# Mathematical Constants, Mathematical Constants II

*by Steven R. Finch*

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**REVIEWED BY OSMO PEKONEN**

How many significant mathematical constants do you know? Most students of mathematics will remember $\pi$ and $e$, of course, and if pressed, the golden mean $\varphi$ and the Euler–Mascheroni constant $\gamma$. Roger Apéry made headlines in 1978 when he proved that $\zeta(3)$, the value of the Riemann zeta function at the odd positive integer 3, is irrational. An inscription on his tombstone at the Père Lachaise Cemetery in Paris proclaims:

$$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\cdots \neq \frac{p}{q}.$$

A generalization of Apéry's work to $\zeta(2k+1)$ for $k>1$ still is "a mystery wrapped in an enigma" (as described in the spring 1979 issue of this magazine). For Catalan's constant

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=0.9159655941\ldots,$$

a similar irrationality result is also unknown. Steven R. Finch's incredible labor of love, an encyclopedia of mathematical constants, begins with such basics, then moves on to more elaborate topics.

Some mathematical constants are known as precise closed-form expressions, but many a constant manifests itself only as an approximation known to a few decimal places or located between bounds whose improvement is a challenge. Chaitin's evasive constant $\Omega$, or the halting probability of a self-delimiting universal computer, exists but cannot be computed. And then there are purely conjectural constants whose very existence has not been established. Some constants discussed arise naturally from authentic mathematical problems, while others look more like recreations. Among the latter, let us mention as an example Trott's constant $E$ = 0.1084101512…, the unique number whose decimal digits coincide with the sequence of coefficients in its expansion as a continued fraction. An open question is whether $E$ is transcendental.

One wonders how these two volumes were born. Published as books 94 and 169 of the *Encyclopedia of Mathematics and Its Applications*, they contain a total of 269 meticulously documented essays from all fields of mathematics. To name a few, in Volume I there are articles on the statistics of continued fractions, chaos in nonlinear systems, prime numbers, sum-free sets, isoperimetric problems, approximation theory, self-avoiding walks and the Ising model (from statistical physics), binary and digital search trees (from theoretical computer science), the Prouhet–Thue–Morse sequence, univalent functions, geometric probability, the traveling salesman problem, and best packings. And in Volume II, one can find essays on number theory, graph theory, combinatorics, inequalities and approximation, elliptic curves and modular forms, Poisson–Voronoi tessellations, random triangles, Brownian motion, Prandtl–Blasius flow (from fluid dynamics), Lyapunov exponents, knots and tangles, continued fractions, Galton–Watson trees, electrical capacitance (from potential theory), Zermelo's navigation problem, and the minimality of soap film surfaces. It appears astonishing to me that a single individual went through all these topics. His achievement can only be compared to the On-Line Encyclopedia of Integer Sequences. Could this be yet another case of obsessive collecting disorder? By no means, for we learn that Steven R. Finch is also an accomplished pianist and composer, so he does have a life beyond formulas and numbers.

Some of the most intriguing formulas of mathematics (like those of Ramanujan) adorn this treasure trove of mathematical gems. Open problems abound. Finch generously gives credit to obscure papers by baptizing some lesser-known constants with the names of half-forgotten authors, so his two books set standards for nomenclature. Honoring the elders, he would like to call $\sqrt{2}$ Pythagoras's constant and $\pi$ Archimedes's constant. Well-known mathematicians who get a lot of hits in the index of names of Volume I include the Borwein brothers, Conway & Sloane, Erdős, Gardner, Gradhsteyn & Ryzhik, Hardy & Littlewood, Knuth, Plouffe, Pólya & Szegő, Ribenboim, Zeilberger, and many others. Finch's mentor Philippe Flajolet (1948–2011) is often cited as well. Volume II unfortunately has no index of names, but many of the same names found in the first volume appear here as well. Finch stops at the gates of string theory, mentioning some results in enumerative geometry due to Kontsevich, Vafa, Witten, and others, but concedes that "much work lies ahead to rigorously confirm everything written here."

Amazingly, much of mathematics seems to be nothing but a story about $\pi$, $e$, $\varphi$, and $\gamma$. These constants keep appearing under different guises throughout the two volumes. Finch would like to demonstrate that other constants as well might be more important and interconnected than we think. "That is, if we work and listen hard enough, the echoes will become audible." Even so, no other mathematical constants seem to gain prominence comparable to that of the classical quartet.

Jet Wimp reviewed Finch's first volume in this magazine (summer issue 2004), asserting that "all mathematicians should own this book." The same applies to the second volume.


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