The prospective reserve of a life insurance contract with modifications in a Non-Markovian setting

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Tämän tutkielman tarkoituksena on tarkastella henkivakuutussopimukseen liittyvää diskontattua tulevaisuuden varantoa (prospective reserve). Tavoitteena on yleistää käsitteitä ja teorioita Markovilaisesta asetelmasta epä-Markovilaiseen tilanteeseen luopumalla Markovilaisista oletuksista. Markov-prosessin lisäykset ovat aina toisistaan ja ajasta riippumattomia, jota emme oleta yleisten hyppyprosessien kohdalla. Henkivakuutussopimuksen tilaa kuvataan diskreetillä hyppyprosessilla, joka voi kertoa esimerkiksi, onko asiakas vielä elossa ja onko sopimus loppunut vai ei.

Tulemme osoittamaan, että tulevaisuuden varannolla on vaihtoehtoinen esitysmuoto takaperoisen stokastisen differentiaaliyhtälön (TSDY) ratkaisuna myös epä-Markovilaisessa tapauksessa. Lisäksi tutkimme epälineaarista varannonkeruuta, jossa varallisuusprosessi saa riippua sopimukseen liittyvästä kerättävästä maksureservistä. Yleensä tämä tapahtuu, kun otetaan huomioon mahdolliset sopimusmuutokset kesken sopimuskauden. Yleinen tapa on hyödyntää jo kerättyä osuutta odotetusta reservistä muutoksista aiheutuvien kulujen hyvittämiseen, ja ylijäämä käsitellään usein asiakkaan varallisuutena. Tällöin kerättävää reserviä voidaan pitää yhtenä varallisuusprosessin osana, joka aiheuttaa iteratiivisen kierteen määritelmässä, ja tulevaisuuden varannon määritelmän oikeellisuutta pitää jatkotarkastella.

Tutkielman tärkeimmät tulokset ovat analoginen jatkumo sekä Thielen yhtälölle että Cantellin Teorialle epä-Markovilaiseen asetelmaan. Thielen yhtälöä käytetään yleisesti henkivakuutusmatematiikassa TSDY-esityksen löytämiseen tulevaisuuden varannolle. Cantellin Teoria taas takaa tarpeelliset ehdot aktuaarisen tasapainon säilyttämiseen sopimusmuutoksissa.

Lopuksi rakennamme paljon teoriaa hyppyprosessien, niiden kompensaattorien sekä kompensoitujen martingaalien ympärille. Esittelemme eksplisiittisen kaavan kompensaattoreille ja todistamme Itô isometriaa vastaavan tuloksen kompensoiduille hyppyprosesseille. Tulemme myös johtamaan eksplisiittisen ratkaisun Martingaaliesitys Teoriaan minkä tahansa integroitavan satunnaismuuttujan ehdolliselle odotusarvolle. Myöhemmissä kappaleissa hyödynnämme tätä tulosta tulevaisuuden varannon TSDY-esityksen validoimiseen.

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In this thesis we inspect the prospective reserve of a life insurance contract. The objective is to generalize the concepts from the Markovian framework into the non-Markovian setting. A Markov process has independent increments which is not assumed for pure jump processes. The changes of the state of the life insurance contract can therefore posses dependencies among themselves.

The prospective reserve will have a backward stochastic differential equation representation even in the non-Markovian setting. Furthermore we will consider the case of non-linear reserving where the payment process is allowed to be depended on the prospective reserve. This occurs under contract modifications where the current premium reserve is utilized to cover the liabilities induced by the modification and the rest is viewed as the assets of the customer. In other words the charged premiums in the life insurance contract are allowed to be calculated utilizing the present expected premium reserve as a part of the payment process. This creates a iterative cycle which questions the validity of the definition of the prospective reserve.

The main theorems in this thesis are analogous extensions of the Thiele equation and the Cantelli Theorem to the non-Markovian setting. The Thiele equation is utilized to prove the BSDE representation for the prospective reserve and the Cantelli Theorem yields means to sustain the actuarial equivalence at contract modifications.

Lastly we construct a lot of theory around jump processes, their compensators and compensated martingales even providing an explicit formula for the stochastic intensities and an Itô type of isometry for the compensated jump processes. We also prove an explicit solution to the Martingale Representation Theorem for a specific type of a stochastic process, which is applied to the prospective reserve.

Contents

CHAPTER 1

Introduction

The actuarial literature concerning a Markovian jump process displaying the contract states is well established. We will be focusing on the more general jump processes without the Markov-assumption in the context of the life insurance mathematics. A common theme in the life insurance is maintaining on average a balance between the liabilities and the assets of the contractor over the life insurance contracts. One utilized tool to prepare for the future insurance compensations as an insurance contractor is called the prospective reserve. This is essentially equivalent to the expected net present value of the contract given the history that has happened all ready during the contract period. One must also consider the present value of the future cash flow by discounting it with respect to the inflation rate. With higher sum insured the risk involved also contributes to proportionally higher net premiums required to possibly counterbalance the increased future liabilities towards the customer.

A more natural way of defining the prospective reserve is straight from the payment process which illustrates the value of the contract at each given time during the contract period. This is referred to linear reserving and we will see in the Chapter 4 that the prospective reserve satisfies a Thiele type of backward stochastic differential equation (BSDE) also in the non-Markovian setting. The foundations to this will be constructed in the Chapter 3 which displays general theory about jump processes without the Poisson process -correspondence. A critical note is that we will only treat life insurance contract with a terminal time T. This means that the contract dissolves upon reaching the predetermined time T if the contract has not been redeemed before that.

According to [**8**, 6.8] the reserve part of the premiums could be considered to being an asset of the insured. Therefore in the case of contract modifications the prospective reserve can be conveniently utilized to finance any expenses induced by the policy alterations. This creates a circular dependence between the cash flow and the net premium reserve and makes the definition of the reserve significantly more intricate. In the latter part of the Chapter 4 we will deliberate this aforementioned phenomenon coined nonlinear reserving. It turns out in the Theorem 4.4 that the BSDE validates the definition of the prospective reserve in the nonlinear case also maintaining the actuarial equivalence, given the generator of the BSDE satisfies some standard conditions. [**6**, Theorem 5.1]

Lastly in the Chapter 5 we will incorporate the contract modifications to the non-Gaussian framework and prove the existence of the prospective reserve. The Extended Cantelli Theorem proves that this is equal with the assumption that the accumulated net premium reserve covers the insurance benefit at the time of the conversion of the insurance policy terms. In the literature this is referred to sum-at-risk being zero at the occurrence of the contract modification. We will introduce a new part to the payment process which regards the expenses associated to the modification. The thesis is constructed base on the research paper published by Christiansen and Djehiche in 2021. [**4**]

CHAPTER 2

Preliminaries

2.1. Probability space

We will start by recalling the definition of a probability space.

Definition 2.1 (σ-algebra). Let Ω be a non-empty set. Then the collection of sets $\mathcal{F}$ is called a σ-algebra in Ω, if

(1) For all $A \in \mathcal{F}$ also $A \subset \Omega$,
(2) $\Omega \in \mathcal{F}$,
(3) For all $A \in \mathcal{F}$ also $A^C \in \mathcal{F}$,
(4) For all $A_1, A_2, \dots \in \mathcal{F}$ also $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ defined in Definition 2.1 is called a *measurable space*. The σ-algebra will act as a domain for the probability measure.

Definition 2.2 (Probability measure). The function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called a probability measure if

(1) $\mathbb{P}(\Omega) = 1$,
(2) For all disjoint sets $A_1, A_2, \dots \in \mathcal{F}$ it holds that

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*.

Definition 2.3 (Complete probability space). A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *complete*, if for all $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 0$ it holds

$$A \in \mathcal{F}, \quad \text{for all } A \subseteq E.$$

2.2. Stochastic processes

Definition 2.4 (Random variable). Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces. A function $f : \Omega \to E$ is called *measurable*, if for all $B \in \mathcal{E}$ it holds that

$$f^{-1}(B) \in \mathcal{F}.$$

In the probability theory a measurable function is called a *random variable*.

Definition 2.5 (Stochastic process). Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces. Assume for some index set I the random variables $X_t : \Omega \to E$, $t \in I$. The family of random variables $(X_t)_{t \in I}$ is called a *stochastic process*.

Remark 2.6. The measurable space $(E, \mathcal{E})$ in Definition 2.5 is called the *state space*.

For the further classification of stochastic processes we will introduce the notion of filtration.

Definition 2.7 (Filtration, [**10**, Definition 3.1.1]). Assume a measurable space $(\Omega, \mathcal{F})$ and an index set I. The collection of σ-algebras $(\mathcal{F}_t)_{t\in I}$ is a *filtration*, if

$$\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F} \quad \text{for all } s, t \in I, t \leq s$$

Definition 2.8. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and its filtration $(\mathcal{F}_t)_{t\geq 0}$ together is a called a *stochastic basis* and is denoted by $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$.

Remark 2.9. A filtration is said to be generated by a process $X = (X_t)_{t\geq 0}$ and is marked $(\mathcal{F}_t^X)_{t\geq 0}$, if for all $t \geq 0$ it holds that $\mathcal{F}_t^X = \sigma(X_s : s \in [0, t])$, where $\sigma(X)$ denotes the smallest σ-algebra such that the mapping X is a random variable. The filtration $(\mathcal{F}_t^X)_{t\geq 0}$ is called a *natural filtration* of the process X.

Definition 2.10 (Augmented filtration). A filtration $(\mathcal{F}_t)_{t\geq 0}$ with respect to a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *augmented*, given for all $E \in \mathcal{F}$ with $\mathbb{P}(E) = 0$ it holds $E \in \mathcal{F}_0$.

Definition 2.11 (Right-continuous filtration). Assume a measurable space $(\Omega, \mathcal{F})$. A filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous on $(\Omega, \mathcal{F})$ if and only if for all $t \geq 0$ it holds

$$\mathcal{F}_t = \bigcap_{s:s>t} \mathcal{F}_s.$$

Definition 2.12 (Usual conditions,[**9**, Definition 2.25]). Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a filtration fulfills the *usual conditions* if it is right-continuous and augmented.

Remark 2.13. As described in [**11**] any probability space can be completed by including the null-sets from Definition 2.3. In the setting of complete probability space any filtration with respect to the probability space can be augmented by adding all of the null sets of the underlying probability space to each of the σ-algebras in the filtration ([**11**]). Moreover any filtration $(\mathcal{F}_t)_{t\geq 0}$ can be expanded to be right-continuous by defining $\mathcal{G}_t := \cap_{s:s>t}\mathcal{F}_s$ for each t, and replacing the filtration $(\mathcal{F}_t)_{t\geq 0}$ with the filtration $(\mathcal{G}_t)_{t\geq 0}$ ([**11**]). This is an expansion, because of the relation $\mathcal{F}_t \subseteq \mathcal{F}_s$ for all $t \leq s$.

Definition 2.14 (Adapted process). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. A process $(X_t)_{t\geq 0}$ is called *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$, if X_t is $\mathcal{F}_t$ measurable for all $t \geq 0$.

Remark 2.15. Note that any process is always adapted with respect to its natural filtration, due to the construction of the filtration.

Definition 2.16 (Progressively measurable process, [**9**, Section 1.1. Definition 1.11]). Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ and state space $(E, \mathcal{E})$. A stochastic process X is *progressively measurable* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for all $t \geq 0$ the mapping $X : [0, t] \times \Omega \to E, (s, \omega) \mapsto X(s, \omega)$ is measurable between the measurable spaces $([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F})$ and $(E, \mathcal{E})$.

DEFINITION 2.17 (Predictable process, [**3**, I D4]). Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. The σ-algebra $\mathcal{P}$ over $(0, \infty) \times \Omega$ defined as

$$\mathcal{P} := \sigma(\{(s,t] \times A : 0 \leq s \leq t, A \in \mathcal{F}_s\})$$

is called a *predictable* σ-algebra. A stochastic process $(X_t)_{t\geq 0}$ is called *predictable*, if it is measurable with respect to the σ-algebra $\mathcal{P}$.

REMARK 2.18. Any left-continuous adapted process is also predictable ([**2**, Proposition 3.5.2 (ii)]). The predictable σ-algebra $\mathcal{P}$ can also be generated by all continuous and $(\mathcal{F}_t)_{t\geq 0}$-adapted processes ([**2**, Proposition 3.5.2 (ii)]).

DEFINITION 2.19 (Counting process,[**12**, I 2.3]). Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. Let $(T_n)_{n=0}^{\infty}$ be a sequence of random variables such that $0 = T_0 < T_1 < T_2 < \dots$ almost surely, and define

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{t \geq T_n\}}$$

Then the process $N = (N_t)_{t\geq 0}$ is called a *counting process*. The state space of the process N is $(\bar{\mathbb{N}}, 2^{\bar{\mathbb{N}}})$, where $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

2.3. Martingales

We denote the expected value of a random variable X by $\mathbb{E}X = \int_\Omega X \, d\mathbb{P}$, assuming it exists. If the expected value is finite for a random variable X, then the random variable X is called *integrable*. For a rigorous construction of the expected value, reader is referred to [**7**, Chapter 6] or [**14**, Chapters 5 & 6].

PROPOSITION 2.20 (Theorem of Radon-Nikodym, [**7**, Theorem 7.2.1]). *Assume a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a signed measure* μ. *If* $\mu \ll \mathbb{P}$ *then there exists an integrable and measurable* $L : \Omega \to \mathbb{R}$ *which satisfies*

$$\mu(A) = \int_A L \, d\mathbb{P} \quad \forall A \in \mathcal{F}.$$

DEFINITION 2.21 (Radon-Nikodym-derivative, [**7**, Definition 7.2.2]). We call the random variable L from the Proposition 2.20 the Radon-Nikodym-derivative and mark

$$L = \frac{d\mu}{d\mathbb{P}}.$$

Moreover it holds that

$$\int_\Omega \mathbb{1}_A \, d\mu = \int_\Omega \mathbb{1}_A L \, d\mathbb{P}, \quad \text{for } A \in \mathcal{F}. \tag{2.1}$$

DEFINITION 2.22 (Conditional expectation, [**14**, Section 9.2.]). Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ and an integrable random variable $X : \Omega \to \mathbb{R}$. An integrable random variable Y fulfilling the properties

(1) Y is $\mathcal{G}$-measurable and
(2) for all $A \in \mathcal{G}$ it holds $\mathbb{E}[Y\mathbb{1}_A] = \mathbb{E}[X\mathbb{1}_A]$

is called *conditional expectation* of X given $\mathcal{G}$. Random variable Y is denoted by $Y = \mathbb{E}[X|\mathcal{G}]$ a.s.

Remark 2.23. The conditional expectation exists and is a.s.-unique as a consequence of the Radon-Nikodym Theorem. Here are some of the properties of the conditional expectation [**14**, Section 9.7.]. The setting is the same as in the Definition 2.22.

(1) If Z is $\mathcal{G}$-measurable and bounded, then it holds that

$$\mathbb{E}[ZX|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}], \quad a.s.$$

We refer to this as *take out what is known.*

(2) If for sub-σ-algebra $\mathcal{H} \subset \mathcal{F}$ it holds $\mathcal{H} \subset \mathcal{G}$, then we have the so called *tower property*:

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]\big|\mathcal{H}\big] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{H}]\big|\mathcal{G}\big] = \mathbb{E}[X|\mathcal{H}], \quad a.s.$$

Definition 2.24 (Martingale). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. An adapted process $X = (X_t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-*martingale* if

(1) $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$.

(2) For all $0 \leq s \leq t$

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \quad a.s.$$

Definition 2.25 (Square-integrable martingale, [**3**, I D4]). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. A martingale $X = (X_t)_{t\geq 0}$ is square-integrable over $[0, c]$, if

$$\mathbb{E}[X_c^2] < \infty.$$

Definition 2.26 (Stopping time, [**3**, A2 Definition 1]). Let $(\Omega, \mathcal{F})$ be a measurable space and let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. A random variable $\tau : \Omega \to [0, \infty]$ is called a $(\mathcal{F}_t)_{t\geq 0}$-*stopping time* if

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$$

for all $t \geq 0$.

Definition 2.27 (Stopped σ-algebra, [**3**, A2 Definition 3]). Let $(\Omega, \mathcal{F})$ be a measurable space, let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and assume a stopping time $\tau : \Omega \to [0, \infty]$. The *past at time* τ is defined as

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Proposition 2.28 (Doob's optional sampling, [**9**, Theorem 3.22]). *Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. Let $(X_t)_{t\geq 0}$ be a right-continuous martingale and let S and T, $S(\omega) \leq T(\omega) \leq N < \infty$, be bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping times. Then*

$$\mathbb{E}(X_T|\mathcal{F}_S) = X_S \quad \mathbb{P} - a.s.$$

Proof. See the proof of Theorem 3.22 in [**9**]. □

Definition 2.29 (Local martingale, [**2**, Definition 2.1.8]). Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ a process X is said to be a local martingale, if for every $\varepsilon > 0$ and $T \in (0, \infty)$ there exists a stopping time τ such that $\mathbb{P}(T < \tau) < \varepsilon$ and the process $X^\tau = (X_{t\wedge\tau})_{t\geq 0}$ is a martingale.

CHAPTER 3

Explicit solution to the Martingale Representation

3.1. The setting

We will recall the definition of a pure jump process.

Definition 3.1 (Jump times, [**3**, Section 2.1]). Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A sequence of random variables $(T_n)_{n\in\mathbb{N}_0}$ is called *jump times* if

(1) $T_n \in [0, \infty]$ for all $n \in \mathbb{N}_0$
(2) $T_0 \equiv 0$
(3) For all $T_n < \infty$ it holds that $T_n < T_{n+1}$.

Definition 3.2 (Pure jump process). Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $X = (X(t))_{t\geq 0}$ with a state space $(E, \mathcal{E})$ is called a *pure jump process*, if the set E is countable and there exists

(1) Jump times $(T_n)_{n\in\mathbb{N}_0}$,
(2) For each $n \in \mathbb{N}_0$ there exists state $s \in E$,

such that

$$X(t, \omega) = s, \quad \forall\, t \in [T_n, T_{n+1}), \mathbb{P}\text{-a.s.}$$

Remark 3.3 ([**3**, Section 2.1]). A jump process $X = (X(t))_{t\geq 0}$ with corresponding jump times $(T_n)_{n\in\mathbb{N}_0}$ is *nonexplosive* if and only if $\lim_{n\to\infty} T_n = \infty$.

We will consider an insurance policy with finitely many different states $S \subset \mathbb{N}_0$, which will be modelled as follows. Let us assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The state of the policy will be indicated by a right-continuous pure jump process $X = (X_t)_{t\in I}$ with left limits, where $X(\omega, t) \in S$ are defined in $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore we will equip the probability space with a natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$ of the process $X = (X(t))_{t\geq 0}$ so that the process X is adapted.

Lemma 3.4. *Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable state space $(S, \mathcal{S})$. Let process X be a jump process with jump times $(T_n)_{n\in\mathbb{N}_0}$. Then the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$ of the process X is right-continuous.*

Proof. We will apply [**3**, A2 Theorem 25]. Indeed the sequence $(T_n, X(T_n))_{n\in\mathbb{N}*}$ can be seen as an S-marked point process, where $0 < T_1 < T_2 < \dots$ denote the jump times of the process X. The definition of a S-marked point process can be found in [**3**, A2 D22]. □

To fulfill the usual conditions we augment the natural filtration while maintaining the right-continuity property.

Proposition 3.5. *Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable state space $(S, \mathcal{S})$. The augmented natural filtration of a càdlàg pure jump process $X = (X_t)_{t\geq 0}$ is right-continuous.*

PROOF. Denote the augmented natural filtration as $\tilde{\mathbb{F}}^X := (\tilde{\mathcal{F}}_t^X)_{t\geq 0}$. Since the filtration $(\mathcal{F}_t)_{t\geq 0}$ is increasing, then the collection $\tilde{\mathbb{F}}^X$ is also increasing. Therefore to conclude the right-continuity of the augmented natural filtration it is sufficient to confirm the equation

$$\tilde{\mathcal{F}}_t^X = \bigcap_{n\in\mathbb{N}^*} \tilde{\mathcal{F}}_{t+\frac{1}{n}}^X, \quad \forall t. \tag{3.1}$$

The inclusion "$\subset$" in (3.1) is true by the definition of the filtration being nested. We will proceed with the proof of the inclusion "$\supset$".

Assume an element $A \in \cap_{n\geq 1}\tilde{\mathcal{F}}_{t+1/n}^X$. By definition it holds for all $n \geq 1$ that

$$A \in \tilde{\mathcal{F}}_{t+1/n}^X.$$

Therefore from augmentation we are able to acquire for all $n \geq 1$ an element $B_n \in \mathcal{F}_{t+1/n}^X$ such that

$$\mathbb{P}(A \triangle B_n) = 0,$$

where $A\triangle B_n := (A \cup B_n) \setminus (A \cap B_n)$ denotes the symmetric difference.

For this paragraph we will follow the proof in [**13**, Answer by Saz]. Let us consider now the set

$$B := \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} B_n.$$

Fix an integer $N \in \mathbb{N}^*$. The objective is to prove that the set B belongs to the σ-algebra $\mathcal{F}_{t+1/N}^X$. Since for all $n \geq N$ it holds

$$B_n \in \mathcal{F}_{t+1/n} \subset \mathcal{F}_{t+1/N}^X$$

we have for all $k \geq N$ that

$$\bigcap_{n=k}^{\infty} B_n \in \mathcal{F}_{t+1/N}^X.$$

Moreover $\left(\bigcap_{n\geq k} B_n\right)_{k\in\mathbb{N}}$ is an increasing sequence so therefore

$$B = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} B_n = \bigcup_{k=N}^{\infty}\bigcap_{n=k}^{\infty} B_n \in \mathcal{F}_{t+1/N}^X. \tag{3.2}$$

Note that the number N was arbitrary, so the equation (3.2) holds for all $N \geq 1$. The Lemma 3.4 yields that the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$ is right-continuous, therefore

$$B \in \bigcap_{N\geq 1} \mathcal{F}_{t+1/N}^X = \mathcal{F}_t^X.$$

Now the objective is to show that $A \in \tilde{\mathcal{F}}_t^X$. Applying basic set operations we may deduce

$$\begin{aligned} A\triangle B &= A\triangle\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} B_n\right) \\ &= \left(A\cup\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} B_n\right)\right)\cap\left(A\cap\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} B_n\right)\right)^C \\ &= \left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(A\cup B_n)\right)\cap\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(A\cap B_n)\right)^C . \end{aligned} \tag{3.3}$$

Furthermore by de Morgan's law [**7**, Lemma 2.5.2 (3)] we receive

$$\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(A\cap B_n)\right)^C = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left((A\cap B_n)^C\right). \tag{3.4}$$

The Lemma of Fatou [**7**, Theorem 2.5.3] yields

$$\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}(A\cup B_n) \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}(A\cup B_n) \tag{3.5}$$

Combining the results of the equations (3.3), (3.4) and (3.5) gives

$$A\triangle B \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left((A\cup B_n)\cap(A\cap B_n)^C\right)$$

Finally the following is true:

$$A\triangle B \subseteq \bigcup_{n=1}^{\infty}\left((A\cup B_n)\cap(A\cap B_n)^C\right).$$

Utilizing the monotonicity and the additivity of the probability measure we receive

$$0 \leq \mathbb{P}(A\triangle B) \leq \mathbb{P}\left(\bigcup_{n\geq 1}[A\triangle B_n]\right) \leq \sum_{n\geq 1}\mathbb{P}(A\triangle B_n) = 0. \tag{3.6}$$

Note that because of the augmentation we have that

$$B \in \mathcal{F}_t^X \subset \tilde{\mathcal{F}}_t^X$$

and because of the equation (3.6) and the σ-algebra $\tilde{\mathcal{F}}_t^X$ is augmented, also

$$A\triangle B \in \tilde{\mathcal{F}}_t^X.$$

Lastly we have that the σ-algebra $\tilde{\mathcal{F}}_t^X$ is closed under set operations, therefore

$$A = B\triangle(A\triangle B) \in \tilde{\mathcal{F}}_t^X,$$

which proves the last inclusion $\bigcap_{n\geq 1}\tilde{\mathcal{F}}_{t+\frac{1}{n}}^X \subset \tilde{\mathcal{F}}_t^X$.

$\square$

The usual conditions are then satisfied because the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ was assumed to be complete. Denote the augmented natural filtration of the process X from now onward as $\mathbb{F}^X := (\tilde{\mathcal{F}}_t^X)_{t\geq 0}$.

To specify the state of process X at time t we may formulate an indicator process. For every state $i \in S$ we define

$$I_i(t,\omega) := \mathbb{1}_{\{X(t)=i\}}(\omega),$$

for all $t \in I$ and $\omega \in \Omega$. In addition to the indicator processes we may also define processes $(N_{ij}(t))_{t\geq 0}$ for $i,j \in S$ such that $i \neq j$ counting the number of jumps occurring from state i to state j until time t. For all states $i,j \in S$ such that $i \neq j$ we assign

$$N_{ij}(t) := \begin{cases} \#\{s \in (0,t] : X(s-) = i, X(s) = j\}, & t > 0 \\ 0, & t = 0. \end{cases} \tag{3.7}$$

Because the process X was assumed to be càdlàg , the processes $(N_{ij}(t))_{t\geq 0}$ are also càdlàg and are counting processes in the sense of the Definition 2.19. The aforementioned processes should be adapted for further purposes, which is indeed the case. Due to the definition of the processes $(N_{ij}(t))_{t\geq 0}$ and $(I_i(t))_{t\geq 0}$ they depend on the information the process X. Therefore, likewise the process X, they are adapted and càdlàg.

Lastly we shall define the jump times $0 = T_0 < T_1 < T_2 < \dots$ of the process X more rigorously. For $n \in \mathbb{N}^*$ the time of the n-th jump of the process X is provided by

$$T_n := \inf\{t \geq 0 : \sum_{i,j:i\neq j} N_{ij}(t) = n\}.$$

3.2. Stieltjes-Lebesgue integral

This section will follow the construction of Stieltjes-Lebesgue integral given in [**3**, A4]. In the beginning we will define a suitable space of integrator functions.

Definition 3.6 (Variation, [**3**] A4 D1). Let $t \geq 0$ and $\mathcal{D}$ denote the set of all partitions $0 = t_0 < t_1 < \dots < t_N = t$ of $[0,t]$. If for a function $f(t)$ it holds that

$$V_f(t) := \sup_{(t_0,\dots,t_N)\in\mathcal{D}:N\in\mathbb{N}} \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| < \infty,$$

then f is called to be *of bounded variation over finite intervals.* Moreover the function V_f is referred to as the *variation* of f.

For further definitions we consider a set of functions which are càdlàg, zero in $t = 0$ and of bounded variation over finite intervals. Such a function is denoted as being BV. For any BV function $f : \mathbb{R}^+ \to \mathbb{R}$ we have the decomposition

$$f(t) = V_f(t) - (-f(t) + V_f(t)), \quad \forall t > 0.$$

Defining the functions

$$a(t) := V_f(t) \quad \text{and} \quad b(t) := -f(t) + V_f(t)$$

we have that they are both right-continuous and non-decreasing.

Following the definitions we set the weight functions of an interval to be

$$\tau_a((s,t]) := a(t) - a(s) \quad \text{and} \quad \tau_b((s,t]) := b(t) - b(s),$$

for $0 \le s < t < \infty$ ([**1**] 7.1). Moreover set $\tau_a(\emptyset) = \tau_b(\emptyset) := 0$. Due to function f being of bounded variation over finite intervals, the measures are σ-finite measures on $((0,\infty), \mathcal{B}((0,\infty)))$. They are indeed measures, since the functions a and b are non-negative non-decreasing functions. Therefore countable additivity in $(0,\infty)$ is achieved ([**1**] 7.1.1).

Parallel to the construction of the Lebesgue integral we define the Stieltjes-Lebesgue outer measure

$$\mu_a^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \tau_a\big((a_i, b_i]\big) : A \subset \bigcup_{i=1}^{\infty} (a_i, b_i] \right\},$$

for any $A \subset \mathbb{R}^+$ ([**1**] Definition 7.1.1). The same construction is used for the outer measure μ_b^* respectively. Finally the Stieltjes-Lebesgue measures $\mu_a, \mu_b : \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+$ are attained in the similar fashion to the Lebesgue measure construction. Reader is advised to refer to [**1**, Chapter 5] for a more rigorous approach for the measure construction.

The Stieltjes-Lebesgue integral is now defined like the Lebesgue integral with respect to the signed measure $\mu_a - \mu_b$. The integral with respect to the variation is defined as

$$\int_{(0,\infty)} u(s)\, |df(s)| := \int_{(0,\infty)} u(s)\, d\mu_a.$$

Definition 3.7. ([**3**] A4) Let f be a BV function and let u be a Borel-measurable function. If

$$\int_{(0,\infty)} |u(s)|\, |df(s)| < \infty,$$

then the function u is *Stieltjes-Lebesgue integrable* with respect to the function f. The *Stieltjes-Lebesgue integral of the function u with respect to the function f* is the difference of the integrals

$$\int_{(0,\infty)} u(s)\, df(s) := \int_{(0,\infty)} u(s)\, d\mu_a(s) - \int_{(0,\infty)} u(s)\, d\mu_b(s).$$

Remark 3.8 ([**3**, A1 Equation (1.6)]). The assumption $\int_{(0,\infty)} |u(s)|\, d\mu_a(s) < \infty$ implies also that $\int_{(0,\infty)} |u(s)|\, d\mu_b(s) < \infty$, because of the inequality $|f(t)| \le V_f(t)$ and therefore $\mu_b(A) \le 2\mu_a(A)$ for all $A \in \mathcal{B}(\mathbb{R})$.

For the jump process N_{ij} we know that it is non-decreasing and $N_{ij}(0) = 0$ by the definition. The variation of the process N_{ij} is then for fixed $i,j \in S$ of the form

$$V_{N_{ij}}(t) = \sup_{(t_0,\dots,t_M)\in\mathcal{D}: M\in\mathbb{N}} \sum_{i=1}^{M} \Big[N_{ij}(t_i) - N_{ij}(t_{i-1}) \Big] = N_{ij}(t_M) - N_{ij}(t_0) = N_{ij}(t).$$

The Stieltjes-Lebesgue measure for the process N_{ij} is therefore $\mu_a((0,t]) = N_{ij}(t)$ and $\mu_b((0,t]) = 0$. Applying the previous results we may derive the Stieltjes-Lebesgue integral of a predictable process Z with respect to the process N_{ij}.

$$\begin{aligned}\int_{(0,\infty)} Z(t)\,dN_{ij}(t) &= \int_{(0,\infty)} Z(t)\,d\mu_a(t)\\ &= \sup_{(t_0,\dots,t_M)\in\mathcal{D}} \sum_{k=1}^{M} Z(t_k)\Big(N_{ij}(t_k) - N_{ij}(t_{k-1})\Big)\\ &= \sum_{n=1}^{\infty} Z(t_n),\end{aligned}$$

where $t_n := \inf\{t \in (0,\infty) : N_{ij}(t) = n\}$ for all $n \in \mathbb{N}$. In order to extend the definition of the integral over Borel sets we define

$$\int_A Z(t)\,dN_{ij}(t) := \int_{(0,\infty)} \mathbb{1}_A(t) Z(t)\,dN_{ij}(t),$$

when ever $A \in \mathcal{B}((0,\infty))$.

3.2.1. Integration by parts. To construct an interpretation for the notation $dN_{ij}^2(t)$ we introduce the integration by parts formula for the Stieltjes-Lebesgue integral.

PROPOSITION 3.9 ([**3**] A4 T2). *Suppose that two càdlàg functions f and g are of bounded variation over finite intervals. Then it holds that*

$$f(t)g(t) = f(0)g(0) + \int_{(0,t]} f(s)\,dg(s) + \int_{(0,t]} g(s-)\,df(s).$$

PROOF. We follow closely the proof given in [**3**]. Firstly we have by the Stieltjes-Lebesgue definition that

$$\big(f(t) - f(0)\big)\big(g(t) - g(0)\big) = \int_{(0,t]} df(x) \int_{(0,t]} dg(y).$$

Using Fubini's theorem we are able to deduce

$$\int_{(0,t]} df(x) \int_{(0,t]} dg(y) = \iint_{(0,t]\times(0,t]} df(x)\,dg(y).$$

Define the sets $D_1 = \{(x,y) : 0 < x \leq y \leq t\}$ and $D_2 = \{(x,y) : 0 < y < x \leq t\}$. By additivity of the integral we receive

$$\iint_{(0,t]\times(0,t]} df(x)\,dg(y) = \iint_{D_1} df(x)\,dg(y) + \iint_{D_2} df(x)\,dg(y).$$

Once again applying Fubini's theorem it follows that

$$\begin{aligned}\iint_{D_1} df(x)\,dg(y) &= \int_{(0,t]} \left(\int_{(0,y]} df(x)\right) dg(y)\\ &= \int_{(0,t]} f(y) - f(0)\,dg(y)\\ &= \int_{(0,t]} f(y)\,dg(y) - f(0)\big(g(t) - g(0)\big).\end{aligned}$$

In a similar fashion

$$\begin{aligned}\iint_{D_2} df(x)\,dg(y) &= \int_{(0,t]} \left(\int_{(0,x)} dg(y)\right) df(x)\\ &= \int_{(0,t]} g(x-) - g(0)\,df(x)\\ &= \int_{(0,t]} g(x-)\,df(x) - g(0)\big(f(t)-f(0)\big).\end{aligned}$$

By combining all of the results

$$\begin{aligned}\big(f(t)-f(0)\big)\big(g(t)-g(0)\big) = &\int_{(0,t]} f(y)\,dg(y) - f(0)\big(g(t)-g(0)\big) +\\ &\int_{(0,t]} g(x-)\,df(x) - g(0)\big(f(t)-f(0)\big)\end{aligned}$$

and therefore

$$\begin{aligned}f(t)g(t) = &\int_{(0,t]} f(y)\,dg(y) + \int_{(0,t]} g(x-)\,df(x) - f(0)g(t) + f(0)g(0) -\\ &g(0)f(t) + g(0)f(0) + f(t)g(0) + f(0)g(t) - g(0)f(0)\\ =&f(0)g(0) + \int_{(0,t]} f(y)\,dg(y) + \int_{(0,t]} g(x-)\,df(x).\end{aligned}$$

□

3.3. Compensated processes

Definition 3.10 (Jump intensity, [**3**, Chapter II D7]). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis and assume an adapted jump process $N = (N(t))_{t\geq 0}$. Let $\lambda = (\lambda(t))_{t\geq 0}$ be a progressively measurable non-negative process with

$$\mathbb{E}\int_0^t \lambda(s)\,ds < \infty \quad \text{for all } t \geq 0.$$

If the equation

$$\mathbb{E}\left[\int_{(0,\infty)} C(s)\,dN(s)\right] = \mathbb{E}\left[\int_0^\infty C(s)\lambda(s)\,ds\right]$$

holds for every nonnegative predictable process $(C(t))_{t\geq 0}$, then N *admits the* $(\mathcal{F}_t)_{t\geq 0}$-*intensity* λ.

Recall the definition of the indicator process as $I_i(t) = \mathbb{1}_{\{X(t)=i\}}$ for all $i \in S$. In the succeeding sections we will present an explicit form for the intensities, simultaneously proving the existence of an intensity for any counting process. Therefore we assume, when ever $i, j \in S$ and $i \neq j$, that for every counting process $(N_{ij}(t))_{0\leq t\leq T}$ there exists an intensity $(\lambda_{ij}(t))_{0\leq t\leq T}$, such that

$$\mathbb{E}\left(\int_0^T \sum_{i,j:i\neq j} I_i(s-)\lambda_{ij}(s)\,ds\right) < \infty.$$

The compensated jump processes of the processes N_{ij} are defined by setting

$$M_{ij}(t) := N_{ij}(t) - \int_0^t I_i(s-)\lambda_{ij}(s)\, ds, \quad 0 \le t \le T$$

and $M_{ij}(0) := 0$ for all $i, j \in S, i \neq j$.

PROPOSITION 3.11 ([**3**, II L2]). *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\ge0})$ be a stochastic basis. Assume that a jump process $N = (N(t))_{t\ge0}$ admits an intensity $(\lambda(t))_{t\ge0}$. Then the process N is nonexplosive and*

$$M := \left(N(t) - \int_0^t \lambda(s)\, ds\right)_{t\ge0} \quad \textit{is a local martingale.}$$

PROOF. Let us prove first that the process N is nonexplosive. Define the process $A = (A(t))_{t\ge0}$ with $A(t) := \int_0^t \lambda(s)\, ds$. Following the proof in [**3**, II L2] we set the hitting times

$$S_n := \inf\{t : A(t) \in (n, \infty)\}, \quad \forall n \in \mathbb{N}_0,$$

with the convention that $\inf \emptyset = \infty$. From the definition of the intensity λ, it follows that the process A is càdlàg and adapted. Furthermore the set (n, ∞) is open, therefore S_n is a stopping time for all n ([**12**, I 1 Theorem 3]). The process $(\mathbb{1}_{\{t\le S_n\}}(t))_{t\ge0}$ is càdlàg and adapted, so it is predictable (Remark 2.18). We have by the definition of the intensity that

$$\mathbb{E}[N(S_n)] = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{t\le S_n\}}(s)\, dN(s)\right] = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{t\le S_n\}}(s)\lambda(s)\, ds\right] = \mathbb{E}[A(S_n)], \tag{3.8}$$

where we used that the counting process N is increasing. Because of the definition of the intensity we have $A(t) < \infty$ a.s., therefore $N(t) < \infty$ a.s. and $S_n \overset{n\uparrow\infty}{\to} \infty$ a.s. and the process N is nonexplosive.

Now we will prove that the process M is a local martingale. The idea is the same as in the proof of the Corollary 6.7 [**5**, 5.6]. At first note that M is adapted and $\mathbb{E}[M(t \wedge S_n)] = 0$ for all t, S_n by (3.8). Assume $0 \le s < t < \infty$. Then the random variables $S_n \wedge t$ and $S_n \wedge s$ are also stopping times. As we previously deduced, it holds for the stopping times S_n that $S_n \to \infty$. To show the local martingale property, it is sufficient to prove that

$$\mathbb{E}[M(S_n \wedge t) - M(S_n \wedge s)|\mathcal{F}_s] = 0, \quad a.s. \tag{3.9}$$

To yield (3.9) it is enough to show that for all $A \in \mathcal{F}_s$ it holds

$$\mathbb{E}\left[\mathbb{1}_A\big(M(S_n \wedge t) - M(S_n \wedge s)\big)\right] = 0, \tag{3.10}$$

because by the definition of the conditional expectation (Definition 2.22) we would then have the preceding (3.9).

By a straight forward calculation we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_A\big(M(S_n\wedge t)-M(S_n\wedge s)\big)\right]\\
&=\mathbb{E}\left[\mathbb{1}_A\left(\int_{(S_n\wedge s,S_n\wedge t]} dM(u)\right)\right]\\
&=\mathbb{E}\left[\int_{(S_n\wedge s,S_n\wedge t]} \mathbb{1}_A(dN(u)-\lambda(u)\,du)\right]\\
&=\mathbb{E}\left[\int_{(0,\infty)} \mathbb{1}_{(S_n\wedge s,S_n\wedge t]}(u)\mathbb{1}_A(dN(u)-\lambda(u)\,du)\right].
\end{aligned}$$

Because of the assumption $\mathbb{1}_A\in\mathcal{F}_s$ and the random times $S_n\wedge s, S_n\wedge t$ and $u\in(0,\infty)$ are stopping times, we have that $\mathbb{1}_A\mathbb{1}_{\{S_n\wedge s<u\}}, \mathbb{1}_A\mathbb{1}_{\{S_n\wedge\geq u\}}\in\mathcal{F}_s$ ([**5**, 5.1 Th 1.16d]). Then the process $\mathbb{1}_{(S_n\wedge s,S_n\wedge t]}(u)\mathbb{1}_A$ is adapted, and being a left-continuous process it is also predictable (Remark 2.18). Therefore by the definition of the intensity we have

$$\mathbb{E}\left[\int_{(0,\infty)} \mathbb{1}_{(S_n\wedge s,S_n\wedge t]}(u)\mathbb{1}_A(dN(u)-\lambda(u)\,du)\right]=0,$$

which proves the equation (3.9). □

The converse is also true for the Proposition 3.11, so in fact we have an equivalency between the definition of the intensity and the martingale characterization.

PROPOSITION 3.12. [**3**, II 3 T9] *Let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. Assume an adapted non-explosive jump process $N=(N(t))_{t\geq 0}$. If there exists a non-negative and progressively measurable process $(\lambda(t))_{t\geq 0}$ such, that*

$$M:=\left(N(t)-\int_0^t\lambda(s)\,ds\right)_{t\geq 0}\qquad \textit{is a local martingale,}$$

then the process N admits the intensity λ.

PROOF. The idea of the proof is the same as in [**5**, 5.6 Theorem 6.5]. We are required to prove that

$$\mathbb{E}\left[\int_{(0,\infty)} C(s)\,dN(s)\right]=\mathbb{E}\left[\int_0^\infty C(s)\lambda(s)\,ds\right] \tag{3.11}$$

for every non-negative predictable process $(C(t))_{t\geq 0}$. At first we will verify the equation (3.11) for the simple processes of the predictable σ-algebra

$$\mathcal{P}:=\sigma(\{(s,t]\times A: 0\leq s\leq t, A\in\mathcal{F}_s\}).$$

Then the Monotone Class Theorem for functions will be applied to generalize the result. Note here that the indicator functions are non-negative.

For $0\leq s\leq t$ set $C(\omega,u)=\mathbb{1}_{A\times(s,t]}(\omega,u)$, where the set A is $\mathcal{F}_s$-measurable. Then it holds by the right-continuity of the process N that

$$\begin{aligned}
\mathbb{E}\left[\int_{(0,\infty)} C(u)\,dN(u)\right]&=\mathbb{E}\left[\mathbb{1}_A\int_{(0,\infty)}\mathbb{1}_{(s,t]}(u)\,dN(u)\right]\\
&=\mathbb{E}[\mathbb{1}_A(N(t)-N(s))].
\end{aligned}$$

By the tower property and "take out what is known" we receive

$$\mathbb{E}[\mathbb{1}_A(N(t) - N(s))] = \mathbb{E}\Big[\mathbb{1}_A\mathbb{E}[N(t) - N(s)|\mathcal{F}_s]\Big]. \tag{3.12}$$

Here we may utilize the assumption of the process M being a local martingale. For a localizing sequence $T_0 \leq T_1 \leq \ldots$ it holds

$$\mathbb{E}[M(t \wedge T_n) - M(s \wedge T_n)|\mathcal{F}_s] = M(s \wedge T_n) - M(s \wedge T_n) = 0 \quad a.s.,$$

which is equal to stating that

$$\mathbb{E}[N(t \wedge T_n) - N(s \wedge T_n)|\mathcal{F}_s] = \mathbb{E}\left[\int_0^{t\wedge T_n} \lambda(u)\, du - \int_0^{s\wedge T_n} \lambda(u)\, du \middle| \mathcal{F}_s\right] \quad a.s.$$

Letting the index $n \to \infty$ implies $T_n \to \infty$ and

$$\mathbb{E}[N(t) - N(s)|\mathcal{F}_s] = \mathbb{E}\left[\int_0^{t} \lambda(u)\, du - \int_0^{s} \lambda(u)\, du \middle| \mathcal{F}_s\right] \quad a.s. \tag{3.13}$$

Combining the results of the equations (3.12) and (3.13) we may complete the first step of the proof:

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_A\mathbb{E}[N(t) - N(s)|\mathcal{F}_s]\Big] \\
&= \mathbb{E}\left[\mathbb{1}_A\mathbb{E}\left[\int_0^{t} \lambda(u)\, du - \int_0^{s} \lambda(u)\, du \middle| \mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_A \int_s^t \lambda(u)\, du\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{A\times(s,t]}(u)\lambda(u)\, du\right].
\end{aligned}$$

The case for $C = \mathbb{1}_{A\times\{0\}}$ where $A \in \mathcal{F}_0$ follows in similar fashion. This proves the (3.11) for simple predictable non-negative processes. For the generalization we notice that the set of non-negative predictable processes $(C(t))_{t\geq 0}$ for which (3.11) holds is a monotone class. To validate this we will provide some arguments. Remark the set of predictable non-negative processes fulfilling (3.11) as $\mathcal{P}^+$.

(1) $\mathbb{1}_{A\times(s,t]} \in \mathcal{P}^+$ for $0 \leq s \leq t < \infty$ and $A \in \mathcal{F}_s$ as was shown.
(2) $\mathcal{P}^+$ is a vector space in $\mathbb{R}$.
(3) Any increasing and bounded sequence of non-negative predictable processes converges into a predictable non-negative process. Moreover by applying the Monotone Convergence Theorem to the Lebesgue integrals and to the Stieltjes-Lebesgue integrals separately on both sides also the equation (3.11) holds.[**5**, 5.6. Th 6.5]

Here the σ-algebra generated by the π-system $\{(s,t] \times A : 0 \leq s \leq t, A \in \mathcal{F}_s\}$ is by definition the predictable σ-algebra $\mathcal{P}$. The Monotone Class Theorem for functions [**7**, Proposition 10.3.3] then implies that the set $\mathcal{P}^+$ contains all $\mathcal{P}$-measurable and bounded process.

□

In the literature the process M is called the *accompanying martingale* of the process N. The process A in the Proposition 3.11 is known as the compensator of the process N. ([**12**, III 5])

3.4. Square integrable martingales

Assume a set $\{(Z_{ij}(t))_{0\leq t\leq T} : i \neq j\}$ of predictable processes on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t^X)_{0\leq t\leq T})$. The focus shifts now on to the stochastic integral

$$\begin{aligned} \int_{(0,t]} Z(s)\, dM(s) &:= \sum_{i,j:i\neq j} \int_{(0,t]} Z_{ij}(s)\, dM_{ij}(s) \\ &:= \sum_{i,j:i\neq j} \int_{(0,t]} Z_{ij}(s)\, (dN_{ij}(s) - I_i(s-)\lambda_{ij}(s)\, ds), \end{aligned} \tag{3.14}$$

for $0 \leq t \leq T$, where $I_i(t) = \mathbb{1}_{\{X(t)=i\}}$. Firstly denote a norm with respect to the jump intensities $\Lambda := \{\lambda_{ij} : i \neq j\}$ as

$$\|Z(t)\|_\Lambda^2 := \sum_{i,j:i\neq j} Z_{ij}^2 I_i(t-)\lambda_{ij}(t), \quad 0 \leq t \leq T. \tag{3.15}$$

We will now show that the stochastic integral in (3.14) under the constrain $\mathbb{E}\left[\int_0^T \|Z(s)\|_\Lambda^2\, ds\right] < \infty$ is a square-integrable martingale with respect to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t^X)_{0\leq t\leq T})$. We will start with a result that can be interpreted as the Itô isometry for a jump process.

Proposition 3.13. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq T})$ fulfill the usual assumptions. Assume for $i, j \in S, i \neq j$ the counting processes $(N_{ij}(t))_{t\geq 0}$ that admit intensities $(\lambda_{ij}(t))_{t\geq 0}$. Let $(Z_{ij}(t))_{t\geq 0}$ be predictable processes such that $\mathbb{E}\left[\int_0^T \|Z(s)\|_\Lambda^2\, ds\right] < \infty$. Then*

$$\mathbb{E}\left(\int_{(0,T]} Z(s)\, dM(s)\right)^2 = \mathbb{E}\int_0^T \|Z(s)\|_\Lambda^2\, ds.$$

Proof. By applying the integration by parts formula (Proposition 3.9) we receive

$$\begin{aligned} &\left(\int_{(0,T]} Z(s)\, dM(s)\right)^2 \\ &= \int_{(0,T]} \int_{(0,s)} Z(u)\, dM(u)\, d\left(\int_{(0,s]} Z(u)\, dM(u)\right) + \\ &\quad \int_{(0,T]} \int_{(0,s]} Z(u)\, dM(u)\, d\left(\int_{(0,s]} Z(u)\, dM(u)\right) \\ &= \int_{(0,T]} \left(\int_{(0,s)} Z(u)\, dM(u) Z(s) + \int_{(0,s)} Z(u)\, dM(u) Z(s) + \right. \\ &\quad \left. \int_{\{s\}} Z(u)\, dM(u) Z(s)\right) dM(s) \\ &= \int_{(0,T]} \left(2\int_{(0,s)} Z(u)\, dM(u)\right) Z(s)\, dM(s) + \\ &\quad \int_{(0,T]} Z(s) \int_{\{s\}} Z(u)\, dM(u)\, dM(s). \end{aligned} \tag{3.16}$$

By right-continuity of the process M we may deduce for the last line of the equation (3.16) that

$$\int_{(0,T]} Z(s) \int_{\{s\}} Z(u)\, dM(u)\, dM(s) = \int_{(0,T]} Z^2(s)\Big(M(s) - M(s-)\Big)\, dM(s). \tag{3.17}$$

Here we will use the definition of the process M to achieve

$$\begin{aligned} M(s) - M(s-) &= N(s) - N(s-) - \left(\int_{(0,s]} \lambda(u)\, du - \int_{(0,s)} \lambda(u)\, du\right) \\ &= N(s) - N(s-) - \int_{\{s\}} \lambda(u)\, du \\ &= \mathbb{1}_{\{dN(s)=1\}}. \end{aligned} \tag{3.18}$$

Now applying the (3.18) to the right hand side of the equation (3.17) we receive

$$\begin{aligned} &\int_{(0,T]} Z^2(s)\Big(M(s) - M(s-)\Big)\, dM(s) \\ &= \int_{(0,T]} Z^2(s)\mathbb{1}_{\{dN(s)=1\}}\,\big(dN(s) - \lambda(s)\, ds\big) \\ &= \sum_{s\in(0,T]:dN(s)=1} Z^2(s). \end{aligned}$$

This is by the definition of the integral

$$\sum_{s\in(0,T]:dN(s)=1} Z^2(s) = \int_{(0,T]} Z^2(s)\, dN(s).$$

Combining all of the previous results we are left with the equality

$$\begin{aligned} &\left(\int_{(0,T]} Z(s)\, dM(s)\right)^2 \\ &= \int_{(0,T]} \left(2\int_{(0,s)} Z(u)\, dM(u)\right) Z(s)\, dM(s) + \int_{(0,T]} Z^2(s)\, dN(s). \end{aligned} \tag{3.19}$$

Note that we have now a predictable non-negative process $(Z^2(t))_{0\le t\le T}$ which in conjunction with the Definition 3.10 of the jump intensity implies

$$\int_{(0,T]} Z^2(s)\, dN(s) = \int_{(0,T]} Z^2(s)\lambda(s)\, ds \tag{3.20}$$

in the equation (3.19). Moreover with the definition of the norm in (3.15) we may further mark

$$\int_{(0,T]} Z^2(s)\lambda(s)\, ds = \int_0^T \|Z(s)\|_\Lambda^2\, ds$$

in the equation (3.19).

Let us inspect the process

$$\int_{(0,T]} \left(2\int_{(0,s)} Z(u)\, dM(u)\right) Z(s)\, dM(s) \tag{3.21}$$

from the equation (3.19). We have that the stochastic integral $\int_{(0,s)} Z(u)\, dM(u)$ is path-wise left-continuous with respect to the variable s:

$$\lim_{t\to s-} \int_{(0,t)} Z(u)\, dM(u) = \int_{(0,s)} Z(u)\, dM(u).$$

Therefore it is also predictable by the Remark 2.18. Moreover the process Z is predictable by assumption, and the measurability is preserved in arithmetic operations, so the process $\left(\int_{(0,s)} Z(u)\, dM(u) Z(s)\right)_{s\in[0,T]}$ is predictable. Also the process M was proven to be a local martingale in the Proposition 3.11. This validates the stochastic integral in (3.21) to be interpreted as a predictable process being integrated with respect to a local martingale M. According to the Theorem 29 in [**12**, IV 2] the stochastic integral in (3.21) is then also a local martingale, given that the predictable process is locally bounded. This last constraint follows directly from the assumption $\mathbb{E}\left[\int_0^T \|Z(s)\|_\Lambda^2\, ds\right] < \infty$.

We have proven that there exists a localizing sequence $\tau_1, \tau_2, \dots$ with $\lim_{n\to\infty} \tau_n \to T$ such that the process

$$\left(\int_{(0,t\wedge\tau_n]} 2\int_{(0,s)} Z(u)\, dM(u) Z(s)\, dM(s)\right)_{t\in(0,T]}$$

is a martingale for all $n \in \mathbb{N}$. Therefore we may deduce that

$$\mathbb{E}\left[\left(\int_{(0,T\wedge\tau_n]} Z(s)\, dM(s)\right)^2\right] = \mathbb{E}\left[\int_0^{T\wedge\tau_n} \|Z(s)\|_\Lambda^2\, ds\right]. \tag{3.22}$$

for all $n \in \mathbb{N}$. Letting $\tau_n \to T$ we achieve

$$\mathbb{E}\left[\left(\int_{(0,T)} Z(s)\, dM(s)\right)^2\right] = \mathbb{E}\left[\int_0^{T} \|Z(s)\|_\Lambda^2\, ds\right]. \tag{3.23}$$

What is left to show is that

$$\mathbb{E}\left[\left(\int_{(0,T)} Z(s)\, dM(s)\right)^2\right] = \mathbb{E}\left[\left(\int_{(0,T]} Z(s)\, dM(s)\right)^2\right]. \tag{3.24}$$

The right hand side can be further disintegrated:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{(0,T]} Z(s)\, dM(s)\right)^2\right] =& \mathbb{E}\left[\left(\int_{(0,T)} Z(s)\, dM(s)\right)^2\right] + \\
& 2\mathbb{E}\left[\int_{(0,T)} Z(s)\, dM(s)(M(T) - M(T-))Z(T)\right] + \qquad (3.25)\\
& \mathbb{E}\left[(M(T) - M(T-))^2 Z^2(T)\right] \qquad (3.26)
\end{aligned}$$

As was proven previously we may deduce for the term (3.26) that

$$\mathbb{E}\left[(M(T) - M(T-))^2 Z^2(T)\right] = \mathbb{E}\left[(N(T) - N(T-))^2 Z^2(T)\right].$$

This is equal to zero, because the probability of a jump $N(T) - N(T-) = 1$ at time T is zero. Similar reasoning is applicable to the term (3.25) and therefore the equality (3.24) holds. Combining the equations (3.24) and (3.23) we have acquired the isometry of the norms:

$$\mathbb{E}\left(\int_{(0,T]} Z(s)\,dM(s)\right)^2 = \mathbb{E}\int_0^T \|Z(s)\|_\Lambda^2\,ds.$$

□

PROPOSITION 3.14. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T})$ fulfill the usual assumptions. Assume for $i, j \in S, i \neq j$ the counting processes $(N_{ij}(t))_{t\ge 0}$ that admit intensities $(\lambda_{ij}(t))_{t\ge 0}$. Let $(Z_{ij}(t))_{t\ge 0}$ be a predictable process such that $\mathbb{E}\left[\int_0^T \|Z(s)\|_\Lambda^2\,ds\right] < \infty$. Then the process*

$$\left(\int_{(0,t]} Z(s)\,dM(s)\right)_{0\le t\le T}$$

as defined in (3.14) is a square-integrable $(\mathcal{F}_t)_{0\le t\le T}$-martingale.

PROOF. The adaptability follows directly from the definition of the processes. The Doob's maximal inequality for martingales gives

$$\mathbb{E}\sup_{t\in[0,T]}\left(\int_{(0,t]} Z(s)\,dM(s)\right)^2 \le 4\mathbb{E}\left(\int_{(0,T]} Z(s)\,dM(s)\right)^2. \tag{3.27}$$

Then the square-integrability is a consequence of the Proposition 3.13 and the assumption of this proposition which yield

$$4\mathbb{E}\left(\int_{(0,T]} Z(s)\,dM(s)\right)^2 = 4\mathbb{E}\int_0^T \|Z(s)\|_\Lambda^2\,ds < \infty.$$

To prove the martingale property we are required to verify that for all t, s such that $0 \le s \le t \le T$ it holds that

$$\mathbb{E}\left[\int_{(0,t]} Z(u)\,dM(u)\middle|\mathcal{F}_s\right] = \int_{(0,s]} Z(u)\,dM(u) \quad a.s. \tag{3.28}$$

By utilizing the measurability of the right hand side with respect to the σ-algebra $\mathcal{F}_s$ it is equivalent to show that

$$\mathbb{E}\left[\int_{(s,t]} Z(u)\,dM(u)\middle|\mathcal{F}_s\right] = 0 \quad a.s. \tag{3.29}$$

Here we may split the predictable process Z in to the positive and negative part and apply the linearity of the integrals:

$$\begin{aligned}
&\mathbb{E}\left[\int_{(s,t]} Z(u)\,dM(u)\middle|\mathcal{F}_s\right]\\
=&\mathbb{E}\left[\int_{(s,t]} Z^+(u) - Z^-(u)\,dM(u)\middle|\mathcal{F}_s\right]\\
=&\mathbb{E}\left[\int_{(s,t]} Z^+(u)\,dM(u)\middle|\mathcal{F}_s\right] - \mathbb{E}\left[\int_{(s,t]} Z^-(u)\,dM(u)\middle|\mathcal{F}_s\right] \quad a.s.
\end{aligned}$$

The proof will be concluded following the proof of the Corollary 6.7 in [**5**, 5.6]. Now since the processes $(Z^+(t))_{0\le t\le T}$ and $(Z^-(t))_{0\le t\le T}$ are non-negative and predictable it is sufficient to prove

$$\mathbb{E}\left[\int_{(s,t]} Z^+(u)\,dM(u)\middle|\mathcal{F}_s\right] = 0 \quad a.s. \tag{3.30}$$

which will yield the desired result (3.29). Due to the definition of the conditional expectation it is equivalent to prove that for all $A \in \mathcal{F}_s$ it holds

$$\mathbb{E}\left[\mathbb{1}_A \int_{(s,t]} Z^+(u)\,dM(u)\right] = 0. \tag{3.31}$$

The left hand side in (3.31) is equal to

$$\begin{aligned}
&\mathbb{E}\left[\int_{(0,T]} \mathbb{1}_{(s,t]}(u)\mathbb{1}_A Z^+(u)\,dM(u)\right] \\
=&\mathbb{E}\left[\sum_{i,j\in S: i\neq j} \int_{(0,T]} \mathbb{1}_{(s,t]}(u)\mathbb{1}_A Z^+(u)\left(dN_{ij}(u) - I_i(s-)\lambda_{ij}\,ds\right)\right].
\end{aligned}$$

Here the process $(\mathbb{1}_{(s,t]}\mathbb{1}_A Z^+(u))_{0\le u\le T}$ is still predictable and non-negative with respect to the variable u, so we may implement the definition of the intensity λ_{ij} to yield

$$\mathbb{E}\left[\int_{(0,T]} \mathbb{1}_{(s,t]}(u)\mathbb{1}_A Z^+(u)\,dN_{ij}(u)\right] = \mathbb{E}\left[\int_{(0,T]} \mathbb{1}_{(s,t]}(u)\mathbb{1}_A Z^+(u)\,I_i(u-)\lambda_{ij}\,du\right]. \tag{3.32}$$

for all $i, j \in S$ such that $i \neq j$. The equation (3.32) proves the equation (3.31) which analogously proves the martingale property (3.28).

□

Proposition 3.15 ([**3**, III T11]). *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T})$ fulfill the usual assumptions. Assume for $i, j \in S, i \neq j$ the counting processes $(N_{ij}(t))_{t\ge 0}$ that admit intensities $(\lambda_{ij}(t))_{t\ge 0}$. Let $(Z_{ij}(t))_{t\ge 0}$ be predictable processes such that the process $J := (\int_{(0,t]} Z(s)\,dM(s))_{0\le t\le T}$ is a càdlàg square-integrable martingale. Then*

$$\mathbb{E}\left[\int_0^T \|Z(s)\|_\Lambda^2\,ds\right] < \infty.$$

Proof. We will utilize the integration by parts formula from the Proposition 3.9 applied to the product J^2, which was formulated in the Proposition 3.13 equation (3.19):

$$\begin{aligned}
J^2(t) &= J^2(0) + \int_{(0,t]} 2J(s-)Z(s)\,dM(s) + \int_{(0,t]} Z^2(s)\,dN(s) \\
&= J^2(0) + \int_{(0,t]} 2J(s-)Z(s)\,dM(s) + \int_{(0,t]} Z^2(s)\,(dN(s) - \lambda(s)\,ds + \lambda(s)\,ds) \\
&= J^2(0) + \int_{(0,t]} \Big(2J(s-) + Z(s)\Big)Z(s)\,dM(s) + \int_{(0,t]} Z^2(s)\lambda(s)\,ds). \qquad (3.33)
\end{aligned}$$

We continue by truncating the stochastic integral with the stopping times $(U_n)_n$ and $(V_n)_n$ where

$$U_n := \inf\{t : M(t-) + \int_0^t \lambda(s)\,ds \geq n\} \quad \text{and} \tag{3.34}$$

$$V_n := \inf\{t : \int_0^Z (s)^2 \lambda(s)\,ds \geq n\},$$

with the convention that $\inf \emptyset = \infty$. Now it holds also that

$$\begin{aligned} J^2(t \wedge U_n \wedge V_m) = J^2(0) &+ \int_{(0,t\wedge U_n \wedge V_m]} \Big(2J(s-) + Z(s)\Big) Z(s)\,dM(s) \\ &+ \int_{(0,t\wedge U_n \wedge V_m]} Z^2(s)\lambda(s)\,ds. \end{aligned} \tag{3.35}$$

The term $H(t) := \int_{(0,t\wedge U_n\wedge V_m]} (2J(s-) + Z(s))Z(s)\,dM(s)$ is a martingale, which we will prove. By the Lemma 3 of [**3**], to prove that H is a martingale it is sufficient to show that $\mathbb{E}[\int_0^t |H(s)|\lambda(s)\,ds] < \infty$ for all $t \in [0,T]$. We shall inspect the condition further. By triangle inequation and monotonicity of the Lebesgue integral we have

$$\begin{aligned} \mathbb{E}[\int_0^t |H(s)|\lambda(s)\,ds] &\leq \mathbb{E}\left[\int_0^{t\wedge U_n \wedge V_m} |2J(s-) + Z(s)||Z(s)|\lambda(s)\,ds\right] \\ &\leq \mathbb{E}\left[\int_0^t \mathbb{1}_{\{s\leq U_n\}} \mathbb{1}_{\{s\leq V_n\}} \Big(|2J(s-)||Z(s)| + |Z(s)|^2\Big)\lambda(s)\,ds\right]. \end{aligned}$$

Due to the definition of the stopping time U_n it holds

$$\mathbb{1}_{\{s\leq U_n\}}|2J(s-)| \leq 2n.$$

We have that

$$\int_0^{t\wedge V_m} 2n|Z(s)|\lambda(s)\,ds \leq \int_0^{t\wedge V_m} 2n(Z^2(s) + 1)\lambda(s)\,ds.$$

It also holds that

$$\int_0^{t\wedge U_n\wedge V_m} (Z^2(s)+1)\lambda(s)\,ds = \int_0^{t\wedge V_m} Z^2(s)\lambda(s)\,ds + \int_0^{t\wedge U_n \wedge V_m} \lambda(s)\,ds \leq m + n.$$

Therefore we may conclude that

$$\mathbb{E}[\int_0^t |H(s)|\lambda(s)\,ds] \leq 2n(m+n) + m < \infty$$

and the process J is a martingale with mean zero.

As a consequence of this we may take expected values on both sides of (3.33) to reach

$$\mathbb{E}\left[\int_{(0,t\wedge U_n\wedge V_m]} Z^2(s)\lambda(s)\,ds\right] = \mathbb{E}[J^2(t\wedge U_n \wedge V_m)] - \mathbb{E}[M^2(0)] \leq \mathbb{E}[J^2(t\wedge U_n\wedge V_m)].$$

Recall the definition of the stopping times $(U_n)_n$ in (3.34). Due to the process $(J(t-))0 \leq t \leq T$ being square-integrable and left continuous it holds $\lim_{n\to\infty} U_n =$

∞. We have reached

$$\mathbb{E}\left[\int_{(0,t\wedge V_m]} Z^2(s)\lambda(s)\,ds\right] \leq \mathbb{E}[J^2(t\wedge V_m)]. \tag{3.36}$$

The square-integrability yields also $\mathbb{E}[J^2(t\wedge V_m)] < \infty$ because by applying Fatou's lemma we have

$$\mathbb{E}[J^2(V_m)] = \mathbb{E}[\lim_{t\to\infty} J^2(t\wedge V_m)] \leq \lim_{t\to\infty} \mathbb{E}[J^2(t\wedge V_m)] \tag{3.37}$$

and we may apply Doob's optional sampling theorem for sub-martingales to reach

$$\lim_{t\to\infty} \mathbb{E}[J^2(t\wedge V_m)] \leq \lim_{t\to\infty} \sup_{0\leq s\leq t} \mathbb{E}[J^2(s)] < \infty. \tag{3.38}$$

Therefore in (3.36) we have

$$\mathbb{E}\left[\int_{(0,t\wedge V_m]} Z^2(s)\lambda(s)\,ds\right] < \infty$$

which implies because of the definition $(V_m)_m$ that

$$\mathbb{E}\left[\int_0^T \|Z(s)\|_\Lambda^2\,ds\right] = \mathbb{E}\left[\int_{(0,t]} Z^2(s)\lambda(s)\,ds\right] < \infty.$$

□

Corollary 3.16. *For a compensated process $M := ((N(t) - \int_0^t \lambda(s)\,ds)(t))_{0\leq t\leq T}$ and a predictable process $(Z(t))_{0\leq t\leq T}$ the following statements are equivalent:*

(1) $\mathbb{E}\left[\int_0^T \|Z(s)\|_\Lambda^2\,ds\right] < \infty$,

(2) *M is a square-integrable martingale.*

If either of the (1) or (2) holds, then we have the isometry

$$\mathbb{E}\left(\int_{(0,T]} Z(s)\,dM(s)\right)^2 = \mathbb{E}\int_0^T \|Z(s)\|_\Lambda^2\,ds. \tag{3.39}$$

Proof. This is a summary of the Propositions 3.13, 3.14 and 3.15.

□

3.5. Preliminary results for stopping times

Recall the notation from the beginning of the chapter. For the augmented natural filtration of the jump process $X := (X(t))_{0\leq t\leq T}$ we mark $\mathbb{F}^X := (\tilde{\mathcal{F}}_t^X)_{0\leq t\leq T}$. The corresponding counting processes $(N_{ij}(t))_{0\leq t\leq T}$ for $i\neq j$ are assigned as

$$N_{ij}(t) := \begin{cases} \#\{s\in(0,t] : X(s-) = i, X(s) = j\}, & t > 0 \\ 0, & t = 0, \end{cases}$$

and the convention $N(t) := \sum_{i,j\in S: i\neq j} N_{ij}(t)$. Lastly the jump times $(T_n)_{n\in\mathbb{N}}$ of the jump process X are defined as $T_n := \inf\{t\geq 0 : N(t) = n\}$.

Proposition 3.17 ([**3**, A2 T23]). *The jump times $0 < T_1 < T_2 < \dots$ of the process $(X(t))_{0\leq t\leq T}$ are stopping times in $\mathbb{F}^X$.*

PROOF. We have that the counting processes N_{ij} are adapted to the filtration, because they share the same information with the jump process. Moreover we have the representation

$$\{T_n \leq t\} = \{N(t) \geq n\} = \{ \sum_{i,j \in S : i \neq j} N_{ij}(t) \geq n\} \in \tilde{\mathcal{F}}_t^X.$$

□

We may then define the stopped σ-algebras with respect to the stopping times as

$$\mathcal{F}_{T_n} := \{A \in \mathcal{F} : A \cap \{T_n \leq t\} \in \tilde{\mathcal{F}}_t^X \text{ for all } t \geq 0\}.$$

PROPOSITION 3.18 ([**3**, A2 T28]). *Assume a jump process X and denote its natural filtration as $(\mathcal{F}_t^X)_{t \geq 0}$. Then for all stopping times τ it holds*

$$\mathcal{F}_\tau = \sigma(X(t \wedge \tau) : t \geq 0). \tag{3.40}$$

PROOF. See the proof of Theorem 28 in [**3**].

□

PROPOSITION 3.19 ([**3**, A2 T30]). *Assume a jump process $(X(t))_{0 \leq t \leq T}$ and its augmented natural filtration $\mathbb{F}^X$. We set $X_n := (X(t))_{t \in [T_n, T_{n+1})}$ with the convention that $T_0 \equiv 0$. Then for the jump times $0 < T_1 < T_2 < \dots$ it holds for all $n \in \mathbb{N}$ that*

$$\mathcal{F}_{T_n} = \sigma(X_0, T_1, X_1, \dots, T_n, X_n).$$

PROOF. To prove the inclusion $\sigma(X_0, T_1, X_1, \dots, T_n, X_n) \subset \mathcal{F}_{T_n}$ it is sufficient to show that

$$\{X_i \in A\} \in \mathcal{F}_{T_n} \tag{3.41}$$

for all i such that $0 \leq i \leq n$ and all $A \subset \mathcal{S}$. We have that the process X is left-continuous and adapted to its natural filtration, so therefore it is also progressively measurable. This implies that the random variables X_{T_n} are $\mathcal{F}_{T_n}$ measurable. Moreover for all $i \leq n$ it holds $\mathcal{F}_{T_i} \subset \mathcal{F}_{T_n}$ and therefore the random variables X_{T_i} are $\mathcal{F}_{T_n}$-measurable. Finally the stopping times T_i are also $\mathcal{F}_{T_n}$-measurable.

The inclusion $\mathcal{F}_{T_n} \subset \sigma(X_0, T_1, X_1, \dots, T_n, X_n)$ is more trivial due to the structure of the σ-algebra $\mathcal{F}_{T_n}$. Indeed the Proposition **??** shows that all of the sets $A \in \mathcal{F}_{T_n}$ are $\sigma(X(t \wedge T_n) : t \geq 0)$-measurable. Therefore they are also $\sigma(N(t \wedge T_n) : t \geq 0)$-measurable. Moreover we have the relationship

$$N(t \wedge T_n) = \sum_{m \geq 1} \mathbb{1}_{\{T_m \leq t \wedge T_n\}} \tag{3.42}$$

and the sets A are also $\sigma(T_m : 1 \leq m \leq n)$-measurable.

□

PROPOSITION 3.20 ([**3**, A2 T32]). *Assume a jump process $(X(t))_{0 \leq t \leq T}$ and its augmented natural filtration $\mathbb{F}^X$. Let T_n denote the jump times of the process X. Then for any bounded stopping time τ it holds*

$$\mathcal{F}_\tau \cap \{T_n \leq \tau < T_{n+1}\} = \mathcal{F}_{T_n} \cap \{T_n \leq \tau < T_{n+1}\} \tag{3.43}$$

PROOF. The Proposition 3.18 implies that the σ-algebra $\mathcal{F}_\tau$ is generated by the sets $\{X(t \wedge \tau) : t \geq 0\}$ and also a similar result holds for $\mathcal{F}_{T_n} = \sigma(X(t \wedge T_n) : t \geq 0)$. Moreover

$$\{X(t \wedge \tau) \in A\} \cap \{T_n \leq \tau < T_{n+1}\} = \{X(t \wedge T_n) \in A\} \cap \{T_n \leq \tau < T_{n+1}\},$$

for $A \subset \mathcal{S}$ which tells that the collections in (3.43) must be equal, since the σ-algebras posses the same generating sets.

□

COROLLARY 3.21. [**3**, A2 T33] *Assume a jump process $(X(t))_{0 \leq t \leq T}$ and the respective jump times $(T_n)_{n \in \mathbb{N}}$. For any $(\mathcal{F}_t^X)_{t \geq 0}$ -stopping time τ there exists a sequence $(R_n)_{n \in \mathbb{N}}$ of random variables with*

$$\tau \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1},$$

when ever $\tau \geq T_n$.

PROOF. From the Proposition 3.20 we receive

$$\mathcal{F}_\tau \cap \{T_n \leq \tau < T_{n+1}\} = \mathcal{F}_{T_n} \cap \{T_n \leq \tau < T_{n+1}\}.$$

The Proposition 3.19 further improves the result with

$$\mathcal{F}_{T_n} \cap \{T_n \leq \tau < T_{n+1}\} = \sigma(X_0, T_1, X_1, \ldots, T_n, X_n) \cap \{T_n \leq \tau < T_{n+1}\}.$$

Based on this we may represent the stopping time τ as

$$\tau \cdot \mathbb{1}_{\{T_n \leq \tau < T_{n+1}\}} = \psi_n(X_0, T_1, X_1, \ldots, T_n, X_n) \cdot \mathbb{1}_{\{T_n \leq \tau < T_{n+1}\}},$$

for some measurable function ψ_n for all n. Now choose $(R_n)_{n \in \mathbb{N}}$ with

$$R_n := (\psi_n(X_0, T_1, X_1, \ldots, T_n, X_n) - T_n)^+$$

as the sequence of random variables which concludes the proof. Note that the Proposition 3.19 states $\mathcal{F}_{T_n} = \sigma(X_0, T_1, X_1, \ldots, T_n, X_n)$ for all n. Therefore the random variables R_n are $\mathcal{F}_{T_n}$-measurable by construction.

□

3.6. Explicit form of the intensity

We will now proceed with an explicit expression for the intensities. Before the representation one technical lemma is still required.

LEMMA 3.22 ([**3**, I 2 E7]). *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ be a stochastic basis. Assume that $(X(t))_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$-progressive process. If for all bounded stopping times T*

$$\mathbb{E}[X_T] = \mathbb{E}[X_0] < \infty,$$

then the process $(X(t))_{t \geq 0}$ is a martingale.

PROOF. [**3**, I E7]

Let $0 \leq s \leq t$ and $A \in \mathcal{F}_s$. Then we choose the stopping times

$$T(\omega) = t \quad \text{and} \quad S(\omega) = \begin{cases} s & \omega \in A, \\ t & \omega \in A^C. \end{cases}$$

The random variable T is a stopping time because $\{T \leq v\} \in \{\emptyset, \Omega\} \subset \mathcal{F}_v$ for all $v \in [0, \infty)$. Moreover for the random variable S it holds for $0 \leq v < s$ that

$\{S \leq v\} = \emptyset \in \mathcal{F}_v$. When $s \leq v < t$ we have that $\{S \leq v\} = A \in F_s \subset F_v$, and for the final interval $v > t$ it is true that $\{S \leq v\} = \Omega \in \mathcal{F}_v$.

The random variables S and T are therefore stopping times, and by the assumption we may deduce that

$$\mathbb{E}[X_T] = \mathbb{E}[X_0] = \mathbb{E}[X_S]. \tag{3.44}$$

Now utilizing the definitions of the stopping times we get

$$\mathbb{E}X_S = \mathbb{E}[X_s \mathbb{1}_A] + \mathbb{E}[X_t \mathbb{1}_{A^C}] \quad \text{and} \quad \mathbb{E}X_T = \mathbb{E}[X_t \mathbb{1}_A] + \mathbb{E}[X_t \mathbb{1}_{A^C}].$$

The equation (3.44) then receives the form

$$\mathbb{E}[X_s \mathbb{1}_A] = \mathbb{E}[X_t \mathbb{1}_A],$$

for all $A \in \mathcal{F}_s$, which is by the definition of the conditional expectation equal to the martingale assumption

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s \quad a.s.$$

□

Following the construction in [**3**, III 2] we assume that the waiting times $T_{n+1} - T_n$ given the history $\mathcal{F}_{T_n}$ admit densities. For the densities we remark

$$\mathbb{P}[T_{n+1} - T_n \in A, X_{T_{n+1}} = j, X_{T_n} = i | \mathcal{F}_{T_n}] = \int_A g^{(n+1)}(x, i, j)\, dx \tag{3.45}$$

and

$$g^{(n+1)}(x) := \sum_{i,j \in S : i \neq j} g^{(n+1)}(x, i, j), \tag{3.46}$$

where S is the finite state space of the jump process $(X(t))_{0 \leq t \leq T}$. Then it also holds in the notation that

$$\int_A g^{(n+1)}(x)\, dx = \mathbb{P}[T_{n+1} - T_n \in A | \mathcal{F}_{T_n}]. \tag{3.47}$$

The following proposition yields a representation for the intensities. It also entails an important corollary in the aspect of the martingale representation in the next chapter.

Proposition 3.23 ([**3**, III T7]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Assume a jump process $(X(t))_{0 \leq t \leq T}$ with a finite state space S and its respective counting processes $N_{ij} = (N_{ij}(t))_{0 \leq t \leq T}$. Mark the augmented natural filtration of X as $(\mathcal{F}_t^X)_{0 \leq t \leq T}$. Let $(T_n)_{n \in \mathbb{N}}$ denote the jump times $T_n = \inf\{t \geq 0 : \sum_{i \neq j \in S} N_{ij}(t) = n\}$. If we define the processes $\lambda_{ij} = (\lambda_{ij}(t))_{0 \leq t \leq T}$ for all $i, j \in S$ such that $i \neq j$ with*

$$\lambda_{ij}(t) := \sum_{n \geq 0} \frac{g^{(n+1)}(t - T_n, i, j)}{1 - \int_0^{t - T_n} g^{(n+1)}(x)\, dx} \mathbb{1}_{\{T_n \leq t < T_{n+1}\}},$$

then the processes $M_{ij} := (M_{ij}(t))_{0 \leq t \leq T}$ where

$$M_{ij}(t) := N_{ij}(t) - \int_0^t \lambda_{ij}(s)\, ds$$

are local martingales for $i, j \in S$ and $i \neq j$, with the localizing sequence $(T_n)_{n \in \mathbb{N}}$.

PROOF. The idea is the same as in the proof of [**3**, III T7]. Lemma 3.22 shows that in order to prove that the processes

$$\left(N_{ij}(t \wedge T_n) - \int_0^{t\wedge T_n} \lambda_{ij}(s)\, ds\right)_{0\le t\le T}$$

are martingales for every n it is sufficient to prove that

$$\mathbb{E}[M_{ij}(S \wedge T_n)] = \mathbb{E}[M_{ij}(0 \wedge T_n)] = 0.$$

holds for every bounded stopping time S for all n. This is equal to showing

$$\mathbb{E}[N_{ij}(S \wedge T_n)] = \mathbb{E}\left[\int_0^{S\wedge T_n} \lambda_{ij}(s)\, ds\right] \tag{3.48}$$

for all n and bounded S.

We shall begin by representing the stopping time $S \wedge T_n$ more conveniently utilizing $\mathcal{F}_{T_n}$-measurable functions. The Corollary 3.21 justifies the existence of $\mathcal{F}_{T_n}$-measurable random variables $(R_n)_{n\in\mathbb{N}}$ such that

$$S \wedge T_{n+1} = (T_n + R_n) \wedge T_{n+1},$$

when ever $S \ge T_n$.

By applying this to the equation (3.48) the right hand side becomes

$$\begin{aligned}
\mathbb{E}\left[\int_0^{S\wedge T_n} \lambda_{ij}(s)\, ds\right] &= \mathbb{E}\left[\sum_{k=0}^{n-1}\left(\mathbb{1}_{\{S\ge T_k\}} \int_{T_k}^{(T_k+R_k)\wedge T_{k+1}} \lambda_{ij}(s)\, ds\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1}\left(\mathbb{1}_{\{S\ge T_k\}} \int_{T_k}^{(T_k+R_k)\wedge T_{k+1}} \sum_{\ell\ge 0} \frac{g^{(\ell+1)}(s-T_\ell, i, j)}{1-\int_0^{s-T_\ell} g^{(\ell+1)}(x)\, dx} \mathbb{1}_{\{T_\ell\le s< T_{\ell+1}\}}\, ds\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1}\left(\mathbb{1}_{\{S\ge T_k\}} \int_{T_k}^{(T_k+R_k)\wedge T_{k+1}} \frac{g^{(k+1)}(s-T_k, i, j)}{1-\int_0^{s-T_k} g^{(k+1)}(x)\, dx}\, ds\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1}\left(\mathbb{1}_{\{S\ge T_k\}} \int_0^{R_k\wedge(T_{k+1}-T_k)} \frac{g^{(k+1)}(s, i, j)}{1-\int_0^{s} g^{(k+1)}(x)\, dx}\, ds\right)\right],
\end{aligned}$$

where we utilized the change of variable with the integral. We may now take advantage of the Tower property and the information that $\{S \ge T_k\}$ is $\mathcal{F}_{T_k}$-measurable to get

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{n-1}\left(\mathbb{1}_{\{S\ge T_k\}} \int_0^{R_k\wedge(T_{k+1}-T_k)} \frac{g^{(k+1)}(s, i, j)}{1-\int_0^{s} g^{(k+1)}(x)\, dx}\, ds\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1}\left(\mathbb{E}\left[\mathbb{1}_{\{S\ge T_k\}} \int_0^{R_k\wedge(T_{k+1}-T_k)} \frac{g^{(k+1)}(s, i, j)}{1-\int_0^{s} g^{(k+1)}(x)\, dx}\, ds\middle|\mathcal{F}_{T_j}\right]\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1}\left(\mathbb{1}_{\{S\ge T_k\}}\mathbb{E}\left[\int_0^{R_k\wedge(T_{k+1}-T_k)} \frac{g^{(k+1)}(s, i, j)}{1-\int_0^{s} g^{(k+1)}(x)\, dx}\, ds\middle|\mathcal{F}_{T_j}\right]\right)\right].
\end{aligned}$$

The inner conditional expectation becomes of our interest. We have by the definition of the conditional expectation that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{R_k\wedge(T_{k+1}-T_k)} \frac{g^{(k+1)}(s,i,j)}{1-\int_0^s g^{(k+1)}(x)\,dx}\,ds\middle|\mathcal{F}_{T_j}\right] \\
&= \int_0^\infty g^{(k+1)}(u)\int_0^{R_k\wedge u} \frac{g^{(k+1)}(s,i,j)}{1-\int_0^s g^{(k+1)}(x)\,dx}\,ds\,du \tag{3.49}
\end{aligned}$$

Fubini's theorem can now be utilized to reverse the order of integration. The integration boundaries become

$$\{(u,s) : 0<u<\infty, 0<s<R_k\wedge u\} = \{(s,u) : 0<s<R_k, s<u<\infty\},$$

so the (3.49) equals after Fubini

$$\begin{aligned}
&\int_0^{R_k} \frac{g^{(k+1)}(s,i,j)}{1-\int_0^s g^{(k+1)}(x)\,dx}\int_s^\infty g^{(k+1)}(u)\,du\,ds \\
&= \int_0^{R_k} g^{(k+1)}(s,i,j)\frac{1-\int_0^s g^{(k+1)}(u)\,du}{1-\int_0^s g^{(k+1)}(x)\,dx}\,ds \\
&= \int_0^{R_k} g^{(k+1)}(s,i,j)\,ds.
\end{aligned}$$

So far we have reached the equality

$$\mathbb{E}\left[\int_0^{S\wedge T_n} \lambda_{ij}(s)\,ds\right] = \mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{1}_{\{S\geq T_k\}}\int_0^{R_k} g^{(k+1)}(s,i,j)\,ds\right]. \tag{3.50}$$

We will now turn our attention towards the left hand side of the equation (3.48). We may inspect

$$\begin{aligned}
\mathbb{E}[N_{ij}(S\wedge T_n)] &= \mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{1}_{\{S\geq T_k\}}\Big(N_{ij}(S\wedge T_{k+1}) - N_{ij}(S\wedge T_k)\Big)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{1}_{\{S\geq T_k\}}\mathbb{1}_{\{R_k\geq T_{k+1}-T_k\}}\mathbb{1}_{\{X_{T_k}=i\}}\mathbb{1}_{\{X_{T_{k+1}}=j\}}\right].
\end{aligned}$$

Once again by the Tower property and the $\mathcal{F}_{T_k}$-measurability of the set $\{S \geq T_k\}$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{1}_{\{S\geq T_k\}}\mathbb{1}_{\{R_k\geq T_{k+1}-T_k\}}\mathbb{1}_{\{X_{T_k}=i\}}\mathbb{1}_{\{X_{T_{k+1}}=j\}}\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{E}\left[\mathbb{1}_{\{S\geq T_k\}}\mathbb{1}_{\{R_k\geq T_{k+1}-T_k\}}\mathbb{1}_{\{X_{T_k}=i\}}\mathbb{1}_{\{X_{T_{k+1}}=j\}}\middle|\mathcal{F}_{T_k}\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{1}_{\{S\geq T_k\}}\mathbb{E}\left[\mathbb{1}_{\{R_k\geq T_{k+1}-T_k\}}\mathbb{1}_{\{X_{T_k}=i\}}\mathbb{1}_{\{X_{T_{k+1}}=j\}}\middle|\mathcal{F}_{T_k}\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{1}_{\{S\geq T_k\}}\mathbb{P}\left(R_k \geq T_{k+1}-T_k, X_{T_k}=i, X_{T_{k+1}}=j\middle|\mathcal{F}_{T_k}\right)\right].
\end{aligned}$$

Remark the definition of the densities $g^{(n)}$, which yields

$$\mathbb{P}\left(R_k \geq T_{k+1}-T_k, X_{T_k}=i, X_{T_{k+1}}=j\middle|\mathcal{F}_{T_k}\right) = \int_0^{R_k} g^{(k+1)}(s,i,j)\,ds.$$

Therefore we may conclude

$$\mathbb{E}[N_{ij}(S\wedge T_n)] = \mathbb{E}\left[\sum_{k=0}^{n-1}\mathbb{1}_{\{S\geq T_k\}}\int_0^{R_k} g^{(k+1)}(s,i,j)\,ds\right]. \tag{3.51}$$

The equations (3.50) and (3.51) together confirm the desired result of (3.48), so the processes M_{ij} are local martingales with a localizing sequence $(T_n)_{n\in\mathbb{N}}$.

□

COROLLARY 3.24. *Given the same assumptions as in the Proposition 3.23, the processes $(N_{ij}(t))_{0\leq t\leq T}$ admit the intensities $(\lambda_{ij}(t))_{0\leq t\leq T}$, where $i,j\in S$ and $i\neq j$.*

PROOF. From the Proposition 3.23 we may deduce that the processes

$$\left(N_{ij}(t\wedge T_n) - \int_0^{t\wedge T_n}\lambda_{ij}(s)\,ds\right)_{0\leq t\leq T}$$

are martingales for all T_n. Therefore by the Proposition 3.12 the processes $(N_{ij}(t))_{0\leq t\leq T}$ admit the intensities $(\lambda_{ij}(t))_{0\leq t\leq T}$.

□

3.7. Martingale representation

In this section we will present an explicit solution to the Martingale Representation Theorem when the representable process is a conditional expectation of a integrable random variable. This theorem is essential in the context of the prospective reserve which is also defined via a conditional expectation of the payment process of the life insurance contract discounted to the present value. We shall present the rigorous definitions in the next chapter. This is only to emphasize the necessity of the following result in regard to the rest of the analyses.

THEOREM 3.25 ([**3**, III 3 T9]). *Assume a jump process $X = (X(t))_{0\leq t\leq T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space S and denote the augmented natural filtration of the process X as $(\mathcal{F}_t)_{0\leq t\leq T}$. Let $(N_{ij}(t))_{0\leq t\leq T}$ be the counting processes of the process X and $(\lambda_{ij}(t))_{0\leq t\leq T}$ be the corresponding intensities. Assume a right-continuous process $(Y(t))_{0\leq t\leq T}$ given by $Y(t) := \mathbb{E}[\zeta|\mathcal{F}_t]$ with ζ being an integrable random variable. Then there exist a.s-unique predictable processes $Z_{ij} = (Z_{ij}(t))_{0\leq t\leq T}$ such that for all $i, j \in S$ with $i \neq j$ it holds*

$$\int_0^t |Z_{ij}(s)|\lambda_{ij}(s)\, ds < \infty, \quad \mathbb{P} - a.s.$$

and the processes $(Z_{ij}(t))_{0\leq t\leq T}$ satisfy

$$Y_t = Y_0 + \sum_{i,j\in S:i\neq j} \int_{(0,t]} Z_{ij}(s)(dN_{ij}(s) - \lambda_{ij}(s)\, ds), \quad \mathbb{P} - a.s. \tag{3.52}$$

REMARK 3.26. With the notation of this chapter the equation (3.52) is simplified into a more general form of

$$Y(t) = Y(0) + \int_{(0,t]} Z(s)\, dM(s) \quad a.s.$$

for all $0 \leq t \leq T$.

PROOF. The proof of the Theorem 3.25 follows the proof given in [**3**, III 3 T9]. At first we will present some notation before the main part of the proof. From the Proposition 3.23 we will use the form of the intensities λ_{ij} where

$$\lambda_{ij}(t) = \sum_{n\geq 0} \frac{g^{(n+1)}(t - T_n, i, j)}{1 - \int_0^{t-T_n} g^{(n+1)}(x)\, dx} \mathbb{1}(T_n \leq t < T_{n+1}). \tag{3.53}$$

Furthermore the a.s.-unique solutions Z_{ij} will have an explicit form of

$$Z_{ij}(t) = \sum_{n\geq 0} \left(f^{(n)}(t - T_n, i, j) - \frac{\sum\limits_{i,j\in S:i\neq j} \int_{t-T_n}^{\infty} f^{(n)}(s, i, j) g^{(n+1)}(s, i, j)\, ds}{1 - \int_0^{t-T_n} g^{(n+1)}(x)\, dx} \right) \mathbb{1}_{\{T_n<t\leq T_{n+1}\}}, \tag{3.54}$$

where for all $n \in \mathbb{N}$ the functions $f^{(n)}$ are $\mathcal{F}_{T_n} \otimes \mathcal{B}((0,\infty))$-measurable mappings with

$$f^{(n)}\big(T_{n+1} - T_n, X(T_n), X(T_{n+1})\big) = Y(T_{n+1}), \tag{3.55}$$

for $T_{n+1} < \infty$.

Now we will utilize the Doob's optional sampling theorem (Proposition 2.28) to the stopping times $T_{n+1} \wedge t$ and T_{n+1}. The theorem argues that the martingale property $\mathbb{E}[X_T|S] = X_S$ holds almost surely for all bounded stopping times $T \geq S$ and right-continuous martingales $(X_t)_{t\geq 0}$. Here we require that $T_n < \infty$. Therefore we have that

$$\begin{aligned} \mathbb{E}[Y(t)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{t\wedge T_{n+1}}] &= \mathbb{E}[Y(t)|\mathcal{F}_{t\wedge T_{n+1}}]\mathbb{1}_{\{t<T_{n+1}\}} \\ &= Y(t \wedge T_{n+1})\mathbb{1}_{\{t<T_{n+1}\}} \quad a.s., \end{aligned}$$

where we applied the $\mathcal{F}_{t\wedge T_{n+1}}$-measurability of the random variable $\mathbb{1}_{\{t<T_{n+1}\}}$. Note that here $t \le T_{n+1}$ and the arriving times T_n are bounded by the terminal time T. Utilizing the Doob's theorem again for the stopping times $t\wedge T_{n+1} \le T_{n+1}$ we receive

$$\begin{aligned} Y(t\wedge T_{n+1})\mathbb{1}_{\{t<T_{n+1}\}} &= \mathbb{E}[Y(T_{n+1})|\mathcal{F}_{t\wedge T_{n+1}}]\mathbb{1}_{\{t<T_{n+1}\}} \\ &= \mathbb{E}[Y(T_{n+1})\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{t\wedge T_{n+1}}] \quad a.s. \end{aligned}$$

Now note that the events $\{T_n \le t\}$ are $\mathcal{F}_{t\wedge T_{n+1}}$-measurable for all n because of the equality

$$\{T_n \le t\} = \{T_n \le t \wedge T_{n+1}\} \in \mathcal{F}_{t\wedge T_{n+1}}.$$

Therefore applying Doob's optional sampling theorem twice in the similar fashion to the previous examinations we may deduce

$$\begin{aligned} &\mathbb{E}[Y(t)\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{t\wedge T_{n+1}}] \\ &= \mathbb{E}[Y(t)|\mathcal{F}_{t\wedge T_{n+1}}]\mathbb{1}_{\{T_n\le t<T_{n+1}\}} \\ &= \mathbb{E}[Y(T_{n+1})|\mathcal{F}_{t\wedge T_{n+1}}]\mathbb{1}_{\{T_n\le t<T_{n+1}\}} \\ &= \mathbb{E}[Y(T_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{t\wedge T_{n+1}}] \quad a.s. \end{aligned}$$

So far we have proven that

$$\mathbb{E}[\mathbb{1}_A Y(t)\mathbb{1}_{\{T_n\le t<T_{n+1}\}}] = \mathbb{E}[\mathbb{1}_A Y(T_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}] \tag{3.56}$$

for all $A \in \mathcal{F}_{t\wedge T_{n+1}}$.

Lastly we will prove the result for the σ-algebra $\mathcal{F}_{T_n}$. Let therefore $C \in \mathcal{F}_{T_n}$. The Proposition 3.20 proves that there exists a set $A \in \mathcal{F}_{t\wedge T_{n+1}}$ such that

$$C \cap \{T_n \le t < T_{n+1}\} = A \cap \{T_n \le t < T_{n+1}\}. \tag{3.57}$$

We may now proceed by combining the equalities (3.56) and (3.57). By the definition of the conditional expectation we may state

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_C Y(t)\mathbb{1}_{\{T_n\le t<T_{n+1}\}}] \\ &= \mathbb{E}[\mathbb{1}_A Y(t)\mathbb{1}_{\{T_n\le t<T_{n+1}\}}] \\ &= \mathbb{E}[\mathbb{1}_A Y(T_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}] \\ &= \mathbb{E}[\mathbb{1}_C Y(T_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}]. \end{aligned}$$

Therefore we have reached the conclusion that

$$\mathbb{E}[Y(t)\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}] = \mathbb{E}[Y(T_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}]. \tag{3.58}$$

We will now consider a different representations for the both sides of the equation (3.58). The next step is to express the process Y with $\mathcal{F}_{T_n}$-measurable processes as in [**3**, III E9]. Fix $t_0 \in [T_n, T_{n+1})$. Then there exists a sequence of $\mathcal{F}_{t_0}$-simple functions $(q_k)_{k\in\mathbb{N}}$ with

$$q_k(\omega) = \sum_{\ell=1}^{\infty} \alpha_\ell \mathbb{1}_{A_\ell}, \quad \alpha_\ell \in \mathbb{R}, A_\ell \in \mathcal{F}_{t_0},$$

such that the sequence converges to the random variable $Y(t_0)$ as $k \to \infty$. But Proposition 3.20 proves that the simple functions have $\mathcal{F}_{T_n}$-measurable counterparts:

For all $A_\ell \in \mathcal{F}_{t_0}$ there exist $C_\ell \in \mathcal{F}_{T_n}$ such that

$$A_\ell \cap \{T_n \le t \le T_{n+1}\} = C_\ell \cap \{T_n \le t \le T_{n+1}\}$$

and therefore there exists a sequence of $\mathcal{F}_{T_n}$-measurable simple functions $(p_k)_{k\in\mathbb{N}}$ with $p_k(\omega) = q_k(\omega)$ for all k. The sequence also converges so denote the limit as $h^{(n)} := \lim_{k\to\infty} p_k$. Moreover by the definition of the process Y and the tower property it holds

$$\mathbb{E}[Y(t)|\mathcal{F}_{T_n}] = \mathbb{E}[\mathbb{E}[\zeta|\mathcal{F}_t]|\mathcal{F}_{T_n}] = \mathbb{E}[\zeta|\mathcal{F}_{T_n}] = Y(T_n), \quad a.s. \tag{3.59}$$

Then for the left side of the (3.58) we have

$$\begin{aligned}\mathbb{E}[Y(t)\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}] &= \mathbb{E}[h_t^{(n)}\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}]\\ &= \mathbb{E}[h_t^{(n)}\mathbb{1}_{\{T_n\le t\}}\mathbb{1}_{\{T_{n+1}>t\}}|\mathcal{F}_{T_n}].\end{aligned}$$

The random variable $\mathbb{1}_{\{T_n\le t\}}h_t^{(n)}$ is $\mathcal{F}_{T_n}$ measurable because the random variables $\mathbb{1}_{\{T_n\le t\}}$ and $h_t^{(n)}$ are $\mathcal{F}_{T_n}$-measurable. We may conclude therefore that

$$\begin{aligned}\mathbb{E}[Y(t)\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}] &= h_t^{(n)}\mathbb{1}_{\{T_n\le t\}}\mathbb{E}[\mathbb{1}_{\{T_{n+1}>t\}}|\mathcal{F}_{T_n}]\\ &= h_t^{(n)}\mathbb{1}_{\{T_n\le t\}}\mathbb{P}[T_{n+1}-T_n > t-T_n|\mathcal{F}_{T_n}]\\ &= h_t^{(n)}\mathbb{1}_{\{T_n\le t\}}\int_{t-T_n}^{\infty} g^{(n+1)}(x)\,dx, \quad a.s. \qquad (3.60)\end{aligned}$$

where we applied the definition of the densities $g^{(n)}$.

Now we turn our attention towards the right side of the equation (3.58). Recall the functions $f^{(n)}$ from the (3.55). We receive a representation for the conditional expectation of the random variables $Y(T_n)$:

$$\mathbb{E}[Y(T_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}] = \mathbb{E}[f^{(n)}(T_{n+1}-T_n, Z_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}].$$

It is convenient to split the indicator function into

$$\mathbb{1}_{\{T_n\le t<T_{n+1}\}} = \mathbb{1}_{\{T_n\le t\}}\mathbb{1}_{\{T_{n+1}>t\}} = \mathbb{1}_{\{T_n\le t\}}\mathbb{1}_{\{T_{n+1}-T_n>t-T_n\}}.$$

The indicator function $\mathbb{1}_{\{T_n\le t\}}$ is $\mathcal{F}_{T_n}$-measurable by the Proposition 3.19. Therefore we can take it out of the conditional expectation and receive

$$\begin{aligned}&\mathbb{E}[Y(T_{n+1})\mathbb{1}_{\{T_n\le t<T_{n+1}\}}|\mathcal{F}_{T_n}]\\ &= \mathbb{E}[f^{(n)}(T_{n+1}-T_n, Z_{n+1})\mathbb{1}_{\{T_n\le t\}}\mathbb{1}_{\{T_{n+1}-T_n>t-T_n\}}|\mathcal{F}_{T_n}]\\ &= \mathbb{1}_{\{T_n\le t\}}\mathbb{E}\left[f^{(n)}(T_{n+1}-T_n, Z_{n+1})\mathbb{1}_{\{T_{n+1}-T_n>t-T_n\}}|\mathcal{F}_{T_n}\right]\\ &= \mathbb{1}_{\{T_n\le t\}}\int_{t-T_n}^{\infty} f^{(n)}(s, Z_{n+1})g^{(n+1)}(s)\,ds\\ &= \mathbb{1}_{\{T_n\le t\}}\sum_{i,j\in S:i\ne j}\int_{t-T_n}^{\infty} f^{(n)}(s,i,j)g^{(n+1)}(s,i,j)\,ds, \quad a.s. \qquad (3.61)\end{aligned}$$

By unifying both of the results (3.61) and (3.60) with the original equation (3.58) we receive

$$h_t^{(n)}\mathbb{1}_{\{T_n\le t\}}\int_{t-T_n}^{\infty} g^{(n+1)}(x)\,dx = \mathbb{1}_{\{T_n\le t\}}\sum_{i,j\in S:i\ne j}\int_{t-T_n}^{\infty} f^{(n)}(s,i,j)g^{(n+1)}(s,i,j)\,ds.$$

This is equivalent with

$$h_t^{(n)} \mathbb{1}_{\{T_n \le t\}} = \frac{\mathbb{1}_{\{T_n \le t\}} \sum_{i,j \in S: i \neq j} \int_{t-T_n}^{\infty} f^{(n)}(s,i,j) g^{(n+1)}(s,i,j)\, ds}{\int_{t-T_n}^{\infty} g^{(n+1)}(x)\, dx}. \tag{3.62}$$

We also have that $Y(T_n) = h_{T_n}^{(n)}$ because of (3.59), the information $\mathbb{E}[Y(t)|\mathcal{F}_{T_n}] = \mathbb{E}[h_t^{(n)}|\mathcal{F}_{T_n}]$ on $t \in [T_n, T_{n+1})$ and the $\mathcal{F}_{T_n}$-measurability of the processes $h^{(n)}$. Therefore we may deduce from the representation (3.62) that

$$\begin{aligned} Y(T_n) &= h_{T_n}^{(n)} \mathbb{1}_{\{T_n \le T_n\}} \\ &= \frac{\mathbb{1}_{\{T_n \le T_n\}} \sum_{i,j \in S: i \neq j} \int_{T_n - T_n}^{\infty} f^{(n)}(s,i,j) g^{(n+1)}(s,i,j)\, ds}{\int_{T_n - T_n}^{\infty} g^{(n+1)}(x)\, dx} \\ &= \sum_{i,j \in S: i \neq j} \int_0^{\infty} f^{(n)}(s,i,j) g^{(n+1)}(s,i,j)\, ds. \end{aligned}$$

In the denominator we used the information that

$$\int_{T_n - T_n}^{\infty} g^{(n+1)}(x)\, dx = \mathbb{P}\big[T_{n+1} - T_n \in [0,\infty] \big| \mathcal{F}_{T_n}\big] = 1.$$

This previous result alongside with the examination in (3.62) yields the formula for the process Y in between the jumps: For all $t \in [T_n, T_{n+1})$ where $T_n < T$ it holds

$$Y(t) - Y(T_n) = \sum_{i,j \in S: i \neq j} \left(\frac{\int_{t-T_n}^{\infty} f^{(n)}(s,i,j) g^{(n+1)}(s,i,j)\, ds}{\int_{t-T_n}^{\infty} g^{(n+1)}(x)\, dx} - \int_0^{\infty} f^{(n)}(s,i,j) g^{(n+1)}(s,i,j)\, ds \right). \tag{3.63}$$

Our objective now is to express the difference $M_t - M_{T_n}$ in terms of the processes Z_{ij} and the intensities λ_{ij}, which were defined in (3.54) and (3.53) respectively. To proceed towards the martingale representation and to clarify the subsequent analyses we shall now shorten our notation with

$$\begin{aligned} a(t) &= \int_t^{\infty} g^{(n+1)}(x)\, dx \quad \text{and} \\ b(t) &= \sum_{i,j \in S: i \neq j} \int_t^{\infty} f^{(n)}(s,i,j) g^{(n+1)}(s,i,j)\, ds. \end{aligned}$$

Due to the definition of the densities $g^{(n)}$ we have $a(0) = 1$. The equation (3.63) becomes then

$$Y(t) - Y(T_n) = \frac{b(t - T_n)}{a(t - T_n)} - \frac{b(0)}{a(0)},$$

when ever $t \in [T_n, T_{n+1})$ and $T_n < T$. By applying the integration by parts formula (Proposition 3.9) to the functions $b(t)$ and $1/a(t)$ it follows that

$$\frac{b(t)}{a(t)} = \frac{b(0)}{a(0)} + \int_{(0,t]} \frac{1}{a(s)}\, db(s) - \int_{(0,t]} \frac{b(s)}{(a(s))^2}\, da(s).$$

Therefore it holds that

$$Y(t)-Y(T_n)=\int_{(0,t-T_n]}\frac{1}{a(s)}\,db(s)-\int_{(0,t-T_n]}\frac{b(s)}{(a(s))^2}\,da(s),\quad \forall t\in[T_n,T_{n+1}). \tag{3.64}$$

Keeping in mind the notation and the constraints $t\in[T_n,T_{n+1})$ and $T_n<T$, the equation (3.64) transforms into

$$\begin{aligned}
Y(t)-Y(T_n)=&\int_{(0,t-T_n]}\frac{1}{\int_s^\infty g^{(n+1)}(x)\,dx}\,d\left(\sum_{i,j\in S:i\neq j}\int_s^\infty f^{(n)}(u,i,j)g^{(n+1)}(u,i,j)\,du\right)-\\
&\int_{(0,t-T_n]}\frac{\sum\limits_{i,j\in S:i\neq j}\int_s^\infty f^{(n)}(u,i,j)g^{(n+1)}(u,i,j)\,du}{\left(\int_s^\infty g^{(n+1)}(x)\,dx\right)^2}\,d\left(\int_s^\infty g^{(n+1)}(x)\,dx\right)\\
=&-\int_{(0,t-T_n]}\frac{\sum\limits_{i,j\in S:i\neq j}f^{(n)}(s,i,j)g^{(n+1)}(s,i,j)}{\int_s^\infty g^{(n+1)}(x)\,dx}\,ds+\\
&\int_{(0,t-T_n]}\frac{\sum\limits_{i,j\in S:i\neq j}\int_s^\infty\left(f^{(n)}(u,i,j)g^{(n+1)}(u,i,j)\right)du\ g^{(n+1)}(s)}{\left(\int_s^\infty g^{(n+1)}(x)\,dx\right)^2}\,ds.
\end{aligned}\tag{3.65}$$

In the last conclusion we utilized the Fundamental Theorem of Calculus which reversed the signs of the terms.

In order to reach the representation which includes the processes Z_ij and λ_{ij}, the difference in (3.65) must still be modified. The first term becomes

$$\begin{aligned}
&\int_{(0,t-T_n]}\frac{\sum\limits_{i,j\in S:i\neq j}f^{(n)}(s,i,j)g^{(n+1)}(s,i,j)}{\int_s^\infty g^{(n+1)}(x)\,dx}\,ds\\
=&\int_{(T_n,t]}\frac{\sum\limits_{i,j\in S:i\neq j}f^{(n)}(s-T_n,i,j)g^{(n+1)}(s-T_n,i,j)}{\int_{s-T_n}^\infty g^{(n+1)}(x)\,dx}\,ds\\
=&\sum_{i,j\in S:i\neq j}\int_{T_n}^t f^{(n)}(s-T_n,i,j)\frac{g^{(n+1)}(s-T_n,i,j)}{\int_{s-T_n}^\infty g^{(n+1)}(x)\,dx}\,ds.
\end{aligned}$$

Because of the definition $g^{(n+1)}(x):=\sum_{i,j\in S:i\neq j}g^{(n+1)}(x,i,j)$ the second term becomes

$$\begin{aligned}
&\int_{(0,t-T_n]}\frac{\sum\limits_{i,j\in S:i\neq j}\int_s^\infty\left(f^{(n)}(u,i)g^{(n+1)}(u,i,j)\right)du\ g^{(n+1)}(s)}{\left(\int_s^\infty g^{(n+1)}(x)\,dx\right)^2}\,ds\\
=&\sum_{i,j\in S:i\neq j}\int_{T_n}^t\frac{\sum\limits_{i,j\in S:i\neq j}\int_{s-T_n}^\infty f^{(n)}(x,i,j)g^{(n+1)}(x,i,j)\,dx}{\int_{s-T_n}^\infty g^{(n+1)}(x)\,dx}\cdot\frac{g^{(n+1)}(s-T_n,i,j)}{\int_{s-T_n}^\infty g^{(n+1)}(x)\,dx}\,ds.
\end{aligned}$$

Therefore we may finally conclude that with $t \in [T_n, T_{n+1})$ and $T_n < T$ it holds that

$$\begin{aligned}
Y(t) - Y(T_n) = &\sum_{i,j\in S:i\neq j} \int_{T_n}^{t} \Bigg[- \Bigg(f^{(n)}(s - T_n) - \frac{\sum_{i,j\in S:i\neq j} \int_{s-T_n}^{\infty} f^{(n)}(x,i,j) g^{(n+1)}(x,i,j)\,dx}{\int_{s-T_n}^{\infty} g^{(n+1)}(u,i,j)\,du} \Bigg) \cdot \\
&\frac{g^{(n+1)}(s - T_n, i, j)}{\int_{s-T_n}^{\infty} g^{(n+1)}(u,i,j)\,du} \Bigg] ds \\
= &- \sum_{i,j\in S:i\neq j} \int_{T_n}^{t} Z_{ij}(s) \lambda_{ij}(s)\,ds.
\end{aligned} \tag{3.66}$$

We will have to consider what happens at the jumps before terminating the proof. The process M is right continuous with left limits. As a consequence of the equations (3.55) and (3.62) it holds that

$$\begin{aligned}
&Y(T_{n+1}) - Y(T_{n+1}-) \\
&= f^{(n)}\big(T_{n+1} - T_n, X(T_n)\big) - \lim_{t\to T_{n+1}-} \frac{\sum_{i,j\in S:i\neq j} \int_{t-T_n}^{\infty} f^{(n)}(s,i,j) g^{(n+1)}(s,i,j)\,ds}{\int_{t-T_n}^{\infty} g^{(n+1)}(x)\,dx} \\
&= f^{(n)}\big(T_{n+1} - T_n, X(T_n)\big) - \frac{\sum_{i,j\in S:i\neq j} \int_{T_{n+1}-T_n}^{\infty} f^{(n)}(s,i) g^{(n+1)}(s,i)\,ds}{\int_{T_{n+1}-T_n}^{\infty} g^{(n+1)}(x)\,dx} \\
&= \sum_{i,j\in S:i\neq j} f^{(n)}\big(T_{n+1} - T_n, i, j\big) - \frac{\sum_{i,j\in S:i\neq j} \int_{T_{n+1}-T_n}^{\infty} f^{(n)}(s,i) g^{(n+1)}(s,i)\,ds}{\int_{T_{n+1}-T_n}^{\infty} g^{(n+1)}(x)\,dx} \\
&= \sum_{i,j\in S:i\neq j} Z_{ij}(T_{n+1}).
\end{aligned}$$

In the final step we utilized the definition of the process Z_{ij} in (3.54). Note that due to the jump occuring at the time T_{n+1} it holds that $N_{ij}(T_{n+1}) - N_{ij}(T_{n+1}-) = 1$ and $\Delta N_{ij}(t) = 0$ for all $t \in [T_n, T_{n+1})$. Therefore due to the right-continuity of the processes Z_{ij} we have

$$\sum_{i,j\in S:i\neq j} Z_{ij}(T_{n+1}) = \sum_{i,j\in S:i\neq j} \int_{(T_n,T_{n+1}]} Z_{ij}(s)\,dN_{ij}(s). \tag{3.67}$$

A direct consequence of the revelations (3.67) and (3.66) is that for $t \in [T_n, T_{n+1})$ such that $T_n < \infty$ it is true that

$$\begin{aligned}
Y(t) - Y(0) &= Y(t) + \sum_{k=1}^{n} \Big(- Y(T_k) + Y(T_k) - Y(T_k-) + Y(T_k-) \Big) - Y(0) \\
&= Y(t) - Y(T_n) + \sum_{k=1}^{n} \Big(Y(T_k) - Y(T_k-) + Y(T_k-) - Y(T_{k-1}) \Big) \\
&= \sum_{i,j \in S: i \neq j} \Bigg(- \int_{T_n}^{t} Z_{ij}(s) \lambda_{ij}(s)\, ds + \sum_{k=1}^{n} \Bigg(\int_{(T_{k-1}, T_k]} Z_{ij}(s)\, dN_{ij}(s) - \\
&\quad \int_{T_{k-1}}^{T_k -} Z_{ij}(s) \lambda_{ij}(s)\, ds \Bigg) \Bigg) \\
&= \sum_{i,j \in S: i \neq j} \Bigg(- \int_{T_n}^{t} Z(s) \lambda(s)\, ds + \int_{(0, T_n]} Z(s) (dN(s) - \lambda(s)\, ds) \Bigg) \\
&= \sum_{i,j \in S: i \neq j} \Bigg(\int_{(T_n, t]} Z_{ij}(s) (dN_{ij}(s) - \lambda_{ij}(s)\, ds) + \int_{(0, T_n]} Z_{ij}(s) (dN_{ij}(s) - \lambda_{ij}(s)\, ds) \Bigg) \\
&= \sum_{i,j \in S: i \neq j} \int_{(0,t]} Z_{ij}(s) \big(dN_{ij}(s) - \lambda_{ij}(s)\, ds \big).
\end{aligned} \tag{3.68}$$

This is equivalent with the desired result of (3.52).

□

COROLLARY 3.27 ([**4**, Proposition 2.4.]). *Assume a jump process $X = (X(t))_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space S and denote the augmented natural filtration of the process X as $(\mathcal{F}_t)_{0 \le t \le T}$. Let $(N_{ij}(t))_{0 \le t \le T}$ be the counting processes of the process X and $(\lambda_{ij}(t))_{0 \le t \le T}$ be the corresponding intensities. Assume an integrable random variable ζ and let $Y = (Y(t))_{t \in [0,T]}$ be a unique right-continuous process such that*

$$Y(t) := \mathbb{E}[\zeta | \mathcal{F}_t].$$

If we define the processes $Z_{ij} = (Z_{ij}(t))_{0 \le t \le T}$ as

$$Z_{ij}(t) := \sum_{n=0}^{\infty} \mathbb{1}_{T_n < t \le T_{n+1}} \left(\mathbb{E}[\zeta | \mathcal{F}_{T_n}, T_{n+1} = t, X(T_{n+1}) = j] - \frac{\mathbb{E}[\zeta \mathbb{1}_{\{t < T_{n+1}\}} | \mathcal{F}_{T_n}]}{\mathbb{E}[\mathbb{1}_{\{t < T_{n+1}\}} | \mathcal{F}_{T_n}]} \right), \tag{3.69}$$

then the process Y satisfies the Martingale representation equation

$$Y_t = Y_0 + \sum_{i,j \in S: i \neq j} \int_{(0,t]} Z_{ij}(s) (dN_{ij}(s) - \lambda_{ij}(s)\, ds), \quad \mathbb{P} - a.s. \tag{3.70}$$

Proof. From the Theorem 3.25 we may extract that the processes $(Z_{ij}(t))_{0\le t\le T}$ satisfying the (3.70) are of the form

(3.71)

$$Z_{ij}(t)=\sum_{n\ge 0}\left(f^{(n)}(t-T_n,i,j)-\frac{\sum\limits_{i,j\in S:i\neq j}\int_{t-T_n}^{\infty}f^{(n)}(s,i,j)g^{(n+1)}(s,i,j)\,ds}{1-\int_0^{t-T_n}g^{(n+1)}(x)\,dx}\right)\mathbb{1}_{\{T_n<t\le T_{n+1}\}}.$$

We shall prove the equivalence to the equation (3.69) by inspecting the parts individually.

For the term $1-\int_0^{t-T_n}g^{(n+1)}(x)\,dx$ it holds

$$\begin{aligned}(1-\int_0^{t-T_n}g^{(n+1)}(x)\,dx)\mathbb{1}_{\{T_n<t\le T_{n+1}\}}&=\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\mathbb{P}(T_{n+1}-T_n\in(t-T_n,\infty)|\mathcal{F}_{T_n})\\&=\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\mathbb{P}(t<T_{n+1}|\mathcal{F}_{T_n})\\&=\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\mathbb{E}[\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}].\end{aligned}$$

Inspecting another term in (3.71) we may deduce by the definition of the functions $f^{(n)}$ that

$$\begin{aligned}&f^{(n)}(t-T_n,i,j)\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\\&=f^{(n)}(T_{n+1}-T_n,X(T_n),X(T_{n+1}))\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\mathbb{1}_{\{X(T_n)=i,X(T_{n+1})=j,t=T_{n+1}\}}\\&=Y(T_{n+1})\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\mathbb{1}_{\{X(T_n)=i,X(T_{n+1})=j,t=T_{n+1}\}}.\end{aligned}$$

Given the assumptions in the Corollary we may proceed with

$$Y(T_{n+1})\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\mathbb{1}_{\{X(T_n)=i,X(T_{n+1})=j,t=T_{n+1}\}}=\mathbb{E}[\zeta|\mathcal{F}_{T_n},T_{n+1}=t,X(T_{n+1})=j]\mathbb{1}_{\{T_n<t\le T_{n+1}\}}.$$

For the final term in (3.71) we have

$$\begin{aligned}&\sum_{i,j\in S:i\neq j}\int_{t-T_n}^{\infty}f^{(n)}(s,i,j)g^{(n+1)}(s,i,j)\,ds\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\\&=\sum_{i,j\in S:i\neq j}\int_{T_{n+1}-T_n}^{\infty}f^{(n)}(s,X(T_n),X(T_{n+1}))g^{(n+1)}(s,i,j)\,ds\\&\quad\times\mathbb{1}_{\{T_n<t\le T_{n+1}\}}\mathbb{1}_{\{X(T_n)=i,X(T_{n+1})=j,t=T_{n+1}\}}\\&=\mathbb{E}[\zeta\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}].\end{aligned}$$

We have successfully proven the equivalences between the different terms of the representations for the processes $(Z_{ij}(t))0\le t\le T$. Therefore the claim follows as a corollary of the Theorem 3.25.

□

Corollary 3.28 ([**4**, Corollary 2.5]). *Assume a jump process $X=(X(t))_{0\le t\le T}$ on $(\Omega,\mathcal{F},\mathbb{P})$. Let $(\mathcal{F}_t)_{t\ge 0}$ be the augmented natural filtration of the process X and $(M_{ij}(t))_{0\le t\le T}$ be the compensated processes associated to the process X. Consider an integrable càdlàg process $(\zeta(t))_{0\le t\le T}$ such that for all $t\in[0,T]$ the random variable $\zeta(t)-\zeta(0)$ is $\mathcal{F}_t$-measurable.*

If we define the processes $(Z_{ij}(t))_{0\le t\le T}$ *as*

$$Z_{ij}(t) := \sum_{n=0}^{\infty} \mathbb{1}_{\{T_n<t\le T_{n+1}\}}\left(\mathbb{E}[\zeta(t-)|\mathcal{F}_{T_n}, T_{n+1}=t, X(T_{n+1})=j] - \frac{\mathbb{E}[\zeta(t-)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}{\mathbb{E}[\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}\right),$$

then the process $(Y(t))_{0\le t\le T}$ *with* $Y(t)=\mathbb{E}[\zeta(t)|\mathcal{F}_t]$ *satisfies for* $t\in[0,T]$ *the SDE*

$$Y(t) = \zeta(t) - \zeta(0) + \sum_{i,j\in S: i\neq j} \int_{(0,t]} Z_{ij}(s)\, dM_{ij}(s).$$

PROOF. Because the random variable $(\zeta(t)-\zeta(0))$ is $\mathcal{F}_t$-measurable for fixed t we have by the take out what is known principle

$$\mathbb{E}[\zeta(0)|\mathcal{F}_t] = \mathbb{E}[\zeta(t)-\zeta(t)+\zeta(0)|\mathcal{F}_t] = \mathbb{E}[\zeta(t)|\mathcal{F}_t] - (\zeta(t)-\zeta(0)). \tag{3.72}$$

We will utilize the Corollary 3.27 to the process $W = (W(t))_{0\le t\le T}$ with $W(t) = \mathbb{E}[\zeta(0)|\mathcal{F}_t]$. Now it holds for

$$Z_{ij}(t) = \sum_{n=0}^{\infty} \mathbb{1}_{T_n<t\le T_{n+1}}\left(\mathbb{E}[\zeta(0)|\mathcal{F}_{T_n}, T_{n+1}=t, X(T_{n+1})=j] - \frac{\mathbb{E}[\zeta(0)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}{\mathbb{E}[\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}\right),$$

that

$$W(t) = W(0) + \int_{(0,t]} Z(s)\, dM(s).$$

Because of the (3.72) the previous equation is equivalent with

$$\mathbb{E}[\zeta(t)|\mathcal{F}_t] = \zeta(t) - \zeta(0) + \int_{(0,t]} Z(s)\, dM(s).$$

Finally we will note that

$$\begin{aligned}
&\mathbb{1}_{T_n<t\le T_{n+1}}\mathbb{E}[\zeta(t-)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}] \\
&= \mathbb{1}_{T_n<t\le T_{n+1}}\mathbb{E}[(\zeta(t) - \mathbb{E}[\zeta(t)-\zeta(t-)|\mathcal{F}_t])\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}] \\
&= \mathbb{1}_{T_n<t\le T_{n+1}}\mathbb{E}[(\zeta(t) - (\zeta(t)-\zeta(0)))\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}] \\
&= \mathbb{1}_{T_n<t\le T_{n+1}}\mathbb{E}[\zeta(0)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]
\end{aligned}$$

and a similar reasoning applies to the

$$\mathbb{E}[\zeta(0)|\mathcal{F}_{T_n}, T_{n+1}=t, X(T_{n+1})=j] = \mathbb{E}[\zeta(t-)|\mathcal{F}_{T_n}, T_{n+1}=t, X(T_{n+1})=j].$$

Therefore we also have that

$$Z_{ij}(t) = \sum_{n=0}^{\infty} \mathbb{1}_{\{T_n<t\le T_{n+1}\}}\left(\mathbb{E}[\zeta(t-)|\mathcal{F}_{T_n}, T_{n+1}=t, X(T_{n+1})=j] - \frac{\mathbb{E}[\zeta(t-)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}{\mathbb{E}[\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}\right)$$

and the claim follows from the Corollary 3.27.

$\square$

CHAPTER 4

Life insurance models

In this chapter we will construct a model to describe the price of a life insurance contract. Recall the essential definitions of the Chapter 3. The contract termination time is assumed to be $T \in (0, \infty)$. The states of the contract are assumed to be a finite set $\mathcal{S}$. If not specifically stated, we all ways assume $i, j \in \mathcal{S}$ with $i \neq j$ when writing the indexes. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with the augmented natural filtration $\mathcal{F}$ of a finite jump process $X = (X(t))_{t\in[0,T]}$ with the state space $\mathcal{S}$. Denote the corresponding counting processes with $N_{ij} = (N_{ij}(t))_{t\in[0,T]}$ that count the jumps occurring from the state i of the policy to the state j. The state of the policy is denote with the jump process X. The counting processes have intensities $(\lambda_{ij}(t))_{t\in[0,T]}$ accordingly. Denote the indicator process of the state of the process X with $I_i(t) := \mathbb{1}_{\{X(t)=i\}}$ for all $t \in [0, T]$. The compensated processes $(M_{ij}(t))_{t\in[0,T]}$ are denote by

$$M_{ij}(t) := N_{ij}(t) - \int_0^t I_i(s-)\lambda_{ij}(s)\, ds.$$

The jump times of the process X are marked with $(T_n)_{n\in\mathbb{N}^*}$.

4.1. Payment process

According to [**4**, 2.3.] a common choice for the payment process is of the form

$$A(t) = \sum_{i\in S} \int_{(0,t]} \mathbb{1}_{\{X(s-)=i\}} \Big(\alpha_i(s)\, ds + a_i(s)\, d\nu(s)\Big) + \sum_{i,j\in S: i\neq j} \int_{(0,t]} \beta_{ij}(s)\, dN_{ij}(s). \tag{4.1}$$

The predictable processes $(\alpha_i(t))_{t\in[0,T]}$ describe the continuous part of the payment in each of the states $i \in S$. The processes $(a_i(t))_{t\in[0,T]}$ are also predictable and represent the lump payments in the state i. They are accompanied by the step process $(\nu(t))_{t\in[0,T]}$ with $\Delta\nu(t) \in \{0, 1\}$ for all t, that determines the rate of the payments, for example monthly or annual arrival of the charges. The final term consists of the predictable processes $(\beta_{ij}(t))_{t\in[0,T]}$ that reflect the immediate transition fee induced by the contract changing states from the state i to the state j. Finally we present a notational convenience that

$$\alpha_{X(t-)}(t) := \sum_{i\in S} \mathbb{1}_{\{X(t-)=i\}}\alpha_i(t)$$
$$a_{X(t-)}(t) := \sum_{i\in S} \mathbb{1}_{\{X(t-)=i\}}a_i(t).$$

The terminal time $T \in [0, \infty) \cup \{\infty\}$ at which the contract ends is assumed to be finite analogous to the previous chapter. All though much research has been done on the case $T = \infty$ we are particularly interest only in the case $T < \infty$ due to the inclusion of backward stochastic differential equations in the later sections.

4.2. Prospective reserve

An essential part of determining the correct price of the life insurance contract is calculating the expected value of the upcoming charges. The result is called the prospective reserve of the payment process $A = (A(t))_{t\in[0,T]}$. We shall note the compensator processes $(\lambda_{ij}(t))_{t\in[0,T]}$ associated to the jump process $(X(t))_{t\in[0,T]}$ collectively with $\Lambda := (\lambda_{ij})_{ij}$.

DEFINITION 4.1 (Prospective reserve, [**4**, Definition 3.1]). Let $X = (X(t))_{t\in[0,T]}$ be a jump process, Λ be the respective compensators of the process X and denote the natural filtration of the X as $(\mathcal{F}_t)_{t\geq 0}$. Assume a payment process $A = (A(t))_{t\in[0,T]}$ and a bounded progressively measurable discount rate $(\delta(t))_{t\in[0,T]}$. The prospective reserve $Y = (Y(t))_{t\in[0,T]}$ of the payment process A is

$$Y(t) := \mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\,du}\, dA(s)\Big|\mathcal{F}_t\right], \quad t \in [0,T]. \tag{4.2}$$

REMARK 4.2. The term $e^{-\int_t^s \delta(u)\,du}$ is utilized to discount the future cash flow. The process $(dA(t))_{t\in[0,T]}$ describes the changes in the balance between the (discounted) benefits and premiums. The conditional expectation averages the amount of prospective reserve required given the history F_t.

4.2.1. Linear reserve. The prospective reserve predicts the expected amount of future premiums with the given information $\mathcal{F}_t$. The cash flow of the life insurance contract may depend on the prospective reserve, in which case we refer to the act of non-linear reserving. We will first discuss about the more simple case which is linear reserving. In that case the payment processes $(a_i(t))_{t\in[0,T]}$, $(\alpha_i(t))_{t\in[0,T]}$ and $(\beta_{ij}(t))_{t\in[0,T]}$ do not depend on the prospective reserve Y. The payments processes are assumed to be predictable and satisfy the condition

$$\mathbb{E}\left[\int_0^T (|\alpha_{X(t-)}(t)|^2 + \|\beta(t)\|_\Lambda^2)\,dt + \int_{(0,T]} |a_{X(t-)}(t)|^2\,d\nu(t)\right] < \infty. \tag{4.3}$$

Here we have utilized the notation

$$\|\beta(t)\|_\Lambda^2 := \sum_{i,j\in S: i\neq j} \beta_{ij}^2(t)\mathbb{1}_{\{X(t-)=i\}}(t)\lambda_{ij}(t), \quad \text{for } t \in (0,T].$$

4.2.2. Formula for the prospective reserve in linear case. We will formulate the prospective reserve utilizing a backwards stochastic differential equation in the case of linear reserving. Before doing so we will establish some essential results and notions. The assumptions relating to the payment processes are assumed to hold, including the condition (4.3).

At first we define the process $\tilde{M}_i := (\tilde{M}_i(t))_{t\in[0,T]}$ with

$$\tilde{M}_i(t) := \sum_{j\in S: j\neq i} \int_0^t \beta_{ij}(s)\,dM_{ij}(s).$$

The processes $\tilde{M}_i$ are square-integrable martingales as a consequence of the Proposition 3.14. Indeed the processes $(\beta_{ij}(t))_{t\in[0,T]}$ are predictable by definition and from the assumption (4.3) it follows that

$$\mathbb{E}\left[\int_0^T \|\beta(s)\|_\Lambda^2\, ds\right] < \infty.$$

With that in mind we may rewrite

$$A(t) = \sum_{i\in S}\int_{(0,t]} \mathbb{1}_{\{X(s-)=i\}}\Big(\alpha_i(s)\, ds + a_i(s)\, d\nu(s)\Big) + \sum_{i,j\in S: i\neq j}\int_{(0,t]} \beta_{ij}(s)\, dN_{ij}(s) \tag{4.4}$$

$$= \sum_{i\in S}\int_{(0,t]} \mathbb{1}_{\{X(s-)=i\}}\Big(\alpha_i(s)\, ds + a_i(s)\, d\nu(s)\Big) \tag{4.5}$$

$$+ \sum_{j\in S: j\neq i}\int_0^t \beta_{ij}(s)\left(dM_{ij}(s) + \mathbb{1}_{\{X(s-)=i\}}\lambda_{ij}(s)\, ds\right) \tag{4.6}$$

$$= \sum_{i\in S}\int_0^t \Big(\alpha_{X(s-)}(s)\, ds + \sum_{j\in S: j\neq i}\beta_{ij}(s)\mathbb{1}_{\{X(s-)=i\}}\lambda_{ij}(s)\, ds\Big) + \int_{(0,t]} a_{X(s-)}\, d\nu(s) + \tilde{M}_{X(t-)}, \tag{4.7}$$

where we have defined $\tilde{M}_{X(t-)} := \sum_{i\in\mathcal{S}} \tilde{M}_i(t)\mathbb{1}_{\{X(t-)=i\}}$.

Therefore we may express the form (4.1) of the payment process $(A(t))_{t\in[0,T]}$ with

$$dA(t) = \gamma_{X(t-)}(t)\, dt + a_{X(t-)}(t)\, d\nu(t) + \sum_{i\in S} d\tilde{M}_i(t) \tag{4.8}$$

where the process $\gamma := (\gamma(t))_{t\in[0,T]}$ is defined as

$$\gamma_i(t) := \alpha_i(t) + \sum_{j\in S: j\neq i}\beta_{ij}(t)\mathbb{1}_{\{X(t-)=i\}}\lambda_{ij}(t), \quad i\in S. \tag{4.9}$$

Recall the fact that the processes $\tilde{M}_i$ are square-integrable martingales. Because of the definition of the discount rate δ the process $\left(e^{-\int_t^s \delta(u)\, du}\right)_{s\in(t,T]}$ is bounded and predictable. Then according to the Theorem 11 in [**12**, IV] the process

$$\left(\int_{(t,T]} e^{-\int_t^s \delta(u)\, du}\, d\tilde{M}_i(s)\right)_{t\in[0,T]}$$

is also a square-integrable martingale. The martingale property implies then that

$$\sum_{i\in S}\mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\, du}\, d\tilde{M}_i(s)\Big|\mathcal{F}_t\right] = \sum_{i\in S}\int_{\emptyset} e^{-\int_t^s \delta(u)\, du}\, d\tilde{M}_i(s) = 0. \tag{4.10}$$

This result combined with the new form of the payment process A in (4.8) justifies for the prospective reserve Y that

$$Y(t) = \mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\, du}\Big(\gamma_{X(s-)}(s)\, ds + a_{X(s-)}(s)\, d\nu(s)\Big)\Big|\mathcal{F}_t\right]. \tag{4.11}$$

From the equation (4.11) we may acquire an equivalent equation

$$Y(t)e^{-\int_0^t \delta(u)\,du} = \mathbb{E}\left[\int_{(t,T]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big)\Big|\mathcal{F}_t\right]$$

and furthermore by adding to the both sides

$$\begin{aligned}Y(t)e^{-\int_0^t \delta(u)\,du} &+ \int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big)\\ &= \mathbb{E}\left[\int_{(0,T]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big)\Big|\mathcal{F}_t\right].\end{aligned}$$

On the left hand side we utilized the $\mathcal{F}_t$-measurability of the stochastic integral induced by the integration limits.

THEOREM 4.3 ([**4**, Proposition 3.2.]). *Let $(X(t))_{t\in[0,T]}$ be a jump process with finite state space S and $(\mathcal{F}_t)_{t\geq 0}$ be its augmented natural filtration. Assume the associated counting processes $(N_{ij}(t))_{t\in[0,T]}$, compensators $\Lambda := (\lambda_{ij})_{ij}$ and compensated processes $(M_{ij}(t))_{t\in[0,T]}$, where $M_{ij}(t) := N_{ij}(t) - \int_0^t \mathbb{1}_{\{X(s-)=i\}}\lambda_{ij}(s)\,ds$ and $M_{ij}(0) \equiv 0$. Assume a payment process $(A(t))_{t\in[0,T]}$ satisfying (4.1). Let the prospective reserve $(Y(t))_{t\in[0,T]}$ be given by $Y(t) = \mathbb{E}[\int_{(t,T]} e^{-\int_t^s \delta(u)\,du}\,dA(s)|\mathcal{F}_t]$, where the process $(\delta(t))_{t\in[0,T]}$ is a predictable and progressively measurable discount rate. Let the combined payment process $(\gamma(t))_{t\in[0,T]}$ be defined as in (4.9).*

Then there exist unique predictable processes $(Z_{ij}(t))_{t\in[0,T]}$ almost surely satisfying

$$\mathbb{1}_{\{X(t-)=i\}}Z_{ij}(t) = \mathbb{1}_{\{X(t-)=i\}}\Big(\beta_{ij}(t) + \mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t)=j] - \mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t)=i]\Big),$$

such that the prospective reserve Y is a solution to the BSDE

$$\begin{cases} dY(t) = & \Big(-\delta(t)Y(t) + \gamma_{X(t-)}(t)\Big)\,dt + a_{X(t-)}(t)\,d\nu(t) + Z(t)\,dM(t),\\ Y(T) = & 0.\end{cases} \tag{4.12}$$

PROOF. The idea is the same as in the proof of the Proposition 3.2 in [**4**]. We shall take advantage of the explicit solution to the Martingale Representation Theorem from the previous chapter. Consider the process $\zeta := (\zeta(t))_{t\in[0,T]}$ with

$$\zeta(t) := \int_{(t,T]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big). \tag{4.13}$$

We have that the process ζ is càdlàg because the processes that define it are càdlàg or predictable. Moreover the process ζ is also integrable, which can be verified:

$$\mathbb{E}|\zeta(t)| \leq \mathbb{E}\left[\int_t^T e^{-\int_0^s \delta(u)\,du}|\gamma_{X(s-)}(s)|\,ds + \int_{(t,T]} e^{-\int_0^s \delta(u)\,du}|a_{X(s-)}(s)|\,d\nu(s)\Big)\right].$$

The process $(\delta(t))_{t\in[0,T]}$ is bounded, therefore there exists $R \in \mathbb{R}$ such that $e^{-\int_0^T \delta(u)\,du} < R$. The payment processes were assumed to be square-integrable in (4.3) so that they are also integrable, therefore we may conclude that $\mathbb{E}|\zeta(t)| < \infty$ for all $t \in [0,T]$.

Lastly for all $t \in [0,T]$ the random variable

$$\zeta(t) - \zeta(0) = -\int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big) \tag{4.14}$$

is $\mathcal{F}_t$-measurable as a result of the measurability of its defining processes.

We may therefore apply the Corollary 3.28 to the process which yields

$$\mathbb{E}[\zeta(t)|\mathcal{F}_t] = \zeta(t) - \zeta(0) + \sum_{i,j\in:i\neq j} \int_{(0,t]} \hat{Z}_{ij}(s)\, dM_{ij}(s) \tag{4.15}$$

for $t \in [0, T]$ and

$$\hat{Z}_{ij}(t) := \sum_{n=0}^{\infty} \mathbb{1}_{\{T_n<t\leq T_{n+1}\}} \Big(\mathbb{E}[\zeta(t-)|\mathcal{F}_{T_n}, T_{n+1} = t, X(T_{n+1}) = j] - \frac{\mathbb{E}[\zeta(t-)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}{\mathbb{E}[\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]} \Big). \tag{4.16}$$

We will now interpret the SDE in (4.15) to yield a solution to the BSDE in (4.3). Recalling the definition of the process $(\zeta(t))_{t\in[0,T]}$ gives

$$\begin{aligned}
\mathbb{E}[\zeta(t)|\mathcal{F}_t] &= \mathbb{E}\left[\int_{(t,T]} e^{-\int_0^s \delta(u)\, du} \Big(\gamma_{X(s-)}(s)\, ds + a_{X(s-)}(s)\, d\nu(s) \Big) \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\, du} \Big(\gamma_{X(s-)}(s)\, ds + a_{X(s-)}(s)\, d\nu(s) \Big) \Big| \mathcal{F}_t \right] e^{-\int_0^t \delta(u)\, du}.
\end{aligned}$$

As was previously established in (4.11) we may utilize the new form of the prospective reserve Y to the last step of the previous equation to reach

$$\mathbb{E}[\zeta(t)|\mathcal{F}_t] = Y(t)e^{-\int_0^t \delta(u)\, du}. \tag{4.17}$$

Let us now inspect the term $\zeta(t) - \zeta(0)$. By the equation (4.14) it holds that

$$\zeta(t) - \zeta(0) = -\int_{(0,t]} e^{-\int_0^s \delta(u)\, du} \Big(\gamma_{X(s-)}(s)\, ds + a_{X(s-)}(s)\, d\nu(s) \Big). \tag{4.18}$$

Only the last term in (4.15) still requires interpretation. For the stochastic process $dM_{ij}(t)$ we may write

$$\int_{(0,t]} \hat{Z}_{ij}(s)\, dM_{ij}(t) = \int_{(0,t]} \hat{Z}_{ij}(s) \big(\mathbb{1}_{\{X(t-)=i\}}\, dM_{ij}(t) \big).$$

Proceeding forward the definition of the processes Z_{ij} in (4.16) yields that on $t \in (T_n, T_{n+1}]$ it holds

$$\begin{aligned}
\mathbb{1}_{\{X(t-)=i\}} Z_{ij}(t) = \mathbb{1}_{\{X(t-)=i\}} \Big(& \mathbb{E}[\zeta(t-)|\mathcal{F}_{T_n}, T_{n+1} = t, X(T_{n+1}) = j] - \\
& \frac{\mathbb{E}[\zeta(t-)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]}{\mathbb{E}[\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}]} \Big).
\end{aligned} \tag{4.19}$$

In an attempt to simplify this we may conclude that on $t \in (T_n, T_{n+1}]$

$$\mathbb{1}_{\{X(t-)=i\}} \mathbb{E}[\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}] = \mathbb{1}_{\{X(t-)=i\}} \mathbb{P}(T_{n+1} > t|\mathcal{F}_{T_n}) = \mathbb{1}_{\{X(t-)=i\}} \tag{4.20}$$

due to the relationship $\{X(t-) = i\} = \{t \leq T_{n+1}\}$.

The Proposition 3.20 argues that on $T_n < t \leq T_{n+1}$ the histories $\mathcal{F}_{T_n}$ and $\mathcal{F}t-$ share identical information due to the jump not happening before the time t and the process X is right-continuous. Therefore it holds in (4.19) that

$$\mathbb{1}_{\{X(t-)=i\}} \mathbb{E}[\zeta(t-)|\mathcal{F}_{T_n}, T_{n+1} = t, X(T_{n+1}) = j] = \mathbb{1}_{\{X(t-)=i\}} \mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = j]. \tag{4.21}$$

Similar arguments can be presented to verify

$$\mathbb{1}_{\{X(t-)=i\}}\mathbb{E}[\zeta(t-)\mathbb{1}_{\{t<T_{n+1}\}}|\mathcal{F}_{T_n}] = \mathbb{1}_{\{X(t-)=i\}}\mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = i]. \tag{4.22}$$

Therefore

$$\mathbb{1}_{\{X(t-)=i\}}Z_{ij}(t) = \mathbb{1}_{\{X(t-)=i\}}\Big(\mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = j] - \mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = i]\Big).$$

In this equation we can use $\zeta(t)$ instead of $\zeta(t-)$ because

$$\zeta(t) - \zeta(t-) = -\int_{(t-,t]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big) = a(t)(\nu(t) - \nu(t-))$$

and the processes a and ν are $\mathcal{F}_{t-}$-measurable being predictable. Then it follows that $\zeta(t) - \zeta(t-)$ is $\mathcal{F}_{t-}$-measurable and

$$\begin{aligned}
&\mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = j] - \mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = i] \\
&= \mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = j] + \big(\zeta(t) - \zeta(t-)\big) - \\
&\qquad \mathbb{E}[\zeta(t-)|\mathcal{F}_{t-}, X(t) = i] - \big(\zeta(t) - \zeta(t-)\big) \\
&= \mathbb{E}[\zeta(t)|\mathcal{F}_{t-}, X(t) = j] - \mathbb{E}[\zeta(t)|\mathcal{F}_{t-}, X(t) = i].
\end{aligned}$$

Now we may utilize the Tower Property and the form (4.17) for the function $\mathbb{E}[\zeta(t)|\mathcal{F}_t]$ to yield

$$\begin{aligned}
\mathbb{E}[\zeta(t)|\mathcal{F}_{t-}, X(t) = j] &= \mathbb{E}[\mathbb{E}[\zeta(t)|\mathcal{F}_t]|\mathcal{F}_{t-}] \\
&= \mathbb{E}[Y(t)e^{-\int_0^t \delta(u)\,du}]|\mathcal{F}_{t-}] \\
&= e^{-\int_0^t \delta(u)\,du}\mathbb{E}[Y(t)|\mathcal{F}_{t-}] \quad a.s. \qquad (4.23)
\end{aligned}$$

This combined with the revelations (4.20), (4.21) and (4.22) together prove the connection between the processes $\hat{Z}$ and Z:

$$\sum_{i,j\in:i\neq j}\int_{(0,t]} \hat{Z}_{ij}(s)\,dM_{ij}(s) = \sum_{i,j\in:i\neq j}\int_{(0,t]} e^{-\int_0^t \delta(u)\,du} Z_{ij}(s)\,dM_{ij}(s). \tag{4.24}$$

So far as a conclusion of the (4.15), (4.17), (4.18) and (4.24) we have proven that

$$\begin{aligned}
Y(t)e^{-\int_0^t \delta(u)\,du} = &-\int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big) + \\
&\sum_{i,j\in S:i\neq j}\int_{(0,t]} e^{-\int_0^t \delta(u)\,du} Z_{ij}(s)\,dM_{ij}(s) \qquad (4.25)
\end{aligned}$$

for the process $(Z_{ij}(t))_{t\in[0,T]}$ satisfying

$$\mathbb{1}_{\{X(t-)=i\}}Z_{ij}(t) = \mathbb{1}_{\{X(t-)=i\}}\Big(\mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t) = j] - \mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t) = i]\Big).$$

The next step in the proof is to display that the backwards stochastic differential equation (4.12) can be derived from the (4.25).

Recall the integration by parts formula for càdlàg processes of bounded variation from the Proposition 3.9. We apply the integration by parts to the product

$Y(t)e^{-\int_0^t \delta(u)\,du}$ on the left hand side of the equation (4.25) to receive

$$Y(t)e^{-\int_0^t \delta(u)\,du} = Y(0)e^{-\int_0^0 \delta(u)\,du} + \int_{(0,t]} e^{-\int_0^(s-)\delta(u)\,du}\,dY(s) + \int_0^t Y(s)\,d\Big(e^{-\int_0^s \delta(u)\,du}\Big)$$
$$= \int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\,dY(s) - \int_0^t Y(s)e^{-\int_0^s \delta(u)\,du}\delta(s)\,ds.$$

Here we utilized the Fundamental Theorem of Calculus to differentiate the inner function of the exponential function. The equation (4.25) is therefore equivalent with

(4.26)
$$\int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\,dY(s) = -\int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big)+$$

$$\sum_{i,j\in S:i\neq j}\int_{(0,t]} Z_{ij}(s)\,dM_{ij}(s) + \int_0^t Y(s)e^{-\int_0^s \delta(u)\,du}\delta(s)\,ds. \tag{4.27}$$

Now the right hand side of the SDE (4.27) can be simplified which will be done next. We have by the definition that

$$\sum_{i,j\in S:i\neq j}\int_{(0,t]} e^{-\int_0^t \delta(u)\,du} Z_{ij}(s)\,dM_{ij}(s) =: \int_{(0,t]} e^{-\int_0^t \delta(u)\,du} Z(s)\,dM(s).$$

Moreover with a straightforward calculation

$$\int_0^t Y(s)e^{-\int_0^s \delta(u)\,du}\delta(s)\,ds - \int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big)$$
$$= \int_0^t e^{-\int_0^s \delta(u)\,du}\Big(\delta(s)Y(s) - \gamma_{X(s-)}(s)\Big)\,ds - \int_{(0,t]} e^{-\int_0^s \delta(u)\,du} a_{X(s-)}(s)\,d\nu(s)$$
$$= \int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\Big[\Big(\delta(s)Y(s) - \gamma_{X(s-)}(s)\Big) - a_{X(s-)}(s)\,d\nu(s)\Big].$$

With this in mind we may once again rewrite the equation (4.27) in an equivalent form

$$\int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\,dY(s) = \int_{(0,t]} e^{-\int_0^s \delta(u)\,du}\Big[\Big(\delta(s)Y(s) - \gamma_{X(s-)}(s)\Big)-$$
$$a_{X(s-)}(s)\,d\nu(s) + Z(s)\,dM(s)\Big]. \tag{4.28}$$

From this by subtracting the equation at the time t from the equation at the time T it follows

$$\int_{(t,T]} e^{-\int_0^s \delta(u)\,du}\,dY(s) = \int_{(t,T]} e^{-\int_0^s \delta(u)\,du}\Big[\Big(\delta(s)Y(s) - \gamma_{X(s-)}(s)\Big)-$$
$$a_{X(s-)}(s)\,d\nu(s) + Z(s)\,dM(s)\Big].$$

By applying the Theorem of Radon-Nikodym to the change of measures this gives the desired BSDE

$$dY(t) = \Big(-\delta(t)Y(t) + \gamma_{X(t-)}(t)\Big)\,dt + a_{X(t-)}(t)\,d\nu(t) + Z(t)\,dM(t),$$

with $Y(T) = 0$ which is provided by the Definition 4.1 of the prospective reserve at the termination time.

□

4.3. Nonlinear reserve

In the previous sections we assumed the act of linear reserving. We shall now extend our analysis to the nonlinear case. The payment processes $(\alpha_i(t))_{t\in[0,T]}$, $(a_i(t))_{t\in[0,T]}$ and $(\beta_{ij}(t))_{t\in[0,T]}$ are now permitted to depend on the prospective reserve $(Y(t-))_{t\in[0,T]}$ and the process $(Z(t))_{t\in[0,T]}$, hence making the definition and existence of the prospective reserve more complicated. In this section we shall discuss some sufficient conditions to make the definition of the prospective reserve valid. Assume now that for all $i, j \in S$ such that $i \neq j$ and

$$\begin{aligned} \alpha_i(t) &:= \alpha_i(t, Y(t-), Z(t)), \\ a_i(t) &:= a_i(t, Y(t-), Z(t)), \\ \beta_i(t) &:= \beta_i(t, Y(t-), Z(t)) \end{aligned} \tag{4.29}$$

the processes $(\alpha_i(t))_{t\in[0,T]}$, $(a_i(t))_{t\in[0,T]}$ and $(\beta_{ij}(t))_{t\in[0,T]}$ are predictable. Moreover denote for all $i \in S$ the combined price processe $(\gamma_i(t))_{t\in[0,T]}$ with

$$\gamma_i(t) := \alpha_i(t) + \sum_{j:j\neq i} \beta_{ij}(t)\mathbb{1}_{\{X(t-)=i\}}\lambda_{ij}(t), \tag{4.30}$$

which is also dependent on the processes $(Y(t))_{t\in[0,T]}$ and $(Z(t))_{t\in[0,T]}$.

Recall the definition of the norm

$$\|Z(t)\|_\Lambda^2 := \sum_{i,j\in S:i\neq j} Z_{ij}^2(t)\mathbb{1}_{\{X(t-)=i\}}(t)\lambda_{ij}(t), \quad t \in (0, T],$$

for processes $(Z_{ij}(t))_{t\in[0,T]}$ and intensities $(\lambda_{ij}(t))_{t\in[0,T]}$ where $i, j \in S$ and $i \neq j$. We require some degree of boundedness on finite intervals from the payment processes. Let $t \in [0, T]$.

For each $y, \bar{y}, z_{ij}, \bar{z}_{ij} \in \mathbb{R}$ there should $\mathbb{P}$-a.s. exist a constant $C \in [0, \infty)$ such that

$$|\gamma_i(t, y, z)(\omega) - \gamma_i(t, \bar{y}, \bar{z})(\omega)| \leq C(|y - \bar{y}| + \|z - \bar{z}\|_\Lambda), \quad i \in S. \tag{4.31}$$

Moreover for each $y, \bar{y}, z_{ij}, \bar{z}_{ij} \in \mathbb{R}$ there should $\mathbb{P} \times \nu$-a.s. exist constants $C_1 \in [0, 1)$ and $C_2 \in [0, \infty)$ such that

$$|a_i(t, y, z)(\omega) - a_i(t, \bar{y}, \bar{z})(\omega)| \leq C_1|y - \bar{y}| + C_2\|z - \bar{z}\|_\Lambda, \quad i \in S. \tag{4.32}$$

In addition to the two limitations the payment process $(\gamma(t, 0, 0))_{t\in[0,T]}$ should be square-integrable:

$$\mathbb{E}\left[\int_0^T |\gamma_i(t, 0, 0)|^2\, dt\right] < \infty, \quad i \in S. \tag{4.33}$$

The assumption (4.31) is referred to as the process $(\gamma_i(t))_{t\in[0,T]}$ being *uniformly Lipschitz continuous* and the stronger condition in (4.32) is called a *firm Lipschitz bound* on the process $(a_i(t))_{t\in[0,T]}$. [**6**, Theorem 5.1] The utilized condition depends on the continuity of the measure that the process is integrated with respect to. These requirements guarantee the existence and uniqueness of a solution to a BSDE involving the generator process that satisfies the requirements. [**6**, Theorem 5.1]

Given these requirements the prospective reserve in the case of nonlinear reserving is well defined and a similar result to the Theorem 4.3 holds:

PROPOSITION 4.4 ([**4**, Proposition 3.5]). *Assume payment processes* $(a_i(t))_{t\in[0,T]}$ *and* $(\gamma_i(t))_{t\in[0,T]}$ *that satisfy (4.29) and (4.30). If all of the conditions (4.31), (4.32) and (4.33) hold for the payment processes, then there exists a unique solution* (Y, Z) *to the backward SDE*

$$\begin{cases} dY(t) = & \Big(- \delta(t)Y(t) + \gamma_{X(t-)}(t)\Big)\, dt + a_{X(t-)}(t)\, d\nu(t) + Z(t)\, dM(t), \\ Y(T) = & 0, \end{cases} \tag{4.34}$$

satisfying simultaneously also

$$Y(t) = \mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s)\Big)\Big|\mathcal{F}_t\right] \tag{4.35}$$

and

$$I_i(t-)Z_{ij}(t) = I_i(t-)\left(\beta_{ij}(t) + \mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t) = j] - \mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t) = i]\right), \quad a.s. \tag{4.36}$$

Moreover the process $(Y(t))_{t\in[0,T]}$ *is adapted and the processes* $(Z_{ij}(t))_{t\in[0,T]}$ *are predictable with*

$$\mathbb{E}\left[\sup_{t\in[0,t]}|Y(t)|^2 + \int_0^T \|Z(t)\|_\Lambda^2\,dt\right] < \infty. \tag{4.37}$$

PROOF. The existence and uniqueness of the solution (Y, Z) with the given data given the assumptions of the Proposition follows from the theory of BSDE's which we will inspect. The Definition 3.1 in [**6**] presents the following form for a BSDE:

$$Y(T) = Y(t) - \int_{(t,T]} F\big(\omega, s, Y(s-), Z(s)\big)\,d\mu(s) + \sum_{i,j\in\mathcal{S}:i\neq j}\int_{(t,T]} Z_{ij}(s)\,dM_{ij}(s). \tag{4.38}$$

Here the number of the processes Z_{ij} must be countable and it must hold $Y(T) \in L^2$. A solution to the BSDE is a pair (Y, Z) such that the processes $Z := \{Z_{ij}\}_{ij}$ are predictable, the process Y is adapted and they satisfy the equation (4.38).

We have for the original BSDE with $Y(T) = 0$ that

$$\begin{aligned}&\Big(- \delta(t)Y(t) + \gamma_{X(t-)}(t)\Big)\,dt + a_{X(t-)}(t)\,d\nu(t) + Z(t)\,dM(t)\\ &= -\delta(t)Y(t)\,dt + \Big(\mathbb{1}_{\{\Delta\nu(t)=0\}}\gamma_{X(t-)}(t) + \mathbb{1}_{\{\Delta\nu(t)\neq 0\}}a_{X(t-)}(t)\Big)\,\big(dt + d\nu(t)\big) + Z(t)\,dM(t).\end{aligned}$$

We will therefore apply the Theorem 5.1 of [**6**] to the generator process $(F(t))_{t\in[0,T]}$ with

$$F(t) := \mathbb{1}_{\{\Delta\nu(t)=0\}}\gamma_{X(t-)}(t) + \mathbb{1}_{\{\Delta\nu(t)>0\}}a_{X(t-)}(t)$$

and $(\mu(t))_{t\in[0,T]}$ such that

$$d\mu(t) := dt + d\nu(t).$$

It states that if given a progressively measurable and predictable function F that has a firm Lipschitz bound, and the assumptions

$$\mathbb{E}\int_{[0,T]} \|F(t,0,0)\|^2 \, d\mu(t) < \infty$$

and

$$C_1 \Delta\mu(t) < 1, \tag{4.39}$$

hold, then there exists a unique solution (Y, Z) to the BSDE

$$\begin{cases} dY(t) = & -\delta(t)Y(t)\,dt + F(t)\,d\mu(t) + Z(t)\,dM(t), \\ Y(T) = & 0. \end{cases} \tag{4.40}$$

The firm Lipschitz bound is presented in the assumptions (4.31) and (4.32). The condition (4.39) holds also due to the definition of the jump process $\Delta\nu(t) \leq 1$ and $C_1 \in [0,1)$ by (4.32). Lastly we have because of the proposed assumption (4.33) and the condition of the payment processes (4.3) that

$$\mathbb{E}\int_{[0,T]} \|F(t,0,0)\|^2 \, d\mu(t) \leq \mathbb{E}\int_0^T |\gamma_i(t,0,0)|^2 \, dt + \mathbb{E}\int_{[0,T]} |a_i(t,0,0)|^2 \, d\nu < \infty. \tag{4.41}$$

By the definition of solution the processes Z_{ij} are predictable and the process Y is adapted. Moreover the Theorem 5.1 in [**6**] states that

$$Y \in \left\{ Y : \Omega \times [0,T] \to \mathbb{R}^K, \mathbb{E} \sup_{t\in[0,t]} \|Y(t)\|^2 < \infty \right\} \tag{4.42}$$

and

$$Z \in \left\{ Z : \Omega \times [0,T] \to \mathbb{R}^{K\times\infty}, \mathbb{E}\int_0^T \|Z(t)\|_\Lambda^2 \, dt < \infty \right\} \tag{4.43}$$

validating the statement (4.37).

We are still required to show that the unique solution (Y, Z) has the forms (4.35) and (4.36). By defining the process $\tilde{Y} := (e^{-\int_0^t \delta(u)\,du} Y(t))_{t\in[0,T]}$ we have by the integration by parts formula that

$$\tilde{Y}(t) = -\int_t^T \delta(t) e^{-\int_0^t \delta(u)\,du} Y(t)\,dt + \int_{(0,t]} e^{-\int_0^t \delta(u)\,du}\, dY(t).$$

The (Y, Z) is a solution to the BSDE in (4.40), therefore we have

$$\begin{aligned} \tilde{Y}(t) &= \int_{(t,T]} e^{-\int_0^s \delta(u)\,du} \Big(-\delta(s)Y(s)\,ds + dY(s) \Big) \\ &= \int_{(t,T]} e^{-\int_0^s \delta(u)\,du} \Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s) + Z(s)\,dM(s) \Big). \end{aligned}$$

The process $\tilde{Y}$ is adapted. Then we may take the conditional expectation with respect to the $\mathcal{F}_t$ from the both sides to yield

$$\tilde{Y}(t) = \mathbb{E}\left[\int_{(t,T]} e^{-\int_0^s \delta(u)\,du} \Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s) + Z(s)\,dM(s) \Big) \Big| \mathcal{F}_t \right].$$

But this is equal to our original prospective reserve equation in the Definition 4.1 if we divide both sides with $e^{-\int_0^t \delta(u)\,du}$. Thus we receive equivalently

$$Y(t) = \mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\,du}\Big(\gamma_{X(s-)}(s)\,ds + a_{X(s-)}(s)\,d\nu(s) + Z(s)\,dM(s)\Big)\Big|\mathcal{F}_t\right]. \tag{4.44}$$

We have reached now a conclusion that the first process in the unique solution (Y, Z) satisfies the definition of the prospective reserve. Therefore we may apply the Theorem 4.3 in this nonlinear case as well to supply a representation for the processes Z_{ij}. We have then

$$I_i(t-)Z_{ij}(t) = I_i(t-)\Big(\beta_{ij}(t) + \mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t) = j] - \mathbb{E}[Y(t)|\mathcal{F}_{t-}, X(t) = i]\Big), \quad a.s.,$$

where we utilized the indicator functions $I_i(t-) := \mathbb{1}_{\{X(t-)=i\}}$.

$\square$

CHAPTER 5

Modifications to the contract

In this chapter we will include the option for modifying the contract mid term. We allow for example the changes in the maturity of the contract or in the value of the contract benefits. A common practice in the field of life insurance is to treat the all ready accumulated prospective reserve as a part of the insured's wealth. It is utilized to cover up the lump sum expenses of the of the new insurance contract. This way the payment processes will become dependent of the amount of prospective reserve in case of a future contract modifications. Therefore we are required to use the nonlinear reserving models. We will keep performing the analyses in a non-Markovian setting, which is more general compared to the Markovian case.

Consider a model where a jump process process $(X(t))_{t\in[0,T]}$ denotes the current state of the policy holder. The jump process $(J(t))_{t\in[0,T]}$ corresponds to the different states of the contract itself due to the possible contract modifications. The objective of this chapter is to extend the Cantelli Theorem from the Markov processes to the non-Markovian framework. If (X, J) is a Markov process and do not jump simultaneously then the Cantelli Theorem states that the contract modifications can be ignored while determining the state-wise prospective reserves if the insurance benefit and the prospective reserve are equal at the time of the contract modification. [**4**, 4.] The equilibrium of the insurance benefit and the prospective reserve is called the risk at sum.

5.1. Fundamentals of the expanded state space

Our aim is to extend the analyses from the previous chapter to hold for the new process (X, J). This requires establishing first some fundamentals concerning the new setting.

Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and fix a real number $T \in (0, \infty)$. Let $(X, J) := (X(t), J(t))_{t\in[0,T]}$ be a càdlàg jump process and equip the probability space with the augmented natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the process (X, J). We will notate the state space of the process $(J(t))_{t\in[0,T]}$ with $\mathcal{J} \subset \mathbb{N}$, which describes the different modes of the contract and is allowed to be countable in contrast to the state space S of the process $(X(t))_{t\in[0,T]}$ only being finite.

The jump times $(\tau_n)_n$ of the process $(J(t))_{t\in[0,T]}$ denoted as

$$\tau_n := \inf\Big\{t \geq 0 : \sum_{i,j\in\mathcal{J}:i\neq j} \#\big\{s \in (0, t] : J(s-) = i, J(s) = j\big\} = n\Big\}$$

are stopping times in $(\mathcal{F}_t)_{t\geq 0}$ by the Proposition 3.17. For the convenience we assume that the process (X, J) does not have simultaneous jumps within its coordinate processes. According to [**4**, 4.1] the assumption is not necessary for the upcoming results but reduces the technicality in the proofs.

Now we will define the indicator processes and the counting processes for each individual process $(X(t))_{t\in[0,T]}$ and $(J(t))_{t\in[0,T]}$. For the processes $(I_i^0(t))_{t\in[0,T]}$, $(I_k^1(t))_{t\in[0,T]}$, $(N_{ij}^0(t))_{t\in[0,T]}$ and $(N_{kl}^0(t))_{t\in[0,T]}$ we set

$$\begin{aligned} I_i^0(t) &:= \mathbb{1}_{\{X(t-)=i\}}, \\ I_k^1(t) &:= \mathbb{1}_{\{J(t-)=k\}}, \\ N_{ij}^0(t) &:= \#\{s \in (0,t] : X(s-) = i, X(s) = j\} \quad \text{and} \\ N_{kl}^0(t) &:= \#\{s \in (0,t] : J(s-) = k, J(s) = l\}. \end{aligned}$$

Furthermore we define the processes $(N^0(t))_{t\in[0,T]}$ and $(N^1(t))_{t\in[0,T]}$ that count the total number of jumps with

$$\begin{aligned} N^0(t) &:= \sum_{i,j\in S: i\neq j} N_{ij}^0(t) \quad \text{and} \\ N^1(t) &:= \sum_{k,l\in \mathcal{J}: k\neq l} N_{kl}^1(t). \end{aligned} \tag{5.1}$$

With $\Lambda^0 := (\lambda_{ij}^0)_{ij}$ and $\Lambda^1 := (\lambda_{kl}^1)_{kl}$ we mark the jump intensities of the processes $(X(t))_{t\in[0,T]}$ and $(J(t))_{t\in[0,T]}$ respectively. We inspected the intensities more in detail in the section *Explicit form of the intensity.* Furthermore we make the assumption regarding the integrability of the intensities:

$$\mathbb{E}\left[\int_0^T \left(\sum_{i,j\in S: i\neq j} I_i^0(t-)\lambda_{ij}^0(t) + \sum_{k,l\in\mathcal{J}: k\neq l} I_i^1(t-)\lambda_{kl}^1(t)\right) dt\right] < \infty. \tag{5.2}$$

Due to the processes $(X(t))_{t\in[0,T]}$ and $(J(t))_{t\in[0,T]}$ not having concurrent jumps we may consider the process $\tilde{X} := (X, J)$ as a state space extension for the process X. All of the previous analyses considering the jump process X can now be applied also to the extended jump process $\tilde{X}$. Lastly we define the compensated martingales $(M_{ij}^0(t))_{t\in[0,T]}$ and $(M_{kl}^1(t))_{t\in[0,T]}$ for both of the processes with

$$\begin{aligned} M_{ij}^0(t) &= N_{ij}^0(t) - \int_0^t I_i^0(s-)\lambda_{ij}^0(s)\, ds \quad \text{where} \quad M_{ij}^0(0) \equiv 0, \\ M_{kl}^1(t) &= N_{kl}^1(t) - \int_0^t I_i^1(s-)\lambda_{kl}^1(s)\, ds \quad \text{where} \quad M_{kl}^1(0) \equiv 0. \end{aligned}$$

5.2. Notation and assumptions

The problems occur when we require the premiums to be equivalent with the premium reserves which is referred to actuarial equivalence. If we relieve the conditions and do not emphasis the actuarial equivalence for the duration of the life insurance contract, the calculations can be executed in a similar fashion to the previous chapter. All though this requires less theory it not a general result and does not take advantage of the full potential to the practical applications. We shall begin by introducing the essential modifications to the notation.

We will utilize a payment process $(A(t))_{t\in[0,T]}$ that satisfies the SDE

$$dA(t) = \big(\alpha_{\tilde{X}(t-)}(t)\,dt + a_{\tilde{X}(t-)}(t)\,d\nu(t)\big) + \sum_{i,j\in S:i\neq j} \beta^0_{ij}(t)\,dN^0_{ij}(t) + \sum_{k,j\in\mathcal{J}:k\neq l} \beta^1_{kl}(t)\,dN^1_{kl}(t) \tag{5.3}$$

for all $t\in[0,T]$ where the process $(\nu(t))_{t\in[0,T]}$ is a counting process. Moreover the processes $(\alpha_i(t))_{t\in[0,T]}$, $(a_i(t))_{t\in[0,T]}$, $(\beta^0_{ij}(t))_{t\in[0,T]}$ and $(\beta^1_{kl}(t))_{t\in[0,T]}$ in (5.3) are assumed to be predictable and square-integrable, which means

$$\mathbb{E}\left[\int_{(0,T]} |A_{\tilde{X}(t-)}(t)|^2\,d\nu(t) + \int_0^T \Big(|\alpha_{\tilde{X}(t-)}(t)|^2 + \|\beta^0(t)\|^2_{\Lambda^0} + \|\beta^1(t)\|^2_{\Lambda^1}\Big)\,dt\right] < \infty.$$

The newly introduced process $(\beta^1_{kl}(t))$ describes the one-off premiums induced by the contract modifications from state k to state l. In order to keep consistent with the definitions we set

$$\gamma_{(i,k)}(t) := \alpha_i(t) + \sum_{j\in S:j\neq i} \beta^0_{ij}(t)\lambda^0_{ij}(t) + \sum_{l\in\mathcal{J}:l\neq k} \beta^1_{kl}(t)\lambda^1_{kl}(t) \tag{5.4}$$

and then the equation for the prospective reserve $(Y(t))_{t\in[0,T]}$ becomes

$$Y(t) = \mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\,du}\Big(\gamma_{\tilde{X}(s-)}(s)\,ds + a_{\tilde{X}(t-)}(s)\,d\nu(s)\Big)\Big|\mathcal{F}_t\right], \quad t\in[0,T], \tag{5.5}$$

where the process $(\delta(t))_{t\in[0,T]}$ is bounded and adapted.

Corollary 5.1. *Under the assumptions of this chapter there exist unique predictable processes $(Z^0_{ij}(t))_{t\in[0,T]}$ and $(Z^1_{kl}(t))_{t\in[0,T]}$ satisfying*

$$\mathbb{E}\left[\int_0^T \big(\|Z^0(s)\|^2_{\Lambda^0} + \|Z^1(s)\|^2_{\Lambda^1}\big)\,ds\right] < \infty \tag{5.6}$$

such that the prospective reserve Y from (5.5) is a solution to the BSDE

$$\begin{cases} dY(t) = & \Big(-\delta(t)Y(t) + \gamma_{\tilde{X}(t-)}(t)\Big)\,dt + a_{\tilde{X}(t-)}(t)\,d\nu(t) + Z^0(t)\,dM^0(t) + Z^1(t)\,dM^1(t),\\ Y(T) = & 0. \end{cases} \tag{5.7}$$

Proof. By repeating the proof of the Theorem 4.3 individually to the processes $(Z^0_{ij}(t))_{t\in[0,T]}$ and $(Z^1_{kl}(t))_{t\in[0,T]}$ we can verify that for all $t\in[0,T]$ it almost surely holds

$$I^0_i(t-)Z^0_{ij}(t) = I^0_i(t-)\sum_{k\in\mathcal{J}} I^1_k(t-)\Big(\mathbb{E}[Y(t)|\mathcal{F}_{t-},\tilde{X}(t)=(j,k)] - \mathbb{E}[Y(t)|\mathcal{F}_{t-},\tilde{X}(t)=(i,k)]\Big),$$

$$I^1_k(t-)Z^1_{kl}(t) = I^1_k(t-)\sum_{i\in\mathcal{S}} I^0_i(t-)\Big(\mathbb{E}[Y(t)|\mathcal{F}_{t-},\tilde{X}(t)=(i,l)] - \mathbb{E}[Y(t)|\mathcal{F}_{t-},\tilde{X}(t)=(i,k)]\Big).$$

By continuing to redo the proof of the Theorem 4.3 with the extended jump process (X,J) we arrive at the BSDE in (5.7).

We shall now concern the inequation (5.6). This follows from the fact that the processes $((\int_0^t Z^0\,dM^0)(t))_{t\in[0,T]}$ and $((\int_0^t Z^1\,dM^1)(t))_{t\in[0,T]}$ are square-integrable martingales which is equivalent in the light of the Corollary 3.16 with (5.6).

□

5.3. Maintaining actuarial equivalence with modifications

Now we will consider preserving the actuarial equivalence in contract modifications. This can be understood like that the prospective reserve $(Y(t))_{t\in[0,T]}$ satisfying the (5.5) with a given modification at a time τ should not be impacted retrospectively on the interval $[0,\tau)$. We proceed by proving first that the jump intensities can be utilized to determine a unique probability measure. According to [**4**, 2.4] it is a common practice in the actuary literature to first define the intensities and then determine the corresponding probability measure.

We require some theory on the Radon-Nikodym-derivative. With that in mind we define a Doléans-Dade exponential within our assumptions. Following [**3**, VI T2] for predictable non-negative processes $(Z_{ij}(t))_{t\in[0,T]}$ we set

$$L(t) := \prod_{i\neq j} L_{ij}(t) \tag{5.8}$$

with

$$L_{ij}(t) := \begin{cases} e^{\int_0^t (1-Z_{ij}(s))\lambda_{ij}(s)\,ds}, & t < T_{1,i,j} \\ e^{\int_0^t (1-Z_{ij}(s))\lambda_{ij}(s)\,ds} \prod_{n\geq 1} Z_{ij}(T_{n,i,j}) \mathbb{1}_{T_{n,i,j}\leq t}, & t \geq T_{1,i,j}. \end{cases} \tag{5.9}$$

According to [**3**, VI T2] the process $(L(t))t \in [0,T]$ is a non-negative supermartingale which implies that it is integrable. Now we prove a formula for the Radon-Nikodym-derivative.

Lemma 5.2 ([**3**, VI T3]). *Assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, adapted counting processes $(N_{ij}(t))_{t\in[0,T]}$ and the corresponding intensities $(\lambda_{ij}(t))_{t\in[0,T]}$. Let $(Z_{ij}(t))_{t\in[0,T]}$ be non-negative predictable processes such that $\mathbb{E}\sum_{i\neq j}\int_0^t Z_{ij}(s)\lambda_{ij}(s)\,ds < \infty$ for all $t \in [0,T]$. Suppose also that $\mathbb{E}[L(1)] = 1$ for $(L(t))_{t\in[0,T]}$ satisfying (5.8).*

If we define a new probability measure $\tilde{\mathbb{P}}$ with the Radon-Nikodym-derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = L(1), \tag{5.10}$$

then each $(N_{ij}(t))_{t\in[0,T]}$ has the corresponding $\tilde{\mathbb{P}}$-intensity

$$\tilde{\lambda}_{ij}(t) = Z_{ij}(t)\lambda_{ij}(t), \quad \text{for } t \in [0,1]. \tag{5.11}$$

Proof. The proof will be done like in [**3**, VI T3] by showing that the Definition 3.10 of the intensity holds for the probability measure $\tilde{\mathbb{P}}$ and intensities $(\lambda_{ij}(t))_{t\in[0,T]}$.

Let $(C(t))_{t\in[0,T]}$ be a non-negative predictable process. We have because of the definition (5.10) of the Radon-Nikodym-derivative that

$$\begin{aligned}\tilde{\mathbb{E}}\int_{[0,1]} C(s)\,dN_{ij}(s) &:= \int_\Omega\int_{[0,1]} C(s)\,dN_{ij}(s)\,d\tilde{\mathbb{P}}\\ &= \int_\Omega L(1)\int_{[0,1]} C(s)\,dN_{ij}(s)\,d\mathbb{P}\\ &= \mathbb{E}\left[L(1)\int_{[0,1]} C(s)\,dN_{ij}(s)\right] \end{aligned} \tag{5.12}$$

and also similarly

$$\tilde{\mathbb{E}}\int_0^1 C(s)\tilde{\lambda}_{ij}(s)\,ds = \mathbb{E}\left[L(1)\int_0^1 C(s)Z_{ij}(s)\lambda_{ij}(s)\,ds\right].$$

The integration by parts formula (Proposition 3.9) is now applicable to the last part of the equation (5.12) yielding

$$\mathbb{E}\left[L(1)\int_{[0,1]} C(s)\,dN_{ij}(s)\right] = \mathbb{E}\int_{(0,1]} L(t)C(t)\,dN_{ij}(t) + \mathbb{E}\int_{(0,1]}\int_{[0,t-]} C(s)\,dN_{ij}(s)\,dL(t). \tag{5.13}$$

We shall inspect the last term more closely. Because the process C is predictable, then the integral

$$(C\cdot N_{ij})(t) := \int_{[0,t-]} C(s)\,dN_{ij}(s)$$

is adapted. It is also left-continuous due to the definition so therefore the Remark 2.18 proves that the stochastic integral $((C\cdot N_{ij})(t))_{t\in[0,T]}$ is a predictable process. Also the exponential martingale $(L(t))_{t\in[0,T]}$ is a local martingale [**3**, VI T2] which proves by the [**12**, IV 2 Theorem 29] that the integral

$$\int_{(0,1]} (C\cdot N_{ij})(t)\,dL(t) \tag{5.14}$$

is a local martingale given that the process $((C\cdot N_{ij})(t))_{t\in[0,T]}$ is bounded. Moreover the processes N_{ij}, C and L are non-negative which yields by the martingale property and the tower propoerty that

$$\mathbb{E}\int_{(0,1\wedge V_n]}\int_{[0,t-]} C(s)\,dN_{ij}(s)\,dL(t) = \mathbb{E}\left[\mathbb{E}\left[\int_{(0,1\wedge V_n]}\int_{[0,t-]} C(s)\,dN_{ij}(s)\,dL(t)\middle|\mathcal{F}_0\right]\right] = 0$$

for some localizing sequence $(V_n)_n$. Now letting $V_n\to T$ we receive

$$\mathbb{E}\left[L(1)\int_{[0,1]} C(s)\,dN_{ij}(s)\right] = \mathbb{E}\int_{(0,1]} L(t)C(t)\,dN_{ij}(t) \tag{5.15}$$

in the equation (5.13). This result is analogous to the Meyer-Dellacherie's Integration Formula for martingales and increasing processes. [**3**, A2 2 T19]

In a similar fashion we may apply the integration by parts formula a bit differently to get

$$\begin{aligned}\mathbb{E}\left[L(1)\int_0^1 C(s)Z_{ij}(s)\lambda_{ij}(s)\,ds\right] &= \mathbb{E}\int_0^1 L(t-)C(t)Z_{ij}(s)\lambda_{ij}(t)\,dt \\ &\quad + \mathbb{E}\int_0^1\int_{[0,t]} C(s)Z_{ij}(s)\lambda_{ij}(s)\,ds\,dL(t),\end{aligned} \tag{5.16}$$

where $\mathbb{E}\int_0^1\int_{[0,t]} C(s)Z_{ij}(s)\lambda_{ij}(s)\,ds\,dL(t) = 0$ because the processes λ_{ij}, C and Z_{ij} are predictable and the Lebesgue measure dt is continuous. Therefore we are left with

$$\mathbb{E}\left[L(1)\int_0^1 C(s)Z_{ij}(s)\lambda_{ij}(s)\,ds\right] = \mathbb{E}\int_0^1 L(t-)C(t)Z_{ij}(s)\lambda_{ij}(t)\,dt. \tag{5.17}$$

As a side note we have that $N_{ij}(0) = 0$ $\mathbb{P}$-a.s. which validates the equality

$$\mathbb{E}\int_{(0,1]} L(t)C(t)\,dN_{ij}(t) = \mathbb{E}\int_{[0,1]} L(t)C(t)\,dN_{ij}(t). \tag{5.18}$$

Because the left-continuous adapted process $(L(t-))t \in [0,T]$ is non-negative by definition and predictable (Remark 2.18) we would have by the definition of the intensity λ_{ij} that

$$\mathbb{E}\int_{[0,1]} L(t-)C(t)Z_{ij}(s)\,dN_{ij}(t) = \mathbb{E}\int_0^1 L(t-)C(t)Z_{ij}(s)\lambda_{ij}(t)\,dt. \tag{5.19}$$

To prove the statement of the Lemma we are still required to show that

$$\mathbb{E}\int_{[0,1]} L(t)C(t)\,dN_{ij}(t) = \mathbb{E}\int_{[0,1]} L(t-)C(t)Z_{ij}(s)\,dN_{ij}(t), \tag{5.20}$$

which will be performed next.

Because the processes N_{ij} are non-explosive (Proposition 3.11) it holds $\lim_{n\to\infty} T_n = T$. Therefore there exists for each $t \in [0,1]$ only a finite index set I of stopping times $(T_{n,i,j})_{n\in I}$ such that $T_{n,i,j} \le t$ for all $n \in I$. Fix a time $t \in [0,1]$ and denote the maximums

$$T_{M,i,j} := \max\{T_{n,i,j} : T_{n,i,j} \le t\}.$$

Then in the definition of the process L we have

$$\begin{aligned}L_{ij}(t) &= e^{\int_0^t (1-Z_{ij}(s))\lambda_{ij}(s)\,ds}\prod_{n=1}^{M} Z_{ij}(T_{n,i,j})\mathbb{1}_{T_{n,i,j}\le t} \\ &= Z_{ij}(T_{M,i,j})\mathbb{1}_{T_{M,i,j}=t}e^{\int_0^{t-} (1-Z_{ij}(s))\lambda_{ij}(s)\,ds}\prod_{n=1}^{M-1}\Big(Z_{ij}(T_n)\mathbb{1}_{T_{n,i,j}\le t}\Big)\mathbb{1}_{T_{n,i,j}<t} \\ &= Z_{ij}(t)L_{ij}(t-).\end{aligned}$$

This implies $L(t) = Z_{ij}(t)L(t-)$ which proves the last missing link (5.20). Therefore

$$\tilde{\mathbb{E}}\int_{[0,1]} C(s)\,dN_{ij}(s) = \tilde{\mathbb{E}}\int_0^1 C(s)\tilde{\lambda}_{ij}(s)\,ds \tag{5.21}$$

and the processes $(\tilde{\lambda}_{ij}(t))_{t\in[0,T]}$ are $\tilde{\mathbb{P}}$-intensities on the interval $[0,1]$.

□

The next proposition is a special case of the Girsanov-Meyer Theorem [**12**, III Theorem 39] with jump processes instead of semimartingales.

PROPOSITION 5.3 ([**4**, Proposition 4.3]). *Assume on a measurable space $(\Omega, \mathcal{F})$ a bivariate jump process $(X, J) := (X(t), J(t))_{t\in[0,T]}$ and let $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration. Let $(\tau_m)_{m\in\mathbb{N}}$ denote the jump times of the process $(J(t))_{t\in[0,T]}$. Then for every contract modification $m \in \mathbb{N}$ there exists a unique probability measure $\mathbb{P}^m$ such that the process (X, J) has $\mathbb{P}^m$-intensities $(\lambda^0_{ij}(t))_{t\in[0,T]}$ and $(\mathbb{1}_{\{t\leq\tau_m\}}(t)\lambda^1_{kl}(t))_{t\in[0,T]}$ respectively to the coordinate processes, for $i, j \in \mathcal{S}, i \neq j; k, l \in \mathcal{J} : k \neq l$. Moreover it holds*

1. $\mathbb{P}^m = \mathbb{P}$ *on* $\mathcal{F}_{\tau_m}$,
2. $\mathbb{P}^m \sim \mathbb{P}$ *on* $\mathcal{F}_{\tau_{m+1}}$ *and*
3. $\mathbb{P}^m \ll \mathbb{P}$ *on* $\mathcal{F}_\infty$.

REMARK 5.4. The measures μ and ν are equal ($\mu = \nu$) if $\mu(A) = \nu(A)$ for all $A \in \mathcal{F}$. The measure μ is absolutely continuous with respect to the ν ($\mu \ll \nu$) if $\nu(E) = 0$ for all $E \in \mathcal{F}$ such that $\mu(E) = 0$. [**7**, Definition 7.1.1] The measures are equivalent ($\mu \sim \nu$) if $\mu \ll \nu$ and $\nu \ll \mu$. [**12**, III 8]

PROOF. First fix $m \in \mathbb{N}$. The assumption (5.2) states that for some $\mathbb{P}$-null set $N \in \Omega$ such that $\mathbb{P}(\Omega \setminus N) = 1$ it holds

$$\int_0^T \left(\sum_{i,j\in S: i\neq j} I^0_i(t-)\lambda^0_{ij}(t) + \sum_{k,l\in\mathcal{J}:k\neq l} I^1_i(t-)\lambda^1_{kl}(t) \right) dt < \infty, \quad \omega \in \Omega \setminus N. \tag{5.22}$$

We may prove the assertions within the set $\Omega \setminus N$ without the loss of generality. Therefore it is possible to redefine the intensities so that the inequation (5.22) holds for all $\omega \in \Omega$. Then we have for all that

$$\mathbb{E} \sum_{k,l\in\mathcal{J}:k\neq l} \int_0^t \mathbb{1}_{\{t\leq\tau_m\}}(s)\lambda^1_{kl}(s)\, ds < \infty. \tag{5.23}$$

We shall now prove the existence of the probability measure $\mathbb{P}^m$ such that the process (X, J) has the intensities $(\lambda^0_{ij}(t), (\mathbb{1}_{\{t\leq\tau_m\}}\lambda^1_{kl})(t))_{t\in[0,T]}$ by applying the Lemma 5.2. Let us glance at the assumptions whether they hold in our case. We have the counting processes $(N^1_{kl}(t))t \in [0, T]$ and the intensities $(\lambda^1_{kl}(t))t \in [0, T]$. The process $((\mathbb{1}_{\{t\leq\tau_m\}})(t))t \in [0, T]$ is non-negative and left-continuous. By the Remark 2.18 it is then predictable. Finally we have the inequality (5.23).

Define the the stochastic exponential in the domain of this Proposition as

$$L(t) := \prod_{k,l\in\mathcal{J}:k\neq l} L_{kl}(t) \tag{5.24}$$

where

$$L_{kl}(t) := \begin{cases} e^{\int_0^t (1-\mathbb{1}_{\{s\le\tau_m\}})\lambda_{ij}(s)\,ds}, & t < \tau_{1,k,l} \\ e^{\int_0^t (1-\mathbb{1}_{\{s\le\tau_m\}})\lambda_{ij}(s)\,ds} \prod_{n\in\mathbb{N}:\tau_{n,k,l}\le t} \mathbb{1}_{\{\tau_{n,k,l}\le\tau_m\}}, & t \ge \tau_{1,k,l} \end{cases} \tag{5.25}$$

$$= e^{\int_0^t (1-\mathbb{1}_{\{s\le\tau_m\}})\lambda_{ij}(s)\,ds}. \tag{5.26}$$

Here we have that

$$\begin{aligned} \mathbb{E}L(1) &= \mathbb{E}\left[e^{\int_0^t (1-\mathbb{1}_{\{1\le\tau_m\}})\lambda_{ij}(s)\,ds}\right] \\ &= \mathbb{P}(\{1>\tau_m\})e^{\int_0^t \lambda_{ij}(s)\,ds} \\ &= 1 \end{aligned}$$

We define the probability measure $\mathbb{P}^m$ as in the Lemma 5.2 such that

$$\mathbb{P}^m(A) := \int_A L(1)\,d\mathbb{P} \tag{5.27}$$

holds. Then the Lemma 5.2 states that the processes $(N^1_{kl}(t))t \in [0,T]$ have the $\mathbb{P}^m$-intensities $(\mathbb{1}_{\{t\le\tau_m\}}(t)\lambda^1_{kl}(t))_{t\in[0,T]}$.

Moreover the new probability measure $\mathbb{P}^m$ satisfies the assertions (1), (2) and (3). Therefore the processes $(\lambda^0_{ij}(t))t \in [0,T]$ are also $\mathbb{P}^m$-intensities of the counting processes $(N^0_{ij}(t))t \in [0,T]$.

□

5.3.1. Extended Cantelli Theorem.

THEOREM 5.5 ([**4**, Theorem 4.4]). *Assume that $m \in \mathbb{N}$. Let $(\Omega,\mathcal{F},\mathbb{P})$ and $(\Omega,\mathcal{F},\mathbb{P}^m)$ be the probability spaces from the Proposition 5.3. Denote the solutions to the BSDE's given by the Corollary 5.1 with (Y,Z^0,Z^1) and $(Y^m,Z^{0,m},Z^{1,m})$ that correspond to the probability measures $\mathbb{P}$ and $\mathbb{P}^m$ respectively. Then the following assertions are equivalent:*

(1) *For all $k \in \mathcal{J}$ we have $\mathbb{P}\times t$-almost everywhere*

$$\mathbb{1}_{\{\tau_m<t\le\tau_{m+1}\}} \sum_{l\in\mathcal{J}:l\ne k} \Big(\beta^1_{kl}(t) + Z^1_{kl}(t)\Big)\mathbb{1}_{\{J(t-)=k\}}\lambda^1_{kl}(t) = 0.$$

.

(2) *For all $t \in [0,\tau_{m+1})$ it holds $\mathbb{P}$-almost surely*

$$\Big(Y(t),Z^0(t),Z^1(t)\Big) = \Big(Y^m(t),Z^{0,m}(t),Z^{1,m}(t)\Big).$$

PROOF. In the Proposition 5.3 we presented the formula for the $\mathbb{P}^m$-intensities of the processes $(N^1_{kl}(t))_{t\in[0,T]}$:

$$\mathbb{1}_{\{t\le\tau_{m+1}\}}(t)\lambda^1_{kl}(t). \tag{5.28}$$

The expression implies that the intensity is zero on the set $\{t : t > \tau_{m+1}\}$. Therefore

$$\begin{aligned}
&\mathbb{P}^m(\{\omega : \tau_{m+1}(\omega) < \infty\}) \\
&= \mathbb{P}^m(\{\omega : \exists t, \sum_{k,l\in\mathcal{J}:k\neq l} N^1_{kl}(t,\omega) = m+1\}) \\
&= 1 - \mathbb{P}^m(\{\omega : \lambda^1_{kl}(t) = 0, \forall t > \tau_{m+1}\}) \\
&= 0 \qquad (5.29)
\end{aligned}$$

as well.

We shall begin by proving the implication (1) $\implies$ (2). Assume that

$$\mathbb{1}_{\{\tau_m < t \leq \tau_{m+1}\}} \sum_{l\in\mathcal{J}:l\neq k} \Big(\beta^1_{kl}(t) + Z^1_{kl}(t)\Big) \mathbb{1}_{\{J(t-)=k\}} \lambda^1_{kl}(t) = 0$$

holds $\mathbb{P}^m$-almost surely for all $k \in \mathcal{J}$. Then the Proposition 5.3 yields that the measures $\mathbb{P}^m$ and $\mathbb{P}$ are equal on the $\mathcal{F}_{\tau_m}$. As a consequence the expected values $\mathbb{E}$ and $\tilde{\mathbb{E}}$ are equal in the sense that for $t \in [0, \tau_m]$ in each of the equations

(5.30)
$$I^0_i(t-) Z^0_{ij}(t) = I^0_i(t-) \sum_{k\in\mathcal{J}} I^1_k(t-) \Big(\mathbb{E}[Y(t)|\mathcal{F}_{t-}, \tilde{X}(t) = (j,k)] - \mathbb{E}[Y(t)|\mathcal{F}_{t-}, \tilde{X}(t) = (i,k)]\Big),$$

(5.31)
$$I^1_k(t-) Z^1_{kl}(t) = I^1_k(t-) \sum_{i\in\mathcal{S}} I^0_i(t-) \Big(\mathbb{E}[Y(t)|\mathcal{F}_{t-}, \tilde{X}(t) = (i,l)] - \mathbb{E}[Y(t)|\mathcal{F}_{t-}, \tilde{X}(t) = (i,k)]\Big)$$

(5.32)
$$Y(t) = \mathbb{E}\left[\int_{(t,T]} e^{-\int_t^s \delta(u)\,du} \Big(\gamma_{\tilde{X}(s-)}(s)\,ds + a_{\tilde{X}(t-)}(s)\,d\nu(s)\Big) \Big| \mathcal{F}_t\right], \quad t \in [0,T],$$

(5.33)
$$\mathbb{E}\left[\int_0^T \left(\|Z^0(s)\|^2_{\Lambda^0} + \|Z^1(s)\|^2_{\Lambda^1}\right) ds\right] < \infty,$$

the expected value $\tilde{\mathbb{E}}$ can be replaced with the $\mathbb{E}$. Moreover the Proposition 3.20 extends this result to the $[0, \tau_{m+1})$. Both of the triplets (Y, Z^0, Z^1) and $(Y^m, Z^{0,m}, Z^{1,m})$ are therefore solutions to the same backward SDE on $[0, \tau_{m+1})$.

Because of the observation (5.29) we have that

$$\mathbb{P}^m(T < \tau_{m+1}) = 1 \qquad (5.34)$$

thus the solutions (Y, Z^0, Z^1) and $(Y^m, Z^{0,m}, Z^{1,m})$ solve the same BSDE on $[0,T]$, $\mathbb{P}^m$-almost surely. Due to the $\mathbb{P}^m$-a.s. uniqueness of the solution to the BSDE the solutions must be $\mathbb{P}^m$-indistinguishable on $[0,T]$, which means that

$$\mathbb{P}^m\Big(Y(t) = Y^m(t), Z^0(t) = Z^{0,m}(t), Z^1(t) = Z^{1,m}(t), t \in [0,T]\Big) = 1.$$

Now we may recall the Proposition 5.3 to conclude that the measures $\mathbb{P}^m$ and $\mathbb{P}$ are equivalent on the σ-algebra $\mathcal{F}_{\tau_{m+1}-}$. Therefore also

$$\mathbb{P}\Big(Y(t) = Y^m(t), Z^0(t) = Z^{0,m}(t), Z^1(t) = Z^{1,m}(t), t \in [0,T]\Big) = 1 \qquad (5.35)$$

due to the case that some of the processes would not be indistinguishable is a $\mathbb{P}^m$-null set.

We shall proceed by validating the implication (2) $\implies$ (1). Assume that the solutions (Y, Z^0, Z^1) and $(Y^m, Z^{0,m}, Z^{1,m})$ are $\mathbb{P}$-indistinguishable on $[0, \tau_{m+1})$. Then for the difference of the BSDE's it holds $\mathbb{P}$-a.s for all $t \in [0, \tau_{m+1})$ that

$$
\begin{aligned}
0 =& Y(t) - Y^m(t) \\
=& \int_0^t \Big(- \delta(s) Y(s) + \gamma_{\tilde{X}(s-)}(s) \Big)\, ds - \int_0^t \Big(- \delta(s) Y^m(s) + \gamma^m_{\tilde{X}(s-)}(s) \Big)\, ds \\
&+ \int_{(0,t]} a_{\tilde{X}(s-)}(s)\, d\nu(s) - \int_{(0,t]} a_{\tilde{X}(s-)}(s)\, d\nu(s) \\
&+ \int_{(0,t]} Z^0(s)\, dM^0(s) - \int_{(0,t]} Z^{0,m}(s)\, dM^{0,m}(s) \\
&+ \int_{(0,t]} Z^1(s)\, dM^1(s) - \int_{(0,t]} Z^{1,m}(s)\, dM^{1,m}(s) \\
=& \int_0^t \Big(- \delta(s) \big(Y(s) - Y^m(s)\big) + \sum_{i,j \in \mathcal{S}: i \neq j} I_i^0(s-) \beta_{ij}^0(s) \lambda_{ij}^0(s) \\
&- \sum_{k,l \in \mathcal{J}: k \neq l} I_k^1(s-) \beta_{kl}^1(s) \mathbb{1}_{\{s \leq \tau_m\}} \lambda_{kl}^1(s) \Big)\, ds \\
&+ \int_{(0,t]} Z^0(s) \Big(dN^0(s) - \lambda^0(s)\, ds \Big) - \int_{(0,t]} Z^{0,m}(s) \Big(dN^0(s) - \mathbb{1}_{\{s \leq \tau_m\}} \lambda^0(s)\, ds \Big) \\
&+ \int_{(0,t]} Z^1(s) \Big(dN^1(s) - \lambda^1(s)\, ds \Big) - \int_{(0,t]} Z^{1,m}(s) \Big(dN^1(s) - \mathbb{1}_{\{s \leq \tau_m\}} \lambda^1(s)\, ds \Big) \\
=& \int_0^t \Big(\sum_{i \neq j} I_i^0(s-) \beta_{ij}^0(s) \lambda_{ij}^0(s) - \sum_{k \neq l} I_k^1(s-) \beta_{kl}^1(s) \mathbb{1}_{\{s \leq \tau_m\}} \lambda_{kl}^1(s) \Big)\, ds \\
&- \int_0^t Z^0(s) \lambda^0(s)\, ds + \int_0^t Z^{0,m}(s) \mathbb{1}_{\{s \leq \tau_m\}} \lambda^0(s)\, ds \\
&- \int_0^t Z^1(s) \lambda^1(s)\, ds + \int_0^t Z^{1,m}(s) \mathbb{1}_{\{s \leq \tau_m\}} \lambda^1(s)\, ds \\
=& \int_0^t (\mathbb{1}_{\{s \leq \tau_m\}} - 1) \Big(\sum_{i \neq j} I_i^0(s-) \beta_{ij}^0(s) \lambda_{ij}^0(s) + \sum_{k \neq l} I_k^1(s-) \beta_{kl}^1(s) \lambda_{kl}^1(s) \Big) \\
&+ \big(\mathbb{1}_{\{s \leq \tau_m\}} - 1\big) Z^0(s) \lambda^0(s) + \big(\mathbb{1}_{\{s \leq \tau_m\}} - 1\big) Z^1(s) \lambda^1(s)\, ds \\
=& \int_0^t \Bigg(\sum_{i \neq j} \Big(I_i^0(s-) \beta_{ij}^0(s) \lambda_{ij}^0(s) + I_i^0(s-) Z_{ij}^0(s) \lambda_{ij}^0(s) \Big) \\
&+ \sum_{k \neq l} \Big(I_k^1(s-) \beta_{kl}^1(s) \lambda_{kl}^1(s) + I_k^1(s-) Z_{kl}^1(s) \lambda_{kl}^1(s) \Big) \Bigg) \big(\mathbb{1}_{\{s \leq \tau_m\}} - 1\big)\, ds \\
=& \int_0^t \Big(\sum_{i \neq j} \big(\beta_{ij}^0(s) + Z_{ij}^0(s) \big) I_i^0(s-) \lambda_{ij}^0(s) \big) + \sum_{k \neq l} \big(\beta_{kl}^1(s) + Z_{kl}^1(s) \big) I_k^1(s-) \lambda_{kl}^1(s) \big) \Big) \\
&\times \big(\mathbb{1}_{\{s \leq \tau_m\}} - 1\big)\, ds.
\end{aligned}
$$

Here it holds that $\mathbb{1}_{\{t>\tau_m\}} = 1 - \mathbb{1}_{\{t\leq\tau_m\}}$. If we consider an additional condition that $t \leq \tau_{m+1}$ then we have

$$0 = \sum_{i\neq j} \big(\beta^0_{ij}(t) + Z^0_{ij}(t)\big) I^0_i(t-)\lambda^0_{ij}(t)\big)\mathbb{1}_{\{\tau_{m+1}\geq t>\tau_m\}} \quad \mathbb{P}\text{-a.s.} \tag{5.36}$$

Therefore the previous equation with the lengthy calculation is equivalent to

$$0 = \int_0^t \sum_{k\neq l} \big(\beta^1_{kl}(t) + Z^1_{kl}(t)\big) I^1_k(t-)\lambda^1_{kl}(t)\big)\mathbb{1}_{\{\tau_{m+1}\geq t>\tau_m\}}\, dt \quad \mathbb{P}\text{-a.s.} \tag{5.37}$$

The required statement (1) now follows from this revelation.

□

Remark 5.6. According to the [**4**, Theorem 4.4] the term $\beta^1_{kl}(t) + Z^1_{kl}(t)$ can be interpreted as the sum-at-risk when modifying the contract from state k to state l. The processes $(\beta^1_{kl}(t))t \in [0, T]$ describe the surrender payment once a modification has been made, and the processes $(Z^1_{ij}(t))t \in [0, T]$ are part of the solution to the BSDE taking care of the adaptability of the prospective reserve Y. The newly proven Theorem 5.5 is therefore indeed analogous to the Cantelli Theorem on the Markovian setting and is a generalization to the non-Markovian frame.

Bibliography

[1] Burk, F. (2007). *A Garden of Integrals.* The Mathematical Association of America.

[2] Bichteler, K. (2002). *Stochastic integration with jumps.* Cambridge University Press.

[3] Brémaud, P. (1981). *Point processes and queues, martingale dynamics.* Springer-Verlag New York Inc.

[4] Christiansen, M. & Djehiche, B. (2020). Nonlinear reserving and multiple contract modifications in life insurance. *Insurance: Mathematics and Economics, 93*, 187–195. https://doi.org/10.1016/j.insmatheco.2020.05.004

[5] Çinlar, E. (2010). *Probability and Stochastics* Springer Science+Business Media.

[6] Cohen, S. & Elliott, R. (2012). Existence, uniqueness and comparisons for BSDEs in general spaces. *The Annals of Probability, 40,* no.5, 2264–2297. http://dx.doi.org/10.1214/11-AOP679

[7] Geiss, C. & Geiss, S. (2021). *An introduction to probability theory.* Lecture notes. Department of Mathematics and Statistics. University of Jyväskylä. Lectured on 6.5.2021.

[8] Gerber, H. (1997). *Life insurance mathematics.* 3rd ed. Springer Science+Business Media Inc.

[9] Karatzas, I. & Shreve, S. (1998). *Brownian motion and stochastic calculus.* 2nd ed. Springer Science+Business Media Inc.

[10] Lamberton, D. & Lapeyre, B. (1996). *Stochastic calculus applied to finance.* CRC Press LLC.

[11] Lowther, G. (8.11.2009). Filtrations and Adapted Processes [blogpost]. Cited 20.10.2021. Received from https://almostsuremath.com/2009/11/08/filtrations-and-adapted-processes/

[12] Protter, P. (2005). *Stochastic Integration and Differential Equations.* 2nd ed. Version 2.1. Springer-Verlag Berlin Heidenberg

[13] Saz (https://math.stackexchange.com/users/36150/saz) (2015, January 15). *Right-continuity of completed filtration.* StackExchange. https://math.stackexchange.com/questions/933371/right-continuity-of-completed-filtration/1185562#1185562. Cited 28.2.2022.

[14] Williams, D. (1991). *Probability with martingales.* Cambridge University Press.