The equivalent photon approximation in one- and two-photon exchange processes

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Abstract

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In this thesis, the equivalent photon approximation and the associated equivalent photon distribution are derived in one- and two-photon exchange processes using a helicity-based method in the framework of quantum field theory. The derivations are presented in more detail than is usually found in the literature. The general forms of the photon distributions for spin-0 and spin-$\frac{1}{2}$ particles are examined using the general vertex structure of quantum electrodynamics and scalar quantum electrodynamics. A total of three photon distributions based on the dipole form factors of the proton are derived and applied to muon-pair production in proton-proton collisions. The invariant-mass spectrum of the muon pair obtained using the equivalent photon approximation is compared to a full calculation of the same observable at total center-of-mass energies $\sqrt{s} = 200\,\text{GeV}$ and $\sqrt{s} = 13\,\text{TeV}$. The full calculation is obtained at leading order without any approximations and numerically integrated using Monte-Carlo methods, and is found to be in good agreement with an experimental measurement by ATLAS. The results show that the distribution neglecting both the mass and the anomalous magnetic moment of the proton gives more accurate results than the distribution neglecting only the anomalous magnetic moment of the proton. The results also indicate that the distribution including both the mass and the anomalous magnetic moment of the proton overestimates the full calculation by less than $4\,\%$, thus providing an excellent approximation to the calculation of the invariant-mass spectrum in the muon-pair production process at both energies.

Keywords: Equivalent photon approximation, two-photon physics, lepton-pair production

Tiivistelmä

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Ekvivalentti fotoniapproksimaatio yhden ja kahden fotonivaihdon prosesseissa
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Tässä opinnäytetyössä johdetaan ekvivalentti fotoniapproksimaatio ja siihen liittyvä ekvivalentti fotonijakauma yhden ja kahden fotonivaihdon prosesseissa helisiteettipohjaista menetelmää käyttäen kvanttikenttäteorian viitekehyksessä. Johtamiset esitetään yksityiskohtaisemmin kuin kirjallisuudessa yleensä. Fotonijakaumien yleisiä muotoja tarkastellaan spin-0- ja spin-$\frac{1}{2}$-hiukkasille käyttäen kvantti- ja skalaarikvanttisähködynamiikan yleisiä verteksirakenteita. Kolme protonin dipolimuototekijöihin perustuvaa fotonijakaumaa johdetaan ja niitä sovelletaan myoniparituottoon protoni-protoni-törmäyksissä. Ekvivalentin fotoniapproksimaation avulla saatua myoniparin invarianttia massaspektriä verrataan saman observaabelin tarkkaan laskuun kokonaismassakeskipiste-energioilla $\sqrt{s} = 200\,\mathrm{GeV}$ ja $\sqrt{s} = 13\,\mathrm{TeV}$. Tarkka lasku lasketaan ilman approksimaatioita johtavaan kertalukuun ja sitä integroidaan numeerisesti Monte-Carlo-menetelmin, ja sen nähdään vastaavan hyvin ATLAS-kokeessa saatuja tuloksia. Tulosten perusteella jakauma, jossa sekä protonin massa että sen anomaalinen magneettinen momentti on jätetty huomiotta, on tarkempi kuin jakauma, jossa vain protonin anomaalinen magneettinen momentti on jätetty huomiotta. Tulokset osoittavat myös, että sekä protonin massan että sen anomaalisen magneettisen momentin huomioiva jakauma yliarvio tarkkaa laskua alle 4 %, tarjoten siten erinomaisen approksimaation invariantin massaspektrin laskemiseen myoniparituotossa molemmilla energioilla.

Avainsanat: Ekvivalentti fotoniapproksimaatio, kahden fotonin fysiikka, leptoniparituotto

Preface

This thesis concerning the equivalent photon approximation is effectively a continuation of my Bachelor's thesis. In my Bachelor's thesis, I derived the well-known semiclassical formula for the equivalent photon distribution. Now, I will derive both the factorization of electromagnetic cross sections and the equivalent photon distribution using field-theoretical methods. As an example, I will also consider muon-pair production and compare the equivalent photon approximation to a full leading-order calculation.

I started working on this thesis during my summer internship at the physics department at the University of Jyväskylä. I am grateful for that opportunity, as it allowed me to solely focus on the thesis.

At the beginning of the summer, I also tried to reconcile the photon distributions obtained from the semiclassical and field-theoretical derivations, but the connection between the two is non-trivial. While the overall behaviour is the same for both distributions, obtaining the semiclassical distribution from the field-theoretical approach would likely require reworking the usual formula connecting cross sections and invariant amplitudes to include dependence on the impact parameter. Due to time constraints, I did not pursue this avenue further.

I want to thank my advisors for their guidance and time spent in helping me and thinking through problems with me. I would also like to thank my friends for all their support. Finally, I want to express my deepest gratitude to my parents.

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Sami Yrjänheikki

Contents

1 Introduction

Quantum electrodynamics (QED) is one of the most successful physical theories and an important building block of the Standard Model of particle physics. It is the fundamental theory of electromagnetism, describing the interactions between photons and leptons, and it has been stringently tested in experiments. [1] All electrically charged particles are subject to electromagnetic interactions mediated by photons. The QED Lagrangian and Feynman rules are briefly reviewed in Appendix B.

Quantum chromodynamics (QCD) is another piece of the Standard Model and it describes the strong interaction. In addition to electric charge, strongly interacting particles carry a color charge. In the Standard Model, quarks and gluons have a color charge and are therefore subject to strong interactions mediated by gluons. [2]

The most common method of computing observables in quantum field theories, such as cross sections and decay widths, is perturbation theory. Quantum electrodynamics is very amenable to perturbation theory, which allows the computation of accurate predictions with relatively little effort. [3] In QCD, factorization theorems [4] are important tools that enable the use of perturbation theory, as they separate the perturbative and non-perturbative aspects of scattering processes involving hadrons. However, the convergence of the perturbative series is usually slower than in QED.

One example of factorization can be found in deep inelastic scattering (DIS) $e^-p \to e^-X$, where a virtual photon emitted by the electron interacts with the proton and breaks it apart. The cross section can be factored into [3, §17.3]

$$\sigma(e^-p \to e^-X) = \sum_f \int \mathrm{d}\xi\, f_{f/p}(\xi)\hat{\sigma}(e^-q_f \to e^-q_f), \tag{1.1}$$

where ξ is the longitudinal momentum fraction of the proton. The function $f_{f/p}(\xi)$ is called the parton distribution function (PDF) of the proton, and $\mathrm{d}\xi\, f_{f/p}(\xi)$ can be interpreted as the differential probability of finding some constituent particle f with longitudinal momentum fraction ξ inside the proton. These constituents, which are summed over in Eq. (1.1), are called partons and are nowadays known to be quarks, antiquarks, and gluons. The PDF contains the physics of long-ranged

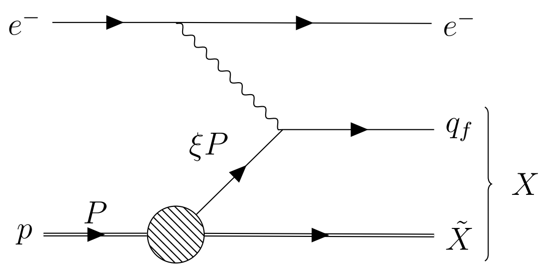

Figure 1. Deep inelastic scattering $e^-p \to e^-X$ in the parton model. The photon emitted from the electron interacts with a quark (momentum ξP) taken from the proton (momentum P).

interactions that cannot be completely determined from perturbative QCD. On the other hand, the cross section $\hat{\sigma}$ of the partonic subprocess $e^-q_f \to e^-q_f$ can be calculated perturbatively. [2, 3]

The interpretation of Eq. (1.1) is that the photon emitted from the electron does not interact with the proton as a whole, but rather with a quark taken from the proton. The interaction between the electron and the quark, mediated by the photon, is represented by the partonic subprocess $e^-q_f \to e^-q_f$. The non-perturbative part, which in this example is the distribution of quarks inside the proton, has been isolated to the PDF. This interpretation is illustrated in Figure 1.

As another example, the cross section of the Drell–Yan process $pp \to e^-e^+X$ can similarly be factored into [3, §17.4]

$$\sigma(pp \to e^-e^+X) = \sum_f \int \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, f_{f/p}(\xi_1) f_{\bar{f}/p}(\xi_2) \hat{\sigma}(q_f\bar{q}_f \to e^-e^+). \tag{1.2}$$

Even though the PDFs cannot be calculated analytically using perturbation theory, they can be determined numerically from experimental data. A key property of PDFs is universality, which means that the PDFs are process-independent. [3, §17.3] This implies that the PDF $f_{f/p}$ is the same in both Eq. (1.1) and Eq. (1.2).

The equivalent photon approximation (EPA) is the analog of collinear factorization in QED. Instead of the proton (or any other charged particle) consisting of quarks and gluons, it is considered as a source of photons in the equivalent photon approximation. This is usually justified by a classical argument, where one notices that the electric and magnetic fields of an ultrarelativistic charged particle resemble those of a pulse of electromagnetic radiation [5, §11.10]. The charged particle can therefore be interpreted as a cloud of photons. It is then these photons which mediate the interaction. Thus, the potentially complicated interaction of the proton with some other charged particle, say, a lepton, can be replaced by an interaction between a photon and the lepton.

The main objective of this thesis is to derive the factorization of electromagnetic cross sections in processes involving one or two photon exchanges, which are analogous to Eqs. (1.1) and (1.2), respectively. By doing this, we will also obtain the photon distribution function $f_{\gamma/p}$ of the proton, and more generally of any charged spin-0 or spin-$\frac{1}{2}$ particle. In the context of the equivalent photon approximation, this photon distribution function is more often called the equivalent photon distribution or the equivalent photon flux.

One can also take the analogy with parton distribution functions further. A numerical photon distribution function of the proton, which includes both elastic and inelastic contributions, has been obtained in Refs. [6, 7]. Evolution equations for the photon and electron distribution functions can also be derived [3, §17.5]. These aspects are outside the scope of this thesis and will not be further considered.

Historically, the equivalent photon approximation was independently discovered by C. F. Weizsäcker [8] and E. J. Williams [9] in 1934 [5, §15.4]. Thus, the equivalent photon approximation is also called the Weizsäcker–Williams method. Originally, the equivalent photon distribution was derived semiclassically by Fourier-transforming the electric and magnetic fields of an ultrarelativistic charged point-like particle; see for example Refs. [5, §15.4] and [9, 10]. However, the focus of this thesis is in applying field-theoretical methods to the equivalent photon approximation.

In high-energy physics, the equivalent photon approximation is often used in studying hadronic interactions and heavy-ion collisions. For example, exclusive muon-pair production in the process $\gamma\gamma \to \mu^+\mu^-$ was studied by the ATLAS collaboration [11] using proton-proton collisions. In this example, the photons which fuse to the muon pair are emitted by the colliding protons. As another example, Drees and Zeppenfeld [12] applied the equivalent photon approximation to the production of supersymmetric particles in electron-proton collisions. More generally, particle production in hadronic collisions is often modelled using the equivalent photon approximation [13–15]. In addition, Monte-Carlo event generators, such as Pythia [16] and STARlight [17], utilize the equivalent photon approximation.

In 2013, d'Enterria and Silveira [18] proposed using equivalent photons from colliding lead ions to observe light-by-light scattering $\gamma\gamma \to \gamma\gamma$ experimentally. Light-by-light scattering is a purely quantum-mechanical process which occurs at one-loop level in QED via fermion box diagrams, and has been proposed as a sensitive probe for effects beyond the standard model [18, 19]. The equivalent photon approximation gives a means of isolating the light-by-light scattering cross section $\sigma(\gamma\gamma \to \gamma\gamma)$ from the lead-lead scattering cross section $\sigma(\mathrm{Pb\,Pb} \to \mathrm{Pb\,Pb}\,\gamma\gamma)$. Direct observations of the light-by-light scattering process were later obtained by the ATLAS [19–21] and CMS [22] collaborations using lead-lead collisions.

Although it has been nearly a century since the initial development of the equivalent photon approximation, the method has seen sustained interest in both applications and further development of the method itself. Early derivations using field-theoretical methods were done two decades later [23, 24] and helicity-based methods another two decades later [25–27]. Numerous improvements have been also been obtained, see for example Refs. [28, 29]. In 1984, Dawson [30] derived an extension of the equivalent photon approximation to the electroweak sector. Dependence on the impact parameter has also been considered in the context of the classical field approximation [31, 32]. Very recently, in 2022, the validity of the equivalent photon approximation was again considered in Ref. [33].

What is missing from the literature is a clear and detailed derivation of the equivalent photon approximation. While the method has been derived in many references using many different methods, it is often difficult to see what approximations have been made and where, leading to a lack of clarity on the applicability of the method. This thesis attempts to partially fill that gap by going through the derivation in enough detail as to show the approximations that ultimately lead to the equivalent photon approximation.

In this thesis, we use a helicity-based approach to derive the factorization of scattering cross sections and subsequently the equivalent photon distribution. The definitive source for the field-theoretical equivalent photon approximation is Ref. [34], which we follow throughout this thesis. We start with the polarization of virtual photons in Section 2. In Section 3, we derive the equivalent photon approximation in processes with one photon exchange and in Section 4 generalize the derivation to processes with two photon exchanges. In Section 5, we consider the general forms of the photon distributions for spin-0 and spin-$\frac{1}{2}$ particles. In Section 6, we apply the equivalent photon approximation to muon-pair production and compare it with a full calculation. Finally, in Section 7, we give some concluding remarks.

1.1 Notation and conventions

Cartesian vectors $\mathbf{v} = (v_x, v_y, v_z)$ are denoted with a bold symbol. Components of a four-vector $u = (u^0, u^1, u^2, u^3)$ are denoted with upper Greek indices as in u^μ, where $\mu = 0, 1, 2, 3$. Indices can be lowered with the metric tensor, $v_\mu \equiv g_{\mu\nu} v^\nu$. Repeated indices are summed over according to the Einstein summation convention,

$$a_\mu b^\mu \equiv a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3.$$

The mostly-minus signature is used for the metric, so that

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

With this signature, $a \cdot b \equiv a_\mu b^\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$ and $p^2 = m^2$, where p is the momentum of an on-shell particle with mass m.

Commutators are denoted with square brackets $[A, B] \equiv AB - BA$ and anticommutators with curly braces $\{A, B\} \equiv AB + BA$. The γ-matrices (see Appendix A) are denoted by γ^μ with $\mu = 0, 1, 2, 3$ and γ-matrices with lowered indices are defined by $\gamma_\mu \equiv g_{\mu\nu} \gamma^\nu$. Contractions of γ-matrices with four-vectors are denoted with the slash notation $\not{a} \equiv a_\mu \gamma^\mu = a^\mu \gamma_\mu$. The generator of Lorentz transformations is defined as $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Time flows from left to right in Feynman diagrams. When two expressions a and b are required to be equal, the notation $a \stackrel{!}{=} b$ is used.

2 Polarization of virtual photons

It is well known that real photons only have two polarization states, which are referred to as the transverse polarization states [2, §6.9] For a virtual photon, however, we will see that we need three polarization states. These states will play an important role in the calculation of scattering cross sections in Sections 3 and 4. Thus, it is useful to first consider these polarization states in detail.

2.1 Helicity eigenvectors

Consider an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, where ω is infinitesimal and $\delta^\mu{}_\nu$ is the Kronecker delta. The defining feature of a Lorentz transformation Λ is $g = \Lambda^T g \Lambda$, which follows[1] from the transformation law of four-vectors $a'^\mu = \Lambda^\mu{}_\nu a^\nu$ and the required invariance of the inner product $a' \cdot b' \stackrel{!}{=} a \cdot b$. To first order in ω,

$$\begin{aligned} g^{\mu\nu} &= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma} = (\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) g^{\rho\sigma} \\ &= \left[\delta^\mu{}_\rho \delta^\nu{}_\sigma + \delta^\mu{}_\rho \omega^\nu{}_\sigma + \delta^\nu{}_\sigma \omega^\mu{}_\rho + \omega^\mu{}_\rho \omega^\nu{}_\sigma\right] g^{\rho\sigma} \\ &\simeq g^{\mu\nu} + g^{\mu\sigma}\omega^\nu{}_\sigma + g^{\rho\nu}\omega^\mu{}_\rho = g^{\mu\nu} + \omega^{\nu\mu} + \omega^{\mu\nu}, \end{aligned}$$

which implies that ω has to be antisymmetric, $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

The most general form of ω reads

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -r_3 & r_2 \\ b_2 & r_3 & 0 & -r_1 \\ b_3 & -r_2 & r_1 & 0 \end{pmatrix},$$

where b_i corresponds to an infinitesimal boost along the i-axis and r_i corresponds to an infinitesimal rotation around the i-axis [35, §C.1]. Thus, the generator of

[1] Since $a' \cdot b' = g_{\rho\sigma} a'^\rho b'^\sigma = g_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu a^\mu b^\nu = (\Lambda^T)_\mu{}^\rho g_{\rho\sigma} \Lambda^\sigma{}_\nu a^\mu b^\nu \stackrel{!}{=} g_{\mu\nu} a^\mu b^\nu = a \cdot b$, it follows that $\Lambda^T g \Lambda = g$.

rotations along the z-axis is defined by[2]

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

since we can obtain the full rotation matrix by exponentiating $-i\theta S_3$.

The helicity operator is defined as the projection of the spin in the direction of momentum $\mathbf{q}$,

$$h \equiv \frac{\mathbf{S} \cdot \mathbf{q}}{|\mathbf{q}|},$$

where $\mathbf{S}$ is the spin operator. Let us first consider the case where the particle is moving along the z-axis. The helicity operator then reduces to $h = S_3$, where S_3 is given in Eq. (2.1). The eigenvalues of $h = S_3$ are $\lambda = 0, \pm 1$. For $\lambda = \pm 1$, the corresponding normalized eigenvectors are

$$\varepsilon^{\mu}_{\pm 1} = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0). \tag{2.2}$$

The eigenvalue $\lambda = 0$ corresponds to two eigenvectors, $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$. We will see in Section 2.2 that it is useful to define their linear combination

$$\varepsilon^{\mu}_{0} = \frac{1}{\sqrt{-q^2}}(|\mathbf{q}|, 0, 0, q^0), \tag{2.3}$$

where $\sqrt{-q^2}$ is a quantity representing the analog of mass of the virtual photon. Normalization in this context means that

$$\varepsilon_\lambda \cdot \varepsilon^{*}_{\lambda'} = (-1)^{\lambda} \delta_{\lambda\lambda'}. \tag{2.4}$$

We can also notice that

$$q \cdot \varepsilon_\lambda = 0, \tag{2.5}$$

where $q^\mu = (q^0, 0, 0, q_z)$. Since Eq. (2.5) is covariant, it holds in all frames, not just in the frame where it was derived. The helicity vectors derived in Eqs. (2.2) and (2.3) can be found in many textbooks, such as in Ref. [2, §6].

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[2] The choice $r_3 = i$ is ultimately fixed by the form of the commutation relation $[S_i, S_j] = i\varepsilon_{ijk} S_k$.

2.2 Polarization tensors

We now choose the polarization vectors of a virtual photon with momentum q to be helicity eigenstates. Thus, in the case where the photon travels along the z-axis, the polarization vectors are given by Eqs. (2.2) and (2.3). However, instead of working directly with these polarization vectors, we will derive manifestly covariant forms for the expressions $\sum_{\lambda=\pm1} \varepsilon_\lambda^\mu (\varepsilon_\lambda^\nu)^*$ and ε_0^μ.

Consider the collision $\gamma(q) + A(k) \to B(k')$ of the virtual photon γ with another particle A in their center-of-mass frame, where the momentum of each particle is shown in parenthesis. Furthermore, we choose the axes in such a way that the beam axis and the z-axis coincide. In this frame, $q^\mu = (E, 0, 0, q_z)$ and $k^\mu = (\tilde{E}, 0, 0, -q_z)$. Consulting Eq. (2.2), we see that $k \cdot \varepsilon_{\pm1} = 0$.

We consider first the transverse polarization vectors $\varepsilon_{\pm1}$. These satisfy the orthogonality relations

$$q \cdot \varepsilon_{\pm1} = k \cdot \varepsilon_{\pm1} = 0 \tag{2.6}$$

as well as the normalization condition (see Eq. (2.4))

$$\varepsilon_{\pm1} \cdot \varepsilon_{\pm1}^* = -1. \tag{2.7}$$

We only need to consider the form appearing in the squared invariant amplitudes, which is the sum

$$P^{\mu\nu} \equiv \sum_{\lambda=\pm1} \varepsilon_\lambda^\mu (\varepsilon_\lambda^\nu)^*. \tag{2.8}$$

Requiring $P^{\mu\nu}$ to be real, we get that

$$P^{\mu\nu} = (P^{\mu\nu})^* = P^{\nu\mu}. \tag{2.9}$$

Since this sum can only depend on the momenta k and q, the most general symmetric form of P is

$$P^{\mu\nu} = a g^{\mu\nu} + b k^\mu k^\nu + c q^\mu q^\nu + d(k^\mu q^\nu + q^\mu k^\nu), \tag{2.10}$$

where a, b, c, and d are scalar functions of k and q. From the orthogonality requirements in Eq. (2.6) it follows that

$$q_\mu P^{\mu\nu} = k_\mu P^{\mu\nu} = 0, \tag{2.11}$$

which in turn give the pair of equations

$$\begin{cases} (a + cq^2 + d(q \cdot k))q^\nu + (b(q \cdot k) + dq^2)k^\nu = 0, \\ (a + bk^2 + d(q \cdot k))k^\nu + (c(q \cdot k) + dk^2)q^\nu = 0. \end{cases}$$

Since these must hold for any k and q, the coefficients of k^ν and q^ν must vanish separately, yielding

$$a + cq^2 + d(q \cdot k) = b(q \cdot k) + dq^2 = a + bk^2 + d(q \cdot k) = c(q \cdot k) + dk^2 = 0. \quad (2.12)$$

The solution of Eq. (2.12) is

$$\begin{aligned} b &= \frac{aq^2}{(q \cdot k)^2 - q^2k^2}, \\ c &= \frac{ak^2}{(q \cdot k)^2 - q^2k^2}, \\ d &= -\frac{a(q \cdot k)}{(q \cdot k)^2 - q^2k^2}. \end{aligned} \quad (2.13)$$

Substituting Eq. (2.13) into Eq. (2.10),

$$P^{\mu\nu} = \frac{a\left[((q \cdot k)^2 - q^2k^2)g^{\mu\nu} + q^2k^\mu k^\nu + k^2q^\mu q^\nu - (q \cdot k)(k^\mu q^\nu + q^\mu k^\nu)\right]}{(q \cdot k)^2 - q^2k^2}. \quad (2.14)$$

The normalization condition in Eq. (2.7) translates to

$$g_{\mu\nu}P^{\mu\nu} = \sum_{\lambda=\pm1} \varepsilon_\lambda \cdot \varepsilon_\lambda^* = -2. \quad (2.15)$$

Applying this to Eq. (2.14) gives $a = -1$ and so

$$P^{\mu\nu} = -g^{\mu\nu} + \frac{(q \cdot k)(k^\mu q^\nu + q^\mu k^\nu) - q^2k^\mu k^\nu - k^2q^\mu q^\nu}{(q \cdot k)^2 - q^2k^2}, \quad (2.16)$$

which gives us a manifestly covariant form for the transverse polarization sum.

Using the orthogonality $q_\mu P^{\mu\nu} = k_\mu P^{\mu\nu} = 0$,

$$\begin{aligned} P_{\mu\rho}P^{\rho\nu} &= P_{\mu\rho}\left(-g^{\rho\nu} + \frac{(q \cdot k)(k^\rho q^\nu + q^\rho k^\nu) - q^2k^\rho k^\nu - k^2q^\rho q^\nu}{(q \cdot k)^2 - q^2k^2}\right) \\ &= -P_{\mu\rho}g^{\rho\nu} = -P_\mu{}^\nu. \end{aligned} \quad (2.17)$$

From this and Eqs. (2.9) and (2.15) it also follows that

$$P_{\mu\nu}P^{\mu\nu} = g^{\nu\rho}P_{\mu\nu}P^{\mu}{}_{\rho} = g^{\nu\rho}P_{\nu\mu}P^{\mu}{}_{\rho} = -g^{\nu\rho}P_{\nu\rho} = 2. \tag{2.18}$$

Equation (2.17) describes, up to a sign, an idempotent operator. That is, applying P twice is the same as applying $-P$ once. Apart from the sign difference, this is characteristic of a projection operator. Furthermore, for any vector $v^\mu = aq^\mu + bk^\mu$, we have again that $P_{\mu\nu}v^\nu = 0$. Based on these observations, we can interpret P as a sign-reversing projection operator to the subspace orthogonal to the vectors q and k.

Next we consider the longitudinal polarization vector ε_0. In addition to the orthogonality $q \cdot \varepsilon_0 = 0$, the longitudinal polarization vector is orthogonal to the transverse polarization vectors:

$$q \cdot \varepsilon_0 = \varepsilon_0 \cdot \varepsilon_{\pm 1} = 0. \tag{2.19}$$

The normalization follows from Eq. (2.4):

$$\varepsilon_0 \cdot \varepsilon_0^* = +1. \tag{2.20}$$

Denoting

$$R^\mu \equiv \epsilon_0^\mu, \tag{2.21}$$

we can parametrize it by

$$R^\mu = ak^\mu + bq^\mu. \tag{2.22}$$

Equation (2.19) can then be cast into the form

$$q_\mu R^\mu = 0; \;\; R_\mu P^{\mu\nu} = 0. \tag{2.23}$$

It should be noted that the parametrization (2.22) automatically satisfies $R_\mu P^{\mu\nu} = 0$. Now, applying $q_\mu R^\mu = 0$ to Eq. (2.22) yields

$$q_\mu R^\mu = a(q \cdot k) + bq^2 = 0,$$

the solution of which is $b = -a(q \cdot k)/q^2$. We now have the overall form of R,

$$R^\mu = a\left(k^\mu - q^\mu \frac{q \cdot k}{q^2}\right).$$

The normalization scalar a can be fixed with Eq. (2.20), which is equivalent to $R^\mu R^*_\mu = +1$. This gives

$$R^\mu R^*_\mu = |a|^2 \left(k^2 - \frac{(q \cdot k)^2}{q^2} \right) = 1. \tag{2.24}$$

By choosing a to be real and positive, we obtain $a = \sqrt{\frac{-q^2}{(q\cdot k)^2 - q^2 k^2}}$ from Eq. (2.24), and therefore

$$R^\mu = \sqrt{\frac{-q^2}{(q \cdot k)^2 - q^2 k^2}} \left(k^\mu - q^\mu \frac{q \cdot k}{q^2} \right). \tag{2.25}$$

Since a was chosen to be real, R^μ is also real.

Using Eqs. (2.16) and (2.25), we can calculate the total polarization sum

$$\begin{aligned}
\sum_{\lambda=0,\pm 1} (-1)^\lambda \varepsilon^\mu_\lambda (\varepsilon^\nu_\lambda)^* &= -P^{\mu\nu} + R^\mu R^\nu \\
&= g^{\mu\nu} - \frac{(q\cdot k)(q^\mu k^\nu + k^\mu q^\nu) - q^2 k^\mu k^\nu - k^2 q^\mu q^\nu}{(q\cdot k)^2 - q^2 k^2} \\
&\quad + \frac{-q^2}{(q\cdot k)^2 - q^2 k^2} \left(k^\mu - q^\mu \frac{q\cdot k}{q^2} \right) \left(k^\nu - q^\nu \frac{q\cdot k}{q^2} \right) \\
&= g^{\mu\nu} - \frac{(q\cdot k)(q^\mu k^\nu + k^\mu q^\nu) - q^2 k^\mu k^\nu - k^2 q^\mu q^\nu}{(q\cdot k)^2 - q^2 k^2} \\
&\quad - \frac{q^2 k^\mu k^\nu - (q\cdot k)(q^\mu k^\nu + k^\mu q^\nu) + \frac{(q\cdot k)^2}{q^2} q^\mu q^\nu}{(q\cdot k)^2 - q^2 k^2} \\
&= g^{\mu\nu} - \frac{(q\cdot k)^2 q^\mu q^\nu}{q^2 \left((q\cdot k)^2 - q^2 k^2 \right)} + \frac{k^2 q^\mu q^\nu}{(q\cdot k)^2 - q^2 k^2} \\
&= g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2},
\end{aligned} \tag{2.26}$$

which we recognize as the tensor structure of the photon propagator in the Landau gauge [3, Eq. (9.58)]. This observation ultimately justifies our choice of the three polarization vectors in Eqs. (2.2) and (2.3) for the polarization of virtual photons.

As a sanity check, we can return to the center-of-mass frame, where $q^\mu = (E, 0, 0, q_z)$ and $k^\mu = (\tilde{E}, 0, 0, -q_z)$. In this case,

$$q \cdot k = E\tilde{E} + q_z^2 \tag{2.27}$$

and

$$(q\cdot k)^2 - q^2k^2 = (E\tilde{E} + q_z^2)^2 - (E^2 - q_z^2)(\tilde{E}^2 + q_z^2) = (E+\tilde{E})^2 q_z^2. \tag{2.28}$$

Using Eqs. (2.2), (2.3), (2.27), and (2.28),

$$P^{\mu\nu} = -g^{\mu\nu} + \frac{(q\cdot k)(q^\mu k^\nu + k^\mu q^\nu) - q^2 k^\mu k^\nu - k^2 q^\mu q^\nu}{(q\cdot k)^2 - q^2k^2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & i & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & i & 0 \end{pmatrix} = \sum_{\lambda=\pm 1} \varepsilon_\lambda^\mu (\varepsilon_\lambda^\nu)^*$$

and

$$R^\mu R^\nu = \frac{-q^2}{(q\cdot k)^2 - q^2k^2}\left(k^\mu - q^\mu \frac{q\cdot k}{q^2}\right)\left(k^\nu - q^\nu \frac{q\cdot k}{q^2}\right)$$

$$= -\frac{1}{E^2 - q_z^2}\begin{pmatrix} q_z^2 & 0 & 0 & Eq_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Eq_z & 0 & 0 & E^2 \end{pmatrix} = \frac{1}{-q^2}\begin{pmatrix} q_z \\ 0 \\ 0 \\ E \end{pmatrix}\begin{pmatrix} q_z & 0 & 0 & E \end{pmatrix}$$

$$= \varepsilon_0^\mu \varepsilon_0^\nu = \varepsilon_0^\mu (\varepsilon_0^\nu)^*.$$

Furthermore, in the real photon limit $q^2 \to 0$ where $E = \sqrt{q^2 + q_z^2} \to q_z$, an explicit calculation shows that

$$-g^{\mu\nu} + \frac{q^\mu \tilde{q}^\nu + \tilde{q}^\mu q^\nu}{q\cdot\tilde{q}} = \begin{pmatrix} -1 + \dfrac{2E^2}{E^2+q_z^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - \dfrac{2q_z^2}{E^2+q_z^2} \end{pmatrix} \tag{2.29}$$

$$\to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = P^{\mu\nu},$$

where $\tilde{q}^\mu \equiv (E, 0, 0, -q_z)$ is the space-reflected version of $q^\mu = (E, 0, 0, q_z)$. The polarization tensor shown on the left-hand side of Eq. (2.29) is another form of the polarization tensor for real photons and can often be found in textbooks; see for example Ref. [36, Eq. (8.59)].

3 Equivalent photon approximation in one-photon exchange processes

In this section, we consider the equivalent photon approximation in a generic process $AB \to BX$ with one photon exchange. The idea of the equivalent photon approximation is that instead of considering the entire process $AB \to BX$ (Figure 2a), we can consider only the photon absorption process $\gamma A \to X$ (Figure 2b). The total cross section $\sigma(AB \to BX)$ is then the photon absorption cross section $\hat{\sigma}(\gamma A \to X)$ folded with the photon distribution $f_{\gamma/B}$. The goal of this section is to derive the photon distribution $f_{\gamma/B}$ from the virtual photon emission process $B \to B\gamma$ (Figure 2c) and to see how the factorization of $\sigma(AB \to BX)$ into $\hat{\sigma}(\gamma A \to X)$ and $f_{\gamma/B}$ arises.

3.1 Kinematics

We consider the process $A(k) + B(p) \to B(p') + X(k')$ shown in Figure 2a. The momentum of each particle is indicated in parenthesis and the conservation of momentum reads

$$k + p = p' + k'. \tag{3.1}$$

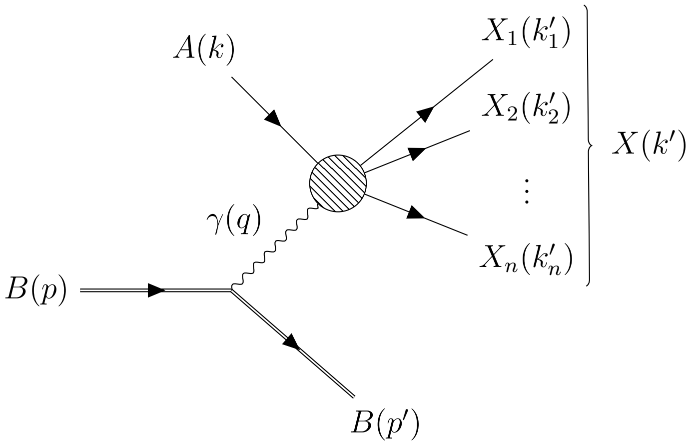
For each momentum vector, we introduce the notation

$$\begin{aligned} k &= (k^0, \mathbf{k}), \\ k' &= (k'^0, \mathbf{k}'), \\ p &= (E, \mathbf{p}), \\ p' &= (E', \mathbf{p}'). \end{aligned}$$

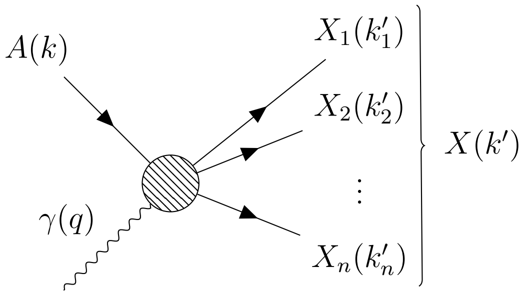
We will assume that the particle B stays intact, so that

$$p^2 = M^2 = (p')^2, \tag{3.2}$$

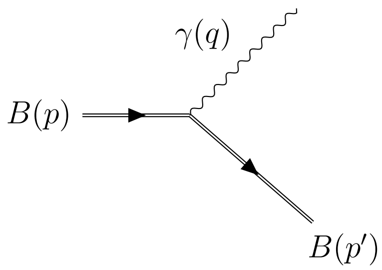
where M is the mass of particle B. This does not apply to particle A, where a fixed but arbitrary n-particle final state X is allowed. The total momentum of the final



(a) Total scattering $AB \to BX$.



(b) Virtual photon absorption $\gamma A \to X$.



(c) Virtual photon emission $B \to B\gamma$.

Figure 2. Feynman diagrams related to the $AB \to BX$ scattering. The momentum of each particle is denoted in parenthesis. The final state $X(k')$ consists of n particles, each with momentum k'_i such that $k' = \sum_{i=1}^{n} k'_i$.

state is denoted by k', while the individual momenta of the final-state particles are denoted by k'_i, where $i = 1, 2, \ldots, n$. Thus, in general[3]

$$m^2 = k^2 \neq (k')^2 = (k'_1 + k'_2 + \cdots k'_n)^2,$$

where m is the mass of particle A.

The momentum-transfer vector $q \equiv (q^0, \mathbf{q}) \equiv p - p' = k' - k$ gives the momentum of the intermediate-state photon. Using Eq. (3.2),

$$q^2 = (p - p')^2 = p^2 - 2p \cdot p' + (p')^2 = 2p^2 - 2p \cdot p' = 2p \cdot (p - p') = 2p \cdot q \qquad (3.3)$$

[3]There are of course situations where $k^2 = (k')^2$ holds, such as when $n = 1$ and $X = A$, but in a general setting we cannot assume anything about such a relation.

and similarly

$$q^2 = 2(p')^2 - 2p \cdot p' = 2p' \cdot (p' - p) = -2p' \cdot q. \tag{3.4}$$

The Mandelstam s for the entire process $AB \to BX$ is given by

$$s = (p + k)^2 = (p' + k')^2. \tag{3.5}$$

We also define a corresponding Mandelstam s for the subprocess $\gamma A \to X$ with

$$\hat{s} = (q + k)^2 = (k')^2. \tag{3.6}$$

Since[4] $q^2 \leq 0$, it is then useful to define

$$Q^2 \equiv -q^2 \geq 0.$$

For further use, we define the kinematical invariant

$$y \equiv \frac{q \cdot k}{p \cdot k}. \tag{3.7}$$

In the target rest frame, where $\mathbf{k} = \mathbf{0}$,

$$y = \frac{q^0 k^0 - \mathbf{q} \cdot \mathbf{k}}{E k^0 - \mathbf{p} \cdot \mathbf{k}} = \frac{q^0}{E} = \frac{E - E'}{E} = 1 - \frac{E'}{E}. \tag{3.8}$$

Thus, we see that in this frame, y gives the fractional energy loss of the projectile particle B. Since[5] $0 \leq E' \leq E$, it then follows from Eq. (3.8) that

$$0 \leq y \leq 1. \tag{3.9}$$

Now,

$$ys = \frac{q \cdot k}{p \cdot k}(p + k)^2 = \frac{q \cdot k}{p \cdot k}\left(M^2 + 2p \cdot k + m^2\right) \tag{3.10}$$

and

$$\hat{s} = (q + k)^2 = q^2 + 2q \cdot k + m^2. \tag{3.11}$$

[4]In any frame, $q^2 = (p - p')^2 = 2M^2 - 2EE' + 2\mathbf{p} \cdot \mathbf{p}'$. In the brick-wall frame where $\mathbf{p} = -\mathbf{p}'$ and $E = E'$, we have $q^2 = 2\left(M^2 - E^2 - |\mathbf{p}|^2\right) = -4\,|\mathbf{p}|^2 \leq 0$.

[5]The statement $E' \leq E$ is equivalent, by Eqs. (3.7) and (3.8), to $q \cdot k \geq 0$. Since $\hat{s} = (q+k)^2 \geq m^2$, Eq. (3.11) and $-q^2 \geq 0$ imply that $2q \cdot k \geq 0$.

By neglecting the masses m and M, and the virtuality q^2, Eqs. (3.10) and (3.11) together imply that

$$\hat{s} \simeq 2q \cdot k \simeq ys. \tag{3.12}$$

Similarly, we find that

$$q^2 = (p - p')^2 = 2M^2 - 2p \cdot p' \simeq -2p \cdot p', \tag{3.13}$$

when $M \simeq 0$. In the limit $q^2 \to 0$, Eq. (3.13) implies that $p \cdot p' \simeq 0$. Since the mass M has been neglected, $E = |\mathbf{p}|$ and $E' = |\mathbf{p}'|$, which gives

$$0 \simeq p \cdot p' = EE' - \mathbf{p} \cdot \mathbf{p}' \simeq EE'(1 - \cos\theta), \tag{3.14}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$. It follows from Eq. (3.14) that $\cos\theta = 1$, which means that $\mathbf{p}$ and $\mathbf{p}'$ are parallel. Assuming that $\mathbf{p}$ is along the z-axis so that $\mathbf{p} = (0, 0, E)$ and $\mathbf{p}' = (0, 0, E')$, Eq. (3.8) gives

$$p' = (E', \mathbf{p}') \simeq (E', 0, 0, E') = (1 - y)(E, 0, 0, E) = (1 - y)p,$$

and therefore

$$q = p - p' \simeq p - (1 - y)p = yp. \tag{3.15}$$

In general, we have

$$Q^2 = -(p - p')^2 = -2M^2 + 2p \cdot p' = -2M^2 + 2(EE' - |\mathbf{p}|\,|\mathbf{p}'|\cos\theta), \tag{3.16}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$. From Eq. (3.16) we see that if $\cos\theta = 1$, then the minimum value Q^2_{min} of Q^2 is obtained. That is to say,

$$Q^2_{\text{min}} = -2M^2 + 2(EE' - |\mathbf{p}|\,|\mathbf{p}'|). \tag{3.17}$$

In order to derive an invariant expression for Q^2_{min}, we work in the center-of-mass frame of the incoming particles A and B such that $\mathbf{k} = -\mathbf{p}$. Using Eq. (3.5),

$$\begin{aligned} s = (p + k)^2 &= \Big((E, \mathbf{p}) + (k^0, -\mathbf{p})\Big)^2 = (E + k^0)^2 \\ &= \Big(\sqrt{M^2 + |\mathbf{p}|^2} + \sqrt{m^2 + |\mathbf{p}|^2}\Big)^2. \end{aligned} \tag{3.18}$$

With Eq. (3.18), we can write $|\mathbf{p}|^2$ in terms of s, m^2 and M^2. The result is

$$|\mathbf{p}|^2 = \frac{1}{4s}\left(s-(m-M)^2\right)\left(s-(m+M)^2\right). \tag{3.19}$$

Similarly, by using Eq. (3.6),

$$\begin{aligned}\hat{s} = (q+k)^2 &= \left((E-E', \mathbf{p}-\mathbf{p}') + (k^0, -\mathbf{p})\right)^2 = (E-E'+k^0)^2 - |\mathbf{p}'|^2 \\ &= \left(\sqrt{M^2+|\mathbf{p}|^2} - \sqrt{M^2+|\mathbf{p}'|^2} + \sqrt{m^2+|\mathbf{p}|^2}\right)^2 - |\mathbf{p}'|^2 .\end{aligned} \tag{3.20}$$

Solving Eq. (3.20) for $|\mathbf{p}'|^2$ and using Eq. (3.19), we find

$$|\mathbf{p}'|^2 = \frac{1}{4s}\left(s-\left(\sqrt{\hat{s}}-M\right)^2\right)\left(s-\left(\sqrt{\hat{s}}+M\right)^2\right). \tag{3.21}$$

Then, using Eqs. (3.19) and (3.21),

$$\begin{aligned}E^2 = M^2 + |\mathbf{p}|^2 &= M^2 + \frac{1}{4s}\left(s-(m-M)^2\right)\left(s-(m+M)^2\right) \\ &= \frac{1}{4s}\left(s-m^2+M^2\right)^2\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}(E')^2 = M^2 + |\mathbf{p}'|^2 &= M^2 + \frac{1}{4s}\left(s-\left(\sqrt{\hat{s}}-M\right)^2\right)\left(s-\left(\sqrt{\hat{s}}+M\right)^2\right) \\ &= \frac{1}{4s}\left(s-\hat{s}+M^2\right)^2 .\end{aligned} \tag{3.23}$$

Plugging Eqs. (3.19) and (3.21)–(3.23) into Eq. (3.17),

$$\begin{aligned}Q^2_{\text{min}} &= -2M^2 + \frac{1}{2s}\left(s-m^2+M^2\right)\left(s-\hat{s}+M^2\right) \\ &\quad - \frac{1}{2s}\sqrt{\left(s-(m-M)^2\right)\left(s-(m+M)^2\right)} \\ &\qquad \times \sqrt{\left(s-(\hat{s}-M)^2\right)\left(s-(\hat{s}+M)^2\right)} \\ &= -2M^2 + \frac{1}{2s}\left(s-m^2+M^2\right)\left(s-\hat{s}+M^2\right) \\ &\quad - \frac{1}{2s}\sqrt{\left(s-(m-M)^2\right)\left(s-(m+M)^2\right)}\sqrt{\left(s-\hat{s}-M^2\right)^2 - 4\hat{s}M^2}.\end{aligned} \tag{3.24}$$

Neglecting the mass of the particle A by setting $m = 0$, Eq. (3.24) reduces to

$$\begin{aligned} Q^2_{\min} \simeq -2M^2 + \frac{1}{2s}(s + M^2)(s - \hat{s} + M^2) \\ - \frac{1}{2s}(s - M^2)\sqrt{(s - \hat{s} - M^2)^2 - 4\hat{s}M^2}, \end{aligned} \tag{3.25}$$

which has been obtained in Ref. [12, Eq. (2.8)] and Ref. [33, Eq. (21)].

Another useful approximation for $Q^2_{\min}$ can be derived in the target rest frame where Eq. (3.8) holds. Following Ref. [37, §2] and expanding to first order in M^2,

$$\begin{aligned} Q^2_{\min} &= -2M^2 + 2(EE' - |\mathbf{p}|\,|\mathbf{p}'|) \\ &= -2M^2 + 2EE'\left(1 - \sqrt{1 - \frac{M^2}{E^2}}\sqrt{1 - \frac{M^2}{(E')^2}}\right) \\ &\simeq -2M^2 + 2EE'\left(1 - \left(1 - \frac{M^2}{2E^2}\right)\left(1 - \frac{M^2}{2(E')^2}\right)\right) \\ &= -2M^2 + 2EE'\left(\frac{M^2}{2E^2} + \frac{M^2}{2(E')^2} - \frac{M^4}{4E^2(E')^2}\right) \\ &\simeq -2M^2 + 2EE'\left(\frac{M^2}{2E^2} + \frac{M^2}{2(E')^2}\right) \\ &= -2M^2 + M^2(1 - y) + \frac{M^2}{1 - y} \\ &= \frac{M^2 y^2}{1 - y}. \end{aligned} \tag{3.26}$$

In particular $Q^2_{\min} \propto M^2$. It should be noted that Eq. (3.26) can also be obtained by substituting $\hat{s} = ys$ (see Eq. (3.12)) into Eq. (3.25) and expanding to first order in M^2.

3.2 General scattering formalism

We begin by writing down the invariant amplitude of the scattering $AB \to BX$ by the exchange of a photon, shown in Figure 2a. We denote the absorptive part containing the vertex[6] $\gamma A \to X$ as ℓ^μ and the emissive part containing the vertex $B \to \gamma B$ as H^ν. Feynman rules of QED, given in Appendix B, tell us that the

[6] While the vertices technically have no notions of incoming or outgoing particles, the language of absorptive and emissive parts is in this context a useful way to differentiate between the two parts of the diagram connected by a photon.

invariant amplitude $\mathcal{M}(AB \to BX)$ is given by

$$i\mathcal{M}(AB \to BX) = (iZ_A e\ell^\mu)\left[-\frac{ig_{\mu\nu}}{q^2}\right](iZ_B e H^\nu) = iZ_A Z_B e^2 \frac{\ell^\mu H_\mu}{q^2}, \tag{3.27}$$

where two factors of iZe have been factored out of the $\gamma A \to X$ and $B \to \gamma B$ vertices. The constants Z_A and Z_B are the electric charges of the particles A and B, respectively, in units of e ($Z = -1$ for the electron and $Z = +1$ for the proton). Squaring the amplitude in Eq. (3.27),

$$|\mathcal{M}(AB \to BX)|^2 = \frac{Z_A^2 Z_B^2 e^4}{q^4} \ell^\mu (\ell^\nu)^* H_\mu H_\nu^*.$$

Finally, we average over the initial-state spins and sum over the final-state spins in order to obtain the unpolarized squared amplitude

$$\overline{|\mathcal{M}(AB \to BX)|^2} = \frac{Z_A^2 Z_B^2 e^4}{q^4} \overline{\ell^\mu (\ell^\nu)^*} \frac{1}{2} \overline{\sum} H_\mu H_\nu^* \equiv \frac{Z_A^2 Z_B^2 e^4}{q^4} \overline{\ell^\mu (\ell^\nu)^*} W_{\mu\nu}, \tag{3.28}$$

where in the last step we defined $W_{\mu\nu} \equiv \frac{1}{2}\overline{\sum} H_\mu H_\nu^*$. The symbol $\overline{\sum}$ indicates summing, but not averaging, over all appropriate initial- and final-state spin states. The bar in $\overline{\ell^\mu (\ell^\nu)^*}$ indicates initial-state averaging and final-state summation. It should also be noted that while the process considered here is quite general, the notation has been chosen to suggest that the virtual photon is emitted by a hadron and absorbed by a lepton.

The differential cross section for the process $AB \to BX$ is then [36, §G]

$$\mathrm{d}\sigma(AB \to BX) = \frac{\overline{|\mathcal{M}(AB \to BX)|^2}}{4\sqrt{(p \cdot k)^2 - p^2 k^2}} \mathrm{dPS}, \tag{3.29}$$

where $k^2 = m^2$ and $p^2 = M^2$. The flux factor $\sqrt{(p \cdot k)^2 - p^2 k^2}$ assumes that $\mathbf{p}$ and $\mathbf{k}$ are collinear. The phase space element dPS is

$$\mathrm{dPS} = (2\pi)^4 \delta^{(4)}(p + k - p' - k') \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'} \mathrm{d}\Pi_X,$$

where the four-dimensional Dirac delta function $\delta^{(4)}$ enforces the conservation of momentum stated in Eq. (3.1). The Lorentz-invariant measure $\mathrm{d}^3\mathbf{p}'/(2\pi)^3 2E'$ gives the phase-space volume element for the final-state particle B with energy E'. The term $\mathrm{d}\Pi_X$ is a product of similar phase-space volume elements of the n particles in the final state X,

$$\mathrm{d}\Pi_X = \prod_{i=1}^{n} \frac{\mathrm{d}^3\mathbf{k}'_i}{(2\pi)^3 2(k'_i)^0}, \tag{3.30}$$

where $(k'_i)^0$ gives the energy of the i^{th} particle in the final state X. Thus, using Eq. (3.28), we can rewrite Eq. (3.29) as

$$\begin{aligned} \mathrm{d}\sigma(AB \to BX) = \frac{Z_A^2 Z_B^2 e^4}{q^4} & \frac{\overline{\ell^\mu (\ell^\nu)^*} W_{\mu\nu}}{4\sqrt{(p\cdot k)^2 - p^2 k^2}} \\ & \times (2\pi)^4 \delta^{(4)}(p + k - p' - k') \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'} \,\mathrm{d}\Pi_X. \end{aligned} \tag{3.31}$$

For simplicity, we consider the case where all momenta of the final state X have been integrated over. To perform this integral in Eq. (3.31), we introduce the absorption tensor

$$\ell^{\mu\nu} \equiv \int \mathrm{d}\Pi_X \overline{\ell^\mu (\ell^\nu)^*} (2\pi)^4 \delta^{(4)}(p + k - p' - k'). \tag{3.32}$$

With the help of Eqs. (3.31) and (3.32), the differential cross section can be written as

$$\mathrm{d}\sigma(AB \to BX) = \frac{Z_A^2 Z_B^2 e^4}{q^4} \frac{\ell^{\mu\nu} W_{\mu\nu}}{4\sqrt{(p\cdot k)^2 - p^2 k^2}} \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'}. \tag{3.33}$$

The integral over k' in Eq. (3.32), which is included in the phase space element $\mathrm{d}\Pi_X$, ensures that $\ell^{\mu\nu}$ can only depend on the momenta k and q.

3.3 Photon absorption

We now consider the virtual photon absorption $\gamma A \to X$ shown in Figure 2b. We define[7] the amplitude

$$i\mathcal{M}_\lambda(\gamma A \to X) = i Z_A e \ell_\mu \varepsilon^\mu_\lambda(q),$$

so that the squared amplitude is

$$|\mathcal{M}_\lambda(\gamma A \to X)|^2 = Z_A^2 e^2 \ell_\mu \ell^*_\nu \varepsilon^\mu_\lambda (\varepsilon^\nu_\lambda)^*. \tag{3.34}$$

[7] The amplitude must be defined when the photon is virtual. However, the subsequent transverse amplitude (3.35) coincides with the amplitude obtained from Feynman rules when the photon is real.

From this we define the spin-averaged transverse absorption amplitude

$$\overline{|\mathcal{M}_T(\gamma A \to X)|^2} \equiv \frac{1}{2} Z_A^2 e^2 \overline{\ell_\mu \ell_\nu^*} \sum_{\lambda=\pm 1} \varepsilon_\lambda^\mu (\varepsilon_\lambda^\nu)^* = \frac{1}{2} Z_A^2 e^2 \overline{\ell_\mu \ell_\nu^*} P^{\mu\nu} \tag{3.35}$$

and the longitudinal absorption amplitude

$$\overline{|\mathcal{M}_L(\gamma A \to X)|^2} \equiv Z_A^2 e^2 \overline{\ell_\mu \ell_\nu^*} \varepsilon_0^\mu (\varepsilon_0^\nu)^* = Z_A^2 e^2 \overline{\ell_\mu \ell_\nu^*} R^\mu R^\nu, \tag{3.36}$$

where Eqs. (2.8) and (2.21) were used to identify the polarization tensors $P^{\mu\nu}$ and R^μ. From these amplitudes we can define the corresponding absorption cross sections

$$\begin{aligned} \mathrm{d}\hat{\sigma}_{T/L}(\gamma A \to X) &= \frac{\overline{\left|\mathcal{M}_{T/L}(\gamma A \to X)\right|^2}}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(q + k - k') \mathrm{d}\Pi_X \\ &= \frac{\overline{\left|\mathcal{M}_{T/L}(\gamma A \to X)\right|^2}}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - k') \mathrm{d}\Pi_X . \end{aligned} \tag{3.37}$$

If the photon is real, the differential cross section $\mathrm{d}\sigma_T(\gamma A \to X)$ in Eq. (3.37) coincides with the cross section of the subprocess $\gamma A \to X$ when $\mathbf{q}$ and $\mathbf{k}$ are collinear. However, Eq. (3.29) has already been written in a frame where $\mathbf{p}$ and $\mathbf{k}$ are collinear, and in general there is no guarantee that $\mathbf{q}$ and $\mathbf{p}$ are collinear. Nevertheless, when Q^2 and M^2 are small, $\mathbf{q}$ and $\mathbf{p}$ are indeed approximately collinear; see Eq. (3.15).

Explicitly, Eqs. (3.35)–(3.37) say that the transverse absorption cross section is

$$\mathrm{d}\hat{\sigma}_T(\gamma A \to X) = \frac{1}{2} Z_A^2 e^2 \frac{\overline{\ell_\mu \ell_\nu^*} P^{\mu\nu}}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - k')\, \mathrm{d}\Pi_X \tag{3.38}$$

and the longitudinal absorption cross section is

$$\mathrm{d}\hat{\sigma}_L(\gamma A \to X) = Z_A^2 e^2 \frac{\overline{\ell_\mu \ell_\nu^*} R^\mu R^\nu}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - k')\, \mathrm{d}\Pi_X . \tag{3.39}$$

Once again we integrate over the momenta of the final state X to obtain the integrated transverse absorption cross section

$$\hat{\sigma}_T(\gamma A \to X) = \frac{1}{2} Z_A^2 e^2 \frac{\ell_{\mu\nu} P^{\mu\nu}}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} \tag{3.40}$$

and the integrated longitudinal absorption cross section

$$\hat{\sigma}_L(\gamma A \to X) = Z_A^2 e^2 \frac{\ell_{\mu\nu} R^\mu R^\nu}{4\sqrt{(q \cdot k)^2 - q^2 k^2}}, \tag{3.41}$$

where Eq. (3.32) was used to identify the absorption tensor $\ell_{\mu\nu}$.

3.4 Factorization of the cross section

As in Section 2.2 for the polarization tensor $P^{\mu\nu}$, we can parametrize $\ell^{\mu\nu}$ as

$$\ell^{\mu\nu} = a g^{\mu\nu} + b k^\mu k^\nu + c q^\mu q^\nu + d\left(k^\mu q^\nu + k^\nu q^\mu\right), \tag{3.42}$$

where the coefficients a, b, c, and d are functions of the scalar invariants m^2, q^2 and $q \cdot k$. The Ward identity $q_\mu \ell^{\mu\nu} = 0$ yields

$$\left(a + cq^2 + d(q \cdot k)\right) q^\nu + \left(b(q \cdot k) + dq^2\right) k^\nu = 0. \tag{3.43}$$

Equation (3.43) must hold for arbitrary momenta, so the coefficients of q^ν and k^ν have to vanish identically. This results in the pair of equations

$$\begin{cases} a + cq^2 + d(q \cdot k) = 0, \\ b(q \cdot k) + dq^2 = 0. \end{cases} \tag{3.44}$$

Solving Eq. (3.44), we obtain $c = b(q \cdot k)^2/q^4 - a/q^2$ and $d = -b(q \cdot k)/q^2$. Plugging these into Eq. (3.42), and using Eqs. (2.25) and (2.26),

$$\begin{aligned} \ell^{\mu\nu} &= a\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + b\left(k^\mu - q^\mu \frac{q \cdot k}{q^2}\right)\left(k^\nu - q^\nu \frac{q \cdot k}{q^2}\right) \\ &= a \sum_{\lambda=0,\pm1} (-1)^\lambda \varepsilon_\lambda^\mu (\varepsilon_\lambda^\nu)^* + b\frac{(q \cdot k)^2 - q^2 k^2}{-q^2} R^\mu R^\nu \\ &= -a \sum_{\lambda=\pm1} \varepsilon_\lambda^\mu (\varepsilon_\lambda^\nu)^* + a\varepsilon_0^\mu (\varepsilon_0^\nu)^* + b\frac{(q \cdot k)^2 - q^2 k^2}{-q^2} R^\mu R^\nu \\ &= \underbrace{-a}_{\equiv \tilde{a}} P^{\mu\nu} + \underbrace{\left[a + b\frac{(q \cdot k)^2 - q^2 k^2}{-q^2}\right]}_{\equiv \tilde{b}} R^\mu R^\nu. \end{aligned} \tag{3.45}$$

Thus, with the obvious choice of $\tilde{a}$ and $\tilde{b}$, we can write

$$\ell^{\mu\nu} = \tilde{a}P^{\mu\nu} + \tilde{b}R^{\mu}R^{\nu}. \tag{3.46}$$

Contracting Eq. (3.46) with $R_\mu R_\nu$,

$$R_\mu R_\nu \ell^{\mu\nu} = \tilde{a}\,\overbrace{R_\mu R_\nu P^{\mu\nu}}^{=0} + \tilde{b}\,\overbrace{R_\mu R_\nu R^\mu R^\nu}^{=1} = \tilde{b}, \tag{3.47}$$

and with $P_{\mu\nu}$,

$$P_{\mu\nu}\ell^{\mu\nu} = \tilde{a}\,\overbrace{P_{\mu\nu}P^{\mu\nu}}^{=2} + \tilde{b}\,\overbrace{P_{\mu\nu}R^\mu R^\nu}^{=0} = 2\tilde{a}, \tag{3.48}$$

where Eqs. (2.18), (2.20), and (2.23) were used. Then, by using Eqs. (3.47) and (3.48),

$$\begin{aligned}
\ell_{\mu\nu}W^{\mu\nu} &= \left(\tilde{a}P_{\mu\nu} + \tilde{b}R_\mu R_\nu\right) W^{\mu\nu} \\
&= \left(\frac{1}{2}P_{\alpha\beta}\ell^{\alpha\beta}P_{\mu\nu} + R_\alpha R_\beta \ell^{\alpha\beta} R_\mu R_\nu\right) W^{\mu\nu} \\
&= \frac{1}{2}P_{\alpha\beta}\ell^{\alpha\beta}P_{\mu\nu}W^{\mu\nu} + R_\alpha R_\beta \ell^{\alpha\beta} R_\mu R_\nu W^{\mu\nu} \\
&\equiv \frac{1}{2}TP_{\alpha\beta}\ell^{\alpha\beta} + LR_\alpha R_\beta \ell^{\alpha\beta},
\end{aligned} \tag{3.49}$$

where in the last line we defined

$$\begin{aligned}
T &\equiv P_{\mu\nu}W^{\mu\nu}, \\
L &\equiv R_\mu R_\nu W^{\mu\nu}.
\end{aligned} \tag{3.50}$$

We can now recognize the remaining tensorial structure in Eq. (3.49) to be the same as in the photon absorption cross sections (3.40) and (3.41). Thus,

$$\begin{aligned}
T\hat{\sigma}_T(\gamma A \to X) + L\hat{\sigma}_L(\gamma A \to X) &= Z_A^2 e^2 \frac{\frac{1}{2}T\ell_{\mu\nu}P^{\mu\nu} + L\ell_{\mu\nu}R^\mu R^\nu}{4\sqrt{(q\cdot k)^2 - q^2k^2}} \\
&= Z_A^2 e^2 \frac{\ell_{\mu\nu}W^{\mu\nu}}{4\sqrt{(q\cdot k)^2 - q^2k^2}}.
\end{aligned} \tag{3.51}$$

Rewriting Eq. (3.33) with Eq. (3.51) in mind,

$$\mathrm{d}\sigma(AB \to BX) = \frac{Z_A^2 Z_B^2 e^4}{q^4} \frac{\ell_{\mu\nu}W^{\mu\nu}}{4\sqrt{(p\cdot k)^2 - p^2k^2}} \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'}$$

$$= \frac{Z_B^2 e^2}{q^4} \sqrt{\frac{(q\cdot k)^2 - q^2k^2}{(p\cdot k)^2 - p^2k^2}} \left[Z_A^2 e^2 \frac{\ell_{\mu\nu} W^{\mu\nu}}{4\sqrt{(q\cdot k)^2 - q^2k^2}} \right] \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'}$$

$$= \frac{Z_B^2 e^2}{q^4} \sqrt{\frac{(q\cdot k)^2 - q^2k^2}{(p\cdot k)^2 - p^2k^2}}$$

$$\times \left[T\hat{\sigma}_T(\gamma A \to X) + L\hat{\sigma}_L(\gamma A \to X) \right] \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'}.$$

We now take the limit $q^2 \to 0$. In this limit, the transverse cross section $\hat{\sigma}_T(\gamma A \to X)$ approaches the cross section $\hat{\sigma}(\gamma A \to X)$ where the photon is now real and therefore transversely polarized. The absorption tensor $\ell^{\mu\nu}$ and the transverse polarization tensor $P^{\mu\nu}$ are finite as $q^2 \to 0$, but R^μ has a singularity at $q^2 \to 0$. This means that $\tilde{b} = R_\mu R_\nu \ell^{\mu\nu}$ (see Eq. (3.47)) must approach zero fast enough to keep the combination $\tilde{b} R^\mu R^\nu$ appearing in Eq. (3.46) finite in the limit $q^2 \to 0$. By Eq. (3.41),

$$\hat{\sigma}_L(\gamma A \to X) \propto R_\mu R_\nu \ell^{\mu\nu} = \tilde{b}, \tag{3.52}$$

so the longitudinal cross section $\hat{\sigma}_L(\gamma A \to X)$ vanishes in the limit $q^2 \to 0$. One can also verify explicitly[8] that $L = R_\mu R_\nu W^{\mu\nu}$ remains finite as $q^2 \to 0$. All in all, in the limit $q^2 \to 0$,

$$\mathrm{d}\sigma(AB \to BX) \simeq \frac{Z_B^2 e^2}{q^4} \sqrt{\frac{(q\cdot k)^2}{(p\cdot k)^2 - p^2k^2}} T\hat{\sigma}(\gamma A \to X) \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'}. \tag{3.53}$$

3.5 Photon emission

In general, the emission tensor $W^{\mu\nu}$ cannot be calculated perturbatively. For instance, the proton has internal structure which is not predicted by perturbative QCD [38]. Instead, it must be parametrized using some unknown form factors that are ultimately determined by experimental data.

For the time being, we keep a very general form for $W^{\mu\nu}$ and only assume symmetry $W^{\mu\nu} = W^{\nu\mu}$. This makes it easier to use different parametrizations further on. Since $W^{\mu\nu}$ depends only on p and q,

$$W^{\mu\nu} = \alpha_1 p^\mu p^\nu + \alpha_2 (p^\mu q^\nu + q^\mu p^\nu) + \alpha_3 q^\mu q^\nu + \alpha_4 g^{\mu\nu}, \tag{3.54}$$

[8] The contraction L is calculated explicitly in Eq. (3.58), which shows that $L \to \alpha_4$ as $q^2 \to 0$, where α_4 is a finite parameter of the emission tensor, defined in Eq. (3.54).

where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are scalar functions of $p^2 = M^2$ and q^2.

We must now calculate $T = P_{\mu\nu}W^{\mu\nu}$ as it appears in the cross section (3.53). From Eq. (2.15) we already know that $g_{\mu\nu}P^{\mu\nu} = -2$. Using Eq. (3.3),

$$\begin{aligned}
p_\mu p_\nu P^{\mu\nu} + M^2 &= \frac{2(q\cdot k)(q\cdot p)(p\cdot k) - q^2(p\cdot k)^2 - k^2(q\cdot p)^2}{(q\cdot k)^2 - q^2k^2} \\
&= \frac{q^2(q\cdot k)(p\cdot k) - q^2(p\cdot k)^2 - \frac{1}{4}q^4k^2}{(q\cdot k)^2 - q^2k^2} \\
&= \frac{1}{4}\frac{q^2}{(q\cdot k)^2 - q^2k^2}\left[4(q\cdot k)(p\cdot k) - 4(p\cdot k)^2 - q^2k^2\right] \\
&= \frac{1}{4}\frac{q^2}{(q\cdot k)^2 - q^2k^2}\Big[4(q\cdot k)(p\cdot k) - 4(p\cdot k)^2 \\
&\qquad\qquad -(q\cdot k)^2 + (q\cdot k)^2 - q^2k^2\Big] \\
&= \frac{1}{4}\frac{-q^2}{(q\cdot k)^2 - q^2k^2}\left[(2p\cdot k - q\cdot k)^2 - \left((q\cdot k)^2 - q^2k^2\right)\right] \\
&= -\frac{q^2}{4}\left[\frac{(2p\cdot k - q\cdot k)^2}{(q\cdot k)^2 - q^2k^2} - 1\right].
\end{aligned} \tag{3.55}$$

It is now useful to use the Lorentz-invariant $y = (q\cdot k)/(p\cdot k)$, defined originally in Eq. (3.7). Then Eq. (3.55) can be written as

$$\begin{aligned}
p_\mu p_\nu P^{\mu\nu} &= -\frac{q^2}{4}\left(\frac{4M^2}{q^2} + \frac{(p\cdot k)^2\left(2 - \frac{q\cdot k}{p\cdot k}\right)^2}{(q\cdot k)^2 - q^2k^2} - 1\right) \\
&= -\frac{q^2}{4}\left(\frac{4M^2}{q^2} + \frac{(2-y)^2}{y^2 - q^2k^2/(p\cdot k)^2} - 1\right).
\end{aligned} \tag{3.56}$$

Using Eqs. (2.11), (2.15), and (3.56),

$$\begin{aligned}
T = P_{\mu\nu}W^{\mu\nu} &= \alpha_1 p_\mu p_\nu P^{\mu\nu} + \alpha_4 g_{\mu\nu}P^{\mu\nu} \\
&= -\alpha_1\frac{q^2}{4}\left(\frac{4M^2}{q^2} + \frac{(2-y)^2}{y^2 - q^2k^2/(p\cdot k)^2} - 1\right) - 2\alpha_4 \\
&\simeq -\alpha_1\frac{q^2}{4}\left(\frac{4M^2}{q^2} + \frac{(2-y)^2}{y^2} - 1\right) - 2\alpha_4 \\
&= -\alpha_1 q^2\left(\frac{M^2}{q^2} + \frac{1-y}{y^2}\right) - 2\alpha_4,
\end{aligned} \tag{3.57}$$

where the second-to-last line follows from setting $q^2 \to 0$ in the denominator of the second term in parenthesis.

For completeness, we note that from Eq. (2.24) it follows that $g_{\mu\nu}R^\mu R^\nu = R_\mu R^\mu = 1$, and from Eq. (3.3),

$$
\begin{aligned}
p_\mu R^\mu &= \sqrt{\frac{-q^2}{(q\cdot k)^2 - q^2k^2}} \left(p\cdot k - \frac{(q\cdot p)(q\cdot k)}{q^2}\right) \\
&= \sqrt{\frac{-q^2}{(q\cdot k)^2 - q^2k^2}} \left(p\cdot k - \frac{1}{2}(q\cdot k)\right) \frac{p\cdot k}{p\cdot k} \\
&= \sqrt{\frac{-q^2}{(q\cdot k)^2 - q^2k^2}} \left(1 - \frac{y}{2}\right)(p\cdot k),
\end{aligned}
$$

so that

$$
\begin{aligned}
L = W_{\mu\nu}R^\mu R^\nu &= \alpha_1 p_\mu p_\nu R^\mu R^\nu + \alpha_4 g_{\mu\nu} R^\mu R^\nu \\
&= -\frac{\alpha_1 q^2 (p\cdot k)^2}{(q\cdot k)^2 - q^2k^2}\left(1 - \frac{y}{2}\right)^2 + \alpha_4.
\end{aligned}
\tag{3.58}
$$

3.6 Equivalent photon distribution

We now consider the rest frame of the projectile, where $\mathbf{p} = \mathbf{0}$. In this frame, $E = M$ and $E' = \sqrt{r^2 + M^2}$, where $r \equiv |\mathbf{p}'| = |\mathbf{0} - \mathbf{p}'| = |\mathbf{q}|$. Then

$$
\begin{aligned}
Q^2 \equiv -q^2 = -(p - p')^2 &= -2M^2 + 2p\cdot p' = -2M^2 + 2EE' - 2\overbrace{\mathbf{p}\cdot\mathbf{p}'}^{=0} \\
&= 2M\sqrt{r^2 + M^2} - 2M^2 = 2M\left(\sqrt{r^2 + M^2} - M\right)
\end{aligned}
\tag{3.59}
$$

and

$$
\begin{aligned}
y = \frac{q\cdot k}{p\cdot k} &= 1 - \frac{p'\cdot k}{p\cdot k} = 1 - \frac{E'k^0 - \mathbf{p}'\cdot\mathbf{k}}{Ek^0 - \mathbf{p}\cdot\mathbf{k}} = 1 - \frac{E'}{E} + \frac{\mathbf{p}'\cdot\mathbf{k}}{Ek^0} \\
&= \frac{E - E'}{E} + \frac{r\,|\mathbf{k}|\cos\theta}{Ek^0} = \frac{q^0}{M} + \frac{r\,|\mathbf{k}|\cos\theta}{Mk^0},
\end{aligned}
\tag{3.60}
$$

where θ is the angle between $\mathbf{p}'$ and $\mathbf{k}$.

We now change variables from r and $\cos\theta$ to Q^2 and y. The Jacobian for this is

$$
J = \frac{\partial(Q^2, y)}{\partial(\cos\theta, r)} = \det\begin{pmatrix} \dfrac{\partial Q^2}{\partial(\cos\theta)} & \dfrac{\partial y}{\partial(\cos\theta)} \\ \dfrac{\partial Q^2}{\partial r} & \dfrac{\partial y}{\partial r} \end{pmatrix}.
\tag{3.61}
$$

It is easy to see from Eq. (3.59) that $\partial Q^2/\partial(\cos\theta) = 0$, which means that by

Eqs. (3.59)–(3.61),

$$J = -\frac{\partial y}{\partial(\cos\theta)}\frac{\partial Q^2}{\partial r} = -\frac{r\,|\mathbf{k}|}{Mk^0}\frac{2Mr}{\sqrt{r^2+M^2}} = -\frac{2\,|\mathbf{k}|\,r^2}{E'k^0}.$$

We can then write the volume element as

$$\mathrm{d}y\,\mathrm{d}Q^2 = |J|\,\mathrm{d}(\cos\theta)\,\mathrm{d}r = \frac{2\,|\mathbf{k}|\,r^2}{E'k^0}\,\mathrm{d}(\cos\theta)\,\mathrm{d}r.$$

Going to spherical coordinates and integrating the trivial azimuthal dependence away, we obtain

$$\begin{aligned}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'} &= \frac{r^2\,\mathrm{d}r\,\mathrm{d}(\cos\theta)\,\mathrm{d}\phi}{(2\pi)^3 2E'} = \frac{r^2\,\mathrm{d}r\,\mathrm{d}(\cos\theta)}{(2\pi)^2 2E'} = \frac{E'k^0}{2\,|\mathbf{k}|\,r^2}\frac{r^2}{(2\pi)^2 2E'}\,\mathrm{d}y\,\mathrm{d}Q^2 \\ &= \frac{1}{4(2\pi)^2}\frac{k^0}{|\mathbf{k}|}\,\mathrm{d}y\,\mathrm{d}Q^2.\end{aligned} \tag{3.62}$$

Still in the projectile's rest frame,

$$(p\cdot k)^2 - p^2k^2 = E^2(k^0)^2 - E^2\left((k^0)^2 - |\mathbf{k}|^2\right) = E^2\,|\mathbf{k}|^2\,,$$

so by using Eq. (3.7),

$$\begin{aligned}\frac{(p\cdot k)^2 - p^2k^2}{(q\cdot k)^2} &= \frac{(p\cdot k)^2}{(p\cdot k)^2}\frac{(p\cdot k)^2 - p^2k^2}{(q\cdot k)^2} = \frac{1}{y^2}\frac{(p\cdot k)^2 - p^2k^2}{(p\cdot k)^2} = \frac{1}{y^2}\frac{E^2\,|\mathbf{k}|^2}{E^2(k^0)^2} \\ &= \frac{1}{y^2}\frac{|\mathbf{k}|^2}{(k^0)^2}.\end{aligned} \tag{3.63}$$

Since $(k^0)^2 = k^2 + |\mathbf{k}|^2 \simeq |\mathbf{k}|^2$ when $k^2 = m^2 \ll |\mathbf{k}|^2$, we can approximate $(k^0)^2/\,|\mathbf{k}|^2 \simeq 1$ and use Eqs. (3.62) and (3.63) to obtain

$$\sqrt{\frac{(q\cdot k)^2}{(p\cdot k)^2 - p^2k^2}}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'} = \frac{y}{4(2\pi)^2}\frac{(k^0)^2}{|\mathbf{k}|^2}\,\mathrm{d}y\,\mathrm{d}Q^2 \simeq \frac{1}{(4\pi)^2}y\,\mathrm{d}y\,\mathrm{d}Q^2. \tag{3.64}$$

Going back to the full differential cross section $\mathrm{d}\sigma(AB\to BX)$ given in Eq. (3.53) and plugging in Eq. (3.64),

$$\frac{\mathrm{d}\sigma(AB\to BX)}{\mathrm{d}y\,\mathrm{d}Q^2} \simeq \frac{Z_B^2e^2}{q^4}\frac{y}{(4\pi)^2}T\hat{\sigma}(\gamma A\to X) = \frac{\alpha}{4\pi}\frac{y}{Q^4}Z_B^2T\hat{\sigma}(\gamma A\to X), \tag{3.65}$$

where $\alpha = e^2/4\pi \approx 1/137$ is the fine-structure constant and $\hat{\sigma}(\gamma A \to X)$ is understood to be evaluated at $\hat{s} \simeq ys$ (see Eq. (3.12)). We can now define the photon distribution

$$f_{\gamma/B}(y, Q^2) \equiv \frac{\alpha}{4\pi} \frac{y}{Q^4} Z_B^2 T, \tag{3.66}$$

where $T = T(y, Q^2, M^2)$, and then rewrite Eq. (3.65) as

$$\frac{\mathrm{d}\sigma(AB \to BX)}{\mathrm{d}y\,\mathrm{d}Q^2} \simeq f_{\gamma/B}(y, Q^2)\hat{\sigma}(\gamma A \to X). \tag{3.67}$$

Integrating Eq. (3.67) over y and Q^2,

$$\begin{aligned} \sigma(AB \to BX) &\simeq \int \mathrm{d}y\,\mathrm{d}Q^2 \frac{\mathrm{d}\sigma(AB \to BX)}{\mathrm{d}y\,\mathrm{d}Q^2} \\ &= \int \mathrm{d}y \int \mathrm{d}Q^2 f_{\gamma/B}(y, Q^2)\hat{\sigma}(\gamma A \to X) \\ &\equiv \int \mathrm{d}y\, f_{\gamma/B}(y)\hat{\sigma}(\gamma A \to X), \end{aligned} \tag{3.68}$$

where on the last line we defined the Q^2-integrated photon distribution

$$f_{\gamma/B}(y) \equiv \int \mathrm{d}Q^2 f_{\gamma/B}(y, Q^2). \tag{3.69}$$

Defining such an integrated distribution is possible, because the integrated cross section $\hat{\sigma}(\gamma A \to X)$ does not depend on Q^2. Using Eq. (3.57), we can write the equivalent photon distribution more explicitly as

$$f_{\gamma/B}(y, Q^2) = \frac{\alpha}{2\pi} \frac{y}{Q^2} Z_B^2 \left[\frac{\alpha_1}{2} \left(\frac{1-y}{y^2} - \frac{M^2}{Q^2} \right) - \frac{\alpha_4}{Q^2} \right], \tag{3.70}$$

where α_1 and α_4 are parameters of the emission tensor $W^{\mu\nu}$ (3.54). Equation (3.70) corresponds[9] to Eq. (6.16b) in Ref. [34].

Equation (3.67) finally gives the claimed factorization of electromagnetic cross sections in one-photon exchange processes, while Eq. (3.70) gives a general formula for the equivalent photon distribution. Given any emission tensor $W^{\mu\nu}$, one can use Eq. (3.70) to calculate the photon distribution.

[9] Ref. [34] uses $\omega = (q \cdot k)/m$ and $E = (p \cdot k)/m$ instead of y. These are connected by $y = \omega/E$ and $\mathrm{d}y = \mathrm{d}\omega/E$. We will see in Eq. (5.10) that $\alpha_1 = 4$ and $\alpha_4 = -Q^2$ for the case considered in [34, Eq. (6.16b)]. One should also note that [34, Eq. (6.16b)] contains the denominator $\omega^2 + Q^2$, which should be approximated to ω^2. The corresponding approximation was done in the second-to-last line in Eq. (3.57).

4 Equivalent photon approximation in two-photon exchange processes

In this section, we consider the equivalent photon approximation in a generic process involving two photon exchanges, depicted in Figure 3. One might assume that the factorization in Eq. (3.68) generalizes to

$$\sigma(AB \to ABX) = \int \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}Q_1^2 \, \mathrm{d}Q_2^2 f_{\gamma/A}(y_1, Q_1^2) f_{\gamma/B}(y_2, Q_2^2) \hat{\sigma}(\gamma\gamma \to X). \quad (4.1)$$

We show in this section that this is indeed the case, although there are some additional complications not present in the one-photon exchange case.

4.1 Kinematics

We consider the process $A(p_1) + B(p_2) \to A(p_1') + B(p_2') + X(k')$ depicted in Figure 3a. As in Section 3.1, the momentum of each particle is shown in parenthesis and we denote

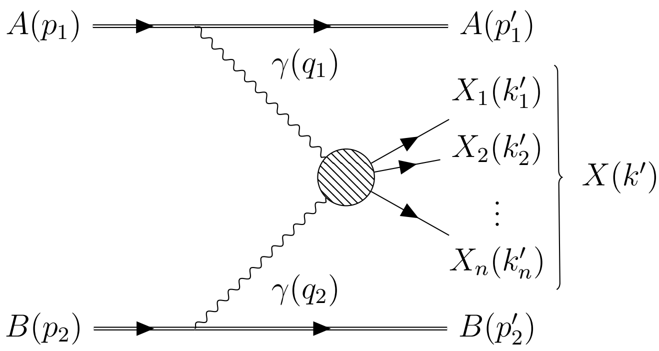
$$\begin{aligned} p_1 &= (E_1, \mathbf{p}_1), \\ p_2 &= (E_2, \mathbf{p}_2), \\ p_1' &= (E_1', \mathbf{p}_1'), \\ p_2' &= (E_2', \mathbf{p}_2'), \\ k' &= (k'^0, \mathbf{k}'). \end{aligned} \quad (4.2)$$

The particles A and B are assumed to stay intact during the scattering process so that

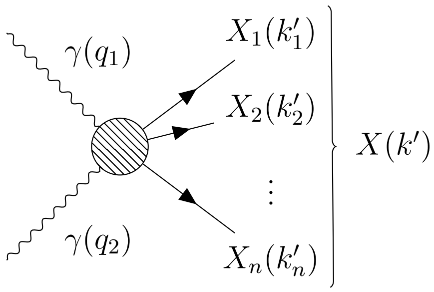
$$p_i^2 = M_i^2 = (p_i')^2,$$

where $i = 1, 2$. Here M_1 and M_2 are the masses of the particles A and B, respectively. The conservation of momentum is given by

$$p_1 + p_2 = p_1' + p_2' + k'.$$



(a) Total scattering $AB \to ABX$.



(b) Virtual two-photon absorption $\gamma\gamma \to X$.

Figure 3. Feynman diagrams related to the scattering $AB \to ABX$. The momentum of each particle is denoted in parenthesis. The final state $X(k')$ consists of n particles, each with momentum k'_i such that $k' = \sum_{i=1}^{n} k'_i$.

The momentum-transfer vectors are defined in the usual way by

$$q_i \equiv (q_i^0, \mathbf{q}_i) \equiv p_i - p'_i.$$

Analogously to Eqs. (3.3) and (3.4),

$$q_i^2 = (p_i - p'_i)^2 = p_i^2 - 2p_i \cdot p'_i + (p'_i)^2 = 2p_i^2 - 2p_i \cdot p'_i = 2p_i \cdot (p_i - p'_i) = 2p_i \cdot q_i \quad (4.3)$$

and

$$q_i^2 = 2(p'_i)^2 - 2p_i \cdot p'_i = 2p'_i \cdot (p'_i - p_i) = -2p'_i \cdot q_i. \quad (4.4)$$

The Mandelstam s for the process $AB \to ABX$ is

$$s = (p_1 + p_2)^2 = (p'_1 + p'_2 + k')^2$$

and for the subprocess $\gamma\gamma \to X$

$$\hat{s} = (q_1 + q_2)^2 = (k')^2.$$

Following Eq. (3.7), we define

$$\begin{aligned} y_1 &\equiv \frac{q_1 \cdot q_2}{p_1 \cdot q_2}, \\ y_2 &\equiv \frac{q_1 \cdot q_2}{p_2 \cdot q_1}. \end{aligned} \tag{4.5}$$

As in Eq. (3.8), y_1 and y_2 can be interpreted as fractional energy losses

$$y_i = 1 - \frac{E_i'}{E_i}$$

in the rest frames of the two respective photons. The rest frames of the photons exist as long as $q_i^2 \neq 0$. As in Eq. (3.9), we also have

$$0 \leq y_i \leq 1. \tag{4.6}$$

By neglecting the masses M_1 and M_2, and taking the limits $q_1^2, q_2^2 \to 0$, the same argument that led to Eq. (3.15) shows that q_i and p_i are approximately proportional. Then it follows from Eq. (4.5) that

$$\begin{aligned} y_1 &\simeq \frac{q_1 \cdot p_2}{p_1 \cdot p_2}, \\ y_2 &\simeq \frac{q_2 \cdot p_1}{p_1 \cdot p_2}. \end{aligned} \tag{4.7}$$

Using Eqs. (4.5) and (4.7), we obtain a useful approximate relation

$$(q_1 \cdot q_2)(p_1 \cdot p_2) \simeq (q_1 \cdot p_2)(q_2 \cdot p_1). \tag{4.8}$$

Finally, as in Eqs. (3.10)–(3.12), by neglecting the masses M_1 and M_2, and taking the limit $q_1^2, q_2^2 \to 0$,

$$\begin{aligned} \hat{s} &= (q_1 + q_2)^2 = q_1^2 + 2q_1 \cdot q_2 + q_2^2 \simeq 2q_1 \cdot q_2 = \frac{q_1 \cdot p_2}{p_1 \cdot p_2} \frac{q_1 \cdot q_2}{p_2 \cdot q_1} 2p_1 \cdot p_2 \\ &\simeq y_1 y_2 (2p_1 \cdot p_2) \simeq y_1 y_2 (p_1 + p_2)^2 = y_1 y_2 s, \end{aligned} \tag{4.9}$$

where we identified y_1 from Eq. (4.7) and y_2 from Eq. (4.5). Like in the one-photon exchange case, it is useful to define

$$Q_i^2 \equiv -q_i^2 \geq 0.$$

Equation (3.26) easily generalizes to

$$Q_{i,\min}^2 \simeq \frac{M_i^2 y_i^2}{1-y_i}, \tag{4.10}$$

while Eqs. (3.24) and (3.25) cannot be directly used due to the increased complexity of the kinematics.

4.2 Scattering cross sections

As in the case of a one photon exchange, we start by considering the cross section $\sigma(AB \to ABX)$ of the entire scattering process, shown in Figure 3a. Using Feynman rules, we obtain the invariant amplitude

$$\begin{aligned} i\mathcal{M}(AB \to ABX) &= (iZ_1 e H_1^\mu)\left[-\frac{ig_{\mu\rho}}{q_1^2}\right](e^2 K^{\rho\sigma})\left[-\frac{ig_{\nu\sigma}}{q_2^2}\right](iZ_2 e H_2^\nu) \\ &= \frac{Z_1 Z_2 e^4}{q_1^2 q_2^2} H_1^\mu K_{\mu\nu} H_2^\nu, \end{aligned} \tag{4.11}$$

where H_1^μ and H_2^ν describe the emission vertices $A \to \gamma A$ and $B \to \gamma B$, respectively, and $K_{\mu\nu}$ describes the two-photon absorption $\gamma\gamma \to X$. The constants Z_1 and Z_2 are the electric charges of the particles A and B in units of e ($Z = -1$ for the electron and $Z = +1$ for the proton). Since the tensors H_1 and H_2 depend only on the photon emission vertex, they are the same as in Eq. (3.27), apart from the different momenta that they depend on. Squaring the amplitude in Eq. (4.11),

$$|\mathcal{M}(AB \to ABX)|^2 = \frac{Z_1^2 Z_2^2 e^8}{q_1^4 q_2^4} H_1^\mu (H_1^\rho)^* H_2^\nu (H_2^\sigma)^* K_{\mu\nu} K_{\rho\sigma}^*.$$

For the spin-averaged amplitude, we can use the same emission tensor as was used in Eq. (3.28) to obtain

$$\overline{|\mathcal{M}(AB \to ABX)|^2} = \frac{Z_1^2 Z_2^2 e^8}{q_1^4 q_2^4} W_1^{\mu\rho} W_2^{\nu\sigma} \overline{K_{\mu\nu} K_{\rho\sigma}^*}. \tag{4.12}$$

The differential cross section is then

$$\begin{aligned} d\sigma(AB \to ABX) = &\frac{\overline{|\mathcal{M}(AB \to ABX)|^2}}{4\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2' - k') \\ &\times \frac{d^3\mathbf{p}_1'}{(2\pi)^3 2E_1'} \frac{d\mathbf{p}_2'}{(2\pi)^3 2E_2'} d\Pi_X, \end{aligned} \tag{4.13}$$

where E_1' and E_2' are the energies of the outgoing particles A and B, respectively, and $d\Pi_X$ is the phase-space volume element of the arbitrary but fixed final state X. This volume element has the same form as it does in Eq. (3.30). In writing Eq. (4.13), we assumed that $\mathbf{p}_1$ and $\mathbf{p}_2$ are collinear. As before, we integrate over the final state X. Only $\overline{K_{\mu\nu} K_{\rho\sigma}^*}$ depends on the momenta of the final state X, so we define

$$K_{\mu\nu\rho\sigma} \equiv \int d\Pi_X \overline{K_{\mu\nu} K_{\rho\sigma}^*} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2' - k'). \tag{4.14}$$

The cross section $d\sigma(AB \to ABX)$ must be real, and since the emission tensors W_1 and W_2 are real, the absorption tensor $K_{\mu\nu\rho\sigma}$ must be real as well. From this it follows that

$$K_{\mu\nu\rho\sigma} = K_{\mu\nu\rho\sigma}^* = K_{\rho\sigma\mu\nu}. \tag{4.15}$$

Thus, the cross section becomes, using Eqs. (4.13) and (4.14),

$$d\sigma(AB \to ABX) = \frac{Z_1^2 Z_2^2 e^8}{q_1^4 q_2^4} \frac{W_1^{\mu\rho} W_2^{\nu\sigma} K_{\mu\nu\rho\sigma}}{4\sqrt{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}} \frac{d^3\mathbf{p}_1'}{(2\pi)^3 2E_1'} \frac{d\mathbf{p}_2'}{(2\pi)^3 2E_2'} \tag{4.16}$$

Next, we consider the two-photon absorption process $\gamma\gamma \to X$, depicted in Figure 3b. The invariant amplitude is defined by

$$i\mathcal{M}_{\lambda_1\lambda_2}(\gamma\gamma \to X) = \varepsilon_{\lambda_1}^{\mu}(q_1) \left(e^2 K_{\mu\nu}\right) \varepsilon_{\lambda_2}^{\nu}(q_2), \tag{4.17}$$

where ε_λ is the polarization vector of the polarization (helicity) state λ. The squared amplitude is then

$$|\mathcal{M}_{\lambda_1\lambda_2}(\gamma\gamma \to X)|^2 = e^4 \varepsilon_{\lambda_1}^{\mu}(q_1) \left(\varepsilon_{\lambda_1}^{\rho}(q_1)\right)^* K_{\mu\nu} K_{\rho\sigma}^* \varepsilon_{\lambda_2}^{\nu}(q_2) \left(\varepsilon_{\lambda_2}^{\sigma}(q_2)\right)^*. \tag{4.18}$$

Summing over the different polarization states, one can use the polarization tensors introduced in Section 2.2 to write the squared amplitude in a covariant form. Here we write down explicitly only the case where both photons are transversely polarized.

In the kinematics of the process $\gamma\gamma \to X$, the transverse polarization tensor (2.16) is

$$P^{\mu\nu}(q_1, q_2) = -g^{\mu\nu} + \frac{(q_1 \cdot q_2)(q_1^\mu q_2^\nu + q_2^\mu q_1^\nu) - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu}{(q_1 \cdot q_2)^2 - q_1^2 q_2^2} = P^{\mu\nu}(q_2, q_1),$$

so we can henceforth simply denote

$$P^{\mu\nu} \equiv P^{\mu\nu}(q_1, q_2) = P^{\mu\nu}(q_2, q_1).$$

In addition, we will introduce the shorthand notation

$$\begin{aligned} R_1^\mu \equiv R^\mu(q_1, q_2) &= \sqrt{\frac{-q_1^2}{(q_1 \cdot q_2)^2 - q_1^2 q_2^2}} \left(q_2^\mu - q_1^\mu \frac{q_1 \cdot q_2}{q_1^2} \right), \\ R_2^\mu \equiv R^\mu(q_2, q_1) &= \sqrt{\frac{-q_2^2}{(q_1 \cdot q_2)^2 - q_1^2 q_2^2}} \left(q_1^\mu - q_2^\mu \frac{q_1 \cdot q_2}{q_2^2} \right) \end{aligned} \tag{4.19}$$

for the longitudinal polarization tensors given in Eq. (2.25). Then, averaging the squared amplitude given in Eq. (4.18) over the transverse polarization states and using the polarization tensor $P^{\mu\nu}$,

$$\begin{aligned} \overline{|\mathcal{M}_{TT}(\gamma\gamma \to X)|^2} &\equiv \frac{1}{4}\overline{\sum} |\mathcal{M}_{\lambda_1\lambda_2}(\gamma\gamma \to X)|^2 \\ &= \frac{1}{4} e^4 \sum_{\lambda_1 = \pm 1} \varepsilon_{\lambda_1}^\mu(q_1) \left(\varepsilon_{\lambda_1}^\rho(q_1)\right)^* \overline{K_{\mu\nu} K_{\rho\sigma}^*} \\ &\quad \times \sum_{\lambda_2 = \pm 1} \varepsilon_{\lambda_2}^\nu(q_2) \left(\varepsilon_{\lambda_2}^\sigma(q_2)\right)^* \\ &= \frac{1}{4} e^4 P^{\mu\rho} \overline{K_{\mu\nu} K_{\rho\sigma}^*} P^{\nu\sigma}. \end{aligned} \tag{4.20}$$

The cross section associated with the amplitude in Eq. (4.20) is defined as

$$\mathrm{d}\hat{\sigma}_{TT}(\gamma\gamma \to X) = \frac{\overline{|\mathcal{M}(\gamma\gamma \to X)_{TT}|^2}}{4\sqrt{(q_1 \cdot q_2)^2 - q_1^2 q_2^2}} (2\pi)^4 \delta^{(4)}(q_1 + q_2 - k') \mathrm{d}\Pi_X, \tag{4.21}$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ are taken to be approximately collinear; see the discussion on page 33. Integrating over the final state X and using Eq. (4.14),

$$\hat{\sigma}_{TT}(\gamma\gamma \to X) = \frac{e^4 P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma}}{16\sqrt{(q_1 \cdot q_2)^2 - q_1^2 q_2^2}}. \tag{4.22}$$

4.3 Two-photon absorption

As in Section 3.4, we must now parametrize the absorption tensor $K^{\mu\nu\rho\sigma}$. Since we integrated over the final state X, it can only depend on the momenta of the photons, q_1 and q_2. In addition, it must obey the symmetry condition given in Eq. (4.15), as well as the Ward identity

$$q_1^\mu K_{\mu\nu\rho\sigma} = q_2^\nu K_{\mu\nu\rho\sigma} = q_1^\rho K_{\mu\nu\rho\sigma} = q_2^\sigma K_{\mu\nu\rho\sigma} = 0. \tag{4.23}$$

The last two conditions in Eq. (4.23) are superfluous, as they follow from the first two conditions and the symmetry $K_{\mu\nu\rho\sigma} = K_{\rho\sigma\mu\nu}$. Due to the more complicated four-index structure, writing the general parametrization is fairly complicated. In the case of a one-photon exchange, the analogous parametrization Eq. (3.42) contained four parameters, which the Ward identity $q_\mu \ell^{\mu\nu} = 0$ reduced to two independent parameters. For $K^{\mu\nu\rho\sigma}$, the initial parametrization has 43 parameters, which are eventually reduced to 8. Thus, deriving a parametrization for $K^{\mu\nu\rho\sigma}$ in terms of the polarization tensors in analogy with Eq. (3.46) is more challenging.

One approach is to first guess the form of $K^{\mu\nu\rho\sigma}$ in terms of P, R_1, and R_2, using Eq. (3.46) as a guide. In particular, we require that each term should be symmetric in the exchange $\mu\nu \leftrightarrow \rho\sigma$ and that each term should individually satisfy the Ward identity. For example, the symmetry requires that a term such as $P^{\mu\nu} R_1^\rho R_2^\sigma$ should be accompanied by another term $R_1^\mu R_2^\nu P^{\rho\sigma}$. The Ward identity in turn excludes certain terms, such as $R_1^\mu R_1^\nu P^{\rho\sigma}$, since $(q_2)_\nu R_1^\nu \neq 0$ in general. All in all, this consideration results in the parametrization

$$\begin{aligned} K^{\mu\nu\rho\sigma} = {} & a_1 P^{\mu\nu} P^{\rho\sigma} + a_2 P^{\mu\rho} P^{\nu\sigma} + a_3 P^{\mu\sigma} P^{\nu\rho} + a_4 \left(P^{\mu\nu} R_1^\rho R_2^\sigma + R_1^\mu R_2^\nu P^{\rho\sigma} \right) \\ & + a_5 P^{\mu\rho} R_2^\nu R_2^\sigma + a_6 R_1^\mu R_1^\rho P^{\nu\sigma} + a_7 \left(P^{\mu\sigma} R_1^\rho R_2^\nu + R_1^\mu R_2^\sigma P^{\nu\rho} \right) \\ & + a_8 R_1^\mu R_2^\nu R_1^\rho R_2^\sigma. \end{aligned} \tag{4.24}$$

Equation (4.24) is certainly symmetric in the exchange $\mu\nu \leftrightarrow \rho\sigma$ and satisfies the Ward identity given in Eq. (4.23). However, it is not obvious that this parametrization is entirely general and is not missing anything. One can use a computer algebra system to check that this is indeed the case. The details are given in Appendix E.

A more systematic method for finding Eq. (4.24) can be found in the literature [25, 34]. We give a brief description of this helicity-based method here. The main idea is to write a formal expansion of $K^{\mu\nu\rho\sigma}$ using some basis tensors $B^{\mu\nu\rho\sigma}_{\lambda_1\lambda_2\lambda_3\lambda_4}$,

$$K^{\mu\nu\rho\sigma} = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4} B^{\mu\nu\rho\sigma}_{\lambda_1\lambda_2\lambda_3\lambda_4}, \tag{4.25}$$

where the sum runs over all polarization states $0, \pm 1$. One convenient choice for the basis is constructed using polarization vectors,

$$B^{\mu\nu\rho\sigma}_{\lambda_1\lambda_2\lambda_3\lambda_4} = (-1)^{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \left(\varepsilon^{\mu}_{\lambda_1}(q_1)\right)^* \left(\varepsilon^{\nu}_{\lambda_2}(q_2)\right)^* \varepsilon^{\rho}_{\lambda_3}(q_1)\varepsilon^{\sigma}_{\lambda_4}(q_2). \tag{4.26}$$

The expansion coefficients $\mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4}$ can then be projected out from the expansion in Eq. (4.25) using Eq. (2.4):

$$\mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \varepsilon^{\mu}_{\lambda_1}(q_1)\varepsilon^{\nu}_{\lambda_2}(q_2) \left(\varepsilon^{\rho}_{\lambda_3}(q_1)\right)^* \left(\varepsilon^{\sigma}_{\lambda_4}(q_2)\right)^* K_{\mu\nu\rho\sigma}. \tag{4.27}$$

Substituting this back into Eq. (4.25) and using Eqs. (2.4), (2.26), and (4.23),

$$\begin{aligned}
K^{\mu\nu\rho\sigma} &= \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} (-1)^{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \left(\varepsilon^{\mu}_{\lambda_1}(q_1)\right)^* \left(\varepsilon^{\nu}_{\lambda_2}(q_2)\right)^* \varepsilon^{\rho}_{\lambda_3}(q_1)\varepsilon^{\sigma}_{\lambda_4}(q_2) \\
&\qquad\qquad \times \varepsilon^{\mu'}_{\lambda_1}(q_1)\varepsilon^{\nu'}_{\lambda_2}(q_2) \left(\varepsilon^{\rho'}_{\lambda_3}(q_1)\right)^* \left(\varepsilon^{\sigma'}_{\lambda_4}(q_2)\right)^* K_{\mu'\nu'\rho'\sigma'} \\
&= K_{\mu'\nu'\rho'\sigma'} \sum_{\lambda_1}(-1)^{\lambda_1} \left(\varepsilon^{\mu}_{\lambda_1}(q_1)\right)^* \varepsilon^{\mu'}_{\lambda_1}(q_1) \sum_{\lambda_2}(-1)^{\lambda_2} \left(\varepsilon^{\nu}_{\lambda_2}(q_2)\right)^* \varepsilon^{\nu'}_{\lambda_2}(q_2) \\
&\quad \times \sum_{\lambda_3}(-1)^{\lambda_3} \left(\varepsilon^{\rho}_{\lambda_3}(q_1)\right)^* \varepsilon^{\rho'}_{\lambda_3}(q_1) \sum_{\lambda_4}(-1)^{\lambda_4} \left(\varepsilon^{\sigma}_{\lambda_4}(q_2)\right)^* \varepsilon^{\sigma'}_{\lambda_4}(q_2) \\
&= K_{\mu'\nu'\rho'\sigma'} \left(g^{\mu\mu'} - \frac{q_1^{\mu}q_1^{\mu'}}{q_1^2}\right) \left(g^{\nu\nu'} - \frac{q_2^{\nu}q_2^{\nu'}}{q_2^2}\right) \left(g^{\rho\rho'} - \frac{q_1^{\rho}q_1^{\rho'}}{q_1^2}\right) \left(g^{\sigma\sigma'} - \frac{q_2^{\sigma}q_2^{\sigma'}}{q_2^2}\right) \\
&= K_{\mu'\nu'\rho'\sigma'} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} \\
&= K^{\mu\nu\rho\sigma}.
\end{aligned}$$

This shows that the tensors chosen in Eq. (4.26) indeed form a basis for $K^{\mu\nu\rho\sigma}$ with the Ward identity being a crucial part of the argument. Furthermore, we once again see that the three polarization vectors defined in Section 2 are sufficient to describe $K^{\mu\nu\rho\sigma}$.

To reduce the number of terms in the expansion (4.25), we note that the expansion coefficients (4.27) have a number of useful properties. Requiring them to be real, one can show that

$$\mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \mathcal{A}^*_{\lambda_1\lambda_2\lambda_3\lambda_4} = \mathcal{A}_{\lambda_3\lambda_4\lambda_1\lambda_2}.$$

Using $\varepsilon_\lambda^* = (-1)^\lambda \varepsilon_{-\lambda}$, which can be seen from Eqs. (2.2) and (2.3), we find that

$$\mathcal{A}_{-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4} = (-1)^{\lambda_1+\lambda_2+\lambda_3+\lambda_4}\mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

Finally, as a consequence of the conservation of angular momentum, the expansion coefficients satisfy the sum rule [25, Eq. (26)]

$$\mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \delta_{\lambda_1-\lambda_2,\lambda_3-\lambda_4}\mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

This sum rule states that the coefficient $\mathcal{A}_{\lambda_1\lambda_2\lambda_3\lambda_4}$ vanishes if $\lambda_1 - \lambda_2 \neq \lambda_3 - \lambda_4$. Using these properties, the expansion (4.25) simplifies considerably. For example, the terms with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = +1$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -1$ can be combined into a single term

$$\mathcal{A}_{++++}B^{\mu\nu\rho\sigma}_{++++} + \mathcal{A}_{----}B^{\mu\nu\rho\sigma}_{----} = \mathcal{A}_{++++}\left(B^{\mu\nu\rho\sigma}_{++++} + B^{\mu\nu\rho\sigma}_{----}\right),$$

where each $+$ and $-$ corresponds to $\lambda = +1$ and $\lambda = -1$, respectively. Now, if $\lambda = \mathcal{S}\cdot\hat{\mathbf{q}}_1$, then $\mathcal{S}\cdot\hat{\mathbf{q}}_2 = \mathcal{S}\cdot(-\hat{\mathbf{q}}_1) = -\lambda$ in center-of-mass frame of the two photons. In terms of the polarization vectors, this implies that

$$\varepsilon_{\pm 1}(q_1) = \varepsilon_{\mp 1}(q_2). \tag{4.28}$$

Using Eq. (4.28) and $\varepsilon_\lambda^* = (-1)^\lambda \varepsilon_{-\lambda}$, one can then show that

$$B^{\mu\nu\rho\sigma}_{++++} + B^{\mu\nu\rho\sigma}_{----} = \frac{1}{2}\left(P^{\mu\nu}P^{\rho\sigma} - P^{\mu\sigma}P^{\nu\rho} + P^{\mu\rho}P^{\nu\sigma}\right).$$

In a similar fashion, one can go through the remaining terms and eventually obtain Eq. (4.24).

The coefficients a_1 through a_8 in Eq. (4.24) can be projected out using the properties of the polarization tensors, Eqs. (2.17) and (2.23). For example,

$$\begin{aligned}
P^{\mu\nu}P^{\rho\sigma}K_{\mu\nu\rho\sigma} &= a_1 P^{\mu\nu}P_{\mu\nu}P^{\rho\sigma}P_{\rho\sigma} + a_2 P^{\mu\nu}P_{\mu\rho}P^{\rho\sigma}P_{\nu\sigma} + a_3 P^{\mu\nu}P_{\mu\sigma}P^{\rho\sigma}P_{\nu\rho} \\
&= 4a_1 + a_2(-P^{\nu}{}_{\rho})(-P^{\rho}{}_{\nu}) + a_3(-P^{\nu}{}_{\sigma})(-P^{\sigma}{}_{\nu}) \\
&= 4a_1 + a_2 P^{\nu\rho}P_{\rho\nu} + a_3 P_{\nu\sigma}P^{\sigma\nu} \\
&= 4a_1 + 2a_2 + 2a_3
\end{aligned}$$

and

$$P^{\mu\nu}R_1^\rho R_2^\sigma K_{\mu\nu\rho\sigma} = a_4 P^{\mu\nu}P_{\mu\nu}R_1^\rho (R_1)_\rho R_2^\sigma (R_2)_\sigma = 2a_4.$$

Going through the rest in exactly the same way, we obtain a system of equations

$$\begin{cases} P^{\mu\nu}P^{\rho\sigma}K_{\mu\nu\rho\sigma} & = 4a_1 + 2a_2 + 2a_3, \\ P^{\mu\rho}P^{\nu\sigma}K_{\mu\nu\rho\sigma} & = 2a_1 + 4a_2 + 2a_3, \\ P^{\mu\sigma}P^{\nu\rho}K_{\mu\nu\rho\sigma} & = 2a_1 + 2a_2 + 4a_3, \\ P^{\mu\nu}R_1^\rho R_2^\sigma K_{\mu\nu\rho\sigma} & = 2a_4, \\ P^{\mu\rho}R_2^\nu R_2^\sigma K_{\mu\nu\rho\sigma} & = 2a_5, \\ R_1^\mu R_1^\rho P^{\nu\sigma}K_{\mu\nu\rho\sigma} & = 2a_6, \\ P^{\mu\sigma}R_1^\rho R_2^\nu K_{\mu\nu\rho\sigma} & = 2a_7, \\ R_1^\mu R_2^\nu R_1^\rho R_2^\sigma K_{\mu\nu\rho\sigma} & = a_8, \end{cases}$$

which has the solution

$$\begin{cases} a_1 & = \frac{1}{8}\left(3P^{\mu\nu}P^{\rho\sigma} - P^{\mu\rho}P^{\nu\sigma} - P^{\mu\sigma}P^{\nu\rho}\right)K_{\mu\nu\rho\sigma}, \\ a_2 & = \frac{1}{8}\left(3P^{\mu\rho}P^{\nu\sigma} - P^{\mu\sigma}P^{\nu\rho} - P^{\mu\nu}P^{\rho\sigma}\right)K_{\mu\nu\rho\sigma}, \\ a_3 & = \frac{1}{8}\left(3P^{\mu\sigma}P^{\nu\rho} - P^{\mu\nu}P^{\rho\sigma} - P^{\mu\rho}P^{\nu\sigma}\right)K_{\mu\nu\rho\sigma}, \\ a_4 & = \frac{1}{2}P^{\mu\nu}R_1^\rho R_2^\sigma K_{\mu\nu\rho\sigma}, \\ a_5 & = \frac{1}{2}P^{\mu\rho}R_2^\nu R_2^\sigma K_{\mu\nu\rho\sigma}, \\ a_6 & = \frac{1}{2}R_1^\mu R_1^\rho P^{\nu\sigma}K_{\mu\nu\rho\sigma}, \\ a_7 & = \frac{1}{2}P^{\mu\sigma}R_1^\rho R_2^\nu K_{\mu\nu\rho\sigma}, \\ a_8 & = R_1^\mu R_2^\nu R_1^\rho R_2^\sigma K_{\mu\nu\rho\sigma}. \end{cases} \tag{4.29}$$

The tensor $K^{\mu\nu\rho\sigma}$ must be finite in the limit $q_1^2, q_2^2 \to 0$. However, R_i is singular as $q_i^2 \to 0$. Looking at Eq. (4.24), this means that the coefficients a_4, a_5, a_6, a_7, and a_8 must compensate and go to zero fast enough to keep each term finite in the limit $q_1^2, q_2^2 \to 0$. Based on Eq. (3.58), we know that after contracting $K_{\mu\nu\rho\sigma}$ with $W_1^{\mu\rho}$ and $W_2^{\nu\sigma}$, the contractions between R and W_1 and W_2 remain finite. With this in mind, we neglect the terms containing factors of R in Eq. (4.24), which leaves

$$K^{\mu\nu\rho\sigma} \simeq a_1 P^{\mu\nu}P^{\rho\sigma} + a_2 P^{\mu\rho}P^{\nu\sigma} + a_3 P^{\mu\sigma}P^{\nu\rho}. \tag{4.30}$$

Using Eq. (4.30),

$$\begin{aligned} W_1^{\mu\rho} W_2^{\nu\sigma} K_{\mu\nu\rho\sigma} &= a_1 W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma} + a_2 W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\rho} P_{\nu\sigma} \\ &\quad + a_3 W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\sigma} P_{\nu\rho}. \end{aligned} \tag{4.31}$$

Since the polarization tensor P and the emission tensors W_1 and W_2 are symmetric in their indices, we can shuffle the indices of the third term in Eq. (4.31) to obtain

$$W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\sigma} P_{\nu\rho} = W_1^{\mu\rho} W_2^{\sigma\nu} P_{\mu\sigma} P_{\rho\nu} = W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma}, \tag{4.32}$$

where the indices ν and σ were renamed in the last step. Combining Eqs. (4.31) and (4.32),

$$W_1^{\mu\rho} W_2^{\nu\sigma} K_{\mu\nu\rho\sigma} = (a_1 + a_3) W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma} + a_2 W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\rho} P_{\nu\sigma}. \tag{4.33}$$

Using Eq. (4.29), we can calculate that

$$\begin{aligned} a_1 + a_3 &= \frac{1}{8} \left(3P^{\mu\nu} P^{\rho\sigma} - P^{\mu\rho} P^{\nu\sigma} - P^{\mu\sigma} P^{\nu\rho}\right) K_{\mu\nu\rho\sigma} \\ &\quad + \frac{1}{8} \left(3P^{\mu\sigma} P^{\nu\rho} - P^{\mu\nu} P^{\rho\sigma} - P^{\mu\rho} P^{\nu\sigma}\right) K_{\mu\nu\rho\sigma} \\ &= \frac{1}{4} \left(P^{\mu\nu} P^{\rho\sigma} - P^{\mu\rho} P^{\nu\sigma} + P^{\mu\sigma} P^{\nu\rho}\right) K_{\mu\nu\rho\sigma}. \end{aligned} \tag{4.34}$$

We have now parametrized the two-photon absorption tensor $K^{\mu\nu\rho\sigma}$ and begun the derivation of the factorization, starting from Eq. (4.31). However, before we can complete this derivation, we must first rewrite Eq. (4.33).

4.4 Transverse projections

At this stage in the one-photon exchange case, the cross section had already been factored. In Eq. (4.33), however, there are two remaining terms instead of one. It turns out that additional work is needed to complete the factorization in the two-photon exchange case.

We first define the transversely projected momentum vectors

$$\begin{aligned} (p_i^\perp)^\mu &\equiv -P^{\mu\nu}(q_1, q_2)(p_i)_\nu, \\ (q_i^\perp)^\mu &\equiv -P^{\mu\nu}(p_1, p_2)(q_i)_\nu. \end{aligned} \tag{4.35}$$

Here we are using the general feature of the polarization tensor P which states that $P^{\mu\nu}(a,b)$ is a (sign-reversing) projection operator to the subspace orthogonal to any given vectors a and b. Using Eq. (2.17),

$$\begin{aligned} p_i^\perp \cdot p_j^\perp &= (-P^{\mu\nu}(q_1,q_2)(p_i)_\nu)(-P_{\mu\rho}(q_1,q_2)p_j^\rho) = -P^\nu{}_\rho(q_1,q_2)(p_i)_\nu p_j^\rho \\ &= (-P_{\nu\rho}(q_1,q_2)p_i^\nu)p_j^\rho = p_i^\perp \cdot p_j \\ &= p_i^\nu(-P_{\nu\rho}(q_1,q_2)p_j^\rho) = p_i \cdot p_j^\perp \end{aligned} \tag{4.36}$$

and completely analogously $q_i^\perp \cdot q_j^\perp = q_i^\perp \cdot q_j = q_i \cdot q_j^\perp$. Furthermore,

$$\begin{aligned} (p_i'^\perp)^\mu &\equiv -P^{\mu\nu}(p_1,p_2)(p_i')_\nu = -P^{\mu\nu}(p_1,p_2)(p_i-q_i)_\nu = P^{\mu\nu}(p_1,p_2)(q_i)_\nu \\ &= -(q_i^\perp)^\mu. \end{aligned} \tag{4.37}$$

Next, we define

$$\begin{aligned} \cos\phi &\equiv -\frac{p_1^\perp \cdot p_2^\perp}{\sqrt{(p_1^\perp)^2(p_2^\perp)^2}}, \\ \cos\phi' &\equiv -\frac{p_1'^\perp \cdot p_2'^\perp}{\sqrt{(p_1'^\perp)^2(p_2'^\perp)}} = -\frac{q_1^\perp \cdot q_2^\perp}{\sqrt{(q_1^\perp)^2(q_2^\perp)^2}}, \end{aligned} \tag{4.38}$$

where the latter form of $\cos\phi'$ follows from Eq. (4.37). It should be noted that both cosines depend on q_1 and q_2, and thus on p_1' and p_2', through Eq. (4.35). In order to better understand these definitions, consider for example the center-of-mass frame of the particles A and B, so that $\mathbf{p}_1 = -\mathbf{p}_2$. Assume further that these particles move along the z-axis, so that

$$\begin{aligned} p_1 &= (E_1, 0, 0, p_z), \\ p_2 &= (E_2, 0, 0, -p_z), \end{aligned} \tag{4.39}$$

where $E_1 = E_2$ when the two particles A and B have the same mass. In this case, a direct computation shows that

$$P^{\mu\nu}(p_1,p_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now write the outgoing momentum vectors of the particles A ($i = 1$) and B

$(i=2)$ in spherical coordinates as

$$p'_i = (E'_i, r_i \cos\phi_i \sin\theta_i, r_i \sin\phi_i \sin\theta_i, r_i \cos\theta_i), \tag{4.40}$$

where E'_i is the energy, $r_i \equiv |\mathbf{p}'_i|$, θ_i is the scattering angle and ϕ_i is the azimuthal angle. Then

$$\begin{aligned} \cos\phi' &= -\frac{p_1'^{\perp} \cdot p_2'^{\perp}}{\sqrt{(p_1'^{\perp})^2 (p_2'^{\perp})^2}} \\ &= -\left[-r_1 r_2 (\cos\phi_1 \sin\theta_1 \cos\phi_2 \sin\theta_2 + \sin\phi_1 \sin\theta_1 \sin\phi_2 \sin\theta_2)\right] \\ &\quad \times \left[r_1^2 \left(\cos^2\phi_1 \sin^2\theta_1 + \sin^2\phi_1 \sin^2\theta_1\right)\right]^{-1/2} \\ &\quad \times \left[r_2^2 \left(\cos^2\phi_2 \sin^2\theta_2 + \sin^2\phi_2 \sin^2\theta_2\right)\right]^{-1/2} \\ &= \frac{(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) \sin\theta_1 \sin\theta_2}{\sin\theta_1 \sin\theta_2} \\ &= \cos(\phi_1 - \phi_2), \end{aligned} \tag{4.41}$$

since $\sin\theta_i \geq 0$ for $0 \leq \theta_i \leq \pi$. The last line of Eq. (4.41) uses a standard trigonometric identity. Thus we see that in this particular frame, the angle ϕ' given by $\cos\phi'$ in Eq. (4.38) has a simple interpretation of being the azimuthal angle between the particles A and B after the scattering. In complete analogy, ϕ can be interpreted as the azimuthal angle between the particles A and B in the center-of-mass frame of the two photons. In other frames the interpretation is less straightforward, but $\cos\phi$ and $\cos\phi'$ are nevertheless manifestly Lorentz-invariant.

In the limit $q_1^2, q_2^2, M_1^2, M_2^2 \to 0$, we have $\cos\phi \simeq \cos\phi'$ [34]. This can be seen by defining $\Delta^{\mu\nu} \equiv P^{\mu\nu}(p_1, p_2) - P^{\mu\nu}(q_1, q_2)$. Then, since

$$\begin{aligned} P^{\mu\nu}(p_1, p_2) &= -g^{\mu\nu} + \frac{(p_1 \cdot p_2)(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) - \overbrace{p_1^2}^{\to 0} p_2^\mu p_2^\nu - \overbrace{p_2^2}^{\to 0} p_1^\mu p_1^\nu}{(p_1 \cdot p_2)^2 - \underbrace{p_1^2 p_2^2}_{\to 0}} \\ &\simeq -g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu}{p_1 \cdot p_2} \end{aligned}$$

and similarly for $P^{\mu\nu}(q_1, q_2)$, we have

$$\Delta^{\mu\nu} \simeq \frac{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu}{p_1 \cdot p_2} - \frac{q_1^\mu q_2^\nu + q_2^\mu q_1^\nu}{q_1 \cdot q_2}. \tag{4.42}$$

Using Eqs. (4.4) and (4.42) as well as $p_1 \cdot p_1' = p_1 \cdot (p_1 - q_1) = M_1^2 - q_1 \cdot p_1 \simeq 0$,

$$\begin{aligned}\Delta^{\mu\nu}(p_1')_\nu &= \frac{p_1^\mu (p_2 \cdot p_1') + p_2^\mu \overbrace{(p_1 \cdot p_1')}^{\to 0}}{p_1 \cdot p_2} - \frac{q_1^\mu (q_2 \cdot p_1') + q_2^\mu \overbrace{(q_1 \cdot p_1')}^{\to 0}}{q_1 \cdot q_2} \\ &\simeq p_1^\mu \left(1 - \frac{q_1 \cdot p_2}{p_1 \cdot p_2}\right) + q_1^\mu \left(1 - \frac{q_2 \cdot p_1}{q_1 \cdot q_2}\right)\end{aligned} \tag{4.43}$$

and analogously

$$\begin{aligned}\Delta^{\mu\nu}(p_2')_\nu &= \frac{p_1^\mu \overbrace{(p_2 \cdot p_2')}^{\to 0} + p_2^\mu (p_1 \cdot p_2')}{p_1 \cdot p_2} - \frac{q_1^\mu \overbrace{(q_2 \cdot p_2')}^{\to 0} + q_2^\mu (q_1 \cdot p_2')}{q_1 \cdot q_2} \\ &\simeq p_2^\mu \left(1 - \frac{q_2 \cdot p_1}{p_1 \cdot p_2}\right) + q_2^\mu \left(1 - \frac{q_1 \cdot p_2}{q_1 \cdot q_2}\right).\end{aligned} \tag{4.44}$$

Then we obtain

$$\Delta^{\mu\nu}(p_1')_\mu(p_1')_\nu = \overbrace{p_1 \cdot p_1'}^{\to 0} \left(1 - \frac{q_1 \cdot p_2}{p_1 \cdot p_2}\right) + \overbrace{(q_1 \cdot p_1')}^{\to 0} \left(1 - \frac{q_2 \cdot p_1}{q_1 \cdot q_2}\right) \simeq 0 \tag{4.45}$$

and

$$\Delta^{\mu\nu}(p_2')_\mu(p_2')_\nu = \overbrace{p_2 \cdot p_2'}^{\to 0} \left(1 - \frac{q_2 \cdot p_1}{p_1 \cdot p_2}\right) + \overbrace{(q_2 \cdot p_2')}^{\to 0} \left(1 - \frac{q_1 \cdot p_2}{q_1 \cdot q_2}\right) \simeq 0. \tag{4.46}$$

Finally,

$$\begin{aligned}\Delta^{\mu\nu}(p_1')_\mu(p_2')_\nu &= (p_1 \cdot p_2') \left(1 - \frac{q_1 \cdot p_2}{p_1 \cdot p_2}\right) + (q_1 \cdot p_2') \left(1 - \frac{q_2 \cdot p_1}{q_1 \cdot q_2}\right) \\ &= (p_1 \cdot p_2 - p_1 \cdot q_2) \left(1 - \frac{q_1 \cdot p_2}{p_1 \cdot p_2}\right) \\ &\quad + (q_1 \cdot p_2 - q_1 \cdot q_2) \left(1 - \frac{q_2 \cdot p_1}{q_1 \cdot q_2}\right) \\ &= p_1 \cdot p_2 - \cancel{q_1 \cdot p_2} - \cancel{q_2 \cdot p_1} + \frac{(q_1 \cdot p_2)(q_2 \cdot p_1)}{p_1 \cdot p_2} \\ &\quad + \cancel{q_1 \cdot p_2} - \frac{(q_1 \cdot p_2)(q_2 \cdot p_1)}{q_1 \cdot q_2} - q_1 \cdot q_2 + \cancel{q_2 \cdot p_1} \\ &\simeq p_1 \cdot p_2 + q_1 \cdot q_2 - p_1 \cdot p_2 - q_1 \cdot q_2 \\ &= 0,\end{aligned} \tag{4.47}$$

where Eq. (4.8) was used to obtain the second-to-last line. On the other hand, since $q_1^\mu P_{\mu\nu}(q_1, q_2) = q_2^\mu P_{\mu\nu}(q_1, q_2) = 0$ and $p_i = q_i + p_i'$, we have

$$p_i^\perp = -P^{\mu\nu}(q_1, q_2)(p_i)_\nu = -P^{\mu\nu}(q_1, q_2)(q_i + p_i')_\nu = -P^{\mu\nu}(q_1, q_2)(p_i')_\nu. \tag{4.48}$$

Using Eqs. (4.45)–(4.48),

$$\begin{aligned}
\cos\phi &= -\frac{p_1^\perp \cdot p_2^\perp}{\sqrt{(p_1^\perp)^2 (p_2^\perp)^2}} \\
&= -\frac{(-P^{\mu\nu}(q_1, q_2)(p_1')_\mu)(p_2')_\nu}{\sqrt{(-P^{\mu\nu}(q_1, q_2)(p_1')_\mu)(p_1')_\nu (-P^{\rho\sigma}(q_1, q_2)(p_2')_\rho)(p_2')_\sigma}} \\
&= -\frac{(-P^{\mu\nu}(p_1, p_2) + \Delta^{\mu\nu})(p_1')_\mu (p_2')_\nu}{\sqrt{(-P^{\mu\nu}(p_1, p_2) + \Delta^{\mu\nu})(p_1')_\mu (p_1')_\nu (-P^{\rho\sigma}(p_1, p_2) + \Delta^{\rho\sigma})(p_2')_\rho (p_2')_\sigma}} \\
&\simeq -\frac{(-P^{\mu\nu}(p_1, p_2))(p_1')_\mu (p_2')_\nu}{\sqrt{(-P^{\mu\nu}(p_1, p_2))(p_1')_\mu (p_1')_\nu (-P^{\rho\sigma}(p_1, p_2))(p_2')_\rho (p_2')_\sigma}} \\
&= -\frac{p_1'^\perp \cdot p_2'^\perp}{\sqrt{(p_1'^\perp)^2 (p_2'^\perp)^2}} = \cos\phi'.
\end{aligned} \tag{4.49}$$

4.5 Factorization of the cross section

Returning back to Eq. (4.33), we can now calculate the two contractions between P, W_1 and W_2 with the help of Eqs. (2.17), (4.36), and (4.38). Using $q_i^\mu P_{\mu\nu} = 0$,

$$\begin{aligned}
W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma} &= (\alpha_1^{(1)} p_1^\mu p_1^\rho + \alpha_4^{(1)} g^{\mu\rho})(\alpha_1^{(2)} p_2^\nu p_2^\sigma + \alpha_4^{(2)} g^{\nu\sigma}) P_{\mu\nu} P_{\rho\sigma} \\
&= \alpha_1^{(1)} \alpha_1^{(2)} p_1^\mu p_1^\rho p_2^\nu p_2^\sigma P_{\mu\nu} P_{\rho\sigma} + \alpha_1^{(1)} \alpha_4^{(2)} p_1^\mu p_1^\rho g^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma} \\
&\quad + \alpha_4^{(1)} \alpha_1^{(2)} g^{\mu\rho} p_2^\nu p_2^\sigma P_{\mu\nu} P_{\rho\sigma} + \alpha_4^{(1)} \alpha_4^{(2)} g^{\mu\rho} g^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma} \\
&= \alpha_1^{(1)} \alpha_1^{(2)} (-P_{\mu\nu} p_1^\mu p_2^\nu)(-P_{\rho\sigma} p_1^\rho p_2^\sigma) + \alpha_1^{(1)} \alpha_4^{(2)} p_1^\mu p_1^\rho P_{\mu\nu} P^\nu{}_\sigma \\
&\quad + \alpha_4^{(1)} \alpha_1^{(2)} p_2^\nu p_2^\sigma P_{\mu\nu} P^\mu{}_\sigma + \alpha_4^{(1)} \alpha_4^{(2)} P_{\mu\nu} P^{\mu\nu} \\
&= \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp \cdot p_2^\perp)^2 + \alpha_1^{(1)} \alpha_4^{(2)} (-P_{\mu\sigma} p_1^\mu p_1^\rho) \\
&\quad + \alpha_4^{(1)} \alpha_1^{(2)} (-P_{\nu\sigma} p_2^\nu p_2^\sigma) + 2\alpha_4^{(1)} \alpha_4^{(2)} \\
&= \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos^2\phi + \alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 \\
&\quad + \alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 2\alpha_4^{(1)} \alpha_4^{(2)},
\end{aligned} \tag{4.50}$$

where $W_i^{\mu\nu} = \alpha_1^{(i)} p_i^\mu p_i^\nu + \alpha_2^{(i)} (p_i^\mu q_i^\nu + q_i^\mu p_i^\nu) + \alpha_3^{(i)} q_i^\mu q_i^\nu + \alpha_4^{(i)} g^{\mu\nu}$ by Eq. (3.54).

Similarly for the second contraction,

$$
\begin{aligned}
W_1^{\mu\nu} W_2^{\rho\sigma} P_{\mu\nu} P_{\rho\sigma} &= (\alpha_1^{(1)} p_1^\mu p_1^\nu + \alpha_4^{(1)} g^{\mu\nu})(\alpha_1^{(2)} p_2^\rho p_2^\sigma + \alpha_4^{(2)} g^{\rho\sigma}) P_{\mu\nu} P_{\rho\sigma} \\
&= \alpha_1^{(1)} \alpha_1^{(2)} p_1^\mu p_1^\nu p_2^\rho p_2^\sigma P_{\mu\nu} P_{\rho\sigma} + \alpha_1^{(1)} \alpha_4^{(2)} p_1^\mu p_1^\nu g^{\rho\sigma} P_{\mu\nu} P_{\rho\sigma} \\
&\quad + \alpha_4^{(1)} \alpha_1^{(2)} g^{\mu\nu} p_2^\rho p_2^\sigma P_{\mu\nu} P_{\rho\sigma} + \alpha_4^{(1)} \alpha_4^{(2)} g^{\mu\nu} g^{\rho\sigma} P_{\mu\nu} P_{\rho\sigma} \\
&= \alpha_1^{(1)} \alpha_1^{(2)} (-P_{\mu\nu} p_1^\mu p_1^\nu)(-P_{\rho\sigma} p_2^\rho p_2^\sigma) + 2\alpha_1^{(1)} \alpha_4^{(2)} (-P_{\mu\nu} p_1^\mu p_1^\nu) \\
&\quad + 2\alpha_4^{(1)} \alpha_1^{(2)} (-P_{\rho\sigma} p_2^\rho p_2^\sigma) + 4\alpha_4^{(1)} \alpha_4^{(2)} \\
&= \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 + 2\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 \\
&\quad + 2\alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 4\alpha_4^{(1)} \alpha_4^{(2)} .
\end{aligned} \tag{4.51}
$$

Defining $\mathcal{C} \equiv W_1^{\mu\rho} W_2^{\nu\sigma} K_{\mu\nu\rho\sigma}$ and plugging Eqs. (4.29), (4.34), (4.50), and (4.51) back into Eq. (4.33),

$$
\begin{aligned}
\mathcal{C} &= W_1^{\mu\rho} W_2^{\nu\sigma} K_{\mu\nu\rho\sigma} = (a_1 + a_3) W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma} + a_2 W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\rho} P_{\nu\sigma} \\
&= \frac{1}{4} \left(P^{\mu'\nu'} P^{\rho'\sigma'} - P^{\mu'\rho'} P^{\nu'\sigma'} + P^{\mu'\sigma'} P^{\nu'\rho'} \right) K_{\mu'\nu'\rho'\sigma'} W_1^{\mu\rho} W_2^{\nu\sigma} P_{\mu\nu} P_{\rho\sigma} \\
&\quad + \frac{1}{8} \left(3P^{\mu'\rho'} P^{\nu'\sigma'} - P^{\mu'\sigma'} P^{\nu'\rho'} - P^{\mu'\nu'} P^{\rho'\sigma'} \right) K_{\mu'\nu'\rho'\sigma'} W_1^{\mu\nu} W_2^{\rho\sigma} P_{\mu\nu} P_{\rho\sigma} \\
&= \frac{1}{4} \left(P^{\mu'\nu'} P^{\rho'\sigma'} - P^{\mu'\rho'} P^{\nu'\sigma'} + P^{\mu'\sigma'} P^{\nu'\rho'} \right) K_{\mu'\nu'\rho'\sigma'} \\
&\qquad \times \left(\alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos^2\phi + \alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + \alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 2\alpha_4^{(1)} \alpha_4^{(2)} \right) \\
&\quad + \frac{1}{8} \left(3P^{\mu'\rho'} P^{\nu'\sigma'} - P^{\mu'\sigma'} P^{\nu'\rho'} - P^{\mu'\nu'} P^{\rho'\sigma'} \right) K_{\mu'\nu'\rho'\sigma'} \\
&\qquad \times \left(\alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 + 2\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + 2\alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 4\alpha_4^{(1)} \alpha_4^{(2)} \right) .
\end{aligned}
$$

Expanding and combining like terms,

$$
\begin{aligned}
\mathcal{C} = &\frac{1}{4} \left(P^{\mu'\nu'} P^{\rho'\sigma'} - P^{\mu'\rho'} P^{\nu'\sigma'} + P^{\mu'\sigma'} P^{\nu'\rho'} \right) K_{\mu'\nu'\rho'\sigma'} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos^2\phi \\
&+ \frac{1}{4} \left(\cancel{P^{\mu'\nu'} P^{\rho'\sigma'}} - P^{\mu'\rho'} P^{\nu'\sigma'} + \cancel{P^{\mu'\sigma'} P^{\nu'\rho'}} \right) K_{\mu'\nu'\rho'\sigma'} \\
&\quad \times \left(\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + \alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 2\alpha_4^{(1)} \alpha_4^{(2)} \right) \\
&+ \frac{1}{8} \left(3P^{\mu'\rho'} P^{\nu'\sigma'} - P^{\mu'\sigma'} P^{\nu'\rho'} - P^{\mu'\nu'} P^{\rho'\sigma'} \right) K_{\mu'\nu'\rho'\sigma'} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \\
&+ \frac{1}{4} \left(3P^{\mu'\rho'} P^{\nu'\sigma'} - \cancel{P^{\mu'\sigma'} P^{\nu'\rho'}} - \cancel{P^{\mu'\nu'} P^{\rho'\sigma'}} \right) K_{\mu'\nu'\rho'\sigma'} \\
&\quad \times \left(\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + \alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 2\alpha_4^{(1)} \alpha_4^{(2)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{8} P^{\mu\nu} P^{\rho\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos^2\phi \\
&\quad - \frac{2}{8} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos^2\phi \\
&\quad + \frac{2}{8} P^{\mu\sigma} P^{\nu\rho} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos^2\phi \\
&\quad + \frac{1}{2} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \left(\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + \alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 2\alpha_4^{(1)} \alpha_4^{(2)} \right) \\
&\quad + \frac{3}{8} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 - \frac{1}{8} P^{\mu\sigma} P^{\nu\rho} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \\
&\quad - \frac{1}{8} P^{\mu\nu} P^{\rho\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 .
\end{aligned}
$$

Using the trigonometric identity $\cos(2\phi) = 2\cos^2\phi - 1$, which also implies that $3 - 2\cos^2\phi = 2 - \cos(2\phi)$,

$$
\begin{aligned}
\mathcal{C} &= \frac{1}{8} P^{\mu\nu} P^{\rho\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \left(2\cos^2\phi - 1 \right) \\
&\quad + \frac{1}{8} P^{\mu\sigma} P^{\nu\rho} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \left(2\cos^2\phi - 1 \right) \\
&\quad + \frac{1}{8} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \left(3 - 2\cos^2\phi \right) \\
&\quad + \frac{1}{2} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \left(\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + \alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 2\alpha_4^{(1)} \alpha_4^{(2)} \right) \\
&= \frac{1}{8} \left(P^{\mu\nu} P^{\rho\sigma} + P^{\mu\sigma} P^{\nu\rho} \right) K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos(2\phi) \\
&\quad + \frac{2}{8} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \\
&\quad - \frac{1}{8} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos(2\phi) \\
&\quad + \frac{1}{2} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \left(\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + \alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 2\alpha_4^{(1)} \alpha_4^{(2)} \right) \\
&= \frac{1}{8} \left(P^{\mu\nu} P^{\rho\sigma} - P^{\mu\rho} P^{\nu\sigma} + P^{\mu\sigma} P^{\nu\rho} \right) K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos(2\phi) \\
&\quad + \frac{1}{4} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} \\
&\qquad \times \left(\alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 + 2\alpha_1^{(1)} \alpha_4^{(2)} (p_1^\perp)^2 + 2\alpha_4^{(1)} \alpha_1^{(2)} (p_2^\perp)^2 + 4\alpha_4^{(1)} \alpha_4^{(2)} \right) \\
&= \frac{1}{8} \left(P^{\mu\nu} P^{\rho\sigma} - P^{\mu\rho} P^{\nu\sigma} + P^{\mu\sigma} P^{\nu\rho} \right) K_{\mu\nu\rho\sigma} \alpha_1^{(1)} \alpha_1^{(2)} (p_1^\perp)^2 (p_2^\perp)^2 \cos(2\phi) \\
&\quad + \frac{1}{4} P^{\mu\rho} P^{\nu\sigma} K_{\mu\nu\rho\sigma} W_1^{\alpha\beta} W_2^{\gamma\delta} P_{\alpha\beta} P_{\gamma\delta},
\end{aligned}
\tag{4.52}
$$

where Eq. (4.51) was used to get to the last line. We now define

$$
T_i \equiv W_i^{\mu\nu} P_{\mu\nu} \tag{4.53}
$$

in complete analogy with Eq. (3.50), and

$$\xi \equiv \frac{1}{8}\left(P^{\mu\nu}P^{\rho\sigma} - P^{\mu\rho}P^{\nu\sigma} + P^{\mu\sigma}P^{\nu\rho}\right) K_{\mu\nu\rho\sigma}\alpha_1^{(1)}\alpha_1^{(2)}(p_1^{\perp})^2(p_2^{\perp})^2. \tag{4.54}$$

Since $P^{\mu\nu}$ and $K_{\mu\nu\rho\sigma}$ depend only on q_1 and q_2, ξ depends only on $q_1, q_2, (p_1^{\perp})^2$, and $(p_2^{\perp})^2$. With Eqs. (4.22), (4.53), and (4.54), we can rewrite Eq. (4.52) as

$$\mathcal{C} = W_1^{\mu\rho}W_2^{\nu\sigma}K_{\mu\nu\rho\sigma} = 4\sqrt{(q_1\cdot q_2)^2 - q_1^2q_2^2}\,\hat{\sigma}(\gamma\gamma\to X)T_1T_2 + \xi\cos(2\phi). \tag{4.55}$$

In Eq. (4.55), all dependence on the angle ϕ is shown explicitly. This is the case because $\hat{\sigma}(\gamma\gamma \to X)$ depends only on q_1 and q_2, T_1 depends only on q_1, q_2, and p_1, T_2 depends only on q_1, q_2, and p_2, and ξ depends only on $q_1, q_2, (p_1^{\perp})^2$, and $(p_2^{\perp})^2$. In particular, none of these terms depend on $p_1^{\perp}\cdot p_2^{\perp}$ and therefore do not contain any dependence on ϕ.

As in the one-photon exchange case, we write the phase space volume element $d^3\mathbf{p}'_i/(2\pi)^3 2E'_i$ in terms of y_i and Q_i^2. Rewriting $\phi_2 = \phi_1 - \phi'$, we have from Eq. (4.40) that

$$p'_1 = (E'_1, r_1\cos\phi_1\sin\theta_1, r_1\sin\phi_1\sin\theta_1, r_1\cos\theta_1) \tag{4.56}$$

and

$$p'_2 = (E'_2, r_2\cos(\phi_1-\phi')\sin\theta_2, r_2\sin(\phi_1-\phi')\sin\theta_2, r_2\cos\theta_2). \tag{4.57}$$

Changing variables first from $\mathbf{p}'_1$ and $\mathbf{p}'_2$ to $r_1, r_2, \theta_1, \theta_2, \phi_1$, and ϕ', the Jacobian reads

$$\begin{aligned} J_1 &\equiv \frac{\partial(\mathbf{p}'_1, \mathbf{p}'_2)}{\partial(r_1, r_2, \theta_1, \theta_2, \phi_1, \phi')} \\ &= \begin{vmatrix} \mathrm{c}\,\phi_1\,\mathrm{s}\,\theta_1 & 0 & r_1\,\mathrm{c}\,\phi_1\,\mathrm{c}\,\theta_1 & 0 & -r_1\,\mathrm{s}\,\phi_1\,\mathrm{s}\,\theta_1 & 0 \\ \mathrm{s}\,\phi_1\,\mathrm{s}\,\theta_1 & 0 & r_1\,\mathrm{s}\,\phi_1\,\mathrm{c}\,\theta_1 & 0 & r_1\,\mathrm{c}\,\phi_1\,\mathrm{s}\,\theta_1 & 0 \\ \mathrm{c}\,\theta_1 & 0 & -r_1\,\mathrm{s}\,\theta_1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\,\tilde{\phi}_2\,\mathrm{s}\,\theta_2 & 0 & r_2\,\mathrm{c}\,\tilde{\phi}_2\,\mathrm{c}\,\theta_2 & -r_2\,\mathrm{s}\,\tilde{\phi}_2\,\mathrm{s}\,\theta_2 & r_2\,\mathrm{s}\,\tilde{\phi}_2\,\mathrm{s}\,\theta_2 \\ 0 & \mathrm{s}\,\tilde{\phi}_2\,\mathrm{s}\,\theta_2 & 0 & r_2\,\mathrm{s}\,\tilde{\phi}_2\,\mathrm{c}\,\theta_2 & r_2\,\mathrm{c}\,\tilde{\phi}_2\,\mathrm{s}\,\theta_2 & -r_2\,\mathrm{c}\,\tilde{\phi}_2\,\mathrm{s}\,\theta_2 \\ 0 & \mathrm{c}\,\theta_2 & 0 & -r_2\,\mathrm{s}\,\theta_2 & 0 & 0 \end{vmatrix} \\ &= r_1^2 r_2^2 \sin\theta_1\sin\theta_2, \end{aligned} \tag{4.58}$$

where we temporarily introduced the notations $\tilde{\phi}_2 \equiv \phi_1 - \phi'$, $\mathrm{s} \equiv \sin$, and $\mathrm{c} \equiv \cos$. We see that the Jacobian (4.58) is the product of two Jacobians associated with spherical coordinates, $r^2 \sin\theta$, even though ϕ_1 appears in both $\mathbf{p}_1'$ and $\mathbf{p}_2'$. Thus,

$$\begin{aligned} \mathrm{d}^3\mathbf{p}_1' \, \mathrm{d}^3\mathbf{p}_2' &= |J_1| \, \mathrm{d}r_1 \, \mathrm{d}r_2 \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \mathrm{d}\phi_1 \, \mathrm{d}\phi' \\ &= r_1^2 r_2^2 \, \mathrm{d}r_1 \, \mathrm{d}r_2 \, \mathrm{d}(\cos\theta_1) \, \mathrm{d}(\cos\theta_2) \, \mathrm{d}\phi_1 \, \mathrm{d}\phi'. \end{aligned} \tag{4.59}$$

Using Eqs. (4.39), (4.56), and (4.57), we can write

$$\begin{aligned} q_1 &= (E_1 - E_1', -r_1 \cos\phi_1 \sin\theta_1, -r_1 \sin\phi_1 \sin\theta_1, p_z - r_1 \cos\theta_1), \\ q_2 &= (E_2 - E_2', -r_2 \cos(\phi_1 - \phi') \sin\theta_2, -r_2 \sin(\phi_1 - \phi') \sin\theta_2, -p_z - r_2 \cos\theta_2). \end{aligned} \tag{4.60}$$

Then, using Eqs. (4.7), (4.39), and (4.60),

$$y_1 = \frac{q_1 \cdot q_2}{p_1 \cdot q_2} \simeq \frac{q_1 \cdot p_2}{p_1 \cdot p_2} = \frac{(E_1 - E_1')E_2 + p_z(p_z - r_1 \cos\theta_1)}{E_1 E_2 + p_z^2}$$

and

$$y_2 = \frac{q_1 \cdot q_2}{p_2 \cdot q_1} \simeq \frac{q_2 \cdot p_1}{p_1 \cdot p_2} = \frac{(E_2 - E_2')E_1 + p_z(p_z + r_2 \cos\theta_2)}{E_1 E_2 + p_z^2},$$

where $E_i' = \sqrt{M_i^2 + r_i^2}$ and $M_i^2 = (p_i')^2$. Similarly,

$$Q_1^2 = -q_1^2 = -(p_1 - p_1')^2 = -2M_1^2 + 2p_1 \cdot p_1' = -2M_1^2 + 2E_1 E_1' - 2p_z r_1 \cos\theta_1$$

and

$$Q_2^2 = -q_2^2 = -(p_2 - p_2')^2 = -2M_2^2 + 2p_2 \cdot p_2' = -2M_2^2 + 2E_2 E_2' + 2p_z r_2 \cos\theta_2.$$

Then it is straightforward to calculate that

$$J_2 \equiv \frac{\partial(y_1, y_2, Q_1^2, Q_2^2)}{\partial(r_1, r_2, \cos\theta_1, \cos\theta_2)} = -\frac{4(E_1 + E_2)^2 p_z^2 r_1^2 r_2^2}{E_1' E_2' (E_1 E_2 + p_z^2)^2} \tag{4.61}$$

and so

$$\mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}Q_1^2 \, \mathrm{d}Q_2^2 = |J_2| \, \mathrm{d}r_1 \, \mathrm{d}r_2 \, \mathrm{d}(\cos\theta_1) \, \mathrm{d}(\cos\theta_2). \tag{4.62}$$

Once again using Eqs. (4.5) and (4.7),

$$\begin{aligned} \frac{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{(q_1 \cdot q_2)^2 - q_1^2 q_2^2} &\simeq \frac{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{(q_1 \cdot q_2)^2} = \frac{(p_1 \cdot p_2)^2}{(p_1 \cdot p_2)^2} \frac{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{(q_1 \cdot q_2)^2} \\ &= \frac{1}{y_1^2 y_2^2} \frac{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{(p_1 \cdot p_2)^2}, \end{aligned} \tag{4.63}$$

where

$$\begin{aligned}\frac{(p_1\cdot p_2)^2-p_1^2p_2^2}{(p_1\cdot p_2)^2}&=\frac{(E_1E_2+p_z^2)^2-(E_1^2-p_z^2)(E_2^2-p_z^2)^2}{(E_1E_2+p_z^2)^2}\\&=\frac{(E_1+E_2)^2p_z^2}{(E_1E_2+p_z^2)^2}.\end{aligned}\tag{4.64}$$

Neglecting the mass M_i, we have $E_i^2=M_i^2+p_z^2\simeq p_z^2$ and therefore

$$\frac{(E_1E_2+p_z^2)^3}{(E_1+E_2)^3p_z^3}\simeq\frac{8p_z^6}{8p_z^6}=1.\tag{4.65}$$

Combining Eqs. (4.58), (4.59), and (4.61)–(4.65),

$$\begin{aligned}\frac{\mathrm{d}^3\mathbf{p}_1'}{2E_1'}\frac{\mathrm{d}^3\mathbf{p}_2'}{2E_2'}&=\frac{r_1^2r_2^2\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}(\cos\theta_1)\,\mathrm{d}(\cos\theta_2)\,\mathrm{d}\phi_1\,\mathrm{d}\phi'}{4E_1'E_2'}\\&\simeq\frac{r_1^2r_2^2}{4E_1'E_2'}\frac{E_1'E_2'(E_1E_2+p_z^2)^2}{4(E_1+E_2)^2p_z^2r_1^2r_2^2}\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}Q_1^2\,\mathrm{d}Q_2^2\,\mathrm{d}\phi_1\,\mathrm{d}\phi'\\&=\frac{(E_1E_2+p_z^2)^2}{16(E_1+E_2)^2p_z^2}\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}Q_1^2\,\mathrm{d}Q_2^2\,\mathrm{d}\phi_1\,\mathrm{d}\phi'\\&\simeq\sqrt{\frac{(p_1\cdot p_2)^2-p_1^2p_2^2}{(q_1\cdot q_2)^2-q_1^2q_2^2}}\frac{(E_1E_2+p_z^2)^3}{16(E_1+E_2)^3p_z^3}y_1y_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}Q_1^2\,\mathrm{d}Q_2^2\,\mathrm{d}\phi_1\,\mathrm{d}\phi'\\&\simeq\frac{1}{16}\sqrt{\frac{(p_1\cdot p_2)^2-p_1^2p_2^2}{(q_1\cdot q_2)^2-q_1^2q_2^2}}\,y_1y_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}Q_1^2\,\mathrm{d}Q_2^2\,\mathrm{d}\phi_1\,\mathrm{d}\phi'.\end{aligned}\tag{4.66}$$

Plugging Eqs. (4.55) and (4.66) into Eq. (4.16),

$$\begin{aligned}\mathrm{d}\sigma(AB\to ABX)&\simeq\frac{Z_1^2Z_2^2e^8}{q_1^4q_2^4}\frac{W_1^{\mu\rho}W_2^{\nu\sigma}K_{\mu\nu\rho\sigma}}{4\sqrt{(p_1\cdot p_2)^2-p_1^2p_2^2}}\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^32E_1'}\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^32E_2'}\\&=\frac{Z_1^2Z_2^2e^4}{q_1^4q_2^4}\frac{4\sqrt{(q_1\cdot q_2)^2-q_1^2q_2^2}\hat{\sigma}(\gamma\gamma\to X)T_1T_2+\xi\cos(2\phi)}{4\sqrt{(q_1\cdot q_2)^2-q_1^2q_2^2}}\\&\quad\times\sqrt{\frac{(q_1\cdot q_2)^2-q_1^2q_2^2}{(p_1\cdot p_2)^2-p_1^2p_2^2}}\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^32E_1'}\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^32E_2'}\\&\simeq\frac{Z_1^2Z_2^2e^4}{q_1^4q_2^4}\frac{4\sqrt{(q_1\cdot q_2)^2-q_1^2q_2^2}\hat{\sigma}(\gamma\gamma\to X)T_1T_2+\xi\cos(2\phi)}{64(2\pi)^6\sqrt{(q_1\cdot q_2)^2-q_1^2q_2^2}}\\&\quad\times y_1y_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}Q_1^2\,\mathrm{d}Q_2^2\,\mathrm{d}\phi_1\,\mathrm{d}\phi'.\end{aligned}\tag{4.67}$$

The integral over the azimuthal angle ϕ_1 can be done trivially. Using Eq. (4.49), we have $\cos(2\phi)\simeq\cos(2\phi')$. Thus, when integrating over ϕ', the term $\xi\cos(2\phi')$ vanishes and we are left with

$$
\begin{aligned}
\mathrm{d}\sigma(AB \to ABX) &= \frac{Z_1^2 Z_2^2}{q_1^4 q_2^4} \frac{e^4 T_1 T_2}{16(2\pi)^4} \hat{\sigma}(\gamma\gamma \to X) y_1 y_2 \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}Q_1^2 \, \mathrm{d}Q_2^2 \\
&= \left[\frac{\alpha}{4\pi} \frac{y_1}{Q_1^4} Z_1^2 T_1 \right] \left[\frac{\alpha}{4\pi} \frac{y_2}{Q_2^4} Z_2^2 T_2 \right] \hat{\sigma}(\gamma\gamma \to X) \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}Q_1^2 \, \mathrm{d}Q_2^2 \\
&= f_{\gamma/A}(y_1, Q_1^2) f_{\gamma/B}(y_2, Q_2^2) \hat{\sigma}(\gamma\gamma \to X) \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}Q_1^2 \, \mathrm{d}Q_2^2,
\end{aligned}
\tag{4.68}
$$

where on the last line we identified the equivalent photon distributions based on Eq. (3.66). Equation (4.68) is an important result, as it establishes the factorization claimed in Eq. (4.1) for two-photon exchange processes when Q_1^2, Q_2^2, M_1^2, and M_2^2 are assumed to be small.

5 Emission tensor structure

This section is concerned with the structure of the emission tensor $W_{\mu\nu}$ defined by the $B \to \gamma B$ vertex. For an elementary particle, the Feynman rules completely determine $W_{\mu\nu}$. For particles with internal structure, the emission tensor must be parametrized with some form factors that are ultimately determined from experimental data. However, Lorentz structure and the Ward identity place significant restrictions on the possible forms of the vertex and, by extension, the emission tensor.

5.1 Scalar particle

When the particle emitting the photon is a spin-0 scalar particle, we can use scalar quantum electrodynamics (scalar QED) to determine the form of the emission tensor $W_{\mu\nu}$. In scalar QED, the spin-$\frac{1}{2}$ Dirac particles found in standard QED are replaced with spin-0 scalar particles.

If the scalar particle is elementary, the vertex is given by the Feynman rule $-ie(p + p')^\mu$ (see Appendix B.2). In the case of a scalar particle with internal structure, we know that the vertex can only depend on the momenta p, p' and q. Only two of these are independent, and here we choose these to be $p + p'$ and $q = p - p'$. In keeping with the Lorentz structure of the elementary vertex, we can write the most general vertex as

$$H^\mu = F(Q^2)(p + p')^\mu + G(Q^2)q^\mu,$$

where F and G are some scalar functions depending on $Q^2 = -q^2$. The Ward identity $q_\mu H^\mu = 0$ further constrains the form of H^μ. Since the emitting particle stays intact, $q \cdot (p + p') = 0$ holds by Eqs. (3.3) and (3.4), and therefore

$$q_\mu H^\mu = Fq \cdot (p + p') + Gq^2 = Gq^2 = 0$$

holds in general only if $G = 0$. We are then left with

$$H^\mu = F(Q^2)(p + p')^\mu = F(Q^2)(2p - q)^\mu.$$

Squaring this yields the emission tensor

$$W_{\mu\nu} = H_\mu H_\nu^* = F^2(Q^2)(2p - q)_\mu (2p - q)_\nu. \tag{5.1}$$

No summing or averaging is necessary here, since the particle is a scalar and therefore has no spin or polarization structure. The function $F(Q^2)$ contains all the information about the structure of the particle. When $F = 1$, we recover the elementary-particle case.

Writing Eq. (5.1) in the form of Eq. (3.54),

$$\begin{aligned} W_{\mu\nu} &= F^2(Q^2)(2p - q)_\mu (2p - q)_\nu = F^2(Q^2)\left[4p_\mu p_\nu - 2(p_\mu q_\nu + p_\nu q_\mu) + q_\mu q_\nu\right] \\ &\overset{!}{=} \alpha_1 p_\mu p_\nu + \alpha_2 (p_\mu q_\nu + q_\mu p_\nu) + \alpha_3 q_\mu q_\nu + \alpha_4 g_{\mu\nu}. \end{aligned}$$

Thus we see that $\alpha_1 = 4F^2(Q^2), \alpha_2 = -2F^2(Q^2), \alpha_3 = F^2(Q^2)$, and $\alpha_4 = 0$. Then the equivalent photon distribution (3.70) reads

$$f_{\gamma/B}(y, Q^2) = \frac{\alpha}{\pi} \frac{y}{Q^2} Z_B^2 F^2(Q^2) \left(\frac{1 - y}{y^2} - \frac{M^2}{Q^2} \right). \tag{5.2}$$

Equation (5.2) gives the general form of the photon distribution for scalar particles. It can, for example, be used in the study of collisions involving lead nuclei, since ^{208}Pb is a spin-0 particle [39].

5.2 Dirac particle

When the emitting particle is a spin-$\frac{1}{2}$ Dirac particle, like the proton, the vertex structure is more complicated compared to the spin-0 case. The vertex rule for an elementary particle is $-ie\gamma^\mu$. Generalizing this to a particle with internal structure, we can write the vertex as $-ie\Gamma^\mu$ where the vertex factor Γ^μ is some 4×4 matrix that contains all the information about the internal structure of the particle. We can therefore write the most general vertex factor Γ^μ as a linear combination of some basis matrices.

A particularly useful basis for 4×4 matrices is given by [3, §3.4]

$$\{\mathbf{1}_4, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5, \gamma^\mu\gamma^5\}. \tag{5.3}$$

The usefulness of this covariant basis stems from their transformation properties under the Lorentz group when combined with a Dirac field ψ in the form $\overline{\psi}\Gamma\psi$. For example, $\overline{\psi}\mathbf{1}_4\psi$ transforms like a scalar while $\overline{\psi}\gamma^\mu\psi$ transforms like a vector. The transformation properties of each basis matrix is listed in Table 1.

Γ	Transformation of $\overline{\psi}\Gamma\psi$
$\mathbf{1}_4$	scalar
γ^μ	vector
$\sigma^{\mu\nu}$	rank-2 tensor
γ^5	pseudoscalar
$\gamma^\mu\gamma^5$	pseudovector

Table 1. Transformation properties of the covariant basis matrices under the Lorentz group [3, §3.4].

To reduce the number of independent terms, we can use the Dirac equation and the Ward identity $q_\mu\Gamma^\mu = 0$. The approach taken here follows Ref. [40, §10.6]. Since the elementary vertex $-ie\gamma^\mu$ transforms like a vector in the form $\overline{\psi}(-ie\gamma^\mu)\psi$, we require $\overline{\psi}(-ie\Gamma^\mu)\psi$ to also transform like a vector. This means that we cannot write Γ^μ directly as a linear combination of the matrices in Eq. (5.3) with scalar coefficients. Instead, these coefficients must have some Lorentz structure to them, which will be provided by the momentum vectors. The only momenta available at the vertex are p and p', or equivalently $q = p - p'$ and $r \equiv p + p'$.

Starting from the identity matrix $\mathbf{1}_4$, we can form two vectors q^μ and r^μ. Moving on to γ^μ, we have the following new combinations:

$$\gamma^\mu, q^\mu \not{q}, r^\mu \not{q}, q^\mu \not{r}, r^\mu \not{r}.$$

Since the vertex factor Γ^μ will always appear in the form $\overline{u}(p')\Gamma^\mu u(p)$, we can use the momentum-space Dirac equations (A.5) and (A.7) to show that

$$\overline{u}(p')\not{q}u(p) = \overline{u}(p')\not{p}u(p) - \overline{u}(p')\not{p}'u(p) = M\overline{u}(p')u(p) - M\overline{u}(p')u(p) = 0$$

and

$$\overline{u}(p')\not{r}u(p) = \overline{u}(p')\not{p}u(p) + \overline{u}(p')\not{p}'u(p) = \overline{u}(p')2Mu(p).$$

With these, we see that

$$\overline{u}(p')\left(Aq^\mu\not{q} + Br^\mu\not{q} + Cq^\mu\not{r} + Dr^\mu\not{r}\right)u(p) = \overline{u}(p')\left(2CMq^\mu + 2DMr^\mu\right)u(p),$$

which can simply be absorbed to the two terms proportional to q^μ and r^μ coming from $\mathbf{1}_4$. Thus, the only new independent term is γ^μ.

Next, we will consider $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, for which we have four new independent combinations:

$$\sigma^{\mu\nu}q_\nu, \sigma^{\mu\nu}r_\nu, \sigma^{\alpha\beta}q_\alpha r_\beta q^\mu, \sigma^{\alpha\beta}q_\alpha r_\beta r^\mu. \tag{5.4}$$

Since for any vector a we have by the Dirac algebra that

$$[\gamma^\mu, \gamma^\nu]a_\nu = [\gamma^\mu, \not{a}] = 2\gamma^\mu\not{a} - \{\gamma^\mu, \not{a}\} = 2\gamma^\mu\not{a} - 2a^\mu,$$

we see that the first two terms in Eq. (5.4) once again reduce to terms already obtained from $\mathbf{1}_4$ and γ^μ. Similarly,

$$[\gamma^\mu, \gamma^\nu]q^\mu r^\nu = [\not{q}, \not{r}] = 2\not{q}\not{r} - \{\not{q}, \not{r}\} = 2\not{q}\not{r} - 2q\cdot r,$$

and since

$$\begin{aligned}
\overline{u}(p')\not{q}\not{r}u(p) &= \overline{u}(p')(\not{p} - \not{p}')(\not{p} + \not{p}')u(p) = \overline{u}(p')(\not{p} - M)(M + \not{p}')u(p)\\
&= \overline{u}(p')\not{p}\not{p}'u(p) + \overline{u}(p')M\not{p}u(p) - \overline{u}(p')\not{p}'Mu(p) - \overline{u}(p')M^2u(p)\\
&= \overline{u}(p')\left(\{\not{p}, \not{p}'\} - \not{p}'\not{p}\right)u(p) - \overline{u}(p')M^2u(p)\\
&= \overline{u}(p')\left(2p\cdot p' - 2M^2\right)u(p),
\end{aligned}$$

we see again that the contributions from the last two terms in Eq. (5.4) are not independent, but can be absorbed to previously obtained terms.

According to Table 1, the remaining basis matrices γ^5 and $\gamma^\mu\gamma^5$ transform like a pseudoscalar and a pseudovector, respectively. This means that they change signs under a parity transformation. Since quantum electrodynamics is a parity-conserving theory, we can rule out any combinations γ^5 and $\gamma^\mu\gamma^5$; see Ref. [41] or Ref. [40, §10.6] for more details.

We now have only three remaining independent terms, so

$$\Gamma^\mu = K_1\gamma^\mu + K_2 r^\mu + K_3 q^\mu, \tag{5.5}$$

where K_1, K_2 and K_3 are functions of the scalar invariants q^2 and M^2. Applying the Ward identity $q_\mu\Gamma^\mu = 0$ to Eq. (5.5) and using $q \cdot r = 0$, we obtain

$$q_\mu\Gamma^\mu = K_1\not{q} + K_2 q\cdot r + K_3 q^2 = K_1\not{q} + K_3 q^2 = 0,$$

which means that out of the three factors K_1, K_2 and K_3, only two are independent. Furthermore, using the Gordon decomposition identity (A.12),

$$\overline{u}(s',p')\gamma^\mu u(s,p) = \overline{u}(s',p')\left[\frac{r^\mu}{2M} - \frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u(s,p),$$

we can rewrite the term proportional to r^μ in Eq. (5.5) in terms of γ^μ and $i\sigma^{\mu\nu}q_\nu$. We will choose the final parametrization following Halzen and Martin[10] [2] as

$$\Gamma^\mu = F_1(Q^2)\gamma^\mu - \frac{i\kappa}{2M}F_2(Q^2)\sigma^{\mu\nu}q_\nu, \tag{5.6}$$

where the elastic electromagnetic form factors F_1 and F_2 contain all the information about the internal structure of the proton. Some authors absorb the constant κ or the mass M into the definition of F_2, see for example Ref. [40, Eq. (10.6.15)] or Ref. [3, Eq. (6.33)]. For a review of the electromagnetic form factors, see Refs. [42, 43].

The constant κ in Eq. (5.6) is the anomalous magnetic moment. Let us take the proton as an example. If the proton had no internal structure, its magnetic moment would be the nuclear magneton $\mu_{\mathrm{N}} = e/2M$ [44, §10.9]. However, due to the proton's internal structure, the measured value of the magnetic moment is actually $2.79\,\mu_{\mathrm{N}}$ [45]. The difference between the measured value and the value predicted by the Dirac equation, in units of μ_{N}, is called the anomalous magnetic moment κ. Thus, for the proton, $\kappa_p = 2.79 - 1 = 1.79$. For the neutron, the predicted value of the magnetic moment is zero since the neutron is electrically neutral. As for the proton, the internal structure of the neutron complicates matter and experiments show that $\kappa_n = \mu_n = -1.91$ [45]. For elementary particles like the electron, the anomalous

[10] When comparing Eq. (5.6) to Eq. (8.13) in Ref. [2], it should be noted that the different sign follows from Halzen and Martin defining $q = p' - p$, while in this thesis we use $q = p - p'$.

magnetic moment is zero[11].

The emission vertex is thus given by

$$H^\mu = \overline{u}(s',p')\Gamma^\mu u(s,p) = \overline{u}(s',p')\left[F_1(Q^2)\gamma^\mu - \frac{i\kappa}{2M}F_2(Q^2)\sigma^{\mu\nu}q_\nu\right]u(s,p) \tag{5.7}$$

and the corresponding emission tensor is

$$\begin{aligned} W_{\mu\nu} &= \overline{H_\mu H_\nu^*} = \frac{1}{2}\overline{\sum} H_\mu H_\nu^* \\ &= \left(F_1^2(Q^2) + \tau\kappa F_2^2(Q^2)\right)(2p-q)_\mu(2p-q)_\nu \\ &\quad + \left(F_1(Q^2) + \kappa F_2(Q^2)\right)^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right) \\ &= \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau}(2p-q)_\mu(2p-q)_\nu + G_M^2(Q^2)\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right), \end{aligned} \tag{5.8}$$

where κ is the anomalous magnetic moment, $\tau \equiv -q^2/4M^2 = Q^2/4M^2$, and $G_E(Q^2)$ and $G_M(Q^2)$ are the Sachs form factors [2, Eq. (8.16)]

$$\begin{aligned} G_E(Q^2) &\equiv F_1(Q^2) - \tau\kappa F_2(Q^2), \\ G_M(Q^2) &\equiv F_1(Q^2) + \kappa F_2(Q^2), \end{aligned}$$

which can be interpreted in the Breit frame[12] as the spatial Fourier transforms of the electric and magnetic moment density distributions, respectively [2, 47]. The derivation of Eq. (5.8) is presented in Appendix C. It should be noted that Eq. (5.8) gives the same[13] tensor as in Ref. [12], up to a factor of $\frac{1}{2}$.

[11]This is only true to first order in α. Loop corrections introduce deviations to the magnetic moments of fermions. The one-loop correction is famously given by $\alpha/2\pi \simeq 0.001$ [3, Eq. (6.59)], obtained originally by Schwinger [46]. This also shows that the anomalous magnetic moments of elementary particles are orders of magnitude smaller than those of the nucleons.

[12]The Breit frame is defined by $\mathbf{p}' = -\mathbf{p}$, where $\mathbf{p}$ and $\mathbf{p}'$ are the incoming and outgoing momenta of the particle, respectively. The Breit frame is also known as the brick-wall frame, since the particle appears to bounce back with no energy transfer to the target. [2]

[13]When comparing the two expressions, it should be noted that the factor $p_\mu p_\nu$ in Ref. [12] reads $(2p-q)_\mu(2p-q)_\nu$ in the notation of this thesis.

Writing Eq. (5.8) in the form of Eq. (3.54),

$$
\begin{aligned}
W^{\mu\nu} &= \frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau}(2p-q)^\mu(2p-q)^\nu + G_M^2(Q^2)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \\
&= \frac{4\left(G_E^2(Q^2)+\tau G_M^2(Q^2)\right)}{1+\tau}p^\mu p^\nu - \frac{2\left(G_E^2(Q^2)+\tau G_M^2(Q^2)\right)}{1+\tau}(p^\mu q^\nu + p^\nu q^\mu) \\
&\quad + \frac{G_E^2(Q^2)-(1-\tau)G_M^2(Q^2)}{1+\tau}q^\mu q^\nu - Q^2 G_M^2(Q^2) g^{\mu\nu} \\
&\overset{!}{=} \alpha_1 p^\mu p^\nu + \alpha_2(p^\mu q^\nu + q^\mu p^\nu) + \alpha_3 q^\mu q^\nu + \alpha_4 g^{\mu\nu},
\end{aligned}
\tag{5.9}
$$

we see that

$$
\begin{aligned}
\alpha_1 &= \frac{4\left(G_E^2(Q^2)+\tau G_M^2(Q^2)\right)}{1+\tau}, \\
\alpha_2 &= \frac{2\left(G_E^2(Q^2)+\tau G_M^2(Q^2)\right)}{1+\tau}, \\
\alpha_3 &= \frac{G_E^2(Q^2)-(1-\tau)G_M^2(Q^2)}{1+\tau}, \\
\alpha_4 &= -Q^2 G_M^2(Q^2).
\end{aligned}
\tag{5.10}
$$

Plugging these into Eq. (3.70),

$$
f_{\gamma/B}(y,Q^2) = \frac{\alpha}{2\pi}\frac{y}{Q^2}Z_B^2\left[\frac{2\left(G_E^2(Q^2)+\tau G_M^2(Q^2)\right)}{1+\tau}\left(\frac{1-y}{y^2}-\frac{M^2}{Q^2}\right) + G_M^2(Q^2)\right].
\tag{5.11}
$$

Equation (5.11) gives the general form of the equivalent photon distribution for spin-$\frac{1}{2}$ particles.

5.3 Dipole form factors

In this section, we apply Eq. (5.11) and calculate the photon distribution of the proton, which will be used in Section 6. We use the so-called dipole form factors [12, 43, 48]

$$
\begin{aligned}
G_E(Q^2) &= \left(1 + Q^2/\Lambda^2\right)^{-2}, \\
G_M(Q^2) &= \mu_p G_E(Q^2),
\end{aligned}
\tag{5.12}
$$

where $\Lambda^2 = 0.71\,\mathrm{GeV}^2$ is an experimentally determined parameter and $\mu_p = 2.79$ [45] is the total magnetic moment of the proton. Using Eq. (5.12) in Eq. (5.11) yields

$$
f_{\gamma/p}(y,Q^2) = \frac{\alpha}{2\pi}\frac{y}{Q^2}\left(1+Q^2/\Lambda^2\right)^{-4}\left[\frac{2(1+\mu_p^2\tau)}{1+\tau}\left(\frac{1-y}{y^2}-\frac{M^2}{Q^2}\right)+\mu_p^2\right],
\tag{5.13}
$$

where $\tau = Q^2/4M^2$. Integrating Eq. (5.13) over Q^2, we obtain the integrated photon distribution (3.69)

$$f_{\gamma/p}(y) = \frac{\alpha y}{2\pi} \int_{Q^2_{\min}}^{Q^2_{\max}} \frac{\mathrm{d}Q^2}{Q^2} \left(1 + \frac{Q^2}{\Lambda^2}\right)^{-4} \left[\frac{2(1+\mu_p^2\tau)}{1+\tau}\left(\frac{1-y}{y^2} - \frac{M^2}{Q^2}\right) + \mu_p^2\right]. \tag{5.14}$$

Since the integrand in Eq. (5.14) is just a rational function of Q^2, it is possible to calculate the integral in terms of elementary functions. However, the full result will not be produced here due to its complexity. Instead, we consider certain special cases.

Setting[14] $Q^2_{\max} = \infty$ and $\mu_p = 1$, the equivalent photon distribution (5.14) reduces to the simpler form

$$f_{\gamma/p}(y) = \frac{\alpha}{2\pi} \int_{Q^2_{\min}}^{\infty} \frac{\mathrm{d}Q^2}{Q^2} \left(1 + Q^2/\Lambda^2\right)^{-4} \left[\frac{1+(1-y)^2}{y} - \frac{2yM^2}{Q^2}\right]. \tag{5.15}$$

This integral is calculated in Appendix D and the result is

$$\begin{aligned} f_{\gamma/p}(y) = & \frac{\alpha}{2\pi}\frac{1+(1-y)^2}{y}\left[\log A - \frac{11}{6} + \frac{3}{A} - \frac{3}{2A^2} + \frac{1}{3A^3}\right] \\ & - \frac{\alpha}{2\pi}\frac{2yM^2}{\Lambda^2}\left[A - 4\log A + \frac{10}{3} - \frac{6}{A} + \frac{2}{A^2} - \frac{1}{3A^3}\right], \end{aligned} \tag{5.16}$$

where $A = 1 + \Lambda^2/Q^2_{\min}$. If we also set $M = 0$ in Eq. (5.16), we obtain a further simplification

$$f_{\gamma/p}(y) = \frac{\alpha}{2\pi}\frac{1+(1-y)^2}{y}\left[\log A - \frac{11}{6} + \frac{3}{A} - \frac{3}{2A^2} + \frac{1}{3A^3}\right], \tag{5.17}$$

which has been previously obtained by Drees and Zeppenfeld [12].

[14] The integrand in Eq. (5.14) contains an overall term $1/Q^2\left(1 + Q^2/\Lambda^2\right)^4 \sim 1/(Q^2)^5$, which means that the integrand decays very quickly as Q^2 grows.

6 Muon-pair production

In this section, we apply the equivalent photon approximation derived in Section 4 to muon-pair production in proton-proton collisions which proceed via two photon exchanges. We also numerically calculate the same cross section without resorting to the equivalent photon approximation to see how they compare.

6.1 Full calculation

The Feynman diagrams associated with the muon-pair production process $pp \to pp\mu^+\mu^-$ are shown in Figure 4. We assume that the process proceeds via two photon exchanges, which means that the two protons cannot interact hadronically with each other.

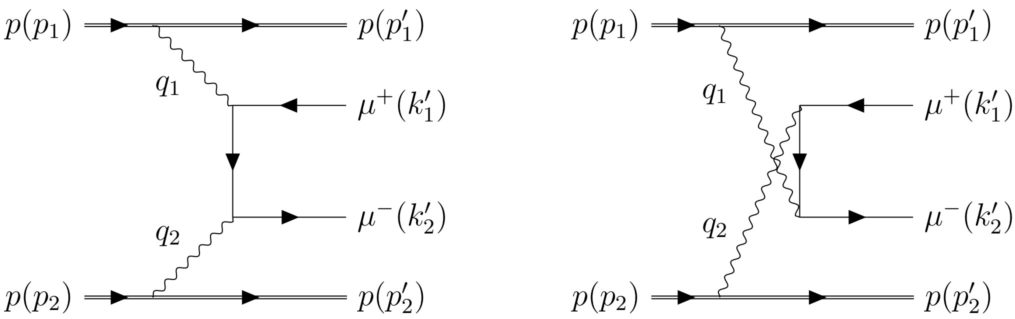


Figure 4. Feynman diagrams for the scattering $pp \to pp\mu^+\mu^-$ when the process is assumed to proceed via two photon exchanges. The momentum of each particle is shown in parenthesis.

The kinematics and associated notation have already been covered in Section 4.1. We are working the center-of-mass frame of the incoming protons where $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}$. Furthermore, we assume that the protons travel along the z-axis. In this frame, $\sqrt{s} = E_1 + E_2$ and

$$p_1 = \left(\frac{\sqrt{s}}{2}, 0, 0, p_z\right),$$
$$p_2 = \left(\frac{\sqrt{s}}{2}, 0, 0, -p_z\right),$$

where $p_z = \sqrt{\frac{s}{4} - M^2}$. We can use the (exact) decomposition obtained in Eq. (4.18) to write the squared amplitude as

$$\overline{|\mathcal{M}(pp \to pp\mu^+\mu^-)|^2} = \frac{e^8}{q_1^4 q_2^4} W_1^{\mu\rho} W_2^{\nu\sigma} \overline{K_{\mu\nu} K^*_{\rho\sigma}}.$$

The tensor $K_{\mu\nu}$ describing the pair creation $\gamma\gamma \to \mu^+\mu^-$ can now be written down explicitly using Feynman rules:

$$K_{\mu\nu} = -i\overline{u}(k_2', s_2) \left[\gamma_\nu \frac{\not{q}_1 - \not{k}_1' + m^2}{(q_1 - k_1')^2 - m^2} \gamma_\mu + \gamma_\mu \frac{\not{q}_2 - \not{k}_1' + m}{(q_2 - k_1')^2 - m^2} \gamma_\nu \right] v(k_1', s_1), \quad (6.1)$$

where s_1 and s_2 are the spin states of the antimuon and muon, respectively, and m is the muon mass. The spinors $\overline{u}$ and v are introduced in Appendix A.

It is instructive to see how the Ward identity

$$q_1^\mu K_{\mu\nu} = q_2^\nu K_{\mu\nu} = 0 \quad (6.2)$$

is realized in Eq. (6.1). We can use Eq. (A.4) and the conservation of momentum $q_1 + q_2 = k_1' + k_2'$ to obtain

$$\begin{aligned}(\not{q}_1 - \not{k}_1' + m)\not{q}_1 &= (\not{q}_1 - \not{k}_1' + m)(\not{q}_1 - \not{k}_1' - m + (\not{k}_1' + m)) \\ &= (q_1 - k_1')^2 - m^2 + (\not{q}_1 - \not{k}_1' + m)(\not{k}_1' + m)\end{aligned} \quad (6.3)$$

and

$$\begin{aligned}\not{q}_1(\not{q}_2 - \not{k}_1' + m) &= (\not{q}_1 - \not{k}_2' + m + (\not{k}_2' - m))(\not{q}_2 - \not{k}_1' + m) \\ &= -(\not{q}_2 - \not{k}_1' - m - (\not{k}_2' - m))(\not{q}_2 - \not{k}_1' + m) \\ &= -(q_2 - k_1')^2 + m^2 + (\not{k}_2' - m)(\not{q}_2 - \not{k}_1' + m).\end{aligned} \quad (6.4)$$

Using the Dirac equations in momentum space (Eqs. (A.5) and (A.7)–(A.9)) as well as Eqs. (6.3) and (6.4),

$$\begin{aligned}q_1^\mu K_{\mu\nu} &= -i\overline{u}(k_2', s_2) \left[\gamma_\nu \frac{\not{q}_1 - \not{k}_1' + m^2}{(q_1 - k_1')^2 - m^2} \not{q}_1 + \not{q}_1 \frac{\not{q}_2 - \not{k}_1' + m}{(q_2 - k_1')^2 - m^2} \gamma_\nu \right] v(k_1', s_1) \\ &= -i\overline{u}(k_2', s_2) \left[\gamma_\nu \frac{(q_1 - k_1')^2 - m^2 + (\not{q}_1 - \not{k}_1' + m)(\not{k}_1' + m)}{(q_1 - k_1')^2 - m^2} \right. \\ &\qquad \left. + \frac{-(q_2 - k_1')^2 + m^2 + (\not{k}_2' - m)(\not{q}_2 - \not{k}_1 + m)}{(q_2 - k_1')^2 - m^2} \gamma_\nu \right] v(k_1', s_1)\end{aligned}$$

$$
\begin{aligned}
&= -i\overline{u}(k_2', s_2)\left[\gamma_\nu\left(1 + \frac{(\not{q}_1 - \not{k}_1' + m)(\not{k}_1' + m)}{(q_1 - k_1')^2 - m^2}\right)\right. \\
&\qquad\qquad \left. - \left(1 - \frac{(\not{k}_2' - m)(\not{q}_2 - \not{k}_1' + m)}{(q_2 - k_1')^2 - m^2}\right)\gamma_\nu\right] v(k_1', s_1) \\
&= -i\overline{u}(k_2', s_2)\gamma_\nu \frac{\not{q}_1 - \not{k}_1' + m}{(q_1 - k_1')^2 - m^2} \overbrace{(\not{k}_1' + m)v(k_1', s_1)}^{=0} \\
&\quad - i\,\underbrace{\overline{u}(k_2', s_2)(\not{k}_2' - m)}_{=0} \frac{\not{q}_2 - \not{k}_1' + m}{(q_2 - k_1')^2 - m^2}\gamma_\nu v(k_1', s_1) = 0.
\end{aligned}
$$

The case $q_2^\nu K_{\mu\nu} = 0$ can be checked in a similar manner.

To simplify the notation, we denote

$$
k_{\mu\nu} \equiv \gamma_\nu \frac{\not{q}_1 - \not{k}_1' + m^2}{(q_1 - k_1')^2 - m^2}\gamma_\mu + \gamma_\mu \frac{\not{q}_2 - \not{k}_1' + m}{(q_2 - k_1')^2 - m^2}\gamma_\nu \tag{6.5}
$$

so that

$$
k_{\mu\nu}^\dagger = \gamma_\mu \frac{\not{q}_1 - \not{k}_1' + m^2}{(q_1 - k_1')^2 - m^2}\gamma_\nu + \gamma_\nu \frac{\not{q}_2 - \not{k}_1' + m}{(q_2 - k_1')^2 - m^2}\gamma_\mu = k_{\nu\mu}.
$$

Then, using standard trace techniques (see Appendix A),

$$
\begin{aligned}
\overline{K_{\mu\nu}K_{\rho\sigma}^*} &= \sum_{s_1 s_2} K_{\mu\nu}K_{\rho\sigma}^* \\
&= \sum_{s_1 s_2} \overline{u}(k_2', s_2)k_{\mu\nu}v(k_1', s_1)\left[\overline{u}(k_2', s_2)k_{\rho\sigma}v(k_1', s_1)\right]^\dagger \\
&= \sum_{s_1 s_2} \overline{u}(k_2', s_2)k_{\mu\nu}v(k_1', s_1)\overline{v}(k_1', s_1)k_{\sigma\rho}u(k_2', s_2).
\end{aligned} \tag{6.6}
$$

Making the matrix structure explicit in Eq. (6.6) and using Eq. (A.10),

$$
\begin{aligned}
\overline{K_{\mu\nu}K_{\rho\sigma}^*} &= \sum_{s_1 s_2} \overline{u}(k_2', s_2)_A(k_{\mu\nu})_{AB}v(k_1', s_1)_B\overline{v}(k_1', s_1)_C(k_{\sigma\rho})_{CD}u(k_2', s_2)_D \\
&= \sum_{s_2} u(k_2', s_2)_D\overline{u}(k_2', s_2)_A(k_{\mu\nu})_{AB}\sum_{s_1} v(k_1', s_1)_B\overline{v}(k_1', s_1)_C(k_{\sigma\rho})_{CD} \\
&= \mathrm{tr}\left\{(\not{k}_2' + m)k_{\mu\nu}(\not{k}_1' - m)k_{\sigma\rho}\right\} \\
&= \mathrm{tr}\left\{(\not{k}_2' + m)\left[\gamma_\nu \frac{\not{q}_1 - \not{k}_1' + m^2}{(q_1 - k_1')^2 - m^2}\gamma_\mu + \gamma_\mu \frac{\not{q}_2 - \not{k}_1' + m}{(q_2 - k_1')^2 - m^2}\gamma_\nu\right]\right. \\
&\qquad \left.\times(\not{k}_1' - m)\left[\gamma_\rho \frac{\not{q}_1 - \not{k}_1' + m^2}{(q_1 - k_1')^2 - m^2}\gamma_\sigma + \gamma_\sigma \frac{\not{q}_2 - \not{k}_1' + m}{(q_2 - k_1')^2 - m^2}\gamma_\rho\right]\right\},
\end{aligned} \tag{6.7}
$$

where the last line follows directly from Eq. (6.5).

We now define $\mathcal{C}' \equiv W_1^{\mu\rho} W_2^{\nu\sigma} \overline{K_{\mu\nu} K_{\rho\sigma}^*}$. Using the Ward identity (6.2) and Eqs. (5.9) and (5.12),

$$\begin{aligned}
\mathcal{C}' &= W_1^{\mu\rho} W_2^{\nu\sigma} \overline{K_{\mu\nu} K_{\rho\sigma}^*} \\
&= \left[\frac{4(1+\frac{\mu_p^2 Q_1^2}{4M^2})}{1+\frac{Q_1^2}{4M^2}} p_1^\mu p_1^\rho - \mu_p^2 Q_1^2 g^{\mu\rho} \right] \left[\frac{4(1+\frac{\mu_p^2 Q_2^2}{4M^2})}{1+\frac{Q_2^2}{4M^2}} p_2^\nu p_2^\sigma - \mu_p^2 Q_2^2 g^{\nu\sigma} \right] \\
&\quad \times \overline{K_{\mu\nu} K_{\rho\sigma}^*} G_E^2(Q_1^2) G_E^2(Q_2^2).
\end{aligned} \tag{6.8}$$

Using Eq. (4.13), and Eq. (4.2) for the notation of the momentum vectors,

$$\begin{aligned}
\mathrm{d}\sigma(pp \to pp\mu^+\mu^-) &= \frac{e^8}{q_1^4 q_2^4} \frac{(2\pi)^4 \mathcal{C}'}{4\sqrt{(p_1 \cdot p_2)^2 - M^4}} \delta^{(4)}(p_1 + p_2 - p_1' - p_2' - k_1' - k_2') \\
&\quad \times \frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3 2E_1'} \frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3 2E_2'} \frac{\mathrm{d}^3\mathbf{k}_1'}{(2\pi)^3 2(k_1')^0} \frac{\mathrm{d}^3\mathbf{k}_2'}{(2\pi)^3 2(k_2')^0} \\
&= \frac{\alpha^4}{64\pi^4 \sqrt{(p_1 \cdot p_2)^2 - M^4}} \frac{\mathcal{C}'}{Q_1^4 Q_2^4} \delta^{(4)}(p_1 + p_2 - p_1' - p_2' - k_1' - k_2') \\
&\quad \times \frac{\mathrm{d}^3\mathbf{p}_1'}{E_1'} \frac{\mathrm{d}^3\mathbf{p}_2'}{E_2'} \frac{\mathrm{d}^3\mathbf{k}_1'}{(k_1')^0} \frac{\mathrm{d}^3\mathbf{k}_2'}{(k_2')^0}.
\end{aligned}$$

Denoting

$$\begin{aligned}
\eta &\equiv \frac{\alpha^4}{64\pi^4 \sqrt{(p_1 \cdot p_2)^2 - M^4}}, \\
\epsilon &\equiv E_1 + E_2 - E_1' - E_2',
\end{aligned} \tag{6.9}$$

the integrated cross section is then

$$\begin{aligned}
\sigma(pp \to pp\mu^+\mu^-) &\equiv \sigma_{\mu^+\mu^-} \\
&= \eta \int \frac{\mathrm{d}^3\mathbf{p}_1'}{E_1'} \frac{\mathrm{d}^3\mathbf{p}_2'}{E_2'} \frac{\mathrm{d}^3\mathbf{k}_1'}{(k_1')^0} \frac{\mathrm{d}^3\mathbf{k}_2'}{(k_2')^0} \frac{\mathcal{C}'}{Q_1^4 Q_2^4} \delta(\epsilon - (k_1')^0 - (k_2')^0) \\
&\quad \times \delta^{(3)}(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{k}_1' + \mathbf{k}_2').
\end{aligned} \tag{6.10}$$

Using the three-dimensional delta function, we can trivially integrate over $\mathbf{k}_2'$. We also write the remaining momentum vectors in spherical coordinates using the notation

$$\begin{aligned}
\mathbf{k}_1' &= r_1(\cos\phi_1 \sin\theta_1, \sin\phi_1 \sin\theta_1, \cos\theta_1), \\
\mathbf{p}_i' &= R_i(\cos\Phi_i \sin\Theta_i, \sin\Phi_i \sin\Theta_i, \cos\Theta_i).
\end{aligned} \tag{6.11}$$

Then, Eq. (6.10) becomes

$$\sigma_{\mu^+\mu^-} = \eta \int \frac{R_1^2 \mathrm{d}R_1 \mathrm{d}\cos\Theta_1 \mathrm{d}\Phi_1}{\sqrt{M^2+R_1^2}} \frac{R_2^2 \mathrm{d}R_2 \mathrm{d}\cos\Theta_2 \mathrm{d}\Phi_2}{\sqrt{M^2+R_2^2}} \frac{r_1^2 \mathrm{d}r_1 \mathrm{d}\cos\theta_1 \mathrm{d}\phi_1}{\sqrt{m^2+r_1^2}\sqrt{m^2+|\mathbf{k}_2'|^2}} \times \frac{\mathcal{C}'}{Q_1^4 Q_2^4} \delta(\epsilon - (k_1')^0 - (k_2')^0), \tag{6.12}$$

where $\mathbf{k}_2' = -\mathbf{p}_1' - \mathbf{p}_2' - \mathbf{k}_1'$ is implicit.

We must still integrate over the remaining delta function. To do this, we use the composition property [49]

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i^*)}{|f'(x_i^*)|}, \tag{6.13}$$

where the sum is over all simple zeroes x_i^* of f. To that end, we define

$$\begin{aligned} f(r_1) &\equiv \epsilon - (k_1')^0 - (k_2')^0 = \epsilon - \sqrt{m^2+r_1^2} - \sqrt{m^2+|\mathbf{k}_2'|^2} \\ &= \epsilon - \sqrt{m^2+r_1^2} - \sqrt{m^2+|\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{k}_1'|^2} \\ &= \epsilon - \sqrt{m^2+r_1^2} - \sqrt{m^2+|\mathbf{p}_1'+\mathbf{p}_2'|^2 + 2(\mathbf{p}_1'+\mathbf{p}_2')\cdot\mathbf{k}_1' + r_1^2}. \end{aligned} \tag{6.14}$$

For convenience, we define

$$\begin{aligned} \rho^2 &\equiv |\mathbf{p}_1'+\mathbf{p}_2'|^2, \\ u &\equiv \frac{(\mathbf{p}_1'+\mathbf{p}_2')\cdot\mathbf{k}_1'}{r_1}. \end{aligned} \tag{6.15}$$

Using Eqs. (6.11) and (6.15),

$$\begin{aligned} u = {} & \cos\theta_1 (R_1\cos\Theta_1 + R_2\cos\Theta_2) \\ & + \sin\theta_1 (R_1 \sin\Theta_1 \cos(\phi_1 - \Phi_1) + R_2 \sin\Theta_2 \cos(\phi_1 - \phi_2)). \end{aligned} \tag{6.16}$$

With these definitions, Eq. (6.14) becomes

$$f(r_1) = \epsilon - \sqrt{m^2+r_1^2} - \sqrt{m^2+\rho^2+2ur_1+r_1^2}, \tag{6.17}$$

with all dependence on r_1 shown explicitly. Solving the equation $f(r_1) = 0$, we obtain two solutions

$$r_1^{\pm} = \frac{-u(\epsilon^2-\rho^2) \pm \epsilon\sqrt{(\epsilon^2-\rho^2)^2 - 4m^2(\epsilon^2-u^2)}}{2(\epsilon^2-u^2)}. \tag{6.18}$$

It is important to notice that Eq. (6.18) is not equivalent to $f(r_1) = 0$ as there are extraneous solutions in Eq. (6.18). This can be taken into account in the numerical integration routine by checking that a solution candidate is actually a solution.

Moving on to $\left|f'(r_1^\pm)\right|$, we can take the derivative of Eq. (6.17) to obtain

$$\begin{aligned} f'(r_1) &= -\frac{r_1\left(\sqrt{m^2+r_1^2}+\sqrt{m^2+\rho^2+2ur_1+r_1^2}\right)}{\sqrt{m^2+r_1^2}\sqrt{m^2+\rho^2+2ur_1+r_1^2}} - \frac{u}{\sqrt{m^2+\rho^2+2ur_1+r_1^2}} \\ &= \frac{r_1\left(\epsilon-\sqrt{m^2+r_1^2}+\sqrt{m^2+\rho^2+2ur_1+r_1^2}-\epsilon\right)}{\sqrt{m^2+r_1^2}\sqrt{m^2+\rho^2+2ur_1+r_1^2}} - \frac{u}{\sqrt{m^2+\rho^2+2ur_1+r_1^2}} \\ &= \frac{r_1(f(r_1)-\epsilon)}{\sqrt{m^2+r_1^2}\sqrt{m^2+\rho^2+2ur_1+r_1^2}} - \frac{u}{\sqrt{m^2+\rho^2+2ur_1+r_1^2}}. \end{aligned}$$

Then, since $f(r_1^\pm) = 0$,

$$\left|f'(r_1^\pm)\right| = \frac{\left|\frac{r_1^\pm \epsilon}{\sqrt{m^2+(r_1^\pm)^2}} + u\right|}{\sqrt{m^2+\rho^2+2ur_1^\pm+(r_1^\pm)^2}}. \tag{6.19}$$

Thus, we obtain using Eq. (6.13) that

$$\delta(\epsilon-(k_1')^0-(k_2')^0) = \delta(f(r_1)) = \frac{\delta(r_1-r_1^+)}{\left|f'(r_1^+)\right|} + \frac{\delta(r_1-r_1^-)}{\left|f'(r_1^-)\right|}. \tag{6.20}$$

Plugging Eq. (6.20) to Eq. (6.12) and calculating the integral over r_1,

$$\begin{aligned} \sigma_{\mu^+\mu^-} = \eta \int & \frac{R_1^2 \mathrm{d}R_1 \mathrm{d}\cos\Theta_1 \mathrm{d}\Phi_1}{\sqrt{M^2+R_1^2}} \frac{R_2^2 \mathrm{d}R_2 \mathrm{d}\cos\Theta_2 \mathrm{d}\Phi_2}{\sqrt{M^2+R_2^2}} \mathrm{d}\cos\theta_1 \mathrm{d}\phi_1 \\ & \times \sum_{r_1^*=r_1^\pm} \left[\frac{\mathcal{C}'}{Q_1^4 Q_2^2} \frac{r_1^2}{\sqrt{m^2+r_1^2}\sqrt{m^2+|\mathbf{k}_2'|^2}} \frac{\Theta(r_1)}{|f'(r_1)|}\right]_{r_1=r_1^*}, \end{aligned} \tag{6.21}$$

where $\Theta(x)$ is the Heaviside theta function defined by

$$\Theta(x) \equiv \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

and the notation $[\cdots]_{r_1=r_1^*}$ means that every occurrence of r_1 inside the square brackets should be replaced by r_1^*. The term $\Theta(r_1)$ originates from the integral over $r_1 \geq 0$, since

$$\int_0^\infty \mathrm{d}x\, \delta(x - x^*) = \Theta(x^*).$$

In order to obtain the cross section that is approximately differential in the invariant mass

$$W_{\mu^+\mu^-} \equiv \sqrt{(k_1' + k_2')^2} \tag{6.22}$$

of the muon-antimuon system, we can introduce a cut function

$$\Theta(W_{\min}^2 \leq W_{\mu^+\mu^-}^2 \leq W_{\max}^2) \equiv \begin{cases} 1, & \text{if } W_{\min}^2 \leq W_{\mu^+\mu^-}^2 \leq W_{\max}^2, \\ 0, & \text{otherwise,} \end{cases} \tag{6.23}$$

to the total cross section integral in Eq. (6.21). The parameters $W_{\min}$ and $W_{\max}$ define an invariant-mass bin, and the cross section in one such bin is then given by

$$\begin{aligned} \frac{\mathrm{d}\sigma_{\mu^+\mu^-}}{\mathrm{d}W_{\mu^+\mu^-}} = \eta \int & \frac{R_1^2 \mathrm{d}R_1 \mathrm{d}\cos\Theta_1 \mathrm{d}\Phi_1}{\sqrt{M^2 + R_1^2}} \frac{R_2^2 \mathrm{d}R_2 \mathrm{d}\cos\Theta_2 \mathrm{d}\Phi_2}{\sqrt{M^2 + R_2^2}} \mathrm{d}\cos\theta_1 \mathrm{d}\phi_1 \\ & \times \sum_{r_1^* = r_1^\pm} \left[\frac{\mathcal{C}'}{Q_1^4 Q_2^2} \frac{r_1^2}{\sqrt{m^2 + r_1^2}\sqrt{m^2 + |\mathbf{k}_2'|^2}} \frac{\Theta(r_1)}{|f'(r_1)|} \right]_{r_1 = r_1^*} \\ & \times \frac{\Theta(W_{\min}^2 \leq W_{\mu^+\mu^-}^2 \leq W_{\max}^2)}{W_{\max} - W_{\min}}. \end{aligned} \tag{6.24}$$

6.2 Numerical methods

With the integral in Eq. (6.24) written as it is, the integration variables are $R_1, \cos\Theta_1, \Phi_1, R_2, \cos\Theta_2, \Phi_2, \cos\theta_1$, and ϕ_1. The cosines $\cos\Theta_1, \cos\Theta_2$, and $\cos\theta_1$ have well-defined integration bounds extending from -1 to 1. Similarly, the integration bounds of the azimuthal angles Φ_1, Φ_2, and ϕ_1 extend from 0 to 2π. For the integration bounds of the radial integration variables R_1 and R_2, we choose to extend them from 0 to $\sqrt{s}/2$. These bounds are not necessarily optimal, but they cover the entire physical region of the phase space. Writing these bounds down explicitly, we have

$$\frac{\mathrm{d}\sigma_{\mu^+\mu^-}}{\mathrm{d}W_{\mu^+\mu^-}} = \eta \int_0^{\frac{\sqrt{s}}{2}} \mathrm{d}R_1 \frac{R_1^2}{\sqrt{M^2+R_1^2}} \int_{-1}^{1} \mathrm{d}\cos\Theta_1 \int_0^{2\pi} \mathrm{d}\Phi_1$$
$$\times \int_0^{\frac{\sqrt{s}}{2}} \mathrm{d}R_2 \frac{R_2^2}{\sqrt{M^2+R_2^2}} \int_{-1}^{1} \mathrm{d}\cos\Theta_2 \int_0^{2\pi} \mathrm{d}\Phi_2 \int_{-1}^{1} \mathrm{d}\cos\theta_1 \int_0^{2\pi} \mathrm{d}\phi_1$$
$$\times \sum_{r_1^*=r_1^\pm} \left[\frac{\mathcal{C}'}{Q_1^4 Q_2^2} \frac{r_1^2}{\sqrt{m^2+r_1^2}\sqrt{m^2+|\mathbf{k}_2'|^2}} \frac{\Theta(r_1)}{|f'(r_1)|} \right]_{r_1=r_1^*}$$
$$\times \frac{\Theta(W_{\min}^2 \leq W_{\mu^+\mu^-}^2 \leq W_{\max}^2)}{W_{\max}-W_{\min}}, \tag{6.25}$$

where $\mathcal{C}' = W_1^{\mu\rho} W_2^{\nu\sigma} \overline{K_{\mu\nu} K_{\rho\sigma}^*}$, η, $r_1^\pm$, and $\left|f'(r_1^\pm)\right|$ are defined in Eqs. (6.9), (6.18), and (6.19).

In principle, Eq. (6.25) could be used directly for numerical integration. However, due to the finite precision of floating-point arithmetic, further work is needed. The issue is in the contraction $\mathcal{C}'$, which contains large canceling terms [50]. Numerical error in the subtraction of two large numbers leads to completely invalid results.

One solution is to rewrite the contraction in a way that avoids these cancellations altogether. This has been done by Vermaseren in Ref. [50]. Another approach, also briefly mentioned by Vermaseren, is to replace the vector p_i in the emission tensor W_i by

$$P_i^\mu \equiv p_i^\mu - \frac{1}{y_i} q_i^\mu, \tag{6.26}$$

where y_i is taken from Eq. (4.7). This replacement is allowed by the Ward identity $q_i^\mu K_{\mu\nu} = 0$, as analytically the replacement $p_i \to P_i$ does nothing. However, doing this replacement improves numerical stability dramatically. The reason for this can be found in Eq. (3.15). Since the bulk of the integral is concentrated around the point $(Q_1^2, Q_2^2) = (0,0)$ where $q_i^\mu \simeq y_i p_i^\mu$, the linear combination P_i becomes very small and eliminates the large numerical error; see also Ref. [51].

While the contraction $\mathcal{C}'$ could in principle be calculated by hand using Eqs. (6.7), (6.8), and (6.26), the large number of terms makes this impractical. Instead, we use FeynCalc [52–54] to calculate and simplify the contraction. Conservation of momentum is used to eliminate the momentum vector k_2' by writing $k_2' = q_1 + q_2 - k_1'$. The resulting expression, written in terms of scalar products of the momentum vectors P_1, P_2, q_1, q_2, and k_1', is converted to a C-style string with the Mathematica function `CForm`. The scalar products are then converted to appropriate variable names and a `C++` source file, comprising of a single function that calculates the value of the contraction, is produced and exported.

Since the contraction $\mathcal{C}'$ is implemented by scalar products of the momentum vectors, the required momentum vectors have to be reconstructed from the integration variables[15]. The three-momentum vector $\mathbf{p}_i'$ can easily be calculated from the values of R_i, $\cos\Theta_i$, and Φ_i using Eq. (6.11). Since $0 \leq \Theta_i \leq \pi$, the value of $\sin\Theta_i \geq 0$ is determined unambiguously by $\sin\Theta_i = \sqrt{1-\cos^2\Theta_i}$. The full momentum vector p_i' can then be calculated using the dispersion relation $E_i' = \sqrt{M^2 + R_i^2}$.

The momentum vector k_1' can be reconstructed similarly, but now r_1 is no longer an integration variable. Instead, it must be calculated using Eq. (6.18). As we mentioned earlier, there are extraneous solutions in Eq. (6.18). To account for this, the solutions $r_1^\pm$ calculated with Eq. (6.18) are checked to be actual solutions by verifying numerically that $f(r_1^\pm) = 0$ holds. In addition, if $r_1^\pm < 0$, that solution is rejected, as is required by the Heaviside theta function $\Theta(r_1)$ appearing in Eq. (6.25). After the momentum vectors p_1', p_2', and k_1' have been reconstructed, the rest follows easily since we can then calculate any scalar products directly.

To compute the integral (6.25) numerically, we use the Gsl [55] implementation of the Vegas [56, 57] algorithm. Vegas is an adaptive Monte-Carlo integration algorithm. For an introduction to Monte-Carlo integration methods, see for example Ref. [58].

In practice, Vegas contains two types of iteration loops. For a given integration grid, Vegas calculates the integral by computing the value of the integrand at N randomly selected points, and it does this for some number n_{iter} of iterations. After these iterations, the value of the integral I and its absolute error estimate ΔI are returned alongside a χ^2-value. The χ^2-value indicates the statistical consistency of the integral value and its error estimate. The closer the χ^2-value is to 1, the more reliable are the results. Vegas then adapts, or rebins, the previous integration grid to concentrate the integration points to regions where the function contributes the most to the integral. After the rebinning procedure, Vegas starts iterating the integral using the new integration grid. [55, 58]

The integration routine is split into two phases. In the first phase (Phase 1), when the integration grid has not adapted sufficiently well to the physically allowed

[15] The alternative is to write all scalar products directly in terms of the integration variables.

Table 2. VEGAS integration parameters for the full calculation (6.25). These parameters are defined by GSL in the `gsl_monte_vegas_params` structure. The values are given for both phases of the integration routine; see the text on page 79. The value of n_{iter} is given in Table 3.

Parameter	Description	Phase 1	Phase 2
`alpha`	Stiffness of rebinning algorithm	2.0	0.1
`iterations`	Number of iterations per rebinning	2	n_{iter}
`mode`	Variance reduction method	stratified	stratified

Table 3. Integration routine parameters for the full calculation (6.25). These parameters control the number of sampling points as well as the stopping condition for the integration routine.

Parameter	Description	$\sqrt{s} = 200\,\text{GeV}$	$\sqrt{s} = 13\,\text{TeV}$
N	Number of sampling points	2×10^9	2×10^9
n_{iter}	Number of iterations per rebinning	15	20
$I_{\text{cut-off}}$	Minimum integral value for transitioning to Phase 2	1	1
$\delta\chi^2$	Maximum allowed value for $\lvert\chi^2 - 1\rvert$	0.1	0.1
δI_{max}	Maximum allowed value for the relative error δI	0.1	0.1
n_{max}	Maximum number of rebinning iterations	200	200

region, only two iterations are calculated before the rebinning procedure. Once the value of the integral rises above some cut-off value $I_{\text{cut-off}}$, the integration grid is deemed sufficiently well-adapted and the second phase (Phase 2) begins. In the second phase, n_{iter} iterations are performed before rebinning the grid.

The integration routine stops when the χ^2-value gets sufficiently close to 1. That is to say, when $|\chi^2 - 1| < \delta\chi^2$ for some $\delta\chi^2$. Additionally, the relative error $\delta I \equiv |\Delta I/I|$ must be smaller than some value δI_{max}. For practical reasons, a fail-safe mechanism is also present. This means that if some maximum number of rebinning iterations n_{max} is exceeded, the integration is stopped and the integral with the χ^2-value closest to 1 thus far is returned.

All integration parameters can be found in Tables 2 and 3. The code and the data files are available on GitHub[16].

It should be noted that the numerical calculation of lepton pair production cross sections in hadron-hadron collisions has also been considered elsewhere. In 1991, the generator `LPAIR` [59] was created based on Ref. [50]. A more modern version of `LPAIR`, called `CepGen`, was developed in Ref. [60]. These generators are more powerful and complex as they include facilities for applying experimental cuts as well as for the consideration of dissociative processes where one or both hadrons break apart.

6.3 Equivalent photon approximation

We now apply the equivalent photon approximation to muon-pair production. In this context, the integral (4.68) reads

$$\sigma^{\mathrm{EPA}}(pp \to pp\mu^+\mu^-) \equiv \sigma^{\mathrm{EPA}}_{\mu^+\mu^-} = \int \mathrm{d}y_1\, \mathrm{d}y_2 f_{\gamma/p}(y_1) f_{\gamma/p}(y_2) \hat{\sigma}(\gamma\gamma \to \mu^+\mu^-). \tag{6.27}$$

The equivalent photon distribution $f_{\gamma/p}(y)$ for the dipole form factors (5.12) was calculated in Eqs. (5.16) and (5.17). In addition, we consider the case where the proton's magnetic moment μ_p is kept in place. We still set $Q^2_{\max} = \infty$ so that by Eq. (5.14),

$$f_{\gamma/p}(y) = \frac{\alpha y}{2\pi} \int_{Q^2_{\min}}^{\infty} \frac{\mathrm{d}Q^2}{Q^2} \left(1 + \frac{Q^2}{\Lambda^2}\right)^{-4} \left[\frac{2(1+\mu_p^2\tau)}{1+\tau}\left(\frac{1-y}{y^2} - \frac{M^2}{Q^2}\right) + \mu_p^2\right]. \tag{6.28}$$

While this integral can be calculated in terms of elementary functions, we do not produce the complicated result here. Instead, we calculate it using Mathematica and use the resulting expression directly for numerical integration.

The integrated cross section $\hat{\sigma}(\gamma\gamma \to \mu^+\mu^-)$ is given by[17] the Breit–Wheeler formula [61, Exercise 3.15]

$$\begin{aligned}\hat{\sigma}_{\gamma\gamma} &\equiv \hat{\sigma}(\gamma\gamma \to \mu^+\mu^-) \\ &= \frac{4\pi\alpha^2}{\hat{s}}\left[\left(1 + \frac{4m^2}{\hat{s}} - \frac{8m^4}{\hat{s}^2}\right)\log\frac{1+\sqrt{1-4m^2/\hat{s}}}{1-\sqrt{1-4m^2/\hat{s}}} - \left(1+\frac{4m^2}{\hat{s}}\right)\sqrt{1-\frac{4m^2}{\hat{s}}}\right],\end{aligned}$$

[16] https://github.com/samiyr/fyss9490

[17] The cross section given in Ref. [61] is written in terms of the relativistic speed v of either the muon or the antimuon in their center-of-mass frame. In this frame, $v = \sqrt{1 - 4m^2/\hat{s}}$.

where $\hat{s}$ is the Mandelstam s associated with the process $\gamma\gamma \to \mu^+\mu^-$, corresponding to the squared invariant mass $W^2_{\mu^+\mu^-}$ of the muon-antimuon-system defined in Eq. (6.22):

$$\hat{s} = (k'_1 + k'_2)^2 = W^2_{\mu^+\mu^-}. \tag{6.29}$$

Based on Eq. (4.6) alone, the integration domain of the integral in Eq. (6.27) would be the unit square $0 \leq y_1, y_2 \leq 1$. However, in the center-of-mass frame where $\mathbf{k}'_1 = -\mathbf{k}'_2 \equiv \mathbf{k}'$,

$$\hat{s} = (k'_1 + k'_2)^2 = 2m^2 + 2\left(m^2 + |\mathbf{k}'|^2 + |\mathbf{k}'|^2\right) \geq 4m^2. \tag{6.30}$$

Using Eqs. (4.9) and (6.30),

$$y_1 y_2 \geq \frac{4m^2}{s}. \tag{6.31}$$

Thus, the integration region is not the unit square $[0,1]^2$, but rather the set

$$\{(y_1, y_2) \in [0,1]^2 : y_1 y_2 \geq 4m^2/s\}, \tag{6.32}$$

which is illustrated in Figure 5. Thus, Eq. (6.27) becomes

$$\sigma^{\mathrm{EPA}}_{\mu^+\mu^-} = \int_0^1 \mathrm{d}y_1 \int_0^1 \mathrm{d}y_2\, f_{\gamma/p}(y_1) f_{\gamma/p}(y_2)\, \hat{\sigma}_{\gamma\gamma}(\hat{s} = y_1 y_2 s)\Theta(y_1 y_2 s - 4m^2), \tag{6.33}$$

where the Heaviside theta function enforces Eq. (6.31).

As in Eq. (6.24), we can obtain the approximately differential cross section $\mathrm{d}\sigma^{\mathrm{EPA}}_{\mu^+\mu^-}/\mathrm{d}W_{\mu^+\mu^-}$ by introducing the same cut as in Eq. (6.23). Thus, by Eqs. (6.29) and (6.33),

$$\begin{aligned} \frac{\mathrm{d}\sigma^{\mathrm{EPA}}_{\mu^+\mu^-}}{\mathrm{d}W_{\mu^+\mu^-}} = &\int_0^1 \mathrm{d}y_1 \int_0^1 \mathrm{d}y_2\, f_{\gamma/p}(y_1) f_{\gamma/p}(y_2)\, \hat{\sigma}_{\gamma\gamma}(\hat{s} = y_1 y_2 s)\Theta(y_1 y_2 s - 4m^2) \\ &\times \frac{\Theta(W^2_{\mathrm{min}} \leq y_1 y_2 s \leq W^2_{\mathrm{max}})}{W_{\mathrm{max}} - W_{\mathrm{min}}}. \end{aligned} \tag{6.34}$$

This integral can easily be calculated using the GSL implementation of the VEGAS algorithm. The integration parameters, defined in Section 6.2, are listed in Tables 4 and 5.

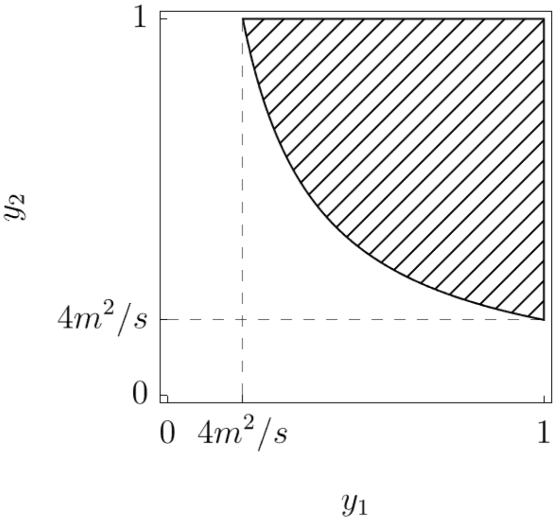


Figure 5. The integration region of the integral in Eq. (6.27). The shaded region corresponds to the set given in Eq. (6.32).

6.4 Results and comparisons

In order to validate the results obtained from the full calculation, we can compare the results obtained from Eq. (6.25) to experimental data. The observable $\mathrm{d}\sigma_{\mu^+\mu^-}/\mathrm{d}W_{\mu^+\mu^-}$ has been measured in the ATLAS experiment [11] at $\sqrt{s} = 13\,\mathrm{TeV}$. Due to practical limitations of particle detectors, experimental results have various kinematical cuts. The cuts in the ATLAS experiment restrict the transverse momentum p_T and pseudorapidity η of the muon and antimuon. When the incoming protons travel along the z-axis, these two kinematical observables are given by [62, §2.4]

$$(p_\mathrm{T})_i = \sqrt{(k_i'^x)^2 + (k_i'^y)^2}$$

and

$$\eta_i = \frac{1}{2}\log\frac{|\mathbf{k}_i'| + k_i'^z}{|\mathbf{k}_i'| - k_i'^z},$$

where $k_i' = (k_i'^0, \mathbf{k}_i') = (k_i'^0, k_i'^x, k_i'^y, k_i'^z)$.

The cuts in the ATLAS experiment are split into two regions of the invariant mass $W_{\mu^+\mu^-}$. In the region $12\,\mathrm{GeV} < W_{\mu^+\mu^-} < 30\,\mathrm{GeV}$, a minimum transverse momentum $p_\mathrm{T} > 6\,\mathrm{GeV}$ is required for the muon and the antimuon. In the region $30\,\mathrm{GeV} < W_{\mu^+\mu^-} < 70\,\mathrm{GeV}$, the cut $p_\mathrm{T} > 10\,\mathrm{GeV}$ is required for both particles. In both regions, the pseudorapidity of both the muon and the antimuon are required to be in the range $|\eta| < 2.4$. Implementing these cuts in the full calculation can be

Table 4. VEGAS integration parameters for the equivalent photon approximation calculation (6.34). These parameters are defined by GSL in the `gsl_monte_vegas_params` structure.

Parameter	Description	Value
`alpha`	Stiffness of rebinning algorithm	1.5
`iterations`	Number of iterations per rebinning	5
`mode`	Variance reduction method	stratified

Table 5. Integration routine parameters for the equivalent photon approximation calculation (6.34). These parameters control the number of sampling points as well as the stopping condition for the integration routine. The parameter values are the same for both values of $\sqrt{s}$.

Parameter	Description	Value
N	Number of sampling points	10^7
$\delta\chi^2$	Maximum allowed value for $\lvert\chi^2 - 1\rvert$	0.1
$\delta I_{\max}$	Maximum allowed value for the relative error δI	0.1
$n_{\max}$	Maximum number of rebinning iterations	200

done by introducing cut functions for the transverse momenta and pseudorapidities of both muons to the integral (6.25) in the exact same way as the invariant-mass cut (6.23) was introduced.

The comparison between the ATLAS data and the full calculation with experimental cuts is shown in Figure 6. The full calculation overestimates the experimentally obtained cross section by $5 - 25\,\%$. This level of agreement is reasonable considering that the full calculation uses the relatively simple dipole form factors (5.12). In particular, they do not take finite-size corrections into account. If the two protons collide too close to each other in the impact parameter, additional hadronic interactions may also occur that can dissociate the proton. Including these finite-size corrections would reduce the cross section, which is consistent with the full calculation overestimating the experimental data. This result gives us confidence in using Eq. (6.25) to evaluate the performance of the equivalent photon approximation, which is based on the same dipole form factors.

We can now compare the equivalent photon approximation with the full calculation. For the equivalent photon approximation cross section $\mathrm{d}\sigma^{\mathrm{EPA}}_{\mu^+\mu^-}/\mathrm{d}W_{\mu^+\mu^-}$ (6.34), we use the three distributions given in Eqs. (5.16), (5.17), and (6.28). All three distributions use the same dipole form factors (5.12), but with differing approx-

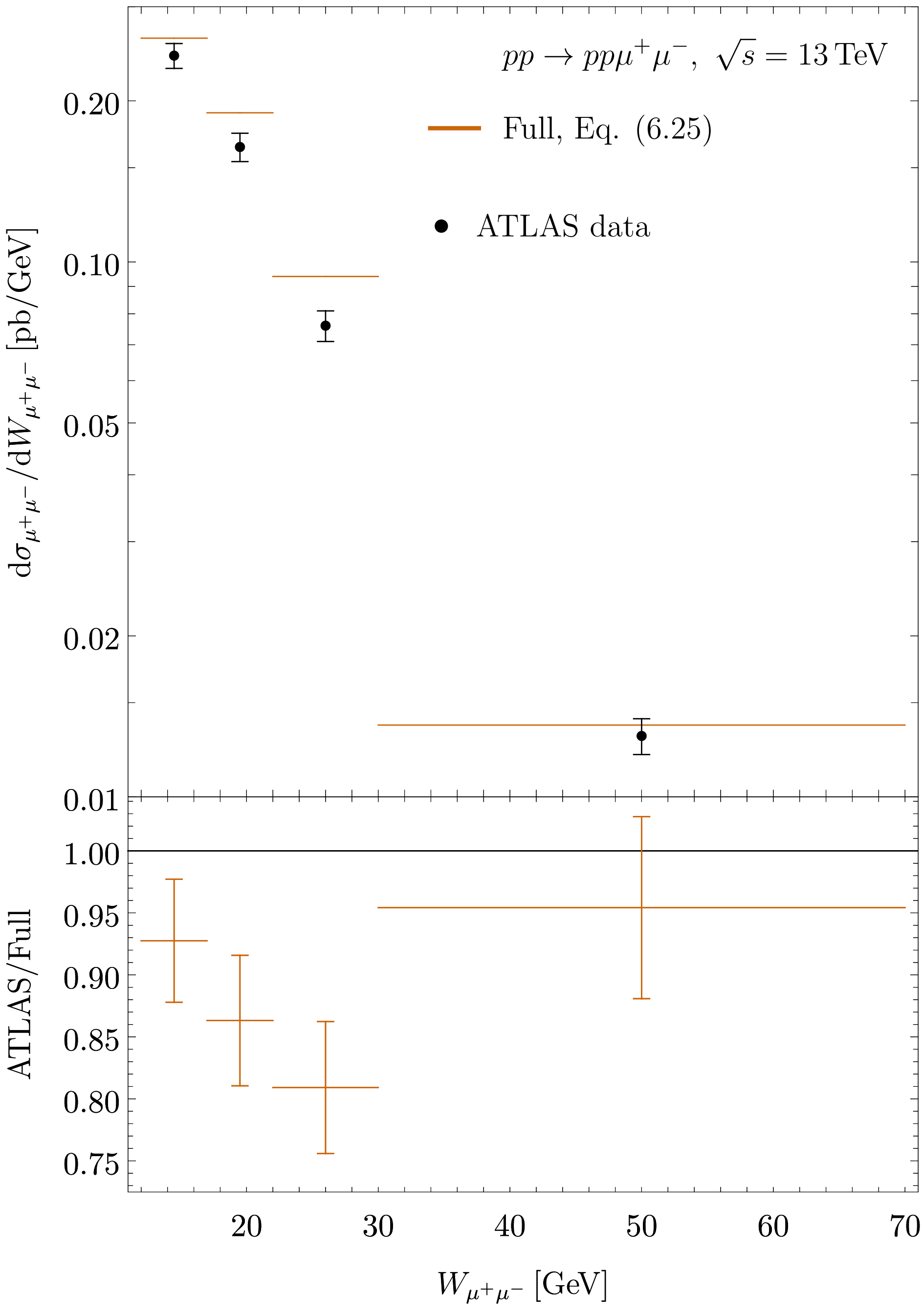


Figure 6. Muon-pair production cross sections in proton-proton collisions at $\sqrt{s} = 13$ TeV. The upper plot shows the cross sections and the lower plot shows the ratio between the ATLAS data given in Ref. [11] and the results obtained from the full calculation (6.25) with experimental cuts. The solid orange line is the full calculation (6.25) and the black data points represent the ATLAS data. The cross sections are binned in the invariant mass $W_{\mu^+\mu^-}$ of the muon pair. The integration parameters are listed in Tables 2 and 3.

imations. In Eq. (6.28) we set $Q^2_{\max} = \infty$. The distribution (5.16) neglects the anomalous magnetic moment of the proton by setting $\mu_p = 1$. Finally, Eq. (5.17) also neglects the mass of the proton. In all distributions, we also approximated $Q^2_{\min}$ using Eq. (4.10).

The results at $\sqrt{s} = 200\,\text{GeV}$ are shown in Figure 7. In the range $12\,\text{GeV} < W_{\mu^+\mu^-} < 50\,\text{GeV}$, the simplest photon distribution (5.17) fares reasonably well by initially overestimating the cross section by 24 % and then underestimating it by 20 %. However, the underestimation continues in the region $50\,\text{GeV} < W_{\mu^+\mu^-} < 70\,\text{GeV}$, with the distribution (5.17) underestimating the cross section by up to 49 %. Interestingly, the less approximative and thus supposedly more accurate distribution (5.16) fares worse by consistently underestimating the cross section by $40 - 89\,\%$. However, the least approximative distribution (6.28) is still the most accurate. This distribution overestimates the cross section only by less than 1 %.

The results at $\sqrt{s} = 13\,\text{TeV}$ are shown in Figure 8. This time, all three distributions approximate the full cross section to within 20 %. The simplest distribution (5.17) overestimates the cross section by $15 - 18\,\%$, while the distribution (5.16), which includes the mass term, underestimates the cross section by $8 - 16\,\%$. The distribution (6.28) is still the most accurate as it overestimates the cross section by less than 4 %.

To see why the distributions (5.16) and (5.17) give more comparable results at $\sqrt{s} = 13\,\text{TeV}$ than they do at $\sqrt{s} = 200\,\text{GeV}$, we must consider Eq. (6.31). As s grows, the product $y_1 y_2$ can become smaller. Thus, at higher values of $\sqrt{s}$, the values of y_1 and y_2 are smaller. Looking at Eq. (5.16), we see that the mass correction term is proportional to y, which makes it insignificant at small values of y in comparison to the term in Eq. (5.17), which behaves like $1/y$ at small values of y.

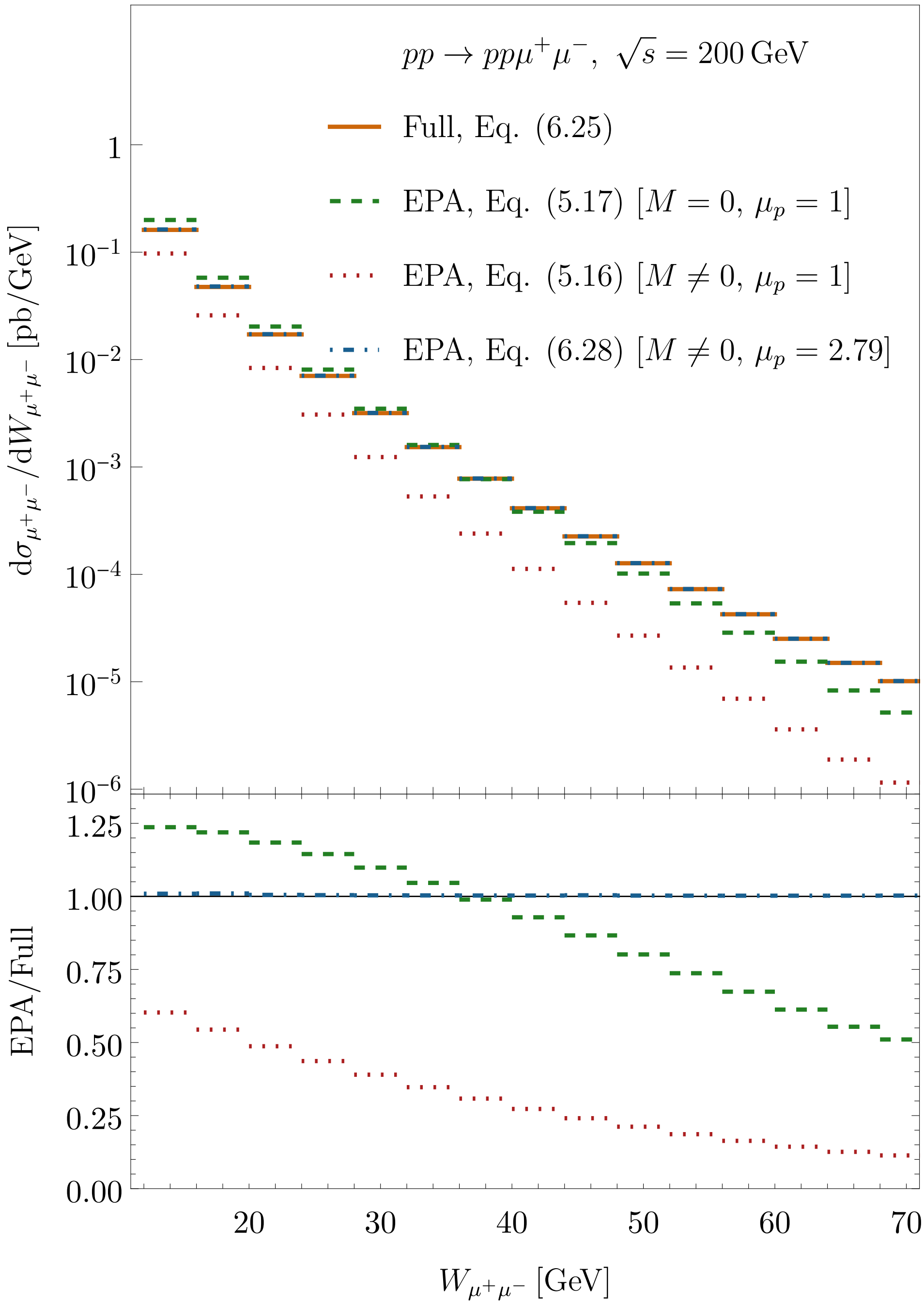


Figure 7. Muon-pair production cross sections in proton-proton collisions at $\sqrt{s} = 200\,\text{GeV}$. The upper plot shows the cross sections and the lower plot shows the ratio between the results obtained using the equivalent photon approximation and the full calculation. The solid orange line is the full calculation (6.25). The non-solid lines use the equivalent photon approximation. The dashed green line uses the photon distribution from Eq. (5.17), the dotted red line uses Eq. (5.16), and the dash-dotted blue line uses Eq. (6.28). The cross sections are binned in the invariant mass $W_{\mu^+\mu^-}$ of the muon pair. The integration parameters are listed in Tables 2–5.

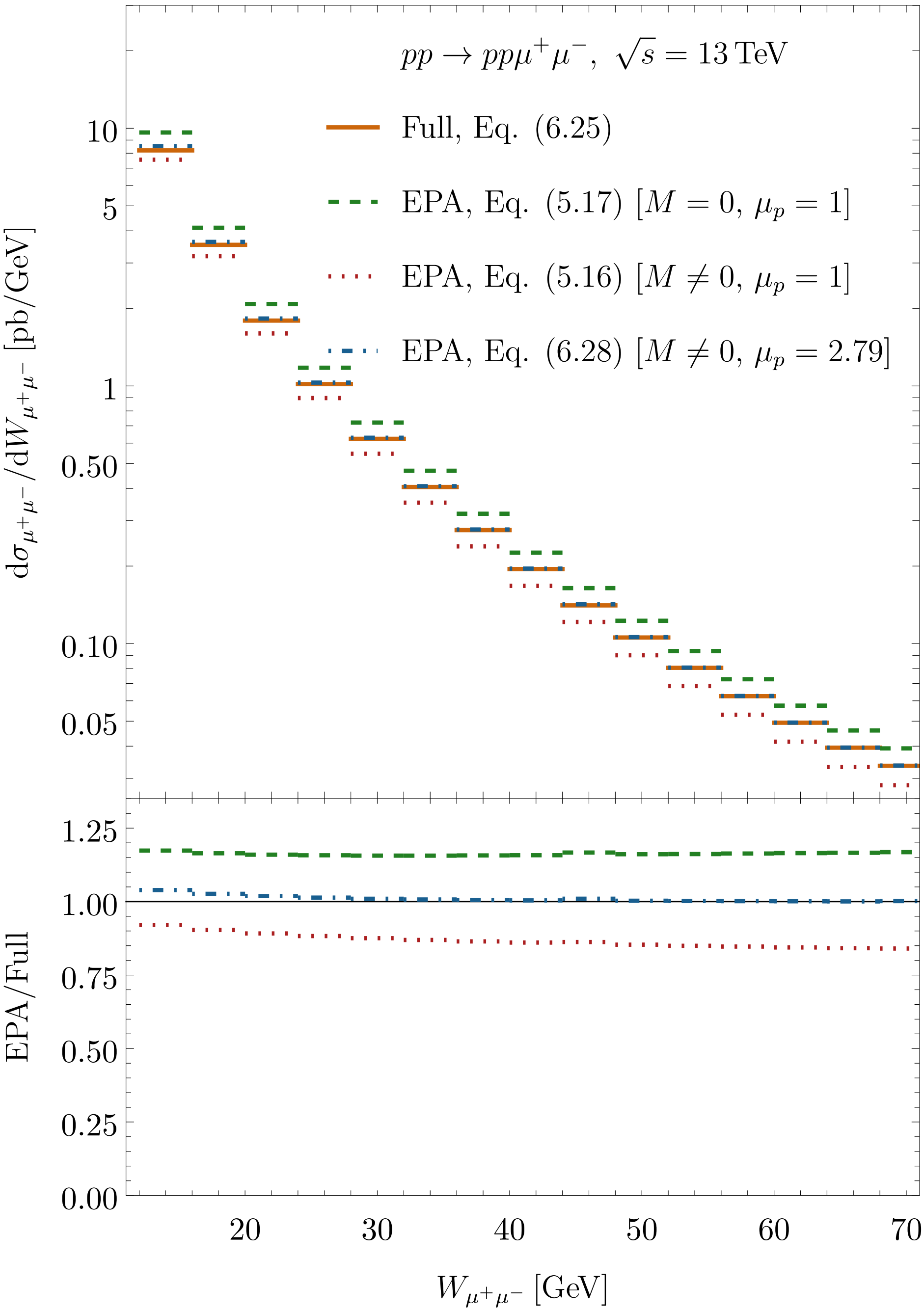


Figure 8. Same as Figure 7, but at $\sqrt{s} = 13\,\text{TeV}$.

7 Conclusions

In this thesis, we derived the factorization of electromagnetic cross sections involving one- and two-photon exchanges in the equivalent photon approximation using a helicity-based method used in Ref. [34]. In the process, we obtained a general expression for the equivalent photon distribution, Eq. (3.70). We also considered the distributions for spin-0 (Eq. (5.2)) and spin-$\frac{1}{2}$ (Eq. (5.11)) particles in Section 5.

Obtaining the factorization in a one-photon exchange process was relatively straightforward using the polarization tensors defined in Section 2.2, since they provide a convenient orthogonal basis (see Eqs. (3.46) and (4.24)). The kinematics are also relatively simple in this case. While the basic idea used in the one-photon exchange case also works in the two-photon exchange case, the generalization to that case presented some additional challenges.

One challenge was obtaining the parametrization of $K^{\mu\nu\rho\sigma}$ (4.24). Unlike the one-photon exchange case, there are four Lorentz-indices to contend with. In the one-photon exchange case, the transition from $\ell^{\mu\nu}$ being written in terms of individual momentum vectors (Eq. (3.42)) to it being written in terms of the polarization tensors (Eq. (3.46)) was done in a few lines in Eq. (3.45). More importantly, the transition was done not systematically but by inspection: looking at the first line in Eq. (3.45), one can notice the same tensor structure that is also found in the polarization tensors, specifically in Eqs. (2.25) and (2.26). Since even writing the general form of $K^{\mu\nu\rho\sigma}$ is a non-trivial task, as we found in Appendix E, trying to rewrite the parametrization in terms of the polarization tensors by inspection alone is essentially hopeless, as can be seen in Eq. (4.29).

Another challenge was dealing with the extra terms in Eq. (4.33). Ultimately, the resolution required the introduction of transverse projections in Section 4.4. A related complication was the fact that the angle ϕ, which arose from the squared matrix element in Eq. (4.50), is different from the angle ϕ', which arose from the kinematics of the phase space in Eq. (4.57). The approximate equality of these two angles was eventually obtained in Eq. (4.49).

We did not consider the case where the momenta of the final state X are not integrated over. The simplicity of $\ell^{\mu\nu}$ in Eq. (3.42) follows from the integration over the phase space of X, which was done in Eq. (3.32). Without this integration, the general structure of $\ell^{\mu\nu}$, and similarly that of $K^{\mu\nu\rho\sigma}$, would be considerably more complicated, as additional momentum vectors would need to be included in the parametrizations.

The integration over the phase space of X becomes significant when experimental cuts are considered. One can add some experimental cuts directly to Eq. (4.68), but these would have to be written in terms of y_1, y_2, Q_1^2, and Q_2^2 alone. This was possible to do for the invariant mass $W_{\mu^+\mu^-}$ of the muon pair in Eq. (6.25) using Eq. (4.9), but for many other cuts it is not. Instead, one would have to use a differential cross section $\mathrm{d}\hat{\sigma}(\gamma\gamma \to X)$ in lieu of the fully integrated cross section $\hat{\sigma}(\gamma\gamma \to X)$ used in Eq. (4.68). For an example of such a cross section, see Ref. [13].

In Section 6, we considered photon-induced muon-pair production in proton-proton collisions as an example of the application of the equivalent photon approximation. We compared the equivalent photon approximation with three photon distributions, all based on dipole form factors of the proton, to a full calculation based on the same dipole form factors. The full calculation was compared to experimental data from the ATLAS experiment with the results agreeing to within 25 %, which is in line with our expectations considering the lack of finite-size corrections in the full calculation.

We compared the equivalent photon approximation and the full calculation at two total center-of-mass energies, $\sqrt{s} = 200\,\mathrm{GeV}$ and $\sqrt{s} = 13\,\mathrm{TeV}$. At $\sqrt{s} = 200\,\mathrm{GeV}$, we found significant differences between the three distributions. While the least approximative distribution (6.28) gave the best accuracy, the distribution (5.17) without the mass term was surprisingly more accurate than the distribution (5.16) with the mass term. At $\sqrt{s} = 13\,\mathrm{TeV}$, these two distributions gave more accurate results overall. This time, the distribution (5.16) with the mass term was slightly more accurate than the distribution (5.17) without it. The distribution (6.28) still provided the most accurate results.

Based on these results, we can attest to the usefulness of the equivalent photon approximation, and specifically of the photon distribution (5.17). Using Eq. (5.17), one can obtain fairly accurate results in muon-pair production with a particularly simple distribution formula. If one is willing to let go of the simple distribution (5.17), it is also easy to obtain very accurate results by using Eq. (6.28). The distribution (5.16), however, is not likely to find any use cases. This is because the distribution (5.16) is, at best, only as accurate as the simpler distribution (5.17), with it potentially being far less accurate in certain situations.

In the future one could, aside from not integrating over the final-state momenta, also test the equivalent photon approximation in other processes. Of course, many processes have already been considered in the literature, such as massive gauge boson production [26, 63], but usually with differing photon distributions. One could also consider finite-size corrections in the equivalent photon approximation. So far, finite-size corrections seem to have been included in the equivalent photon approximation by some semiclassical argument or using additional modeling; a purely field-theoretical derivation of an impact-parameter dependent photon distribution was not found in the general literature by the author during the preparation of this thesis.

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A Dirac equation

The Dirac equation is a relativistic wave equation describing massive spin-$\frac{1}{2}$ particles, also known as Dirac particles. In covariant notation, the Dirac equation reads [3]

$$i\gamma^\mu \partial_\mu \psi = m\psi, \tag{A.1}$$

where the γ-matrices fulfill the Dirac algebra[18]

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{A.2}$$

This algebra follows naturally from the Dirac ansatz and by requiring that the squared equation coincide with the Klein-Gordon equation. The γ-matrices are a set of four 4×4-matrices. While the γ-matrices are not four-vectors, it is conventional to use similar notation. In particular, lowering the index is done by the metric tensor, $\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu$. The γ-matrices have a number of useful properties, most notably [2]

$$\begin{aligned} (\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0, \\ (\gamma^0)^2 &= \mathbf{1}_4. \end{aligned} \tag{A.3}$$

Another consequence of the Dirac algebra is

$$\not{a}\not{a} = a_\mu a_\nu \gamma^\mu\gamma^\nu = \frac{1}{2} a_\mu a_\nu \left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) = a_\mu a_\nu g^{\mu\nu} = a^2, \tag{A.4}$$

where the slash notation $\not{a} \equiv a_\mu \gamma^\mu$ was used.

One set of solutions to Eq. (A.1) is given by plane waves of the form $\psi(x) = u(p)e^{-ip\cdot x}$, where $u(p)$ is a four-component spinor with momentum p. Substituting this into the Dirac equation (A.1) yields the Dirac equation in momentum space,

$$\not{p}u(p) = mu(p). \tag{A.5}$$

[18] To be precise, the algebra should read $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$, where $\mathbf{1}_4$ is the 4×4 identity matrix. However, this identity matrix is usually not written explicitly.

The adjoint of Eq. (A.5) can be derived by taking the hermitian conjugate of Eq. (A.5) and using Eq. (A.3), yielding

$$\begin{aligned} mu^\dagger(p) &= (\not{p}u(p))^\dagger = p_\mu(\gamma^\mu u(p))^\dagger = p_\mu u^\dagger(p)(\gamma^\mu)^\dagger \\ &= u^\dagger(p)\gamma^0\not{p}\gamma^0 = \overline{u}(p)\not{p}\gamma^0, \end{aligned} \tag{A.6}$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$ is the Dirac adjoint of ψ. Multiplying Eq. (A.6) by γ^0 from the right, we obtain the adjoint equation

$$\overline{u}(p)\not{p} = m\overline{u}(p). \tag{A.7}$$

Another set of solutions, the negative-energy solutions, are given by plane waves of the form $\psi(x) = v(p)e^{ip\cdot x}$. These spinors are usually associated with antiparticles and they satisfy the momentum-space Dirac equation [3]

$$\not{p}v(p) = -mv(p) \tag{A.8}$$

and the corresponding adjoint equation

$$\overline{v}(p)\not{p} = -m\overline{v}(p). \tag{A.9}$$

The spinors also satisfy the spin-sum formulas [3]

$$\begin{aligned} \sum_s u(k,s)\overline{u}(k,s) &= \not{k} + m, \\ \sum_s v(k,s)\overline{v}(k,s) &= \not{k} - m. \end{aligned} \tag{A.10}$$

Using Eq. (A.3), we can obtain another useful identity,

$$\begin{aligned} \left(\overline{f_1}\gamma^\mu f_2\right)^\dagger &= \left(f_1^\dagger\gamma^0\gamma^\mu f_2\right)^\dagger = f_2^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger f_1 = f_2^\dagger\gamma^0\gamma^\mu f_1 \\ &= \overline{f_2}\gamma^\mu f_1, \end{aligned} \tag{A.11}$$

where f_1, f_2 are some spinors (any combination of u and v). Equation (A.11) easily generalizes to

$$\left(\overline{f_1}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n} f_2\right)^\dagger = \overline{f_2}\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1} f_1.$$

A.1 Gordon decomposition identity

A useful consequence of Eqs. (A.5) and (A.7) is the Gordon decomposition identity. For simplicity, we denote $\overline{u}' \equiv \overline{u}(p')$ and $u \equiv u(p)$. Using Eqs. (A.2), (A.5), and (A.7),

$$\begin{aligned}
\overline{u}' i\sigma^{\mu\nu}(p'-p)_\nu u &= \frac{1}{2}\overline{u}'[\gamma^\mu,\gamma^\nu](p-p')_\nu u = \frac{1}{2}\overline{u}'\gamma^\mu(\not{p}-\not{p}')u - \frac{1}{2}\overline{u}'(\not{p}-\not{p}')\gamma^\mu u \\
&= \frac{1}{2}\overline{u}'\gamma^\mu(m-\not{p}')u - \frac{1}{2}\overline{u}'(\not{p}-m)\gamma^\mu u \\
&= m\overline{u}'\gamma^\mu u - \frac{1}{2}\overline{u}' p'_\nu \gamma^\mu\gamma^\nu u - \frac{1}{2}\overline{u}' p_\nu \gamma^\nu\gamma^\mu u \\
&= m\overline{u}'\gamma^\mu u - \frac{1}{2}\overline{u}' p'_\nu \left(2g^{\mu\nu} - \gamma^\nu\gamma^\mu\right) u - \frac{1}{2}\overline{u}' p_\nu \left(2g^{\nu\mu} - \gamma^\mu\gamma^\nu\right) u \\
&= m\overline{u}'\gamma^\mu u - \overline{u}'(p+p')^\mu u + \frac{1}{2}\overline{u}'\not{p}'\gamma^\mu u + \frac{1}{2}\overline{u}'\gamma^\mu\not{p}u \\
&= m\overline{u}'\gamma^\mu u - \overline{u}'(p+p')^\mu u + m\overline{u}'\gamma^\mu u \\
&= 2m\overline{u}'\left[\gamma^\mu - \frac{(p+p')^\mu}{2m}\right]u,
\end{aligned}$$

which gives the Gordon decomposition identity

$$\overline{u}'\gamma^\mu u = \overline{u}'\left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u = \overline{u}'\left[\frac{(p+p')^\mu}{2m} - \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u. \tag{A.12}$$

A.2 Trace techniques

The following standard trace identities are taken from Ref. [3, §A.3]:

$$\begin{aligned}
\operatorname{tr}\mathbf{1}_4 &= 4, \\
\operatorname{tr}(\text{odd number of }\gamma\text{-matrices}) &= 0, \\
\operatorname{tr}(\gamma_\mu\gamma_\nu) &= 4g_{\mu\nu}, \\
\operatorname{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) &= 4\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right).
\end{aligned} \tag{A.13}$$

These can be derived using the Dirac algebra, Eq. (A.2), and properties of the trace, namely linearity and cyclicity.

For traces containing six or more γ-matrices, one can use the Dirac algebra to write the traces in terms of traces containing a product of four γ-matrices. However, in practice, this is best left to a computer algebra system.

Denoting $\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, we can use Eq. (A.13) to derive

$$\begin{aligned}
\mathrm{tr}(\gamma_\mu\gamma_\nu\sigma_{\rho\lambda}) &= \frac{i}{2}\,\mathrm{tr}\left(\gamma_\mu\gamma_\nu\left[\gamma_\rho, \gamma_\lambda\right]\right) = \frac{i}{2}\,\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda) - \frac{i}{2}\,\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho) \\
&= 2i\left[g_{\mu\nu}g_{\rho\lambda} - g_{\mu\rho}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\rho} - (g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda})\right] \\
&= 4i(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})
\end{aligned} \tag{A.14}$$

and therefore

$$\begin{aligned}
\mathrm{tr}(\not{a}\gamma_\nu\sigma_{\rho\lambda}b^\lambda) &= a^\mu b^\lambda\,\mathrm{tr}(\gamma_\mu\gamma_\nu\sigma_{\rho\lambda}) = 4i((a\cdot b)g_{\nu\rho} - a_\rho b_\nu), \\
\mathrm{tr}(\gamma_\mu\not{a}\sigma_{\rho\lambda}b^\lambda) &= a^\nu b^\lambda\,\mathrm{tr}(\gamma_\mu\gamma_\nu\sigma_{\rho\lambda}) = 4i(a_\rho b_\mu - (a\cdot b)g_{\mu\rho}).
\end{aligned} \tag{A.15}$$

B Quantum electrodynamics

Quantum electrodynamics (QED) couples charged spin-$\frac{1}{2}$ Dirac particles to a photon field. In the case of a single particle species, this theory is described by the Lagrangian

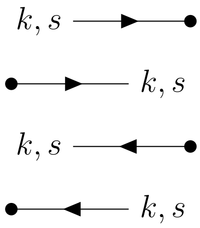
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{\partial} - m)\psi - e\overline{\psi}\gamma^{\mu}\psi A_{\mu}, \quad \text{(B.1)}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, e is the elementary charge and $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. The Lagrangian in Eq. (B.1) is invariant under a local U(1) gauge transformation

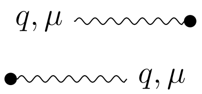
$$\begin{aligned} \psi &\to \psi' = e^{i\alpha(x)}\psi, \\ A_\mu &\to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha(x), \end{aligned}$$

where $\alpha(x)$ is a real-valued function. [3] The Feynman rules for QED in the Feynman gauge [3, §A.1] are listed below. In all cases, the momentum flows from left to right.

External fermions



External photons

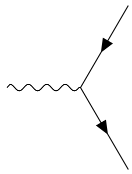


Propagators

$$k \longrightarrow k \;=\; \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} \qquad \text{(fermion propagator)}$$

$$q,\mu \sim\sim\sim q,\nu \;=\; -\frac{ig^{\mu\nu}}{q^2+i\epsilon} \qquad \text{(photon propagator)}$$

Vertices

μ  $= iZe\gamma^\mu$ (QED vertex)

B.1 The Ward identity

One important consequence of the gauge invariance of QED is the so-called Ward identity. Consider some QED process which includes a photon with momentum q, shown in Figure 9. The amplitude can be written as $\mathcal{M} = \varepsilon_\mu(q)\mathcal{M}^\mu$, where $\varepsilon_\mu(q)$ is the polarization vector of the photon, $\mathcal{M}^\mu$ is the amplitude of the rest of the diagram, and all relevant diagrams have been summed over. The Ward identity then states that [3]

$$q_\mu \mathcal{M}^\mu = 0.$$

Current conservation, which follows from gauge invariance, is implemented at the amplitude level by the Ward identity.

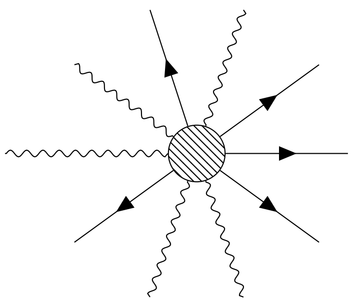
q, μ 

Figure 9. A diagram of some generic QED process with invariant amplitude $\mathcal{M} = \varepsilon_\mu(q)\mathcal{M}^\mu$, where $\varepsilon_\mu(q)$ is the polarization vector of a photon with momentum q and $\mathcal{M}^\mu$ is the invariant amplitude of the rest of the diagram to which the photon is connected.

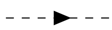
B.2 Scalar quantum electrodynamics

In scalar QED, scalar particles are coupled to photons, instead of Dirac particles as is the case in standard QED. Scalar QED is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi,$$

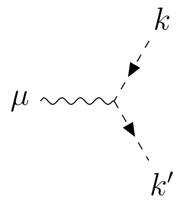
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field tensor, $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative and m is the mass of the scalar particle. The Feynman rules for scalar QED are listed below. [3, Problem 9.1]

Propagators

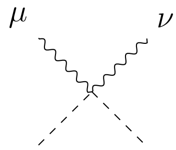
k  k $= \dfrac{i}{k^2 - m^2 + i\epsilon}$ (scalar propagator)

q, μ q, ν $= -\dfrac{ig^{\mu\nu}}{q^2 + i\epsilon}$ (photon propagator)

Vertices



$= -ie(k + k')^\mu$ (sQED vertex)



$= 2ie^2 g^{\mu\nu}$ (seagull vertex)

C Calculation of the emission tensor for Dirac particles

This section shows in detail the calculation of the emission tensor $W_{\mu\nu}$ starting from Eq. (5.7). Here we denote the spinors as $u \equiv u(p,s)$ and $u' \equiv u(p',s')$, and similarly for $\overline{u}$ and $\overline{u}'$. Squaring the vertex of the form $H_\mu = \overline{u}'\Gamma_\mu u$,

$$\begin{aligned} H_\mu H_\nu^* &= \overline{u}'\Gamma_\mu u \left[\overline{u}'\Gamma_\nu u\right]^\dagger = \overline{u}'\Gamma_\mu u \left[(u')^\dagger \gamma^0 \Gamma_\nu u\right]^\dagger \\ &= \overline{u}'\Gamma_\mu u \left[u^\dagger \Gamma_\nu^\dagger (\gamma^0)^\dagger u'\right] = \overline{u}'\Gamma_\mu u u^\dagger \gamma^0\gamma^0\Gamma_\nu^\dagger\gamma^0 u' \\ &= \overline{u}'\Gamma_\mu u \overline{u}\left(\gamma^0\Gamma_\nu^\dagger\gamma^0\right) u'. \end{aligned} \tag{C.1}$$

Using standard trace techniques, the matrix structure of Eq. (C.1) can be made explicit by writing

$$H_\mu H_\nu^* = \overline{u}'_A (\Gamma_\mu)_{AB} u_B \overline{u}_C \left(\gamma^0\Gamma_\nu^\dagger\gamma^0\right)_{CD} u'_D = u'_D \overline{u}'_A (\Gamma_\mu)_{AB} u_B \overline{u}_C \left(\gamma^0\Gamma_\nu^\dagger\gamma^0\right)_{CD}.$$

Summing over the final-state spin s', averaging over the initial-state spin s, and using the spin-projection operator

$$\sum_s u_A \overline{u}_B = (\not{p} + M)_{AB}$$

from Eq. (A.10), we obtain the emission tensor

$$\begin{aligned} W_{\mu\nu} = \overline{H_\mu H_\nu^*} &= \frac{1}{2}\sum_{s'} u'_D \overline{u}'_A (\Gamma_\mu)_{AB} \sum_s u_b \overline{u}_C \left(\gamma^0\Gamma_\nu^\dagger\gamma^0\right)_{CD} \\ &= \frac{1}{2}(\not{p}' + M)_{DA}(\Gamma_\mu)_{AB}(\not{p} + M)_{BC}\left(\gamma^0\Gamma_\nu^\dagger\gamma^0\right)_{CD} \\ &= \frac{1}{2}\operatorname{tr}\left[(\not{p}' + M)\Gamma_\mu(\not{p} + M)\gamma^0\Gamma_\nu^\dagger\gamma^0\right]. \end{aligned}$$

Since

$$\gamma^0\sigma_{\mu\nu}^\dagger\gamma^0 = -\frac{i}{2}\gamma^0(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)^\dagger\gamma^0 = -\frac{i}{2}\gamma^0(\gamma_\nu^\dagger\gamma_\mu^\dagger - \gamma_\mu^\dagger\gamma_\nu^\dagger)\gamma^0 = \sigma_{\mu\nu},$$

we have, using the explicit form of Γ_μ found in Eq. (5.7), that

$$\gamma^0\Gamma_\mu^\dagger\gamma^0 = F_1\gamma^0\gamma_\mu^\dagger\gamma^0 + \frac{i\kappa}{2M}F_2\gamma^0\sigma_{\mu\nu}^\dagger\gamma^0 q^\nu = F_1\gamma_\mu + \frac{i\kappa}{2M}F_2\sigma_{\mu\nu}q^\nu.$$

Therefore

$$\begin{aligned}
W_{\mu\nu} &= \frac{1}{2}\,\mathrm{tr}\left[(\not{p}' + M)\left(F_1\gamma_\mu - \frac{i\kappa}{2M}F_2\sigma_{\mu\lambda}q^\lambda\right)\right.\\
&\qquad\left.\times(\not{p} + M)\left(F_1\gamma_\nu + \frac{i\kappa}{2M}F_2\sigma_{\nu\rho}q^\rho\right)\right]\\
&= \frac{1}{2}F_1^2\,\mathrm{tr}\left[(\not{p}' + M)\gamma_\mu(\not{p} + M)\gamma_\nu\right]\\
&\quad + \frac{i\kappa}{4M}F_1F_2\,\mathrm{tr}\left[(\not{p}' + M)\gamma_\mu(\not{p} + M)\sigma_{\nu\rho}q^\rho\right]\\
&\quad - \frac{i\kappa}{4M}F_1F_2\,\mathrm{tr}\left[(\not{p}' + M)\sigma_{\mu\lambda}q^\lambda(\not{p} + M)\gamma_\nu\right]\\
&\quad + \frac{\kappa^2}{8M^2}F_2^2\,\mathrm{tr}\left[(\not{p}' + M)\sigma_{\mu\lambda}q^\lambda(\not{p} + M)\sigma_{\nu\rho}q^\rho\right]\\
&\equiv \frac{1}{2}F_1^2\,\mathrm{tr}_1 + \frac{i\kappa}{4M}F_1F_2(\mathrm{tr}_2 - \mathrm{tr}_3) + \frac{\kappa^2}{8M^2}F_2^2\,\mathrm{tr}_4\,.
\end{aligned} \tag{C.2}$$

The first trace tr_1 has the form of the standard leptonic tensor. Replacing the momentum p' in favor of $q = p - p'$ and using Eq. (3.3), we have [3, §5]

$$\begin{aligned}
\mathrm{tr}_1 \equiv \mathrm{tr}\left[(\not{p}' + M)\gamma_\mu(\not{p} + M)\gamma_\nu\right] &= 4(p_\mu p'_\nu + p_\nu p'_\mu - (p\cdot p' - M^2)g_{\mu\nu})\\
= 4(2p_\mu p_\nu - p_\mu q_\nu - p_\nu q_\mu) + 2q^2 g_{\mu\nu}.
\end{aligned} \tag{C.3}$$

For the three remaining traces, the fact that the trace of an odd product of γ-matrices is zero is extensively used to simplify the expressions without explicitly writing the vanishing traces. Thus, using Eqs. (3.3), (3.4), and (A.15),

$$\begin{aligned}
\mathrm{tr}_2 &\equiv \mathrm{tr}\left[(\not{p}' + M)\gamma_\mu(\not{p} + M)\sigma_{\nu\rho}q^\rho\right]\\
&= M\,\mathrm{tr}(\not{p}'\gamma_\mu\sigma_{\nu\rho}q^\rho) + M\,\mathrm{tr}(\gamma_\mu\not{p}\sigma_{\nu\rho}q^\rho)\\
&= 4iM((q\cdot p')g_{\mu\nu} - q_\mu p'_\nu) + 4iM(q_\mu p_\nu - (q\cdot p)g_{\mu\nu})\\
&= 4iM\left(-\frac{1}{2}q^2 g_{\mu\nu} - q_\mu p'_\nu\right) + 4iM\left(q_\mu p_\nu - \frac{1}{2}q^2 g_{\mu\nu}\right)\\
&= 4iMq_\mu(p_\nu - p'_\nu) - 4iMq^2 g_{\mu\nu}\\
&= 4iM(q_\mu q_\nu - q^2 g_{\mu\nu}).
\end{aligned} \tag{C.4}$$

Similarly,

$$\begin{aligned}
\mathrm{tr}_3 &\equiv \mathrm{tr}\left[(\not{p}' + M)\sigma_{\mu\lambda}q^\lambda(\not{p} + M)\gamma_\nu\right] \\
&= M\,\mathrm{tr}(\not{p}'\sigma_{\mu\lambda}q^\lambda\gamma_\nu) + M\,\mathrm{tr}(\sigma_{\mu\lambda}q^\lambda\not{p}\gamma_\nu) \\
&= M\,\mathrm{tr}(\gamma_\nu\not{p}'\sigma_{\mu\lambda}q^\lambda) + M\,\mathrm{tr}(\not{p}\gamma_\nu\sigma_{\mu\lambda}q^\lambda) \\
&= 4iM((q\cdot p)g_{\mu\nu} - q_\nu p_\mu) + 4iM(q_\nu p'_\mu - (q\cdot p')g_{\mu\nu}) \\
&= 4iM\left(\frac{1}{2}q^2 g_{\mu\nu} - q_\nu p_\mu\right) + 4iM\left(q_\nu p'_\mu + \frac{1}{2}q^2 g_{\mu\nu}\right) \\
&= 4iMq^2 g_{\mu\nu} - 4iMq_\nu(p_\mu - p'_\mu) \\
&= 4iM(q^2 g_{\mu\nu} - q_\mu q_\nu).
\end{aligned} \tag{C.5}$$

For the last trace,

$$\begin{aligned}
\mathrm{tr}_4 &\equiv \mathrm{tr}\left[(\not{p}' + M)\sigma_{\mu\lambda}q^\lambda(\not{p} + M)\sigma_{\nu\rho}q^\rho\right] \\
&= \mathrm{tr}(\not{p}'\sigma_{\mu\lambda}q^\lambda\not{\sigma}_{\nu\rho}q^\rho) + M^2\,\mathrm{tr}(\sigma_{\mu\lambda}q^\lambda\sigma_{\nu\rho}q^\rho) \\
&= -\frac{1}{4}\,\mathrm{tr}(\not{p}'[\gamma_\mu,\not{q}]\not{p}[\gamma_\nu,\not{q}]) - \frac{1}{4}M^2\,\mathrm{tr}([\gamma_\mu,\not{q}][\gamma_\nu,\not{q}]) \\
&= -\frac{1}{4}p_\alpha q_\beta q_\lambda p'_\rho\Big[\mathrm{tr}(\gamma_\rho\gamma_\mu\gamma_\beta\gamma_\alpha\gamma_\nu\gamma_\lambda) - \mathrm{tr}(\gamma_\rho\gamma_\mu\gamma_\beta\gamma_\alpha\gamma_\lambda\gamma_\nu) \\
&\qquad\qquad - \mathrm{tr}(\gamma_\rho\gamma_\beta\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\lambda) + \mathrm{tr}(\gamma_\rho\gamma_\beta\gamma_\mu\gamma_\alpha\gamma_\alpha\gamma_\nu)\Big] \\
&\quad - \frac{M^2}{2}\left[\mathrm{tr}(\gamma_\mu\not{q}\gamma_\nu\not{q}) - q^2\,\mathrm{tr}(\gamma_\mu\gamma_\nu)\right] \\
&= -4p_\alpha q_\beta q_\lambda p'_\rho\Big[g_{\nu\rho}(g_{\alpha\mu}g_{\beta\lambda} - g_{\alpha\beta}g_{\lambda\mu}) + g_{\alpha\nu}(g_{\beta\lambda}g_{\mu\rho} - g_{\beta\rho}g_{\lambda\mu}) - g_{\alpha\lambda}g_{\beta\nu}g_{\mu\rho} \\
&\qquad\qquad + g_{\alpha\rho}(g_{\beta\nu}g_{\lambda\mu} - g_{\beta\lambda}g_{\mu\nu}) + g_{\alpha\lambda}g_{\beta\rho}g_{\mu\nu} + g_{\alpha\beta}g_{\lambda\rho}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu}g_{\lambda\rho}\Big] \\
&\quad - 2M^2\left[q_\alpha q_\beta(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\alpha\nu}) - q^2 g_{\mu\nu}\right],
\end{aligned}$$

where FeynCalc was used on the last line to simplify the trace calculation. Using Eqs. (3.3) and (3.4), the trace simplifies to

$$\begin{aligned}
\mathrm{tr}_4 = -4\Big[&p'_\nu(p_\mu q^2 - (q\cdot p)q_\mu) + p_\nu(q^2 p'_\mu - (q\cdot p')q_\mu) - (q\cdot p)q_\nu p'_\mu \\
&+ (p\cdot p')(q_\nu q_\mu - q^2 g_{\mu\nu}) + (q\cdot p)(q\cdot p')g_{\mu\nu} + (q\cdot p)(q\cdot p')g_{\mu\nu} - p_\mu q_\nu(q\cdot p')\Big] \\
&- 4M^2(q_\mu q_\nu - q^2 g_{\mu\nu}) \\
= -4\Big[&p_\mu p'_\nu q^2 - \frac{1}{2}p'_\nu q_\mu q^2 + p_\nu p'_\mu q^2 + \frac{1}{2}p_\nu q_\mu q^2 - \frac{1}{2}q^2 q_\nu p'_\mu \\
&+(p\cdot p')(q_\mu q_\nu - q^2 g_{\mu\nu}) - \frac{1}{4}q^4 g_{\mu\nu} - \frac{1}{4}q^4 g_{\mu\nu} + \frac{1}{2}p_\mu q_\nu q^2\Big] - 4M^2(q_\mu q_\nu - q^2 g_{\mu\nu})
\end{aligned}$$

$$
\begin{aligned}
&= -4\left[p_\mu p'_\nu q^2 + p_\nu p'_\mu q^2 + \frac{1}{2}(p_\nu - p'_\nu)q_\mu q^2 + \frac{1}{2}(p_\mu - p'_\mu)q_\nu q^2 \right.\\
&\qquad \left. + \left(M^2 - \frac{1}{2}q^2\right)(q_\mu q_\nu - q^2 g_{\mu\nu}) - \frac{1}{2}q^4 g_{\mu\nu}\right] - 4M^2(q_\mu q_\nu - q^2 g_{\mu\nu})\\
&= -4q^2(p_\mu p'_\nu + p_\nu p'_\mu) - 4q_\mu q_\nu q^2 - 4M^2(q_\mu q_\nu - q^2 g_{\mu\nu})\\
&\quad + 2q^2(q_\mu q_\nu - q^2 g_{\mu\nu}) + 2q^4 g_{\mu\nu} - 4M^2(q_\mu q_\nu - q^2 g_{\mu\nu})\\
&= -4q^2(p_\mu p'_\nu + p_\nu p'_\mu) - 2q_\mu q_\nu q^2 - 8M^2(q_\mu q_\nu - q^2 g_{\mu\nu}).
\end{aligned}
$$

Further rewriting $p' = p - q$, we obtain

$$
\mathrm{tr}_4 = -4q^2(2p_\mu p_\nu - p_\mu q_\nu - p_\nu q_\mu) - 2q_\mu q_\nu q^2 - 8M^2(q_\mu q_\nu - q^2 g_{\mu\nu}). \tag{C.6}
$$

Plugging Eqs. (C.3)–(C.6) back into Eq. (C.2),

$$
\begin{aligned}
W_{\mu\nu} &= 2F_1^2(2p_\mu p_\nu - p_\mu q_\nu - p_\nu q_\mu) + F_1^2 q^2 g_{\mu\nu} - 2\kappa F_1 F_2(q_\mu q_\nu - q^2 g_{\mu\nu})\\
&\quad - \frac{\kappa^2}{4M^2}F_2^2\left(2q^2(2p_\mu p_\nu - p_\mu q_\nu - p_\nu q_\mu) + q_\mu q_\nu q^2 + 4M^2(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\\
&= F_1^2\left((2p-q)_\mu(2p-q)_\nu - q_\mu q_\nu\right) + F_1^2 q^2 g_{\mu\nu} - 2\kappa F_1 F_2(q_\mu q_\nu - q^2 g_{\mu\nu})\\
&\quad - \frac{\kappa^2 q^2}{4M^2}F_2^2(2p-q)_\mu(2p-q)_\nu - \kappa^2 F_2^2(q_\mu q_\nu - q^2 g_{\mu\nu})\\
&= \left(F_1^2 - \frac{\kappa^2 q^2}{4M^2}F_2^2\right)(2p-q)_\mu(2p-q)_\nu + F_1^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\\
&\quad + 2\kappa F_1 F_2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right) + \kappa^2 F_2^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\\
&= \left(F_1^2 - \frac{\kappa^2 q^2}{4M^2}F_2^2\right)(2p-q)_\mu(2p-q)_\nu + (F_1 + \kappa F_2)^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\\
&\equiv \left(F_1^2 + \tau\kappa^2 F_2^2\right)(2p-q)_\mu(2p-q)_\nu + (F_1 + \kappa F_2)^2\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right),
\end{aligned} \tag{C.7}
$$

where on the last step we defined $\tau \equiv -q^2/4M^2 = Q^2/4M^2$. We can also write Eq. (C.7) using the so-called Sachs form factors [2]

$$
\begin{aligned}
G_E(Q^2) &\equiv F_1(Q^2) - \tau\kappa F_2(Q^2),\\
G_M(Q^2) &\equiv F_1(Q^2) + \kappa F_2(Q^2).
\end{aligned} \tag{C.8}
$$

Inverting Eq. (C.8),

$$F_1(Q^2) = \frac{G_E(Q^2) + \tau G_M(Q^2)}{1+\tau},$$

$$F_2(Q^2) = \frac{G_M(Q^2) - G_E(Q^2)}{\kappa(1+\tau)}.$$

From this it follows that

$$\begin{aligned} F_1^2(Q^2) + \tau\kappa^2 F_2^2(Q^2) &= \frac{\left(G_E(Q^2) + \tau G_M(Q^2)\right)^2}{(1+\tau)^2} + \frac{\tau\left(G_M(Q^2) - G_E(Q^2)\right)^2}{(1+\tau)^2} \\ &= \frac{(1+\tau)G_E^2(Q^2) + \tau(1+\tau)G_M^2(Q^2)}{(1+\tau)^2} \\ &= \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau}. \end{aligned}$$

Thus, Eq. (C.7) finally becomes

$$W_{\mu\nu} = \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau}(2p-q)_\mu(2p-q)_\nu + G_M^2(Q^2)\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right).$$

D Integration of the equivalent photon distribution for the dipole form factors

This section goes through the calculation of the integral

$$I \equiv \int_{Q^2_{\rm min}}^{\infty} \frac{\mathrm{d}Q^2}{Q^2} \left(1 + Q^2/\Lambda^2\right)^{-4} \left[\frac{1+(1-y)^2}{y} - \frac{2yM^2}{Q^2}\right], \tag{D.1}$$

which is a part of Eq. (5.15). Splitting the integral into two, we need to calculate the integrals

$$I_n \equiv \int_{Q^2_{\rm min}}^{\infty} \frac{\mathrm{d}Q^2}{(Q^2)^n} \left(1 + Q^2/\Lambda^2\right)^{-4}, \tag{D.2}$$

where $n = 1, 2$. Setting $u = 1 + Q^2/\Lambda^2$ and defining $B \equiv 1 + Q^2_{\rm min}/\Lambda^2 > 1$, we can change variables in Eq. (D.2) to obtain

$$I_n = \int_B^{\infty} \frac{\mathrm{d}u\,\Lambda^2}{u^4\left[\Lambda^2(u-1)\right]^n} = \frac{1}{(\Lambda^2)^{n-1}} \int_B^{\infty} \frac{\mathrm{d}u}{u^4(u-1)^n}.$$

Using partial fractions,

$$I_1 = -\int_B^{\infty} \mathrm{d}u \left(\frac{1}{1-u} + \frac{1}{u}\right) - \frac{1}{B} - \frac{1}{2B^2} - \frac{1}{3B^3}. \tag{D.3}$$

The remaining integral in Eq. (D.3) is

$$\begin{aligned} -\int_B^{\infty} \mathrm{d}u \left(\frac{1}{1-u} + \frac{1}{u}\right) &= -\left[\log u - \log(1-u)\right]_B^{\infty} \\ &= \lim_{u\to\infty} \log\frac{1-u}{u} + \log B - \log(1-B) \\ &= \log(-1) + \log B - \log(-1) - \log(B-1) \\ &= \log B - \log(B-1). \end{aligned} \tag{D.4}$$

Defining $A \equiv 1 + \Lambda^2/Q^2_{\rm min} > 1$, it is easy to see that $B = A/(A-1)$. Using this and Eq. (D.4), Eq. (D.3) becomes

$$\begin{aligned}
I_1 &= \log B - \log(B-1) - \frac{1}{B} - \frac{1}{2B^2} - \frac{1}{3B^3} \\
&= \log\frac{A}{A-1} - \log\frac{1}{A-1} - \frac{A-1}{A} - \frac{(A-1)^2}{2A^2} - \frac{(A-1)^3}{3A^3} \\
&= \log A - \log(A-1) - \log 1 + \log(A-1) - \frac{11A^3 - 18A^2 + 9A - 2}{6A^3} \\
&= \log A - \frac{11}{6} + \frac{3}{A} - \frac{3}{2A^2} + \frac{1}{3A^3}.
\end{aligned} \tag{D.5}$$

Similarly, we can use partial fractions, $B = A/(A-1)$, and Eq. (D.4) to obtain

$$\begin{aligned}
\Lambda^2 I_2 &= \int_B^\infty \frac{\mathrm{d}u}{u^4(u-1)^2} \\
&= \int_B^\infty \frac{\mathrm{d}u}{(u-1)^2} + 4\int_B^\infty \mathrm{d}u\left(\frac{1}{1-u} + \frac{1}{u}\right) + \frac{3}{B} + \frac{1}{B^2} + \frac{1}{3B^3} \\
&= \frac{1}{B-1} - 4\left(\log B - \log(B-1)\right) + \frac{3}{B} + \frac{1}{B^2} + \frac{1}{3B^3} \\
&= A - 1 - 4\log A + \frac{3(A-1)}{A} + \frac{(A-1)^2}{A^2} + \frac{(A-1)^3}{3A^3} \\
&= A - 1 - 4\log A + \frac{13A^3 - 18A^2 + 6A - 1}{3A^3} \\
&= A - 4\log A + \frac{10}{3} - \frac{6}{A} + \frac{2}{A^2} - \frac{1}{3A^3}.
\end{aligned} \tag{D.6}$$

Combining Eqs. (D.5) and (D.6), Eq. (D.1) is thus given by

$$\begin{aligned}
I &= \int_{Q^2_{\rm min}}^\infty \frac{\mathrm{d}Q^2}{Q^2}\left(1 + Q^2/\Lambda^2\right)^{-4}\left[\frac{1+(1-y)^2}{y} - \frac{2yM^2}{Q^2}\right] \\
&= \frac{1+(1-y)^2}{y} I_1 - 2yM^2 I_2 \\
&= \frac{1+(1-y)^2}{y}\left[\log A - \frac{11}{6} + \frac{3}{A} - \frac{3}{2A^2} + \frac{1}{3A^3}\right] \\
&\quad - \frac{2yM^2}{\Lambda^2}\left[A - 4\log A + \frac{10}{3} - \frac{6}{A} + \frac{2}{A^2} - \frac{1}{3A^3}\right].
\end{aligned}$$

E Parametrization of the two-photon absorption tensor

In this appendix we show that Eq. (4.24) holds. Since the final state X has been integrated over, $K^{\mu\nu\rho\sigma}$ can only depend on q_1 and q_2. In addition, it has the symmetry $K^{\mu\nu\rho\sigma} = K^{\rho\sigma\mu\nu}$ (see Eq. (4.15)). Therefore, the only available tensors are q_1^μ, q_2^μ, and $g^{\mu\nu}$, out of which we can construct $K^{\mu\nu\rho\sigma}$. There are three types of combinations: two metric tensors, one metric tensor and two vectors, and four vectors. Of the first type, there are three terms:

$$g^{\mu\nu}g^{\rho\sigma}, g^{\mu\rho}g^{\nu\sigma}, g^{\mu\sigma}g^{\nu\rho}.$$

For the terms with one metric tensor and two vectors, we must consider the symmetry $K^{\mu\nu\rho\sigma} = K^{\rho\sigma\mu\nu}$ in the exchange of indices $\mu\nu \leftrightarrow \rho\sigma$. For example, the term $g^{\mu\nu}q_1^\rho q_1^\sigma$ is not symmetric in the exchange $\mu\nu \leftrightarrow \rho\sigma$. In order to keep the symmetry of $K^{\mu\nu\rho\sigma}$, there must be an accompanying term $g^{\rho\sigma}q_1^\mu q_1^\nu$. The term $g^{\mu\rho}q_1^\nu q_1^\sigma$, on the other hand, is automatically symmetric in the exchange $\mu\nu \leftrightarrow \rho\sigma$.

A systematic approach to symmetrization that can be easily implemented on a computer algebra system is based on permutations. We construct an ordered list of permutations l_1 based on the permutations of the indices (μ, ν, ρ, σ) in some specific order. The ordering itself does not matter, only that the same one is always used. As an example, the default ordering of the MATHEMATICA function `Permutations` gives

$$l_1 = ((\mu,\nu,\rho,\sigma),(\mu,\nu,\sigma,\rho),(\mu,\rho,\nu,\sigma),\ldots,(\sigma,\rho,\nu,\mu)). \tag{E.1}$$

We then construct another ordered list l_2 based on the permutations of (ρ, σ, μ, ν):

$$l_2 = ((\rho,\sigma,\mu,\nu),(\rho,\sigma,\nu,\mu),(\rho,\mu,\sigma,\nu),\ldots,(\nu,\mu,\sigma,\rho)). \tag{E.2}$$

The ordered lists l_1 and l_2 contain the same elements but in different orders.

From these two lists of indices we construct two more lists, L_1 and L_2, containing terms of the form

$$g^{\mu_1\mu_2} q_i^{\mu_3} q_j^{\mu_4}, \tag{E.3}$$

where the indices are applied in order. Based on Eqs. (E.1) and (E.2),

$$L_1 = (g^{\mu\nu} q_1^{\rho} q_1^{\sigma}, g^{\mu\nu} q_1^{\rho} q_2^{\sigma}, \dots, g^{\sigma\rho} q_2^{\nu} q_2^{\mu})$$

and

$$L_2 = (g^{\rho\sigma} q_1^{\mu} q_1^{\nu}, g^{\rho\sigma} q_1^{\mu} q_2^{\nu}, \dots, g^{\nu\mu} q_2^{\sigma} q_2^{\rho}).$$

It should be noted that the lists L_1 and L_2 initially contain duplicate elements, but these can easily be removed. Again, L_1 and L_2 have the same elements but in different orders.

The next step is finding a permutation between L_1 and L_2. That is, a function that maps the elements of L_1 to elements of L_2 in order. For the lists we have considered here, that permutation is given by

$$P = (1\ 12)(2\ 22)(3\ 23)\cdots(18\ 19). \tag{E.4}$$

The permutation in Eq. (E.4) is denoted using transpositions $(i\ j)$, which in this context means that the i^{th} element of L_1 equals the j^{th} element of L_2. We can use this permutation to construct pairs of symmetric elements,

$$\begin{pmatrix} g^{\mu\nu} q_1^{\rho} q_1^{\sigma} & g^{\rho\sigma} q_1^{\mu} q_1^{\nu} \\ g^{\mu\nu} q_1^{\rho} q_2^{\sigma} & g^{\rho\sigma} q_1^{\mu} q_2^{\nu} \\ \vdots & \vdots \\ g^{\sigma\rho} q_2^{\nu} q_2^{\mu} & g^{\nu\mu} q_2^{\sigma} q_2^{\rho} \end{pmatrix}. \tag{E.5}$$

Here, the first column is L_1 and the second column is L_1 under the permutation P. Summing the terms in Eq. (E.5) yields a list of 24 symmetric terms of the form shown in Eq. (E.3).

Similarly, one can construct the 16 symmetrized terms of the form

$$q_i^{\mu_1} q_j^{\mu_2} q_k^{\mu_3} q_l^{\mu_4}.$$

All in all, there are $3 + 24 + 16 = 43$ terms in the parametrization of $K^{\mu\nu\rho\sigma}$. These terms can of course be constructed by hand, but using a computer algebra system eliminates the otherwise very real possibility of careless errors.

We now have a parametrization $\tilde{K}^{\mu\nu\rho\sigma}$ of $K^{\mu\nu\rho\sigma}$ with 43 unknown parameters, denoted by $\{\beta_i\}_{i=1}^{43}$. The Ward identity (Eq. (4.23)) places further restrictions on the form of $\tilde{K}^{\mu\nu\rho\sigma}$. The two contractions give a number of equations from the remaining independent terms. This is analogous to how in the one-photon exchange case, the single Ward identity $q_\mu \ell^{\mu\nu} = 0$ resulted in two equations in Eq. (3.44). One can then solve these equations generated by the Ward identity, which reduces the number of unknown parameters to 15.

To see that $\tilde{K}^{\mu\nu\rho\sigma}$ is indeed equivalent to Eq. (4.24), one can calculate the unknown parameters $\{a_i\}_{i=1}^{8}$ in Eq. (4.24) using $\tilde{K}^{\mu\nu\rho\sigma}$. For example,

$$\begin{aligned} a_1 &= \frac{1}{8}\left(3P^{\mu\nu}P^{\rho\sigma} - P^{\mu\rho}P^{\nu\sigma} - P^{\mu\sigma}P^{\nu\rho}\right)\tilde{K}_{\mu\nu\rho\sigma} \\ &= -(\beta_{34}+\beta_{37})q_1^2 q_2^2 - (\beta_6+\beta_{26})(q_1\cdot q_2) - (\beta_{29}+\beta_{32}q_1^2+\beta_9+\beta_{10})\frac{q_1^2 q_2^2}{q_1\cdot q_2}. \end{aligned}$$

This gives the parameters $\{a_i\}$ in terms of $\{\beta_j\}$. Substituting the parameters $\{a_i\}$ written in terms of $\{\beta_j\}$ back into Eq. (4.24), one can then show that $K^{\mu\nu\rho\sigma} = \tilde{K}^{\mu\nu\rho\sigma}$.

The MATHEMATICA code which constructs the parametrization $\tilde{K}^{\mu\nu\rho\sigma}$ and shows its equivalence with Eq. (4.24) using the methods described here is available on GITHUB[19].

[19] https://github.com/samiyr/fyss9490