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# A remark on two notions of flatness for sets in the Euclidean space

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**Abstract.** In this note we compare two ways of measuring the  $n$ -dimensional “flatness” of a set  $S \subset \mathbb{R}^d$ , where  $n \in \mathbb{N}$  and  $d > n$ . The first is to consider the classical Reifenberg-flat numbers  $\alpha(x, r)$  ( $x \in S$ ,  $r > 0$ ), which measure the minimal scaling-invariant Hausdorff distances in  $B_r(x)$  between  $S$  and  $n$ -dimensional affine subspaces of  $\mathbb{R}^d$ . The second is an “intrinsic” approach in which we view the same set  $S$  as a metric space (endowed with the induced Euclidean distance). Then we consider numbers  $\mathfrak{a}(x, r)$  that are the scaling-invariant Gromov–Hausdorff distances between balls centered at  $x$  of radius  $r$  in  $S$  and the  $n$ -dimensional Euclidean ball of the same radius. As main result of our analysis we make rigorous a phenomenon, first noted by David and Toro, for which the numbers  $\mathfrak{a}(x, r)$  behaves as the square of the numbers  $\alpha(x, r)$ . Moreover, we show how this result finds application in extending the Cheeger–Colding intrinsic-Reifenberg theorem to the biLipschitz case. As a by-product of our arguments, we deduce analogous results also for the Jones’ numbers  $\beta$  (i.e. the one-sided version of the numbers  $\alpha$ ).

## 1. Introduction and main results

In this short note we consider two ways of measuring the “flatness” of a set in the Euclidean space. The first one is by considering its best approximation by affine planes: more precisely, given a set  $S \subset \mathbb{R}^d$  and  $n \in \mathbb{N}$ , with  $n < d$ , one defines

$$(1.1) \quad \alpha(x, r) := r^{-1} \inf_{\Gamma} \mathfrak{d}_H(S \cap B_r(x), \Gamma \cap B_r(x)) \quad \text{for every } r > 0 \text{ and } x \in S,$$

where  $\mathfrak{d}_H$  is the Hausdorff distance and where the infimum is taken among all the  $n$ -dimensional affine planes  $\Gamma$  containing  $x$ .

The second more metric-oriented approach is to use the *Gromov–Hausdorff distance*, in particular given  $S \subset \mathbb{R}^d$  and  $n \in \mathbb{N}$ , with  $n < d$ , we set

$$\mathfrak{a}(x, r) := r^{-1} \mathfrak{d}_{\text{GH}}(B_r^{(S, \mathfrak{d}_{\text{Eucl}})}(x), B_r^{\mathbb{R}^n}(0)) \quad \text{for every } r > 0 \text{ and } x \in S,$$

where  $(S, \mathfrak{d}_{\text{Eucl}})$  is the metric space obtained by endowing  $S$  with the Euclidean distance. It follows immediately from the definition of  $\mathfrak{d}_{\text{GH}}$  that

$$(1.2) \quad \mathfrak{a}(x, r) \leq \alpha(x, r) \quad \text{for every } r > 0 \text{ and } x \in S.$$

Moreover, it is easy to build many examples for which

$$\alpha(x, r) \leq 4\mathbf{a}(x, r),$$

with  $\mathbf{a}(x, r) \neq 0$  and arbitrary small (one can take  $S$  to be a segment with a very small interval removed from its center). This shows that in general (1.2) cannot be improved and leads to the intuition the quantities  $\alpha(x, r)$  and  $\mathbf{a}(x, r)$  are in some sense equivalent. However, there are non-trivial cases in which the stronger inequality

$$\mathbf{a}(x, r) \leq 100\alpha(x, r)^2$$

holds. The key example is the one of a very thin triangle: let  $P, Q \in \mathbb{R}^2$  be the two points in the upper half plane that are at distance 1 from the origin and at distance  $\varepsilon \in (0, \frac{1}{2})$  from the  $x$ -axis and let  $S \subset \mathbb{R}^2$  be the union of the two closed segments joining  $P$  and  $Q$  to the origin  $O$ . It can be immediately seen that  $\alpha(O, 1) \geq \varepsilon$ , while projecting  $S$  orthogonally onto the  $x$ -axis easily shows that  $\mathbf{a}(O, r) \leq 4\varepsilon^2$ .

The aim of this note is to explore the above phenomenon, that is to clarify to which extent and in which cases the quantities  $\mathbf{a}(x, r)$  behave like the square of the quantities  $\alpha(x, r)$ .

To state our main result, we need the following notation: fixed  $\varepsilon \in (0, \frac{1}{2})$ ,  $S \subset \mathbb{R}^d$  and  $n \in \mathbb{N}$  with  $n < d$ , for every  $i \in \mathbb{Z}$  we set

$$(1.3) \quad \alpha_i := \sup_{x \in S \cap B_1(0)} \alpha(x, 2^{-i}), \quad \mathbf{a}_i := \sup_{x \in S \cap B_{1-\varepsilon}(0)} \mathbf{a}(x, 2^{-i}),$$

where we neglected the dependence on  $\varepsilon, n$  and  $S$ . Our main result reads as follows:

**Theorem A.** *For every  $n \in \mathbb{N}$  there exists  $\delta(n) > 0$  such that the following holds. Let  $S \subset \mathbb{R}^d$ , with  $d > n$ ,  $\varepsilon \in (0, \frac{1}{2})$  and define the numbers  $\alpha_i, \mathbf{a}_i$  as in equation (1.3). Suppose that  $\alpha_i \leq \delta$  for every  $i \geq \bar{i} - 2$ , for some  $\bar{i} \in \mathbb{N}$  with  $2^{-\bar{i}} < \varepsilon$ . Then*

$$(1.4) \quad \sum_{i \geq \bar{i}} \mathbf{a}_i^\lambda < C_\lambda \sum_{i \geq \bar{i}-2} \alpha_i^{2\lambda} \quad \text{for all } \lambda > 0,$$

where  $C_\lambda$  is a positive constant depending only on  $\lambda$  and  $n$ . In particular, for every  $\lambda > 0$  it holds that

$$(1.5) \quad \sum_{i \geq 0} \alpha_i^{2\lambda} < +\infty \implies \sum_{i \geq 0} \mathbf{a}_i^\lambda < +\infty.$$

Theorem A will follow from a “weak” version of the inequality  $\mathbf{a}(x, r) \lesssim \alpha(x, r)^2$  (see Theorem 3.1), which as said above cannot hold in its “strong” form.

It has to be said that the fact that the numbers  $\mathbf{a}(x, r)$  “behave” as the square of numbers  $\alpha(x, r)$  was already noted, at least at an informal level, by David and Toro (see [8]). However, to the author’s best knowledge, both the statement and the proof of Theorem A are new.

**The Jones’ numbers  $\beta$ .** We will also prove the analogue of Theorem A for the “one sided”-version of the numbers  $\alpha(x, r)$ : given a set  $S \subset \mathbb{R}^d$  and  $n \in \mathbb{N}$ , with  $n < d$ , we set

$$(1.6) \quad \beta(r, x) := r^{-1} \inf_{\Gamma} \sup_{y \in S \cap B_r(x)} \mathbf{d}(y, \Gamma) \quad \text{for every } r > 0 \text{ and } x \in S,$$

where the infimum is taken among all the  $n$ -dimensional affine planes  $\Gamma$  containing  $x$ . The numbers  $\beta(x, r)$  are usually refer to as  $L^\infty$ -Jones' numbers (see for example [9], [2] and [11]). It is immediate from the definition that  $\beta(x, r) \leq \alpha(x, r)$  for every  $x \in S$  and  $r > 0$ .

We then define the following metric analogue of  $\beta(x, r)$ : for a set  $S \subset \mathbb{R}^d$  and  $n \in \mathbb{N}$ , with  $n < d$ , we set

$$b(r, x) := r^{-1} \inf \{ \delta : \text{there exists a } \delta\text{-isometry } f : (S \cap B_r^{\mathbb{R}^d}(x), d_{\text{Eucl}}) \rightarrow (B_r^{\mathbb{R}^n}(0), d_{\text{Eucl}}) \}$$

for every  $r > 0$  and  $x \in S$  (see Section 3 for the definition of  $\delta$ -isometry). As for the numbers  $\alpha(x, r)$  and  $a(x, r)$  we have the immediate inequality

$$(1.7) \quad b(x, r) \leq 2\beta(x, r) \quad \text{for every } r > 0 \text{ and } x \in S.$$

Similarly to the numbers  $\alpha_i, a_i$  we define for a given  $S \subset \mathbb{R}^d$ ,  $n \in \mathbb{N}$  with  $n < d$ , a fixed  $\varepsilon \in (0, \frac{1}{2})$  and every  $i \in \mathbb{Z}$ ,

$$(1.8) \quad \beta_i := \sup_{x \in S \cap B_1(0)} \beta(x, 2^{-i}), \quad b_i := \sup_{x \in S \cap B_{1-\varepsilon}(0)} b(x, 2^{-i}),$$

where we neglected the dependence on  $\varepsilon, n$  and  $S$ . Then we can prove the following:

**Theorem B.** *For every  $n \in \mathbb{N}$  there exists  $\delta(n) > 0$  such that the following holds. Let  $S \subset \mathbb{R}^d$ , with  $d > n$ ,  $\varepsilon \in (0, \frac{1}{2})$  and define the numbers  $\beta_i, b_i$  as in (1.8). Suppose that  $\alpha_i \leq \delta$  for every  $i \geq \bar{i} - 2$ , for some  $\bar{i} \in \mathbb{N}$  with  $2^{-\bar{i}} < \varepsilon$  (where  $\alpha_i$  are as in (1.3)), then*

$$(1.9) \quad \sum_{i \geq \bar{i}} b_i^\lambda < C_\lambda \sum_{i \geq \bar{i} - 2} \beta_i^{2\lambda} \quad \text{for all } \lambda > 0,$$

where  $C_\lambda$  is a positive constant depending only on  $\lambda$  and  $n$ .

In particular, Theorem B implies that, whenever  $\limsup_{i \rightarrow +\infty} \alpha_i < \delta$  (with  $\delta$  as in the statement of the theorem), we have that

$$(1.10) \quad \sum_{i \geq 0} \beta_i^{2\lambda} < +\infty \implies \sum_{i \geq 0} b_i^\lambda < +\infty,$$

for every  $\lambda > 0$ .

As for Theorem A, Theorem B will be deduced from a “weak” version of the inequality  $b(x, r) \lesssim \beta(x, r)^2$ , which is also contained in Theorem 3.1.

**Converse inequalities.** In the last section we will prove that, contrary to the inequality “ $a(x, r) \lesssim \alpha^2(x, r)$ ”, the opposite estimate  $\alpha^2(x, r) \lesssim a(x, r)$  holds in full generality. Moreover, we will also prove that  $\beta(x, r)^2 \leq b(x, r)$  holds, provided  $B_r(x) \cap S$  contains  $n$  “sufficiently” independent points. This combined with the results in Theorem A and B shows that the intrinsic “Gromov–Hausdorff” approach and the “extrinsic” Hausdorff approach to measuring “flatness” in the Euclidean space are in a sense equivalent up to a square factor.

**Motivations and application to Reifenberg’s theorem.** We now explain the role, consequences and motivations of the results in this note.

It is essential to recall that the quantities  $\alpha(x, r)$  and  $\beta(x, r)$  coupled with smallness or summability conditions as in (1.5), (1.10) (also called Dini conditions) are tightly linked to

parametrization and rectifiability results for sets in the Euclidean space. The more classical are the celebrated Reifenberg theorem ([15]) and the rectifiability results of Jones ([11]), but there are also more recent and sophisticated works containing variants, generalizations and refinements of these type of statements (see for example [16], [7], [8], [9] and the references therein). It also worth to mention the works in [14], [10] and [1], which contain similar results, but where  $L^p$ -versions of the  $\beta$ -Jones' numbers are considered.

There has been recently a growing interest in extending statements for sets in  $\mathbb{R}^d$  as above (or rectifiability results in general), to the setting of metric spaces. The most notable instance of this is the intrinsic-Reifenberg theorem by Cheeger and Colding ([5]), which has recently found many applications especially in the theory of singular metric spaces with synthetic curvature conditions (see for example [5], [13] and [12]).

If one is interested in extending to the setting of metric spaces results in  $\mathbb{R}^d$  involving the quantities  $\alpha(x, r)$  and  $\beta(x, r)$  (or variants of them), it is more convenient to consider instead the numbers  $\mathbf{a}(x, r)$  and  $\mathbf{b}(x, r)$ . This is because the quantities  $\alpha(x, r)$ ,  $\beta(x, r)$  are confined to the Euclidean space, while their ‘‘Gromov–Hausdorff’’ counterparts are immediately generalized to the metric setting. For this reason it is essential to have a good understanding of the relation between the numbers  $\alpha(x, r)$ ,  $\beta(x, r)$  and the numbers  $\mathbf{a}(x, r)$ ,  $\mathbf{b}(x, r)$ . This is the point in which Theorem A and Theorem B find their relevance.

Indeed, our Theorem A shows that it makes sense to interpret the numbers  $\alpha(x, r)$  as the square of  $\mathbf{a}(x, r)$ , at least when one is interested in their decay behavior. To explain further the consequence of this fact we now show that Theorem A is crucial if one wants to extend the biLipschitz version of Reifenberg theorem to the metric setting. Let us also say that this problem is what originated the writing of this note.

We first need to recall the classical Reifenberg's theorem:

**Theorem 1.1** ([15]). *For every  $n, d \in \mathbb{N}$  with  $n < d$  there exists a constant  $\delta = \delta(n, d)$  such that the following holds. Let  $S \subset \mathbb{R}^d$  be closed, containing the origin and such that*

$$\alpha_i < \delta \quad \text{for all } i \in \mathbb{N}_0,$$

where  $\alpha_i$  are as in (1.3). Then there exists a biHölder homeomorphism

$$F : \Omega \rightarrow S \cap B_{1/2}^{\mathbb{R}^n}(0),$$

where  $\Omega$  is an open set in  $\mathbb{R}^n$ .

It was proven by Toro that if we require, besides smallness, also a fast decay of the number  $\alpha_i$  as  $i \rightarrow +\infty$ , the biHölder regularity of the map  $F$  can be improved to biLipschitz. In particular, we have the following:

**Theorem 1.2** ([16]). *For every  $\varepsilon > 0$ ,  $n, d \in \mathbb{N}$  with  $n < d$ , there exists a constant  $\delta = \delta(n, d, \varepsilon) > 0$  such that the following holds. Let  $S \subset \mathbb{R}^d$  be closed, containing the origin and such that*

$$(1.11) \quad \sum_{i \geq 0} \alpha_i^2 < \delta,$$

where  $\alpha_i$  are as in (1.3). Then there exists a  $(1 + \varepsilon)$ -biLipschitz homeomorphism

$$F : \Omega \rightarrow S \cap B_{1/2}^{\mathbb{R}^n}(0),$$

where  $\Omega$  is an open set in  $\mathbb{R}^n$ .

It is a remarkable result by Cheeger and Colding that Reifenberg’s theorem can be generalized to metric spaces:

**Theorem 1.3** ([5]). *For every  $n \in \mathbb{N}$  there exists  $\varepsilon = \varepsilon(n) > 0$  such that the following holds. Let  $(Z, d)$  be a complete metric space, let  $z_0 \in Z$  and define*

$$a_i := \sup_{z \in B_{2/3}(z_0)} 2^i d_{\text{GH}}(B_{2^{-i}}(z), B_{2^{-i}}^{\mathbb{R}^n}(0)), \quad i \in \mathbb{N}_0.$$

Suppose that

$$a_i \leq \varepsilon \quad \text{for all } i \in \mathbb{N}_0.$$

Then there exists a biHölder homeomorphism  $F : \Omega \rightarrow B_{1/2}(z_0)$ , where  $\Omega$  is an open set in  $\mathbb{R}^n$ .

As said above this result found recently a wide range of applications, in particular in the study of regularity of singular metric spaces. It is therefore natural to ask whether also an analogous of Theorem 1.2 holds in the metric setting. A careful analysis of the arguments in [5] shows that, with little modifications, they can be adapted to prove the following biLipschitz version of Theorem 1.3:

**Theorem 1.4.** *For every  $n \in \mathbb{N}$  and  $\varepsilon > 0$  there exists  $\delta = \delta(n, \varepsilon) > 0$  such that the following holds. Under the notations and assumptions of Theorem 1.3 suppose that*

$$(1.12) \quad \sum_{i \geq 0} a_i < \delta.$$

Then there exists a  $(1 + \varepsilon)$ -biLipschitz homeomorphism  $F : \Omega \rightarrow B_{1/2}(z_0)$ , where  $\Omega$  is an open set in  $\mathbb{R}^n$ .

Comparing the above with Theorem 1.2, the presence of the summability assumption (which is stronger than square summability) might lead to think that something stronger than Theorem 1.4 should hold, at least for “nicer” metric spaces, like subsets of the Euclidean space. Indeed, if one restricts its attention only to (1.2) (which as we said, cannot be improved), the summability assumption (1.12) for a subset  $S$  of the Euclidean space (when regarded as metric space  $(S, d_{\text{Eucl}})$ ) appears stronger than (1.11). Therefore it may seem that Theorem 1.2 is in a sense missing some of the information contained in the theorem of Toro.

However the key observation is that Theorem A implies that

$$(1.13) \quad \text{Theorem 1.4 is stronger than Theorem 1.2,}$$

where by “stronger” we mean that any set  $S$  satisfying the hypotheses of Theorem 1.2 also satisfies the hypotheses of Theorem 1.4 (when regarded as metric space  $(S, d_{\text{Eucl}})$ ).

Finally, Theorem A says also something about the sharpness of Theorem 1.4, indeed it is well known that the power two in (1.11) cannot be replaced by any higher order power (see for example [16]), in particular (1.5) implies that also the power one in (1.12) cannot be improved. This observation together with (1.13) suggests that Theorem 1.4 is in a sense the correct generalization of Theorem 1.2.

It is worth to mention that another instance (besides Theorem 1.4) where a summability condition on the Gromov–Hausdorff distances is natural and necessary in metric spaces was

already observed by Colding (see [6, Section 4.5]). Roughly said he proves that the summability of the Gromov–Hausdorff distance from a cone on dyadic scales on a Riemannian manifold is necessary and sufficient to have uniqueness of the tangent cone. Moreover, he points out the discrepancy between this summability assumption in comparison with the square summability assumption in Theorem 1.2 by Toro. As for the biLipschitz Reifenberg above, our Theorem A explains this discrepancy.

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## 2. Preliminary results

We gather in this section some elementary and well known results that will be needed in the sequel.

In what follows and in all this note we denote by  $d_H$  the Hausdorff distance between sets in  $\mathbb{R}^d$ . Moreover, given an affine plane  $\Gamma$  in  $\mathbb{R}^d$  and a point  $x \in \mathbb{R}^d$  we denote by  $d(x, \Gamma)$  the distance between  $x$  and  $\Gamma$ .

The following elementary Lemma is well known in literature (see e.g. [9, Lemma 12.62]).

**Lemma 2.1.** *For every  $n \in \mathbb{N}$  with  $n$  there exists a constant  $C = C(n) > 0$  such that the following holds. Suppose that  $\Gamma_1, \Gamma_2$  are two affine  $n$ -dimensional planes in  $\mathbb{R}^d$ , with  $d > n$ , such that there exist points  $\{x_i\}_{i=0}^n \subset \Gamma_2 \cap B_1^{\mathbb{R}^d}(0)$  satisfying*

$$|x_i - x_0 - e_i| \leq \frac{1}{10} \quad \text{for every } i = 1, \dots, n,$$

$$d(x_i, \Gamma_1) \leq \varepsilon \quad \text{for every } i = 0, \dots, n,$$

where  $e_1, \dots, e_d$  are orthonormal vectors in  $\mathbb{R}^d$  and  $\varepsilon \in (0, \frac{1}{100})$ . Then

$$d_H(\Gamma_1 \cap B_4(0), \Gamma_2 \cap B_4(0)) \leq C\varepsilon.$$

The following result is also standard and can be easily proved applying Gram–Schmidt orthogonalization procedure together with a straightforward computation in coordinates (see for example [8, Lemma 7.11]).

**Lemma 2.2.** *For every  $n \in \mathbb{N}$  exists a constant  $C = C(n) > 0$  such that the following holds. Let  $f : B_r^{\mathbb{R}^n}(0) \rightarrow \mathbb{R}^d$ ,  $d > n$ , be an  $(\varepsilon r)$ -isometry (see Section 3) with  $\varepsilon \in (0, 1)$ , then there exists an isometry  $I : \mathbb{R}^n \rightarrow \mathbb{R}^d$  such that  $I(0) = f(0)$  and satisfying*

$$|I - f| \leq C\sqrt{\varepsilon r} \quad \text{on } B_r^{\mathbb{R}^n}(0).$$

We now define a notion of distance between affine planes in  $\mathbb{R}^d$ .

**Definition 2.3.** Let  $\Gamma_1, \Gamma_2$  be two affine  $n$ -dimensional planes. We put

$$d(\Gamma_1, \Gamma_2) := d_H(\tilde{\Gamma}_1 \cap B_1(0), \tilde{\Gamma}_2 \cap B_1(0)),$$

where  $\tilde{\Gamma}_i$  is the  $n$  dimensional plane parallel to  $\Gamma_i$  and passing through the origin.

Notice that the function  $d$  just defined clearly satisfies

$$d(\Gamma_1, \Gamma_3) \leq d(\Gamma_1, \Gamma_2) + d(\Gamma_2, \Gamma_3).$$

The following elementary lemma says that if two affine planes are sufficiently close with respect to the distance  $d$ , then they are not orthogonal to each other.

**Lemma 2.4.** *Let  $\Gamma_1, \Gamma_2$  two  $n$ -dimensional affine planes in  $\mathbb{R}^d$  such that  $d(\Gamma_1, \Gamma_2) < 1$ . Write  $\Gamma_i$  as  $p_i + V_i$  where  $p_i \in \mathbb{R}^d$  and  $V_i$  is a  $n$ -dimensional subspace of  $\mathbb{R}^d$ . Then*

$$V_1^\perp \oplus V_2 = \mathbb{R}^d.$$

*In particular, for every  $p \in \Gamma_1$  there exists  $q \in \Gamma_2$  such that  $\Pi(q) = p$ , where  $\Pi$  is the orthogonal projection onto  $\Gamma_1$ .*

*Proof.* It is enough to prove that  $V_1^\perp \cap V_2 = \{0\}$ . Suppose  $v \in V_1^\perp \cap V_2$ . Then we can regard  $V_1, V_2$  as affine planes through the origin and parallel to  $\Gamma_1, \Gamma_2$ . Therefore by hypothesis

$$|v| = d(v, V_1) \leq d(\Gamma_1, \Gamma_2)|v|$$

and thus  $v = 0$ . □

The following simple technical result will be the main tool for the proof of Theorem 3.1.

**Lemma 2.5.** *Let  $\Gamma_1, \Gamma_2$  two  $n$ -dimensional affine planes in  $\mathbb{R}^d$ . Then for any  $x \in \Gamma_1$  and any  $y \in \mathbb{R}^d$  (different from  $x$ )*

$$|x - y|^2 \leq |\Pi(x) - \Pi(y)|^2 + |x - y|^2 \left( d(\Gamma_1, \Gamma_2) + \frac{d(y, \Gamma_1)}{|x - y|} \right)^2,$$

where  $\Pi$  denotes the orthogonal projection onto  $\Gamma_2$ .

*Proof.* Let  $\alpha := d(\Gamma_1, \Gamma_2)$ . Up to translating both the plane  $\Gamma_1$  and the points  $x, y$  by the vector  $\Pi(x) - x$ , we can suppose  $x \in \Gamma_2$  and  $x = 0$ . Let now  $p$  be the orthogonal projection of  $y$  onto  $\Gamma_1$ . Since both  $\Gamma_1$  and  $\Gamma_2$  contain the origin, we have that

$$d(p, \Gamma_2) \leq d_H(\Gamma_2 \cap B_{|p|}(0), \Gamma_1 \cap B_{|p|}(0)) \leq |p|\alpha \leq |y|\alpha.$$

Therefore  $d(y, \Gamma_2) \leq d(y, p) + d(p, \Gamma_2) = d(y, p) + d(p, \Gamma_2) \leq d(y, \Gamma_1) + |y|\alpha$ . Then by Pythagoras' theorem

$$\begin{aligned} |y|^2 &= |\Pi(y) - y|^2 + |\Pi(y)|^2 = d(y, \Gamma_2)^2 + |\Pi(y)|^2 \\ &\leq (d(y, \Gamma_1) + |y|\alpha)^2 + |\Pi(y) - \Pi(x)|^2, \end{aligned}$$

since  $\Pi(x) = 0$ . This concludes the proof. □

We conclude with two results about the numbers  $\alpha(x, r)$  and  $\beta(x, r)$  (recall their definition in (1.1), (1.6)), which are well known in the literature. The first one shows that there exists a plane which realizes  $\beta(x, r)$  (i.e. that minimizes (1.6)) and at the same time almost realizes  $\alpha(x, r)$ . The second is a classical tilting estimates, which says that the orientation of such realizing plane do not vary too much from scale to scale and between points close to each other.



**Proposition 2.6** (Realizing plane). *For every  $n \in \mathbb{N}$  there exists  $C = C(n) \geq 1$  such that the following holds. Let  $S \subset \mathbb{R}^d$  with  $d > n$ , let  $x \in S$  and  $r > 0$  be such that  $\alpha(x, r) \leq \frac{1}{100}$ . Then there exists an  $n$ -dimensional affine plane  $\Gamma_x^r$  that realizes  $\beta(x, r)$  and such that*

$$r^{-1}d_H(S \cap B_r(x), \Gamma_x^r \cap B_r(x)) \leq C\alpha(x, r).$$

*Proof.* The existence of two planes  $\Gamma$  and  $\Gamma'$  that realize  $\beta(x, r)$  and  $\alpha(x, r)$  respectively follows by compactness. Without loss of generality we can assume that  $x = 0$  and  $r = 1$ . Since  $0 \in \Gamma'$ , there exist orthonormal vectors  $e_1, \dots, e_n \in \Gamma'$  and points  $x_1, \dots, x_n \in S \cap B_1(0)$  such that  $|x_i - e_i| \leq \alpha(0, 1)$ ,  $i = 1, \dots, n$ . Moreover, there exist points  $y_0, y_1, \dots, y_n \in \Gamma$  such that  $|y_i - x_i|, |y_0| \leq \beta(0, 1) \leq \alpha(0, 1)$ ,  $i = 1, \dots, n$ . In particular,

$$|y_i - y_0 - e_i| \leq 4\alpha(0, 1), \quad i = 1, \dots, n,$$

and we can apply Lemma 2.1 to deduce that  $d_H(\Gamma \cap B_1(0), \Gamma' \cap B_1(0)) \leq C\alpha(0, 1)$ , which concludes the proof.  $\square$

**Proposition 2.7** (Tilting estimate). *For any  $n \in \mathbb{N}$  there exist constants  $\alpha = \alpha(n)$  and  $C = C(n) > 0$  such that the following holds. Let  $S \subset \mathbb{R}^d$  with  $n > d$  and let  $r > 0$  and  $x, y \in S$  be such that  $\alpha(x, r), \alpha(y, r) \leq \alpha(n)$  and  $|x - y| < \frac{1}{4}r$ . Then it holds*

$$\begin{aligned} d(\Gamma_x^r, \Gamma_x^{2r}) &\leq C(\beta(x, r) + \beta(x, 2r)), \\ d(\Gamma_x^r, \Gamma_y^r) &\leq C(\beta(x, r) + \beta(y, r)) \end{aligned}$$

for any choice of realizing planes  $\Gamma_x^r, \Gamma_y^r, \Gamma_x^{2r}$  (as given by Proposition 2.6).

*Proof.* We prove only the second, since the first is analogous.

As usual, the scaling and translation invariant nature of the statement allows us to assume that  $r = 1$  and  $x$  to be the origin. Then there exist orthonormal vectors  $e_1, \dots, e_n \in \Gamma_x^1$  and points  $x = x_0, x_1, \dots, x_n \in S \cap B_1(0)$  such that

$$\left| x_i - \frac{1}{2}e_i \right| \leq C(n)\alpha(x, 1), \quad i = 1, \dots, n.$$

Moreover, (if  $\alpha(x, 1)$  is small enough)  $x_i \in B_1(y)$ , hence there exist points  $y_0, \dots, y_n \in \Gamma_y^1$  such that  $|x_i - y_i| \leq \beta(y, 1)$ ,  $i = 0, \dots, n$ . Finally, there exist points  $z_1, \dots, z_n \in \Gamma_x^1$  such that  $|z_i - x_i| \leq \beta(x, 1)$ ,  $i = 1, \dots, n$ . Putting all together we have

$$\left| y_i - y_0 - \frac{1}{2}e_i \right| \leq C\alpha(x, 1) + 2\beta(y, 1) + \beta(x, 1), \quad i = 1, \dots, n,$$

and

$$d(y_i, \Gamma_x^1) \leq \beta(0, 1) + \beta(y, 1), \quad i = 0, \dots, n,$$

hence (if  $\alpha(x, 1), \alpha(y, 1)$  are small enough) we can apply Lemma 2.1 to deduce that

$$d_H(\Gamma_x^1 \cap B_1(0), \Gamma_y^1 \cap B_1(0)) \leq C(\beta(x, 1) + \beta(y, 1)).$$

Observing that  $d(x, \Gamma_y^1) \leq \beta(x, 1) + \beta(y, 1)$  and recalling that  $x$  is the origin concludes the proof.  $\square$

### 3. Proof of the main theorems: $\mathbf{a} \leq \alpha^2$ and $\mathbf{b} \leq \beta^2$

Both Theorem A and Theorem B will be deduced as corollaries of the following more precise result.

**Theorem 3.1.** *For every  $n \in \mathbb{N}$  there exist constants  $C = C(n) > 0$  and  $\varepsilon = \varepsilon(n) > 0$  such that the following holds. Let  $i \in \mathbb{N}_0$ ,  $S \subset \mathbb{R}^d$  with  $d > n$  and assume that  $\alpha_j \leq \varepsilon$  for every  $j \geq i - 2$  (where  $\beta_j$  are as in (1.3)), then*

$$(A) \quad \mathbf{a}(x, 2^{-i}) \leq C \left( \sup_{j \in \mathbb{N}_0} \frac{(\beta_{i-2} + \cdots + \beta_{i+j})^2}{2^j} \right) \vee C \alpha_i^2 \quad \text{for all } x \in S, |x| \leq 1 - 2^{-i},$$

$$(B) \quad \mathbf{b}(x, 2^{-i}) \leq C \sup_{j \in \mathbb{N}_0} \frac{(\beta_{i-2} + \cdots + \beta_{i+j})^2}{2^j} \quad \text{for all } x \in S, |x| \leq 1 - 2^{-i},$$

where  $\beta_j$  are as in (1.8).

Inequalities (A) and (B) should be thought as weak versions of the formal inequalities “ $\mathbf{a}(x, r) \leq C\alpha(x, r)^2$ ” and “ $\mathbf{b}(x, r) \leq C\beta(x, r)^2$ ” that are not true in general since, as we saw in the introduction, (1.2) and (1.7) cannot be improved.

*Proof of Theorem A and Theorem B, given Theorem 3.1.* Let  $\varepsilon \in (0, \frac{1}{2})$  and  $\bar{i} \in \mathbb{N}$  be as in the hypotheses of Theorem A and Theorem B. Since  $\varepsilon > 2^{-\bar{i}}$ , from Theorem 3.1 and the definition of the numbers  $\mathbf{a}_i$  we have

$$\begin{aligned} (\mathbf{a}_i)^\lambda &\leq \sup_{\substack{x \in S, \\ |x| \leq 1 - 2^{-i}}} \mathbf{a}(x, 2^{-i})^\lambda \\ &\leq C \left( \sup_{j \in \mathbb{N}_0} \frac{(\beta_{i-2} + \cdots + \beta_{i+j})^{2\lambda}}{2^{\lambda j}} \right) \vee C \alpha_i^{2\lambda} \\ &\leq C \alpha_i^{2\lambda} + C \sum_{j \geq 0} \frac{(j+3)^{2\lambda-1} \vee 1}{2^{\lambda j}} (\beta_{i-2}^{2\lambda} + \cdots + \beta_{i+j}^{2\lambda}) \quad \text{for all } i \geq \bar{i}. \end{aligned}$$

An analogous estimate holds for  $\mathbf{b}_i$  for all  $i \geq \bar{i}$ . Recalling that  $\beta_i \leq \alpha_i$ , we obtain

$$\begin{aligned} \sum_{i \geq \bar{i}} \mathbf{a}_i^\lambda &\leq C \sum_{i \geq \bar{i}} \alpha_i^{2\lambda} + C \sum_{i \geq \bar{i}} \sum_{j \geq 0} \frac{(j+3)^{2\lambda-1} \vee 1}{2^{\lambda j}} (\alpha_{i-2}^{2\lambda} + \cdots + \alpha_{i+j}^{2\lambda}) \\ &\leq C \sum_{i \geq \bar{i}} \alpha_i^{2\lambda} + C \sum_{j \geq 0} \frac{(j+3)^{2\lambda-1} \vee 1}{2^{\lambda j}} \sum_{i \geq \bar{i}} (\alpha_{i-2}^{2\lambda} + \cdots + \alpha_{i+j}^{2\lambda}) \\ &\leq C \sum_{i \geq \bar{i}} \alpha_i^{2\lambda} + C \left( \sum_{j \geq 0} \frac{(j+3)^{2\lambda} \vee (j+3)}{2^{\lambda j}} \right) \left( \sum_{i \geq \bar{i}-2} \alpha_i^{2\lambda} \right), \end{aligned}$$

which proves (1.4). The exact same computations yields also (1.9).  $\square$

Before passing to the proof of Theorem 3.1 we recall the definition of  $\delta$ -isometry and how it can be used to estimate the Gromov–Hausdorff distance. Given two metric spaces

$(X_i, d_i)$ ,  $i = 1, 2$ , and a number  $\delta > 0$ , we say that a map  $f : X_1 \rightarrow X_2$  is a  $\delta$ -isometry if  $|d_2(f(x), f(y)) - d_1(x, y)| < \delta$  for every  $x, y \in X_1$ . It holds that

$$d_{\text{GH}}((X_1, d_1), (X_2, d_2)) \leq 2 \inf\{\delta > 0 : \text{there exists } \delta\text{-isometry } f : X_1 \rightarrow X_2 \\ \text{with } f(X_1) \text{ } \delta\text{-dense in } X_2\},$$

see for example [4] for a proof.

*Proof of Theorem 3.1.* Observe that it is sufficient to consider the case  $x = 0$  and  $i = 0$  for both (A) and (B), since the conclusion then follows by translating and scaling.

We define

$$\theta := C^2 \sup_{j \in \mathbb{N}_0} \frac{(\beta_{-2} + \cdots + \beta_j)^2}{2^j}, \quad \theta' := \max(\theta, C\alpha_0^2),$$

where  $C$  is a big enough constant depending only on  $n$ , to be determined later. Before proceeding we make the following observation:

$$(3.1) \quad C^2(\beta_{-2} + \cdots + \beta_j)^2 > \lambda > 0 \implies \theta > \frac{2\lambda}{2^j}.$$

Along the proof, for a given  $x \in S$  and  $r > 0$  we will denote by  $\Gamma_x^r$  one of the realizing planes given by Proposition 2.6 (the choice of the particular plane is not relevant).

**Proof of (B).** Let  $\Pi$  be the orthogonal projection onto  $\Gamma_0^1$ . It is sufficient to show that

$$(3.2) \quad \Pi : S \cap B_1(0) \rightarrow \Gamma_0^1 \cap B_1(0) \quad \text{is a } \theta\text{-isometry}$$

with respect to the Euclidean distance.

Choose  $x, y \in S \cap B_1(0)$  distinct and observe that there exists a unique integer  $j \geq 0$  such that

$$(3.3) \quad \frac{1}{2^j} \leq |x - y| < \frac{1}{2^{j-1}}.$$

Applying Proposition 2.7 multiple times (assuming  $\alpha_j \leq \alpha(n)$  for every  $j \geq i - 2$ , with  $\alpha(n)$  as in the statement of Proposition 2.7), we have

$$(3.4) \quad d(\Gamma_0^1, \Gamma_x^{2^{-j+1}}) \leq d(\Gamma_0^1, \Gamma_0^2) + d(\Gamma_0^2, \Gamma_0^{2^2}) + d(\Gamma_0^{2^2}, \Gamma_x^{2^2}) \\ + d(\Gamma_x^{2^2}, \Gamma_x^{2^1}) + \cdots + d(\Gamma_x^{2^{-j+2}}, \Gamma_x^{2^{-j+1}}) \\ \leq D(\beta(0, 0) + \beta(0, 2) + \beta(0, 2^2) + \beta(x, 2^2) + \cdots + \beta(x, 2^{-j+1})) \\ \leq D(\beta_{-2} + \cdots + \beta_{j-1})$$

for some constant  $D$  depending only on  $n$ . We consider now two cases: when

$$(D + 4)(\beta_{-2} + \cdots + \beta_{j-1}) > 1$$

or the opposite. In the first case, assuming that  $C \geq D + 4$ , from (3.1) we have  $\theta \geq \frac{2}{2^j}$  and therefore

$$\|\Pi(x) - \Pi(y)\| - |x - y| \leq |x - y| \leq \frac{2}{2^j} < \theta,$$

that is what we wanted. Hence we can suppose that  $(D + 4)(\beta_{-2} + \dots + \beta_{j-1}) \leq 1$ . Since from (3.3) it holds that  $|x - y| \geq 2^{-j}$ , we have that

$$(3.5) \quad d(y, \Gamma_x^{2^{-j+1}}) \leq 4\beta(x, 2^{-j+1})|x - y|.$$

We can now apply Lemma 2.5 to the planes  $\Gamma_0^1, \Gamma_x^{2^{-j+1}}$ , that coupled with (3.4) and (3.5) gives

$$|\Pi(x) - \Pi(y)| \geq |x - y| \sqrt{1 - (D(\beta_{-2} + \dots + \beta_{j-1}) + 4\beta_{j-1})^2}.$$

Hence

$$|x - y| - |\Pi(x) - \Pi(y)| \leq |x - y| \left(1 - \sqrt{1 - ((D + 4)(\beta_{-2} + \dots + \beta_{j-1}))^2}\right).$$

Thanks to the assumption  $(D + 4)(\beta_{-2} + \dots + \beta_{j-1}) \leq 1$ , we can use the elementary inequality  $1 - \sqrt{1 - x} \leq x$ , valid for  $0 \leq x \leq 1$ , to finally obtain

$$\begin{aligned} \|x - y| - |\Pi(x) - \Pi(y)\| &\leq |x - y| ((D + 4)(\beta_{-2} + \dots + \beta_{j-1}))^2 \\ &\stackrel{(3.3)}{\leq} \frac{((D + 4)(\beta_{-2} + \dots + \beta_{j-1}))^2}{2^{j-1}} \leq \theta, \end{aligned}$$

where we have used the definition of  $\theta$  and assuming  $C \geq 2(D + 4)$ . This concludes the proof of (3.2) and thus the proof of (B).

**Proof of (A).** In view of (3.2), we only need to show that  $\Pi$  is also  $\theta'$ -surjective.

**Claim.** Let  $C' = C'(n) \geq 1$  be the constant given by Proposition 2.6. Then for every  $p \in \Gamma_0^1 \cap B_1(0)$  and  $x \in S \cap B_1(0)$  such that  $C'\alpha_0 \geq |p - \Pi(x)| \geq \theta'$  it holds

$$B_{3/4|p - \Pi(x)|}(p) \cap \Pi(S \cap B_1(0)) \neq \emptyset.$$

Before proving the claim, we show that it implies that  $\Pi$  is  $\theta'$ -surjective. Indeed, suppose it is not, i.e. there exists  $p \in \Gamma_0^1 \cap B_1(0)$  such that

$$R := \sup\{r : B_r(p) \cap \Pi(S \cap B_1(0)) = \emptyset\} \geq \theta'.$$

Since  $d_H(\Gamma_0^1 \cap B_1(0), S \cap B_1(0)) \leq C'\alpha_0$  (recall that  $\Gamma_0^1$  was chosen as a realizing plane as given by Proposition 2.6), there exists  $x \in S \cap B_1(0)$  such that  $|x - p| \leq C'\alpha_0$  and in particular  $|\Pi(x) - p| \leq C'\alpha_0$ . Therefore  $R \leq C'\alpha_0$ . This implies, from the definition of  $R$ , that there exists a point  $x' \in S \cap B_1(0)$  such that  $\theta' \leq R \leq |\Pi(x') - p| \leq \min(\frac{5}{4}R, C'\alpha_0)$ . However, the Claim gives that

$$\emptyset \neq B_{3/4|\Pi(x') - p|}(p) \cap \Pi(S \cap B_1(0)) \subset B_{\frac{15}{16}R}(p) \cap \Pi(S \cap B_1(0))$$

that contradicts the minimality of  $R$ .

**Proof of the Claim.** Set  $R := |p - \Pi(x)|$ . To make the proof more easy to follow, we first explain the intuition behind it. The key idea is that near  $x$  the set  $S$  is distributed in a horizontal manner, near a plane passing through  $x$ . We can then move along this plane towards  $p$  and thus find a point  $y$  in  $S \cap B_1(0)$  such that  $|\Pi(y) - p| \sim \frac{R}{2}$ . However, since  $p$  can be near the boundary of  $B_1(0)$ , in this movement we might go outside the ball  $B_1(0)$ .

To avoid this issue, we consider a point  $q$  such that  $|p - q| \sim \frac{R}{2}$  but placed radially towards the origin and then find a point  $y$  (using the idea described above of moving horizontally near  $x$ ) that projects near  $q$ .

Start by noticing that (if  $\alpha_0$  is small enough with respect to  $n$ )  $R \leq C'\alpha_0 < \frac{1}{4}$ . Therefore there exists a unique integer  $j \geq 2$  such that

$$(3.6) \quad \frac{1}{2^{j+1}} \leq R < \frac{1}{2^j}.$$

Since by assumption  $\theta \leq \theta' \leq R \leq \frac{1}{2^j}$ , from (3.1) we have

$$(C(\beta_{-2} + \dots + \beta_j))^2 \leq \frac{1}{2}.$$

Define now the point  $q \in \Gamma_0^1 \cap B_1(0)$  as

$$q = p - \frac{p}{|p|} \frac{R}{2}.$$

Then

$$(3.7) \quad |q| = \left| |p| - \frac{R}{2} \right| \leq 1 - \frac{R}{2};$$

indeed,  $|p| < 1$  and  $R < 1$ . Moreover,

$$|p - q| = \frac{R}{2} \quad \text{and} \quad |q - \Pi(x)| \leq |p - \Pi(x)| + |p - q| = \frac{3}{2}R.$$

Consider the plane  $\Gamma_x^{2^{-j}}$ , arguing as in (3.4) we can show that

$$d(\Gamma_0^1, \Gamma_x^{2^{-j}}) \leq C(\beta_{-2} + \dots + \beta_j) < 1,$$

provided  $C$  is big enough. Then by Proposition 2.4 there exists a point  $e \in \Gamma_x^{2^{-j}}$  such that  $\Pi(e) = q$ . Applying Lemma 2.5, we obtain

$$|e - x|^2 \leq |q - \Pi(x)|^2 + \frac{|e - x|^2}{2}$$

that implies

$$|e - x| \leq \sqrt{2}|q - \Pi(x)| \leq 3R \leq \frac{1}{2^{j-2}}.$$

Therefore there exists  $y \in S \cap B_{2^{-j+2}}(x)$  such that  $|y - e| \leq \alpha_{j-2} 2^{-j+2} < \frac{R}{4}$  (provided  $\alpha_{j-2}$  is small enough). Thus

$$|\Pi(y) - p| \leq |\Pi(y) - \Pi(e)| + |p - q| \leq |y - e| + \frac{R}{2} < \frac{3}{4}R,$$

that means  $\Pi(y) \in B_{3/4R}(p)$ . It remains to prove that  $y \in B_1(0)$ . First we observe that from (3.6) and the assumption  $R \leq C'\alpha_0$  we have

$$|y - x| \leq \frac{4}{2^j} \leq 8R \leq 8C'\alpha_0.$$

Hence, since  $x \in B_1(0)$ ,

$$d(y, \Gamma_0^1) \leq |x - y| + d(x, \Gamma_0^1) \leq 9C'\alpha_0.$$

From previous computations we know that

$$|\Pi(y) - q| = |\Pi(y) - \Pi(e)| \leq |y - e| \leq \frac{R}{4},$$

therefore from (3.7),  $|\Pi(y)| \leq |q| + \frac{R}{4} \leq 1 - \frac{R}{4}$ . From the Pythagoras Theorem we obtain

$$\begin{aligned} |y|^2 &= |\Pi(y)|^2 + d(y, \Gamma_0^1)^2 \\ &\leq \left(1 - \frac{R}{4}\right)^2 + (9C')^2 \alpha_0^2 \\ &= 1 + R \left( \frac{R}{16} + \frac{(9C')^2 \alpha_0^2}{R} - \frac{1}{2} \right). \end{aligned}$$

Thus to conclude it is enough to show that

$$\frac{R}{16} + \frac{(9C')^2 \alpha_0^2}{R} < \frac{1}{2}.$$

Since by assumption  $R \geq \theta'$  and by definition  $\theta' \geq C\alpha_0^2$ , we deduce that

$$\frac{\alpha_0^2}{R} \leq \frac{1}{C}.$$

Therefore recalling that  $R < 1$ , the above inequality is satisfied as soon as  $C < 4(9C')^2$ . This concludes the proof.  $\square$

#### 4. Converse inequalities: $\alpha^2 \leq \mathbf{a}$ , $\beta^2 \leq \mathbf{b}$

As explained in the introduction and in Section 3, the inequalities “ $\mathbf{a}(x, r) \leq C\alpha(x, r)^2$ ” and “ $\mathbf{b}(x, r) \leq C\beta(x, r)^2$ ” are not true in general, but hold only in their weaker formulations contained in Theorem 3.1. In this final section we will prove that the opposite inequality  $\alpha(x, r)^2 \leq C\mathbf{a}(x, r)$  do hold in general, together with a weaker version of  $\beta(x, r)^2 \leq C\mathbf{b}(x, r)$ .

**Proposition 4.1.** *Let  $S \subset \mathbb{R}^d$  and  $n \in \mathbb{N}$  with  $n < d$ . Then*

$$\alpha(x, r)^2 \leq C\mathbf{a}(x, r) \quad \text{for every } x \in S, r > 0,$$

where  $\alpha(x, r)$  is as in (1.1) and  $C = C(n) > 0$ .

*Proof.* Straightforward from Lemma 2.2  $\square$

Given a set of  $n + 1$  points  $x_0, \dots, x_n \in \mathbb{R}^d$ , we denote by  $\text{Vol}_n(x_0, \dots, x_n)$  the volume of the  $n$ -dimensional simplex with vertices  $x_0, \dots, x_n$ . It is well known that for every  $n \in \mathbb{N}$  there exists a polynomial  $P_n : \mathbb{R}^{(n+1)n/2} \rightarrow \mathbb{R}$  such that

$$(4.1) \quad \text{Vol}_n(x_0, \dots, x_n)^2 = P_n(\{|x_i - x_j|^2\}_{0 \leq i < j \leq n}),$$

see for example [3, Section 40] for a proof.

The following results states that  $\beta(x, r) \leq C\sqrt{\mathbf{b}(x, r)}$ , provided  $B_r(x) \cap S$  contains  $n$  points which are “sufficiently independent” in the sense that they span a simplex with large volume.

**Proposition 4.2.** *Let  $S \subset \mathbb{R}^d$  and  $n \in \mathbb{N}$  with  $n < d$ . Then*

$$\beta(x, r) \leq C \left( \frac{\sqrt{\mathbf{b}(x, r)}}{V_n} \wedge V_n^{\frac{1}{n}} \right) \quad \text{for every } x \in S, r > 0,$$

where

$$V_n = \sup_{\{x_i\}_{i=0}^n \subset S \cap B_r(x)} r^{-n} \text{Vol}_n(x_0, \dots, x_n),$$

$\beta(x, r)$  is as in (1.6) and  $C = C(n) > 0$ . In particular, if  $\alpha(x, r) < \frac{1}{8}$  ( $\alpha(x, r)$  is as in (1.1)), then

$$\beta(x, r) \leq 100C \sqrt{\mathbf{b}(x, r)} \quad \text{for every } x \in S, r > 0,$$

*Proof.* After a rescaling we can consider only the case  $r = 1$ . We can also suppose that  $V_n > 0$ , otherwise there is nothing to prove. Fix  $\varepsilon > 0$ . There exists a map

$$f : S \cap B_1(0) \rightarrow B_1^{\mathbb{R}^n}(0)$$

that is a  $(\mathbf{b}(x, 1) + \varepsilon)$ -isometry. Moreover, there exist  $\{x_i\}_{i=0}^n \subset S \cap B_1(x)$  such that

$$\text{Vol}_n(x_0, \dots, x_n) > V_n - \varepsilon.$$

Let  $x_{n+1} \in S \cap B_1(x)$  be arbitrary and observe that, since  $f(x_0), \dots, f(x_n), f(x_{n+1}) \in \mathbb{R}^n$ , we must have  $\text{Vol}_{n+1}(f(x_0), \dots, f(x_n), f(x_{n+1})) = 0$ . From (4.1) and the fact that  $P_{n+1}$  is locally Lipschitz, it follows that

$$\begin{aligned} \text{Vol}_{n+1}(x_0, \dots, x_n, \bar{x})^2 &\leq C(n) \sup_{0 \leq i < j \leq n} \|f(x_i) - f(x_j)\|^2 - |x_i - x_j|^2 \\ &\leq 4C(n)(\mathbf{b}(x, 1) + \varepsilon). \end{aligned}$$

Therefore, denoted by  $\Gamma$  the  $n$ -dimensional plane spanned by  $x_0, \dots, x_n$ , it holds

$$d(x_{n+1}, \Gamma) = \frac{\text{Vol}_{n+1}(x_0, \dots, x_n, x_{n+1})}{\text{Vol}_n(x_0, \dots, x_n)} \leq C(n) \frac{\sqrt{\mathbf{b}(x, 1) + \varepsilon}}{V_n - \varepsilon}.$$

Moreover, it is clear that there exists a constant  $C'(n) > 0$  such that

$$\text{Vol}_{n+1}(x_0, \dots, x_n, x_{n+1}) \leq C' V_n^{\frac{n+1}{n}}.$$

From the arbitrariness of  $x_{n+1} \in S \cap B_1(x)$  and  $\varepsilon > 0$  the conclusion follows.  $\square$

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