Backward Stochastic Differential Equations in Dynamics of Life Insurance Solvency Risk

Onni Hinkkanen

Master's Thesis in Mathematics

University of Jyväskylä
Department of Mathematics and Statistics
Autumn 2022

Abstract

In this thesis we describe the dynamics of solvency level in life insurance contracts. We do this by representing the underlying sources of risk and the solvency level as the solution to a forward-backward stochastic differential equation system. We start by introducing Brownian motion, stochastic integration, stochastic differential equations, and backward stochastic differential equations. With these notions described we can start constructing the model for solvency risk. Afterwards we also give a link to partial differential equation theory and a Monte Carlo example for obtaining explicit representations for the processes involved.

We will denote the net value of the contract by a process N, which will depend on underlying economic and demographic variables. We say that the contract is solvent at time t if $N_t \geq 0$. We can express the change in solvency probability at the expiry time T as

$$\mathbb{P}(N_T \geq 0|\mathcal{F}_t) - \mathbb{P}(N_T \geq 0|\mathcal{F}_0) = \int_0^t U_r^\top dM_r^X = \int_0^t Z_r^\top dB_r,$$

where the filtration $(\mathcal{F}_t)_{t\geq 0}$ describes the information available at time t, M_r^X is the martingale part from Doob's decomposition of the process X. Furthermore, the progressively measurable processes U and Z represent the contributions of the aforementioned underlying variables to the overall solvency risk, and the effects the Brownian driver B has on the solvency level, respectively.

More technically, the forward-backward system we study is of the form

$$\begin{cases} d(X_s, V_s^-)^\top = \tilde{\mu}(s, X_s, V_s^-)ds + \tilde{\sigma}(s, X_s)dB_s, & (X_t, V_t^-)^\top = (v, x)^\top \\ -dY_s = -Z_s^\top dB_s, & Y_T = \Psi\left(X_T^{(t,x)}, V_T^{-(t,x,v)}\right), \end{cases}$$

where $\tilde{\mu}$ and $\tilde{\sigma}$ are used in defining the process X and contain the information on actuarial assumptions, V^- is the retrospective reserve, which describes the present value of assets that belong to the insurance contract at each time t, and Ψ is a terminal condition, which in our case is not continuous. Under some Lipschitz, boundedness and continuity conditions it will yield a unique, square integrable solution $(X_s, V_s^-, Y_s, Z_s)_{s\in[t,T]}$ which we use for the description of solvency level in two different viewpoints; one considering the effects of the underlying demographic variables and the other studying the contributions of the Brownian driver.

Backwards Stochastic Differential Equations in Life Insurance Solvency Risk Dynamics

Tiivistelmä

Tässä tutkielmassa tutkimme henkivakuutussopimuksen solvenssiriskin dynamiikkaa esittämällä taustalla vaikuttavat riskitekijät sekä solvenssistatuksen ratkaisuina etu-takaperoiseen stokastiseen differentiaaliyhtälösysteemiin (*forward-backward stochastic differential equation system*). Solvenssitaso kuvaa tilanteessamme vakuutusyhtiön kykyä hoitaa vastattavansa minä tahansa aikana, toisin sanottuna yhtiön kykyä maksaa myymänsä vakuutusten korvaukset.

Ensiksi esittelemme stokastisen prosessin, stokastisen integraalin, stokastisen differentiaaliyhtälön sekä takaperoisen stokastisen differentiaaliyhtälön käsitteet. Näiden avulla rakennamme mallin solvenssiriskin analysoimiselle. Lopuksi yhdistämme solvenssiriskin laskemisen osittaisdifferentiaaliyhtälöteoriaan sekä esittelemme Monte Carlo -esimerkin, jonka avulla voimme löytää eksplisiittiset muodot mallissa käytettäville prosesseille.

Kuvaamme henkivakuutuksen arvoa prosessilla N, joka määritellään siten, että se riippuu taustalla vaikuttavista ekonomisista ja demograafisista muuttujista. Sanomme, että sopimus on vakavarainen (*solvent*) ajassa t, jos $N_t \geq 0$; toisin sanottuna sopimuksen arvo on positiivinen ajanhetkessä t. Voimme ilmaista muutoksen solvenssitodennäköisyydessä vakuutussopimuksen päättymishetkellä T muodossa

$$\mathbb{P}(N_T \geq 0|\mathcal{F}_t) - \mathbb{P}(N_T \geq 0|\mathcal{F}_0) = \int_0^t U_r^\top dM_r^X = \int_0^t Z_r^\top dB_r,$$

jossa filtraatio $(\mathcal{F}_t)_{t\geq 0}$ kuvaa ajanhetkellä t saatavilla olevaa informaatiota, M_r^X on Doobin dekomposition mukainen martingaaliosa prosessista X, jolla kuvataan taustalla vaikuttavia muuttujia. Lisäksi prosessit U ja Z ovat progressiivisesti mitalliset; U kuvailee taustamuuttujien vaikutusta solvenssiriskiin ja Z kuvailee stokastisen integraation taustalla olevan Brownin liikkeen B vaikutusta sopimuksen solvenssitasoon. Prosessin Z voidaan katsoa jakautuvan vielä eteenpäin kahteen osaan, joista toinen kuvailee prosessin X heilahteluja ja toinen kuvailee niiden vaikutusta solvenssiriskiin.

Etu-takaperoinen systeemi, jota tutkimme, on muodossa

$$\begin{cases} d(X_s, V_s^-)^\top = \tilde{\mu}(s, X_s, V_s^-)ds + \tilde{\sigma}(s, X_s)dB_s, & (X_t, V_t^-)^\top = (v, x)^\top \\ -dY_s = -Z_s^\top dB_s, & Y_T = \Psi\left(X_T^{(t,x)}, V_T^{-(t,x,v)}\right), \end{cases}$$

missä X on edellä mainittu prosessi, joka määritellään funktioiden $\tilde{\mu}$ ja $\tilde{\sigma}$ avulla, jotka kuvailevat taustalla vaikuttavia aktuaarisia oletuksia, prosessi V^- kuvailee sopimukseen liittyvien varojen arvoa kullakin ajanhetkellä t ja satunnaismuuttuja Ψ on prosessin Y päätearvo hetkellä T. Tietyillä Lipschitz-, rajoittuneisuus- ja jatkuvuusehdoilla systeemi saa yksikäsitteisen, neliöintegroituvan ratkaisun, jonka avulla pystymme kuvailemaan solvenssiriskiä kahdelta eri näkökulmalta. Nämä näkökulmat ovat prosessien U ja Z määrittelytapojen mukaiset.

Contents

1. Introduction

The main focus of this thesis is to study the dynamics of solvency status of a life insurance contract with the help of Backward Stochastic Differential Equation theory. We will use the shorthand BSDE for these equations. The goal is to describe the change in solvency probability at expiry as

$$\mathbb{P}(N_T \geq 0|\mathcal{F}_t) - \mathbb{P}(N_T \geq 0|\mathcal{F}_0) = \int_0^t U_r^\top dM_r^X = \int_0^t Z_r^\top dB_r, \tag{1.1}$$

where the martingale M_r^X describes the effects of the random fluctuations of underlying monetary and demographic assumptions, U represents the contributions of these variables on the risk, and the process Z describes the effects the Brownian motion has on the solvency level. In our context, solvency level describes the ability of the insurance company to pay out its liabilities at any given time. This is regulated on the portfolio level by the European Union's 'Solvency II' regime. However, the viewpoint of a single contract is also valuable.

To this end, we start by laying out the notation used in the thesis. After that, in Section 3 we introduce the basic notions of probability theory, such as the stochastic basis, and stochastic processes with some examples. One of the most important class of these processes is the martingale. We will be working with a time set $I = [0, T]$ with a finite terminal time T, since the setting lends itself naturally to our focus on BSDEs, which are usually introduced with a terminal condition on a finite time horizon.

One other very important process is the Brownian motion, which will also be introduced in this section. It is the fundamental stochastic process on which we build the notion of stochastic integrals, Stochastic Differential Equations and BSDEs. We also introduce the usual conditions and augmentation of the natural filtration of stochastic processes, since they will be of technical importance on the later theory. We will only state their importance and not focus on the counterexamples for cases when these conditions fail.

Stochastic integration is a way to generalize the Rieman-Stieltjes integral to a setting with random fluctuations. We introduce this generalization in Section 4. Here both the integrator and the integrand will be a stochastic process. The main focus will be the case where one integrates over the Brownian motion. We will focus on a certain space of suitable integrands, namely $\mathcal{L}_2$, although other spaces can be considered as well.

After this, we are ready to start working towards the Stochastic Differential Equation, SDE in short, in Section 5. It is a natural extension of the deterministic differential equation to the setting where we have stochastic integrals. Similarly to the 'usual' differential equation, the SDEs have a starting value and will consider a stochastic process which describes the randomness with respect to the time variable. The random behaviour shall include two terms, both a Lebesgue integral and a stochastic integral. One can model particular random behaviour with the help of an SDE, and by solving the SDE, one may find an explicit form for the modeled behaviour.

The famous Itô formula is also introduced in this section. It is analogous to the chain rule in Leibnitz-Newtonian calculus. It is of enormous help in solving SDEs and BSDEs and will be applied numerously throughout the thesis.

In Section 6 the definition and a uniqueness and existence result will be presented for the general BSDE. The concept was first introduced by Bismut in [**2**]. The theory was built on later by multiple mathematicians including Pardoux and Peng in [**16**] and [**17**], El Karoui, Peng and Quenez in [**6**] and Pardoux in [**15**]. The aforementioned section only holds a few results, since everything we necessitate for our main goal is introduced in Section 7. There we state the assumptions for our setting and prove the results we need later on.

Lastly, we concentrate on the actuarial view of BSDEs in Section 8. We will construct the forward-backward stochastic differential equations needed to model prospective liabilities and retrospective reserve of a life insurance contract. After that, we prove the solutions are unique and yield an explicit formula which we can use to study the dynamics of solvency risk in said contract. We also link the study of backward stochastic differential equations and partial differential equations in a specific case with the help of the Thiele equation for life insurance and the Feynman-Kac formula.

2. Notation

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$. We will use the following notation in the thesis for $p, q \in (1, \infty)$:

- $\mathcal{L}^0$ the space of simple processes,
- $\mathcal{L}_0(\mathbb{F})$ the space of $\mathbb{F}$-progressively measurable processes,
- $\mathcal{L}_{p,q}(\mathbb{F})$ the space of processes X with $X \in \mathcal{L}_0(\mathbb{F})$ and $\mathbb{E}\left[\left(\int_0^T |X_s|^p ds\right)^{\frac{q}{p}}\right] < \infty$, where we abbreviate $\mathcal{L}_p(\mathbb{F}) := \mathcal{L}_{p,p}(\mathbb{F})$,
- $\mathcal{L}_p^{loc}(\mathbb{F}) := \{X \in \mathcal{L}_0(\mathbb{F}) : \int_0^T |X_s|^p ds < \infty, \text{a.s.}\}$,
- $a \wedge b := \min\{a, b\}$,

If the process takes values in, say $\mathbb{R}^d$, and we want to emphasize the dimension, we will use the notations $\mathcal{L}_p(\mathbb{F}, \mathbb{R}^d)$ for each of the spaces stated above. If the filtration or the sigma-algebra is obvious, it may be omitted.

3. Stochastic Processes

In this section we will introduce the notion of a stochastic process and some definitions concerning stochastic processes. We will define a significant class of processes, the martingale. Martingales are processes for which the conditional expectation of the next value in the sequence at a particular time is equal to the present value. This means, in essence, that the expectation of the sequence of random variables does not depend on its history. This property is used, for example, in fair game theory, mathematical finance, risk theory and many other fields of study. Later we will take a look at an important stochastic process, the Brownian Motion. It is used widely in stochastic analysis and forms the basis of the stochastic integral, which we will define later.

Definition 3.1. Let $I = [0, T]$ for some $0 < T < \infty$. We call a family of random variables $X := (X_t)_{t\in I} := (X(t,\omega))_{t\in I,\omega\in\Omega}$ with $X : I \times \Omega \to \mathbb{R}$ a stochastic process with an index set I.

We only choose the finite time horizon, since our main focus is in Backward Stochastic Differential Equations which have a finite terminal time. One can choose other sets of time, for instance $I = [0, \infty)$ or $I = \mathbb{N}$ for discrete processes. We will, however, not pursue this further.

Definition 3.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of sigma-algebras $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ is called a *filtration*, if for all $0 \le s \le t; s, t \in I$ it holds that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$. We call the quadruplet $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ a stochastic basis.

Definition 3.3. Let $I = [0, T]$ and $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ be a stochastic basis. We say it satisfies the usual conditions, if

(i) the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete,
(ii) all null-sets of $\mathcal{F}$ are contained in $\mathcal{F}_0$,
(iii) the filtration $\mathbb{F}$ is right-continuous, i.e., $\bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t$ for all $t, t+\epsilon \in [0, T]$.

Definition 3.4. Let $X = (X_t)_{t\in I}$ be a stochastic process on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$. The process X is said to be

- *measurable*, if the function $X : I \times \Omega \to \mathbb{R}, X(t,\omega) := X_t(\omega)$ is $(\mathcal{B}(I) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable,
- *progressively measurable* with respect to the filtration $\mathbb{F}$, or $\mathbb{F}$-measurable, provided that for all $s \in I$ the function $X : [0, s] \times \Omega \to \mathbb{R}$ with $X(t,\omega) := X_t(\omega)$, is $(\mathcal{B}([0, s]) \otimes \mathcal{F}_s, \mathcal{B}(\mathbb{R}))$-measurable,
- *adapted* with respect to the filtration $\mathbb{F}$ given that for all $t \in I$ it holds that X_t is $\mathcal{F}_t$-measurable,
- *a modification* of a process $Y = (Y_t)_{t\in I}$ provided that
$$\mathbb{P}(X_t = Y_t) = 1 \quad \text{for all } t \in I.$$

Proposition 3.5. *An adapted process such that all trajectories are left-continuous (or right-continuous) is progressively measurable.*

Proof. The idea of the proof is to check the measurability for step stochastic processes approximating the process itself and then use pointwise convergence. For technical details, see eg. [**9**, Proposition 2.1.11] □

Definition 3.6. Let $X = (X_t)_{t\in[0,T]}$ be a $\mathbb{F}$-progressively measurable stochastic process on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$, $\mathcal{F}^{X_t}$ be the natural sigma-algebra generated by X_t and $\mathcal{F}_t^X := \sigma(\mathcal{F}^{X_s} : s \in [0,t])$ be a sigma-algebra. Now $\mathbb{F}^X := (\mathcal{F}_t^X)_{0\le t\le T}$ is the natural filtration generated by the stochastic process X.

Definition 3.7. Let $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ be a stochastic basis. The process $M = (M_t)_{t\in I}$ is called a *martingale* if the following properties hold:

(i) M is adapted to the filtration $\mathbb{F}$,
(ii) M is integrable, i.e.,
$$\mathbb{E}|X_t| < \infty \text{ for all } t \in I,$$
(iii) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ a.s. for all $0 \le s \le t \le T$.

Definition 3.8. Let $M = (M_t)_{t\in I}$ be a martingale. It belongs to the space of continuous and square integrable martingales starting at zero, denoted by $\mathcal{M}_2^{c,0}$, if

(i) $\mathbb{E}(M_t)^2 < \infty$ for all $t \in I$, i.e. M is square integrable,
(ii) the paths $t \mapsto M_t(\omega)$ are continuous for all $\omega \in \Omega$,
(iii) $M_0 = 0$.

3.1. Brownian Motion.

In this subsection we introduce the Brownian motion. The definitions and propositions follow [**9**]. The proofs can be read there. They have been omitted from the thesis due to not being the main focus.

The Brownian motion can intuitively be thought to be the limit of a (scaled) random walk. It has some important properties, such as normally distributed and independent increments. The Brownian motion has continuous paths which are nowhere differentiable. It is also a Markov process and a martingale.

Definition 3.9 (Brownian motion)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $B = (B(t,\omega))_{t\ge0,\omega\in\Omega}$ is called a (standard) Brownian motion provided that the following conditions hold:

(i) $\mathbb{P}(B(0) = 0) = 1$,
(ii) All paths $t \to B(t,\omega)$ are continuous,
(iii) The random variable $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$ for all $0 \le s < t < \infty$. That means for all $A \in \mathcal{B}(\mathbb{R})$ we have
$$\mathbb{P}(B(t) - B(s) \in A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{x^2}{2(t-s)}}\,dx,$$
(iv) $B(t)$ has independent increments, i.e., for any $n \in \mathbb{N}, 0 \le t_1 < t_2 < \cdots < t_n$ and $A_1, A_2, \dots A_n \in \mathcal{B}(\mathbb{R})$ one has
$$\mathbb{P}(B(t_1) \in A_1, B(t_2) - B(t_1) \in A_2, \dots, B(t_n) - B(t_{n-1}) \in A_n) =$$
$$\mathbb{P}(B(t_1) \in A_1)\mathbb{P}(B(t_2) - B(t_1) \in A_2) \cdots \mathbb{P}(B(t_n) - B(t_{n-1}) \in A_n).$$

Proposition 3.10. *Let $B = (B(t))_{t\ge0}$ be a stochastic process for which all paths are continuous and $B_0 = 0$. Then the following assertions are equivalent:*

(i) The process B is a standard Brownian motion
(ii) The process B is a Gaussian process with $\mathbb{E}B_t = 0$ and $\mathbb{E}B_tB_s = \min\{t,s\}$

Definition 3.11 ($\mathbb{F}$-Brownian motion)**.** Let $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ be a stochastic basis and $B = (B_t)_{t\in I}, B_t : \Omega \to \mathbb{R}$ be an adapted stochastic process. The process is called a (standard) $(\mathcal{F}_t)_{t\in I}$-Brownian motion, if the following conditions hold:

(i) $B_0 = 0$,
(ii) the paths $t \to B(t, \omega)$ are continuous for all $\omega \in \Omega$,
(iii) for all $0 \leq s < t$ such that $s, t \in I$ we have $B_t - B_s \sim \mathcal{N}(0, t - s)$,
(iv) for all $0 \leq s < t$ the random variable $B_t - B_s$ is independent from the sigma-algebra $\mathcal{F}_s$.

Proposition 3.12. *Let $B = (B_t)_{t\in I}$ be a Brownian motion as defined in Definition 3.9 and $(\mathcal{F}_t^B)_{t\in I}$ be its natural filtration given by $(\mathcal{F}_t^B)_{t\in I} := \sigma(B_s : s \in [0, t])$. Now B is a $(\mathcal{F}_t^B)_{t\in I}$-Brownian motion in the sense of Definition 3.11.*

Proposition 3.13 ([**21**] Corollary 2.1.10)**.** *The augmented natural filtration of a Brownian motion satisfies the usual conditions.*

Definition 3.14 (Multidimensional Brownian motion)**.**

(i) We call $B = (B_1, \dots, B_d)^\top$ a (standard) d-dimensional Brownian motion if $B_1, \dots, B_d$ are independent, one-dimensional Brownian motions.
(ii) Let $B = (B_t)_{t\in I}$ with $B_t = (B_t^1, \dots, B_t^d)^\top$. We call this a (standard) d-dimensional $\mathbb{F}$-Brownian motion provided that $(B_t^1)_{t\geq 0}, \dots, (B_t^d)_{t\geq 0}$ are independent processes, each of which is also a $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

4. Stochastic Integration

Stochastic integrals were originally introduced by Kiyosi Itô in 1944 in his paper 'Stochastic Integral' in the Proceedings of the Imperial Academy (Tokyo). The stochastic integral, or Itô integral, is a generalization of the Rieman-Stieltjes integral for random functions, such as the Brownian motion. In the stochastic integral the integrand and the integrator are stochastic processes. It has a wide range of applications in every area of science involving random fluctuations.

In the aforementioned paper, Itô also proved a theorem, later dubbed Itô's lemma or Itô's formula. It is another very important tool in stochastic analysis, mathematical finance and virtually every field utilizing the stochastic integral. It is analogous to the chain rule in Leibnitz-Newtonian calculus.

We construct the stochastic integral in the same way Itô did, as a limit of simple stochastic processes. We state the existence and uniqueness of the integral in an appropriate space and hint at an extension, which will not be pursued further. Later we introduce the famous Itô's lemma.

From this section onwards we always assume that the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ satisfies the usual conditions, since it is integral to the Itô integral, and that the process $B = (B_t)_{t\in I}$ is an $\mathbb{F}$-Brownian motion.

4.1. The Stochastic Integral. We start by constructing the stochastic integral for simple processes.

Definition 4.1. A stochastic process $L = (L_t)_{t\in I}$ is a *simple process* or a *step stochastic process* if we find for any $n \in \mathbb{N}$

(i) a time net $0 = t_0 < t_1 < \cdots < t_n = T$,
(ii) $\mathcal{F}_{t_i}$-measurable random variables $\xi_i : \Omega \to \mathbb{R}, i = 0, 1, \dots, n-1$ with $\sup_{i,\omega} |\xi_i(\omega)| < \infty$,

such that

$$L_t(\omega) = \sum_{i=1}^{n} \xi_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)}(t).$$

We shall denote the space of simple processes by $\mathcal{L}^0$.

Definition 4.2. Let $L \in \mathcal{L}^0$ and $t \in I$. We define the stochastic integral at time t by

$$I_t(L) := \sum_{i=1}^{n} \xi_{i-1}(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}). \tag{4.1}$$

Note that the integral itself is a stochastic process.

Proposition 4.3 (Linearity). *Let $L, K \in \mathcal{L}^0$ and $a, b \in \mathbb{R}$. One has that*

$$I_t(aL + bK) = aI_t(L) + bI_t(K).$$

Proof. Without loss of generality, we can assume the representations $L_t = \sum_{i=1}^{n} \xi_{i-1}\mathbb{1}_{[t_{i-1},t_i)}$ and $K_t = \sum_{i=1}^{n} \eta_{i-1}\mathbb{1}_{[t_{i-1},t_i)}$ for L and K, respectively. We get from the definition of simple processes that $aL + bK = (aL_t + bK_t)_{t\in I}$ with

$$\begin{aligned} aL_t + bK_t &= a\sum_{i=1}^{n} \xi_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)}(t) + b\sum_{i=1}^{n} \eta_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)}(t) \\ &= \sum_{i=1}^{n} a\xi_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)}(t) + b\eta_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)(t)} \\ &= \sum_{i=1}^{n} (a\xi_{i-1}(\omega) + b\eta_{i-1}(\omega))\mathbb{1}_{[t_{i-1},t_i)}(t), \end{aligned}$$

where ξ_i and η_i are the random variables for L and K, respectively. Now for the integral it holds

$$\begin{aligned} I_t(aL + bK) &= \sum_{i=1}^{n} (a\xi_{i-1} + b\eta_{i-1})(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}) \\ &= a\sum_{i=1}^{n} \xi_{i-1}(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}) + b\sum_{i=1}^{n} \eta_{i-1}(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}) \\ &= aI_t(L) + bI_t(K), \end{aligned}$$

which concludes the proof. □

Lemma 4.4. *Let $L \in \mathcal{L}^0$ and $I_t(L)$ be its integral. Then $\mathbb{E}I_t(L) = 0$ and*

$$\mathbb{E}(|I_t(L)|^2) = \int_0^t \mathbb{E}(|L_s|^2)ds,$$

where $t \in I$.

The proof of the Lemma is given in [**12**, Lemma 4.3.2].

Definition 4.5. Let $\mathcal{L}_2$ be the space of all progressively measurable processes $L = (L_t)_{t\in I}$ with $\mathbb{E}\int_0^T L_t^2 dt < \infty$.

Lemma 4.6. *Let $L \in \mathcal{L}_2$. There exist a sequence $(L^n)_{n=0}^{\infty}$ of simple processes in $\mathcal{L}^0$ such that*

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |L_t - L_t^n|^2 dt = 0.$$

We say that $L^n \to L$ with respect to the L_2 mean.

The proof of the above Lemma can be found in [**12**, Lemma 4.3.3].

Now we can construct the stochastic integral for processes in $\mathcal{L}_2$. Let $L \in \mathcal{L}_2$ and $(L^n)_{n=0}^{\infty}$ be the sequence on simple processes as in Lemma 4.6. For each n the integral is defined by Definition 4.2. By Lemma 4.4 we have that

$$\begin{aligned} \mathbb{E}(|I_t(L^n) - I_t(L^m)|^2) &= \int_0^t \mathbb{E}(|L_s^n - L_s^m|^2)ds \\ &\to 0, \quad \text{as } n, m \to \infty. \end{aligned}$$

Thus $(I_t(L^n))_{n=0}^{\infty}$ is a Cauchy sequence in probability. Because of this the limit is unique and we can define

$$I_t(L) := \lim_{n\to\infty} I_t(L^n). \tag{4.2}$$

Next we show that $I_t(L)$ is well-defined. Let $(K^n)_{n=0}^{\infty}$ be another sequence such that $K^n \to L$. Now by the linearity of I and Lemma 4.4

$$\begin{aligned}
\mathbb{E}(|I_t(L^n) - I_t(K^m)|^2) &= \mathbb{E}(|I_t(L^n - K^m)|^2) \\
&= \int_0^t \mathbb{E}(|L_s^n - K_s^m)|^2)ds \\
&= \int_0^t \mathbb{E}|(L_s^n - L_s) - (K_s^m - L_s)|^2 ds \\
&\le 2\int_0^t \mathbb{E}[|L_s^n - L_s|^2 + |K_s^m - L_s|^2]ds \\
&\to 0 \quad \text{as } n, m \to \infty,
\end{aligned}$$

where we also use the inequality $(x-y)^2 \le 2(x^2+y^2)$. From this it follows that $\lim_{n\to\infty} I_t(L^n) = \lim_{m\to\infty} I_t(K^m)$. Hence the limit $I_t(L)$ is well-defined.

Definition 4.7. A continuous modification of the limit defined in (4.2) is called the *stochastic integral* or the *Itô integral* at time t. It is denoted by $\int_0^t L_s dB_s$.

Theorem 4.8. *Let $L \in \mathcal{L}_2$ and $X = (X_t)_{t\in I}$ with*

$$X_t = \int_0^t L_s dB_s, \quad t \in I.$$

Now there exists a modification of the stochastic process $\left(\int_0^t L_s dB_s\right)_{t\in I}$, which has continuous paths on the interval I.

Proof. Cf. [**12**, Theorem 4.6.2]. □

Let us introduce some properties of the integral, mainly linearity, martingality and Itô isometry.

Proposition 4.9. *Let $L, K \in \mathcal{L}_2$ and $a, b \in \mathbb{R}$. Now*

(i) $I_t(aL + bK) = aI_t(L) + bI_t(K)$ a.s. for all $t \in I$,
(ii) $(I_t(L))_{t\in I} \in \mathcal{M}_2^{c,0}$,
(iii) $\mathbb{E}|I_t(L)|^2 = \mathbb{E}\int_0^t L_s^2 ds$ for all $t \in I$.

The proof is given in [**21**, Theorem 2.2.7].

Remark 4.10. The property (iii) is called Itô isometry.

Since the Itô integral I is linear, we get the next Corollary.

Corollary 4.11. *For any $L, K \in \mathcal{L}_2$ we get the following equality*

$$\mathbb{E}\int_0^t K_s dB_s \int_0^t L_s dB_s = \mathbb{E}\int_0^t K_s L_s ds.$$

The stochastic integral can be uniquely (up to indistinguishability) extended to the space $\mathcal{L}_2^{loc}$ and sustain mostly the same properties. Notably, the stochastic integral in the space is a local martingale, cf. Proposition 4.9 (ii). This extension is, however, not needed in the thesis and will remain a curiosity in the context of this thesis.

4.2. Itô's Formula. We are ready to introduce the celebrated Itô formula. To this end, we first introduce Itô processes and then introduce the formula. No examples will be given in this section since it will be utilized heavily later.

Definition 4.12. A continuous and adapted process $X = (X_t)_{t\geq 0}, X_t : \Omega \to \mathbb{R}$ is called an *Itô process* if there exists a process $L \in \mathcal{L}_2$, a progressively measurable process a with $\int_0^T |a_s|ds < \infty$ a.s., and an $\mathcal{F}_0$-measurable random variable X_0 such that

$$X_t = X_0 + \int_0^t L_s dB_s + \int_0^t a_s ds.$$

Definition 4.13. Let $f(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function with continuous partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$. We denote the space of these processes by $C^{1,2}([0,\infty) \times \mathbb{R})$.

Theorem 4.14. *Let $X = (X_t)_{t\geq 0}$ be an Itô process with*

$$X_t = X_0 + \int_0^t L_s dB_s + \int_0^t a_s ds, \quad t \in [0,\infty),$$

$f \in C^{1,2}$ and $\mathbb{E}\int_0^T |\frac{\partial f}{\partial x}(s,X_s)L_s|^2 ds < \infty$ for all $T > 0$. Then $f(t,X_t)$ is also an Itô process and

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)a_s ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s,X_s)L_s dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2} L_s^2 ds. \end{aligned}$$

The proof is given in [**21**, Theorem 2.3.2]. Next we introduce the multidimensional Itô formula in the spirit of [**21**].

Theorem 4.15. *Let B be a d-dimensional $\mathbb{F}$-Brownian motion. Furthermore, let a^i be progressively measurable processes for which $\int_0^T |a_s^i|ds < \infty$ a.s., $L^{i,j} \in \mathcal{L}_2, 1 \leq i \leq d_1, 1 \leq j \leq d$. Define $a := (a^1,\ldots,a^{d_1})^T$ and $L := (L^{i,j})_{1\leq i\leq d_1, 1\leq j\leq d}$, which take values in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_1\times d}$ respectively. Let $X = (X^1,\ldots,X^{d_1})^T$ be a d_1-dimensional Itô process, i.e.*

$$X_t^i = X_0^i + \sum_{j=1}^d \int_0^t L_s^{i,j} dB_s^j + \int_0^t a_s^i ds, \quad i = 1,\ldots,d_1.$$

Now for a function $f \in C^{1,2}(I \times \mathbb{R}^{d_1})$ with $\mathbb{E}\int_0^T |\frac{\partial f}{\partial X_i}(s,X_s)L_s^{i,j}|^2 ds < \infty$ for all $1 \leq i \leq d_1$, all $1 \leq j \leq d$ and all $T > 0$, it holds that

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t \left[\frac{\partial f}{\partial t}(s,X_s) + \sum_{i=1}^{d_1} \frac{\partial f}{\partial x_i}(s,X_s)a_s^i + \frac{1}{2}\sum_{i,j=1}^{d_1}\sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)L_s^{i,k}L_s^{j,k}\right] ds \\ &+ \sum_{i=1}^{d_1}\sum_{j=1}^{d}\int_0^t \frac{\partial f}{\partial x_i}(s,X_s)L_s^{i,j}dB_s^j. \end{aligned}$$

It is noteworthy to state that there are many formulations of the famous lemma, these are just two examples. One can derive the lemma for different time intervals, backward processes et cetera.

5. Stochastic Differential Equations

Again, a Stochastic Differential Equation is an analogue to the Leibnitz-Newtonian calculus' differential equation. It plays an important role in stochastic modeling, economics, biology and other fields of study. The perturbation by the Brownian motion proves to be useful in many applications. The theory of stochastic differential equations was originally motivated by K. Itô's desire to construct diffusion processes by solving stochastic differential equations.

5.1. The One-Dimensional Case. In this section we focus on stochastic differential equations (SDE) of the form

$$dX_t = \sigma(t, X_t)dB_t + \lambda(t, X_t)dt, \quad X_a = \xi. \tag{5.1}$$

Equation (5.1) can be understood as a Stochastic Integral Equation (SIE)

$$X_t = \xi + \int_a^t \sigma(s, X_s)dB_s + \int_a^t \lambda(s, X_s)ds, \quad 0 \leq a \leq t \leq T, \tag{5.2}$$

Definition 5.1. Let $\xi \in \mathbb{R}$, $\sigma, \lambda : I \times \mathbb{R} \to \mathbb{R}$ be measurable functions. Now the process $X = (X_t)_{t \in I}$ is a solution to the SDE (5.1) provided that

(i) X is jointly measurable, i.e. $(\mathcal{B}(I) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable,
(ii) $(\sigma(s, X_s))_{s \in I} \in \mathcal{L}_2$ so that $\int_a^t \sigma(s, X_s)dB_s$ is an Itô integral,
(iii) For almost all sample paths of $\lambda(t, X_t)$ it holds that $\int_0^T |\lambda(s, X_s)|ds < \infty$ a.s.,
(iv) Equation (5.2) holds almost surely for all $t \in I$.

We need to impose more restrictions to the functions σ and λ if we want to ensure unique solutions.

Definition 5.2. Let $g(t, x) : I \times \mathbb{R} \to \mathbb{R}$ be a measurable function. It satisfies the Lipschitz condition in the variable x if there exists a constant $K > 0$ such that

$$|g(t, x) - g(t, y)| \leq K|x - y| \quad \text{for all } t \in I; x, y \in \mathbb{R}.$$

Definition 5.3. Let $g(t, x) : I \times \mathbb{R} \to \mathbb{R}$ be a measurable function. It satisfies the linear growth condition in the variable x if there exists a constant $K > 0$ such that

$$|g(t, x)| \leq K(1 + |x|), \quad \text{for all } t \in I; x \in \mathbb{R}.$$

Proposition 5.4. *Let $\sigma(t, x), \lambda(t, x) : I \times \mathbb{R} \to \mathbb{R}$ be measurable functions that satisfy the Lipschitz and linear growth conditions. Let ξ be an $\mathcal{F}_a$-measurable random variable such that $\mathbb{E}(|\xi|^2) < \infty$. Then the Equation (5.2) has a unique continuous solution X.*

The proof is given in [**12**, Theorem 10.3.5]. Let us proceed with an example of solving a linear SDE.

Example 5.5. Let $x_0, \lambda_1, \lambda_2, \sigma_1, \sigma_2 \in \mathbb{R}$. Let us consider the SDE

$$dX_t = (\sigma_1 X_t + \sigma_2)dB_t + (\lambda_1 X_t + \lambda_2)dt.$$

Clearly both the integrands satisfy the Lipschitz and linear growth conditions and x_0 is a constant, hence by Proposition 5.4 we get a unique solution. Define the process $Y = (Y_t)_{t\in I}, Y_t := X_t \exp\{(\frac{\sigma_1^2}{2} - \lambda_1)t - \sigma_1 B_t\}$ and the function $f(t,x,y) := x\exp\{(\frac{\sigma_1^2}{2} - \lambda_1)t - \sigma_1 y\}$. Now we can use the Itô formula (4.15) for two processes:

$$
\begin{aligned}
Y_t = x_0 &+ \int_0^t (\frac{\sigma_1^2}{2} - \lambda_1)Y_s ds \\
&+ \int_0^t (\sigma_1 X_s + \sigma_2)\frac{Y_s}{X_s} dB_s \\
&+ \int_0^t (\lambda_1 X_s + \lambda_2)\frac{Y_s}{X_s} ds \\
&+ \int_0^t (-\sigma_1) Y_s dB_s \\
&+ \frac{1}{2}\int_0^t \sigma_1^2 Y_s ds \\
&+ \int_0^t (\sigma_1 X_s + \sigma_2)\frac{-\sigma_1 Y_s}{X_s} ds \\
&= x_0 + \int_0^t \frac{\sigma_2 Y_s}{X_s} dB_s + \int_0^t \frac{\lambda_2 - \sigma_1\sigma_2}{X_s} Y_s ds \\
&= x_0 + \int_0^t \sigma_2 \exp\{(\frac{\sigma_1^2}{2} - \lambda_1)s - \sigma_1 B_s\} dB_s \\
&+ \int_0^t (\lambda_2 - \sigma_1\sigma_2)\exp\{(\frac{\sigma_1^2}{2} - \lambda_1)s - \sigma_1 B_s\} ds
\end{aligned}
$$

for all $t \in I$ almost surely. From the definition of Y_t we get $X_t = Y_t \exp\{(\lambda_1 - \frac{\sigma_1^2}{2})t + \sigma_1 B_t\}$ and thus we have found the solution X_t.

5.2. The Multidimensional Case. Here we fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ (which we recall was assumed to satisfy the usual conditions) and a d-dimensional $\mathbb{F}$-Brownian motion B. Now the SDE can be stated as

$$X_t = \eta + \int_0^t \sigma(\omega, s, X_s) dB_s + \int_0^t \lambda(\omega, s, X_s) ds, \quad 0 \le t \le T, \quad \text{a.s.} \tag{5.3}$$

where X is d_1-dimensional, η is $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^{d_1}))$-measurable and the restrictions of the functions $\lambda : [0,T]\times\Omega\times\mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$ and $\sigma : [0,T]\times\Omega\times\mathbb{R}^{d_1} \to \mathbb{R}^{d_1\times d}$ on $[0,t]$ are, for every $t \in [0,T]$, $\big(\mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{d_1}), \mathcal{B}(\mathbb{R}^{d_1})\big)$ and $\big(\mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{d_1}), \mathcal{B}(\mathbb{R}^{d_1\times d})\big)$ measurable, respectively. We say that a progressively measurable (d_1-dimensional) process X is the solution to the SDE (5.3) if $\sigma(\cdot,\cdot,X) \in \mathcal{L}_2^{loc}(\mathbb{F}, \mathbb{R}^{d_1\times d})$ and $\lambda(\cdot,\cdot,X) \in \mathcal{L}_1^{loc}(\mathbb{F}, \mathbb{R}^{d_1})$ for all (ω, t) and the equation (5.3) holds almost surely, where $\mathcal{L}_p^{loc}(\mathbb{F}, \mathbb{R}^d)$ stands for the space of $\mathbb{F}$-progressively measurable processes X for which $\int_0^T |X_s|^p ds < \infty$ a.s.

Definition 5.6. For the rest of the thesis let

(i) $\mathcal{L}_{p,q}(\mathbb{F}, \mathbb{R}^d)$ be the space of progressively measurable processes X which take values in $\mathbb{R}^d$ with and $\mathbb{E}\left[\left(\int_0^T |X_s|^p ds\right)^{\frac{q}{p}}\right] < \infty$, where we abbreviate $\mathcal{L}_p(\mathbb{F}) := \mathcal{L}_{p,p}(\mathbb{F})$,

(ii) $\mathcal{L}_p(\mathcal{F}_0, \mathbb{R}^d)$ be the space of random variables X with $\mathbb{E}|X|^p < \infty$.

Assumption 5.7. *Let the following hold:*

- *σ and λ are uniformly Lipschitz continuous, i.e. there exists a constant $L \geq 0$ such that, for (t, ω),*

$$|\sigma_t(x_1) - \sigma_t(x_2)| + |\lambda_t(x_1) - \lambda_t(x_2)| \leq L|x_1 - x_2|, \text{ for all } x_1, x_2 \in \mathbb{R}^{d_1}, \lambda \otimes \mathbb{P} - a.s.,$$

  where λ, by abuse of notation, denotes the Lebesgue measure.
- *$\eta \in \mathcal{L}_2(\mathcal{F}_0, \mathbb{R}^{d_1})$, $\sigma(\cdot, \cdot, 0) \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d_1 \times d})$ and $\lambda(\cdot, \cdot, 0) \in \mathcal{L}_{1,2}(\mathbb{F}, \mathbb{R}^{d_1})$ for all (ω, t).*

Theorem 5.8 ([**21**] Theorem 3.3.1)**.** *Under Assumption 5.7, there exists a unique solution X to the SDE (5.3) for which $X \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d_1})$. With these assumptions there also exists a continuous modification of X.*

6. Backward Stochastic Differential Equations

Backward Stochastic Differential Equations, BSDEs in short, are, in contrast to SDEs, defined by not a starting value, but a terminal one. The history of the theory is rich, despite its rather short existence, from linear BSDEs used in the model behind the Black and Scholes formula, according to [**18**], to more general, multidimensional nonlinear BSDEs. The first published paper on the latter appeared in 1990, see [**16**].

We shall present an existence and uniqueness result in this section that is somewhat limited in generality. In the later sections we will go more into detail about the setting needed for the main results of the thesis. For more complete and general BSDE theory, see e.g. [**21**] and [**18**].

Let us look at a BSDE of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \tag{6.1}$$

and call a *pair* of processes (Y, Z) a solution, if they are progressively measurable and satisfy some conditions introduced below. The function f is called the generator of the BSDE and the random variable ξ the terminal condition.

Assumption 6.1. *Let the generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function and let the following assumptions hold*

(i) $\xi \in L^2(\mathbb{P})$.

(ii) $f(\cdot, \cdot, y, z)$ *is progressively measurable for all* $y, z \in \mathbb{R}$.

(iii) There exists a constant $L > 0$ *such that*

$$|f(t, \omega, y_1, z_1) - f(t, \omega, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|)$$

for all $y_1, y_2, z_1, z_2 \in \mathbb{R}$ *and all* $(t, \omega) \in [0, T] \times \Omega$.

(iv) $\mathbb{E} \int_0^T f^2(t, 0, 0)dt < \infty$.

Theorem 6.2 ([**9**] Theorem 5.3.2)**.** *Let $\mathcal{S}_2$ be the space of adapted and continuous processes X such that $\mathbb{E} \sup_{0 \leq t \leq T} |X_t|^2 < \infty$ and the Assumptions 6.1 hold. Then the BSDE (6.1) has a unique solution $(Y, Z) \in \mathcal{S}_2 \times \mathcal{L}_2$.*

Let us examine an illustrative example before introducing multidimensional BSDEs.

Example 6.3. Assume the linear BSDE

$$Y_t = \xi + \int_t^T (a_s + b_s Y_s + c_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \tag{6.2}$$

where $(b_s)_{s\in[0,T]}$ and $(c_s)_{s\in[0,T]}$ are bounded and $\mathbb{F}$-progressively measurable, $(a_s)_{s\in[0,T]} \in \mathcal{L}_2$ and $\xi \in L^2(\mathbb{P})$. Let

$$\Gamma_t = 1 + \int_0^t \Gamma_s b_s ds + \int_0^t \Gamma_s c_s dB_s. \tag{6.3}$$

We claim that the BSDE (6.2) has a unique solution and that for the Y-process it holds that

$$Y_t = \mathbb{E}\left[\frac{\Gamma_T}{\Gamma_t}\xi + \int_t^T \frac{\Gamma_s}{\Gamma_t} a_s ds | \mathcal{F}_t\right].$$

First we check that Assumption 6.1 holds. Let us denote the driver by $f(s,y,z) = a_s + b_s y + c_s z$.

(i) By assumption $\xi \in L^2(\mathbb{P})$.
(ii) We have that the processes a, b and c are progressively measurable. Hence f is progressively measurable.
(iii) Since the processes b and c are bounded, we find an upper bound for which f is uniformly Lipschitz.
(iv) For the function f we have $\mathbb{E}\int_0^T f^2(t,0,0)dt = \mathbb{E}\int_0^T a_t^2 dt < \infty$ by assumption.

Since all the assumptions hold, Theorem 6.2 yields the unique solution. Next, we have

$$\begin{aligned} Y_t - Y_0 &= \xi + \int_t^T (a_s + b_s Y_s + c_s Z_s)ds - \int_t^T Z_s dB_s \\ &- \xi - \int_0^T (a_s + b_s Y_s + c_s Z_s)ds + \int_0^T Z_s dB_s \\ &= - \int_0^t (a_s + b_s Y_s + c_s Z_s)ds + \int_0^t Z_s dB_s \end{aligned}$$

for all $t \in [0,T]$ a.s., which implies

$$Y_t = Y_0 - \int_0^t (a_s + b_s Y_s + c_s Z_s)ds + \int_0^t Z_s dB_s$$

for all $t \in [0,T]$ a.s. Now, let us apply Itô's formula 4.15 for $g(t,x,y) := xy$,

$$\begin{aligned} g(t,\Gamma_t,Y_t) &= Y_0 + \int_0^t Y_s b_s \Gamma_s ds - \int_0^t \Gamma_s (a_s + b_s Y_s + c_s Z_s)ds \\ &+ \int_0^t Y_s c_s \Gamma_s dB_s + \int_0^t \Gamma_s Z_s dB_s + \int_0^t c_s \Gamma_s Z_s ds \\ &= Y_0 - \int_0^t a_s \Gamma_s ds + \int_0^t \Gamma_s (c_s Y_s + Z_s) dB_s \end{aligned}$$

for all $t \in [0,T]$ a.s., hence we obtain

$$\begin{aligned} \Gamma_t Y_t - \Gamma_T Y_T &= g(t,\Gamma_t,Y_T) - g(T,\Gamma_T,Y_T) \\ &= Y_0 - \int_0^t a_s \Gamma_S ds + \int_0^t \Gamma_s (c_s Y_s + Z_s) dB_s \\ &- Y_0 + \int_0^T a_s \Gamma_S ds - \int_0^T \Gamma_s (c_s Y_s + Z_s) dB_s \\ &= \int_t^T a_s \Gamma_s ds - \int_t^T \Gamma_s (c_s Y_s + Z_s) dB_s \end{aligned}$$

for all $t \in [0,T]$ a.s. from which we deduce

$$\Gamma_t Y_t = \Gamma_T Y_T + \int_t^T a_s \Gamma_s ds - \int_t^T \Gamma_s (c_s Y_s + Z_s) dB_s$$

for all $t \in [0,T]$ a.s. Now we can take conditional expectation w.r.t. $\mathcal{F}_t$ on both sides to get

$$\Gamma_t Y_t = \mathbb{E}\left[\Gamma_T Y_T + \int_t^T a_s \Gamma_s ds \middle| \mathcal{F}_t\right]. \tag{6.4}$$

for all $t \in [0,T]$ a.s. This holds because $\mathbb{E}[\Gamma_t Y_t|\mathcal{F}_t] = \Gamma_t Y_t$ and because the conditional expectation of the Itô integral is zero since the integral is a martingale. For this we show that we can take the conditional expectation of $\int_t^T a_s \Gamma_s ds$. By [**9**, Theorem 4.3.2] we have for the process Γ that $\mathbb{E}\sup_{0\le t\le T}|\Gamma_t|^p < \infty$ for $p \ge 2$, so especially $\mathbb{E}\int_0^T |\Gamma_t|^2 dt < \infty$. Now we can use Hölder's inequality A.3 to attain

$$\mathbb{E}\int_t^T |a_s\Gamma_s| ds \le \left(\mathbb{E}\int_0^T a_s^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \Gamma_s^2 ds\right)^{\frac{1}{2}} < \infty$$

from which we derive the ability to take the conditional expectation. This also implies the Itô integrals are local martingales. Next, to show that the conditional expectation of the Itô integral is zero, we use Burkholder-Davis-Gundy inequalities A.5 and Lemma A.4. Let us treat the integrals $\int_t^T \Gamma_s Z_s dB_s$ and $\int_t^T \Gamma_s Y_s c_s dB_s$ separately. We get for the first integral

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\Big|\int_0^t \Gamma_s Z_s dB_s\Big| &\le \alpha_1 \mathbb{E}\sqrt{\int_0^T \Gamma_s^2 Z_s^2 ds} \\
&\le \alpha_1 \mathbb{E}\left[\sup_{t\in[0,T]}|\Gamma_t|\sqrt{\int_0^T Z_s^2 ds}\right] \\
&\le \alpha_1 \sqrt{\mathbb{E}\sup_{t\in[0,T]}|\Gamma_t|^2}\sqrt{\mathbb{E}\int_0^T Z_s^2 ds} < \infty,
\end{aligned}$$

hence by Lemma A.4 the integral is a martingale. The proof is similar for the other integral. Finally, we divide both sides of (6.4) by Γ_t to get

$$Y_t = \mathbb{E}\left[\frac{\Gamma_T}{\Gamma_t}\xi + \int_t^T \frac{\Gamma_s}{\Gamma_t} a_s ds | \mathcal{F}_t\right].$$

This is allowed since $\Gamma_t \neq 0$. This holds because (6.3) is a forward SDE, and from Example 5.5 we deduce that Γ is not zero.

Next we introduce a more general uniqueness and existence result for a multidimensional (nonlinear) BSDE and the assumptions required to obtain well-posedness.

Theorem 6.4 ([**21**] Theorem 4.3.1)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ be a stochastic basis, B be a d-dimensional Brownian motion on it, where we assume the filtration $\mathbb{F}$ is the augmented natural filtration generated by the Brownian motion. Consider the following nonlinear BSDE:*

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \le t \le T, \quad a.s. \tag{6.5}$$

Here we recall $\mathcal{L}_2(\mathbb{F}, \mathbb{R}^s)$ is the space of $\mathbb{F}$-progressively measurable processes X taking values in $\mathbb{R}^s$ with $\mathbb{E}\int_0^T |X_s|^2 ds < \infty$. In addition, let the following assumptions hold:

(i) $f : [0,T]\times\Omega\times\mathbb{R}^{d_2}\times\mathbb{R}^{d_2\times d} \to \mathbb{R}^{d_2}$ *is* $\big(\mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{d_2}) \otimes \mathcal{B}(\mathbb{R}^{d_2\times d}), \mathcal{B}(\mathbb{R}^{d_2})\big)$*-progressively measurable for all* $t \in [0,T]$*,*

(ii) $|f(t,\omega,y_1,z_1) - f(t,\omega,y_2,z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|)$ *for every* $y_1, y_2 \in \mathbb{R}^{d_2}, z_1, z_2 \in \mathbb{R}^{d_2\times d}$ *and all* $(t,\omega) \in [0,T] \times \Omega$*,*

(iii) $\xi \in \mathcal{L}_2(\mathcal{F}_T, \mathbb{R}^{d_2})$ *and* $f(\cdot,\cdot,0,0) \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d_2})$*.*

Now the BSDE (6.5) has a unique solution $(Y,Z) \in \mathcal{L}_2(\mathbb{F},\mathbb{R}^{d_2}) \times \mathcal{L}_2(\mathbb{F},\mathbb{R}^{d_2\times d})$*.*

7. BSDE results needed for the main result

In this section we will collect the necessary conditions and results on which the main result of this thesis is built.

Proposition 7.1 ([**16**] Lemma 2.1)**.** *$\xi \in L^2(\Omega, \mathcal{F}_1, \mathbb{P}; \mathbb{R}^d), f \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^d)$ and $g \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$. There exists a unique solution $(Y, Z) \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^d) \times \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$ for the BSDE*

$$Y_t = \xi - \int_t^1 f(s)ds - \int_t^1 [g(s) + Z_s]dB_s, \quad 0 \le t \le 1. \tag{7.1}$$

Proof. Let us define

$$Y_t = \mathbb{E}\left[Y_1 - \int_t^1 f(s)ds \middle| \mathcal{F}_t\right], \quad 0 \le t \le 1.$$

Now it follows from the Martingale Representation Theorem A.6 that there exists a process $\bar{Z} \in \mathcal{L}_2(\mathbb{F})$ such that

$$\mathbb{E}\left[Y_1 - \int_t^1 f(s)ds \middle| \mathcal{F}_t\right] = Y_0 + \int_0^1 \bar{Z} dB_s. \tag{7.2}$$

Finally, let $Z_t = \bar{Z}_t - g(t)$, $0 \le t \le 1$. One can easily see that the pair (Y, Z) solves (7.1) and is of the constructed form. □

Next we want to consider a similar BSDE of more general form. To obtain uniqueness, we need to impose some stricter restrictions.

Proposition 7.2 ([**16**] Proposition 2.2)**.** *Let $\xi \in L^2(\Omega, \mathcal{F}_1, \mathbb{P}; \mathbb{R}^d), g \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$ and $f : \Omega \times (0,1) \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ be a mapping for which the following hold:*

(i) $\mathcal{P}$ is the sigma-algebra of $\mathbb{F}$-progressively measurable subsets of $\Omega \times (0,1)$,
(ii) f is $(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d\times k}), \mathcal{B}(\mathbb{R}^d))$-measurable,
(iii) $f(\cdot, 0) \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^d)$,
(iv) $|f(t, y_1) - f(t, y_2)| \le c|y_1 - y_2|$ for some $c > 0$ and all $y_1, y_2 \in \mathbb{R}^{d\times k}$, and (t, ω) a.e.

Then there exists a unique pair $(Y, Z) \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^d) \times \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$ which solves the following BSDE:

$$Y_t = \xi - \int_t^1 f(s, Z_s)ds - \int_t^1 [g(s) - Z_s]dB_s. \tag{7.3}$$

Before the proof we note that the conditions (iii) and (iv) together imply that $f(\cdot, y(\cdot)) \in \mathcal{L}_2(\mathbb{F}, \mathbb{R}^d)$ whenever $y \in \mathcal{L}_2(\mathcal{F}, \mathbb{R}^{d\times k})$.

Proof. *Uniqueness.* Let us first look at the one-dimensional case, the multidimensional case follows. Let (Y_1, Z_1) and (Y_2, Z_2) be two solutions to the BSDE. Now, by applying Itô's formula to the function $h(t, x, y) = |x - y|^2$, we get

$$\begin{aligned}
|Y_1(t) - Y_2(t)|^2 = &-2\int_t^1 (Y_1(s) - Y_2(s))f(s, Z_1(s))ds \\
&- 2\int_t^1 (Y_1(s) - Y_2(s))(g(s) + Z_1(s))dB_s \\
&+ 2\int_t^1 (Y_1(s) - Y_2(s))f(s, Z_2(s))ds \\
&+ 2\int_t^1 (Y_1(s) - Y_2(s))(g(s) + Z_2(s))dB_s \\
&- \int_t^1 [g(s) + Z_1(s)]^2 + [g(s) + Z_2(s)]^2 - 2[g(s) + Z_1(s)][g(s) + Z_2(s)]ds \\
= &-2\int_t^1 (Y_1(s) - Y_2(s))(Z_1(s) - Z_2(s))dB_s \\
&- 2\int_t^1 (f(s, Z_1(s)) - f(s, Z_2(s)))(Y_1(s) - Y_2(s))ds \\
&- \int_t^1 |Z_1(s) - Z_2(s)|^2 ds,
\end{aligned}$$

where $\frac{\partial h}{\partial t} = 0$ and $h(1, Y_1, Y_2) = 0$. Next we show that $\sup_{0\le t\le 1} |Y_1(t) - Y_2(t)|^2$ is $\mathbb{P}$-integrable. To this end, from the above application of Itô's formula we deduce

$$\begin{aligned}
|Y_1(t) - Y_2(t)|^2 \le\ &2 \sup_{0\le t\le 1} \left| \int_t^1 (Y_1(s) - Y_2(s))(Z_1(s) - Z_2(s))dB_s \right| \\
&+ 2 \sup_{0\le t\le 1} \left| \int_t^1 \left(f(s, Z_1(s)) - f(s, Z_2(s))\right)(Y_1(s) - Y_2(s))ds \right| \\
&+ 2 \sup_{0\le t\le 1} \left| \int_t^1 |Z_1(s) - Z_2(s)|^2 ds \right|
\end{aligned}$$

Here we take the supremum and the expectation:

$$
\begin{aligned}
\mathbb{E}\sup_{0\le t\le 1}|Y_1(t)-Y_2(t)|^2 &\le 2\mathbb{E}\sup_{0\le t\le 1}\left|\int_t^1 (Y_1(s)-Y_2(s))(Z_1(s)-Z_2(s))dB_s\right| \\
&+ 2\mathbb{E}\sup_{0\le t\le 1}\left|\int_t^1 \left(f(s,Z_1(s))-f(s,Z_2(s))\right)(Y_1(s)-Y_2(s))ds\right| \\
&+ 2\mathbb{E}\sup_{0\le t\le 1}\left|\int_t^1 |Z_1(s)-Z_2(s)|^2 ds\right| \\
&\le 4\mathbb{E}\sup_{0\le t\le 1}\left|\int_0^t (Y_1(s)-Y_2(s))(Z_1(s)-Z_2(s))dB_s\right| \\
&+ 2\mathbb{E}\int_t^1 |\left(f(s,Z_1(s))-f(s,Z_2(s))\right)(Y_1(s)-Y_2(s))|ds \\
&+ 2\mathbb{E}\int_0^1 |Z_1(s)-Z_2(s)|^2 ds \\
&\le 4\alpha_1\mathbb{E}\sqrt{\int_0^1 (Y_1(s)-Y_2(s))^2(Z_1(s)-Z_2(s))^2 ds} \\
&+ 2c\mathbb{E}\int_0^1 |Y_1(s)-Y_2(s)||Z_1(s)-Z_2(s)|ds \\
&+ 2\mathbb{E}\int_0^1 |Z_1(s)-Z_2(s)|^2 ds \\
&\le 4\alpha_1\left(\mathbb{E}\sup_{0\le t\le 1}|Y_1(s)-Y_2(s)|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^1 |Z_1(s)-Z_2(s)|^2 ds\right)^{\frac{1}{2}} \\
&+ 2c\left(\mathbb{E}\sup_{0\le t\le 1}|Y_1(s)-Y_2(s)|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^1 |Z_1(s)-Z_2(s)|^2 ds\right)^{\frac{1}{2}} \\
&+ 2\mathbb{E}\int_0^1 |Z_1(s)-Z_2(s)|^2 ds.
\end{aligned}
$$

Note that for $B(t) := \int_t^1 (Y_1(s)-Y_2(s))(Z_1(s)-Z_2(s))dB_s$ it holds that

$$\sup_{0\le t\le 1}|B(t)| \le \sup_{0\le t\le 1}|B(0)-B(t)| + |B(0)| \le 2\sup_{0\le t\le 1}|B(0)-B(t)|.$$

Thus we use the BDG inequality, since $(B(0)-B(t))_{t\in[0,1]}$ is a local martingale, and Hölder's inequality. We now set $x := \left(\mathbb{E}\sup_{0\le t\le 1}|Y_1(s)-Y_2(s)|^2\right)^{\frac{1}{2}}$ and $A := \left(\mathbb{E}\int_0^1 |Z_1(s)-Z_2(s)|^2 ds\right)^{\frac{1}{2}}$ and simplify to $x^2 \le (4\alpha_1+2c)xA + 2A^2$. Furthermore, we can solve the second order inequality for x and deduct that $x<\infty$, since $\alpha_1, c<\infty$ and from our assumption concerning Z, we have that $A<\infty$.

We still need to show measurability. From (7.3) we see that Y_1 and Y_2 have continuous paths:

$$t \mapsto \int_t^1 f(s,Z_s)ds$$

is a.s. continuous since $\mathbb{E}\int_0^1 |f(s,Z_s)|ds < \infty$ and

$$t \mapsto \int_t^1 (g(s) - Z_s)dB_s \tag{7.4}$$

is a.s. continuous as well, since $\mathbb{E}\int_0^1 |g(s) - Z_s|^2 ds < \infty$. Then, by continuity, we get

$$\sup_{0\le s\le 1} |Y_1(s) - Y_2(s)|^2 = \sup_{s\in[0,1]\cap\mathbb{Q}} |Y_1(s) - Y_2(s)|^2,$$

which is measurable. Furthermore, we have $Z_1 - Z_2 \in \mathcal{L}_2(\mathbb{F})$, hence the Itô integral above is $\mathbb{P}$-integrable and its expectation is zero. Rearranging and using the linearity of expectation, the properties of the map f and the inequality $-2ab \le a^2 + b^2$ we get

$$\begin{aligned}
&\mathbb{E}|Y_1(t) - Y_2(t)|^2 + \mathbb{E}\int_t^1 |Z_1(s) - Z_2(s)|^2 ds \\
&= -2\mathbb{E}\int_t^1 \frac{1}{\sqrt{2}}(f(s,Z_1(s)) - f(s,Z_2(s)))\sqrt{2}(Y_1(s) - Y_2(s))ds \\
&\le 2\mathbb{E}\int_t^1 \frac{1}{\sqrt{2}}c|Z_1(s) - Z_2(s)|\sqrt{2}|Y_1(s) - Y_2(s)|ds \\
&= 2\mathbb{E}\int_t^1 \frac{1}{\sqrt{2}}|Z_1(s) - Z_2(s)|\sqrt{2}c|Y_1(s) - Y_2(s)|ds \\
&\le \frac{1}{2}\mathbb{E}\int_t^1 |Z_1(s) - Z_2(s)|^2 ds + 2c^2\mathbb{E}\int_t^1 |Y_1(s) - Y_2(s)|^2 ds,
\end{aligned}$$

which we rearrange again to obtain

$$\mathbb{E}|Y_1(t) - Y_2(t)|^2 + \frac{1}{2}\mathbb{E}\int_t^1 |Z_1(s) - Z_2(s)|^2 ds \le 2c^2\int_t^1 \mathbb{E}|Y_1(s) - Y_2(s)|^2 ds, \tag{7.5}$$

where we also use Fubini's theorem to change the order of integration, and especially

$$\mathbb{E}|Y_1(t) - Y_2(t)|^2 \le 2c^2\int_t^1 \mathbb{E}|Y_1(s) - Y_2(s)|^2 ds.$$

We may now apply Grönwall's lemma A.7 for $\phi(t) = \mathbb{E}|Y_1(t) - Y_2(t)|^2$ and $B = 0$ to obtain

$$\mathbb{E}|Y_1(t) - Y_2(t)|^2 \le B\exp\left(\int_t^1 2c^2\right) = 0.$$

Finally we apply this to (7.5) to see the uniqueness of Z as well. Hence the solution is unique.

Existence. Let $Z_0(t) \equiv 0$. By defining an approximating sequence $\{(Y_n(t), Z_n(t)); 0 \le t \le 1\}_{n\ge1}$ in $\mathcal{L}_2(\mathbb{F},\mathbb{R}^d) \times \mathcal{L}_2(\mathbb{F},\mathbb{R}^{d\times k})$, with the help of Lemma 7.1, recursively by

$$Y_n(t) = \xi - \int_t^1 f(s, Z_{n-1}(s))ds - \int_t^1 [g(s) + Z_n(s)]dB_s, \tag{7.6}$$

one can use a kind of Picard iteration. Lemma 7.1 is useful here, since every iteration of (7.6) is of the form required in the Lemma. By using Itô's formula and the same inequalities as above we obtain

$$\mathbb{E}(|Y_{n+1}(t)-Y_n(t)|^2)+\mathbb{E}(\int_t^1 |Z_{n+1}(s)-Z_n(s)|^2 ds)$$
$$\leq K\mathbb{E}\int_t^1 |Y_{n+1}(s)-Y_n(s)|^2 ds+\frac{1}{2}\mathbb{E}\int_t^1 |Z_n(s)-Z_{n-1}(s)|^2 ds,$$

where $K=2c^2$. Let $u_n(t)=\mathbb{E}\int_t^1 |Y_n(s)-Y_{n-1}(s)|^2 ds$ and $v_n(t)=\mathbb{E}\int_t^1 |Z_n(s)-Z_{n-1}(s)|^2 ds$ for $n\geq 1$ and $Y_0(t)\equiv 0$. Since it holds that

$$-\frac{d}{dt}(u_{n+1}(t)e^{Kt})=e^{Kt}\mathbb{E}|Y_{n+1}(t)-Y_n(t)|^2-Ke^{Kt}u_{n+1}(t),$$

we get from the previous inequality

$$-\frac{d}{dt}(u_{n+1}(t)e^{Kt})+e^{Kt}v_{n+1}\leq \frac{1}{2}e^{Kt}v_n(t). \tag{7.7}$$

Integrating from t to 1, we get

$$-u_{n+1}(1)e^K+u_{n+1}(t)e^{Kt}+\int_t^1 e^{Ks}v_{n+1}(s)ds\leq \frac{1}{2}\int_t^1 e^{Ks}v_n(s)ds, \tag{7.8}$$

where we notice $u_{n+1}(1)\equiv 0$ and by dividing by the factor e^{Kt} we obtain

$$u_{n+1}(t)+\int_t^1 e^{K(s-t)}v_{n+1}(s)ds\leq \frac{1}{2}\int_t^1 e^{K(s-t)}v_n(s)ds.$$

We have by setting $t=0$ that

$$\int_0^1 e^{Kt}v_{n+1}(t)dt\leq \frac{1}{2}\int_0^1 e^{Kt}v_n(t)dt, \tag{7.9}$$

where we also omit the term $u_{n+1}(t)$. By iterating the previous inequality we have that

$$\int_0^1 e^{Kt}v_{n+1}(t)dt\leq e^K 2^{-n}\bar{c}, \tag{7.10}$$

where $\bar{c}=\mathbb{E}\int_0^1 |Z_1(t)|^2 dt=\sup_{0\leq t\leq 1} v_1(t)$. Also,

$$u_{n+1}(0)\leq e^K 2^{-n}\bar{c}. \tag{7.11}$$

From (7.7) and the fact that $(d/dt)u_{n+1}(t)\leq 0$ we obtain

$$v_{n+1}(0)\leq Ku_{n+1}(0)+\frac{1}{2}v_n(0)\leq 2^{-n}\bar{K}+\frac{1}{2}v_n(0),$$

where $\bar{K}=\bar{c}Ke^K$. Iterating, it follows that

$$v_{n+1}(0)\leq 2^{-n}(n\bar{K}+v_1(0)). \tag{7.12}$$

Since the square roots of the right hand sides of (7.11) and (7.12) are bounded by a constant times $2^{-\delta n}\geq 2^{-\frac{n}{2}}$ with $\delta<\frac{1}{2}$, we get $\sum_{n=1}^{\infty}2^{-\delta n}<\infty$. For (7.12) this is because $n2^{-n}\leq b2^{-2\delta n}\leq b2^{-\delta n}$ for some b which depends on the choice of δ. From this we conclude that $(Y_n)_{n\geq 1}$ and $(Z_n)_{n\geq 1}$ are Cauchy sequences in $\mathcal{L}_2(\mathbb{F},\mathbb{R}^d)$ and

$\mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$, respectively. Then, from (7.6) we get that $(Y_n)_{n\geq 1}$ is also a Cauchy sequence in $L^2(\Omega; C(0,1;\mathbb{R}^d))$, since

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq 1}|Y_n(t)-Y_m(t)|^2\\
&= \mathbb{E}\sup_{0\leq t\leq 1}\left|\int_t^1 f(s,Z_{m-1}(s))-f(s,Z_{n-1}(s))ds+\int_t^1 Z_m(s)-Z_n(s)dB_s\right|^2\\
&\leq 2\mathbb{E}\sup_{0\leq t\leq 1}\left|\int_t^1 f(s,Z_{m-1}(s))-f(s,Z_{n-1}(s))ds\right|^2\\
&+2\mathbb{E}\sup_{0\leq t\leq 1}\left|\int_t^1 Z_m(s)-Z_n(s)dB_s\right|^2\\
&\leq 2\mathbb{E}\sup_{0\leq t\leq 1}\left[\int_t^1 |f(s,Z_{m-1}(s))-f(s,Z_{n-1}(s))|\,ds\right]^2\\
&+4\alpha_1^2\mathbb{E}\int_0^1 |Z_m(s)-Z_n(s)|^2\,ds\\
&<\epsilon
\end{aligned}
$$

for some ϵ and all $n,m\geq N_\epsilon$, since Z is a Cauchy sequence in $\mathcal{L}_2(\mathbb{F},\mathbb{R}^{d\times k})$ and f is Lipschitz. For the second inequality we used Burkholder-Davis-Gundy inequality once again with the same idea as in the uniqueness part of this proof. By passing to the limit in (7.6) as n tends to infinity, one can show that the pair (Y,Z) defined by

$$Y=\lim_{n\to\infty} Y_n,\quad Z=\lim_{n\to\infty} Z_n,$$

where the limit is taken in $L^2(\Omega; C(0,1;\mathbb{R}^d))$ and $\mathcal{L}_2(\mathbb{F},\mathbb{R}^{d\times k})$, respectively, solves the equation (7.3).

□

We are now ready to study the (still) more general BSDE

$$Y_t=\xi-\int_t^1 f(s,Y_s,Z_s)ds-\int_t^1[g(s,Y_s)+Z_s]dB_s. \tag{7.13}$$

Theorem 7.3 ([**16**] Theorem 3.1)**.** *Let the following hold:*

(i) $\xi\in L^2(\Omega,\mathcal{F}_1,\mathbb{P};\mathbb{R}^d)$,
(ii) $\mathcal{P}$ *is the sigma-algebra of* $\mathbb{F}$*-progressively measurable subsets of* $\Omega\times(0,1)$*,*
(iii) $f:\Omega\times(0,1)\times\mathbb{R}^d\times\mathbb{R}^{d\times k}\to\mathbb{R}^d$ *is* $(\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^{d\times k}),\mathcal{B}(\mathbb{R}^d))$*-measurable,*
(iv) $g:\Omega\times(0,1)\times\mathbb{R}^d\to\mathbb{R}^{d\times k}$ *is* $(\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d),\mathcal{B}(\mathbb{R}^{d\times k}))$*-measurable,*
(v) $f(\cdot,0,0)\in\mathcal{L}_2(\mathbb{F},\mathbb{R}^d),\quad g(\cdot,0,0)\in\mathcal{L}_2(\mathbb{F},\mathbb{R}^{d\times k})$,
(vi) There exists $c>0$ *such that* $|f(t,x_1,y_1)-f(t,x_2,y_2)|\leq c(|x_1-x_2|+|y_1-y_2|)$, $|g(t,x_1)-g(t,x_2)|\leq c|x_1-x_2|$ *for all* $x_1,x_2\in\mathbb{R}^d, y_1,y_2\in\mathbb{R}^{d\times k},(t,\omega)-a.e.$

There exists a unique pair $(Y,Z)\in\mathcal{L}_2(\mathbb{F},\mathbb{R}^d)\times\mathcal{L}_2(\mathbb{F},\mathbb{R}^{d\times k})$ *which solves the BSDE (7.13).*

Again, before the proof, we note that the conditions (iv) and (v) imply that $f(\cdot,Y(\cdot),Z(\cdot))\in\mathcal{L}_2(\mathbb{F},\mathbb{R}^d)$ and $g(\cdot,Y(\cdot),Z(\cdot))\in\mathcal{L}_2(\mathbb{F},\mathbb{R}^{d\times k})$ when $Y\in\mathcal{L}_2(\mathbb{F},\mathbb{R}^d)$ and $Z\in\mathcal{L}_2(\mathbb{F},\mathbb{R}^{d\times k})$, which will be used to apply Proposition 7.2.

Proof. *Uniqueness.* Consider again the one-dimensional case first, the multidimensional case follows. Let $(Y_1, Z_1), (Y_2, Z_2)$ be two solutions. Applying Itô's formula in the same way as in Proposition 7.3 and taking expectations on both sides we acquire

$$\begin{aligned}
&\mathbb{E}|Y_1(t) - Y_2(t)|^2 + \mathbb{E}\int_t^1 |Z_1(s) - Z_2(s)|^2 ds \\
&= -2\mathbb{E}\int_t^1 (Y_1(s) - Y_2(s))(f(s, Y_1(s), Z_1(s)) - f(s, Y_2(s), Z_2(s)))ds \\
&- \mathbb{E}\int_t^1 |g(s, Y_1(s)) - g(s, Y_2(s))|^2 ds \qquad (7.14)\\
&- 2\mathbb{E}\int_t^1 (Z_1(s) - Z_2(s))(g(s, Y_1(s)) - g(s, Y_2(s)))ds \\
&\le \hat{c}\mathbb{E}\int_t^1 |Y_1(s) - Y_2(s)|^2 ds + \frac{1}{2}\mathbb{E}\int_t^1 |Z_1(s) - Z_2(s)|^2 ds
\end{aligned}$$

for some $\hat{c}$. The result follows.
Existence. We use again a type of Picard iteration similarly to when we proved Proposition 7.3. Let $Y_0(t) \equiv 0$ and $\{(Y_n(t), Z_n(t)); 0 \le t \le 1\}_{n\ge1}$ be a sequence in $\mathcal{L}_2(\mathbb{F}, \mathbb{R}^d) \times \mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$ defined recursively by

$$Y_n(t) = \xi - \int_t^1 f(s, Y_{n-1}(s), Z_n(s))ds - \int_t^1 [g(s, Y_{n-1}(s)) + Z_n(s)]dB_s, \quad 0 \le t \le 1. \tag{7.15}$$

By applying Itô's lemma in the same way, using some inequalities, the properties of the maps f and g and Grönwall's lemma, one can get

$$\begin{aligned}
&\mathbb{E}|Y_{n+1}(s) - Y_n(s)|^2 + \mathbb{E}\int_t^1 |Z_{n+1}(s) - Z_n(s)|ds \qquad (7.16)\\
&\qquad \le \tilde{c}\left(\mathbb{E}\int_t^1 |Y_{n+1}(s) - Y_n(s)|^2 ds + \mathbb{E}\int_t^1 |Y_n(s) - Y_{n-1}(s)|^2 ds\right)
\end{aligned}$$

for some $\tilde{c}$. Define $u_n(t) = \mathbb{E}\int_t^1 |Y_n(s) - Y_{n-1}(s)|^2 ds$. It follows from (7.16) that

$$-\frac{d}{dt}u_{n+1}(t) - \tilde{c}u_{n+1}(t) \le \tilde{c}u_n(t), \quad u_{n+1}(1) = 0,$$

or

$$u_{n+1}(t) \le \tilde{c}\int_t^1 e^{\tilde{c}(s-t)}u_n(s)ds.$$

Iterating yields

$$u_{n+1}(0) \le \frac{(\tilde{c}e^{\tilde{c}})^n}{n!}u_1(0).$$

This together with (7.16) implies that $(Y_n)_{n\ge1}$ and $(Z_n)_{n\ge1}$ are Cauchy sequences in $\mathcal{L}_2(\mathbb{F}, \mathbb{R}^d)$ and $\mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$, respectively. Then, from (7.15) we get that $(Y_n)_{n\ge1}$ is also a Cauchy sequence in $L^2(\Omega; C(0, 1; \mathbb{R}^d))$, and by passing to the limit in (7.15) as n tends to infinity, we are given that the pair (Y, Z) defined by

$$Y = \lim_{n\to\infty} Y_n, \quad Z = \lim_{n\to\infty} Z_n,$$

where the limit is taken in $L^2(\Omega; C(0,1;\mathbb{R}^d))$ and $\mathcal{L}_2(\mathbb{F}, \mathbb{R}^{d\times k})$, respectively, solves the equation (7.13). The proof of the sequence being Cauchy in the corresponding L^2 space would be similar to what was done in the proof of Proposition 7.2 and will thus be omitted.

$\square$

8. The actuarial setting; application of Backward Stochastic Differential Equations in life insurance solvency risk

In this section the goal is to describe the dynamics of solvency probability of a life insurance contract. The section is largely based on [**4**]. We want to show that the change in solvency probability of the contract at expiry at time T can be expressed as

$$\mathbb{P}(N_T \geq 0|\mathcal{F}_t) - \mathbb{P}(N_T \geq 0|\mathcal{F}_0) = \int_0^t U_r^\top dM_r^X = \int_0^t Z_r^\top dB_r, \tag{8.1}$$

where M_r^X is the martingale part, according to Doob's decomposition, of a diffusion process X that describes the underlying economic and demographic variables such as mortality rate, interest rate as well as medical and lifestyle assumptions. The process U will be progressively measurable and vector valued and will represent the contribution of different economic and demographic variables to the solvency risk. The process Z, which will be the control process of a BSDE, will describe the effects the underlying Brownian motion has on the solvency level.

An insurance company's solvency level describes its ability to meet its long-term debts and financial obligations. It is a measurement of the company's financial health. For an insurer, the solvency level depends on, among other things, the portfolio and the risks of the contracts provided.

Understanding the solvency level of a contract is vital for an insurer since the European solvency regime 'Solvency II' imposes a minimum solvency probability for the insurer's portfolio. This section focuses on a single contract, which still grants some insight on the whole portfolio.

We will introduce the process N for the net value of a contract, which will be defined by the difference in prospective liabilities and retrospective reserve, in a time-continuous framework. The process will depend on underlying demographic and economic variables, modeled as a diffusion process. Liabilities can be understood as assets or contracts for which the company has a monetary duty to others, e.g. debts or insurance contract benefits to be paid. In our case, we focus on the liabilities of the insurance contract. Thus, prospective liabilities describe the liabilities in the future, which are affected by future events that may or may not happen. Retrospective reserve describes the present value of assets that belong to the insurance contract.

One can usually describe the *prospective liabilities* of a life insurance contract at time t with

$$V_t^+ = f(X_T)e^{-\int_t^T k(u,X_u)du} + \int_t^T g(s, X_s)e^{-\int_t^s k(u,X_u)du}ds, \tag{8.2}$$

where the time horizon $T > 0$ is finite and the functions f, g, k are suitable. Here the factor $\exp\{-\int_t^T k(u, X_u)du\}$ describes the discounts given and the functions f and g correspond to the terminal payment at time T and the continuous payments in the time interval $(0, T)$, respectively, both to the policy holder. Furthermore, the process X describes the underlying actuarial assumptions, and it will be assumed to be n-dimensional and of the form

$$X_t = x_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t \lambda(s, X_s)ds, \quad t \in (0, T] \tag{8.3}$$

with $x_0 \in \mathbb{R}^n$. The process V_t^+ is now the pathwise solution to the SDE

$$dV_t^+ = \big(k(t, X_t)V_t^+ - g(t, X_t)\big)\, dt, \quad t \in [0, T) \tag{8.4}$$

with $V_T^+ = f(X_T)$. Since V_t^+ depends on the future of the process X, it is in general not adapted to the natural filtration generated by X.

Example 8.1 (Prospective reserve of a life insurance)**.** Let us assume that $X_t = (X_t^1, X_t^2)^T$ where $X_t^1 = \phi(t)$ and $X_t^2 = \mu(a + t)$ are stochastic processes that describe interest rate and mortality intensity of a policyholder at age a at contract inception $t = 0$, respectively. The process X is a stochastic process and its randomness describes uncertainties on returns on investments and changes in life expectancy. Let us denote with $T > 0$ the termination time of the contract, $b(t)$ the annuity payment rate, $c(t)$ the death benefit at time t and B_T the lump sum survival benefit at time T. By letting

$$\begin{aligned} k(t, x_1, x_2) &= x_1 + x_2, \\ g(t, x_1, x_2) &= b(t) + c(t)x_2, \\ f(x_1, x_2) &= B_T \end{aligned}$$

one gets for the process V^+ the form

$$V_t^+ = B_T e^{-\int_t^T (\phi(u)+\mu(a+u))du} + \int_t^T (b(s) + c(s)\mu(a + s))e^{-\int_t^s (\phi(u)+\mu(a+u))du} ds.$$

One calls V_t^+ the *prospective reserve at time t in state alive* in classical life insurance, where ϕ and μ are modeled deterministically. However, when ϕ and μ are stochastic processes, the term prospective reserve might be misleading, since V_t^+ is not in general adapted to the filtration $\mathcal{F}_t$, so that the value of the process in the future is unknown and reserving capital cannot be done at time t.

Let us instead solve the SDE (8.4) with initial value v_0 instead of the terminal value;

$$dV_t^- = \big(k(t, X_t)V_t^- - g(t, X_t)\big)\, dt, \quad t \in (0, T], V_0^- = v_0.$$

One obtains a stochastic process V^- for which

$$V_t^- = v_0 e^{\int_0^t k(u,X_u)du} - \int_0^t g(s, X_s)e^{\int_s^t k(u,X_u)du} ds.$$

This process is, unlike V^+, adapted to the natural filtration generated by X.

Example 8.2 (Retrospective reserve of a life insurance)**.** Using the notation from the previous example, let the initial value be the lump sum premium that the policy holder pays at time 0, i.e. $v_0 = -B_0$, where the negative sign is because the payment goes in the other direction than the annuity payments, the death benefit and the survival benefit. In equation form this means

$$V_t^- = -B_0 e^{\int_0^t (\phi(u)+\mu(a+u))du} - \int_0^t (b(s) + c(s)\mu(a + s))e^{\int_s^t (\phi(u)+\mu(a+u))du} ds.$$

Here V^- represents the *retrospective reserve of the contract at time t in state alive* and, as previously stated, is, in contrast to V^+, adapted to the natural filtration generated by X. It corresponds to the present value of the assets belonging to the contract at time t.

Because one can interpret V_t^+ as prospective liabilities of the insurance contract and V_t^- as the accrued assets, both at time t, the following definition can be made:

Definition 8.3. Let the processes V^+ and V^- be defined as above. One calls

$$N_t := V_t^- - V_t^+, \quad t \in [0,T]$$

the *net value* of the life insurance contract at time t.

Lemma 8.4. *$N_t = N_0 e^{\int_0^t k(u,X_u)du}$ for all $t \in [0,T]$. Furthermore, the random variable N_0 is not $\mathcal{F}_0$-measurable.*

Proof. One has by definition that

$$N_0 = v_0 - f(X_T)e^{-\int_0^T k(u,X_u)du} - \int_0^T g(s,X_s)e^{-\int_0^s k(u,X_u)du}ds$$

so that

$$\begin{aligned} N_0 e^{\int_0^t k(u,X_u)du} &= v_0 e^{\int_0^t k(u,X_u)du} - e^{\int_0^t k(u,X_u)du} f(X_T)e^{-\int_0^T k(u,X_u)du} \\ &- e^{\int_0^t k(u,X_u)du} \int_0^T g(s,X_s)e^{-\int_0^s k(u,X_u)du}ds \\ &= v_0 e^{\int_0^t k(u,X_u)du} - f(X_T)e^{-\int_t^T k(u,X_u)du} \\ &- \int_0^T g(s,X_s)e^{\int_0^t k(u,X_u)du}e^{-\int_0^s k(u,X_u)du}ds \\ &= v_0 e^{\int_0^t k(u,X_u)du} - f(X_T)e^{-\int_t^T k(u,X_u)du} \\ &- \int_0^t g(s,X_s)e^{\int_s^t k(u,X_u)du}ds - \int_t^T g(s,X_s)e^{-\int_t^s k(u,X_u)du}ds \\ &= V_t^- - V_t^+ =: N_t. \end{aligned}$$

The second assertion follows from the fact that N_0 depends on the path of the process X_t from time $t=0$ to time $t=T$. □

Because N_t describes the value of the contract at time t, the interesting part is whether it is positive or negative.

Example 8.5 (Solvency level)**.** The insurance regulation Solvency II states that the insurer's assets shall be greater than or equal to the insurer's liabilities over a one-year period with at least probability 99.5%. Let us set the time horizon as $T=1$ and use the notation from before, interpreting v_0 as the insurer's equity at time zero and $f(X_1)$ as the value of the insurer's liabilities at time 1 on the timeline $[0,\infty)$, the solvency rule can be written as

$$\mathbb{P}(N_1 \geq 0 | X_0) \geq 0.995.$$

It is worthwhile to note that the probability restrictions are imposed on the portfolio level, but observing the contract-level dynamics helps us understand how each contract contributes to the overall solvency level. This thesis is focused on the equation (8.1). Here X is a solution to an SDE and as such it is a diffusion process and therefore a Markov process. Due to this nature, we obtain

$$\mathbb{P}(N_T \geq 0|\mathcal{F}_t) = \mathbb{P}(N_T \geq 0|X_t, V_t^-) \tag{8.5}$$

by [**7**, Thm. 2.3]. Subsequently we obtain the following Corollary.

Lemma 8.6. *The distribution function of V_t^+ conditional on X_t is given by*

$$\mathbb{P}(V_t^+ \leq v|X_t = x) = \mathbb{P}(N_T \geq 0|X_t = x, V_t^- = v)$$

for all $t \in [0, T], x \in \mathbb{R}$.

Proof. From the representation of the process N from Lemma 8.4 we see that the sign of the process only depends on the value N_0. Hence it follows that the events $\{N_T \geq 0\}$ and $\{N_t \geq 0\}$ are equal. Furthermore, the events $\{N_t \geq 0\}$ and $\{V_t^+ \leq v\}$ are equal, conditional on $V_t^- = v$ by Definition 8.3. Moreover, we have by [**7**, Theorem 2.3] that

$$\begin{aligned}
\mathbb{P}(V_t^+ \leq v|X_t = x) &= \mathbb{P}(V_t^+ \leq v|X_t = x, V_t^- = v) \\
&= \mathbb{P}(N_t \geq V_t^- - v|X_t = x, V_t^- = v) \\
&= \mathbb{P}(N_t \geq 0|X_t = x, V_t^- = v),
\end{aligned}$$

since the events $\{V_t^+ \leq v\}$ and $\{N_t \geq V_t^- - v\}$ are equal, $\{X_t = x\} \in \sigma(X_t)$ and $\{X_t = x, V_t^- = v\} \in \mathcal{F}_t$.

□

The previous Lemma gives an additional motivation to calculate (8.1), in our case by solving a BSDE, since the conditional distribution is important in accounting when evaluating the prospective liabilities. Hence our results can be used in calculating the conditional distribution as well.

In the following subsections, the behaviour of the solvency probability $\mathbb{P}(N_T \geq 0|\mathcal{F}_t)$ will be formulated as the solution of a forward-backward system. First, the forward-backward system for the prospective liabilities will be given, after which an SDE for the pair (X_t, V_t^-) will be given. Lastly, a BSDE with the terminal condition $\mathbb{1}(N_T \geq 0)$ will be constructed to study the solvency process.

8.1. A forward-backward system for the prospective liabilities. For completeness, we describe first the evolution of prospective liabilities by a forward-backward system. Let us introduce the BSDE in our context. The general BSDE is of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \tag{8.6}$$

Now the solution to (8.6) is a pair of adapted processes (Y, Z) with a terminal condition $Y_T = \xi$. We recall that the function f is called the *driver* and the process Z is known as the control process of the BSDE. Note that we are working with multi-dimensional processes X, B and Z. In our scope, we can view the process Y_s as the value of the contingent claim at time s and Z represents the portfolio process of the hedging strategy.

Firstly, for the rest of the thesis, by a solution of a BSDE we mean a pair of processes (Y_s, Z_s) for which $(Y_s)_{s\in[0,T]}$ is a continuous $\mathbb{R}$-valued adapted process and $(Z_s)_{s\in[0,T]}$ is a square-integrable $\mathbb{R}^n$-valued progressively measurable process.

Secondly, let us state the assumptions for this section, which differ from the assumptions set out in the previous sections. Hereafter we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an n-dimensional Brownian motion denoted by $B = (B_t^1, \dots, B_t^n)^\top$ with the corresponding augmented natural filtration $\mathbb{F}$. The functions μ, σ, k and g are the functions introduced before.

Assumption 8.7. *Let $\mu : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be globally Lipschitz, Borel measurable functions that are at most of linear growth:*

$$\begin{aligned} |\mu(s,x) - \mu(s,y)| + |\sigma(s,x) - \sigma(s,y)| &\le L|x-y|, \\ |\mu(s,x)|^2 + |\sigma(x)|^2 &\le L^2(1+|x|^2), \end{aligned}$$

for every $s \in [0,T]$ and $x, y \in \mathbb{R}^n$ and for some $L > 0$.

Assumption 8.8. *Let k and g be Borel measurable functions for which it holds that*

$$\begin{aligned} |k(s,x) - k(s,y)| + |g(s,x) - g(s,y)| &\le L|x-y|, \\ |g(s,x)| &\le L(1+|x|), \\ |k(s,x)| &\le L, \end{aligned}$$

for all $s \in [0,T]$, $x, y \in \mathbb{R}^n$ and for some $L > 0$.

Here Assumption 8.7 concerns the process X from (8.3) which describes the actuarial assumptions and Assumption 8.8 is used for the functions with which the prospective and retrospective liabilities are defined.

We recall that the process for prospective liabilities is given by

$$V_s^{+(t,x)} = f(X_T^{(t,x)})e^{-\int_s^T k(u,X_u^{(t,x)})du} + \int_s^T g(r, X_r^{(t,x)})e^{-\int_s^r k(u,X_u^{(t,x)})du} dr, \tag{8.7}$$

where the superscript (t,x) means that the process X begins at x at time t. The BSDE for $V_s^{+(t,x)}$ is given by

$$Y_s = f(X_T) + \int_s^T (-k(r,X_r)Y_r + g(r,X_r))dr - \int_s^T Z_r^\top dB_r, \tag{8.8}$$

where we let $Y_s := V_s^{+(t,x)}$. This combined with the process X is commonly referred to as an *uncoupled forward-backward system.*

Proposition 8.9. *Under Assumptions 8.7, 8.8 and for functions $f \in L^2$ the forward-backward system*

$$\begin{cases} dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dB_s, & X_t = x, \qquad t \le s \le T \\ -dY_s = (-k(s,X_s)Y_s + g(s,X_s))ds - Z_s^\top dB_s, & Y_T = f(X_T), t \le s \le T \end{cases}$$

admits a unique square-integrable solution $\left(X_s^{(t,x)}, Y_s^{(t,x)}, Z_s^{(t,x)}\right)$. Furthermore, the prospective liabilities are given by

$$V_s^{+(t,x)} = Y_s^{(t,x)} + \int_s^T e^{-\int_s^r k(u,X_u^{(t,x)})du} Z_r^{(t,x)\top} dB_r. \tag{8.9}$$

Lastly, the random variable $Y_t^{(t,x)}$ is deterministic and can be represented by

$$v(t,x) := Y_t^{(t,x)} = \mathbb{E}\left(Y_t^{(t,x)}\right) = \mathbb{E}\left(V_t^{+(t,x)}\right).$$

PROOF. The functions $h(s,y,z) = -k(s,x(s))y + g(s,x(s))$ and $m(s,y,z) = -z$ implement the measurability and boundedness conditions as functions of y and z required in Theorem 7.3. Thus there exists a unique solution to the BSDE part of the forward-backward system. The process X is a unique solution to the forward SDE due to Proposition 5.4. Therefore the triplet $\left(X_s^{(t,x)}, Y_s^{(t,x)}, Z_s^{(t,x)}\right)$ is unique. The statement regarding the form of the solution stems from [**18**, Remark 5.38] and is a consequence of the linearity of the BSDE. The fact that $Y_t^{(t,x)}$ is deterministic is given by [**6**, Proposition 4.2]. □

The control process Z can be obtained from the Martingale Representation Theorem A.6, however, the form will not be explicit:

$$f\left(X_T^{(t,x)}\right) = \mathbb{E}\left(f\left(X_T^{(t,x)}\right)\right) + \int_t^T Z_s^{(t,x)\top} dB_s.$$

With a little stricter assumptions, however, we will be able to get a deterministic representation of the control process Z because of the Markovian nature of the setting. To prove this, we give the following Lemma, without proof. Before this, however, we introduce a few definitions and assumptions set out in [**20**]. In the following we consider a forward-backward system of the form

$$\begin{cases} dX_s^{(t,x)} = \mu(s,X_s^{(t,x)})ds + \sigma(s,X_s^{(t,x)})dB_s, & X_t = x, \qquad t \le s \le T \\ -dY_s^{(t,x)} = h(s,X_s^{(t,x)},Y_s^{(t,x)},Z_s^{(t,x)})ds - Z_s^{(t,x)\top} dB_s, & Y_T^{(t,x)} = f(X_T^{(t,x)}), t \le s \le T, \end{cases} \tag{8.10}$$

where we omit the starting value (t,x) for convenience. We make these stricter assumptions because we could not verify the validity of the results with the more relaxed assumptions made in [**4**].

Assumption 8.10.

(i) *Let $\sigma(t,x) > 0$ for all (t,x),*
(ii) *μ, σ are bounded and belong to $C_b^{0,1}([0,T] \times \mathbb{R}^k)$,*
(iii) *σ is uniformly Hölder-α continuous in t for $\alpha > \frac{1}{2}$,*
(iv) *For the driver $h \in [0,T] \times \mathbb{R}^3$ it holds that $h(t,x,y,z) = h_1(t,x,y) + h_2(t,x)z$, where h_1, h_2 are continuous and uniformly Lipschitz continuous in x, y. Moreover, h_2 is bounded,*
(v) *f is Lebesgue measurable and $|f(x)| \le \Phi(x) := K(1+|x|^{p_0})$ for some constant K and some $p_0 \ge 1$.*

We introduce the following notation.

Definition 8.11. Let $\mathcal{O}$ be an open subset on $[0,T]\times\mathbb{R}^k$ for some k.

- $C^{0,1}(\mathcal{O})$ is the space of all continuous $\phi:\mathcal{O}\to\mathbb{R}$ that are continuously differentiable in the space variable in $\mathcal{O}$,
- $C_b^{0,1}(\mathcal{O})$ is the space of those $\phi\in C^{0,1}(\mathcal{O})$ such that all the partial derivatives in $\mathcal{O}$ are uniformly bounded,
- $\Gamma^n := \{(t,x) : \max_{t\le s\le T}|\sigma(s,\eta_s^{(t,x)})|\ge\frac{1}{n}\}$, where $\eta^{(t,x)}$ is the solution to the integral equation
$$\eta_s^{(t,x)} = x + \int_t^s \mu(s,\eta_s^{(t,x)})ds$$
and $n\ge 0$,
- $u(t,x) := Y_t^{(t,x)}$, $u_x(t,x) = \nabla Y_t^{(t,x)}$.

Lemma 8.12. *[[**20**] Theorem 5.2 (see also [**8**])] Consider a FBSDE of the form (8.10). Let Assumption 8.10 hold. Then:*

(i) $u\in C^{0,1}(\Gamma^n)$

(ii) *u is locally Lipschitz continuous in x and there exists a constant C_n depending on K,t,α,Φ and n such that*
$$|u_x(t,x)|\le\frac{C_n\Phi(x)}{\sqrt{T-t}}\quad\forall(t,x)\in\Gamma^n.$$
Here $u_x(t,x)$ is understood to be a general derivative if u is not differentiable in x at (t,x).

(iii) *Understanding u_x as in (ii) we have*
$$Z_t = u_x(t,X_t)\sigma(t,X_t).$$

For the rest of the thesis we shall use the notation $d(r,x):=u_x(r,x)$ to be consistent with the source material.

Remark 8.13. Let us have a BSDE of the form
$$\begin{cases}-dY_s = h(s,X_s^{(t,x)},Y_s,Z_s)ds - Z_s^\top dB_s\\ Y_T = f(X_T^{(t,x)}),\end{cases}$$
where X is the diffusion and f gives us the terminal condition. If h *only* depends on the time and the diffusion, i.e. $h(s,x,y,z)=h(s,x)$, we get that
$$Y_s^{(t,x)} = \mathbb{E}\left[f(X_T^{(t,x)}) + \int_s^T h(r,X_r^{(t,x)})dr\middle|\mathcal{F}_s\right],$$
since $\mathbb{E}\int_s^T Z_r dB_r = 0$. The last claim follows from [**20**, Subsections 5.1.1 and 5.1.2; especially equation (5.1.8)].

We can now interpret (8.9) in an actuarial scope. Let us now apply Lemma 8.12 and assume that the probability measure corresponds to a (risk-neutral) market measure. Viewing (8.9) as

$$\underbrace{V_s^{+(t,x)}}_{\text{prospective liabilities}} = \underbrace{Y_s^{(t,x)}}_{\substack{\text{present value of the}\\ \text{prospective liabilities}}} + \int_s^T \underbrace{e^{-\int_s^r k(u,X_u^{(t,x)})du} d\left(r, X_r^{(t,x)}\right)^\top}_{\text{hedging strategy}} \underbrace{\sigma\left(r, X_r^{(t,x)}\right) dB_r}_{\substack{\text{market price changes}\\ \text{in the underlyings}}},$$

or under the real world probability measure as

$$\underbrace{V_s^{+(t,x)}}_{\text{prospective liabilities}} = \underbrace{Y_s^{(t,x)}}_{\substack{\text{best estimate for the}\\ \text{prospective liabilities}}} + \int_s^T \underbrace{e^{-\int_s^r k(u,X_u^{(t,x)})du}}_{\text{discounting factor}} \underbrace{d\left(r, X_r^{(t,x)}\right)^\top}_{\text{sensitivity factor}} \underbrace{\sigma\left(r, X_r^{(t,x)}\right) dB_r}_{\substack{\text{random fluctuations}\\ \text{in the underlyings}}}.$$

This interpretation of $Y_s^{(t,x)}$ as the best estimate for the prospective liabilities at time s makes sense, since it holds that $Y_s^{(t,x)} = \mathbb{E}(V_s^{+(t,x)}|\mathcal{F}_s)$ (remember that the expectation of an Itô integral is zero). The Itô integral describes the randomness around the best estimate. The parts can be understood as follows:

- The discounting factor $e^{-\int_s^r k(u,X_u^{(t,x)})du}$, which can include biometric and financial discounting.
- A sensitivity factor $d\left(r, X_r^{(t,x)}\right)$ which describes the effects that the random fluctuations of the process X have on the prospective liabilities.
- The martingale part $\sigma\left(r, X_r^{(t,x)}\right) dB_r$ of the integrator dX_r describing the random fluctuations of the process X, according to Doob's decomposition.

Example 8.14. We can now view Example 8.1 in a new light with the help of the previous remark. We obtain

$$\begin{aligned} V_s^{+(t,x)} &= \mathbb{E}(V_s^{+(t,x)}|\mathcal{F}_s) \\ &\quad + \int_s^T \underbrace{e^{-\int_s^r \phi(u)du}}_{\text{financial discounting}} \underbrace{e^{-\int_s^r \mu(a+u)du}}_{\text{survival rate}} \underbrace{d\left(r, X_r^{(t,x)}\right)^\top}_{\text{sensitivity factor}} \underbrace{\mathrm{d}\begin{pmatrix} M_\mu(a+r) \\ M_\phi(r) \end{pmatrix}}_{\substack{\text{random fluctuations}\\ \text{in the underlyings}}}, \end{aligned}$$

where $M_\mu(a+r)$ and $M_\phi(r)$ denote the martingale parts of the diffusion processes $\mu(a+r)$ and $\phi(r)$ according to Doob's decomposition.

This decomposition has similarities to the classical surplus decomposition formula. These similarities are discussed in detail in [**4**, Subsection 3.2].

8.2. A link between the BSDE and a PDE. For completeness, in this subsection we want to link the BSDE (8.8) to a corresponding partial differential equation, the Thiele differential equation (for more in-depth deliberation cf. [**14**] and [**19**]), which describes the reserve the insurer must provide and is given in our case by

$$\begin{cases} \partial_t u(t,x) + \mathcal{A}u(t,x) + (-k(t,x)u(t,x) + g(t,x)) = 0 \\ u(T,x) = f(x), \end{cases} \tag{8.11}$$

where u, k, g and f are the functions used in the last subsection, with the partial differential operator $\mathcal{A}$ being defined as

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{n} \left[\sigma\sigma^\top\right]_{ij}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n}\mu_i(t,x)\partial_{x_i}.$$

Here $\partial_t = \frac{\partial}{\partial t}$ and $\partial_{x_i} = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$.

This link is of interest because one can study the regularity of the functions u and d given in the last subsection with the tools of partial differential equations and functional analysis, or vice versa study (some) partial differential equations by simulating random paths of a stochastic process.

We may now link the BSDE (8.8) and the PDE (8.11) with BSDE theory. For the definition of a viscosity solution, see Definition A.8.

Proposition 8.15. *Under Assumptions 8.7 and 8.8 hold, the following holds.*

(i) The function u is continuous in $(t,x) \in [0,T] \times \mathbb{R}^n$ and it is the unique viscosity solution of the partial differential equation (8.11) which grows at most polynomially at infinity.

(ii) If the coefficients σ, μ, k, g and the terminal condition f are three times continuously differentiable with bounded derivatives, then u is a classical solution of (8.11) and belongs in $C^{1,2}([0,T] \times \mathbb{R}^n)$.

(iii) When (ii) holds, the solution of the BSDE (8.8) is given by $Y_s^{(t,x)} = u\left(s, X_s^{(t,x)}\right)$ and $Z_s^{(t,x)} = \sigma\left(s, X_s^{(t,e)}\right)^\top \nabla u\left(s, X_s^{(t,x)}\right)$ for any $t \leq s \leq T$ where $\nabla u = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient.

Proof. The assertion (i) is [**18**, Theorem 5.37]. Assertion (ii) is [**17**, Theorem 3.2]. Assertion (iii) is a special case of Lemma 8.12. It is also proved in [**6**, Corollary 4.1]. □

Corollary 8.16. *Vice versa, if $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ is a classical solution of (8.11), we get from Itô's lemma that $Y_s^{(t,x)} = u(s, X_s^{(t,x)})$ and $Z_s^{(t,x)} = \sigma\left(s, X_s^{(t,x)}\right)^\top \nabla u\left(s, X_s^{(t,x)}\right)$. We have at time t, i.e. at the inception of the contract, that $Y_t^{(t,x)} = u(t,x)$ and $Z_t^{(t,x)} = \sigma(t,x)^\top \nabla u(t,x)$ by the definition of the process X.*

Proof. See [**6**, Proposition 4.3] for the application of Itô's lemma. The form at the inception of the contract is obvious due to x being the initial value of the process X. □

From this link we recover the Feynman-Kac formula in the form

$$V_t^{+(t,x)} = \mathbb{E}\left[f\left(X_T^{(t,x)}\right)e^{-\int_t^T k\left(u, X_u^{(t,x)}\right)du} + \int_t^T g\left(r, X_r^{(t,x)}\right)e^{-\int_t^T k\left(u, X_u^{(t,x)}\right)du}dr\right],$$

at the inception of the contract, cf. [**18**, Remark 5.38 and Subsection 3.8.1].

8.3. An SDE for the economic variables and the retrospective reserve. We can now represent the evolution of the retrospective reserve with a (forward) SDE. This is the forward component of our forward-backward system which is to be constructed later on.

Proposition 8.17. *Under Assumptions 8.7 and 8.8, for every initial condition $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$ the SDE*

$$\begin{pmatrix} X_s \\ V_s^- \end{pmatrix} = \begin{pmatrix} x \\ v \end{pmatrix} + \int_t^s \begin{pmatrix} \mu(u, X_u) \\ k(u, X_u)V_u^- - g(u, X_u) \end{pmatrix} du + \int_s^t \begin{pmatrix} \sigma(u, X_u) \\ 0 \end{pmatrix} dB_u \tag{8.12}$$

with $0 \le t \le s \le T$ has a unique, square integrable strong solution $\left(X_s^{(t,x)}, V_s^{-(t,x,v)}\right)^\top$.

Proof. This follows directly from Proposition 5.8. For an alternate proof, see [**10**, Chapter 5, Theorem 2.9]. □

Here we can use differential notation for (8.12) as

$$\begin{cases} d(X_s, V_s^-)^\top = \tilde{\mu}(s, X_s, V_s^-)ds + \tilde{\sigma}(s, X_s)dB_s, \quad 0 \le t \le s \le T \\ (X_t, V_t^-)^\top = (x, v)^\top \end{cases}$$

with

$$\tilde{\mu}(s, x, v) = \begin{pmatrix} \mu(s, x) \\ k(s, x)v - g(s, x) \end{pmatrix} \in \mathbb{R}^{n+1},$$
$$\tilde{\sigma}(s, x) = \begin{pmatrix} \sigma(s, x) \\ 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times n}.$$

We note that the function $u(t, x, v) = Y_t^{(t,x,v)}$ is the natural candidate for a viscosity solution to the degenerate terminal value problem

$$\begin{cases} \partial_t w(t, x, v) + \mathcal{A}w(t, x, v) + (k(t, x)v - g(t, x))\partial_v w(t, x, v) = 0 \\ w(T, x, v) = \Psi(x, v), \end{cases} \tag{8.13}$$

where

$$\mathcal{A}u(t, x) = \sum_{i=1}^n \mu^i(t, x)\partial_{x_i} u(t, x) + \frac{1}{2}\sum_{i,j=1}^n \left[\sigma(t, x)\sigma(t, x)^\top\right]_{ij} \partial_{x_i x_j} u(t, x).$$

8.4. A forward-backward system for the net value. Now we are ready to combine the forward and backward components to acquire the forward-backward system which describes the evolution of the equity position $\mathbb{P}(N_T \ge 0 | X_t = x, V_t^- = v)$. The backward element is given by

$$\begin{cases} -dY_s = -Z_s^\top dB_S \\ Y_T = \Psi\left(X_T^{(t,x)}, V_T^{-(t,x,v)}\right), \end{cases} \tag{8.14}$$

where the terminal condition $\Psi(x, v) = \mathbb{1}(v - f(x))$ defined by $\mathbb{1} := \mathbb{1}_{[0,\infty)}$ is bounded and Borel-measurable but *not* continuous, the driver of the BSDE is zero and the process X describes the actuarial assumptions.

Proposition 8.18. *Suppose that Assumptions 8.7 and 8.8 hold. These combined with the boundedness of Ψ yield a unique, square-integrable solution*

$\left(\left(X_s^{(t,x)}, V_s^{-(t,x)}\right), Y_s^{(t,x,v)}, Z_s^{(t,x,v)}\right)$ *to the forward-backward system*

$$\begin{cases} d(X_s, V_s^-)^\top = \tilde{\mu}(s, X_s, V_s^-)ds + \tilde{\sigma}(s, X_s)dB_s, & (X_t, V_t^-)^\top = (v, x)^\top \\ -dY_s = -Z_s^\top dB_s, & Y_T = \Psi\left(X_T^{(t,x)}, V_T^{-(t,x,v)}\right). \end{cases} \tag{8.15}$$

Proof. The statement follows directly from Proposition 8.17 and Theorem 7.3. The forward part and the backward part are unique according to the above proposition and theorem, so the triplet is unique as well. □

Once again, the process Z will be given by the martingale representation theorem as the unique square-integrable process for which

$$\Psi(X_T, V_T^-) = \mathbb{E}(\Psi(X_T, V_T^-)) + \int_t^T Z_s^\top dB_s,$$

since $\mathbb{E}\Psi^2 \leq 1$ for all (x, v). Let us formulate a theorem which helps us understand the role of the process Z in our context.

Theorem 8.19 ([**4**] Theorem 3.13)**.** *Assume the Assumptions 8.7 and 8.8 hold and $\Psi = \mathbb{1}(v - f(x))$. Then the following holds.*

(i) *The solvency probability given the information at time s is given by the process $Y_s^{(t,x,v)}$:*
$$Y_s^{(t,x,v)} = \mathbb{P}(N_T \geq 0 | \mathcal{F}_s)$$
for every $t \leq s \leq T$;

(ii) *The change in this probability between times s_1 and s_2 is dependent only on the control process $Z^{(t,x,v)}$:*
$$\mathbb{P}(N_T \geq 0 | \mathcal{F}_{s_2}) - \mathbb{P}(N_T \geq 0 | \mathcal{F}_{s_1}) = \int_{s_1}^{s_2} Z_r^{(t,x,v)^\top} dB_r$$
for every $t \leq s_1 \leq s_2 \leq T$.

Proof.

(i) We can use Proposition 7.1 to see that, since the driver of the BSDE is zero and $\mathbb{E}\Psi^2 < \infty$, we get a unique solution (Y, Z), for which the process Y takes the form
$$Y_s^{(t,x,v)} = \mathbb{E}\left[\Psi(X_T^{(t,x)}, V_T^{-(t,x,v)}) \middle| \mathcal{F}_s\right],$$
which we can interpret as the probability $\mathbb{P}(N_T \geq 0 | \mathcal{F}_s)$, since $V_T^{-(t,x,v)} - f(X_T^{(t,x)}) = V_T^- - V_T^+ = N_T$ by Lemma 8.4, hence $\Psi(X_T^{(t,x)}, V_T^{-(t,x,v)}) = \mathbb{1}(N_T \geq 0)$. For this we recall $V_T^+ = f(X_T)$.

(ii) From the integral version of (8.14) we see that
$$Y_s^{(t,x,v)} = \Psi(X_T^{(t,x)}, V_T^{-(t,x,v)}) - \int_s^T Z_r^{(t,x,v)^\top} dB_r$$

for every $t \leq s \leq T$, hence

$$\begin{aligned} Y_{s_2}^{(t,x,v)} - Y_{s_1}^{(t,x,v)} &= \left(\Psi(X_T^{(t,x)}, V_T^{-(t,x,v)}) - \int_{s_2}^{T} Z_r^{(t,x,v)^\top} dB_r \right) \\ &\quad - \left(\Psi(X_T^{(t,x)}, V_T^{-(t,x,v)}) - \int_{s_1}^{T} Z_r^{(t,x,v)^\top} dB_r \right) \\ &= \int_{s_1}^{s_2} Z_r^{(t,x,v)^\top} dB_r, \end{aligned}$$

which was the assertion.

□

Similarly to the previous subsection, we acquire deterministic functions u and d, under stricter assumptions, such that we can rephrase Theorem 8.19 as follows.

Corollary 8.20. *Under Assumption 8.10 in addition to the assumptions of Theorem 8.19 we have*

(i) There exists a measurable, continuous and deterministic function $u : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ for which the solvency probability given the information at time s is given by

$$\mathbb{P}(N_T \geq 0 | \mathcal{F}_s) = u\left(s, X_s^{(t,x)}, V_s^{-(t,x,v)}\right)$$

for every $t \leq s \leq T$.

(ii) There exists a measurable, continuous and deterministic function $d : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ for which the change in probability between times s_1 and s_2 has the representation

$$\mathbb{P}(N_T \geq 0 | \mathcal{F}_{s_2}) - \mathbb{P}(N_T \geq 0 | \mathcal{F}_{s_1}) = \int_{s_1}^{s_2} d\left(r, X_r^{(t,x)}, V_r^{-(t,x,v)}\right)^\top \tilde{\sigma}\left(r, X_r^{(t,x)}\right) dB_r$$

for every $t \leq s_1 \leq s_2 \leq T$, where we recall that $\tilde{\sigma}(s, x) = \binom{\sigma(s,x)}{0}$.

(iii) One can interpret u as giving the probability that the terminal theoretical equity position is non-negative given the information at time s:

$$u(s, x', v') = \mathbb{P}\left(N_T \geq 0 | X_s^{(t,x)} = x', V_s^{-(t,x,v)} = v'\right).$$

(iv) Additionally, one can view u as the distribution function of the prospective liabilities V_t^+ conditional on the information at time s:

$$u(s, x', v') = \mathbb{P}\left(V_t^{+(t,x,v)} \leq v' | X_s^{(t,x)} = x'\right).$$

Proof. (i) and (ii) follow straight from Lemma 8.12. From Lemma 8.6 we see that (iii) and (iv) are equal. To obtain (iii) from (i) we use (8.5). □

Corollary 8.21. *From the previous theorem and corollary we instantly obtain (8.1), which we recall to be*

$$\mathbb{P}(N_T \geq 0 | \mathcal{F}_t) - \mathbb{P}(N_T \geq 0 | \mathcal{F}_0) = \int_0^t U_r^\top dM_r^X = \int_0^t Z_r^\top dB_r,$$

by letting $s_1 = 0, s_2 = t$ and by setting $U_r = d\left(r, X_r^{(t,x)}, V_r^{-(t,x,v)}\right)$ and $dM_r^X = \tilde{\sigma}\left(r, X_r^{(t,x)}\right) dB_r$. This was the main focus of the section.

Since we have $\mathbb{P}(N_T \geq 0|\mathcal{F}_T) = \mathbb{E}\left[\mathbb{1}(N_T \geq 0)|\mathcal{F}_T\right] = \mathbb{1}(N_T \geq 0)$ a.s., we can rearrange (ii) from Corollary 8.20 to obtain

$$\underbrace{\mathbb{1}(N_T \geq 0)}_{\text{solvency status}} = \underbrace{\mathbb{P}\left(N_T \geq 0|\mathcal{F}_s\right)}_{\substack{\text{estimate for the}\\\text{solvency status}}} + \int_s^T \underbrace{d\left(r, X_r^{(t,x)}, V_r^{-(t,x,v)}\right)^\top}_{\text{sensitivity factor}} \underbrace{\tilde{\sigma}\left(r, X_r^{(t,x)}\right) dB_r}_{\substack{\text{random fluctuations}\\\text{in the underlyings}}}.$$

Here the solvency status at time T is a Bernoulli random variable that is $\mathcal{F}_T$-measurable, where 1 means solvent and 0 means nonsolvent. The second term is an estimate of the solvency status at the end of the monitoring period made with information available at time s and the Itô integral describes the dynamics of solvency status and can be viewed as having been decomposed to the following parts, similarly to Subsection 8.1:

- The integrator $\tilde{\sigma}\left(r, X_r^{(t,x)}\right) dB_r$ corresponds to the martingale part of $dX_r^{(t,x)}$ and describes the random fluctuations of the process X, once again according to Doob's decompostion.
- The sensitivity factor $d\left(r, X_r^{(t,x)}, V_r^{-(t,x,v)}\right)^\top$ describes the effects the aforementioned fluctuations have on the solvency risk.

Thus the measurable, continuous and deterministic function d contains all the information on the risk of not having a positive balance at contract expiration. Additionally, since the expected value of the Itô integral is zero, the estimate $\mathbb{P}\left(N_T \geq 0|\mathcal{F}_s\right)$ for solvency status is unbiased. The measurable, continuous and deterministic function u describes both the solvency level at time s and the distribution of the prospective liabilities at time s, both conditional to the process X. Once again, these are only existence results not yielding any information on how to find these processes.

8.5. Numerical methods. In this subsection we present two numerical methods for the study of the control process, one of which is a PDE method and the other one based on Malliavin calculus. These methods will be presented without the utmost rigor since they are not the main scope of this thesis. We will work with Assumptions 8.7, 8.8 and 8.10, which were the strictest made.

8.5.1. *The PDE method.* In the case that the PDE (8.13) has a classical solution and the terminal condition Ψ is continuously differentiable, one would be able to apply Itô's lemma to show that $Y_s = u(s, X_s, V_s^-)$ and $Z_s = \tilde{\sigma}(s, X_s)^\top \nabla u(s, X_s, V_s^-)$ is the solution to the BSDE (8.15). This connection would allow one to use PDE-based methods to analyze the control process Z. In our setting, however, the terminal condition Ψ is not differentiable as it is not even continuous, so this approach is futile. This is because applying Itô's lemma would require the differentiability of Ψ since it relies on the Taylor expansion of $u(s, x, v) - u(0, x, v)$. Because of this reason, the connection in this case is more delicate.

In our case we require that the process $(X_s, V_s^-)^\top$ has a probability density. This existence is nontrivial, however. In general, we consider the differential operator

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n}[\sigma(t,x)\sigma(t,x)^\top]_{ij}\partial_{x_i x_j} + \sum_{i=1}^{n}\mu^i(t,x)\partial_{x_i}u(t,x) + (k(t,x)v - g(t,x))\partial_v, \quad (8.16)$$

with the same coefficients, mainly the ones with the strictest assumptions, as in the previous subsection. It acts on $C^\infty(\mathbb{R}^n \times \mathbb{R})$ and can be continuously extended to other function spaces. Now we can rewrite the PDE (8.13) as

$$-\partial_t u = \mathcal{L}u. \quad (8.17)$$

Next we recall the notion of a *fundamental solution*, as one does in [**4**] where it is based on [**10**, Chapter 5.7].

Definition 8.22. A fundamental solution to our PDE is a (nonnegative) function $p(t,x,v;s,\xi,\eta)$ defined for $0 \le t < s \le T$, $x,\xi \in \mathbb{R}^n$ and $v,\eta \in \mathbb{R}$ such that for every $f \in C(\mathbb{R}^n \times \mathbb{R})$ with compact support, and $t \in (0,T]$ the function

$$u(t,x,v) = \int_{\mathbb{R}^{n+1}} p(t,x,v;s,\xi,\eta)f(\xi,\eta)d\xi d\eta$$

is bounded, belongs to the class $C^{1,2}$, satisfies (8.17) and

$$\lim_{t\uparrow s} u(t,x,v) = f(x,v)$$

for any $x \in \mathbb{R}^n$ and $v \in \mathbb{R}$.

Furthermore, we require that for fixed (t,x,v) the map $(s,\xi,\eta) \mapsto p(t,x,v;s,\xi,\eta)$ and that for fixed (s,ξ,η) the map $(t,x,v) \mapsto p(t,x,v;s,\xi,\eta)$, both belong to the class $C^{1,2}$. Lastly, we assume boundedness of the first-order spatial derivatives with respect to x and v in the sense that

$$|\partial_{x_j}p(t,x,v;s,\xi,\eta)|, |\partial_v p(t,x,v;s,\xi,\eta)| \le (s-t)^\alpha \Gamma(t,x,v;s,\xi,\eta) \quad (8.18)$$

for some $\alpha < 0$ and a function Γ for which $\int_{\mathbb{R}^{n+1}} |\Gamma(t,x,v;s,\xi,\eta)|d\xi d\eta < \infty$ uniformly in x,v and in t,s. This condition can be acquired via Aronson-type estimates, c.f. [**13**, Corollary 3.25].

When defined this way, the function p can be thought as the transition density of the process (X,V^-) in the sense that

$$\mathbb{P}\left(\left(X_s^{(t,x)}, V_s^{-(t,x,v)}\right) \in A\right) = \int_A p(t,x,v;s,\xi,\eta)d\xi d\eta$$

for Borel measurable sets $A \subseteq \mathbb{R}^n \times \mathbb{R}$.

Next we give the following theorem for the density p. The proof can be read in [**4**].

Theorem 8.23 ([**4**] Theorem 4.1)**.** *Assume that (8.17) has a fundamental solution p such that the conditions (8.18) are fulfilled. Furthermore, let $u \in C^{1,2}([t,T) \times \mathbb{R}^n \times \mathbb{R}) \cap L^\infty([t,T] \times \mathbb{R}^n \times \mathbb{R})$ be a classical solution to the PDE (8.17) with a terminal condition $\Psi \in L^\infty(\mathbb{R}^n \times \mathbb{R})$. Then, $Y_s^{(y,x,v)} = u\left(s, X_s^{(t,x)}, V_s^{-(t,x,v)}\right)$ for $s \in [t,T]$ and $Z_s^{(t,x,v)} = \tilde{\sigma}^\top\left(s, X_s^{(t,x)}\right)\nabla u\left(s, X_s^{(t,x)}, V_s^{-(t,x,v)}\right)$ for $s \in [t,T)$.*

This theorem is substantial since it gives us an opportunity to use PDE theory to study the functions that give us the solvency risk.

8.5.2. *The Malliavin method.* Like in the previous section, if we had a continuous terminal condition and continuous coefficients for the BSDE, there would be well-established numerical methods for simulating the solution (Y, Z) cf. [**5**, Chapter 5]. However, the details are more intricate in our setting. We are also particularly interested in the control process Z so applying methods based on viscosity solutions are not of great use. For a discussion on these methods, we refer the reader to [**1**] for a seminal paper. We shall modify the M_ϵ-method set out in [**3**, Section 5]. In addition to the previous assumptions, let σ be uniformly positive definite for the rest of the thesis.

The goal is to find a representation of Z_s that is easy to simulate. The challenge in our case stems from the fact that there are two sources of degeneracy: the terminal condition Ψ is discontinuous and the diffusion coefficient $\tilde{\sigma}$ is not positive definite. The idea is to regularize the diffusion coefficient and find a representation of the control process Z in the form

$$Z_s^{(t,x)} = \sigma_\epsilon\left(t, X_s^{\epsilon,(t,x)}\right)^\top \mathbb{E}\left(\Psi\left(X_T^{\epsilon,(t,x)}, V_T^{-\epsilon,(t,x,v)}\right) U_T^{\epsilon,s,(t,x,v)}\right),$$

where U is a suitable, easy to simulate process. The processes $(X^\epsilon, V^{-\epsilon})$ are approximations of (X, V^-) and σ_ϵ is an approximation of σ.

Let us sketch the construction we use, without technical details. For the details we refer the reader to [**4**].

(i) *Augmented Brownian motion.* Let $(B'_s)_{t\le s\le T}$ be a one-dimensional Brownian motion that is independent of $(B_s)_{t\le s\le T}$ and defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define an $(n+1)$-dimensional Brownian motion by $\tilde{B}_s = \binom{B_s}{B'_s}$ and denote the corresponding filtration by $(\tilde{\mathcal{F}}_s)_{t\le s\le T}$.

(ii) *Regularization of the diffusion coefficients.* Let

$$\sigma_0(t,x) = \begin{pmatrix} \sigma(t,x) & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

and

$$\sigma_\epsilon(t,x) = \begin{pmatrix} \sigma(t,x) & 0 \\ 0 & \epsilon \end{pmatrix},$$

where $\epsilon > 0$. We have that $\sigma_\epsilon(t,x)$ is positive definite for any $(t,x) \in [0,T] \times \mathbb{R}^n$, since we assumed $\sigma(t,x)$ is positive definite. Furthermore, $\sigma_\epsilon(t,x) \to \sigma_0(t,x)$ uniformly.

(iii) *Perturbation of the BSDE.* Consider the perturbed version of the forward-backward system

$$\begin{cases} (X_s^\epsilon, V_s^{-,\epsilon})^\top = (x,v)^\top + \int_t^s \tilde{\mu}(u, X_u^\epsilon, V_u^{-,\epsilon})du + \int_t^s \sigma_\epsilon(u, X_u^\epsilon) d\tilde{B}_u \\ Y_s^\epsilon = \Psi(X_T^\epsilon, V_T^{-,\epsilon}) - \int_s^T Z_u^{\epsilon\top} d\tilde{B}_u. \end{cases} \tag{8.19}$$

In order to simplify the notation, we omit the initial condition (t,x,v) and denote the solution of (8.19) by $(X^\epsilon, Y^\epsilon, Z^\epsilon)$. Also introduce the forward-backward system

$$\begin{cases} (X_s^0, V_s^{-,\epsilon})^\top = (x,v)^\top + \int_t^s \tilde{\mu}(u, X_u^0, V_u^{-,0})du + \int_t^s \sigma_0(u, X_u^0) d\tilde{B}_u \\ Y_s^0 = \Psi(X_T^0, V_T^{-,0}) - \int_s^T Z_u^{0\top} d\tilde{B}_u \end{cases} \tag{8.20}$$

with solution (X^0, Y^0, Z^0). We write $Z^\epsilon = \binom{Z^{\epsilon,1}}{Z^{\epsilon,2}}$ with $Z^{\epsilon,1} \in \mathbb{R}^n$ and $Z^{\epsilon,2} \in \mathbb{R}$.

(iv) *Limit behaviour.* We have that, when $\epsilon \to 0$,

$$\mathbb{E}\left(\sup_{t\leq s\leq T}\left|X_s^\epsilon - X_s^0\right|^2\right) \to 0,$$

$$\mathbb{E}\left(\sup_{t\leq s\leq T}\left|Y_s^\epsilon - Y_s^0\right|^2\right) + \mathbb{E}\left(\sup_{t\leq s\leq T}\left|Z_s^\epsilon - Z_s^0\right|^2\right) \to 0.$$

The first limit follows from classical SDE continuity results as X corresponds to the forward component of the system and the second limit stems from analogous BSDE results, e.g. [**18**, Theorem 5.11]. Since $\sigma_0(t, X_s^0)d\tilde{B}_s = \sigma(t, X_s^0)dB_s$ as well as SDE solutions being unique in our setting, we must have that $X_s = X_s^0$ for $t \leq s \leq T$ almost surely. The solutions of the BSDE are unique, hence it must hold that Z^0 is of the form $Z_s^0 = \begin{pmatrix} Z_s \\ 0 \end{pmatrix}$, where Z_s is the solution to (8.14). This means that (8.20) is actually the same as

$$\begin{cases} (X_s, V_s^-)^\top = (x, v)^\top + \int_t^s \tilde{\mu}(u, X_u, V_u^-)du + \int_t^s \sigma(u, X_u)dB_u \\ Y_s = \Psi(X_T, V_T^-) - \int_s^T Z_u^\top dB_u \end{cases}$$

which is precisely the same as the formulation in Proposition 8.18.

(v) *Approximation of Z_s.* The following convergence stems from our construction, as $\epsilon \to 0$,

$$\mathbb{E}\left(\int_t^T \left|Z_s^{\epsilon,1} - Z_s\right|^2 ds\right) \to 0.$$

In concrete applications we would need to find precise convergence estimates.

(vi) *Computation of $Z^{\epsilon,1}$.* With help from Malliavin calculus, one can represent the process $Z^{\epsilon,1}$ in a way that lends itself well to numerical calculation. We find the following formulae:

$$Z_s^{\epsilon,1} = \sigma_\epsilon(s, X_s^\epsilon)^\top \mathbb{E}(\Psi(X_T^\epsilon, V_T^{-\epsilon})U_T^{\epsilon,s}|\mathcal{F}_s), \tag{8.21}$$

where

$$U_T^{\epsilon,s} = (\nabla X_s^\epsilon)^{-1^\top}\left[\frac{1}{T-s}\int_s^T \left([\sigma_\epsilon(u, X_u^\epsilon)]^{-1}\nabla X_u^\epsilon\right)^\top d\tilde{B}_u\right] \tag{8.22}$$

and $\nabla X^\epsilon = (\nabla_1 X^\epsilon, \ldots, \nabla_{n+1}X^\epsilon) \in \mathbb{R}^{(n+1)\times(n+1)}$ with column vectors $\nabla_i X^\epsilon \in \mathbb{R}^{n+1}$ for $i = 1, \ldots, n+1$. Each of these column vectors satisfy the linear SDE

$$\nabla_i X_s^\epsilon = e_i + \int_t^s \text{grad } \tilde{\mu}(u, X_u^\epsilon)\nabla_i X_u^\epsilon du + \sum_{j=1}^{n+1}\int_t^s \left[\text{grad } \sigma_\epsilon^j(u, X_u^\epsilon)\right]\nabla_i X_u^\epsilon d\tilde{B}_{u,j}, \tag{8.23}$$

where $e_i = (0, \ldots, 1, \ldots, 0)^\top$ where the value 1 is in the ith row and σ_ϵ^j denotes the jth column of σ_ϵ and by grad we mean the gradient operator $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ in order to avoid confusion, since the equations contain nabla processes as well. These formulae are made clearer in [**4**, Appendix 1].

The representation of (8.21) is due to [**20**, Equation (3.4) in the proof of Theorem 3.2]. We get the following link to partial differential equations:

Remark 8.24. The forward-backward system (8.19) corresponds to the non-degenerate PDE

$$\begin{cases} -\partial_t u_\epsilon = \sum_{i,j=1}^{n+1} \left[\sigma_\epsilon \sigma_\epsilon^\top\right]_{ij} \partial_i \partial_j u_\epsilon + \sum_{i=1}^{n+1} \tilde{\mu} \partial_i u_\epsilon \\ u_\epsilon(T, x, v) = \Psi(x, v). \end{cases}$$

If the terminal function Ψ is not regular, one can typically approximate it by a sequence of smooth functions in a suitable space. The PDE can be further analyzed using standard methods; the key being that one will obtain the same solution as with the BSDE methods laid out in the thesis.

I conclude the thesis by referring the reader to [**4**, Section 5] for an illustration of the PDE toy model, as a concrete example.

References

[1] Barles, G. (1997). *Convergence of numerical schemes for degenerate parabolic equations arising in finance theory.* Numerical Methods in Finance. Cambridge University Press.

[2] Bismut, J.-M. (1973). *Conjugate convex functions in optimal stochastic control.* Journal of Mathematical Analysis and Applications **44**(2), 384-404.

[3] Cvitanić, J., Ma, J. & Zhang, J. (2003). *Efficient computation of hedging portfolios for options with discontinuous payoffs.* Mathematical Finance **13**(1) 135-151.

[4] Christiansen, M.C. & Fahrenwaldt, M. A. (2016). *Dynamics of solvency risk in life insurance liabilities.* Scandinavian Actuarial Journal **2016**(9) 763-792.

[5] Delong, Ł. (2013). *Backward stochastic differential equations with jumps and their actuarial and financial applications.* EAA series. Springer-Verlag.

[6] El Karoui, N. Peng, S. & Quenez, M. (1997). *Backward stochastic differential equations in finance.* Mathemathical Finance **7**(1) 1-71.

[7] Geiss, C. & Geiss, S. (2021). *Advanced Markov Processes.* Lecture notes at University of Jyväskylä.

[8] Geiss, C. & Geiss, S. & Gobet, E. (2012). *Generalized fractional smoothness and L_p-variation of BSDEs with non-Lipshitz terminal condition.* Stochastic Processes and their Applications. Volume 122, issue 5, p. 2078-2116.

[9] Geiss, S. (2021). *Stochastic Differential Equations.* Lecture notes at University of Jyväskylä.

[10] Karatzas, I. & Shreve, S. E. (1991). *Brownian motion and stochastic calculus.* 2nd edition, Volume 113. Graduate texts in mathematics. Springer-Verlag.

[11] Klebaner, F. (2005). *Introduction to Stochastic Calculus with Applications.* Imperial College Press.

[12] Kuo, H.-H. (2006). *Introduction to Stochastic Integration.* Springer.

[13] Kusuoka, S. & Stroock, D. (1985). *Applications of Malliavin calculus, part II.* Journal of the Faculty of Science, University of Tokyo **32**, 1-76.

[14] Norberg, R. (1991). *Reserves in life and pension insurance* Scandinavian Actuarial Journal **1991**(1), 3-24.

[15] Pardoux, E. (1999). *BSDEs, weak convergence and homogenization of semilinear PDEs.* Clarke, F.H., Stern, R.J., Sabidussi, G. (eds) Nonlinear Analysis, Differential Equations and Control. NATO Science Series, vol 528. Springer, Dordrecht. 503-549.

[16] Pardoux, E. & Peng, S. (1990). *Adapted solution of a backward stochastic differential equation.* Systems Control Letters **14**(1), 55-61.

[17] Pardoux, E. & Peng, S. (1992). *Backward stochastic differential equations and quasilinear parabolic partial differential equations. In Stochastic Partial Differential Equations and Their Applications.* Vol. 176, Lecture notes in control and information sciences. Springer.

[18] Pardoux, E. & Răşcanu, A. (2014). *Stochastic differential equations, backward SDEs, partial differential equations.* Vol 69, stochastic modelling and applied probability. Springer.

[19] Steffensen, M. (2000). *A no arbitrage approact to Thiele's differential equation* Insurance Mathematics and Economics **27**(2), 201-214.

[20] Zhang, J. (2005). *Representation of solutions to BSDEs associated with a degenerate FSDE. Annals of Applied Probability* **15**(3), 1798-1831.

[21] Zhang, J. (2017). *Backward Stochastic Differential Equations, From Linear to Fully Nonlinear Theory.* Springer.

Appendix A.

Definition A.1 (Stopping Time)**.** Assume a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. The map $\tau : \Omega \to [0, \infty]$ is called *a stopping time* w.r.t. the filtration if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Definition A.2 (Local Martingale)**.** Let $M = (M_t)_{t\geq 0}$ be a continuous and adapted process with $M_0 = 0$. It is called *a local martingale* provided that there exists an increasing sequence of stopping times $(\tau_n)_{n=0}^{\infty}$ with $\lim_{n\to\infty} \tau_n = \infty$ such that $(M_{t\wedge\tau_n})_{t\geq 0}$ is a martingale for all $n = 0, 1, 2, \ldots$.

Proposition A.3 (Hölder's inequality)**.** *Assume a measurable space $(\Omega, \mathcal{F}, \mu)$ and measurable maps $f, g : \Omega \to \mathbb{R}$. If $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\int_\Omega |fg| d\mu \leq \left(\int_\Omega |f|^p d\mu \right)^{\frac{1}{p}} \left(\int_\Omega |f|^q d\mu \right)^{\frac{1}{q}}.$$

Lemma A.4 ([11] Thm. 7.21)**.** *A continuous local martingale $M = (M_t)_{t\geq 0}$ which is bounded by an integrable upper boung G such that*

$$\sup_{t\geq} |M_t| \leq G, \quad \mathbb{E}G < \infty$$

is a martingale.

Theorem A.5 (Burkholder-Davis-Gundy inequalities)**.** *For any $0 < p < \infty$ there exist constants $\alpha_p, \beta_p > 0$ such that for a process $L \in \mathcal{L}_2^{loc}$ one has that*

$$\beta_p \left\| \sqrt{\int_0^T L_t^2 dt} \right\|_p \leq \left\| \sup_{t\in[0,T]} \left| \int_0^t L_s dB_s \right| \right\|_p \leq \alpha_p \left\| \sqrt{\int_0^T L_t^2 dt} \right\|_p.$$

Moreover, one has $\alpha_p \leq c\sqrt{p}$ for $p \in [2, \infty)$ for some absolute $c > 0$.

Theorem A.6 (Martingale Representation Theorem)**.** *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be complete, $B = (B_t)_{t\in[0,T]}$ be a Brownian motion, $\mathcal{F}_t^0 = \sigma\{B_s : 0 \leq s \leq t\}$ be the smallest sigma-algebra such that all $B_s, s \in [0, t]$ are measurable, $\mathcal{F}_t^B$ be its completion and $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\geq 0}$ be the associated augmented natural filtration. For any random variable $\xi \in \mathcal{L}_2(\mathcal{F}_T^B)$ there exists a unique $\eta \in \mathcal{L}_2(\mathbb{F}^B)$ such that*

$$\xi = \mathbb{E}\xi + \int_0^T \eta_s dB_s.$$

Consequently, for any $\mathbb{F}^B$-martingale M for which $\mathbb{E}M_T^2 < \infty$ there exists a unique $\eta \in \mathcal{L}_2(\mathbb{F}^B)$ such that the following holds:

$$M_t = M_0 + \int_0^t \eta_s dB_s.$$

The latter assertion is obvious due to the fact that $\mathbb{E}M_t = \mathbb{E}M_0 = M_0$.

Theorem A.7. *[Grönwall's Lemma]. Let $B \geq 0$, $\phi : [0, T] \to \mathbb{R}$ be a bounded nonnegative function and $C : [0, T] \to \mathbb{R}$ be a nonnegative measurable function with the property that*

$$\phi(t) \leq B + \int_0^t C(s)\phi(s) ds$$

for all $t \in [0,T]$*. Then one has* $\phi(t) \le B \exp\left(\int_0^t C(s)ds\right)$ *for all* $t \in [0,T]$.

Definition A.8 ([**4**] Definition 3.8)**.** Consider the general terminal value problem

$$\begin{cases} -\partial_t u(t,x) + F(t,x,D_x u(t,x), D_x^2 u(t,x)) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n \\ u(T,x) = \Psi(x), \quad x \in \mathbb{R}_n \end{cases} \tag{A.1}$$

for some bounded and measurable funtion Ψ. Here $\partial_t = \frac{\partial}{\partial t}$ and $\partial_{x_i} = \frac{\partial}{\partial x_i}$ for $i = 1,\dots,n$, $D_x u$ stands for the collection of the first derivatives $\partial_{x_i} u$ and $D_x^2 u$ is the collection of second-order derivatives $\partial_{x_i}\partial_{x_j} u$. Let F be a continuous function. We say that u is a

(1) *viscosity subsolution of* (A.1) if for each $\omega \in C^\infty([0,T) \times \mathbb{R}^n)$,
$$-\partial_t \omega(t_0,x_0) + F(t_0,x_0,D_x\omega(t_0,x_0), D_x^2\omega(t_0,x_0)) \le 0$$
at every $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$ that is a strict maximiser of $u - \omega$ on $[0,T] \times \mathbb{R}^n$ with $u(t_0,X_0) = \omega(t_0,x_0)$

(2) *viscosity supersolution of* (A.1) if for each $\omega \in C^\infty([0,T) \times \mathbb{R}^n)$,
$$-\partial_t \omega(t_0,x_0) + F(t_0,x_0,D_x\omega(t_0,x_0), D_x^2\omega(t_0,x_0)) \ge 0$$
at every $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$ that is a strict minimizer of $u - \omega$ on $[0,T] \times \mathbb{R}^n$ with $u(t_0,X_0) = \omega(t_0,x_0)$,

(3) viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.