of the theorem, we apply Proposition 6.3 to $(X, d) = (\mathbb{H}^1, d)$, $\Omega = B$, and $h = \|f\|$, where $f : B \to f(B) \subset \mathbb{H}^1$ is K-quasiconformal, and we combine the result with Proposition 5.4.

For the proof of the necessity part of Theorem 6.13, let $K \geq 1$ and assume that $p > p(K)$ for a constant $0 < p(K) < 3$ to be determined. We suppose that (6.14) holds for all K-quasiconformal mappings, and we will apply this condition to a particular such map in order to deduce that μ has to be an α-Carleson measure. The choice of the map in question is inspired by the proof of [5, Corollary 4.5] and it involves a quasiconformal *Heisenberg radial stretch map*, see [58,6,7,70] for different contexts in which such stretch maps have arisen. The only relevant information for us is that there exists a K-quasiconformal map $f_K : \mathbb{H}^1 \to \mathbb{H}^1$ with $f_K(\partial B(0, r)) = \partial B(0, r^{\beta(K)})$ for some $\beta(K) \geq 1$ (this follows by considering the inverse of the map f_k studied in [6, Section 4.1]).

Let us verify the α-Carleson measure condition for μ at an arbitrary point $\omega_0 \in \partial B$. Since (6.14) holds for all K-quasiconformal maps on B, it holds in particular for left translations, which shows that $\mu(B) < \infty$. For this reason, it suffices to verify the α-Carleson measure condition of μ for $0 < r < r_0$, where $r_0 \in (0, 1)$ is as in Remark 4.21 concerning the corkscrew property of B. Thus let us fix $\omega_0 \in \partial B$ and $0 < r < r_0$. We will apply condition (6.14) to the map

$$f := f_{\omega_0,r} : \mathbb{H}^1 \setminus \{a\} \to \mathbb{H}^1 \setminus \{0\}, \quad f(y) := f_K(I(a^{-1} \cdot y)),$$

for suitably chosen $a = a(\omega_0, r) \in \mathbb{H}^1 \setminus \overline{B}$, where f_K is the K-quasiconformal radial stretch map discussed above and

$$I(y) = -\frac{1}{\|y\|^4} \left(y_z(|y_z|^2 + iy_t), y_t\right)$$

is the 1-quasiconformal inversion at ∂B. Since left translations are 1-quasiconformal as well, it follows that $f|_B$ is K-quasiconformal and (6.14) is applicable. The point a can be chosen using the exterior corkscrew condition of B such that

$$\frac{r}{M_0} \leq d(a, \partial B) \leq d(a, \omega_0) \leq r, \tag{6.16}$$

recall Remark 4.21. It follows from the formula for the inversion that $\|I(y)\| = 1/\|y\|$ for all $y \in \mathbb{H}^1 \setminus \{0\}$, and hence

$$\|f(y)\| = \|I(a^{-1} \cdot y)\|^{\beta(K)} = \frac{1}{d(y, a)^{\beta(K)}}, \quad y \in \mathbb{H}^1 \setminus \{a\}.$$

For all $y \in B(\omega_0, r) \cap B$, we know by (6.16) that $d(y, a) \leq 2r$, and hence

$$1 = d(y, a)^{\beta(K)} \|f(y)\| \leq 2^{\beta(K)} r^{\beta(K)} \|f(y)\|, \quad \text{for all } y \in B(\omega_0, r) \cap B.$$

Hence, by (6.14),

$$\begin{aligned} \mu(B(\omega_0, r) \cap B) &\leq 2^{\alpha p \beta(K)} r^{\alpha p \beta(K)} \int_B \|f(y)\|^{\alpha p} \, d\mu(y) \\ &\lesssim_{\alpha,p,K} r^{\alpha p \beta(K)} \left(\int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega) \right)^{\alpha}. \end{aligned}$$

Thus the α-Carleson measure property of μ will follow if we manage to show that

$$\int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega) \lesssim_{p,K} r^{3-\beta(K)p}. \tag{6.17}$$

To upper bound the integral, we decompose the domain of integration as follows:

$$\partial B \subset \bigcup_{j=0}^{\infty} A_j, \quad \text{where } A_j := \left\{ \omega \in \partial B : \frac{r}{M_0} 2^j \leq d(\omega, a) < \frac{r}{M_0} 2^{j+1} \right\}.$$

The number of nonempty A_j depends on ω_0 and r, so we need estimates that do not depend on this number. Indeed,

$$\int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega) = \int_{\partial B} d(\omega, a)^{-\beta(K)p} \, d\mathcal{S}^3(\omega) \lesssim \sum_{j=0}^{\infty} r^{-\beta(K)p} 2^{-j\beta(K)p} \, \mathcal{S}^3(A_j).$$

From (6.16), it follows that

$$A_j \subset B\left(\omega_0, \left(\frac{1}{M_0} + 1\right) 2^{j+1} r\right) \cap \partial B, \quad j = 0, 1, \ldots,$$

and hence, by the 3-regularity of $\mathcal{S}^3|_{\partial B}$, the integral can be further estimated as follows:

$$\begin{aligned} \int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega) &\lesssim \sum_{j=0}^{\infty} r^{-\beta(K)p} 2^{-j\beta(K)p} \, \mathcal{S}^3 \left(B\Big(\omega_0, \Big(\frac{1}{M_0} + 1\Big) 2^{j+1} r\Big) \cap \partial B \right) \\ &\lesssim r^{3-\beta(K)p} \sum_{j=0}^{\infty} 2^{j(3-\beta(K)p)}. \end{aligned}$$

This yields (6.17) provided that $p > 3/\beta(K)$, so that the above series is a finite constant depending on K and p. This concludes the proof with $p(K) := 3/\beta(K)$. □

6.2.1. Applications of Theorem 6.13

We apply the characterization of Carleson measures on $B \subset \mathbb{H}^1$ to relate Hardy spaces and quasiconformal mappings integrable on the unit ball (a counterpart of the Bergman spaces), thus generalizing Theorem 9.1 in [5]. For a map $f : B \to \mathbb{H}^1$ and $0 < p < \infty$ we define

$$\|f\|_{A^p} := \left(\int_B \|f(q)\|^p \, dq \right)^{\frac{1}{p}},$$

and write $f \in A^p$ if $\|f\|_{A^p} < \infty$.

Theorem 6.18. *Let f be a quasiconformal mapping $f : B \to f(B) \subset \mathbb{H}^1$.*

(1) *If $f \in H^p$, then $f \in A^{\frac{4}{3}p}$.*
(2) *If $f \in A^p$, then $f \in H^{p'}$ for all $0 < p' < \frac{3}{4}p$.*

The proof of the theorem employs a quantity that is inspired by [5, Theorem 3.3]. Given $f : B \subset \mathbb{H}^1 \to \mathbb{H}^1$, we define

$$M(r,f) := \sup_{q\in\Sigma_r} \|f(q)\|, \quad 0 \le r < 1,$$

where $\Sigma_r := \{q \in B : d(q, \partial B) = 1 - r\}$. The set Σ_r is different from $\partial B(0,r)$ (it contains, e.g., the point $(0,0,1-(1-r)^2) = (0,0,r(2-r))$), and its definition is tailored to play along well with the measure $\mathcal{S}^3|_{\partial B}$.

Proposition 6.19. *Let $0 < p < \infty$ and $K \ge 1$. Then every K-quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1$ with $f(0) = 0$ satisfies*

$$\int_{\partial B} \|f^*(\omega)\|^p \, d\mathcal{S}^3(\omega) \le C \int_0^1 (1-r)^2 M(r,f)^p \, dr, \tag{6.20}$$

for a constant C that depends only on p and K.

If $f : B \to f(B) \subset \mathbb{H}^1$ is an arbitrary quasiconformal map, then

$$\int_0^1 (1-r)^2 M(r,f)^p \, dr < \infty \quad \textit{implies} \quad f \in H^p. \tag{6.21}$$

We first show how Proposition 6.19 implies Theorem 6.18.

Proof of Theorem 6.18 based on Proposition 6.19 . Let $f \in H^p$ be quasiconformal on $B \subset \mathbb{H}^1$. Since the Lebesgue measure on $\mathbb{H}^1$ is 4-Ahlfors regular, we obtain by (6.2) for $s = 3$ that its restriction to B is α-Carleson for $\alpha = \frac{4}{3}$ and hence $f \in A^{\frac{4}{3}p}$ by Corollary 6.15.

In order to show the second assertion of the theorem, we may without loss of generality assume that $0 \notin f(B)$, using compositions with suitable left translations if necessary. Now for every quasiconformal map $f : B \to f(B) \subset \mathbb{H}^1 \setminus \{0\}$, there exist constants $0 < \lambda < 1$ and $C > 1$ such that

$$\|f(q)\| \le C\|f(y)\|, \quad y \in B(q, \lambda d(q, \partial B)), \quad q \in B.$$

This Harnack property follows for instance from [1, Proposition 3.12], observing that $d(f(y), \partial f(B)) \le \|f(y)\|$ since $0 \notin f(B)$. Hence it holds that

$$\|f(q)\| \lesssim \left(\fint_{B(q,\lambda d(q,\partial B))} \|f(y)\|^p \, dy \right)^{\frac{1}{p}} \lesssim \frac{1}{d(q,\partial B)^{\frac{4}{p}}},$$

where in the last inequality we also use that $f \in A^p$. Therefore,

$$\int_0^1 (1-r)^2 M(r,f)^{p'}\,dr \lesssim \int_0^1 (1-r)^{2-\frac{4}{p}p'}\,dr$$

and the integral is finite if $3-\frac{4}{p}p' > 0$. By Proposition 6.19 this yields that $f \in H^{p'}$. $\square$

Proof of Proposition 6.19. We prove the first part of the proposition under the assumption $f(0)=0$. If the integral on the left-hand side of (6.20) was computed with respect to the measure σ, which arises naturally from the modulus formula (2.6), then the proof of the estimate would follow almost verbatim the first part in the proof of [5, Theorem 3.3] with "n" replaced by "4". The main challenge is to prove the stronger inequality with σ replaced by $\mathcal{S}^3|_{\partial B}$. Analogously as in the proof of [5, Theorem 3.3], we define

$$E_\lambda := \{\omega \in \partial B :\ \|f^*(\omega)\| > \lambda\}.$$

As $M(0,f)=0$, and since the sets Σ_r, $0<r<1$, are topological spheres foliating B by Proposition 6.24, there exists a unique $r(\lambda) \in (0,1)$ such that

$$2\,M(r(\lambda),f) = \lambda \tag{6.22}$$

whenever $E_\lambda \neq \emptyset$. Since

$$\begin{aligned}\int_{\partial B} \|f^*(\omega)\|^p\,d\mathcal{S}^3(\omega) &= p\int_0^\infty \mathcal{S}^3\left(\{\omega\in\partial B:\ \|f^*(\omega)\|>\lambda\}\right)\lambda^{p-1}\,d\lambda \\ &\overset{(6.22)}{\leq} \mathcal{S}^3(\partial B)\,2^p\,M(1/2,f)^p \\ &\quad + p\int_{\{\lambda\in(0,\infty]:\,1/2<r(\lambda)<1\}} \mathcal{S}^3\left(\{\omega\in\partial B:\ \|f^*(\omega)\|>\lambda\}\right)\lambda^{p-1}\,d\lambda,\end{aligned}$$

it suffices to prove that

$$\mathcal{S}^3(E_\lambda) \lesssim_K (1-r(\lambda))^3, \quad \text{for all } \lambda>0 \text{ such that } 1/2<r(\lambda)<1. \tag{6.23}$$

Indeed, if we manage to show (6.23), the proof can be concluded exactly as below [5, (3.7)]. In order to establish (6.23), we divide each relevant E_λ into two "good" parts at safe distance from the characteristic points, and a "bad" part close to the characteristic points. To state the definition, for every $\omega \in \partial B \setminus \{z=0\}$ and λ as before, we let $s_\omega := s_{\lambda,\omega} \in (0,1)$ be such that

$$\gamma(s_\omega,\omega) \in \Sigma_{r(\lambda)}.$$

Similarly as in Lemma and Definition 5.13, s_ω, one can make a Borel measurable choice $\omega \mapsto s_\omega$. Then we define

$$G_{0,\lambda} := \{(\sqrt{\cos\alpha}e^{\mathrm{i}\varphi}, \sin\alpha) \in E_\lambda : \cos\alpha \geq 1/2\},$$
$$G_\lambda := \{\omega = (\sqrt{\cos\alpha}e^{\mathrm{i}\varphi}, \sin\alpha) \in E_\lambda \setminus G_{0,\lambda} : 1 - s_\omega \leq \cos\alpha\},$$

and

$$B_\lambda := \{\omega = (\sqrt{\cos\alpha}e^{\mathrm{i}\varphi}, \sin\alpha) \in E(\lambda) \setminus G_{0,\lambda} : 1 - s_\omega > \cos\alpha\}.$$

First, we observe that (6.23) holds with "E_λ" replaced by "$G_{0,\lambda}$". Using the modulus formula (2.6) and $1 - s_\omega \sim 1 - r$ for $\omega = (\sqrt{\cos\alpha}e^{\mathrm{i}\varphi}, \sin\alpha)$ with $\cos\alpha \geq 1/2$ (cf., e.g., Lemma A.2), this can be shown exactly as in the proof of [5, Theorem 3.3]. The resulting estimate is a priori stated in terms of the measure σ, but $\sigma(G_{0,\lambda}) \sim \mathcal{S}^3(G_{0,\lambda})$ since the measures σ and $\mathcal{S}^3|_{\partial B}$ are comparable on the parametric region $\{|\cos\alpha| \geq 1/2\}$.

Second, we prove (6.23) with "E_λ" replaced by "G_λ". We let Γ_λ be the family of radial segments in B that connect $\Sigma_{r(\lambda)}$ to G_λ. We use the same modulus argument as in [5], but more subtle estimates for $1 - s_\omega$. Indeed, the standard modulus argument yields

$$\operatorname{mod}_4(\Gamma_\lambda) \geq \int \left(\int_{s_\omega}^{1} \frac{1}{s}\, ds \right)^{-3} \cos^2\alpha \, d\alpha d\varphi = \int \left(\cos^{-1/2}\alpha \int_{s_\omega}^{1} \frac{1}{s}\, ds \right)^{-3} \sqrt{\cos\alpha}\, d\alpha d\varphi.$$

Since $1 - s_\omega \leq \cos\alpha$ for $\omega = (\sqrt{\cos\alpha}e^{\mathrm{i}\varphi}, \sin\alpha) \in G_\lambda$ and $r(\lambda) > 1/2$, we obtain by Lemma A.4 and the definition of s_ω that

$$\int_{s_\omega}^{1} \frac{1}{s}\, ds \lesssim (1 - s_\omega) \lesssim (1 - r(\lambda))\sqrt{\cos\alpha}.$$

This shows that

$$1 \gtrsim \operatorname{mod}_4(f(\Gamma_\lambda)) \sim_K \operatorname{mod}_4(\Gamma_\lambda) \gtrsim (1 - r(\lambda))^{-3}\, \mathcal{S}^3(G_\lambda),$$

as desired for (6.23).

Finally, we show that (6.23) holds also with "E_λ" replaced by the bad set "B_λ". If $\omega = (\sqrt{\cos\alpha}e^{\mathrm{i}\varphi}, \sin\alpha) \in B_\lambda$, then

$$\cos\alpha \leq 1 - s_\omega \lesssim (1 - r(\lambda))^2,$$

where we invoke Lemma A.4 for the last estimate. Thus we have the crude estimate:

$$\begin{aligned} \mathcal{S}^3(B_\lambda) &\leq 2\pi \int_{\{\alpha \in (-\pi/2, \pi/2) : \cos\alpha \leq (1 - r(\lambda))^2\}} \sqrt{\cos\alpha}\, d\alpha \\ &\lesssim (1 - r(\lambda)) \left| \left\{ \alpha \in (-\pi/2, \pi/2) : \cos\alpha \leq (1 - r(\lambda))^2 \right\} \right|. \end{aligned}$$

Since $r(\lambda) \geq 1/2$ by assumption, we only have to consider $\alpha \in (-\pi/2, \pi/2)$ with $\cos\alpha \leq 1/4$, so that α is either close to $\pi/2$ or to $-\pi/2$. In the first case, we have

$$\left|\alpha - \tfrac{\pi}{2}\right| \lesssim \left|\cos\alpha - \cos\tfrac{\pi}{2}\right| = \cos\alpha \leq (1 - r(\lambda))^2,$$

and in the second case we obtain analogously that α lies in an interval of length $(1-r(\lambda))^2$ around $-\pi/2$. This shows as desired that

$$\mathcal{S}^3(B_\lambda) \lesssim (1 - r(\lambda))^3.$$

Summing the upper bounds for $\mathcal{S}^3(G_{0,\lambda})$, $\mathcal{S}^3(G_\lambda)$ and $\mathcal{S}^3(B_\lambda)$ yields (6.23) and thus yields the first part of the proposition. By Theorem 1.4, the established inequality (6.20) shows that (6.21) holds if $f(0) = 0$. The full statement of the proposition can be reduced to this one. Indeed, if f satisfies the assumption in (6.21), but $f(0) \neq 0$, then consider the new map $\widetilde{f} := L_{f(0)^{-1}} \circ f$, which has the desired property $\widetilde{f}(0) = 0$. Since

$$M(r, \widetilde{f}) = \sup_{q \in \Sigma_r} \|f(0)^{-1} f(q)\| \leq \sup_{q \in \Sigma_r} \|f(q)\| + \|f(0)^{-1}\| = M(r, f) + \|f(0)^{-1}\|$$

for all $r \in [0, 1)$, it follows that $\widetilde{f}$ satisfies the assumption in (6.21) and by the first part of the proof, we conclude that $\widetilde{f} \in H^p$. Then it also follows that $f \in H^p$. $\square$

We conclude this section by proving the topological result that we applied earlier.

Proposition 6.24. *The sets*

$$\Sigma_r := \{q \in B : \, d(q, \partial B) = 1 - r\}, \quad 0 < r < 1,$$

are topological spheres.

Proof. We fix $0 < r < 1$ and analyze the intersection of Σ_r with planes parallel to the xy-plane. Clearly,

$$\Sigma_r \cap \left(\mathbb{R}^2 \times \{t\}\right) = \emptyset \quad \text{for } t \in (-\infty, -1 + (1 - r)^2) \cup (1 - (1 - r)^2, \infty),$$

so it suffices to consider $t \in [-1 + (1 - r)^2, 1 - (1 - r)^2]$. Since rotations R_φ about the t-axis are isometries for the Korányi metric, and the set B is invariant under such rotations, we observe that

$$\Sigma_r \cap \left(\mathbb{R}^2 \times \{t\}\right) = \{R_\varphi(x, 0, t) : \, (x, 0, t) \in \Sigma_r \text{ and } \varphi \in [0, 2\pi)\}.$$

We claim that the function

$$\delta : \mathbb{R} \to [0, +\infty), \quad \delta(x) := d\left((x, 0, t), \partial B\right)$$

is strictly monotone increasing on the interval $(-\sqrt[4]{1-t^2}, 0)$, has a local maximum at $x = 0$, and then decreases strictly monotonically on $(0, 1 + \sqrt[4]{1+t^2})$. To see this, we fix

$$x \in (0, +\sqrt[4]{1+t^2}) \quad \text{and} \quad v \in (-x, x). \tag{6.25}$$

We denote

$$\mathbf{x}_- := (-x, 0, t), \quad \mathbf{v} := (v, 0, t), \quad \mathbf{x}_+ := (x, 0, t)$$

By rotational symmetry, $\delta(-x)$ equals $\delta(x)$. We obtain that

$$B(\mathbf{x}_-, \delta(x)) \cup B(\mathbf{x}_+, \delta(x)) \subset B. \tag{6.26}$$

We aim to deduce that $B(\mathbf{v}, \delta(x))$ lies in the convex hull of $B(\mathbf{x}_-, \delta(x))$ and $B(\mathbf{x}_+, \delta(x))$. Indeed, the latter two balls are left translates of $B(\mathbf{v}, \delta(x))$ by $(-x - v, 0, 0)$ and $(x - v, 0, 0)$, respectively and therefore, every $p \in B(\mathbf{v}, \delta(x))$ lies on a line segment

$$\ell_{v,p} := \{(s, 0, 0) \cdot p : \, s \in [-x - v, x - v]\}$$

starting in $B(\mathbf{x}_-, \delta(x))$ and ending in $B(\mathbf{x}_+, \delta(x))$. By convexity of B, the segment $\ell_{v,p}$ is entirely contained in B, and in particular, $B(\mathbf{v}, \delta(x)) \subset B$. Since $\overline{B}$ is strictly convex, we obtain in fact that the closure of $B(\mathbf{v}, \delta(x))$ is contained in B, and hence $d(\mathbf{v}, \partial B) > \delta(x)$, as desired. Since this argument works for arbitrary points x and v as in (6.25), we conclude that the distance function δ has the claimed strict monotonicity properties. This implies that Σ_r is foliated by circles. More precisely, there exists a function ρ_r such that

$$\Sigma_r = \bigcup_{t \in [-1+(1-r)^2, 1-(1-r)^2]} \{(x, y) \in \mathbb{R}^2 : \, x^2 + y^2 = \rho_r(t)^2\} \times \{t\}, \tag{6.27}$$

and $\rho_r(t) = 0$ for $t = -1 + (1-r)^2$ or $t = 1 - (1-r)^2$.

To conclude the argument, we will show that ρ_r is a continuous function. To this end, let $t \in [-1 + (1-r)^2, 1 - (1-r)^2]$ be arbitrary, and let $(t_n)_n$ be a sequence of points in $(-1 + (1-r)^2, 1 - (1-r)^2)$ converging to t. Since Σ_r is compact (it is closed because $d(\cdot, \partial B)$ is continuous), the sequence $((\rho_r(t_n), 0, t_n))_n \subset \Sigma_r$ has a subsequence that converges to a point $(x', 0, t') \in \Sigma_r$. Since the original sequence $(t_n)_n$ converges to t, we have $t' = t$. Moreover, since $\rho_r(t_n) \geq 0$ for all n, we have $x' \geq 0$, and $(x', 0, t) \in \Sigma_r$ implies that $x' = \rho_r(t)$. Repeating the same reasoning for every subsequence of $(t_n)_n$, we obtain $\lim_{n \to \infty} \rho_r(t_n) = \rho_r(t)$, as desired. Hence ρ_r is continuous and the proposition follows by (6.27). $\square$

Data availability

No data was used for the research described in the article.

Appendix A. Radial curves and nontangential regions

The purpose of the Appendix is to prove Proposition 2.15, which states that there exists $\kappa > 0$ such that for every $\omega \in \partial B \setminus \{z = 0\}$, the radial segment $\gamma(s, \omega)$ is contained in the nontangential approach region $\Gamma_\kappa(\omega)$ for all $s \in (0, 1)$. This can be easily verified for $\omega = (e^{\mathrm{i}\varphi}, 0)$, in which case $\gamma(\cdot, \omega)$ is a horizontal line and $d(\gamma(s, \omega), \omega) = d(\gamma(s, \omega), \partial B)$ for all $s \in [0, 1]$. It is also straightforward to check in the limiting case that unit segments on the vertical axis are contained in $\Gamma_\kappa(\omega)$ for $\omega = (0, 0, \pm 1)$ and any choice of κ. For arbitrary ω, the curve $\gamma(s, \omega)$ stays close to the horizontal normal of ∂B through ω for some time $s \in [s(\omega), 1]$ since it is obtained by a flow along a vector field tangential to $\nabla_H \| \cdot \|$, see [8, (3.1)]. The distance estimates for the remaining curve segment $\gamma(\cdot, \omega)|_{[0,s(\alpha)]}$ are similar to those for the vertical line segment. This is reminiscent of the construction of John curves in [60, (1.3)], but our focus lies on verifying the John property for the given radial curves. This is achieved by estimating $d(\gamma(s, \omega), \omega)$ (Lemma A.2) and $d(\gamma(s, \omega), \partial B)$ (Lemma A.4); the complete proof of Proposition 2.15 is given at the end.

First, we compute the distance between points in ∂B and in the paraboloid

$$P_\alpha := \{(x, y, t) \in \mathbb{H}^1 : \tfrac{t}{x^2+y^2} = \tan\alpha\}, \quad \alpha \in (-\pi/2, \pi/2).$$

Let $s > 0$, $p = (s\sqrt{\cos\alpha}e^{\mathrm{i}\varphi}, \sin\alpha) \in \partial B(0, s) \cap P_\alpha$, and $\widetilde{\omega} \in \partial B$, $\widetilde{\omega} = (\sqrt{\cos\widetilde{\alpha}}e^{\mathrm{i}\widetilde{\varphi}}, \sin\widetilde{\alpha})$ be arbitrary. By applying a rotation about the t-axis, we obtain

$$d(p, \widetilde{\omega}) = \|(-s\sqrt{\cos\alpha}, 0, -s^2\sin\alpha) \cdot (\sqrt{\cos\widetilde{\alpha}}\cos(\widetilde{\varphi} - \varphi), \sqrt{\cos\widetilde{\alpha}}\sin(\widetilde{\varphi} - \varphi), \sin\widetilde{\alpha})\|.$$

Denoting $\phi := \widetilde{\varphi} - \varphi$, a straightforward computation yields that

$$\begin{aligned} & d(p, \widetilde{\omega})^4 \qquad\qquad (\mathrm{A.1}) \\ &= \left((\sqrt{\cos\widetilde{\alpha}}\cos\phi - s\sqrt{\cos\alpha})^2 + \cos\widetilde{\alpha}\sin^2\phi\right)^2 \\ &\quad + \left(\sin\widetilde{\alpha} - s^2\sin\alpha + 2s\sqrt{\cos\alpha}\sqrt{\cos\widetilde{\alpha}}\sin\phi\right)^2 \\ &= 1 + s^4 + s^2(6\cos\alpha\cos\widetilde{\alpha} - 2\sin\alpha\sin\widetilde{\alpha}) \\ &\quad - 4s\sqrt{\cos\alpha}\sqrt{\cos\widetilde{\alpha}}\left(\cos(\widetilde{\alpha} + \phi) + s^2\cos(\alpha - \phi)\right). \end{aligned}$$

We will estimate $d(\gamma(s, \omega), \omega)$ in two different ways, which will yield better estimates depending on which range of parameters we consider.

Lemma A.2. *If* $\omega = (\sqrt{\cos\alpha}\cos\varphi, \sqrt{\cos\alpha}\sin\varphi, \sin\alpha)$ *with* $\varphi \in [0, 2\pi)$, $\alpha \in (-\pi/2, \pi/2)$, *then*

$$d(\gamma(s, \omega), \omega) \lesssim \min\left\{\frac{1-s}{\sqrt{\cos\alpha}}, \sqrt{1-s} + \sqrt[4]{1-s}\sqrt[4]{\cos\alpha}\right\}, \quad s \in (0, 1].$$

Proof. The bound by the first expression in the minimum follows if we estimate the distance between $\gamma(s,\omega)$ and ω from above by the length of the radial curve segment that connects the two points:

$$d(\gamma(s,\omega),\omega) \le \mathrm{length}(\gamma(\cdot,\omega)|_{[s,1]}) \overset{(3.27)}{=} \int_s^1 \frac{1}{\sqrt{\cos\alpha}}\, d\sigma = \frac{1-s}{\sqrt{\cos\alpha}}.$$

To prove the second bound for $d(\gamma(s,\omega),\omega)$, we recall the formula for radial curves provided by Theorem 2.1 and apply (A.1) to $p=\gamma(s,\omega)$ and $\widetilde{\omega}=\omega$. This yields:

$$\begin{aligned} d\left(\gamma(s,\omega),\omega\right)^4 &= 1+s^4+s^2\,\left[6\cos^2\alpha-2\sin^2\alpha\right] \\ &\quad -4s\cos\alpha\,\left[\cos\left(\alpha+\tan\alpha\ln s\right)+s^2\cos\left(\alpha-\tan\alpha\ln s\right)\right]. \end{aligned} \tag{A.3}$$

It is convenient to write this formula as

$$d(\gamma(s,\omega),\omega)^4 = \sin^2\alpha(1-s^2)^2+\cos^2\alpha(1-s)^4-\Lambda$$

where

$$\Lambda := 4s\cos\alpha\left[-\sin\alpha\sin(\tan\alpha\ln s)(1-s^2)-\cos\alpha(1+s^2)(1-\cos(\tan\alpha\ln s))\right].$$

To conclude the proof, it suffices to find a suitable upper bound for $|\Lambda|$. If $s\in(0,1/4]$, then we simply use $|\Lambda|\lesssim s\le(1-s)^2$, which yields in that case $d(\gamma(s,\omega),\omega)\lesssim\sqrt{1-s}$. On the other hand, if $s\in(1/4,1]$, then by the mean value theorem, we find that

$$\begin{aligned} |\Lambda| \lesssim& \cos\alpha(1-s^2)\left|\sin(\tan\alpha\ln s)-\sin(\tan\alpha\ln 1)\right| \\ &+\cos^2\alpha(1+s^2)\left|\cos(\tan\alpha\ln s)-\cos(\tan\alpha\ln 1)\right| \\ \lesssim&(1-s)|\ln s-\ln 1|+\cos\alpha|\ln s-\ln 1| \\ \lesssim&(1-s)^2+(1-s)\cos\alpha, \end{aligned}$$

which yields the desired estimate in that case. □

Lemma A.4. *There exists $s_0\in(0,1)$ such that for all $s\in[s_0,1]$ the following holds. If $\omega=(\sqrt{\cos\alpha}\cos\varphi,\sqrt{\cos\alpha}\sin\varphi,\sin\alpha)$ with $\varphi\in[0,2\pi]$, $\alpha\in(-\pi/2,\pi/2)$, $|\alpha|\ge c$, then*

$$d(\gamma(s,\omega),\partial B)^4 \gtrsim_c \begin{cases} (1-s)^2, & \text{if } 1-s\ge\cos\alpha, \\ \frac{(1-s)^4}{\cos^2\alpha}, & \text{if } 1-s\le\cos\alpha. \end{cases}$$

Proof. Using formula (A.1) for an arbitrary point $\widetilde{\omega}=(\sqrt{\cos\widetilde{\alpha}}e^{i\widetilde{\varphi}},\sin\widetilde{\alpha})\in\partial B$, we write

$$d(\gamma(s,\omega),\widetilde{\omega})^4 = \left(\sqrt{1+s^4+2s^2\cos(\alpha+\widetilde{\alpha})}\right)^2+\left(2s\sqrt{\cos\alpha}\sqrt{\cos\widetilde{\alpha}}\right)^2$$

$$- 4s\sqrt{\cos\alpha}\sqrt{\cos\widetilde{\alpha}}\left(\cos(\widetilde{\alpha}+\phi)+s^2\cos(\alpha-\phi)\right), \tag{A.5}$$

where $\phi = \widetilde{\varphi} - \varphi + \tan\alpha \ln s$. We claim that for all $\phi \in \mathbb{R}$ it holds that

$$[\cos(\widetilde{\alpha}+\phi)+s^2\cos(\alpha-\phi)]^2 \leq 1+s^4+2s^2\cos(\alpha+\widetilde{\alpha}). \tag{A.6}$$

Factoring out the left-hand side, the claim is easily seen to be equivalent to

$$2s^2\cos(\widetilde{\alpha}+\phi)\cos(\alpha-\phi) \leq \sin^2(\widetilde{\alpha}+\phi)+2s^2\cos(\alpha+\widetilde{\alpha})+s^4\sin^2(\alpha-\phi).$$

Using trigonometric formulas, it follows that (A.6) is further equivalent to

$$\begin{aligned} 2s^2\cos\left((\widetilde{\alpha}+\phi)+(\alpha-\phi)\right)+2s^2\sin(\widetilde{\alpha}+\phi)\sin(\alpha-\phi) \leq \sin^2(\widetilde{\alpha}+\phi)+2s^2\cos(\alpha+\widetilde{\alpha}) \\ + s^4\sin^2(\alpha-\phi), \end{aligned}$$

which is clearly true since $0 \leq \left(\sin(\widetilde{\alpha}+\phi)-s^2\sin(\alpha-\phi)\right)^2$ holds for all ϕ. Thus, claim (A.6) is proven. Assume now that $\widetilde{\omega} \in \partial B$ realizes the distance $d(\gamma(s,\omega),\partial B)$. Inserting (A.6) in (A.5), we find that

$$\begin{aligned} d(\gamma(s,\omega),\partial B)^4 &\overset{(A.6)}{\geq} \left(\sqrt{1+s^4+2s^2\cos(\alpha+\widetilde{\alpha})}-2s\sqrt{\cos\alpha\cos\widetilde{\alpha}}\right)^2 \\ &\geq \left(\frac{1-2s^2\cos(\alpha-\widetilde{\alpha})+s^4}{\sqrt{1+2s^2\cos(\alpha+\widetilde{\alpha})+s^4}+2s\sqrt{\cos\alpha\cos\widetilde{\alpha}}}\right)^2 \\ &\geq \frac{(1-s^2)^4}{\left(\cos\alpha+s^2\cos\widetilde{\alpha}+2s\sqrt{\cos\alpha\cos\widetilde{\alpha}}+|\sin\alpha-s^2\sin\widetilde{\alpha}|\right)^2}. \end{aligned} \tag{A.7}$$

The denominator of the last expression can be bounded from above, recalling that $|\alpha| \geq c$ and that

$$d(\gamma(s,\omega),\partial B) = d(\gamma(s,\omega),\widetilde{\omega}) \geq |s\sqrt{\cos\alpha}-\sqrt{\cos\widetilde{\alpha}}| \geq |\sqrt{\cos\alpha}-\sqrt{\cos\widetilde{\alpha}}|-(1-s)\sqrt{\cos\alpha}.$$

Since $1-s \leq d(\gamma(s,\omega),\partial B)$, this yields the following estimates

$$\begin{aligned} |\sqrt{\cos\alpha}-\sqrt{\cos\widetilde{\alpha}}| &\lesssim d(\gamma(s,\omega),\partial B) \\ |\cos\alpha-\cos\widetilde{\alpha}| &\lesssim \left(\sqrt{\cos\alpha}+d(\gamma(s,\omega),\partial B)\right)d(\gamma(s,\omega),\partial B) \\ |\alpha-\widetilde{\alpha}| &\lesssim_c \left(\sqrt{\cos\alpha}+d(\gamma(s,\omega),\partial B)\right)d(\gamma(s,\omega),\partial B). \end{aligned}$$

This, in turn, yields the following estimate for the denominator in (A.7) above,

$$
\begin{aligned}
&\Big|\cos\alpha + s^2\cos\widetilde{\alpha} + 2s\sqrt{\cos\alpha\cos\widetilde{\alpha}} + |\sin\alpha - s^2\sin\widetilde{\alpha}|\Big| \\
&\qquad \le s^2(\cos\alpha - |\sin\alpha|) + 2s\cos\alpha + \cos\alpha + |\sin\alpha| \\
&\qquad + s^2|\cos\alpha - \cos\widetilde{\alpha}| + s^2|\sin\alpha - \sin\widetilde{\alpha}| + 2s\sqrt{\cos\alpha}|\sqrt{\cos\widetilde{\alpha}} - \sqrt{\cos\alpha}| \\
&\qquad \lesssim_c \cos\alpha + (1-s^2)|\sin\alpha| + \big(\sqrt{\cos\alpha} + d(\gamma(s,\omega),\partial B)\big)\, d(\gamma(s,\omega),\partial B) \\
&\qquad =: I_1 + I_2 + I_3.
\end{aligned}
$$

To conclude the proof, it suffices to observe that if $1-s \ge \cos\alpha$, then $I_2 \gtrsim \max\{I_1, I_3\}$, while if $1-s \le \cos\alpha$, then $I_1 \gtrsim \max\{I_2, I_3\}$. The first inequality is a consequence of the direct estimate relying on $|\alpha| \ge c$, $s \ge s_0$ and on the following observation:

$$1 - s \ge \sqrt{\cos\alpha}\sqrt{1-s} \gtrsim \sqrt{\cos\alpha}\, d(\gamma(s,\omega),\partial B).$$

Thus $I_2 \gtrsim \max\{I_1, I_3\}$. If $1-s \le \cos\alpha$, the inequality $I_1 \gtrsim \max\{I_2, I_3\}$ follows similarly. $\square$

With Lemmas A.2 and A.4 at hand, we are ready to prove Proposition 2.15.

Proof of Proposition 2.15. The goal is to find $\kappa > 0$ such that for all $\omega = (\sqrt{\cos\alpha}e^{i\varphi}, \sin\alpha)$,

$$d(\gamma(s,\omega),\omega) \le (1+\kappa) d(\gamma(s,\omega),\partial B), \quad \text{for all } s \in (0,1). \tag{A.8}$$

We fix a constant $0 < c < \pi/2$. If $|\alpha| \le c$ (and hence $\cos\alpha \sim_c 1$), Lemma A.2 yields

$$d(\gamma(s,\omega),\omega) \lesssim_c 1 - s = 1 - \|\gamma(s,\omega)\| \le d(\gamma(s,\omega),\partial B), \quad s \in (0,1). \tag{A.9}$$

If $|\alpha| \ge c$, we first prove the estimate under the assumption that $s \in [s_0, 1)$ for $s_0 < 1$ as in Lemma A.4. In this situation, we discuss separately the cases $1 - s \le \cos\alpha$ and $1 - s \ge \cos\alpha$, using the two different bounds provided by Lemma A.2. If $1 - s \le \cos\alpha$, we deduce

$$d(\gamma(s,\omega),\omega) \lesssim \frac{1-s}{\sqrt{\cos\alpha}} \lesssim_c d(\gamma(s,\omega),\partial B) \quad \text{for all } s \in [s_0, 1). \tag{A.10}$$

If $1 - s \ge \cos\alpha$, we obtain

$$d(\gamma(s,\omega),\omega) \lesssim \sqrt{1-s} \lesssim_c d(\gamma(s,\omega),\partial B) \quad \text{for all } s \in [s_0, 1). \tag{A.11}$$

Finally, if $|\alpha| \ge c$ and $s \in (0, s_0)$, then we simply use the crude estimate

$$d(\gamma(s,\omega),\omega) \lesssim 1 \le \frac{1}{1-s_0} d(\gamma(s,\omega),\partial B). \tag{A.12}$$

Combining the estimates (A.9) - (A.12), we find κ such that (A.8) holds as desired. $\square$

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