DEPARTMENT OF PHYSICS, UNIVERSITY OF JYVÄSKYLÄ
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RIGIDITY AND TRANSIENT WAVE DYNAMICS OF RANDOM NETWORKS

BY
MARKKU KELLOMÄKI

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Preface

The studies reviewed in this thesis have been carried out during the years 1993-1997 at the Department of Physics in the University of Jyväskylä. I express my best thanks to my supervisor, Prof. Jussi Timonen, for challenging topics of research and inspiring guidance. I am much indebted to Dr. Jan Åström for fruitful collaboration and being a great friend. I thank Dr. Seppo Valkealahti and Dr. Markku Kataja for help in both technical and physical problems. I wish to thank the Solid State and Paper Technology groups for inspiring scientific cooperation. The excellent working conditions at the Department, and at VTT Energy (during the finalising of this thesis) were most valuable. Financial supports from KTM, OPM, Tekniikan edistämissäätiö and University of Jyväskylä are gratefully acknowledged.

I want to express my warm thanks to my longtime friend and colleague Mr. Antti Koponen for many shared moments and good humour during our studies and thesis works.

Finally, I am most grateful to you my beloved wife Marjaana and son Joona for your love and support. You made it possible for me to manage the hard work.

Jyväskylä, May 19, 1998

Markku Kellomäki

Abstract

This thesis work deals with two major topics: the rigidity of random line networks and transient wave-front propagation in random discrete media. Rigidity means the ability of a mechanical system to store elastic energy when it is deformed. Random line networks composed of springs are found to be nonrigid at their basic configuration. This is confirmed both by a relaxation method and by a topological algorithm. These networks are found to become rigid at large strains for which the nonlinear stress-strain behaviour is analysed. A reinforced random line network is found to have a transition between rigid and nonrigid phases. The numerical evidence does not support a pure second-order phase transition.

The velocity and amplitude of the leading front of elastic waves propagating in various one- and two-dimensional networks are studied. The leading front is defined in this work as the first displacement maxima of the points initially at rest in the network. In perfect lattices the amplitude first oscillates and then decays following a power law. With increasing disorder the early-time behaviour of the front changes to an exponential decay. The decay coefficient is a universal power-law as a function of the dilution parameter. Two limits of wave-front propagation dynamics are found. If the longitudinal and transverse velocities are equal, disorder is small, and wavelengths are large, the effective-medium approximation correctly estimates the wave-front

behaviour. Conversely, if the two modes have different velocities, the system is more disordered, and the wavelengths are small, the propagation of the leading front takes place along effectively one-dimensional paths of propagation. In this limit the amplitude usually decays faster and the velocity is higher than in the effective-medium limit.

Roughening of the leading wave front is analysed in both elastic percolation lattices and TLM lattices. Roughening means increase of the width of an interface as a function of time and the scale of observation. The kinetics of the roughening of the leading front is found to be partly described by a scaling argument based on Huygens' principle. Finally, conclusions of the results of this thesis are drawn and their implications discussed.

List of publications

I Rigidity and dynamics of random spring networks

M. Kellomäki, J. Åström, and J. Timonen

Physical Review Letters **77**, 2730 (1996).

https://doi.org/10.1103/PhysRevLett.77.2730

II Elastic waves in random-fibre networks

J. Åström, M. Kellomäki, and J. Timonen

Journal of Physics A **30**, 6601 (1997).

https://doi.org/10.1088/0305-4470/30/19/004

III Propagation and kinetic roughening of wave fronts in disordered lattices

J. Åström, M. Kellomäki, M. Alava, and J. Timonen

Physical Review E **56**, 6042 (1997).

https://doi.org/10.1103/PhysRevE.56.6042

IV Early-time dynamics of wave fronts in disordered triangular lattices

M. Kellomäki, J. Åström, and J. Timonen

Physical Review E **57**, R1255 (1998).

https://doi.org/10.1103/PhysRevE.57.R1255

V Elastic-wave fronts in one- and two-dimensional random media

M. Kellomäki, J. Åström, and J. Timonen

submitted to Physical Review E.

The author's contribution

The author of this thesis has written papers I, IV, and V, and done all the numerical and analytical work reported in these papers, except the one-dimensional segment chain model for the wave-front velocity. The author did the analysis of one of the major topics (the effective-medium model) of paper II, as well as the amplitude studies and TLM simulations of paper III. The author also actively participated in the finalising of papers II and III.

Contents

Chapter 1

Introduction

1.1 Disordered media

An ordered medium is such that its physical properties are regular whereas at least some of the properties of a disordered medium vary stochastically. Perfectly ordered media appear seldom in nature. However, by assuming that the medium is ordered, even in cases in which the true structure is disordered, the behaviour of many physical quantities have been understood in great detail. Despite these previous successes, properties of disordered media that cannot be understood by models with no disorder have become a growing field in condensed matter physics and materials science. A profound understanding of the mechanical properties of such media is important for several reasons. There are still many fundamental physical problems that remain unsolved. Much progress has been made in understanding, e.g., the elastic constants of a material near the rigidity transition, the fracture properties and the localization of classical waves. Research of these fundamental problems provides new insight and quantitative results which can be used in applications. In industry there are demands for controlling the scales of disorder in order to get certain end-use characteristics for manufactured materials. For example, the most familiar fibrous compound, ordinary paper, has a stochastic structure which is tried to optimize for many different purposes. There is also need for nondestructive testing of materials. This can be done, e.g., by studying how an ultrasonic pulse propagates in the material.

To understand the characteristics of disordered media, structural modelling of them is often inevitable. In statistical physics a particularly popular approach for studying mechanical properties of discrete disordered media has been to construct simple network models. The spider web or trabecular bone, e.g., are natural network structures. Manufactured materials like fibrous composites are formed as regular or random networks. Network models are usually made as simple as possible in order to enable large numerical statistics or analytical treatment, while maintaining the most important physical properties of the medium. For example, rigidity transition of glasses and gels have been succesfully studied within these models, and random networks of fibres have been found to explain many geometrical and mechanical properties of fibrous composites, such as thin paper sheets.

1.2 Outline of this work

This thesis concentrates on the study of mechanical properties of two-dimensional disordered media. Throughout the work the following approach has been chosen. Mechanical properties are studied within two-dimensional network models. Under simplifying assumptions some parts of the mechanical behaviour of these models are solved analytically. The results of these calculations are then compared with numerical simulations in order to test their validity. This Introduction outlines the general framework of the problems addressed, and reviews the results reported in publications [I-V]. Moreover, this Introduction tries to be complementary to the papers in the sense that we explain here in more detail those aspects of the work that are not covered or are only briefly mentioned in the publications.

In Chapter 2 we review our studies on the rigidity of random line networks composed of Hooke springs. Nonlinear stress-strain behaviour at large strains is studied. Rigidity transition in networks with reinforcing second-nearest neighbour bonds is analysed, and some preliminary results are reviewed. The results of this Chapter are partly reported in publication [I], but most of them are previously unpublished. That is why a self-sustained presentation of the results has been chosen.

In Chapter 3 we present our results for the dynamics of elastic-wave fronts in two-dimensional disordered media. The central issues of this Chapter are the velocity and amplitude of the leading front of a pulse of elastic waves. Propagation of the leading front is first discussed in perfect lattices. Next the early-time dynamics of the front is analysed for increasing disorder. The results for two different limiting behaviours of the front, the effective medium and the effectively one-dimensional behaviour, are presented. Finally, we discuss the roughening of the leading front. The results of this chapter are already reported in publications [II-V]. Therefore, most of the results are only briefly discussed without going into the details that can be found in these publications. This is true in particular for the limit of effectively one-dimensional propagation, which is an important part of this thesis, but is thoroughly reviewed in publication [V]. Finally, in Chapter 4, final conclusions on the topics covered in this thesis are drawn.

Chapter 2

Rigidity of random spring networks

2.1 A brief review

Random networks composed of springs have been very popular models of various physical phenomena during the last few decades. The introduction of these models to amorphous materials[1] was quite natural because of the success of perfect spring lattices in describing crystalline solids. The first ideas of random networks date back to the 1930's[2]. Since then a lot of results have been obtained on the structure and mechanical properties of covalent glasses (see Ref. [3] and references therein for a comprehensive review), even in quite complex three-dimensional problems[4]. Spring models are often very suitable for these materials since forces other than those associated with bond stretching (central forces between nearest neighbours) and bond bending (which can be understood as second-nearest neighbour central forces) are weak[5].

The spring networks have also become quite popular in the research of fracture of disordered media[6, 7, 8] (for a review see e.g. Ref. [9]). In these studies the network has not been considered as a model of any particular material, but rather the simplest model that includes the essential features of a brittle medium. The requirement of simplicity has been motivated by the possibility of making large statistical analyses. Answers to many fundamental fracture problems can be found in this way. However, in many cases this approach oversimplifies the real problems. This is why attempts are now being made[10, 11] to fill the gap between the fracture mechanics in statistical physics and in engineering. Spring networks have also been succesfully used to describe powders[12], gel polymers and gelling solutions[13, 14]. They have as well been applied in studying elasticity of compressed emulsions[15] and random compounds made of thin fibres[1]. In the light of these examples, the range of applicability of random spring-network models seems to be very large.

The stability of structures has been an inspiring subject of research in physics. With stability or *rigidity* we mean here the ability of a mechanical system to store elastic energy in deformations of the system. A nonrigid system can be deformed without any change in the elastic energy. In many materials, e.g. amorphous glasses[4], there can occur a transition between a nonrigid and a rigid phase. Concepts of percolation theory[16, 17] have proved to be powerful in stydying this type of a phase transition. Percolation of a physical property means that it extends continuously, at least via a single path, from one edge of the system to the opposite edge. If rigidity percolates through a system, displacements at one edge of the system produce forces that eventually act on the opposite edge via the rigid paths. *Rigidity percolation* in two dimensions is a completely different phenomenon in networks with only central-force interactions in comparison with networks with also noncentral forces (e.g. bond-bending forces). In the latter type of networks rigidity already percolates if there is a singly connected path across the network. In central-force networks rigidity percolation is a long-range (nonlocal) phenomenon. Adding a single bond to the network can affect the rigidity of a large area of the network - even across the whole network. This is why rigidity percolation in the central-force networks has been and still is a tricky problem.

Maxwell[18] was among the first to study the conditions under which mechanical structures made of rigid bars (infinitely stiff springs), joined together with pivots at their ends, would be stable. He used the method of *constraint counting* to determine the stability without any detailed calculations. It is based on determining whether there are enough constraints (bars) to freeze all the internal degrees of freedom (associated with the pivot joints). An example of how this method is applied will be given in Sec. 2.2. The elastic properties of random networks of springs have been actively studied during the 1980's and 1990's[3, 5, 13, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, I, IV, V]. This period of 15 years has witnessed a rapid improvement in computational facilities. There has been an exponential growth in the computationally feasible system size. Nevertheless, even the order of the rigidity percolation in two-dimensional networks is still controversial. In other respects progress has been made, e.g. the behaviour of elastic constants away from the rigidity threshold has been found to obey effective-medium theory[4, 29].

Most of the studies of central-force rigidity percolation have applied very CPU-intensive ($O(N^3)$ algorithms) relaxation methods. In studying the finite-size scaling of elastic moduli near the rigidity threshold, big enough system sizes have not been achieved. In addition, geometrical aspects such as statistics of the rigid backbone (percolating rigid cluster without dangling ends) have been difficult to explore using approximate numerical methods. Recently there has been significant progress in this subject. The determination of *generic rigidity*[19] was made possible by the introduction of powerful integer algorithms[5, 25, 26]. A network is generically rigid, if it has at least one pathway across the network along which all the internal degrees of freedom are constrained for all but at most a numerable subset of configurations of spring lengths and angles. In other words, only the connectivity or topology of a given network is considered whereas the conventional concept of rigidity (non-generic rigidity) also takes into account the specific geometry defined by bond lengths and angles. For example, the geometry shown in Fig. 2.1 (two collinear bonds) is conventionally rigid when it is stretched but nonrigid when it is compressed. On the other hand, it is generically nonrigid because for all but two initial angles (0 and 180 degrees) between the bonds it cannot store elastic energy. More formally, the network is generically rigid

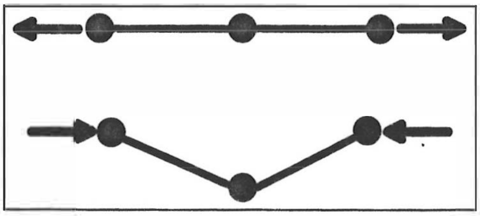

Figure 2.1: Two collinear bonds carry load when stretched but buckle when compressed.

if it has at least one cluster of sites and bonds extending across the network in which there are as many linearly independent constraints (bonds) as there are internal degrees of freedom (all other degrees of freedom than the two translations and the rotation of the whole cluster). A linearly dependent or redundant bond is such that when it is added to the network, it does not constrain any further degrees of freedom. To determine whether a given network is generically rigid or not, the number of redundant bonds must be found. It was found very recently that this problem is equivalent to testing whether the network fulfils a theorem in graph theory[30]:

Theorem 1 (Laman) *The edges of a graph G=(V,E) are independent in two dimensions* $\Leftrightarrow$ *no subgraph G'=(V',E') has more than 2N'-3 edges.*

In this theorem V denotes a set of N vertices (sites of the network) and E denotes a set of B edges (bonds). To determine the overconstrained regions (regions having redundant bonds), this theorem must be applied recursively to all subgraphs in the network. This can be done very efficiently by using a numerical algorithm, the Pebble Game[5, 25]. This algoritm is particularly powerful because it gives *exact* answers and scales in the slowest case as $\propto N^{1.2}$.

In addition to the rigidity percolation in ordinary random networks, also percolation under tension has been studied[31, 32]. It has been found that the percolation threshold is a function of the tension and approaches that of the connectivity percolation as tension increases. Finally, a separate splay-rigid phase of random spring networks[33, 34, 35] has also been analysed.

2.2 Rigidity of random line networks composed of springs

It is important to understand which factors affect the mechanical properties, e.g. stiffness, of planar fibrous compounds of which the most familiar example is paper in all its forms. Since the 1950's[36] these properties have been studied with planar random line-network models, of which we show three examples in Fig. 2.2. The random geometry of the network is constructed by placing N_f one-dimensional straight lines of length L on a rectangle whose area is $A = L_x L_y$. The distributions of the line centres and the orientation of the lines are both random. The density of the network is defined as

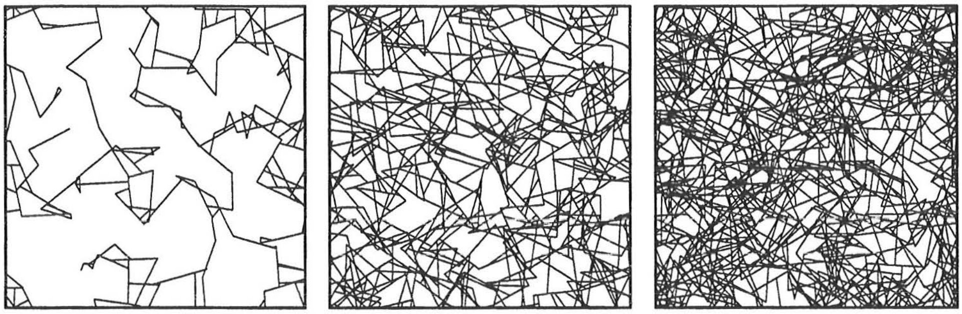

Figure 2.2: Random line networks of different densities. From the left: $q = 1.1q_c$, $q = 3q_c$, and $q = 5q_c$.

$q = N_f L^2/A$, which is the average number of lines on an area L^2. Densities are typically expressed in terms of the connectivity percolation threshold $q_c \approx 5.71$[37]. For densities $q > q_c$ there exists at least one connected path of lines from one side to the other. Lines are bonded together at their crossing points (nodes). Only one pair of lines can cross at any single point because they have zero width. As the dangling ends of lines are removed for simplicity, the coordination number of each node is two, three or four. In Fig. 2.2 we show three examples of these networks for different densities. The leftmost network has a density that is just above the percolation threshold, while the other two are denser. This kind of networks have previously been found to describe many geometrical[38, 39, 40, 41, 42], mechanical[36, 43, 44, 45, 46, 47, I, II], and electrical[48] properties of fibrous composites, and percolation of cracks in rock[49]. In this work the segments of lines between two neighbouring nodes are assumed to be linear Hookean springs with spring constants K_j. In order to keep the elastic modulus E (a material parameter) of the lines (describing the fibres of the material) constant on the segments, we define $K_j = ES/L$, where S is the cross section of the line. Usually we set $S = 1$. This kind of a model corresponds to the limiting case of extremely flexible fibres, which happens, e.g., when the fibres are very slender or wet (wood fibres)[50].

For the static elastic properties of these random line networks all the necessary quantities have now been defined. When the *dynamics* of these networks is considered, also the mass distribution has to be defined. The mass distribution of the lines is simplified by placing an equal mass M on every node and assuming the segments themselves to be massless. The dynamics of the network can now be described by the Hamiltonian

$$H = \frac{1}{2} M \sum_i \dot{\mathbf{u}}_i^2 + \frac{1}{2} \sum_{i,j} K_{ij} (\Delta l_{ij})^2 , \tag{2.1}$$

where $\mathbf{u}_i$ is the displacement of node i from its equilibrium position in the undeformed network, K_{ij} is the spring constant of the segment connecting nodes i and j, and Δl_{ij} is the deviation of spring length from its unstressed value (no linearisation is made).

2.2.1 Constraint-counting calculation

The random-line network composed of springs is divided (in 2D) into rigid (triangles) and nonrigid (other polygons) substructures. The rigidity of this network can be analysed by applying the constraint-counting method[5, 18, 29]. The number of unconstrained or floppy modes in a d-dimensional network can be written as

$$F = dN - N_c = dN - (N_b - N_r) , \tag{2.2}$$

where N is the number of sites, N_c is the number of linearly independent constraints (bonds), N_b is the total number of bonds, and N_r is the number of linearly dependent (redundant) bonds. According to the definition above, a network is *totally* generically rigid if $F = 0$. The condition $F = 0$ is not a necessary condition for the rigidity percolation, but if there is a rigidity threshold, the network can usually be made totally rigid. We define the floppy mode ratio f as

$$f = \frac{F}{dN} = \frac{dN - (\frac{1}{2}N\langle z\rangle - dNn_r)}{dN} = 1 - \frac{\langle z\rangle}{z^*} + n_r\,, \tag{2.3}$$

where $\langle z\rangle$ is the average site coordination number, $z^* := 2d$ is the coordination needed to constrain all of the d degrees of freedom of a site (a single bond belongs to two sites), and $n_r := N_r/(dN) \geq 0$. In two dimensions $z^* = 4$. For the random line network the total number of bonds $N_b = 4N - 2N_f$ because all sites (crossing points of two lines) have initially four bonds and because the two dangling ends of each line are removed. This leads to the result

$$\langle z\rangle = N_b/N = 4(1 - N_f/(2N)) < 4 = z^*\,. \tag{2.4}$$

Equations (2.3) and (2.4) mean that $f > 0$ for any number of lines N_f (or any finite density q) which suggests that the random line network composed of springs may not be rigid. Moreover, the coordination z_i of every node i is always $z_i \leq z^*$ (c.f. above), which means that there are no over-constrained nodes (sites). This ensures that the Maxwell approximation[18] ($n_r = 0$) is *exact* for this network. Using the result $N = q^2A/(\pi L^2)$ (valid for high q) from statistical geometry[39], we can show that, in this network,

$$f = \frac{\pi}{2q} = \frac{\pi}{2\cdot 5.71}\left(\frac{q}{q_c}\right)^{-1}\,. \tag{2.5}$$

Rigid clusters are locally composed of triangles[22] with a common side. If two triangles had a common side, there would necessarily be intersections of three different lines, which contradicts the definition of our geometrical network model. Hence in this network triangles are always elastically isolated from each other, and rigidity cannot percolate at any finite density q of the network. Random line networks composed of springs are qualitatively similar to diluted central-force square lattices. Both have the same $\langle z \rangle$ in the high q or perfect lattice limit, and lack a finite rigidity threshold.

2.2.2 Numerical results

We confirmed the lack of rigidity of random line network composed of springs by MD-like simulated annealing. Simulations were done by applying a 1% uniaxial strain in the x direction. After letting the network to stabilize for a long time (500 times the time of flight of a wave front), the kinetic energy of the nodes was quenched by depleting the velocities of the nodes each time they passed a local potential minimum. This procedure also removed the elastic energy thus indicating that it was only stored in the vibrations, and the final state of the network was unstressed (according to the strict convergence criteria of Refs.[13, 14, 51]) although it underwent a macroscopic deformation. Thus the lack of conventional rigidity is strongly supported, if not rigorously 'proved'.

We also studied the generic rigidity of networks having the same topology as the random line network. We found that the analytically calculated floppy-mode ratio (Eq. (2.5)) was exactly the same as that given by the Pebble-Game algorithm[5, 25]. The fact that $n_r = 0$ exactly was directly confirmed by the Pebble Game using a considerable amount of realizations of the random line network. Floppy modes have zero frequency as they undergo aperiodic unconstrained motions. The existence of zero-frequency (floppy) modes in the random line network composed of springs was numerically confirmed by computing the density of vibrational states as a Fourier transform of the velocity time series. In Fig. 2.3 we show the eigenfrequency spectrum $g(\omega)$ extracted from the velocity-velocity autocorrelation function of data obtained by the equation-of-motion method[4, 52]. The Hanning data window[53] was applied to the truncated time series before doing the Fourier transform. In order to make the calculation of the floppy-mode ratio

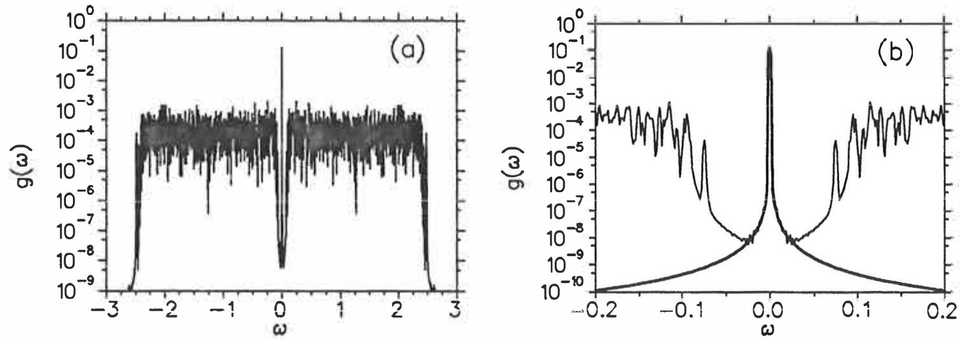

Figure 2.3: Eigenfrequency spectrum of a random line network composed of springs ($q = 3q_c$). Figure (b) is an enlarged version of the spectrum of Fig. (a) near the frequency origin. The bold line denotes the spectrum of the Hanning data window.

f from the spectrum easier, the spring constants of the segments were fixed to the same value $K_{ij} := K$. This operation does not change f as it only changes the distribution of the elastic modes, but not their total number. It is evident from Fig. 2.3 that the zero-frequency modes form a high peak, which is separated by a gap from the elastic modes. The resolution function (Fourier transform of the data window) fits perfectly this peak. The value of f calculated as the ratio of the areas under the resolution function and the spectrum did not coincide with Eq. (2.5), even though quite long time series were used. Further work is still needed to get more reliable results. However, as one can observe from Fig. 2.3, and from simulations of the frequency response to sinusoidal waves[I], all wave modes in random line networks composed of springs are floppy or localized; there are no acoustic modes.

2.2.3 Nonlinear stress-strain behaviour at large strains

In Refs. [31, 32] results were reported for how nonrigid central-force networks begin to carry load (become rigid) when they are subjected to a tension. The rigidity threshold of these networks has been found to become smaller as strain is increased. Their stress-strain behaviour is nonlinear and asymmetric with respect to the applied strain. As we demonstrated above, random line networks composed of springs are not rigid in their equilibrium (unstressed) configuration. It was therefore interesting to study how these networks become rigid when strained and how they behave at large strains. In trying to compute the stress-strain behaviour numerically, we ran into problems with conventional (conjugate-gradient) relaxation methods. Due to a large spectrum of force constants and the nontrivial geometry of the network, convergence problems became severe. This is why we chose to use the simulated annealing method presented in the previous subsection. Our computations were carried out by applying smoothly a strain in the x direction. The network was then relaxed and the elastic energy and stresses on the segments were recorded. After that the strain was increased and the same procedure repeated. In this way we were able to construct the stress-strain curve. In Fig. 2.4 we show a typical elastic energy E_p as a function of strain ϵ. Stress σ as a function of ϵ can be obtained by numerically differentiating this curve. It is evident from this figure that there is a threshold strain, $\epsilon = \epsilon_+ \approx 0.02$ for elongation and $\epsilon = \epsilon_- \approx -0.2$ for compression, that is required for the network to carry load. Diode effects[5] of atypical elastic structures (like the one shown in Fig. 2.1) lead to $|\epsilon_+| \ll |\epsilon_-|$. It can be shown[54] that the curve $\sigma(\epsilon)$ is quite nonlinear beyond the onset of rigidity. It is obvious that it is also quite asymmetric with respect to $\epsilon = 0$. During the simulations we also monitored the segment-stress distribution $n(\sigma)$.

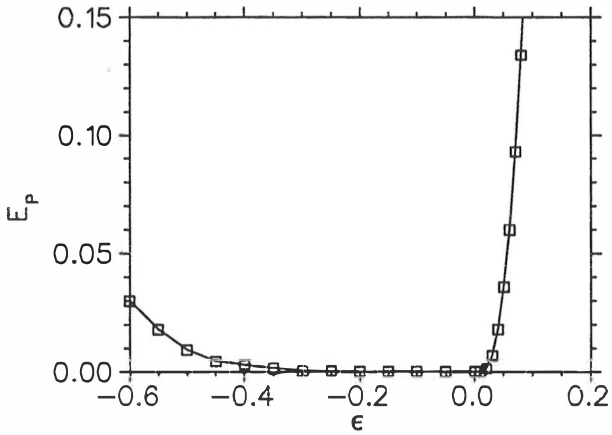

Figure 2.4: Elastic energy E_p vs strain ϵ for a random line network composed of springs ($q = 3q_c$).

The distribution for $\epsilon > \epsilon_+$ is very similar to that for elongated random line networks composed of elastic beams[44]. They are both asymmetric so that the positive stresses related to elongation of line segments are dominating. The distribution for $\epsilon < \epsilon_-$ is symmetric. These results are very useful in analysing the stress-transfer mechanisms in fibrous compounds[55].

2.3 Rigidity transition in a reinforced network

As already noted above, the nature of the rigidity transition in two-dimensional central-force networks is still an open problem. We considered this problem by modifying the previously used network model. The geometry of the extended model is the same as in the random line network, but new bonds (springs) are added between second-nearest nodes on the lines with a probability p. This model, which we call the reinforced random line network, describes a situation in which parts of the flexible lines (fibers) are made rigid.

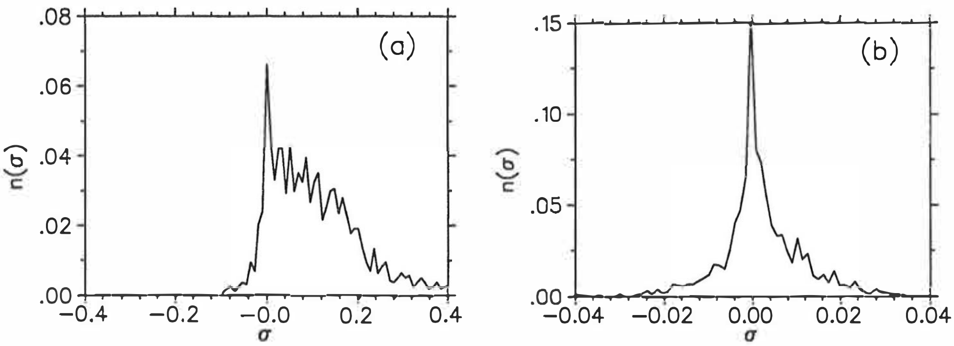

Figure 2.5: Normalized segment-stress distributions $n(\sigma)$ for a random line network composed of springs ($q = 3q_c$). (a) $\epsilon = 0.4 > \epsilon_+$ (elongation), (b) $\epsilon = -0.6 < \epsilon_-$ (compression).

As we have in this model a maximum coordination number $z_{max} = 8 > z^*$ for the nodes, the network can become rigid for a sufficiently large p. It is in fact a better model for stiff fibrous materials than the simple random line network composed of springs. Making a constraint-counting calculation we get an estimate for the rigidity threshold $p = p_c$,

$$p_c = \frac{1}{2(q/\pi - 1)} . \tag{2.6}$$

From Eq. (2.6) we find that as $q \to \infty$, the rigidity threshold $p_c \to 0$. This indicates that this model can be tuned arbitrarily close to the rigidity threshold so that adding of only a few bonds can make it rigid.

We have studied with the Pebble Game the generic rigidity transition in the reinforced networks. In Fig. 2.6 we show df/dp for $6L \times 6L$ networks with $q = 4q_c$. If the floppy-mode ratio f is interpreted as the free energy of the system[5, 25], it is evident from this figure that the constraint-counting calculation gives a rather good estimate for p_c. The curve df/dp is quite interesting because it consists of a number of steps. The points in this curve are separated by one bond which implies that at the steps of the curve, big

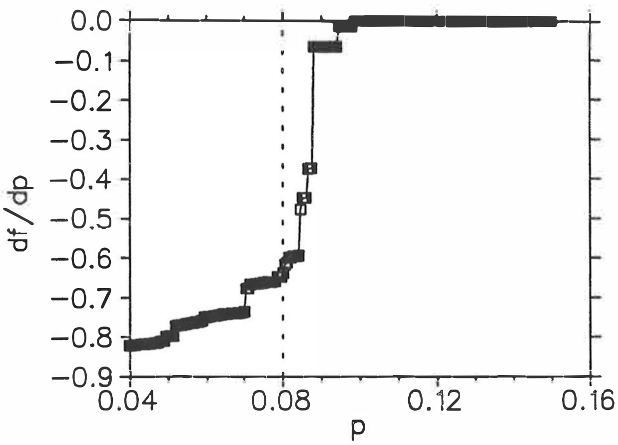

Figure 2.6: df/dp vs. nearest neighbour probability p ($q = 4q_c$, $A = 6L \times 6L$). The vertical dashed line displays the rigidity threshold $p_c = 0.080$ given by Eq. (2.6). After [56].

rigid clusters are formed at once. This behaviour is typical of the rigidity transition in spring networks, which is known to have long-range effects(c.f. Sec. 2.1). However, the rigid clusters that are formed can consist several thousand bonds and nodes, and the size of these clusters increases as the density or the network area increases. This behaviour might be related to a first order transition which is already observed in Cayley trees[28], random-bond networks[57] and percolating clusters in diluted triangular lattices[27]. However, this question still needs further analysis.

Chapter 3

Dynamics of transient wave fronts

3.1 A brief review

The properties of waves in homogeneous media have been succesfully described with simple wave equations and dispersion relations. The situation is quite different in the case where a pulse is injected into a disordered medium. After many scattering events the pulse loses its phase coherence. Hence, a propagating wave cannot be only characterized by a frequency and a direction of propagation. Anderson found in 1958[58] that wave functions of electrons are confined to restricted areas if the potential of the surrounding medium is strongly disordered. The electron is then *localized*, in that the envelope of its wave function decays exponentially away from some point in space. The inverse of the related decay constant is the *localization length*. Since Anderson's work there has been a lot of activity in this field. We briefly summarize here the results[59, 60, 61, 62, 63, 64, 65, 66] most relevant to this work.

There are many length scales relevant to propagation of waves in a disordered medium[66]. For length scales smaller than a couple of mean free paths (average distance of coherent propagation between two scattering events), propagation takes place as in a homogeneous medium. After several mean free paths, the transport of energy appears to be diffusive due to accumulating scattering effects. For even longer scales there are three possibilities. If scattering is weak, the wave can diffuse classically. A decrease in the diffusion constant can occur in the case of strong scattering. This phenomenon is called anomalous diffusion. The third possibility, if scattering is strong enough, is that the diffusion constant vanishes: i.e. the wave is localized. It has been found that in one- and two-dimensional random media all wave modes are localized even in the weak-scattering limit[63, 67, 68], except the case in 1D in which the defects are pair correlated[69, 70]. In three dimensions the situation is different: there can exist both localized and extended modes[66]. The frequencies that separate the localized and extended regimes are called mobility edges. The localization behaviour of classical waves, e.g. of acoustic waves which we are interested in, has been studied extensively during the last few years[71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81]. While much of the published results have dealt with continuous random media, there is growing interest also in the study of discrete cases, such as disordered two-dimensional spring lattices[82, 83], elastic percolation networks[84], granular media[85] and even macroscopic engineering structures[86, 87].

Basic features of pulse propagation in homogeneous media have long since been understood[88, 89, 90]. The effects of dispersion and emissive properties of media on the velocity and attenuation of a signal have been studied quite extensively[91, 92, 93, 94]. An interesting topic in these studies has been the *precursors* (e.g. Brillouin and Sommerfeld precursors) of a pulse, which precede the main body of the pulse and behave in a different manner. There are still many interesting new aspects[95, 96], and novel applications of transient wave fronts[97], that are now emerging. Recently there has also been some interest in disordered wave fronts[98] and in phenomena in highly dispersive excitable media[99, 100].

Despite the vast literature on both localization and propagation of waves, there are still many unknown features related to the case in which a pulse is injected into a *disordered* medium. More specifically, *transient phenomena* such as propagation of the leading wave front, have not yet attracted much attention. The leading wave front, which will be defined below, consists of the disturbances that propagate fastest. As the main signal is usually left significantly behind the leading wave front, the latter resembles precursors in homogeneous media described above. Despite the apparent similarity, we have shown[I, II, III, IV, V] that the properties of the leading wave front cannot be described by similar continuum theories. Precursors in homogeneous media arise from global properties, such as the dispersion relation, whereas in disordered media the dynamics of the leading front strongly reflects local properties, such as the geometry of the paths in the medium. In this thesis we study extensively both the asymptotic and the early-time behaviours of the leading front of elastic waves in various one- and two-dimensional lattices and networks. Both periodic and stochastic networks are analysed in order to find the phenomena arising from disorder. We have found, e.g., that in addition to the ordinary elastic waves, in disordered media the leading wave front also includes an exponentially decaying transient that propagates along effectively independent one-dimensional paths. To study localization and roughening of classical-wave fronts we have also used Lattice-Boltzmann wave automata. We found that the leading front propagates rather undisturbed through a medium with randomly distributed scatterers while the rest of the wave motion diffuses forward very slowly.

3.2 Wave-front velocity and amplitude

The main topic of this thesis is the dynamics of the leading front of elastic waves in disordered discrete networks. A possible definition of that front is that it is the surface in front of which, at a given instant of time, the medium is completely at rest[88, 101]. In our studies[I, II, III, IV, V] we defined the location of the leading wave front more practically through the first maxima in the local displacements. In the following we shall often call it the 'wave front' for simplicity. This kind of definition is motivated by two facts: the first displacement maximum is the first unambiguously definable event when an elastic wave passes a lattice point, and it is very convenient to handle numerically. In Fig. 3.1 we show the leading front of an elastic wave propagating in a bond-diluted square lattice composed of elastic beams. Two

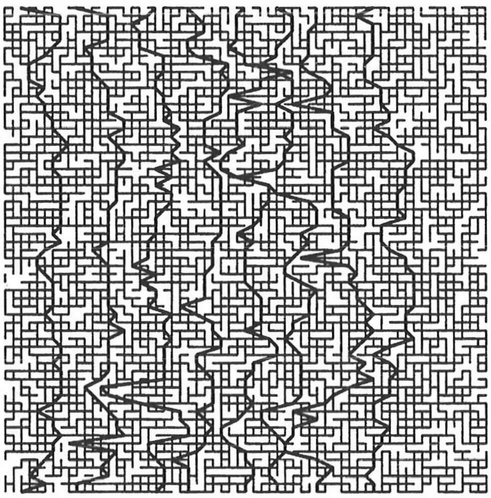

Figure 3.1: Transient elastic wave fronts (thick lines) at different times in a two-dimensional bond-diluted square lattice ($p = 0.7$). After [III].

quantities of major interest in our studies have been the *wave-front velocity*, which is the average velocity of the first local displacement maxima, and the *wave-front amplitude*, which is the average (over a narrow vertical strip) of the values of these maxima.

We have numerically studied propagation of the wave front of a plane-wave pulse induced on the left boundary of the network. The displacement pulse $f(t)$ that was launched into the network at $x = 0$, was of the form

$$f(t) = \begin{cases} A_S \cos(\omega_S t), & -\pi/2\omega_S \leq t \leq \pi/2\omega_S \\ 0, & \text{otherwise} \end{cases}, \tag{3.1}$$

where A_S is the (initial) amplitude and ω_S the driving frequency. The frequency spectrum $g(\omega, \omega_S)$ at $x = 0$ of the pulse is given by

$$g(\omega, \omega_S) = \frac{A_S \omega_S}{\pi(\omega_S^2 - \omega^2)} \cos\left(\frac{\pi\omega}{2\omega_S}\right), \tag{3.2}$$

where ω is the frequency of the component wave. Notice that $g(\omega, \omega_S)$ has its global maximum at $\omega = 0$, and oscillates symmetrically between positive and negative values as $|\omega|$ increases. Furthermore, increasing ω_S increases the bandwidth of the pulse and reduces the number of oscillations near $\omega = 0$. In the following Sections propagation of the wave front of a pulse defined by Eq. (3.1) is analysed.

3.2.1 Perfect lattices

A proper understanding of the behaviour of the leading wave front in perfect lattices is essential before one can study the effects of randomness. The simplest lattice is a regular chain of equal springs K (lattice constant l and mass points M). The dispersion relation of this lattice is of the form

$$k(\omega) = \begin{cases} (2/l) \arcsin(\omega/2\omega_0), & |\omega| \leq 2\omega_0 \\ (\pi/l) - i(2/l) \text{arcosh}(\omega/2\omega_0), & |\omega| > 2\omega_0 \end{cases}, \tag{3.3}$$

where ω is the frequency, $k = k(\omega)$ is the wave number and $\omega_0 := \sqrt{K/M}$. In analysing the time evolution of the pulse we have applied the stationary-phase method[88, 89, 90]. The basic idea of this method is to expand the phase $\omega t - k(\omega)x$ of a plane wave component (ω) of the pulse as a Taylor series near the wave numbers that propagate with the group velocity v_G. A careful inspection[V] of the pulse given by Eq. (3.1), and the dispersion relation Eq. (3.3), revealed that the leading front of the dispersed pulse is the biggest one, and belongs to a stationary wave group that travels with the highest group velocity $v_G(\omega = 0) = c(\omega = 0) = l\sqrt{K/M} =: c$. Applying the stationary-phase method we found that the amplitude of the front is given by

$$A(x) \propto \int_{-\Delta\omega(x)/2}^{\Delta\omega(x)/2} g(\omega, \omega_S) e^{-\alpha(\omega)x} d\omega \,, \tag{3.4}$$

where $\Delta\omega(x) \propto x^{-1/3}$ and $\alpha(\omega) = \mathrm{Im}\{k(\omega)\}$. From the functional behaviour of $g(\omega)$ (c.f. Eq. (3.2)) we can conclude that Eq. (3.4) leads to an oscillatory early-time behaviour for the wave-front amplitude. In the limit $x \to \infty$ the spectrum behaves as $g(\omega, \omega_S) \approx A_S/(\pi\omega_S)$ within the band $\Delta\omega$, and this leads to an asymptotic power-law decay $A(x) \propto u(t = x/c, x) \propto x^{-\frac{1}{3}}$.

In numerical simulations of spring chains[V] we found the expected oscillatory early-time behaviour and the asymptotic power-law decay. These results are quite general. They apply also to any other Bravais lattice with a dispersion relation such that the global maximum of v_G is at $\omega = 0$, and with ω being a monotonous function of $|k|$. As an example we show in Fig. 3.2 the wave-front amplitude for a perfect triangular lattice. It is evident from this figure that both the early-time and asymptotic behaviours of the amplitude are satisfactorily explained by theory. This behaviour of the amplitude should be observable in e.g. single monatomic fcc crystals in which dispersion relations similar to the two-dimensional case considered here appear. For a single crystal of lead, e.g., one can estimate that, in the [100] direction, the oscillating part of the wave-front amplitude will penetrate to a depth of $x_D = 10$ mm, provided that the width of the positive part of the

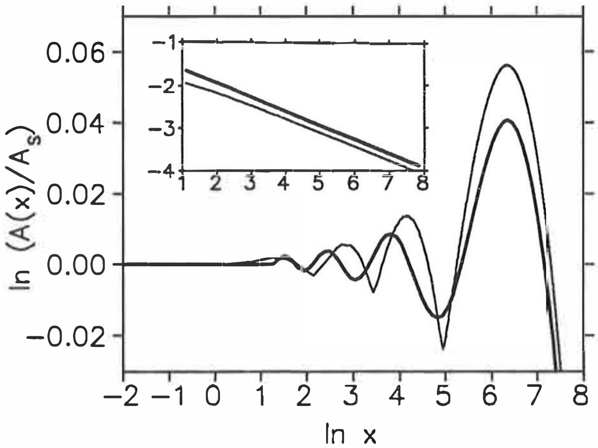

Figure 3.2: Early-time behaviour of the simulated (thin solid line) and calculated (bold solid line, Eq. (3.4)) wave-front amplitude A vs. distance x in a perfect triangular lattice with $K = 1.0$, $M = 1.0$, $a = 1.0$, and $\omega_S = 0.15$ (a longitudinal pulse propagating in the [11] direction). The inset shows the asymptotic behaviour of the amplitude for $\omega_S = 5.0$. After [IV].

pulse spectrum is 5.4 GHz around $\omega = 0$[IV, 102].

Dispersion can also give rise to an exponential decay of the leading wave front in certain elastic structures. For example, if the structure consists of coupled oscillators it acts as a pass-band filter. We have found[V] that even in this case the fastest propagation modes utilize the lowest frequencies. As these modes are below the pass band, the wave-front amplitude decays exponentially. This kind of structures appear locally in randomly bond-diluted square lattices[V].

3.2.2 Early-time dynamics of random networks

We have analysed the early-time dynamics of the leading front in bond-diluted square lattices[III]. As long as the wave front travels along an unbroken horizontal chain of bonds in the lattice, the amplitude remains more or less constant. When the front meets a missing bond, this must be circumvented. Part of the front is thus delayed, and we can assume that the amplitude is reduced to a fraction γ of its value at each missing bond. Assuming that γ remains constant, the amplitude is found to decay as $A(x) = A_0 \exp(-\alpha x)$ with the decay constant α given by

$$\alpha = \alpha_0(1-p)\,, \tag{3.5}$$

where p is the bond probability, and $\alpha_0 = (1-\gamma)$. We have verified numerically[III] that Eq. (3.5) is valid for the early-time behaviour of the wave-front amplitude when $p < 0.95$. A cross-over from an exponential to a less rapid decay can be seen for $p > 0.95$

In a triangular spring network a similar behaviour can be expected. However, rigidity of a spring network is a non-local property, whereas it is a local property in a beam network, which is analogous to a bond-bending network. Therefore, missing bonds soften the spring network on a larger scale. The value of α is thus expected to grow as a function of $1-p$ more rapidly than in Eq. (3.5). In numerical simulations this expectation was confirmed[IV]. The early-time decay of the wave-front amplitude was found to be exponential with a decay constant obeying a power law,

$$\alpha = \alpha_0(1-p)^{1.7}\,, \tag{3.6}$$

where α_0 is a constant. This result is valid for bond probabilities $p < 0.95$. For $p \simeq 1$ we found that the behaviour of the wave-front amplitude is determined by dispersion[IV] (c.f. Sec. 3.2.1). The cross-over from the dispersion-dominated to the disorder-dominated behaviour takes place smoothly in a range of missing bond concentration $0 \leq 1-p \leq 0.05$.

3.2.3 Effective-medium limit

For a sufficiently low degree of disorder the effective medium approximation has turned out to be quite accurate in analytical calculations of disordered media. In this approximation one considers a "typical element" which is embedded in an "effective" homogeneous background (average of the surrounding system). Macroscopic behaviour is then obtained by averaging this system over the distributions of the elements. This kind of an approach is justified when the defect density is low and the relevant fields are slowly varying. The latter requirement means in the case of elastic waves the long-wavelength limit.

When dealing with the fastest transient response to a driving force, there is a further condition on the applicability of the effective medium approximation. If there are more than one wave mode available in the system, all modes should have rather similar propagation speeds. Otherwise, if any exchange of energy between the modes is possible, only the fastest mode will occur in the leading wave front. This kind of situation cannot be described by the simple effective medium approximation.

We calculated the effective Young's modulus within this approximation[III] for bond-diluted square lattices composed of elastic beams, in analogy with the conductance of random resistor networks[16]. In random bond dilution the mass density of the network is independent of p because the number of sites (mass points) of the lattice is constant. Taking into account that the Poisson ratio is zero for a square lattice, the velocity of induced longitudinal waves is given by

$$v_l = \sqrt{\frac{Ew^2l}{m}(p - ((2-2p) + (2p-1)\frac{l}{w})(1-p))}\,, \tag{3.7}$$

where E is the Young's modulus of the beams, w is their width, l their length, and m is the mass of the site. Similarly, we derived a solution for the transverse waves. By numerical simulations [III] we confirmed that both of these solutions are valid for low driving frequencies, isotropic bond stiffnesses (beam aspect ratio $w/l \approx 1$), and for not too low p, as expected. The effective medium approximation treats the medium as a homogeneous continuum, and the amplitude of a propagating wave is expected to remain a constant. However, as the diluted square lattice is intrinsically discrete, at least dispersive decay of the wave front should always be observed. In numerical simulations the relation $A(x) \propto x^{-1/3}$ has proved to be accurate even for $p \simeq 0.60 \ll 1$. It is very interesting that no asymptotic exponential decay related to localization was detected even when the level of disorder was very high (p is close to the percolation threshold $p_c = 0.5$).

The effective medium approximation has also been valuable in calculating various elastic properties of random fibrous compounds, e.g. of paper[36]. When analysing the acoustic-wave fronts in random line networks composed of elastic beams[II], we first calculated the Young's modulus and Poisson ratio of the network. Using results from statistical geometry, the mass density of the network was calculated. The acoustic speed was thereby found to be given by

$$v = \sqrt{\frac{\pi E w^2 L}{2qm}} \sqrt{\frac{3a\pi}{16}} \sqrt{1 - b\frac{q_c}{q}}, \tag{3.8}$$

where L is the line length, q is the network density, and a and b are constants having different values for longitudinal and transverse waves (m and w as in Eq. (3.7)). Numerically we found that Eq. (3.8) is valid for long wavelengths. The condition for the beam segment aspect ratio, $w/l \approx 1$, where l is the average length of the segment, was not so stringent here as for diluted square lattices. The amplitude of the front was found to decay, in this limit, by a power law. The behaviour of random line networks composed of springs was found to be completely different. The effective medium approach could not be used as these networks are not macroscopically rigid (i.e. their Young's modulus is zero).

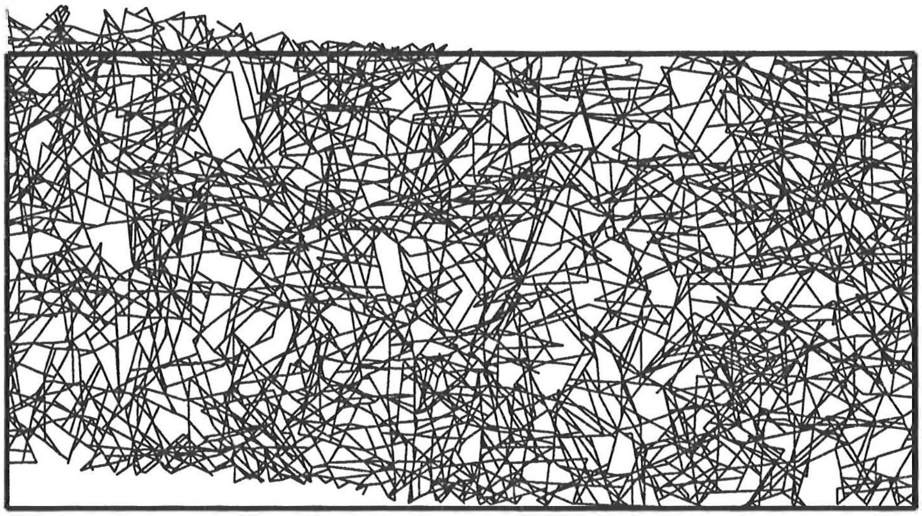

Figure 3.3: The displacements at $t = \pi/2\omega$ in a network with $q = 6q_c$, $m = 0.01$, $w = 0.15$; a sinusoidal transverse wave with a frequency of $\omega = 0.0125$. After[II].

In Fig. 3.3 we show an example of a situation in a random line network composed of beams in which the effective medium approximation is valid. In this figure a long-wavelength transverse wave is launched into the network from its left boundary. The whole structure of the network takes collectively part in the wave motion.

3.2.4 One-dimensional propagation

In the cases where the assumptions of the effective medium approximation do not hold, we have discovered and analysed interesting new phenomena related to propagation of the leading wave front along effectively one-dimensional paths. In Fig. 3.4 we show a typical example of a one-dimensional path of propagation in a random line network composed of springs.

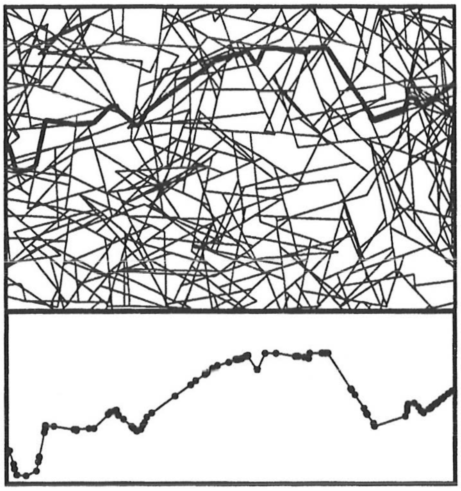

Figure 3.4: A minimum-decay path in a two-dimensional random line network composed of springs. After [V].

A necessary condition for the effectively one-dimensional propagation of the front is different axial and transverse velocities. In square lattices and random line networks composed of elastic beams, this condition is fulfilled when the aspect ratio of the beams is $w/l \ll 1$, or $w/l \gg 1$. An additional condition for the square (and triangular) lattices is disorder in the form of, e.g., random bond dilution. In random line networks composed of springs, there is no transverse mode and propagation always takes place along effectively one-dimensional paths. The properties of these effectively one-dimensional modes differ from those of the usual acoustic modes in many respects. The wave-front amplitude usually decays faster along one-dimensional paths. The decay law is exponential except in the square lattices. The wave-front velocity associated with one-dimensional propagation is typically greater than for the acoustic fronts. It also depends in a different way on the parameters of the medium, such as p and q characterizing the stochastic geometry. Acoustic fronts reflect global properties of the network (e.g. elastic constants of the system), whereas the 'one-dimensional' fronts carry information about local structures.

In one-dimensional paths of a bond-diluted square lattice dispersion leads to an exponential attenuation by band-pass filtering in certain local geometries. Elastic scattering by disorder along the paths of propagation causes localization of a part of the pulse, which results in an exponential decay. However, due to the discreteness of the disordered paths, a kind of dispersion is also present. Due to dispersion, the bandwidth $\Delta\omega$ of the front around $\omega = 0$ becomes narrower; i.e. the front will contain increasingly less high-frequency components as it propagates (c.f. Sec. 3.2.1). As components with lower frequencies have longer localization lengths[V], the apparent exponential decay rate caused by localization decreases. Decay of the front can also arise from the geometrical structure of the paths. If the pulse is continuously forced to split into axial and transverse components, and if these are not equally fast, the slower mode is continuously left behind the leading front. Similarly, if the path is not rigid, the transverse component excites floppy modes. In both cases the loss of energy from the leading wave front leads to an exponential decay. Moreover, the local wave-front velocity is always that of the faster mode. A comprehensive review of our results for the limit of one-dimensional propagation can be found in Ref.[V].

3.3 Roughening of wave fronts

After encountering defects along its path of propagation, an initially straight wave front will roughen (c.f. Fig. 3.1), i.e. the average width of the front increases. This phenomenon is related to the kinetic roughening of growing interfaces. Interface-growth models usually give rise to self-affine interfaces [103, 104], confined to a few universality classes[103] depending on the symmetries of the models. The scaling of self-affine interface growth can be characterized by an initial regime, with a roughness (or wave-front width r) proportional to $r \propto t^{\beta}$, where t is time, and by an asymptotic regime, characterized by $r \propto L^{\chi}$, where L is the linear system size.

The aim of our analysis was to find out if the wave-front roughening can be characterized by scaling resulting from local diffusive behaviour and from the Huygens' principle. In our studies we concentrated on the case of randomly bond-diluted square lattices composed of elastic beams. When the stiffness of bonds was extremely anisotropic (i.e. $w/l \ll 1$ or $w/l \gg 1$), we could solve the roughness $r(t)$ of the leading wave front analytically. We found that $\beta = \frac{1}{2}$ (diffusive early-time growth) and that the front is trapped by missing bonds in a finite time. The width r of the front reaches a system-size independent value $\sqrt{p}/(1-p)$, where p is the bond probability, i.e. the front does not roughen. The early-time behaviour of the leading wave front is nevertheless similar to that of the random deposition model[105]. These results were also confirmed by numerical simulations.

In the case in which the bond stiffness was isotropic ($w/l = 1$), we estimated $r(t)$ by arguments based on the Huygens' principle and uncorrelated noise generated by random vacancies. Our scaling calculations gave $\beta = \frac{1}{2}$ and $\chi = \frac{2}{3}$. Numerical simulations confirmed the value of β, but gave a different value for the roughness exponent $\chi \simeq 0.5$. This means that our argumentation is not precisely valid for the bond-diluted square lattices of elastic beams. To verify whether our arguments hold for the solution of the classical wave equation, we also made simulations with the Transmission Line Matrix (or shortly: TLM) Lattice-Boltzmann method[106, 107, 108] (c.f. Refs. [109, 110, 111] for other wave automata). We considered with this method a situation in which a plane-wave pulse is launched into a two-dimensional space, where there are randomly placed point scatterers. However, the results of these simulations were similar to the elastic lattice: $\beta = \frac{1}{2}$ and $\chi \simeq 0.5$. We are quite sure that our scaling arguments fail because of two reasons. The first reason is that the defects in these lattices consist of only one bond or site. As the minimum wavelength that can be described by a lattice model is two lattice constants, not all the components of the pulse can propagate according to the Huygens' principle around the defects. Therefore, lattices with bigger defects, e.g. 5×5 lattice points, should be analysed, which is a supercomputer-scale numerical problem. The second reason for the failure of the scaling calculation is dispersion. Dispersion in square lattices composed of elastic beams was already discussed above (c.f. Sec. 3.2.1). It has been shown[106] that unphysical dispersion at high frequencies occurs in the TLM method due to discretization of space and time. A new scaling argument taking into account dispersion might be useful in explaining the observed roughening of the leading wave front.

In Fig. 3.5 we show a wave intensity map in a TLM lattice at two instants of time. This simulation was done by injecting a continuous sinusoidal wave into the central column of the lattice. The concentration of reflecting lattice points was 10%. Periodic boundary conditions were imposed on both x and y directions. From this figure one can observe that the leading wave front takes away from the main pulse. It travels quite undisturbed through the medium while the rest of the wave motion diffuses forward very slowly. This phenomenon is similar to that in bond-diluted square lattices composed of beams in which no exponential decay of the front was observed regardless of the disorder. This example once again demonstrates that the leading wave

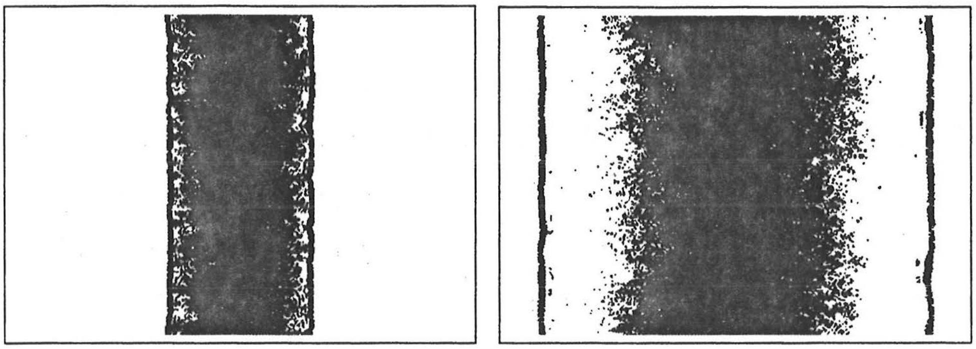

Figure 3.5: Thresholded intensity map of a TLM lattice ($p = 0.9$) at an earlier (on the left) and a later (on the right) moment of time. The black areas represent lattice points at which intensity of waves is greater than a constant threshold value.

front has a very special character in the wave dynamics of disordered media.

Chapter 4

Conclusions

We have studied several aspects of rigidity and transient-wave dynamics of disordered media. The random line network composed of springs turned out to be an interesting subject of research for several reasons. Firstly, it is a nontrivial example of networks in which the Maxwellian approximation in constraint counting is exact. The network model was found to be very close to the rigidity transition in the high-density limit. This property makes it particularly interesting in the study of the nature of the transition: the rigidity 'explodes' over the network when adding a small amount of reinforcement springs. Further work on quantities such as cluster statistics, relevant to rigidity percolation, is now needed. The large-strain behaviour of these networks is also important. It has recently been shown[112] that the widely applied effective medium approximation[36] for the elastic constants of random line networks composed of beams does not take into account the axial stress transfer, which in some cases is expected to be the dominant mechanism. Our results for the stress distribution in random spring networks at large strains confirm this expectation: the stress distribution found in the beam networks are reproduced in our networks even though there is no shear-lag stress transfer mechanism (basic assumption in [36]) in the latter networks. This result has now motivated the construction of a revised effective medium approximation[55], which is hoped to give quantitative results for the elastic constants. This kind of theory would have implications, e.g., in understanding the mechanical behaviour of manufactured paper.

The study of the dynamics of the leading wave front has revealed a rich phenomenology. For perfect lattices we found that the early-time behaviour of the wave-front amplitude can be oscillatory. The length scale in which this phenomenon happens can be macroscopic. The consequence of this phenomenon is that a wave front can propagate a considerable distance before it begins to decay. From an experimental point of view it may be quite difficult to measure this effect: a large single crystal at a very low temperature should be used in order to minimize other attenuation effects. Our results[IV] show that a low concentration of defects (e.g. missing bonds) does not change the qualitative behaviour, which may improve possibilities of detecting the phenomenon.

The longitudinal acoustic-wave fronts in two-dimensional random line networks composed of beams correspond to the $S0$ Lamb wave mode in thin plates. This mode has been widely used in nondestructive testing of elastic constants of paper and board. We have found[113] that the basis weight dependence of the ultrasound speed of both longitudinal and transverse $S0$ waves in paper sheets follows the effective medium result for random line networks composed of beams[II]. Also the Poisson ratio given by the velocities was found to be consistent with the calculated value, $\sigma = \frac{1}{3}$. These observations have motivated us to study by numerical simulations[114] how the fibre orientation distribution and the anisotropic drying shrinkage affect the angular dependence of ultrasound speed.

The most interesting finding of this work was the effectively one-dimensional propagation of the leading wave front. The general conditions for this behaviour were found and both analytical and numerical results for the amplitude and speed were obtained. The leading front propagating along the effectively one-dimensional paths carries different information than the usual acoustic wave following it. This property might be useful in some experimental applications. In Ref. [V] we speculated about the determination of the free fibre length and fibre orientation in paper sheets.

Our results for the roughening of wave fronts were promising but incomplete: the scaling arguments for the asymptotic roughness failed. Further work is clearly needed on this problem. In the TLM simulations we demonstrated that the leading front, which does not suffer from any backscattered waves, propagates rather undisturbed through the medium. The rest of the wave motion seems to diffuse very slowly. This phenomenon also deserves a more detailed analysis.

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[II] Publication II of this thesis.

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