

JYX



**This is a self-archived version of an original article. This version may differ from the original in pagination and typographic details.**

**Author(s):** Fotakis, Jan A.; Molnár, Etele; Niemi, Harri; Greiner, Carsten; Rischke, Dirk H.

**Title:** Multicomponent relativistic dissipative fluid dynamics from the Boltzmann equation

**Year:** 2022

**Version:** Published version

**Copyright:** © 2022 American Physical Society

**Rights:** In Copyright

**Rights url:** <http://rightsstatements.org/page/InC/1.0/?language=en>

**Please cite the original version:**

Fotakis, J. A., Molnár, E., Niemi, H., Greiner, C., & Rischke, D. H. (2022). Multicomponent relativistic dissipative fluid dynamics from the Boltzmann equation. *Physical Review D*, 106(3), Article 036009. <https://doi.org/10.1103/PhysRevD.106.036009>

## Multicomponent relativistic dissipative fluid dynamics from the Boltzmann equation

Jan A. Fotakis<sup>1,\*</sup>, Etele Molnár<sup>1,2</sup>, Harri Niemi<sup>1,3,4</sup>, Carsten Greiner<sup>1</sup>, and Dirk H. Rischke<sup>1,5</sup>

<sup>1</sup>*Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität, Max-von-Laue-Straße 1, D-60438 Frankfurt am Main, Germany*

<sup>2</sup>*Incubator of Scientific Excellence—Centre for Simulations of Superdense Fluids, University of Wrocław, pl. M. Borna 9, PL-50204 Wrocław, Poland*

<sup>3</sup>*Department of Physics, University of Jyväskylä, P.O. Box 35, FI-40014 University of Jyväskylä, Finland*

<sup>4</sup>*Helsinki Institute of Physics, P.O. Box 64, FI-00014 University of Helsinki, Finland*

<sup>5</sup>*Helmholtz Research Academy Hesse for FAIR, Campus Riedberg, Max-von-Laue-Straße 12, D-60438 Frankfurt am Main, Germany*



(Received 22 March 2022; accepted 19 July 2022; published 10 August 2022)

We derive multicomponent relativistic second-order dissipative fluid dynamics from the Boltzmann equations for a reactive mixture of  $N_{\text{spec}}$  particle species with  $N_q$  intrinsic quantum numbers (e.g., electric charge, baryon number, and strangeness) using the method of moments. We obtain the continuity equations for multiple conserved charges as well as the conservation equations for the total energy and momentum in the single-fluid approximation. These  $4 + N_q$  conservation laws are closed by deriving the second-order equations of motion for the dissipative quantities in the  $(10 + 4N_q)$ -moment approximation. The resulting fluid-dynamical equations are formally similar to those of a single-component system, but feature different thermodynamic relations and transport coefficients. We derive general relations for all transport coefficients and compute them explicitly in the ultrarelativistic limit.

DOI: [10.1103/PhysRevD.106.036009](https://doi.org/10.1103/PhysRevD.106.036009)

### I. INTRODUCTION

Determining properties of strong-interaction matter from experimental data measured in high-energy heavy-ion experiments at BNL-RHIC and CERN-LHC is largely based on relativistic fluid-dynamical modeling, see, e.g., Refs. [1–5]. Consequently, relativistic fluid dynamics has become an indispensable tool in the description of the dynamical evolution of relativistic nuclear collisions.

The state of the art of fluid-dynamical modeling of relativistic nuclear matter is based on the relativistic second-order dissipative fluid-dynamical theory of Israel and Stewart [6]. This theory and method is based on the pioneering works of Grad [7] and Müller [8] and was originally formulated for a simple fluid, i.e., a fluid with a single conserved charge. On the other hand, one of the basic features of the fluid created in high-energy nuclear collisions is its multicomponent nature. For example, hadronic matter produced in nuclear collisions consists of a multitude of different types of hadrons, where each hadron species carries multiple intrinsic quantum numbers like baryon number  $B$ , electric charge  $Q$ , and strangeness  $S$ . Therefore, a multicomponent extension of relativistic fluid-dynamical theories that explicitly accounts for multiple

conserved charges is required for a proper description of heavy-ion collisions.

Previous attempts to derive second-order fluid-dynamical equations of motion for relativistic multicomponent mixtures include the pioneering work by Prakash *et al.* [9], which is an extension of Israel-Stewart theory to multicomponent mixtures, and the works by Monnai and Hirano [10,11], which generalize the equations of Ref. [9] by including additional second-order terms as well as providing Onsager's reciprocity relations [12,13] for the transport coefficients. More recent developments by Kikuchi *et al.* [14] apply the renormalization-group method to rederive the second-order equations of motion including additional second-order terms in the dissipative quantities resulting from the nonlinear part of the collision integral.

Furthermore it has been shown in Refs. [15–18] that many features of multicomponent systems depend on the detailed coupling between the diffusion currents associated with different conserved charges. In addition, the mapping between the state of the fluid and the corresponding momentum distribution of particles plays an important role [19,20].

In this work, we present a derivation of multicomponent relativistic second-order dissipative fluid dynamics for a reactive mixture of  $N_{\text{spec}}$  species with  $N_q$  conserved quantum numbers by generalizing the method of moments established

\*fotakis@itp.uni-frankfurt.de

for single-component systems by Denicol *et al.* [21]. By summing the dynamical equations of motion describing the individual particle species we obtain a reduced set of equations corresponding to the so-called “single-fluid” description of a multicomponent fluid. We derive the continuity equations for each conserved quantum charge as well as the conservation laws for total energy and momentum in this single-fluid approximation. These  $4 + N_q$  equations of motion are closed by providing second-order equations of motion in the  $(10 + 4N_q)$ -moment approximation for the dissipative quantities. The latter equations are formally similar to the relaxation equations of a single-component system but feature different transport coefficients, which contain the microscopic interactions of all components. Our approach reproduces the results of Ref. [21] for a single-component fluid in the 14-moment approximation, i.e., for  $N_{\text{spec}} = N_q = 1$ .

This paper is organized as follows. In Sec. II, we introduce the Boltzmann equation, thermodynamic quantities in local equilibrium, as well as fluid-dynamical quantities, both in and out of equilibrium. The definition of the local rest frame, the matching conditions, and the conservation equations are also given. In Sec. III we derive the equations of motion for the irreducible moments from the Boltzmann equation, linearize the collision term, and discuss the Navier-Stokes limit and the order-of-magnitude approximation. Finally, the second-order dissipative fluid-dynamical equations of motion in the  $(10 + 4N_q)$ -moment approximation are derived and discussed. We conclude this work with a summary in the final section. Details of the calculations are delegated to several appendixes. They also contain explicit expressions for all second-order transport coefficients, as well as an explicit calculation of the transport coefficients in the ultrarelativistic limit.

Throughout this paper we adopt natural units,  $\hbar = c = k_B = 1$ , and work in flat Minkowski space-time with metric tensor  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . The timelike fluid four-velocity is denoted by  $u^\mu = \gamma(1, \mathbf{v})^T$ , with normalization  $u^\mu u_\mu \equiv 1$ , where  $\mathbf{v}$  is the three-velocity and  $\gamma = (1 - \mathbf{v}^2)^{-1/2}$ . In the local rest (LR) frame of the fluid,  $u_{\text{LR}}^\mu = (1, \mathbf{0})^T$ . The rank-two projection operator onto the three-space orthogonal to  $u^\mu$  is defined as  $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ . We define the projection of any four-vector  $A^\mu$  onto the three-dimensional subspace orthogonal to  $u^\mu$  as  $A^{(\mu)} \equiv \Delta_\nu^\mu A^\nu$ . The generalization to projection tensors of rank  $2\ell$ , denoted by  $\Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell}$ , is constructed using the elementary projection operator  $\Delta_\nu^\mu$ . The irreducible symmetric, traceless, and orthogonal projection of a rank- $\ell$  tensor  $A^{\mu_1 \dots \mu_\ell}$  is denoted as  $A^{(\mu_1 \dots \mu_\ell)} \equiv \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} A^{\nu_1 \dots \nu_\ell}$ . For example, the rank-four symmetric, traceless, and orthogonal projection operator is defined as  $\Delta_{\alpha\beta}^{\mu\nu} \equiv \frac{1}{2}(\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta}$ , hence  $A^{(\mu\nu)} \equiv \Delta_{\alpha\beta}^{\mu\nu} A^{\alpha\beta}$ .

The four-momentum of a particle of species  $i$  is denoted by  $k_i^\mu = (k_i^0, \mathbf{k}_i)^T$ , which is normalized to the

corresponding species rest mass squared,  $k_i^\mu k_{i,\mu} = m_i^2$ . The energy of a particle of species  $i$  is defined as  $E_{i,\mathbf{k}} = k_i^\mu u_\mu$ , and coincides with the on-shell energy  $k_i^0 = \sqrt{\mathbf{k}_i^2 + m_i^2}$  in the LR frame of the fluid. The orthogonal projection of the four-momentum is  $k_i^{(\mu)} \equiv \Delta_\nu^\mu k_i^\nu$ , and in the LR frame it reduces to the three-momentum  $\mathbf{k}_i$ .

The comoving derivative  $D \equiv u^\mu \partial_\mu$  of any four-vector  $A^\mu = (A^0, \mathbf{A})^T$  is denoted by  $\dot{A}^\mu \equiv u^\nu \partial_\nu A^\mu = DA^\mu$ , while the space-time four-gradient is  $\nabla_\nu A^\mu \equiv \Delta_\nu^\alpha \partial_\alpha A^\mu$ . Note that in the LR frame these relativistic space-time derivatives reduce to the usual time and three-space derivatives,  $\partial_t A^\mu$  and  $\nabla \otimes \mathbf{A}$ . Thus, the four-derivative is decomposed as  $\partial_\mu \equiv u_\mu D + \nabla_\mu$ , hence the relativistic Cauchy-Stokes decomposition reads  $\partial_\mu u_\nu \equiv u_\mu \dot{u}_\nu + \nabla_\mu u_\nu = u_\mu \dot{u}_\nu + \frac{1}{3} \theta \Delta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$ . Here we have defined the expansion scalar,  $\theta \equiv \nabla_\mu u^\mu$ , the shear tensor  $\sigma^{\mu\nu} \equiv \nabla^{(\mu} u^{\nu)} = \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3} \theta \Delta^{\mu\nu}$ , and the vorticity  $\omega^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu)$ , such that  $\sigma^{\mu\nu} u_\nu = \omega^{\mu\nu} u_\nu = 0$ .

Moreover, we label the conserved charge types in the system with the letter  $q$ , which will be treated as an index running over  $B$  (baryon number),  $Q$  (electric charge), and  $S$  (strangeness) for the case a strong-interaction system. For notational convenience, we employ the following notation for the sums over charge types:

$$\sum_q^{\{B,Q,S\}} \equiv \sum_{q=B,Q,S}.$$

## II. REACTIVE MIXTURES IN KINETIC THEORY AND FLUID DYNAMICS

In this section we first introduce the Boltzmann equation for a reactive mixture with special emphasis on the collision term. Before we discuss fluid-dynamical quantities in local equilibrium and out of equilibrium, we study thermodynamic quantities in local equilibrium. This is followed by a discussion of the matching conditions and the choices for the local rest frame. Finally, we list the conservation equations of second-order dissipative fluid dynamics for a multicomponent fluid.

### A. The Boltzmann equation for a reactive mixture

A mixture of  $N_{\text{spec}}$  different (elementary) particle species (i.e., different chemical components) is characterized by the single-particle distribution functions for each particle species  $i$ ,  $f(x, k_i) \equiv f_{i,\mathbf{k}}$ , where we label the particle species by a lower index  $i$ . The space-time evolution of the distribution function of species  $i$  is determined by the relativistic Boltzmann equation [22,23],

$$k_i^\mu \partial_\mu f_{i,\mathbf{k}} \equiv C_i(x, k_i) = \sum_{j=1}^{N_{\text{spec}}} C_{ij}[f], \quad (1)$$

where we neglect any external forces and assume binary collisions only for the sake of simplicity. For binary inelastic, i.e., reactive collisions, the initial and final particles species may be different,  $i + j \rightarrow a + b$ , such that the collision term reads

$$C_{ij}[f] = \frac{1}{2} \sum_{a,b=1}^{N_{\text{spec}}} \int dK'_j dP'_a dP'_b (W_{ab \rightarrow ij}^{pp' \rightarrow kk'} f_{a,\mathbf{p}} f_{b,\mathbf{p}'} \tilde{f}_{i,\mathbf{k}} \tilde{f}_{j,\mathbf{k}'} - W_{ij \rightarrow ab}^{kk' \rightarrow pp'} f_{i,\mathbf{k}} f_{j,\mathbf{k}'} \tilde{f}_{a,\mathbf{p}} \tilde{f}_{b,\mathbf{p}'}), \quad (2)$$

where  $\tilde{f}_{i,\mathbf{k}} = 1 - a_i f_{i,\mathbf{k}}/g_i$ , with  $a_i = \pm 1$  for fermions/bosons, and  $a_i \rightarrow 0$  for classical particles, respectively. Here,  $g_i$  is the spin degeneracy of particle species  $i$ , while the Lorentz-invariant integration measure is  $dK_i = d^3\mathbf{k}_i/[(2\pi)^3 k_i^0]$ . Here, the factor 1/2 corrects for the double counting when integrating over the momenta of particles in the initial and final states [22].

The transition probabilities respect certain symmetry properties under exchange of particles in the initial and final states,  $W_{ij \rightarrow ab}^{kk' \rightarrow pp'} = W_{ji \rightarrow ba}^{pp' \rightarrow kk'}$ , as well as the bilateral normalization property of the microscopic processes [22],

$$\sum_{a,b=1}^{N_{\text{spec}}} \int dP'_a dP'_b W_{ab \rightarrow ij}^{pp' \rightarrow kk'} = \sum_{a,b=1}^{N_{\text{spec}}} \int dP'_a dP'_b W_{ij \rightarrow ab}^{kk' \rightarrow pp'}. \quad (3)$$

In the absence of a reaction threshold, this relation is invariant under time reversal,  $W_{ij \rightarrow ab}^{kk' \rightarrow pp'} = W_{ab \rightarrow ij}^{pp' \rightarrow kk'}$ , i.e., microscopic reversibility or detailed balance. In the case of a binary process the transition probability is [24,25]

$$W_{ij \rightarrow ab}^{kk' \rightarrow pp'} = \frac{(2\pi)^4}{16} |\mathcal{M}_{ij \rightarrow ab}(\sqrt{s}, \Omega)|^2 \times \delta^{(4)}(k_i^\mu + k_j^\mu - p_a^\mu - p_b^\mu), \quad (4)$$

where  $|\mathcal{M}_{ij \rightarrow ab}(\sqrt{s}, \Omega)|^2$  is the Lorentz-invariant transition probability averaged over incoming and summed over outgoing spin states. It only depends on the total center-of-momentum (CM) energy squared,  $s \equiv (k_i^\mu + k_j^\mu)^2 = (p_a^\mu + p_b^\mu)^2$ , and the solid angle  $\Omega$  under which outgoing particles are scattered with respect to the direction of the incoming particles, while the  $\delta^{(4)}$ -function ensures the conservation of energy and momentum in each binary collision. The differential cross section in the CM frame, where  $\mathbf{k}_i + \mathbf{k}_j = \mathbf{p}_a + \mathbf{p}_b \equiv 0$  and  $k_i^0 + k_j^0 = p_a^0 + p_b^0 \equiv \sqrt{s}$ , is defined via the invariant transition probability as

$$d\sigma_{ij \rightarrow ab}(\sqrt{s}, \Omega) = \frac{1}{64\pi^2 s} \frac{p_{ab}}{p_{ij}} |\mathcal{M}_{ij \rightarrow ab}(\sqrt{s}, \Omega)|^2 d\Omega, \quad (5)$$

where the incoming  $p_{ij}$  and outgoing  $p_{ab}$  momenta in the CM frame are

$$p_{ij} \equiv \frac{1}{2\sqrt{s}} \sqrt{(s - (m_i + m_j)^2)(s - (m_i - m_j)^2)}. \quad (6)$$

We note that in the elastic limit  $p_{ij} = p_{ab}$ . We define the total (integrated over angles) cross section as

$$\sigma_{\text{tot},ij \rightarrow ab} \equiv 2\pi\gamma_{ab} \int_{-1}^1 d \cos \vartheta \frac{d\sigma_{ij \rightarrow ab}(\sqrt{s}, \Omega)}{d\Omega}. \quad (7)$$

Here, the symmetry factor  $\gamma_{ab} \equiv 1 - \delta_{ab}/2$  accounts for the double counting when integrating over the momenta of indistinguishable particles in the final state. In the case of isotropic scattering the differential cross section is

$$\frac{d\sigma_{ij \rightarrow ab}(\sqrt{s})}{d\Omega} \stackrel{\text{isotropic}}{=} \frac{\sigma_{\text{tot},ij \rightarrow ab}(\sqrt{s})}{4\pi\gamma_{ab}}. \quad (8)$$

In the elastic limit the transition rates in Eq. (2) are defined as [22]

$$W_{ij \rightarrow ab}^{kk' \rightarrow pp'} \equiv \gamma_{ij} (\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}) W_{ij}^{kk' \rightarrow pp'}, \quad (9)$$

where

$$W_{ij}^{kk' \rightarrow pp'} \equiv \frac{(2\pi)^4}{16} |\mathcal{M}_{ij \rightarrow ij}(\sqrt{s}, \Omega)|^2 \delta^{(4)}(k_i^\mu + k_j^\mu - p_i^\mu - p_j^\mu). \quad (10)$$

Note that in the above expression, we have already introduced the symmetry factor, even though we have not yet integrated over the momentum. For later convenience, we can use Eqs. (5) and (8) to rewrite Eq. (9) in the case of isotropic elastic scattering in terms of the total cross section defined in Eq. (7),

$$W_{ij \rightarrow ab}^{kk' \rightarrow pp'} \equiv (\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}) (2\pi)^6 s \frac{\sigma_{\text{tot},ij}}{4\pi} \times \delta^{(4)}(k_i^\mu + k_j^\mu - p_i^\mu - p_j^\mu). \quad (11)$$

## B. Local equilibrium and associated thermodynamic quantities

In general the single-particle distribution function  $f_{i,\mathbf{k}}$  for every species  $i$  can be decomposed into an equilibrium part,  $f_{i,\mathbf{k}}^{(0)}$ , and an out-of-equilibrium part,  $\delta f_{i,\mathbf{k}}$ , as

$$f_{i,\mathbf{k}} = f_{i,\mathbf{k}}^{(0)} + \delta f_{i,\mathbf{k}}, \quad (12)$$

where the local-equilibrium distribution function of species  $i$  is given by the Jüttner distribution function [26],

$$f_{i,\mathbf{k}}^{(0)} = g_i \left[ \exp\left(\frac{E_{i,\mathbf{k}} - \mu_i}{T}\right) + a_i \right]^{-1}. \quad (13)$$

Due to detailed balance, the collision integral vanishes identically for the local-equilibrium distribution function [22]. Here,  $T \equiv 1/\beta$  is the temperature and  $\mu_i$  is the chemical potential of species  $i$  as defined in the local rest frame. The exact form of the nonequilibrium part of the distribution function  $\delta f_{i,\mathbf{k}}$  will be clarified later.

In various cases of interest such as in high-energy particle physics or relativistic heavy-ion collisions there are inelastic collisions where the particle number corresponding to a given species is not conserved due to particle creation and annihilation processes (i.e., various chemical reactions). Such strong-interaction matter is therefore described by a few conserved intrinsic quantum numbers, such as electric charge, baryon number, and strangeness.

This means that in local equilibrium the chemical potential  $\mu_i$  of a given particle  $i$  may be expressed in terms of  $N_q$  chemical potentials of conserved quantum ‘‘charges,’’

$$\mu_i(\{\mu_q\}) \equiv \sum_q^{\{B,Q,S\}} q_i \mu_q = B_i \mu_B + Q_i \mu_Q + S_i \mu_S, \quad (14)$$

where  $\{\mu_q\} \equiv \{\mu_B, \mu_Q, \mu_S\}$ , with  $\mu_B$ ,  $\mu_Q$ , and  $\mu_S$  being the baryon, electric, and strangeness chemical potentials, respectively, while  $B_i$ ,  $Q_i$ , and  $S_i$  are the baryon number, electric charge, and strangeness of the respective particle species  $i$ .

Now, we introduce the ratio of chemical potential over temperature corresponding to particle species  $i$  as  $\alpha_i \equiv \mu_i/T$ , as well as the ratio of chemical potential over temperature for the conserved quantum charges,  $\alpha_q \equiv \mu_q/T$ . Applying the chain rule, we obtain from Eq. (14)

$$d\alpha_i(\{\alpha_q\}) \equiv \sum_q^{\{B,Q,S\}} \frac{\partial \alpha_i}{\partial \alpha_q} d\alpha_q = \sum_q^{\{B,Q,S\}} q_i d\alpha_q, \quad (15)$$

where the intrinsic quantum number of particle species  $i$  can also be obtained as  $q_i \equiv \partial \alpha_i(\{\alpha_q\}) / \partial \alpha_q$ . Note that the infinitesimal change in any variable  $A$ , here denoted by  $dA$ , can be interchangeably used for the comoving derivative,  $DA$ , as well as the space-time four-gradient,  $\nabla^\mu A$ .

In local thermodynamic equilibrium, we define the following rank- $n$  tensor moments of given power  $r \geq 0$  in energy  $E_{i,\mathbf{k}}^r$  for any given particle species  $i$  as

$$\mathcal{I}_{i,r}^{\mu_1 \dots \mu_n} \equiv \int dK_i E_{i,\mathbf{k}}^r k_i^{\mu_1} \dots k_i^{\mu_n} f_{i,\mathbf{k}}^{(0)} = \langle E_{i,\mathbf{k}}^r k_i^{\mu_1} \dots k_i^{\mu_n} \rangle_{i,0}, \quad (16)$$

where the angular brackets are the abbreviation of the integrals,

$$\langle \dots \rangle_{i,0} \equiv \int dK_i (\dots)_{i,\mathbf{k}} f_{i,\mathbf{k}}^{(0)}. \quad (17)$$

Following Ref. [6], we expand the equilibrium moments (16) in terms of the flow velocity and the associated orthogonal projection operator, which leads to the following expression:

$$\begin{aligned} \mathcal{I}_{i,r}^{\mu_1 \dots \mu_n} &= \sum_{m=0}^{[n/2]} (-1)^m \frac{n!}{2^m m! (n-2m)!} \\ &\times \Delta^{(\mu_1 \mu_2 \dots \mu_{2m-1} \mu_{2m})} u^{\mu_{2m+1}} \dots u^{\mu_n} I_{i,r+n,m}, \end{aligned} \quad (18)$$

where  $n$  and  $m$  are natural numbers, while  $[n/2] \leq n/2$  denotes the largest integer divisible by two. The coefficient  $n!/[2^m m! (n-2m)!]$  counts the number of distinct terms in the symmetrized tensor products  $\Delta^{(\mu_1 \mu_2 \dots \mu_{2m-1} \mu_{2m})} \times u^{\mu_{2m+1}} \dots u^{\mu_n}$ . The coefficients  $I_{i,r+n,m}$  are thermodynamic integrals which only depend on  $\alpha_i$  and  $\beta$ ,

$$I_{i,nm}(\alpha_i, \beta) = \frac{(-1)^m}{(2m+1)!!} \langle E_{i,\mathbf{k}}^{n-2m} (\Delta_{\mu\nu} k_i^\mu k_i^\nu)^m \rangle_{i,0}, \quad (19)$$

where  $(2m+1)!! \equiv (2m+1)!/(2^m m!)$  is the double factorial of an odd integer.

The total derivative of the thermodynamic integrals with respect to the variables  $\alpha_i$  and  $\beta$  reads

$$\begin{aligned} dI_{i,nm} &= \left( \frac{\partial I_{i,nm}}{\partial \alpha_i} \right)_\beta d\alpha_i + \left( \frac{\partial I_{i,nm}}{\partial \beta} \right)_{\alpha_i} d\beta \\ &\equiv J_{i,nm} d\alpha_i - J_{i,n+1,m} d\beta, \end{aligned} \quad (20)$$

where we have defined the auxiliary thermodynamic integrals

$$\begin{aligned} J_{i,nm}(\alpha_i, \beta) &\equiv \left( \frac{\partial I_{i,nm}}{\partial \alpha_i} \right)_\beta = \frac{(-1)^m}{(2m+1)!!} \\ &\times \int dK_i E_{i,\mathbf{k}}^{n-2m} (\Delta_{\mu\nu} k_i^\mu k_i^\nu)^m f_{i,\mathbf{k}}^{(0)} \tilde{f}_{i,\mathbf{k}}^{(0)}. \end{aligned} \quad (21)$$

An integration by parts with  $df_{i,\mathbf{k}}^{(0)}/dE_{i,\mathbf{k}} = -\beta f_{i,\mathbf{k}}^{(0)} \tilde{f}_{i,\mathbf{k}}^{(0)}$  leads to the following relation between the thermodynamic integrals:

$$\beta J_{i,nm} = I_{i,n-1,m-1} + (n-2m) I_{i,n-1,m}. \quad (22)$$

Furthermore, with Eq. (15) we obtain from Eq. (20) an expression for the total derivative of the thermodynamic integrals of species  $i$  in terms of the conserved quantum charges,

$$\begin{aligned} dI_{i,nm} &= \sum_q^{\{B,Q,S\}} \left( \frac{\partial I_{i,nm}}{\partial \alpha_q} \right)_\beta d\alpha_q + \left( \frac{\partial I_{i,nm}}{\partial \beta} \right)_{\{\alpha_q\}} d\beta \\ &\equiv \sum_q^{\{B,Q,S\}} q_i J_{i,nm} d\alpha_q - J_{i,n+1,m} d\beta, \end{aligned} \quad (23)$$



from which follows

$$\left(\frac{\partial I_{i,nm}}{\partial \alpha_q}\right)_\beta = q_i J_{i,nm}. \quad (24)$$

For later use we define the thermodynamic integrals summed over all particle species, which we denote by  $I_{nm}$  and  $J_{nm}$ . Similarly, the thermodynamic integrals of conserved quantum charges,  $I_{nm}^q$ ,  $J_{nm}^q$ , as well as the auxiliary thermodynamic quantities,  $I_{nm}^{qq'}$ ,  $J_{nm}^{qq'}$ , are defined as follows:

$$I_{nm} \equiv \sum_{i=1}^{N_{\text{spec}}} I_{i,nm}, \quad J_{nm} \equiv \sum_{i=1}^{N_{\text{spec}}} J_{i,nm} = \sum_{i=1}^{N_{\text{spec}}} \left(\frac{\partial I_{i,nm}}{\partial \alpha_i}\right)_\beta, \quad (25)$$

$$I_{nm}^q \equiv \sum_{i=1}^{N_{\text{spec}}} q_i I_{i,nm}, \quad J_{nm}^q \equiv \sum_{i=1}^{N_{\text{spec}}} q_i J_{i,nm} = \left(\frac{\partial I_{nm}}{\partial \alpha_q}\right)_\beta, \quad (26)$$

$$I_{nm}^{qq'} \equiv \sum_{i=1}^{N_{\text{spec}}} q_i q'_i I_{i,nm}, \quad J_{nm}^{qq'} \equiv \sum_{i=1}^{N_{\text{spec}}} q_i q'_i J_{i,nm} = \left(\frac{\partial I_{nm}}{\partial \alpha_{q'}}\right)_\beta. \quad (27)$$

Now using Eq. (23) together with the above definitions, the total differential of the thermodynamic integrals summed over all particle species as well as that of the thermodynamic integral of a specific conserved charge read

$$dI_{nm}(\{\alpha_q\}, \beta) = \sum_q^{\{B,Q,S\}} J_{nm}^q d\alpha_q - J_{n+1,m} d\beta, \quad (28)$$

$$dI_{nm}^q(\{\alpha_q\}, \beta) = \sum_{q'}^{\{B,Q,S\}} J_{nm}^{qq'} d\alpha_{q'} - J_{n+1,m}^q d\beta. \quad (29)$$

In terms of physical quantities, we identify the thermodynamic integral of particle species  $i$  with indices  $n = 1$  and  $m = 0$  as the particle density,  $n_i = I_{i,10}$ , while that with indices  $n = 2$  and  $m = 0$  is the energy density,  $e_i = I_{i,20}$ . From Eqs. (20) and (23), we then obtain the standard thermodynamic relations

$$dn_i \equiv \left(\frac{\partial n_i}{\partial \alpha_i}\right)_\beta d\alpha_i + \left(\frac{\partial n_i}{\partial \beta}\right)_{\alpha_i} d\beta = \sum_q^{\{B,Q,S\}} q_i J_{i,10} d\alpha_q - J_{i,20} d\beta, \quad (30)$$

$$de_i \equiv \left(\frac{\partial e_i}{\partial \alpha_i}\right)_\beta d\alpha_i + \left(\frac{\partial e_i}{\partial \beta}\right)_{\alpha_i} d\beta = \sum_q^{\{B,Q,S\}} q_i J_{i,20} d\alpha_q - J_{i,30} d\beta. \quad (31)$$

From these results and Eqs. (25)–(27), or directly from Eq. (29), we may express the total differential of the density of conserved charge  $q$  as

$$\begin{aligned} dn_q &\equiv \sum_{i=1}^{N_{\text{spec}}} q_i dn_i = \sum_{q'}^{\{B,Q,S\}} \frac{\partial n_q}{\partial \alpha_{q'}} d\alpha_{q'} + \frac{\partial n_q}{\partial \beta} d\beta \\ &= \sum_{q'}^{\{B,Q,S\}} J_{10}^{qq'} d\alpha_{q'} - J_{20}^q d\beta \\ &\equiv \sum_{q'}^{\{B,Q,S\}} (\mathcal{T}^{-1})_{qq'} d\alpha_{q'} + (\mathcal{T}^{-1})_{q0} d\beta, \end{aligned} \quad (32)$$

and the total differential of the energy density as

$$\begin{aligned} de &\equiv \sum_{i=1}^{N_{\text{spec}}} de_i = \sum_{q'}^{\{B,Q,S\}} \frac{\partial e}{\partial \alpha_{q'}} d\alpha_{q'} + \frac{\partial e}{\partial \beta} d\beta \\ &= \sum_{q'}^{\{B,Q,S\}} J_{20}^{qq'} d\alpha_{q'} - J_{30} d\beta \\ &\equiv \sum_{q'}^{\{B,Q,S\}} (\mathcal{T}^{-1})_{0q'} d\alpha_{q'} + (\mathcal{T}^{-1})_{00} d\beta, \end{aligned} \quad (33)$$

where, we have defined the following inverse matrix:

$$(\mathcal{T}^{-1})_{qq'} \equiv \begin{pmatrix} \frac{\partial e}{\partial \beta} & \frac{\partial e}{\partial \alpha_{q'}} \\ \frac{\partial n_q}{\partial \beta} & \frac{\partial n_q}{\partial \alpha_{q'}} \end{pmatrix} = \begin{pmatrix} -J_{30} & J_{20}^B & J_{20}^Q & J_{20}^S \\ -J_{20}^B & J_{10}^{BB} & J_{10}^{BQ} & J_{10}^{BS} \\ -J_{20}^Q & J_{10}^{QB} & J_{10}^{QQ} & J_{10}^{QS} \\ -J_{20}^S & J_{10}^{SB} & J_{10}^{SQ} & J_{10}^{SS} \end{pmatrix}. \quad (34)$$

Equations (32) and (33) can be solved for  $d\beta$  and  $d\alpha_q$ ,

$$d\beta = \mathcal{T}_{00} de + \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{0q'} dn_{q'}, \quad (35)$$

$$d\alpha_q = \mathcal{T}_{q0} de + \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{qq'} dn_{q'}. \quad (36)$$

Note that the relations (32) and (33), or equivalently (35) and (36), encode the thermodynamic response of the medium to perturbations and contain information about the chemical composition and/or the equation of state. These thermodynamic relations will be used later in the equations of motion.

### C. Equilibrium fluid-dynamical quantities

The equilibrium moments (16), for  $r = 0$  and for the tensor ranks  $\ell = 1$  and  $\ell = 2$ , define the partial particle four-current and energy-momentum tensor,

$$N_{i,0}^\mu \equiv \mathcal{I}_{i,0}^\mu = \int dK_i k_i^\mu f_{i,\mathbf{k}}^{(0)} \equiv \langle k^\mu \rangle_{i,0}, \quad (37)$$

$$T_{i,0}^{\mu\nu} \equiv \mathcal{I}_{i,0}^{\mu\nu} = \int dK_i k_i^\mu k_i^\nu f_{i,\mathbf{k}}^{(0)} \equiv \langle k^\mu k^\nu \rangle_{i,0}. \quad (38)$$

The tensor decomposition (18) of these quantities with respect to an arbitrary timelike normalized flow velocity  $u^\mu$  and the projection operator  $\Delta^{\mu\nu}$  reads

$$N_{i,0}^\mu \equiv I_{i,10} u^\mu = n_i u^\mu, \quad (39)$$

$$T_{i,0}^{\mu\nu} \equiv I_{i,20} u^\mu u^\nu - I_{i,21} \Delta^{\mu\nu} = e_i u^\mu u^\nu - P_i \Delta^{\mu\nu}. \quad (40)$$

Tensor-projecting these quantities leads to the partial particle density, energy density, and pressure of species  $i$ ,

$$n_i \equiv N_{i,0}^\mu u_\mu = \langle E_{\mathbf{k}} \rangle_{i,0} = I_{i,10} \equiv \mathcal{I}_{i,1}, \quad (41)$$

$$e_i \equiv T_{i,0}^{\mu\nu} u_\mu u_\nu = \langle E_{\mathbf{k}}^2 \rangle_{i,0} = I_{i,20} \equiv \mathcal{I}_{i,2}, \quad (42)$$

$$\begin{aligned} P_i &\equiv -\frac{1}{3} T_{i,0}^{\mu\nu} \Delta_{\mu\nu} = -\frac{1}{3} \langle \Delta_{\mu\nu} k^\mu k^\nu \rangle_{i,0} \\ &= I_{i,21} \equiv -\frac{1}{3} (m_i^2 \mathcal{I}_{i,0} - \mathcal{I}_{i,2}). \end{aligned} \quad (43)$$

The sum over all particle species  $i$  of the partial equilibrium moments leads to the total particle four-current, the conserved charge four-currents, and the energy-momentum tensor of the mixture,

$$N_0^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} N_{i,0}^\mu = \sum_{i=1}^{N_{\text{spec}}} n_i u^\mu \equiv n u^\mu, \quad (44)$$

$$N_{q,0}^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} q_i N_{i,0}^\mu = \sum_{i=1}^{N_{\text{spec}}} q_i n_i u^\mu \equiv n_q u^\mu, \quad (45)$$

$$T_0^{\mu\nu} \equiv \sum_{i=1}^{N_{\text{spec}}} T_{i,0}^{\mu\nu} = \sum_{i=1}^{N_{\text{spec}}} (e_i u^\mu u^\nu - P_i \Delta^{\mu\nu}) \equiv e u^\mu u^\nu - P \Delta^{\mu\nu}. \quad (46)$$

Note that in our case the conserved net-charge four-currents are simply the electric, the baryon, and the strangeness four-current. The primary thermodynamic quantities of the mixture, i.e., the total number density, net-charge density, total energy density, and total pressure, are obtained by summing over all constituents

$$n = \sum_{i=1}^{N_{\text{spec}}} n_i, \quad n_q = \sum_{i=1}^{N_{\text{spec}}} q_i n_i, \quad e = \sum_{i=1}^{N_{\text{spec}}} e_i, \quad P = \sum_{i=1}^{N_{\text{spec}}} P_i. \quad (47)$$

The particle and net-charge number as well as the energy are extensive thermodynamic quantities, while the total pressure of the mixture follows Dalton's law of partial pressures.<sup>1</sup>

<sup>1</sup>These relations hold for systems which can be described by kinetic theory; however, they are violated once the Stoßzahlansatz [22] does not apply, i.e., when long-range interactions or multiparticle correlations become non-negligible.

An equation of state determines these thermodynamic quantities as functions of temperature and chemical potentials, i.e.,  $n_q = n_q(T, \mu_B, \mu_Q, \mu_S)$ ,  $e = e(T, \mu_B, \mu_Q, \mu_S)$ , and  $P = P(T, \mu_B, \mu_Q, \mu_S)$ .

Note that in local thermodynamic equilibrium the individual particle four-currents,  $N_{0,i}^\mu = n_i u^\mu$ , as well as the energy current of species  $i$ ,  $T_{0,i}^{\mu\nu} = e_i u^\mu$ , are parallel to each other. Therefore, all of these currents lead to the same local rest frame of the fluid. Out of equilibrium the fluid-dynamical four-velocity can no longer be uniquely defined. Nonetheless, without any loss of generality, a common flow velocity tied to a chosen local rest frame can still be defined. The difference of fluid-dynamical quantities from their local-equilibrium form will be discussed next.

#### D. Out-of-equilibrium fluid-dynamical quantities

Out of equilibrium, the distribution function differs from its local-equilibrium form by  $\delta f_{i,\mathbf{k}} = f_{i,\mathbf{k}} - f_{i,\mathbf{k}}^{(0)}$ . Introducing a similar notation for the momentum integrals as in Eq. (17),

$$\langle \cdots \rangle_{i,\delta} \equiv \int dK_i (\cdots)_i \delta f_{i,\mathbf{k}}, \quad (48)$$

and

$$\langle \cdots \rangle_i \equiv \int dK_i (\cdots)_i f_{i,\mathbf{k}} = \langle \cdots \rangle_{i,0} + \langle \cdots \rangle_{i,\delta}, \quad (49)$$

cf. Refs. [21,27], we define the irreducible moments of tensor-rank  $\ell$  and energy-rank  $r$  of the deviation of the single-particle distribution function from equilibrium for a given particle species  $i$ ,

$$\begin{aligned} \rho_{i,r}^{\mu_1 \cdots \mu_\ell} &\equiv \Delta_{\nu_1 \cdots \nu_\ell}^{\mu_1 \cdots \mu_\ell} \int dK_i E_{i,\mathbf{k}}^r k_i^{\mu_1} \cdots k_i^{\mu_\ell} \delta f_{i,\mathbf{k}} \\ &= \langle E_{\mathbf{k}}^r k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \rangle_{i,\delta}. \end{aligned} \quad (50)$$

Furthermore, we expand the distribution function  $f_{i,\mathbf{k}}$  around  $f_{i,\mathbf{k}}^{(0)}$  as in Ref. [21],

$$\begin{aligned} \delta f_{i,\mathbf{k}} &\equiv f_{i,\mathbf{k}}^{(0)} \tilde{f}_{i,\mathbf{k}}^{(0)} \phi_{i,\mathbf{k}} \\ &= f_{i,\mathbf{k}}^{(0)} \tilde{f}_{i,\mathbf{k}}^{(0)} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_{i,n}^{\mu_1 \cdots \mu_\ell} k_{i,\langle \mu_1} \cdots k_{i,\mu_\ell \rangle} \mathcal{H}_{i,\mathbf{k},n}^{(\ell)}, \end{aligned} \quad (51)$$

where the irreducible tensors orthogonal to the four-flow are  $k_i^{\langle \mu_1} \cdots k_i^{\mu_\ell \rangle} = \Delta_{\nu_1 \cdots \nu_\ell}^{\mu_1 \cdots \mu_\ell} k_i^{\nu_1} \cdots k_i^{\nu_\ell}$ . These tensors form a complete and orthogonal basis in momentum space. The coefficient  $\mathcal{H}_{i,\mathbf{k},n}^{(\ell)}$  is a polynomial in energy of order  $N_\ell$  defined as

$$\mathcal{H}_{i,\mathbf{k}n}^{(\ell)} = \frac{(-1)^\ell}{\ell! J_{i,2\ell}} \sum_{m=n}^{N_\ell} a_{i,mn}^{(\ell)} \sum_{r=0}^m a_{i,mr}^{(\ell)} E_{i,\mathbf{k}}^r. \quad (52)$$

In principle, the expansion in polynomials in energy is an infinite series, i.e.,  $N_\ell \rightarrow \infty$ . However, we have already introduced parameters  $N_\ell < \infty$  (for each  $\ell \geq 0$ ) at this point since we will truncate the series later on in order to derive a fluid-dynamical theory. The coefficients  $a_{i,mn}^{(\ell)}$  are calculated via the Gram-Schmidt orthogonalization procedure and can be expressed in terms of thermodynamic integrals, see Ref. [21] for more details.

Thus, similarly to the equilibrium moments we define the out-of-equilibrium particle four-current and energy-momentum tensor for particle species  $i$  as

$$\begin{aligned} N_i^\mu &\equiv \int dK_i k_i^\mu (f_{i,\mathbf{k}}^{(0)} + \delta f_{i,\mathbf{k}}) = N_{i,0}^\mu + \rho_{i,0}^\mu \\ &= \langle k^\mu \rangle_{i,0} + \langle k^\mu \rangle_{i,\delta} \equiv \langle k^\mu \rangle_i, \end{aligned} \quad (53)$$

$$\begin{aligned} T_i^{\mu\nu} &\equiv \int dK_i k_i^\mu k_i^\nu (f_{i,\mathbf{k}}^{(0)} + \delta f_{i,\mathbf{k}}) = T_{i,0}^{\mu\nu} + \rho_{i,0}^{\mu\nu} \\ &= \langle k^\mu k^\mu \rangle_{i,0} + \langle k^\mu k^\nu \rangle_{i,\delta} \equiv \langle k^\mu k^\nu \rangle_i, \end{aligned} \quad (54)$$

where  $N_{i,0}^\mu$  and  $T_{i,0}^{\mu\nu}$  were defined in Eqs. (39) and (40), respectively.

The tensor decompositions with respect to an arbitrary timelike normalized flow velocity  $u^\mu$ , summed over all particle species, lead to the total fluid-dynamical quantities of the mixture,

$$\begin{aligned} N^\mu &\equiv \sum_{i=1}^{N_{\text{spec}}} N_i^\mu = \sum_{i=1}^{N_{\text{spec}}} [(n_i + \rho_{i,1})u^\mu + V_i^\mu] \\ &\equiv (n + \delta n)u^\mu + V^\mu, \end{aligned} \quad (55)$$

$$\begin{aligned} N_q^\mu &\equiv \sum_{i=1}^{N_{\text{spec}}} q_i N_i^\mu = \sum_{i=1}^{N_{\text{spec}}} [q_i(n_i + \rho_{i,1})u^\mu + q_i V_i^\mu] \\ &\equiv (n_q + \delta n_q)u^\mu + V_q^\mu, \end{aligned} \quad (56)$$

$$\begin{aligned} T^{\mu\nu} &\equiv \sum_{i=1}^{N_{\text{spec}}} T_i^{\mu\nu} \\ &= \sum_{i=1}^{N_{\text{spec}}} [(e_i + \rho_{i,2})u^\mu u^\nu - (P_i + \Pi_i)\Delta^{\mu\nu} + 2W_i^{(\mu} u^{\nu)} + \pi_i^{\mu\nu}] \\ &\equiv (e + \delta e)u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu}. \end{aligned} \quad (57)$$

The net-particle density, the net-charge density, the energy density, and the isotropic pressure of the out-of-equilibrium mixture are

$$n + \delta n \equiv N^\mu u_\mu = \sum_{i=1}^{N_{\text{spec}}} \langle E_{\mathbf{k}} \rangle_i \equiv \sum_{i=1}^{N_{\text{spec}}} (n_i + \rho_{i,1}), \quad (58)$$

$$n_q + \delta n_q \equiv N_q^\mu u_\mu = \sum_{i=1}^{N_{\text{spec}}} q_i \langle E_{\mathbf{k}} \rangle_i \equiv \sum_{i=1}^{N_{\text{spec}}} q_i (n_i + \rho_{i,1}), \quad (59)$$

$$e + \delta e \equiv T^{\mu\nu} u_\mu u_\nu = \sum_{i=1}^{N_{\text{spec}}} \langle E_{\mathbf{k}}^2 \rangle_i \equiv \sum_{i=1}^{N_{\text{spec}}} (e_i + \rho_{i,2}), \quad (60)$$

$$\begin{aligned} P + \Pi &\equiv -\frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu} = -\sum_{i=1}^{N_{\text{spec}}} \frac{1}{3} \langle \Delta_{\mu\nu} k^\mu k^\nu \rangle_i \\ &= \sum_{i=1}^{N_{\text{spec}}} \left[ P_i - \frac{1}{3} (m_i^2 \rho_{i,0} - \rho_{i,2}) \right] \equiv \sum_{i=1}^{N_{\text{spec}}} (P_i + \Pi_i), \end{aligned} \quad (61)$$

with an equation of state relating the equilibrium quantities. Note that the latter were defined in Eq. (47), while the partial pressure appearing in the last equation was defined in Eq. (43). Hence, it follows that the nonequilibrium correction to the pressure, the so-called bulk viscous pressure of particle species  $i$ , is

$$\Pi_i \equiv -\frac{1}{3} \langle \Delta_{\alpha\beta} k^\alpha k^\beta \rangle_{i,\delta} = -\frac{1}{3} (m_i^2 \rho_{i,0} - \rho_{i,2}). \quad (62)$$

The net-particle diffusion, the net-charge diffusion, and the energy-momentum diffusion currents are

$$V^\mu \equiv \Delta^\mu_\nu N^\nu = \sum_{i=1}^{N_{\text{spec}}} \langle k^{(\mu} \rangle_i = \sum_{i=1}^{N_{\text{spec}}} \rho_{i,0}^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} V_i^\mu, \quad (63)$$

$$V_q^\mu \equiv \Delta^\mu_\nu N_q^\nu = \sum_{i=1}^{N_{\text{spec}}} q_i \langle k^{(\mu} \rangle_i = \sum_{i=1}^{N_{\text{spec}}} q_i \rho_{i,0}^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} q_i V_i^\mu, \quad (64)$$

$$W^\mu \equiv \Delta^\mu_\alpha T^{\alpha\beta} u_\beta = \sum_{i=1}^{N_{\text{spec}}} \langle E_{\mathbf{k}} k^{(\mu} \rangle_i = \sum_{i=1}^{N_{\text{spec}}} \rho_{i,1}^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} W_i^\mu. \quad (65)$$

Finally, the shear-stress tensor of the mixture is

$$\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} = \sum_{i=1}^{N_{\text{spec}}} \langle k^{(\mu} k^{\nu)} \rangle_i = \sum_{i=1}^{N_{\text{spec}}} \rho_{i,0}^{\mu\nu} \equiv \sum_{i=1}^{N_{\text{spec}}} \pi_i^{\mu\nu}. \quad (66)$$

Equations (58)–(66) represent the fluid-dynamical fields of the mixture, which (similarly as in chemical solutions) is a combination of multiple particle species and where the number of particles of an individual species may or may not be conserved. Originally, these fields constitute  $14N_{\text{spec}}$  variables (10 for each energy-momentum tensor  $T_i^{\mu\nu}$  and 4 for each particle current  $N_i^\mu$ ). We assume that the mixture can be treated as a single fluid such that its space-time evolution can be entirely determined in terms of the total energy-momentum tensor  $T^{\mu\nu}$  and the charge four-currents



$N_q^\mu$ . This approach reduces the number of unknown fluid-dynamical fields to  $10 + 4N_q$ .<sup>2</sup>

Further, the out-of-equilibrium part of the distribution function  $\delta f_{i,\mathbf{k}}$  was expanded in terms of an infinite set of independent irreducible moments  $\rho_{i,n}^{\mu_1 \dots \mu_\ell}$ , each of which obeys an equation of motion derived from the relativistic Boltzmann equation (1) (see Sec. III). The dissipative fluid-dynamical fields, Eqs. (62)–(66), are defined in terms of (some of) these moments, and thus are solutions of these equations of motion. The crucial step in the derivation of fluid dynamics using the method of moments is to truncate the infinite set of equations of motion for the irreducible moments, and thus also the series in Eq. (51), in a well-defined manner. To this end, the sums over tensor rank  $\ell$  and over powers of energy  $n$  in Eq. (51) are truncated. The latter is already implied by the truncation parameter  $N_\ell$ , which depends on the respective tensor rank  $\ell$  of the moment. The lowest possible truncation in  $\ell$  and  $n$  is to account for the lowest-order irreducible moments which explicitly appear in the energy-momentum tensor (57) and the charge four-flow (56), namely  $\rho_{i,0}, \rho_{i,1}, \rho_{i,2}, \rho_{i,0}^\mu, \rho_{i,1}^\mu$ , and  $\rho_{i,0}^{\mu\nu}$ . This leads to the truncation  $\ell \leq 2$ , and  $N_0 = 2$ ,  $N_1 = 1$ , and  $N_2 = 0$  in the series in Eq. (51). This is the so-called  $(10 + 4N_q)$ -moment approximation. As mentioned above, these moments, however, further depend on other moments, which may also be of higher tensor rank (e.g.,  $\ell > 2$ ). In Sec. III C we discuss how to further truncate the set of equations of motion.

### E. Matching conditions and local rest frames

In local equilibrium the thermodynamic state of matter is completely determined by a few scalar fields, namely a common temperature  $T$  and the chemical potentials of the constituent particle species  $\alpha_i$ , which are in turn given by the chemical potentials of the conserved charges  $\alpha_q$  via Eq. (14). A common way to determine these thermodynamic variables in an arbitrary state (which is not too far from local equilibrium) is to demand that the net-charge densities and the total energy density are the same as in some fictitious local-equilibrium reference state. These are the so-called Landau matching conditions [28],

$$\begin{aligned} (N_q^\mu - N_{q,0}^\mu)u_\mu &\equiv \sum_{i=1}^{N_{\text{spec}}} q_i (N_i^\mu - N_{i,0}^\mu)u_\mu \\ &= \sum_{i=1}^{N_{\text{spec}}} q_i \rho_{i,1} \equiv \delta n_q = 0, \end{aligned} \quad (67)$$

<sup>2</sup>Note that, naively counting the number of unknowns, there are actually  $15 + 5N_q$  degrees of freedom: 5 degrees of freedom (d.o.f.s) for the  $N_q$  charge four-currents ( $n_q, \delta n_q$ , and 3 components of  $V_q^\mu$ ), and 4 scalar ( $e, \delta e, P, \Pi$ ), 6 vector ( $u^\mu$  and  $W^\mu$ ), and 5 tensor d.o.f.s ( $\pi^{\mu\nu}$ ) for the energy-momentum tensor. However, one d.o.f. is reduced by the equation of state,  $P = P(e, \{n_q\})$ . Furthermore,  $4 + N_q$  additional d.o.f.s are reduced by the matching conditions, see Sec. II E.

$$\begin{aligned} (T^{\mu\nu} - T_0^{\mu\nu})u_\mu u_\nu &\equiv \sum_{i=1}^{N_{\text{spec}}} (T_i^{\mu\nu} - T_{i,0}^{\mu\nu})u_\mu u_\nu \\ &= \sum_{i=1}^{N_{\text{spec}}} \rho_{i,2} \equiv \delta e = 0, \end{aligned} \quad (68)$$

where  $(N^\mu - N_0^\mu)u_\mu \equiv \sum_{i=1}^{N_{\text{spec}}} (N_i^\mu - N_{i,0}^\mu)u_\mu = \sum_{i=1}^{N_{\text{spec}}} \rho_{i,1} \equiv \delta n \neq 0$ , since the total number of particles is not necessarily conserved. Furthermore, Landau's matching condition for the energy, Eq. (68), also leads to a simplification of the bulk viscous pressure in Eqs. (61) and (62),

$$\Pi \equiv -\frac{1}{3} \sum_{i=1}^{N_{\text{spec}}} m_i^2 \rho_{i,0} = \sum_{i=1}^{N_{\text{spec}}} \Pi_i. \quad (69)$$

We note that using the matching conditions (67) and (68), we can express some scalar moments by the others, and thus reduce the number of scalar moments of the multi-component mixture by  $N_q + 1$ .

The number of independent unknowns is further reduced once we choose a local rest frame, or equivalently a definition for the fluid-dynamical flow velocity. The definition of Landau and Lifshitz [28] leads to the so-called Landau frame, or energy frame, and requires that the total energy-momentum diffusion current of the mixture is zero,

$$W^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} W_i^\mu = 0. \quad (70)$$

This directly implies that  $T^{\mu\nu}u_\nu = e u^\mu$ , meaning that the flow velocity  $u^\mu$  is the timelike eigenvector of the energy-momentum tensor with eigenvalue  $e$ . This choice reduces the total number of unknowns by three and leads to additional constraints between the remaining species-specific vector fields  $W_i^\mu$ , i.e., there are only  $N_{\text{spec}} - 1$  independent energy-momentum diffusion fluxes in the mixture of  $N_{\text{spec}}$  different species. Note that Eq. (70) also implies that if the fluid consists of a single component, i.e.,  $N_{\text{spec}} = 1$ , there is no energy-momentum diffusion present in this frame. Unless stated otherwise, the Landau frame is our choice for the local rest frame.

More traditionally, one may use Eckart's definition [29] to fix the local rest frame by demanding that the overall diffusion of one of the conserved net charges, say that of charge  $q^*$ , in the mixture is zero,

$$\tilde{V}_{q^*}^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} \tilde{V}_{q^*,i}^\mu = \sum_{i=1}^{N_{\text{spec}}} q_i^* \tilde{V}_i^\mu = 0, \quad (71)$$

where quantities in this particular  $q^*$ -charge frame are denoted by a tilde. However, in high-energy heavy-ion collisions, where there are multiple conserved charges,

which are not necessarily nonvanishing in all regions of space-time, the definition of the rest frame according to Eckart is less suitable.

Nevertheless, in case of a single nonvanishing conserved charge  $q$ , the Landau and Eckart reference frames are essentially equivalent, hence choosing one over the other is a matter of taste. Namely, the energy-momentum diffusion current in the Eckart frame can be related to the charge diffusion current in the Landau frame via

$$\tilde{W}^\mu \equiv -h_q V_q^\mu, \quad (72)$$

where we introduced the enthalpy per charge,  $h_q = (e + P)/n_q$ , see Appendix B for more details.

### F. Conservation equations

Due to the fact that in binary collisions the net charges as well as the energy and momentum of particles are conserved, the equations of fluid dynamics of a mixture are derived from the Boltzmann equation (1) as

$$\partial_\mu N_q^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} q_i \partial_\mu N_i^\mu = \sum_{i=1}^{N_{\text{spec}}} q_i \int dK_i C_i = 0, \quad (73)$$

$$\partial_\mu T^{\mu\nu} \equiv \sum_{i=1}^{N_{\text{spec}}} \partial_\mu T_i^{\mu\nu} = \sum_{i=1}^{N_{\text{spec}}} \int dK_i k_i^\nu C_i = 0, \quad (74)$$

where there are  $N_q$  independent charge-conservation laws. Note that due to inelastic collisions the number of particles of species  $i$  is no longer conserved and the individual particle species satisfy rate equations,  $\partial_\mu N_i^\mu \neq 0$ . On the other hand, for purely elastic collisions the numbers of particles are conserved, and the momentum integral over each partial collision term vanishes separately,  $\partial_\mu N_i^\mu = 0$ .

With Eq. (56) the  $N_q$  charge-conservation equations (73) assume the form

$$\begin{aligned} \partial_\mu N_q^\mu &\equiv \sum_{i=1}^{N_{\text{spec}}} q_i D(n_i + \rho_{i,1}) + \sum_{i=1}^{N_{\text{spec}}} q_i (n_i + \rho_{i,1}) \theta + \sum_{i=1}^{N_{\text{spec}}} q_i \partial_\mu V_i^\mu \\ &= Dn_q + n_q \theta + \partial_\mu V_q^\mu = 0, \end{aligned} \quad (75)$$

where in the last step we used the first Landau matching condition (67).

The conservation of energy of the mixture is obtained by projecting Eq. (74) onto  $u_\nu$  and inserting Eq. (57),

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &\equiv \sum_{i=1}^{N_{\text{spec}}} D(e_i + \rho_{i,2}) + \sum_{i=1}^{N_{\text{spec}}} (e_i + \rho_{i,2} + P_i + \Pi_i) \theta \\ &\quad + \sum_{i=1}^{N_{\text{spec}}} \partial_\mu W_i^\mu - \sum_{i=1}^{N_{\text{spec}}} W_i^\mu D u_\mu - \sum_{i=1}^{N_{\text{spec}}} \pi_i^{\mu\nu} \sigma_{\mu\nu} \\ &= D e + (e + P + \Pi) \theta - \pi^{\mu\nu} \sigma_{\mu\nu} = 0, \end{aligned} \quad (76)$$

where we have imposed the second Landau matching condition (68) and also fixed the local rest frame according to Landau's convention, Eq. (70).

Using these conservation equations to replace  $dn_q$  and  $de$  in Eqs. (35) and (36) leads to the comoving derivatives of the inverse temperature and the charge chemical potentials multiplied by the inverse temperature,

$$\begin{aligned} D\beta &= -\mathcal{T}_{00}[(e + P + \Pi)\theta - \pi^{\mu\nu} \sigma_{\mu\nu}] \\ &\quad - \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{0q'} [n_q \theta + \partial_\mu V_q^\mu], \end{aligned} \quad (77)$$

$$\begin{aligned} D\alpha_q &= -\mathcal{T}_{q0}[(e + P + \Pi)\theta - \pi^{\mu\nu} \sigma_{\mu\nu}] \\ &\quad - \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{qq'} [n_{q'} \theta + \partial_\mu V_{q'}^\mu]. \end{aligned} \quad (78)$$

Finally, projecting Eq. (74) onto  $\Delta_\beta^\mu$  (which gives the momentum conservation of the mixture), and using Eqs. (68) and (70) leads to

$$\begin{aligned} \Delta_\beta^\mu \partial_\alpha T^{\alpha\beta} &\equiv \sum_{i=1}^{N_{\text{spec}}} (e_i + P_i + \Pi_i) D u^\mu - \nabla^\mu \sum_{i=1}^{N_{\text{spec}}} (P_i + \Pi_i) \\ &\quad + \Delta_\beta^\mu \partial_\alpha \sum_{i=1}^{N_{\text{spec}}} \pi_i^{\alpha\beta} \\ &= (e + P + \Pi) D u^\mu - \nabla^\mu (P + \Pi) + \Delta_\beta^\mu \partial_\alpha \pi^{\alpha\beta} = 0. \end{aligned} \quad (79)$$

This leads to an expression for the acceleration  $D u^\mu$  of the fluid. Noting that in local equilibrium the Gibbs-Duhem relation holds in the form

$$\beta dP = \sum_q^{\{B,Q,S\}} n_q d\alpha_q - (e + P) d\beta, \quad (80)$$

we obtain

$$\begin{aligned} D u^\mu &= -\frac{1}{\beta} \nabla^\mu \beta + \frac{1}{\beta(e + P)} \sum_q^{\{B,Q,S\}} n_q \nabla^\mu \alpha_q \\ &\quad - \frac{1}{e + P} (\Pi D u^\mu - \nabla^\mu \Pi + \Delta_\beta^\mu \partial_\alpha \pi^{\alpha\beta}). \end{aligned} \quad (81)$$

Let us recount the unknown variables and equations of the mixture. There are  $N_q$  conservation equations (75) for  $n_q$  and  $V_q^\mu$ , representing  $4N_q$  variables. The conservation of energy and momentum of the mixture provides the four equations (76) and (79), entailing  $e$ ,  $\Pi$ ,  $u^\mu$ , and  $\pi^{\mu\nu}$ , which represent 10 variables in total, since the equation of state already defines the pressure in terms of  $e$  and the  $n_q$ 's.

Thus, in a dissipative mixture of  $N_q$  conserved charges we have only  $4 + N_q$  conservation equations for a total of  $10 + 4N_q$  unknown fields. The additional equations for the dissipative fields  $\Pi$ ,  $V_q^\mu$ , and  $\pi^{\mu\nu}$  will be derived from the Boltzmann equation in the next section.

### III. SECOND-ORDER DISSIPATIVE FLUID-DYNAMICAL EQUATIONS OF MOTION

In this section, we first derive the equations of motion for the irreducible tensor moments for particle species  $i$ . For a single-component fluid, these equations were first given in Ref. [21]. Here, we extend them towards multicomponent fluids. Then, we linearize the collision term and discuss the Navier-Stokes limit as well as the order-of-magnitude approximation, which provides a simple, and yet effective way to close the set of equations of motion. Finally, we derive and discuss the second-order dissipative fluid-dynamical equations of motion.

#### A. Equations of motion for the irreducible moments

The equations of motion for the irreducible moments  $\rho_{i,r}^{\mu_1 \dots \mu_\ell}$  are obtained directly from the Boltzmann equation (1) by multiplying it with  $E_{i,\mathbf{k}}^{r-1} k_i^{\mu_1} \dots k_i^{\mu_\ell}$ , integrating over momentum space, and taking the comoving derivative. Then, projecting onto  $\Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell}$ , we obtain the equations of motion for the irreducible moments, i.e., an equation for the comoving derivative  $\dot{\rho}_{i,r}^{\langle \mu_1 \dots \mu_\ell \rangle} \equiv \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} D \rho_{i,r}^{\nu_1 \dots \nu_\ell}$ . The irreducible moments of the collision term (2) are defined as

$$C_{i,r-1}^{\langle \mu_1 \dots \mu_\ell \rangle} \equiv \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} \sum_{j=1}^{N_{\text{spec}}} \int dK_i E_{i,\mathbf{k}}^{r-1} k_i^{\nu_1} \dots k_i^{\nu_\ell} C_{ij}[f]. \quad (82)$$

After some calculation the equation of motion for the irreducible moments of tensor-rank zero reads

$$\begin{aligned} \dot{\rho}_{i,r} - C_{i,r-1} &= \alpha_{i,r}^{(0)} \theta - \sum_{q'}^{\{B,Q,S\}} \left( J_{i,r+1,0} \mathcal{T}_{0q'} - \sum_q^{\{B,Q,S\}} q_i J_{i,r,0} \mathcal{T}_{qq'} \right) (\nabla_\mu V_{q'}^\mu - V_{q'}^\mu \dot{u}_\mu) \\ &+ \frac{\theta}{3} \left[ m_i^2 (r-1) \rho_{i,r-2} - (r+2) \rho_{i,r} - 3 \left( J_{i,r+1,0} \mathcal{T}_{00} - \sum_q^{\{B,Q,S\}} q_i J_{i,r,0} \mathcal{T}_{q0} \right) \Pi \right] \\ &+ r \rho_{i,r-1}^\mu \dot{u}_\mu - \nabla_\mu \rho_{i,r-1}^\mu + \left[ (r-1) \rho_{i,r-2}^{\mu\nu} + \left( J_{i,r+1,0} \mathcal{T}_{00} - \sum_q^{\{B,Q,S\}} q_i J_{i,r,0} \mathcal{T}_{q0} \right) \pi^{\mu\nu} \right] \sigma_{\mu\nu}. \end{aligned} \quad (83)$$

This equation is different from Eq. (35) of Ref. [21], because thermodynamic relations are modified in mixtures with multiple conserved charges as compared to a single-component fluid. Similarly, the transport coefficient  $\alpha_{i,r}^{(0)}$  has additional contributions given by the sums  $\sum_q^{\{B,Q,S\}}$  when compared to Eq. (42) of Ref. [21],

$$\begin{aligned} \alpha_{i,r}^{(0)} &= - \left[ I_{i,r,0} + (r-1) I_{i,r,1} + \sum_{q'}^{\{B,Q,S\}} \left( J_{i,r+1,0} \mathcal{T}_{0q'} - \sum_q^{\{B,Q,S\}} q_i J_{i,r,0} \mathcal{T}_{qq'} \right) n_{q'} \right] \\ &- \left( J_{i,r+1,0} \mathcal{T}_{00} - \sum_q^{\{B,Q,S\}} q_i J_{i,r,0} \mathcal{T}_{q0} \right) (e + P). \end{aligned} \quad (84)$$

The equation of motion for the irreducible moments of tensor-rank one is very similar to Eq. (36) of Ref. [21], except for the first-order gradient term  $\sum_q^{\{B,Q,S\}} \alpha_{i,r,q}^{(1)} \nabla^\mu \alpha_q$ ,

$$\begin{aligned} \dot{\rho}_{i,r}^{\langle \mu \rangle} - C_{i,r-1}^{\langle \mu \rangle} &= \sum_q^{\{B,Q,S\}} \alpha_{i,r,q}^{(1)} \nabla^\mu \alpha_q + r \rho_{i,r-1}^{\mu\nu} \dot{u}_\nu - \frac{1}{3} \nabla^\mu (m_i^2 \rho_{i,r-1} - \rho_{i,r+1}) + \alpha_{i,r}^h \nabla^\mu \Pi \\ &- \Delta_\lambda^\mu (\nabla_\nu \rho_{i,r-1}^{\lambda\nu} + \alpha_{i,r}^h \partial_\nu \pi^{\nu\lambda}) + \frac{1}{3} [m_i^2 (r-1) \rho_{i,r-2}^\mu - (r+3) \rho_{i,r}^\mu] \theta \\ &+ \frac{1}{3} [m_i^2 r \rho_{i,r-1} - (r+3) \rho_{i,r+1} - 3 \alpha_{i,r}^h \Pi] \dot{u}^\mu + \rho_{i,r,\nu} \omega^{\mu\nu} \\ &+ \frac{1}{5} [m_i^2 (2r-2) \rho_{i,r-2,\nu} - (2r+3) \rho_{i,r,\nu}] \sigma^{\mu\nu} + (r-1) \rho_{i,r-2}^{\mu\nu\lambda} \sigma_{\nu\lambda}, \end{aligned} \quad (85)$$

where the transport coefficients are

$$\alpha_{i,r,q}^{(1)} = q_i J_{i,r+1,1} + \alpha_{i,r}^h \frac{n_q}{\beta}, \quad \alpha_{i,r}^h = -\beta \frac{J_{i,r+2,1}}{e + P}. \quad (86)$$

Finally, the equation of motion for the irreducible moments of tensor-rank two is formally unchanged from Eq. (37) of Ref. [21], only the species-specific index  $i$  is introduced here,

$$\begin{aligned} \dot{\rho}_{i,r}^{(\mu\nu)} - C_{i,r-1}^{(\mu\nu)} &= 2\alpha_{i,r}^{(2)} \sigma^{\mu\nu} + \frac{2}{15} [m_i^A (r-1) \rho_{i,r-2} - (2r+3) m_i^2 \rho_{i,r} + (r+4) \rho_{i,r+2}] \sigma^{\mu\nu} \\ &+ \frac{2}{5} \dot{u}^{(\mu} [m_i^2 r \rho_{i,r-1}^{(\nu)} - (r+5) \rho_{i,r+1}^{(\nu)}] - \frac{2}{5} \nabla^{(\mu} (m_i^2 \rho_{i,r-1}^{(\nu)} - \rho_{i,r+1}^{(\nu)}) \\ &+ r \rho_{i,r-1}^{\mu\lambda} \dot{u}_\lambda - \Delta_{\alpha\beta}^{\mu\nu} \nabla_\lambda \rho_{i,r-1}^{\alpha\beta\lambda} + (r-1) \rho_{i,r-2}^{\mu\nu\lambda\kappa} \sigma_{\lambda\kappa} + 2 \rho_{i,r}^{\lambda(\mu} \omega_{\lambda}^{\nu)} \\ &+ \frac{1}{3} [m_i^2 (r-1) \rho_{i,r-2}^{\mu\nu} - (r+4) \rho_{i,r}^{\mu\nu}] \theta + \frac{2}{7} [m_i^2 (2r-2) \rho_{i,r-2}^{\lambda(\mu} - (2r+5) \rho_{i,r}^{\lambda(\mu)}] \sigma_{\lambda}^{\nu)}, \end{aligned} \quad (87)$$

where the coefficient  $\alpha_{i,r}^{(2)}$  is also formally the same as in Eq. (44) of Ref. [21],

$$\alpha_{i,r}^{(2)} = (r-1) I_{i,r+2,2} + I_{i,r+2,1}. \quad (88)$$

These are the equations of motion for the irreducible moments up to tensor-rank two for any particle species  $i$ . One can show that in the case of a single-component fluid they reduce to the equations found in Ref. [21].

Furthermore, note that since the conserved quantities in fluid dynamics contain no tensors of rank higher than two, the higher-rank tensors,  $\rho_{i,r}^{\mu_1 \dots \mu_\ell} = 0$  for  $\ell \geq 3$ , in the equations of motion (85) and (87) will be neglected in the following (see Sec. III C).

### B. The linearized collision integral

Further progress requires the linearization of the collision integral (2) in the quantities  $\phi_{i,\mathbf{k}} = \delta f_{i,\mathbf{k}} / [f_{i,\mathbf{k}}^{(0)} \tilde{f}_{i,\mathbf{k}}^{(0)}]$ , such that it simplifies to

$$C_i(x, k) \simeq \frac{1}{2} \sum_{j,a,b=1}^{N_{\text{spec}}} \int dK'_j dP'_a dP'_b W_{ij \rightarrow ab}^{kk' \rightarrow pp'} f_{i,\mathbf{k}}^{(0)} f_{j,\mathbf{k}'}^{(0)} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)} (\phi_{a,\mathbf{p}} + \phi_{b,\mathbf{p}'} - \phi_{i,\mathbf{k}} - \phi_{j,\mathbf{k}'}), \quad (89)$$

where the bilateral normalization condition holds and the equality  $f_{i,\mathbf{k}}^{(0)} f_{j,\mathbf{k}'}^{(0)} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)} = f_{a,\mathbf{p}}^{(0)} f_{b,\mathbf{p}'}^{(0)} \tilde{f}_{i,\mathbf{k}}^{(0)} \tilde{f}_{j,\mathbf{k}'}^{(0)}$  was used [22,23].

Using the linearized collision integral (89) one can show that the corresponding irreducible moments (82) of the collision integral can be expressed in terms of a linear combination of irreducible moments,  $\rho_{i,r}^{\nu_1 \dots \nu_\ell}$ , in a similar way as in Eq. (50) of Ref. [21],

$$C_{i,r-1}^{(\mu_1 \dots \mu_\ell)} \equiv - \sum_{s=1}^{N_{\text{spec}}} \sum_{n=0}^{N_\ell} \sum_{m=0}^{\infty} (\mathcal{A}_{is, rn})_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_\ell} \rho_{s,n}^{\nu_1 \dots \nu_m} = - \sum_{s=1}^{N_{\text{spec}}} \sum_{n=0}^{N_\ell} (\mathcal{A}_{is, rn}^{(\ell)}) \rho_{s,n}^{\mu_1 \dots \mu_\ell}. \quad (90)$$

Here, we have defined the following tensors:

$$\begin{aligned} (\mathcal{A}_{is, rn})^{\mu_1 \dots \mu_\ell \nu_1 \dots \nu_m} &= \frac{1}{2} \sum_{j,a,b=1}^{N_{\text{spec}}} \int dK_i dK'_j dP'_a dP'_b W_{ij \rightarrow ab}^{kk' \rightarrow pp'} f_{i,\mathbf{k}}^{(0)} f_{j,\mathbf{k}'}^{(0)} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)} E_{i,\mathbf{k}}^{r-1} k_i^{\langle \mu_1} \dots k_i^{\mu_\ell \rangle} \\ &\times [\delta_{si} \mathcal{H}_{s, \mathbf{k}n}^{(m)} k_s^{\langle \nu_1} \dots k_s^{\nu_m \rangle} + \delta_{sj} \mathcal{H}_{s, \mathbf{k}'n}^{(m)} k_s^{\langle \nu_1} \dots k_s^{\nu_m \rangle} \\ &- \delta_{sa} \mathcal{H}_{s, \mathbf{p}n}^{(m)} p_s^{\langle \nu_1} \dots p_s^{\nu_m \rangle} - \delta_{sb} \mathcal{H}_{s, \mathbf{p}'n}^{(m)} p_s^{\langle \nu_1} \dots p_s^{\nu_m \rangle}], \end{aligned} \quad (91)$$

where the polynomials  $\mathcal{H}_{s,kn}^{(m)}$  were defined in Eq. (52). These tensors can be decomposed and projected, hence one finally obtains the collision matrix

$$\mathcal{A}_{is, rn}^{(\ell)} = \frac{1}{2\ell + 1} \Delta_{\mu_1 \dots \mu_\ell}^{\nu_1 \dots \nu_\ell} (\mathcal{A}_{is, rn})_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell}, \quad (92)$$

cf. Eq. (A18) of Ref. [21]. Note that in the case of a single-component system, i.e.,  $N_{\text{spec}} = 1$ , the above equation reduces to the diagonal components of  $\mathcal{A}_{is, rn}^{(\ell)}$ . Hence, the particle species labels  $i$  and  $s$  can be dropped and we simply recover the result of Ref. [21],  $C_{i, r-1}^{(\mu_1 \dots \mu_\ell) \text{single}} \rightarrow C_{r-1}^{(\mu_1 \dots \mu_\ell)} = -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1 \dots \mu_\ell}$ .

The inverse of the collision matrix defines the so-called relaxation-time matrix,

$$\tau_{si, nr}^{(\ell)} \equiv (\mathcal{A}^{-1})_{si, nr}^{(\ell)}, \quad (93)$$

where the matrix elements are proportional to the inverse of the mean free path between collisions  $\lambda_{\text{mfp}}^{-1}$ . Therefore, multiplying both sides of Eq. (90) by the relaxation-time matrix we obtain an important relation between the moments of the linearized collision integral and the irreducible moments,

$$\begin{aligned} \sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_\ell} \tau_{si, nr}^{(\ell)} C_{i, r-1}^{(\mu_1 \dots \mu_\ell)} &\equiv -\sum_{i, s'=1}^{N_{\text{spec}}} \sum_{r, n'=0}^{N_\ell} \tau_{si, nr}^{(\ell)} \mathcal{A}_{is', rn'}^{(\ell)} \rho_{s', n'}^{\mu_1 \dots \mu_\ell} \\ &= -\rho_{s, n}^{\mu_1 \dots \mu_\ell}. \end{aligned} \quad (94)$$

The infinite set of equations of motions for the irreducible moments contains infinitely many degrees of freedom. In order to close the equations of motion for the mixture treated as a single fluid, the number of degrees of freedom must be reduced and infinite sums must be truncated at some finite number.

One of the key features of transient fluid dynamics is that the corresponding equations of motion have a single time scale that controls the transient behavior, e.g., relaxation towards equilibrium. On the other hand, the Boltzmann equation has infinitely many such time scales. Even in a single-component gas the modes corresponding to the same tensor rank  $\ell$  are coupled to each other and their dynamics depends on these infinitely many scales. In the case of a mixture the situation is even more complicated as the modes corresponding to different particle species are also coupled. The reason for this is that due to interactions between particles of different species the corresponding moments are correlated, see Refs. [30,31].

One possible way to reduce the number of degrees of freedom would be to generalize the approach of Ref. [21] developed for a single-component system, and diagonalize the collision matrix to determine the slowest microscopic timescales, i.e., the relaxation times that are relevant in our

approximation, and the corresponding modes that dominate the long-time dynamics of the fluid in the transient regime. This has the advantage that the relaxation times that appear in the equations of motion could be explicitly identified as real microscopic timescales. However, the downside of this method is the appearance of terms that are of second or higher order in gradients, denoted as  $\mathcal{O}(\text{Kn}^2)$  in Ref. [21]. These terms can violate stability and causality of the theory. In principle, this can be cured by introducing further independent dynamical variables, cf. for instance Ref. [32], but this is beyond the scope of the current work. Instead, as discussed below, we will employ a slightly simpler approach, where the problematic  $\mathcal{O}(\text{Kn}^2)$  terms do not appear.

### C. The Navier-Stokes limit and the order-of-magnitude approximation

The explicit relation between the irreducible tensor of a given rank and the corresponding fluid-dynamical gradients can be derived by multiplying Eqs. (83), (85), and (87) by the corresponding relaxation-time matrices,  $\tau_{si, nr}^{(\ell)}$ , and using Eq. (94). In this way, the following equations of motion for the irreducible moments of tensor rank  $\ell = 0, 1$ , and 2 are obtained,

$$\sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \tau_{si, nr}^{(0)} \dot{\rho}_{i, r} + \rho_{s, n} = -\zeta_{s, n} \theta + \mathcal{O}(2), \quad (95)$$

$$\sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_1} \tau_{si, nr}^{(1)} \dot{\rho}_{i, r}^{(\mu)} + \rho_{s, n}^\mu = \sum_q^{\{B, Q, S\}} \kappa_{s, n, q} \nabla^\mu \alpha_q + \mathcal{O}(2), \quad (96)$$

$$\sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si, nr}^{(2)} \dot{\rho}_{i, r}^{(\mu\nu)} + \rho_{s, n}^{\mu\nu} = 2\eta_{s, n} \sigma^{\mu\nu} + \mathcal{O}(2), \quad (97)$$

where  $\mathcal{O}(2)$  denote all remaining second- and higher-order terms from the corresponding equations of motion for the irreducible moments. These are terms which are at least of quadratic order in the Knudsen number,  $\mathcal{O}(\text{Kn}^2)$ , or in the inverse Reynolds number,  $\mathcal{O}(\text{Rn}^{-2})$ , or of linear order in their product,  $\mathcal{O}(\text{Kn} \text{Rn}^{-1})$ . Here, we have defined the species-specific bulk-viscosity, diffusion, and shear-viscosity coefficients as

$$\zeta_{s, n} \equiv -\sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \tau_{si, nr}^{(0)} \alpha_{i, r}^{(0)}, \quad (98)$$

$$\kappa_{s, n, q} \equiv \sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_1} \tau_{si, nr}^{(1)} \alpha_{i, r, q}^{(1)}, \quad (99)$$

$$\eta_{s, n} \equiv \sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si, nr}^{(2)} \alpha_{i, r}^{(2)}. \quad (100)$$



Note that the definition of the bulk-viscosity coefficient differs by a factor of  $-m_s^2/3$  compared to Eq. (63) of Ref. [21].

From here on, we will make the assumption that the irreducible moments  $\rho_{i,r}$ ,  $\rho_{i,r}^{(\mu)}$ , and  $\rho_{i,r}^{(\mu\nu)}$  are of the same order, irrespective of the particle species. This implies that the sum over all species of these irreducible moments, i.e.,  $\rho_r$ ,  $\rho_r^{(\mu)}$ , and  $\rho_r^{(\mu\nu)}$  are also of the same order as the species-specific irreducible moments.

The so-called ‘‘order-of-magnitude approximation’’ is based on the first-order solution of the moment equations (95)–(97), which is equivalent to the Navier-Stokes limit. In this limit, the irreducible moments are algebraically related to terms of first order in Knudsen number, also called thermodynamic forces, multiplied by the corresponding transport coefficients,

$$\rho_{s,n} = -\bar{\zeta}_{s,n}\theta + \mathcal{O}(2), \quad (101)$$

$$\rho_{s,n}^\mu = \sum_q^{\{B,Q,S\}} \kappa_{s,n,q} \nabla^\mu \alpha_q + \mathcal{O}(2), \quad (102)$$

$$\rho_{s,n}^{\mu\nu} = 2\eta_{s,n}\sigma^{\mu\nu} + \mathcal{O}(2), \quad (103)$$

while all tensor moments of rank higher than two are at least of second order,  $\rho_{s,n}^{\mu_1 \dots \mu_\ell} \sim \mathcal{O}(2)$  for any  $\ell > 2$ , see Ref. [21] for details.

In principle, all irreducible moments are of first order in inverse Reynolds number,  $\mathcal{O}(\text{Rn}^{-1})$ , and thus formally independent of the power counting in Knudsen number. The order-of-magnitude approximation, which is based on the Navier-Stokes limit (101)–(103), assumes that the irreducible moments are of first order in Knudsen number,  $\mathcal{O}(\text{Kn})$ , i.e., the regime where the Knudsen and the inverse Reynolds numbers are of the same magnitude. This defines a power-counting scheme, similar to the one described in Ref. [21], which helps to assign a certain order to the various terms in the equations of motion. Then, all  $\mathcal{O}(2)$  terms on the right-hand sides of Eqs. (95)–(97), as well as the comoving derivatives on the left-hand sides are of second order in Knudsen number. The order-of-magnitude approximation is very similar to the ‘‘order-of-magnitude method’’ in nonrelativistic fluid dynamics [33].

Using the approximation (101)–(103) while summing over all particle species, and for the moment omitting  $\mathcal{O}(2)$  terms, we obtain the Navier-Stokes relations for the mixture. From Eq. (69) together with Eq. (101) the total bulk viscous pressure of the mixture reads

$$\Pi \equiv - \sum_{s=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \rho_{s,0} = \sum_{s=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \zeta_{s,0} \theta \equiv -\zeta \theta. \quad (104)$$

Similarly, we obtain from Eq. (64) together with Eq. (102) for the conserved charge currents

$$\begin{aligned} V_q^\mu &\equiv \sum_{s=1}^{N_{\text{spec}}} q_s \rho_{s,0}^\mu = \sum_{q'}^{\{B,Q,S\}} \sum_{s=1}^{N_{\text{spec}}} q_s \kappa_{s,0,q'} \nabla^\mu \alpha_{q'} \\ &\equiv \sum_{q'}^{\{B,Q,S\}} \kappa_{qq'} \nabla^\mu \alpha_{q'}. \end{aligned} \quad (105)$$

Finally, the shear-stress tensor of the mixture follows from Eq. (66) together with Eq. (103),

$$\pi^{\mu\nu} \equiv \sum_{s=1}^{N_{\text{spec}}} \rho_{s,0}^{\mu\nu} = \sum_{s=1}^{N_{\text{spec}}} 2\eta_{s,0} \sigma^{\mu\nu} \equiv 2\eta \sigma^{\mu\nu}. \quad (106)$$

Note that the first-order thermodynamic forces are the same for all particle species and for the mixture. Due to this fact, we obtain the first-order transport coefficients of the mixture: the bulk viscosity  $\zeta$ , the diffusion coefficients  $\kappa_{qq'}$ , and the shear viscosity  $\eta$ ,

$$\zeta \equiv - \sum_{s=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \zeta_{s,0}, \quad \kappa_{qq'} \equiv \sum_{s=1}^{N_{\text{spec}}} q_s \kappa_{s,0,q'}, \quad \eta \equiv \sum_{s=1}^{N_{\text{spec}}} \eta_{s,0}. \quad (107)$$

Therefore, using Eqs. (101)–(103) together with the relativistic Navier-Stokes relations (104)–(106) we readily obtain the following algebraic relations between the species-specific irreducible moments and the primary dissipative quantities of the mixture,

$$\rho_{s,n} = \bar{\zeta}_{s,n} \Pi + \mathcal{O}(2), \quad (108)$$

$$\rho_{s,n}^\mu = \sum_q^{\{B,Q,S\}} \bar{\kappa}_{s,n}^{(q)} V_q^\mu + \mathcal{O}(2), \quad (109)$$

$$\rho_{s,n}^{\mu\nu} = \bar{\eta}_{s,n} \pi^{\mu\nu} + \mathcal{O}(2), \quad (110)$$

where we introduced the normalized transport coefficients for each species,

$$\bar{\zeta}_{s,n} = \frac{\zeta_{s,n}}{\zeta}, \quad \bar{\kappa}_{s,n}^{(q)} = \sum_{q'}^{\{B,Q,S\}} \kappa_{s,n,q'} (\kappa^{-1})_{q'q}, \quad \bar{\eta}_{s,n} = \frac{\eta_{s,n}}{\eta}. \quad (111)$$

Here,  $(\kappa^{-1})_{q'q}$  is the inverse of the diffusion-coefficient matrix defined in Eq. (107). We will use Eqs. (108)–(110) to close the equations of motion (83)–(87) in the next section.

#### D. The equations of motion in $(10 + 4N_q)$ -moment approximation

In the  $(10 + 4N_q)$ -moment approximation, we truncate the infinite set of moment equations (83), (85), and (87) in

the following way. We first multiply these equations with the corresponding relaxation-time matrices  $\tau_{si,rr}^{(\ell)}$ . Equation (85) is also multiplied by  $q_s$ , and all equations are summed over species. In what follows, we only consider the set of equations for  $n = 0$ . In all terms, we then substitute  $\rho_{i,r}$ ,  $\rho_{i,r}^\mu$ , and  $\rho_{i,r}^{\mu\nu}$  by the dissipative quantities  $\Pi$ ,  $V_q^\mu$ , and  $\pi^{\mu\nu}$  using Eqs. (108)–(110). Note that in this substitution the  $\mathcal{O}(2)$  terms in Eqs. (108)–(110) become  $\mathcal{O}(3)$  terms and can be neglected together with other higher-order terms. In this way, we finally obtain a closed set of equations of motion for

the dissipative quantities  $\Pi$ ,  $V_q^\mu$ , and  $\pi^{\mu\nu}$ . Further, as discussed in Sec. II D, the  $(10 + 4N_q)$ -moment approximation is the lowest-order truncation of the series in Eq. (51), where  $\ell \leq 2$ , and  $N_0 = 2$ ,  $N_1 = 1$ , and  $N_2 = 0$ . It should be noted that, for  $\ell \leq 2$ , such a truncation in powers of energy (i.e.,  $N_\ell < \infty$ ) neglects infinitely many contributions of order  $\mathcal{O}(\text{Kn})$  in Eq. (51). As shown in Ref. [21], however, systematically increasing the parameters  $N_0$ ,  $N_1$ , and  $N_2$  the values of the corresponding transport coefficients exhibit rapid convergence.

The resulting equation of motion for the bulk viscous pressure is

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} - \sum_q^{\{B,Q,S\}} \ell_{\Pi V}^{(q)} \nabla_\mu V_q^\mu - \sum_q^{\{B,Q,S\}} \tau_{\Pi V}^{(q)} V_q^\mu \dot{u}_\mu - \sum_{q,q'}^{\{B,Q,S\}} \lambda_{\Pi V}^{(q,q')} V_q^\mu \nabla_\mu \alpha_{q'}, \quad (112)$$

where, we have defined the relaxation time and the bulk-viscosity coefficient as

$$\tau_\Pi = - \sum_{s,i=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \sum_{r=0}^{N_0} \tau_{si,0r}^{(0)} \bar{\zeta}_{i,r}, \quad (113)$$

$$\zeta = - \sum_{s=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \zeta_{s,0} \equiv \sum_{s,i=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \sum_{r=0}^{N_0} \tau_{si,0r}^{(0)} \alpha_{i,r}^{(0)}. \quad (114)$$

All second-order transport coefficients appearing in Eq. (112) are listed in Appendix C 1.

Similarly, the equations of motion for the charge diffusion currents read

$$\begin{aligned} \sum_q^{\{B,Q,S\}} \tau_{q'q} \dot{V}_q^{(\mu)} + V_{q'}^\mu &= \sum_q^{\{B,Q,S\}} \kappa_{q'q} \nabla^\mu \alpha_q - \sum_q^{\{B,Q,S\}} \tau_{q'q} V_{q,\nu} \omega^{\nu\mu} - \sum_q^{\{B,Q,S\}} \delta_{VV}^{(q',q)} V_q^\mu \theta - \sum_q^{\{B,Q,S\}} \lambda_{VV}^{(q',q)} V_{q,\nu} \sigma^{\mu\nu} \\ &- \ell_{V\Pi}^{(q')} \nabla^\mu \Pi + \ell_{V\pi}^{(q')} \Delta^{\mu\nu} \nabla_\lambda \pi_\nu^\lambda + \tau_{V\Pi}^{(q')} \Pi \dot{u}^\mu - \tau_{V\pi}^{(q')} \pi^{\mu\nu} \dot{u}_\nu \\ &+ \sum_q^{\{B,Q,S\}} \lambda_{V\Pi}^{(q',q)} \Pi \nabla^\mu \alpha_q - \sum_q^{\{B,Q,S\}} \lambda_{V\pi}^{(q',q)} \pi^{\mu\nu} \nabla_\nu \alpha_q, \end{aligned} \quad (115)$$

where the relaxation-time matrix and the diffusion-coefficient matrix are

$$\tau_{q'q} = \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_1} q_s' \tau_{si,0r}^{(1)} \bar{\kappa}_{i,r}^{(q)}, \quad (116)$$

$$\kappa_{q'q} = \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_1} q_s' \tau_{si,0r}^{(1)} \alpha_{i,r,q}^{(1)}, \quad (117)$$

and the second-order transport coefficients are listed in Appendix C 2.

The diffusion-coefficient matrix  $\kappa_{q'q}$  has been evaluated for several hadronic and partonic systems in Refs. [16–18]. In general, this matrix couples the diffusion current of a specific charge to *all* gradients of the charge chemical potentials via  $V_{q'}^\mu \sim \sum_q^{\{B,Q,S\}} \kappa_{q'q} \nabla^\mu \alpha_q + \mathcal{O}(2)$ . Due to this coupled diffusion, the density gradients in one charge could lead to the local separation in another charge, as demonstrated in Ref. [16]. From the above equations of motion one can see that such a coupling is also present in various second-order terms.

The equation of motion for the shear-stress tensor follows in a similar manner,

$$\begin{aligned} \tau_\pi \dot{\pi}^{(\mu\nu)} + \pi^{\mu\nu} = & 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\lambda^{(\mu} \omega^{\nu)\lambda} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_\pi \pi^{\lambda(\mu} \sigma_\lambda^{\nu)} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} \\ & - \sum_q^{\{B,Q,S\}} \tau_{\pi V}^{(q)} V_q^{(\mu} \dot{u}^{\nu)} + \sum_q^{\{B,Q,S\}} \mathcal{E}_{\pi V}^{(q)} \nabla^{(\mu} V_q^{\nu)} + \sum_{q,q'}^{\{B,Q,S\}} \lambda_{\pi V}^{(q,q')} V_q^{(\mu} \nabla^{\nu)} \alpha_{q'}, \end{aligned} \quad (118)$$

where the relaxation time and the coefficient of the shear viscosity are given by

$$\tau_\pi = \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} \bar{\eta}_{i,r}, \quad (119)$$

$$\eta = \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} \alpha_{i,r}^{(2)}, \quad (120)$$

while the remaining second-order transport coefficients are given in Appendix C 3. The equations of motion (112), (115), and (118) are of relaxation type and are identical to those found in Refs. [10,11,14]. For more details, we refer the reader to the discussion in Appendix A.

As a simple example of a relativistic multicomponent system, we discuss an ultrarelativistic, ideal gas with elastic, isotropic hard-sphere interactions and multiple conserved charges in Appendix F. While the transport coefficients cannot be further reduced to simple and convenient forms, one may easily prove that one obtains well-known results in the single-component limit [21].

#### IV. CONCLUSIONS AND OUTLOOK

In this paper, we have presented the derivation of relativistic second-order dissipative fluid dynamics for multicomponent systems in the  $(10 + 4N_q)$ -moment approximation from the relativistic Boltzmann equation using the method of moments. Starting from the relativistic Boltzmann equation for a multicomponent system we have obtained the equations of motion for the irreducible moments for particle species  $i$ . In the single-fluid approximation for the mixture, the sum of the dynamical equations of motion reduces to  $4 + N_q$  conservation equations that are closed by providing  $6 + 3N_q$  relaxation-type equations of motion for the dissipative quantities. In such a mixture, where the constituents in general carry multiple quantum charges (e.g., a proton carrying electric charge as well as baryon number), the equation of state depends on multiple chemical potentials and temperature. With the help of a new approximation scheme, the so-called order-of-magnitude approximation, we have derived a second-order dissipative theory that does not contain terms of second order in Knudsen number, which are known to render the equations of motion parabolic and thus acausal [21]. Furthermore, the irreducible moments of the deviation of the single-particle distribution of each particle species from equilibrium are

directly proportional to the total bulk viscous pressure  $\Pi$ , the conserved charge-diffusion currents  $V_q^\mu$ , and the total shear-stress tensor  $\pi^{\mu\nu}$  via Eqs. (108)–(110).

Similar to other works which treat multicomponent systems [10,11,14], in this theory the existence of multiple conserved charges is manifest in the equations of motion Eqs. (112), (115), and (118). As expected we obtained exactly the same equations of motion as found in earlier works [10,11,14].

Further, the coupled charge transport becomes explicit in the appearance of mixing terms in the equations of motions, e.g., a dissipative current  $(\Pi, V_q^\mu, \pi^{\mu\nu})$  is coupled to *any other* gradient in chemical potential or diffusion current. As a prominent example, instead of a diffusive Navier-Stokes term with only one diffusion coefficient  $\kappa$  as in a single-component system, we obtain a Navier-Stokes term entailing a matrix of diffusion coefficients  $\kappa_{qq'}$ , which explicitly couples every diffusion current to all gradients in chemical potential. The appearance of a charge-coupled Navier-Stokes term and potential implications for the transport of charge was discussed in Ref. [16] in the case of relativistic nuclear matter.

The advantage of our derivation compared to other theories is that it yields explicit expressions for the transport coefficients in terms of the linearized collision term. Since the mutual interactions of all particle species is contained in the collision term, the multicomponent nature of the mixture is naturally encoded in the transport coefficients.

In the future, this theory will be used to revisit the transport of coupled charge in heavy-ion collisions initiated in Ref. [16] in a more realistic manner. Especially, we expect that it will be relevant for the discussion of physics of compressed baryonic matter at the future FAIR and NICA facilities or for the interpretation of recent results of the isobar run at RHIC. We expect that the coupling of diffusion currents or the charge

gradients to the bulk viscous pressure and the shear-stress tensor may be important in future studies. Now that the explicit expressions of the transport coefficients have been derived, they can be evaluated for nuclear systems. To this end, equations of state from lattice QCD for nonvanishing chemical potentials may be used [34,35]. At the same time, Eq. (51) provides an expression for the so-called  $\delta f$ -correction needed for the freeze-out of the system at the end of the fluid-dynamical phase during the simulation of a heavy-ion collision.

### ACKNOWLEDGMENTS

The authors thank G. S. Denicol and P. Huovinen for fruitful discussions. They acknowledge support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the CRC-TR 211 ‘‘Strong-interaction matter under extreme conditions,’’ Project No. 315477589-TRR 211. J. A. F. acknowledges support from the Helmholtz Graduate School for Heavy-Ion Research. E. M. was also supported by the program Excellence Initiative–Research University of the University of Wrocław of the Ministry of Education and Science. D. H. R. is supported by the State of Hesse within the Research Cluster ELEMENTS: Exploring the Universe from Microscopic to Macroscopic Scales (Project ID 500/10.006). This project is supported by the European Research Council under Project No. ERC-2018-ADG-835105 YoctoLHC. This publication is part of a project that has received funding from the European Union’s Horizon 2020 research and innovation program under Grant Agreement STRONG-2020, No. 824093. This research was funded as a part of the CoE in Quark Matter of the Academy of Finland.

### APPENDIX A: COMPARISON TO OTHER WORKS

In this appendix, we perform a comparison of the second-order relaxation equations found in this paper, Eqs. (112), (115), and (118), to earlier derivations of Monnai and Hirano [10] and Kikuchi, Tsumura, and Kunihiro [14].

Our second-order relaxation equation (112) for the bulk viscosity contains 8 terms in total, while Eq. (69) of Ref. [10] contains 13 terms. Here, we recall this equation noting that  $u_E^\mu = u^\mu$ , while  $J = q$  and  $K = q'$ ,

$$\begin{aligned}
V_J^\mu &= \kappa_{V_J} \nabla^\mu \frac{\mu_J}{T} + \sum_{K \neq J} \kappa_{V_J V_K} \nabla^\mu \frac{\mu_K}{T} - \tau_{V_J} \Delta^{\mu\nu} D V_\nu^J - \sum_{K \neq J} \tau_{V_J V_K} \Delta^{\mu\nu} D V_\nu^K \\
&+ \sum_K \chi_{V_J V_K}^c V_K^\mu \nabla_\nu u_E^\nu + \sum_K \chi_{V_J V_K}^d V_K^\nu \nabla_\nu u_E^\mu + \sum_K \chi_{V_J V_K}^e V_K^\nu \nabla^\mu u_E^\nu \\
&+ \sum_K \chi_{V_J \pi}^{aK} \pi^{\mu\nu} \nabla_\nu \frac{\mu_K}{T} + \chi_{V_J \pi}^c \pi^{\mu\nu} D u_\nu^E + \chi_{V_J \pi}^d \Delta^{\mu\nu} \nabla^\rho \pi_{\nu\rho} + \sum_K \chi_{V_J \Pi}^{aK} \Pi \nabla^\mu \frac{\mu_K}{T} + \chi_{V_J \Pi}^c \Pi D u_E^\mu + \chi_{V_J \Pi}^d \nabla^\mu \Pi \\
&+ \chi_{V_J \pi}^b \pi^{\mu\nu} \nabla_\nu \frac{1}{T} + \chi_{V_J \Pi}^b \Pi \nabla^\mu \frac{1}{T} + \sum_{K,L} \chi_{V_J V_K}^{aL} V_K^\mu D \frac{\mu_L}{T} + \sum_K \chi_{V_J V_K}^b V_K^\mu D \frac{1}{T} + \kappa_{V_J W} \left( \frac{1}{T} D u_E^\mu + \nabla^\mu \frac{1}{T} \right). \quad (A3)
\end{aligned}$$

$$\begin{aligned}
\Pi &= -\zeta \nabla_\mu u_E^\mu - \tau_\Pi D \Pi + \chi_{\Pi\Pi}^c \Pi \nabla_\mu u_E^\mu + \chi_{\Pi\pi} \pi_{\mu\nu} \nabla^{(\mu} u_E^{\nu)} \\
&+ \sum_{J,K} \chi_{\Pi V_J}^{aK} V_J^\mu \nabla_\mu \frac{\mu_K}{T} + \sum_J \chi_{\Pi V_J}^c V_J^\mu D u_E^\mu + \sum_J \chi_{\Pi V_J}^d \nabla^\mu V_J^\mu \\
&+ \sum_J \chi_{\Pi\Pi}^{aJ} \Pi D \frac{\mu_J}{T} + \chi_{\Pi\Pi}^b \Pi D \frac{1}{T} - \zeta_{\Pi\delta e} D \frac{1}{T} \\
&+ \sum_J \zeta_{\Pi\delta n_J} D \frac{\mu_J}{T} + \sum_J \chi_{\Pi V_J}^b V_J^\mu \nabla_\mu \frac{1}{T}. \quad (A1)
\end{aligned}$$

The difference between this equation and ours is due to the difference in the thermodynamic forces and the way the comoving derivatives and space-time four-gradients are employed. In our derivation the comoving derivatives,  $D\beta = D\frac{1}{T}$  and  $D\alpha_q = D\frac{\mu_q}{T}$ , do not appear explicitly since they were replaced using Eqs. (77) and (78), while the space-time four-gradient of the inverse temperature,  $\nabla^\mu \beta$ , is given by Eq. (81). The terms that are expressed differently are in the third line of the above equation (A1). Now, collecting these various terms one can show that Eq. (A1) reduces to Eq. (112).

Similarly, we recall Eq. (77) of Ref. [14], which contains 11 terms,

$$\begin{aligned}
\Pi &= -\zeta \theta - \tau_\Pi \frac{\partial}{\partial \tau} \Pi - \sum_{A=1}^M \ell_{\Pi J}^A \nabla \cdot J_A + \kappa_{\Pi\Pi} \Pi \theta + \kappa_{\Pi\pi} \pi_{\rho\sigma} \sigma^{\rho\sigma} \\
&+ b_{\Pi\Pi\Pi} \Pi^2 + \sum_{A,B=1}^M b_{\Pi J J}^{AB} J_A^\rho J_{B,\rho} + b_{\Pi\pi\pi} \pi^{\rho\sigma} \pi_{\rho\sigma} \\
&+ \sum_{A=1}^M \kappa_{\Pi J}^{(1)A} J_{A,\rho} \nabla^\rho T + \sum_{A,B=1}^M \kappa_{\Pi J}^{(2)BA} J_{A,\rho} \nabla^\rho \frac{\mu_B}{T}, \quad (A2)
\end{aligned}$$

where, we note that  $V_A = V_q$  while  $A = q$  and  $B = q'$ . From these the terms in the second line are of second order in dissipative quantities, i.e., of second order in inverse Reynolds number, originating from the nonlinear part of the collision integral. Note that such second-order terms were also obtained in Refs. [21,36], but are neglected in our study. The remaining 2 terms are formally the same, which can be seen using Eq. (81).

The relaxation equation (115) for the conserved charge current contains 12 terms, while Eq. (70) of Ref. [10] listed below contains 19 terms,

After closer inspection, we observe that the first line contains two sums that are equivalent to our sums over charges, while the additional 4 terms in the last line can be incorporated into already existing terms. Furthermore, the last remaining term,  $\kappa_{V_q W}(\beta D u^\mu + \nabla^\mu \beta)$ , may be expressed using Eq. (81), and hence is fully accounted for in our approach.

On the other hand, Eq. (78) of Ref. [14] contains 14 terms, of which the last 2 are of second order in inverse Reynolds number, while the other terms are formally the same,

$$\begin{aligned}
J_A^\mu &= \sum_{B=1}^M \lambda_{AB} \frac{T^2}{\hbar^2} \nabla^\mu \frac{\mu_B}{T} - \sum_{B=1}^M \tau_J^{AB} \Delta^{\mu\rho} \frac{\partial}{\partial \tau} J_{B,\rho} - \ell_{J\Pi}^A \nabla^\mu \Pi - \ell_{J\pi}^A \Delta^{\mu\rho} \nabla_\nu \pi^\nu{}_\rho \\
&+ \kappa_{J\Pi}^{(1)A} \Pi \nabla^\mu T + \sum_{B=1}^M \kappa_{J\Pi}^{(2)AB} \Pi \nabla^\mu \frac{\mu_B}{T} + \sum_{B=1}^M \kappa_{JJ}^{(1)AB} J_B^\mu \theta + \sum_{B=1}^M \kappa_{JJ}^{(2)AB} J_{B,\rho} \sigma^{\mu\rho} + \kappa_{JJ}^{(3)AB} J_{B,\rho} \omega^{\mu\rho} \\
&+ \kappa_{J\pi}^{(1)A} \pi^{\mu\rho} \nabla_\rho T + \sum_{B=1}^M \kappa_{J\pi}^{(2)AB} \pi^{\mu\rho} \nabla_\rho \frac{\mu_B}{T} + \sum_{B=1}^M b_{\Pi J}^{AB} \Pi J_B^\mu + \sum_{B=1}^M b_{J\pi}^{AB} J_{B,\rho} \pi^{\rho\mu}. \tag{A4}
\end{aligned}$$

The relaxation equation (118) for the shear-stress tensor contains 10 terms, while Eq. (71) of Ref. [10] contains 12 terms

$$\begin{aligned}
\pi^{\mu\nu} &= 2\eta \nabla^{\langle\mu} u_E^{\nu\rangle} - \tau_\pi D \pi^{\langle\mu\nu\rangle} + \chi_{\pi\pi}^c \pi^{\mu\nu} \nabla_\rho u_E^\rho + \chi_{\pi\pi}^d \pi^{\rho\langle\mu} \nabla_\rho u_E^{\nu\rangle} + \chi_{\pi\Pi} \Pi \nabla^{\langle\mu} u_E^{\nu\rangle} \\
&+ \sum_J \chi_{\pi V_J}^c V_J^{\langle\mu} D u_E^{\nu\rangle} + \sum_J \chi_{\pi V_J}^d \nabla^{\langle\mu} V_J^{\nu\rangle} + \sum_{J,K} \chi_{\pi V_J}^{aJ} V_J^{\langle\mu} \nabla^{\nu\rangle} \frac{\mu_K}{T} \\
&+ \sum_J \chi_{\pi V_J}^b V_J^{\langle\mu} \nabla^{\nu\rangle} \frac{1}{T} + \sum_J \chi_{\pi\pi}^{aJ} \pi^{\mu\nu} D \frac{\mu_J}{T} + \chi_{\pi\pi}^b \pi^{\mu\nu} D \frac{1}{T}. \tag{A5}
\end{aligned}$$

Here the last 3 terms that are expressed using  $D\beta$ ,  $D\alpha_q$ , and  $\nabla^\mu \beta$ , may once again be incorporated into other terms.

Finally, Eq. (79) of Ref. [14] contains 13 terms. The last 3 are of second order in inverse Reynolds number, while the remaining 10 terms are formally similar to ours,

$$\begin{aligned}
\pi^{\mu\nu} &= 2\eta \sigma^{\mu\nu} - \tau_\pi \Delta^{\mu\nu\rho\sigma} \frac{\partial}{\partial \tau} \pi_{\rho\sigma} - \sum_{A=1}^M \ell_{\pi J}^A \nabla^{\langle\mu} J_A^{\nu\rangle} + \kappa_{\pi\Pi} \Pi \sigma^{\mu\nu} + \sum_{A=1}^M \kappa_{\pi J}^{(1)A} J_A^{\langle\mu} \nabla^{\nu\rangle} T + \sum_{A,B=1}^M \kappa_{\pi J}^{(2)BA} J_A^{\langle\mu} \nabla^{\nu\rangle} \frac{\mu_B}{T} \\
&+ \kappa_{\pi\pi}^{(1)} \pi^{\mu\nu} \theta + \kappa_{\pi\pi}^{(2)} \pi^{\lambda\langle\mu} \sigma^{\nu\rangle}{}_\lambda + \kappa_{\pi\pi}^{(3)} \pi^{\lambda\langle\mu} \omega^{\nu\rangle}{}_\lambda + b_{\pi\Pi} \Pi \pi^{\mu\nu} + \sum_{A,B=1}^M b_{\pi J}^{AB} J_A^{\langle\mu} J_B^{\nu\rangle} + b_{\pi\pi} \pi^{\lambda\langle\mu} \pi^{\nu\rangle}{}_\lambda. \tag{A6}
\end{aligned}$$

## APPENDIX B: ECKART FRAME

In most textbooks and relevant publications the local rest frame and the fluid four-velocity are chosen according to Eckart [29], since this choice intuitively follows the non-relativistic interpretation of physical quantities. On the other hand, all our results are given relative to the local rest frame of Landau. In this appendix, we will elaborate on the differences.

We may choose to define a different timelike normalized flow vector,  $\tilde{u}^\mu$ , and hence a local frame of reference different from the previously chosen local rest frame (the Landau frame) given by  $u^\mu$ . The  $\tilde{u}$ -frame is related to the  $u$ -frame by a Lorentz transformation. If, we assume that the difference between the frame vectors is small,  $\tilde{u}^\mu - u^\mu \sim \mathcal{O}(1)$ , we may write

$$\tilde{u}^\mu = u^\mu + w^\mu + \mathcal{O}(2). \tag{B1}$$

Computing the normalization of  $\tilde{u}^\mu$  up to order  $\mathcal{O}(1)$ ,

$$\tilde{u}^\mu \tilde{u}_\mu = u^\mu u_\mu + 2u^\mu w_\mu + w^\mu w_\mu = 1 + 2u^\mu w_\mu + \mathcal{O}(2), \tag{B2}$$

and demanding that  $\tilde{u}^\mu$  is also normalized, we conclude that  $w^\mu$  must be orthogonal to  $u^\mu$ ,  $u^\mu w_\mu = 0$ . The projection operator onto the three-space orthogonal to  $\tilde{u}^\mu$  is

$$\begin{aligned}
\tilde{\Delta}^{\mu\nu} &\equiv g^{\mu\nu} - \tilde{u}^\mu \tilde{u}^\nu = \Delta^{\mu\nu} - 2u^{(\mu} w^{\nu)} - w^\mu w^\nu \\
&= \Delta^{\mu\nu} - 2u^{(\mu} w^{\nu)} + \mathcal{O}(2). \tag{B3}
\end{aligned}$$

The tensor decomposition of the primary fluid-dynamical quantities with respect to  $\tilde{u}^\mu$  leads to results similar to the tensor decompositions listed in Eqs. (56)–(57),

$$N_q^\mu \equiv \tilde{n}_q \tilde{u}^\mu + \tilde{V}_q^\mu = \tilde{n}_q (u^\mu + w^\mu) + \tilde{V}_q^\mu, \tag{B4}$$

$$\begin{aligned}
T^{\mu\nu} &\equiv \tilde{e} \tilde{u}^\mu \tilde{u}^\nu - (\tilde{P} + \tilde{\Pi}) \tilde{\Delta}^{\mu\nu} + \tilde{\pi}^{\mu\nu} + 2\tilde{W}^{(\mu} \tilde{u}^{\nu)} \\
&= \tilde{e} u^\mu u^\nu - (\tilde{P} + \tilde{\Pi}) \Delta^{\mu\nu} + \tilde{\pi}^{\mu\nu} + 2(\tilde{e} + \tilde{P}) w^{(\mu} u^{\nu)} \\
&\quad + 2\tilde{W}^{(\mu} u^{\nu)} + \mathcal{O}(2), \tag{B5}
\end{aligned}$$

where we have explicitly applied Landau's matching conditions from Eqs. (67), (68). We have also made use of



Eq. (B1) and the fact that all dissipative quantities are  $\sim \mathcal{O}(1)$ . The physical quantities follow from similar projection operations as in Eqs. (58)–(66).

Furthermore, one can show that neglecting corrections of order  $\mathcal{O}(2)$ , the net-particle density, conserved net-charge density, energy density, equilibrium and bulk pressures, as well the shear-stress components are equal in both local rest frames,

$$\tilde{n}_q = n_q, \quad \tilde{e} = e, \quad \tilde{P} = P, \quad \tilde{\Pi} = \Pi, \quad \tilde{\pi}^{\mu\nu} = \pi^{\mu\nu}. \quad (\text{B6})$$

Note that the quantities without tilde are taken in the Landau frame and are not identical with the corresponding quantities when the frame is not yet specified.

The choice of the local rest frame changes the diffusion currents orthogonal to the flow velocity, i.e., the so-called peculiar velocities. In Eckart's definition of the local rest frame, for *one* conserved charge  $q$  it is required that

$$N_q^\mu \equiv n_q \tilde{u}^\mu \quad \text{and} \quad \tilde{V}_q^\mu \equiv 0, \quad (\text{B7})$$

where  $n_q = N_q^\mu \tilde{u}_\mu$  and  $\tilde{V}_q^\mu = N_q^\nu \tilde{\Delta}_\nu^\mu$ . This means that according to the definition of Eckart, there is no net-charge diffusion current of charge type  $q$  in its own local rest frame. Now, comparing this with the form of  $N_q^\mu$  in the Landau frame leads to

$$N_q^\mu \equiv n_q (u^\mu + w^\mu) \stackrel{!}{=} n_q u^\mu + V_q^\mu. \quad (\text{B8})$$

Thus, for an observer in Landau's local rest frame the particles are diffusing with peculiar four-velocities proportional to the net-charge diffusion current,

$$w^\mu = \frac{V_q^\mu}{n_q}. \quad (\text{B9})$$

Since the dissipative quantity  $V_q^\mu \sim \mathcal{O}(1)$ , this is consistent with our assumption that  $w^\mu \sim \mathcal{O}(1)$ . Up to terms of order  $\mathcal{O}(1)$ , the energy-momentum tensor (B5) reads

$$\begin{aligned} T^{\mu\nu} &\equiv e \tilde{u}^\mu \tilde{u}^\nu - (P + \Pi) \tilde{\Delta}^{\mu\nu} + \pi^{\mu\nu} + 2 \tilde{W}^{(\mu} \tilde{u}^{\nu)} \\ &= e u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu} + 2(e + P) w^{(\mu} u^{\nu)} \\ &\quad + 2 \tilde{W}^{(\mu} u^{\nu)}. \end{aligned} \quad (\text{B10})$$

Using this result together with Eq. (B9) and comparing it to the energy-momentum tensor in the Landau frame,

$$T^{\mu\nu} = e u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}, \quad (\text{B11})$$

leads to the total energy-momentum diffusion current  $\tilde{W}^\mu = \Delta_\alpha^\mu T^{\alpha\beta} u_\beta$  in the Eckart frame,

$$\tilde{W}^\mu \equiv -\frac{e + P}{n_q} V_q^\mu \equiv -h_q V_q^\mu. \quad (\text{B12})$$

This relates the total energy-momentum flux seen in the Eckart frame to the diffusion flux of charge  $q$  observed in the Landau frame.

Finally, we give the four-flow of the charges  $q'$ , of which the diffusion currents were *not* chosen to vanish by Eckart's choice (B7) (i.e.,  $q' \neq q$ ). Using the fact that the charge densities  $n_{q'}$  are equal in both frames, we obtain the condition

$$N_{q'}^\mu = n_{q'} (u^\mu + w^\mu) + \tilde{V}_{q'}^\mu \stackrel{!}{=} n_{q'} u^\mu + V_{q'}^\mu. \quad (\text{B13})$$

Employing Eq. (B9), we arrive at an expression for the diffusion currents as observed in the Eckart frame:

$$\tilde{V}_{q'}^\mu = V_{q'}^\mu - \frac{n_{q'}}{n_q} V_q^\mu. \quad (\text{B14})$$

Note that for the case  $q = q'$ , we again recover the requirement by the Eckart frame definition for the charge  $q$ , Eq. (B7).

## APPENDIX C: TRANSPORT COEFFICIENTS

In this appendix, we list all second-order transport coefficients in the equations of motion (112), (115), and (118).

### 1. The coefficients in the bulk viscosity equation

The second-order transport coefficients in the equation of motion (112) for the bulk viscous pressure are

$$\begin{aligned} \delta_{\Pi\Pi} &= \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \frac{m_s^2}{9} \tau_{si,0r}^{(0)} \left[ m_i^2 (r-1) \bar{\zeta}_{i,r-2} - (r+2) \bar{\zeta}_{i,r} - 3 \left( J_{i,r+1,0} \mathcal{T}_{00} - \sum_q^{\{B,Q,S\}} q_i J_{i,r,0} \mathcal{T}_{q0} \right) \right] \\ &\quad + \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \frac{m_s^2}{3} \tau_{si,nr}^{(0)} \sum_q^{\{B,Q,S\}} \frac{\partial \bar{\zeta}_{i,r}}{\partial \alpha_q} \left[ \mathcal{T}_{q0} (e + P) + \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{qq'} n_{q'} \right] \\ &\quad + \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \frac{m_s^2}{3} \tau_{si,nr}^{(0)} \frac{\partial \bar{\zeta}_{i,r}}{\partial \beta} \left[ \mathcal{T}_{00} (e + P) + \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{0q'} n_{q'} \right], \end{aligned} \quad (\text{C1})$$

$$\lambda_{\Pi r} = - \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \frac{m_s^2}{3} \tau_{si,0r}^{(0)} \left[ (r-1) \bar{\eta}_{i,r-2} + J_{i,r+1,0} \mathcal{T}_{00} - \sum_q^{\{B,Q,S\}} q_i J_{i,r,0} \mathcal{T}_{q0} \right], \quad (\text{C2})$$

$$\mathcal{E}_{\text{PIV}}^{(q)} = - \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \frac{m_s^2}{3} \tau_{si,0r}^{(0)} \left( \bar{\kappa}_{i,r-1}^{(q)} + J_{i,r+1,0} \mathcal{T}_{0q} - \sum_{q'}^{\{B,Q,S\}} q'_i J_{i,r,0} \mathcal{T}_{q'q} \right), \quad (\text{C3})$$

$$\tau_{\text{PIV}}^{(q)} = \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \frac{m_s^2}{3} \tau_{si,0r}^{(0)} \left( r \bar{\kappa}_{i,r-1}^{(q)} + \beta \frac{\partial \bar{\kappa}_{i,r-1}^{(q)}}{\partial \beta} + J_{i,r+1,0} \mathcal{T}_{0q} - \sum_{q'}^{\{B,Q,S\}} q'_i J_{i,r,0} \mathcal{T}_{q'q} \right), \quad (\text{C4})$$

$$\lambda_{\text{PIV}}^{(q,q')} = - \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_0} \frac{m_s^2}{3} \tau_{si,0r}^{(0)} \left( \frac{\partial \bar{\kappa}_{i,r-1}^{(q)}}{\partial \alpha_{q'}} + \frac{n_{q'}}{e+P} \frac{\partial \bar{\kappa}_{i,r-1}^{(q)}}{\partial \beta} \right). \quad (\text{C5})$$

## 2. The coefficients in the charge diffusion equations

The coefficients in the equations of motion (115) for the charge diffusion currents are

$$\begin{aligned} \delta_{VV}^{(q',q)} = & - \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \frac{1}{3} [m_i^2 (r-1) \bar{\kappa}_{i,r-2}^{(q)} - (r+3) \bar{\kappa}_{i,r}^{(q)}] \\ & - \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \sum_{q''}^{\{B,Q,S\}} \frac{\partial \bar{\kappa}_{i,r}^{(q)}}{\partial \alpha_{q''}} \left[ \mathcal{T}_{q''0} (e+P) + \sum_{q'''}^{\{B,Q,S\}} \mathcal{T}_{q''q'''} n_{q'''} \right] \\ & - \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \frac{\partial \bar{\kappa}_{i,r}^{(q)}}{\partial \beta} \left[ \mathcal{T}_{00} (e+P) + \sum_{q''}^{\{B,Q,S\}} \mathcal{T}_{0q''} n_{q''} \right], \end{aligned} \quad (\text{C6})$$

$$\lambda_{VV}^{(q',q)} = - \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \frac{1}{5} [m_i^2 (2r-2) \bar{\kappa}_{i,r-2}^{(q)} - (2r+3) \bar{\kappa}_{i,r}^{(q)}], \quad (\text{C7})$$

$$\mathcal{E}_{\text{VII}}^{(q')} = \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \left[ \frac{1}{3} (m_i^2 \bar{\zeta}_{i,r-1} - \bar{\zeta}_{i,r+1}) - \alpha_{i,r}^h \right], \quad (\text{C8})$$

$$\mathcal{E}_{V\pi}^{(q')} = - \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} (\bar{\eta}_{i,r-1} + \alpha_{i,r}^h), \quad (\text{C9})$$

$$\tau_{\text{VII}}^{(q')} = \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \frac{1}{3} \left[ m_i^2 r \bar{\zeta}_{i,r-1} - (r+3) \bar{\zeta}_{i,r+1} - 3\alpha_{i,r}^h + m_i^2 \beta \frac{\partial \bar{\zeta}_{i,r-1}}{\partial \beta} - \beta \frac{\partial \bar{\zeta}_{i,r+1}}{\partial \beta} \right], \quad (\text{C10})$$

$$\tau_{V\pi}^{(q')} = - \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \left( r \bar{\eta}_{i,r-1} + \beta \frac{\partial \bar{\eta}_{i,r-1}}{\partial \beta} + \alpha_{i,r}^h \right), \quad (\text{C11})$$

$$\lambda_{\text{VII}}^{(q',q)} = - \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \frac{1}{3} \left[ m_i^2 \left( \frac{\partial \bar{\zeta}_{i,r-1}}{\partial \alpha_q} + \frac{n_q}{e+P} \frac{\partial \bar{\zeta}_{i,r-1}}{\partial \beta} \right) - \frac{\partial \bar{\zeta}_{i,r+1}}{\partial \alpha_q} - \frac{n_q}{e+P} \frac{\partial \bar{\zeta}_{i,r+1}}{\partial \beta} \right], \quad (\text{C12})$$

$$\lambda_{V\pi}^{(q',q)} = \sum_{s,i=1}^{N_{\text{spec}}} q'_s \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \left( \frac{\partial \bar{\eta}_{i,r-1}}{\partial \alpha_q} + \frac{n_q}{e+P} \frac{\partial \bar{\eta}_{i,r-1}}{\partial \beta} \right). \quad (\text{C13})$$

### 3. The coefficients in the shear-stress equation

The coefficients in the equation of motion (118) for the shear-stress tensor are

$$\delta_{\pi\pi} = -\frac{1}{3} \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} [m_i^2(r-1)\bar{\eta}_{i,r-2} - (r+4)\bar{\eta}_{i,r}] - \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} \left\{ \sum_q^{\{B,Q,S\}} \frac{\partial \bar{\eta}_{i,r}}{\partial \alpha_q} \left[ \mathcal{T}_{q0}(e+P) + \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{qq'} n_{q'} \right] + \frac{\partial \bar{\eta}_{i,r}}{\partial \beta} \left[ \mathcal{T}_{00}(e+P) + \sum_{q'}^{\{B,Q,S\}} \mathcal{T}_{0q'} n_{q'} \right] \right\}, \quad (\text{C14})$$

$$\tau_{\pi\pi} = -\frac{2}{7} \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} [m_i^2(2r-2)\bar{\eta}_{i,r-2} - (2r+5)\bar{\eta}_{i,r}], \quad (\text{C15})$$

$$\lambda_{\pi\Pi} = \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} \frac{2}{15} [m_i^4(r-1)\bar{\zeta}_{i,r-2} - (2r+3)m_i^2\bar{\zeta}_{i,r} + (r+4)\bar{\zeta}_{i,r+2}], \quad (\text{C16})$$

$$\tau_{\pi V}^{(q)} = -\frac{2}{5} \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} \left[ m_i^2 r \bar{\kappa}_{i,r-1}^{(q)} - (r+5)\bar{\kappa}_{i,r+1}^{(q)} + m_i^2 \beta \frac{\partial \bar{\kappa}_{i,r-1}^{(q)}}{\partial \beta} - \beta \frac{\partial \bar{\kappa}_{i,r+1}^{(q)}}{\partial \beta} \right], \quad (\text{C17})$$

$$\varrho_{\pi V}^{(q)} = -\frac{2}{5} \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} (m_i^2 \bar{\kappa}_{i,r-1}^{(q)} - \bar{\kappa}_{i,r+1}^{(q)}), \quad (\text{C18})$$

$$\lambda_{\pi V}^{(q,q')} = -\frac{2}{5} \sum_{s,i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} \left[ m_i^2 \left( \frac{\partial \bar{\kappa}_{i,r-1}^{(q)}}{\partial \alpha_{q'}} + \frac{n_{q'}}{e+P} \frac{\partial \bar{\kappa}_{i,r-1}^{(q)}}{\partial \beta} \right) - \frac{\partial \bar{\kappa}_{i,r+1}^{(q)}}{\partial \alpha_{q'}} - \frac{n_{q'}}{e+P} \frac{\partial \bar{\kappa}_{i,r+1}^{(q)}}{\partial \beta} \right]. \quad (\text{C19})$$

### APPENDIX D: COMPARISON TO A SINGLE-COMPONENT FLUID

In previous works the transport coefficients in the equations of motion were calculated in a similar manner as explained in this paper. In order to facilitate a comparison and for the sake of completeness, we recall the notational convention and provide some basic relations.

For an arbitrary function of energy,  $F(E_{i,\mathbf{k}})$ , the irreducible tensors satisfy the following orthogonality condition [22]:

$$\int dK_i F(E_{i,\mathbf{k}}) k_i^{\mu_1} \dots k_i^{\mu_\ell} k_i^{\nu_1} \dots k_i^{\nu_n} = \frac{\ell! \delta_{\ell n}}{(2\ell+1)!!} \Delta^{\mu_1 \dots \mu_\ell \nu_1 \dots \nu_n} \int dK_i F(E_{i,\mathbf{k}}) (\Delta_{\alpha\beta} k_i^\alpha k_i^\beta)^\ell. \quad (\text{D1})$$

Therefore, for a given species  $i$ , any irreducible moment of tensor rank  $\ell$  of arbitrary order  $r \leq 0$  may be expressed as a linear combination of irreducible moments of the same tensor rank  $\ell$ , but with different power of energy  $n$  as

$$\rho_{i,r}^{\mu_1 \dots \mu_\ell} = \sum_{n=0}^{N_\ell} \rho_{i,n}^{\mu_1 \dots \mu_\ell} \mathcal{F}_{i,-r,n}^{(\ell)}, \quad (\text{D2})$$

where for  $r, n \geq 0$ ,  $\mathcal{F}_{-r,n}^{(\ell)} = \delta_{rn}$ . Therefore, for  $r \rightarrow -r$ , we obtain

$$\rho_{i,-r}^{\mu_1 \dots \mu_\ell} = \sum_{n=0}^{N_\ell} \rho_{i,n}^{\mu_1 \dots \mu_\ell} \mathcal{F}_{i,r,n}^{(\ell)}, \quad (\text{D3})$$

where using Eqs. (21) and (52) we defined the following coefficient similar to Eq. (66) of Ref. [21]:

$$\begin{aligned} \mathcal{F}_{i,\pm r,n}^{(\ell)} &\equiv \frac{\ell!}{(2\ell+1)!!} \int dK_i E_{i,\mathbf{k}}^{\mp r} \mathcal{H}_{i,\mathbf{k}n}^{(\ell)} (\Delta_{\alpha\beta} k_i^\alpha k_i^\beta)^\ell f_{i,\mathbf{k}}^{(0)} \tilde{f}_{i,\mathbf{k}}^{(0)} \\ &= \sum_{n'=n}^{N_\ell} \sum_{m=0}^{n'} \frac{J_{i,\mp r+m+2\ell,\ell}}{J_{i,2\ell,\ell}} a_{i,n'n}^{(\ell)} a_{i,n'm}^{(\ell)}. \end{aligned} \quad (\text{D4})$$

Therefore, using these results one can also show that the expansion coefficients are related as

$$\sum_{m=0}^{n'} \frac{J_{i,r+m+2\ell,\ell}}{J_{i,2\ell,\ell}} a_{i,n'}^{(\ell)} a_{i,n'm}^{(\ell)} = \delta_{rn'} \delta_{rn}, \quad (\text{D5})$$

which in the case of a single-component system is equivalent to the matrix equation provided in Appendix E of Ref. [21].

Truncating these expressions in the  $(10 + 4N_q)$ -moment approximation, hence using Eq. (D3) with the summation limits  $N_0 = 2$ ,  $N_1 = 1$ ,  $N_2 = 0$  for the various tensor ranks, we obtain the following relations:

$$\rho_{i,-r} \equiv \sum_{n=0}^{N_0} \rho_{i,n} \mathcal{F}_{i,r,n}^{(0)} \approx -\frac{3}{m_i^2} \Pi_i \mathcal{F}_{i,r,0}^{(0)} + \rho_{i,1} \mathcal{F}_{i,r,1}^{(0)} + \rho_{i,2} \left( \frac{1}{m_i^2} \mathcal{F}_{i,r,0}^{(0)} + \mathcal{F}_{i,r,2}^{(0)} \right), \quad (\text{D6})$$

$$\rho_{i,-r}^\mu \equiv \sum_{n=0}^{N_1} \rho_{i,n}^\mu \mathcal{F}_{i,r,n}^{(1)} \approx V_i^\mu \mathcal{F}_{i,r,0}^{(1)} + W_i^\mu \mathcal{F}_{i,r,1}^{(1)}, \quad (\text{D7})$$

$$\rho_{i,-r}^{\mu\nu} \equiv \sum_{n=0}^{N_2} \rho_{i,n}^{\mu\nu} \mathcal{F}_{i,r,n}^{(2)} \approx \pi_i^{\mu\nu} \mathcal{F}_{i,r,0}^{(2)}. \quad (\text{D8})$$

Furthermore, using Eqs. (D6)–(D8) for irreducible moments with positive  $r$  we obtain similar relations by replacing  $-r \rightarrow r$ .

On the other hand, summing Eqs. (D6)–(D8) over species, the irreducible moments of the mixture also lead to the expressions for a single-component fluid in the Landau frame,

$$\rho_{-r} \equiv \sum_{i=1}^{N_{\text{spec}}} \rho_{i,-r} = \sum_{i=1}^{N_{\text{spec}}} \sum_{n=0}^{N_0} \rho_{i,n} \mathcal{F}_{i,r,n}^{(0)} \xrightarrow{\text{single}} -\frac{3}{m^2} \Pi \gamma_r^{(0)} + \mathcal{O}(1), \quad (\text{D9})$$

$$\rho_{-r}^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} \rho_{i,-r}^\mu = \sum_{i=1}^{N_{\text{spec}}} \sum_{n=0}^{N_1} \rho_{i,n}^\mu \mathcal{F}_{i,r,n}^{(1)} \xrightarrow{\text{single}} V^\mu \gamma_r^{(1)} + \mathcal{O}(1), \quad (\text{D10})$$

$$\rho_{-r}^{\mu\nu} \equiv \sum_{i=1}^{N_{\text{spec}}} \rho_{i,-r}^{\mu\nu} = \sum_{i=1}^{N_{\text{spec}}} \sum_{n=0}^{N_2} \rho_{i,n}^{\mu\nu} \mathcal{F}_{i,r,n}^{(2)} \xrightarrow{\text{single}} \pi^{\mu\nu} \gamma_r^{(2)} + \mathcal{O}(1), \quad (\text{D11})$$

where the coefficients  $\gamma_r^{(\ell)}$  are listed in Ref. [21] for the case of a single-component fluid,  $i = N_{\text{spec}} = 1$ .

Notice that moments with negative power of energy are expressed as a linear combination of moments with positive  $r$  which represent the coupling between moments even for simple fluids with a single conserved charge. On the other hand in mixtures the summations over all particle species lead to further couplings, which renders the above expressions rather difficult to use. To circumvent this we have introduced the order-of-magnitude approximation in Sec. III C to express the negative moments and their sums.

Furthermore, in order to compare the irreducible moments to Eqs. (108)–(110), we have to demand

$$\Pi \bar{\zeta}_{1,-r} = \Pi \gamma_r^{(0)}, \quad \pi^{\mu\nu} \bar{\eta}_{1,-r} = \pi^{\mu\nu} \gamma_r^{(2)}, \quad (\text{D12})$$

and since we are dealing with a single charge, say  $q$ , then  $V_q^\mu = qV^\mu$  and hence

$$V_q^{\mu} \bar{\kappa}_{1,-r}^{(q)} = qV^\mu \gamma_r^{(1)}. \quad (\text{D13})$$

## APPENDIX E: PERFORMING THE COLLISION INTEGRALS

In order to evaluate the transport coefficients first we need to calculate the irreducible moments (90) of the collision term (2). These moments are related to the entries of the collision matrix defined in Eqs. (91)–(92). For the sake of convenience we define the following tensor, similarly as in Ref. [21]:

$$\mathcal{X}_{sijab,r}^{\mu_1 \dots \mu_\ell \nu_1 \dots \nu_{\ell+m}} \equiv \mathcal{L}_{sijab,r}^{\mu_1 \dots \mu_\ell \nu_1 \dots \nu_{\ell+m}} - \mathcal{G}_{sijab,r}^{\mu_1 \dots \mu_\ell \nu_1 \dots \nu_{\ell+m}}, \quad (\text{E1})$$

where the loss term is

$$\mathcal{L}_{sijab,r}^{\mu_1 \dots \mu_\ell \nu_1 \dots \nu_{\ell+m}} \equiv \int dK_i dK'_j f_{i,\mathbf{k}}^{(0)} f_{j,\mathbf{k}'}^{(0)} E_{i,\mathbf{k}}^{r-1} k_i^{\mu_1} \dots k_i^{\mu_\ell} (k_i^{\nu_1} \dots k_i^{\nu_{\ell+m}} \delta_{is} + k_j^{\nu_1} \dots k_j^{\nu_{\ell+m}} \delta_{js}) \int dP_a dP'_b W_{ij \rightarrow ab}^{kk' \rightarrow pp'} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)}, \quad (\text{E2})$$

while the gain term is

$$\mathcal{G}_{sijab,r}^{\mu_1 \dots \mu_\ell \nu_1 \dots \nu_{\ell+m}} \equiv \int dK_i dK'_j f_{i,\mathbf{k}}^{(0)} f_{j,\mathbf{k}'}^{(0)} E_{i,\mathbf{k}}^{r-1} k_i^{\mu_1} \dots k_i^{\mu_\ell} \int dP_a dP'_b W_{ij \rightarrow ab}^{kk' \rightarrow pp'} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)} (p_a^{\nu_1} \dots p_a^{\nu_{\ell+m}} \delta_{as} + p_b^{\nu_1} \dots p_b^{\nu_{\ell+m}} \delta_{bs}). \quad (\text{E3})$$

Therefore, once we have evaluated the corresponding  $\mathcal{X}$ -tensors, the elements of the collision matrix  $\mathcal{A}_{is, rn}^{(\ell)}$  in Eq. (92) can be obtained by calculating the following projections:

$$\mathcal{A}_{is, rn}^{(\ell)} = \frac{1}{2} \sum_{n'=n}^{N_\ell} \sum_{m=0}^{n'} \frac{(-1)^\ell a_{s, n' n}^{(\ell)} a_{s, n' m}^{(\ell)}}{\ell! (2\ell + 1) J_{s, 2\ell, \ell}} \sum_{j, a, b=1}^{N_{\text{spec}}} u_{\nu_{\ell+1}} \dots u_{\nu_{\ell+m}} \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} (\mathcal{X}_{sijab, r})_{\mu_1 \dots \mu_\ell}^{\nu_1 \dots \nu_{\ell+m}}. \quad (\text{E4})$$

In order to evaluate the  $dP_a dP'_b$  integrals of the loss term it is useful to choose the center-of-momentum (CM) frame to perform the integration over the transition rate  $W_{ij \rightarrow ab}^{kk' \rightarrow pp'}$ . The total momentum involved in binary collisions,  $P_T^\mu \equiv k_i^\mu + k_j^\mu$ , defines the Mandelstam variable

$$s \equiv (k_i^\mu + k_j^\mu)^2 = (p_a^\mu + p_b^\mu)^2 \equiv P_T^\mu P_{T, \mu}. \quad (\text{E5})$$

The CM frame is defined such that

$$\sqrt{s} \equiv k_i^0 + k_j^0 = p_a^0 + p_b^0 \equiv P_T^0, \quad (\text{E6})$$

$$0 \equiv \mathbf{k}_i + \mathbf{k}_j = \mathbf{p}_a + \mathbf{p}_b \equiv \mathbf{P}_T. \quad (\text{E7})$$

In the following, we use the following substitutions:

$$x \equiv p_a^0 + p_b^0, \quad \frac{dx}{x} = \frac{|\mathbf{p}| d|\mathbf{p}|}{p_a^0 p_b^0}, \quad (\text{E8})$$

and

$$|\mathbf{p}| = \frac{1}{2x} \sqrt{(x^2 - m_+)(x^2 - m_-)}, \quad (\text{E9})$$

where  $|\mathbf{p}| = |\mathbf{p}_a| = |\mathbf{p}'_b|$ , and  $m_\pm \equiv (m_a \pm m_b)^2$ . Therefore, the second integral in Eq. (E2) leads to

$$\mathcal{P}_{ab} \equiv \int dP_a dP'_b W_{ij \rightarrow ab}^{kk' \rightarrow pp'} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)} = \frac{1}{2} \frac{1}{16(2\pi)^2} |\bar{\mathcal{M}}_{ij \rightarrow ab}|^2 \tilde{f}_{a, \sqrt{s}}^{(0)} \tilde{f}_{b, \sqrt{s}}^{(0)} \sqrt{\left(1 - \frac{m_+}{s}\right) \left(1 - \frac{m_-}{s}\right)}, \quad (\text{E10})$$

where, we introduced the notation

$$\tilde{f}_{i, \sqrt{s}}^{(0)} \equiv 1 - \frac{a_i}{\exp\left(\beta \sqrt{m_i^2 + \frac{s}{4} \left(1 - \frac{m_+}{s}\right) \left(1 - \frac{m_-}{s}\right)} - \alpha_i\right) + a_i}. \quad (\text{E11})$$

Now using this result the remaining integral in Eq. (E2) can be calculated. For later use we define the angle-integrated transition probability:

$$|\bar{\mathcal{M}}_{ij \rightarrow ab}(\sqrt{s})|^2 \equiv \int_{S^2} d\Omega |\mathcal{M}_{ij \rightarrow ab}(\sqrt{s}, \Omega)|^2 = 2\pi \int_0^\pi d\vartheta \sin \vartheta |\mathcal{M}_{ij \rightarrow ab}(\sqrt{s}, \vartheta)|^2, \quad (\text{E12})$$

where  $\vartheta$  is the scattering angle in the reaction plane defined as

$$\cos \vartheta = \frac{(k^\mu - k'^\mu)(p_\mu - p'_\mu)}{(k^\mu - k'^\mu)^2}. \quad (\text{E13})$$



The integral in Eq. (E3) is more tedious. Here, we restrict ourselves to isotropic scattering processes, hence all integrals only depend on the normalized total momentum  $\tilde{P}_T^\mu \equiv (k_i^\mu + k_j^\mu)/\sqrt{s}$ . We also introduce the corresponding projection operator  $\Delta_P^{\mu\nu} = g^{\mu\nu} - \tilde{P}_T^\mu \tilde{P}_T^\nu$  orthogonal to the total momentum,  $\Delta_P^{\mu\nu} \tilde{P}_{T,\nu} = 0$ . Therefore, similarly to the thermodynamic integrals in Eq. (18), we decompose the integrals in terms of the normalized total momentum and the associated orthogonal projection operator,

$$\begin{aligned} \Theta_a^{\nu_1 \dots \nu_n} &\equiv \int dP_a dP'_b W_{ij \rightarrow pp'}^{kk'} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)} P_a^{\nu_1} \dots P_a^{\nu_n} \\ &= \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{2^{-m} n!}{m!(n-2m)!} \Delta_P^{(\nu_1 \nu_2} \dots \Delta_P^{\nu_{2m-1} \nu_{2m}} \tilde{P}_T^{\nu_{2m+1}} \dots \tilde{P}_T^{\nu_n)} C_{a,nm}, \end{aligned} \quad (\text{E14})$$

where the coefficients are

$$\begin{aligned} C_{a,nm} &\equiv \frac{(-1)^m}{(2m+1)!!} \int dP_a dP'_b W_{ij \rightarrow pp'}^{kk'} \tilde{f}_{a,\mathbf{p}}^{(0)} \tilde{f}_{b,\mathbf{p}'}^{(0)} (\Delta_{P,\alpha\beta} P_a^\alpha P_a^\beta)^m (\tilde{P}_{T,\alpha} P_a^\alpha)^{n-2m} \\ &= \frac{(-1)^m}{(2m+1)!!} \frac{1}{16(2\pi)^2} |\bar{\mathcal{M}}_{ij \rightarrow ab}|^2 \tilde{f}_{a,\sqrt{s}}^{(0)} \tilde{f}_{b,\sqrt{s}}^{(0)} \frac{s^{n/2}}{2^{n+1}} \left[ \left(1 - \frac{m_+}{s}\right) \left(1 - \frac{m_-}{s}\right) \right]^{\frac{2m+1}{2}} \\ &\quad \times \left[ \frac{4m_a^2}{s} + \left(1 - \frac{m_+}{s}\right) \left(1 - \frac{m_-}{s}\right) \right]^{\frac{n-2m}{2}}. \end{aligned} \quad (\text{E15})$$

## APPENDIX F: CLASSICAL, ULTRARELATIVISTIC SYSTEM WITH HARD-SPHERE INTERACTIONS IN $(10 + 4N_q)$ -MOMENT APPROXIMATION

In this appendix we evaluate the transport coefficients of the theory in the  $(10 + 4N_q)$ -moment approximation ( $N_0 = 2$ ,  $N_1 = 1$ , and  $N_2 = 0$ ) for a classical ( $a_i \rightarrow 0$ ), ultrarelativistic ( $m_i/T \rightarrow 0$ ) multicomponent gas with elastic hard-sphere interactions, for which the transition rate is given by Eq. (11). Equivalently, in Eqs. (E10) and (E15) we can just replace

$$\frac{1}{16(2\pi)^2} |\bar{\mathcal{M}}_{ij \rightarrow ab}|^2 = (\delta_{ia} \delta_{jb} + \delta_{ib} \delta_{ja}) s \sigma_{\text{tot}}, \quad (\text{F1})$$

with  $\sigma_{\text{tot},ij} \equiv \sigma_{\text{tot}} = \text{const}$ . With this replacement, we obtain for Eqs. (E10) and (E15)

$$\mathcal{P}_{ab} = \frac{1}{2} (\delta_{ia} \delta_{jb} + \delta_{ib} \delta_{ja}) s \sigma_{\text{tot}}, \quad (\text{F2})$$

$$C_{a,nm} = \frac{1}{2} (\delta_{ia} \delta_{jb} + \delta_{ib} \delta_{ja}) \frac{(-1)^m}{(2m+1)!!} \frac{s^{(n+2)/2}}{2^n} \sigma_{\text{tot}}. \quad (\text{F3})$$

Furthermore, according to Eqs. (47),  $P = \sum_{i=1}^{N_{\text{spec}}} P_i$  is the total pressure of the system, and it fulfills the ideal gas laws  $e = 3P$  and  $n_i = \beta P_i$ .

### 1. Collision matrix and its inverse

In the following we evaluate the collision matrix (E4) for the vector and tensor moments. The scalar moments play no role since their transport coefficients are proportional to mass, and therefore vanish in the ultrarelativistic case. In these calculations we make use of the ultrarelativistic limit of Eq. (E5), leading to  $s = 2k_{i,\mu} k_j^\mu$ .

Furthermore, in the classical limit, we express the thermodynamic integrals in terms of the partial pressure of the respective species,

$$J_{i,nm} = I_{i,nm} = \frac{1}{2} \frac{(n+1)!}{(2m+1)!!} \frac{1}{\beta^{n-2}} P_i, \quad (\text{F4})$$

where the thermodynamic integrals are related by

$$I_{i,n+2,m} = m_i^2 I_{i,nm} + (2m+3) I_{i,n+2,m+1}. \quad (\text{F5})$$

Following Eq. (D5) the relevant expansion coefficients in the  $(10 + 4N_q)$ -moment approximation are

$$a_{i,0}^{(\ell)} = 1, \quad \forall \ell \geq 0, \quad (\text{F6})$$

$$a_{i,11}^{(1)} = \frac{\beta}{2}, \quad a_{i,10}^{(1)} = -\frac{4}{\beta} a_{i,11}^{(1)} = -2. \quad (\text{F7})$$

### a. Tensor moments ( $\ell = 2$ )

The only relevant contraction of the  $\mathcal{X}$ -tensor reads

$$\sum_{j,a,b=1}^{N_{\text{spec}}} \Delta_{\mu_1\mu_2\nu_1\nu_2} \mathcal{X}_{sijab,r}^{\mu_1\mu_2\nu_1\nu_2} = \frac{4}{9} \sigma_{\text{tot}} \sum_{j=1}^{N_{\text{spec}}} \left[ (2\delta_{is} - \delta_{js}) I_{i,r+4,0} I_{j,10} + \frac{2}{3} (\delta_{is} + \delta_{js}) I_{i,r+3,0} I_{j,20} \right], \quad (\text{F8})$$

and therefore the elements of the collision matrix (E4) for the tensor moments read in  $(10 + 4N_q)$ -moment approximation (note that  $N_2 = 0$ )

$$\mathcal{A}_{is,00}^{(2)} = \frac{1}{2} \frac{a_{s,00}^{(2)} a_{s,00}^{(2)}}{10I_{s,42}} \sum_{j,a,b=1}^{N_{\text{spec}}} \Delta_{\mu_1\mu_2\nu_1\nu_2} \mathcal{X}_{sijab,0}^{\mu_1\mu_2\nu_1\nu_2} = \frac{\sigma_{\text{tot}}}{45I_{s,42}} \sum_{j=1}^{N_{\text{spec}}} \left[ (2\delta_{is} - \delta_{js}) I_{i,40} I_{j,10} + \frac{2}{3} (\delta_{is} + \delta_{js}) I_{i,30} I_{j,20} \right]. \quad (\text{F9})$$

Expressing this in terms of the pressure, we obtain

$$\mathcal{A}_{is,00}^{(2)} = \frac{\sigma_{\text{tot}} \beta}{5} \sum_{j=1}^{N_{\text{spec}}} \frac{P_i P_j}{P_s} (4\delta_{is} - \delta_{js}) = \frac{\sigma_{\text{tot}} \beta}{5} (4\delta_{is} P - P_i), \quad (\text{F10})$$

and in the single-component limit ( $N_{\text{spec}} \rightarrow 1$ ), where  $P_i \equiv P_0 = n_0/\beta$ , we reproduce the result from Ref. [21]:

$$\mathcal{A}_{11,00}^{(2)} \rightarrow \mathcal{A}_{00}^{(2)} = \frac{3}{5} \sigma_{\text{tot}} n_0. \quad (\text{F11})$$

Here, Eq. (F10) defines the entries of an  $N_{\text{spec}}$ -dimensional rectangular, regular matrix. The elements of its inverse are

$$\tau_{si,00}^{(2)} = \frac{5}{\sigma_{\text{tot}} \beta} \frac{1}{12P^2} (3\delta_{si} P + P_s). \quad (\text{F12})$$

### b. Vector moments ( $\ell = 1$ )

The relevant contractions of the  $\mathcal{X}$ -tensor for the vector moments are

$$\sum_{j,a,b=1}^{N_{\text{spec}}} \Delta_{\mu_1\nu_1} \mathcal{X}_{sijab,r}^{\mu_1\nu_1} = \sigma_{\text{tot}} \sum_{j=1}^{N_{\text{spec}}} (\delta_{js} - \delta_{is}) \left( \frac{1}{3} I_{i,r+1,0} I_{j,20} + I_{i,r+2,0} I_{j,10} \right), \quad (\text{F13})$$

$$\sum_{j,a,b=1}^{N_{\text{spec}}} \Delta_{\mu_1\nu_1} u_{\nu_2} \mathcal{X}_{sijab,r}^{\mu_1\nu_1\nu_2} = \sigma_{\text{tot}} \sum_{j=1}^{N_{\text{spec}}} \left[ \frac{2}{9} (2\delta_{js} - \delta_{is}) I_{i,r+1,0} I_{j,30} + \frac{4}{9} (\delta_{js} + \delta_{is}) I_{i,r+2,0} I_{j,20} + \frac{6}{9} (\delta_{js} - 2\delta_{is}) I_{i,r+3,0} I_{j,10} \right], \quad (\text{F14})$$

thus the elements of the collision matrix (E4) read in the  $(10 + 4N_q)$ -moment approximation (where  $N_1 = 1$ ):

$$\begin{aligned} \mathcal{A}_{is,r0}^{(1)} &= -\frac{1}{2} \sum_{n'=0}^1 \sum_{m=0}^{n'} \frac{a_{s,n'0}^{(1)} a_{s,n'm}^{(1)}}{3I_{s,21}} \sum_{j,a,b=1}^{N_{\text{spec}}} u_{\nu_2} \cdots u_{\nu_{1+m}} \Delta_{\mu_1\nu_1} \mathcal{X}_{sijab,r}^{\mu_1\nu_1 \cdots \nu_{1+m}} \\ &= -\frac{\sigma_{\text{tot}}}{6I_{s,21}} \sum_{j=1}^{N_{\text{spec}}} \left\{ (a_{s,00}^{(1)} a_{s,00}^{(1)} + a_{s,10}^{(1)} a_{s,10}^{(1)}) (\delta_{js} - \delta_{is}) \left( \frac{1}{3} I_{i,r+1,0} I_{j,20} + I_{i,r+2,0} I_{j,10} \right) \right. \\ &\quad \left. + a_{s,10}^{(1)} a_{s,11}^{(1)} \left[ \frac{2}{9} (2\delta_{js} - \delta_{is}) I_{i,r+1,0} I_{j,30} + \frac{4}{9} (\delta_{js} + \delta_{is}) I_{i,r+2,0} I_{j,20} + \frac{6}{9} (\delta_{js} - 2\delta_{is}) I_{i,r+3,0} I_{j,10} \right] \right\}, \quad (\text{F15}) \end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_{is,r1}^{(1)} &= -\frac{1}{2} \sum_{m=0}^1 \frac{a_{s,11}^{(1)} a_{s,1m}^{(1)}}{3I_{s,21}} \sum_{j,a,b=1}^{N_{\text{spec}}} u_{\nu_2} \cdots u_{\nu_{1+m}} \Delta^{\mu_1 \nu_1} \mathcal{X}_{sijab,m}^{\mu_1 \nu_1 \cdots \nu_{1+m}} \\
&= -\frac{\sigma_{\text{tot}}}{6I_{s,21}} \sum_{j=1}^{N_{\text{spec}}} \left\{ a_{s,11}^{(1)} a_{s,10}^{(1)} (\delta_{js} - \delta_{is}) \left( \frac{1}{3} I_{i,r+1,0} I_{j,20} + I_{i,r+2,0} I_{j,10} \right) \right. \\
&\quad \left. + a_{s,11}^{(1)} a_{s,11}^{(1)} \left[ \frac{2}{9} (2\delta_{js} - \delta_{is}) I_{i,r+1,0} I_{j,30} + \frac{4}{9} (\delta_{js} + \delta_{is}) I_{i,r+2,0} I_{j,20} + \frac{6}{9} (\delta_{js} - 2\delta_{is}) I_{i,r+3,0} I_{j,10} \right] \right\}. \quad (\text{F16})
\end{aligned}$$

After some calculations, the relevant matrix elements read in terms of pressure:

$$\mathcal{A}_{is,00}^{(1)} = -\frac{4}{9} \sigma_{\text{tot}} \beta \sum_{j=1}^{N_{\text{spec}}} \frac{P_i P_j}{P_s} (\delta_{js} - 2\delta_{is}) = \frac{4}{9} \sigma_{\text{tot}} \beta (2\delta_{is} P - P_i), \quad (\text{F17})$$

$$\mathcal{A}_{is,10}^{(1)} = -\frac{1}{2} \sigma_{\text{tot}} \sum_{j=1}^{N_{\text{spec}}} \frac{P_i P_j}{P_s} (\delta_{js} - \delta_{is}) = \frac{1}{2} \sigma_{\text{tot}} (\delta_{is} P - P_i), \quad (\text{F18})$$

$$\mathcal{A}_{is,01}^{(1)} = -\frac{1}{18} \sigma_{\text{tot}} \beta^2 \sum_{j=1}^{N_{\text{spec}}} \frac{P_i P_j}{P_s} (\delta_{js} + \delta_{is}) = -\frac{1}{18} \sigma_{\text{tot}} \beta^2 (\delta_{is} P + P_i), \quad (\text{F19})$$

$$\mathcal{A}_{is,11}^{(1)} = -\frac{1}{2} \sigma_{\text{tot}} \beta \sum_{j=1}^{N_{\text{spec}}} \frac{P_i P_j}{P_s} (\delta_{js} - \delta_{is}) = \frac{1}{2} \sigma_{\text{tot}} \beta (\delta_{is} P - P_i). \quad (\text{F20})$$

Equations (F17)–(F20) are the elements of the four  $N_{\text{spec}}$ -dimensional rectangular block matrices of the  $(2N_{\text{spec}} \times 2N_{\text{spec}})$ -matrix  $\mathcal{A}^{(1)}$ . Its single-component limit ( $N_{\text{spec}} = 1$ ) is consistent with Ref. [21]:

$$\mathcal{A}^{(1)} \equiv \begin{pmatrix} \mathcal{A}_{11,00}^{(1)} & \mathcal{A}_{11,01}^{(1)} \\ \mathcal{A}_{11,10}^{(1)} & \mathcal{A}_{11,11}^{(1)} \end{pmatrix} = \frac{1}{9} \sigma_{\text{tot}} \beta P_0 \begin{pmatrix} 4 & -\beta \\ 0 & 0 \end{pmatrix}. \quad (\text{F21})$$

We observe that the  $\mathcal{A}^{(1)}$  matrix is singular even in the single-component limit. This is due to the momentum-conservation equation. In order to construct the inverse matrix  $\tau^{(1)}$ , we follow the steps presented in Appendix G by introducing the reduced matrix  $\tilde{\mathcal{A}}^{(1)}$  (therefore effectively removing the irreducible moment  $\rho_{1,1}^\mu$ ), inverting it, and adding zero elements corresponding to the originally removed rows and columns to that inverse, yielding  $\tau^{(1)}$ . We illustrate this procedure in the single-component limit. The reduced matrix and its inverse then just consist of one entry,

$$\tilde{\mathcal{A}}^{(1)} = \frac{4}{9} \sigma_{\text{tot}} \beta P_0, \quad \text{and} \quad \tilde{\tau}^{(1)} = \frac{9}{4 \sigma_{\text{tot}} \beta P_0}. \quad (\text{F22})$$

Adding zero elements yields the final inverse

$$\tau^{(1)} = \frac{9}{4 \sigma_{\text{tot}} \beta P_0} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{F23})$$

In the multicomponent case, we find the following entries of the  $2N_{\text{spec}}$ -dimensional rectangular inverse matrix:

$$\tau_{si,00}^{(1)} = \frac{9}{68 \beta \sigma_{\text{tot}} P^2} (8\delta_{si} P + 9P_s), \quad (\text{F24})$$

$$\tau_{si,01}^{(1)} = \frac{2}{17 \sigma_{\text{tot}} P} (\delta_{si} - \delta_{sN_{\text{spec}}}), \quad (\text{F25})$$

$$\tau_{si,10}^{(1)} = \frac{18}{17 \sigma_{\text{tot}} \beta^2 P^2} (1 - \delta_{sN_{\text{spec}}}) (P_s - \delta_{si} P), \quad (\text{F26})$$

$$\tau_{si,11}^{(1)} = \frac{32}{17 \sigma_{\text{tot}} \beta P} \delta_{si} (1 - \delta_{sN_{\text{spec}}}). \quad (\text{F27})$$

We note that the elements  $\tau_{si,nr}^{(1)}$  are indeed constructed in a way that they vanish in the cases  $n = 1$  and  $s = N_{\text{spec}}$ , or  $r = 1$  and  $i = N_{\text{spec}}$  (i.e., the row and column which was originally removed from  $\mathcal{A}^{(1)}$ ). We remind the reader that adding these zeros simplifies our notation in this work.

## 2. Transport coefficients

Now that the collision matrix has been determined, we can proceed to calculate the transport coefficients of the theory. We remark that the scalar moments have not been discussed since the bulk viscous pressure vanishes identically in the ultrarelativistic (massless) case,  $\Pi \equiv \sum_{s=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \rho_{s,0} = 0$ . The coefficients in the  $(10 + 4N_q)$ -moment approximation ( $N_0 = 2$ ,  $N_1 = 1$ ,  $N_2 = 0$ ) have been defined in Sec. III D and Appendix C. We evaluate them in the ultrarelativistic scenario, where all mass terms vanish. Since the transport coefficients are defined via the coefficients listed in Eqs. (98)–(100), these have to be evaluated first. For this, we introduce shorthand notations for the charge concentration of type  $q$ , and the concentration of the charge combination  $qq'$  in the system, respectively:

$$\mathbf{c}_q \equiv \sum_{j=1}^{N_{\text{spec}}} q_j \frac{P_j}{P}, \quad \text{and} \quad \mathbf{c}_{qq'} \equiv \sum_{j=1}^{N_{\text{spec}}} q_j q'_j \frac{P_j}{P}. \quad (\text{F28})$$

Further, the derivatives in temperature and chemical potential of the weighted partial pressures of a classical gas read

$$\frac{\partial}{\partial \beta} \left( \frac{P_i}{P} \right) = 0 \quad \text{and} \quad \frac{\partial}{\partial \alpha_q} \left( \frac{P_i}{P} \right) = \frac{P_i}{P} (q_i - \mathbf{c}_q), \quad (\text{F29})$$

and from this the derivatives in the charge concentration follow

$$\frac{\partial \mathbf{c}_q}{\partial \beta} = 0 \quad \text{and} \quad \frac{\partial \mathbf{c}_q}{\partial \alpha_{q'}} = \mathbf{c}_{qq'} - \mathbf{c}_q \mathbf{c}_{q'}. \quad (\text{F30})$$

respectively. The relevant expressions for the vector and tensor moments are then obtained from Eqs. (98)–(100) as

$$\eta_{s,0} \equiv \sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_2} \tau_{si,0r}^{(2)} \alpha_{s,r}^{(2)} = \frac{4 P_s}{3 P} \frac{1}{\sigma_{\text{tot}} \beta}, \quad (\text{F31})$$

$$\kappa_{s,0,q} \equiv \sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_1} \tau_{si,0r}^{(1)} \alpha_{i,r,q}^{(1)} = \frac{8}{17 \sigma_{\text{tot}}} \frac{P_s}{P} \left( q_s - \frac{77}{128} \mathbf{c}_q \right), \quad (\text{F32})$$

$$\kappa_{s,1,q} \equiv \sum_{i=1}^{N_{\text{spec}}} \sum_{r=0}^{N_1} \tau_{si,1r}^{(1)} \alpha_{i,r,q}^{(1)} = \frac{26}{17 \sigma_{\text{tot}} \beta} (1 - \delta_{sN_{\text{spec}}}) \frac{P_s}{P} (q_s - \mathbf{c}_q). \quad (\text{F33})$$

From this, the shear viscosity and the diffusion-coefficient matrix immediately follow:

$$\eta \equiv \sum_{s=1}^{N_{\text{spec}}} \eta_{s,0} = \frac{4}{3} \frac{1}{\sigma_{\text{tot}} \beta}, \quad (\text{F34})$$

$$\kappa_{qq'} \equiv \sum_{s=1}^{N_{\text{spec}}} q_s \kappa_{s,0,q'} = \frac{8}{17 \sigma_{\text{tot}}} \left( \mathbf{c}_{qq'} - \frac{77}{128} \mathbf{c}_q \mathbf{c}_{q'} \right). \quad (\text{F35})$$

It should be noted that the derivative of the above diffusion coefficients in inverse temperature, and therefore also from its inverse, vanishes,

$$\frac{\partial}{\partial \beta} \kappa_{qq'} = 0 \quad \text{and} \quad \frac{\partial}{\partial \beta} (\kappa^{-1})_{qq'} = 0. \quad (\text{F36})$$

We note that the result for the shear viscosity is the same as in Ref. [21], while the diffusion coefficients are different since they depend on the various charges in the system, which are not taken into account in a single-component approach. However, in the limit where there is only one conserved particle species in the system (i.e.,  $N_q = 1$  and  $q_s = 1$  and therefore  $\mathbf{c}_q = \mathbf{c}_{qq'} = 1$ ), the obtained expression is equivalent to the value derived in Ref. [21],  $\kappa = 3/(16\sigma_{\text{tot}})$ . We also remark that from the above equation we read off that the diffusion-coefficient matrix is symmetric,  $\kappa_{qq'} = \kappa_{q'q}$  as shown by the Onsager reciprocal relations [12,13]. The relevant weighted base coefficients then read

$$\bar{\eta}_{i,0} \equiv \frac{\eta_{i,0}}{\eta} = \frac{P_i}{P}, \quad (\text{F37})$$

$$\bar{\eta}_{i,-1} \equiv \mathcal{F}_{i,1,0}^{(2)} \bar{\eta}_{i,0} = \frac{\beta P_i}{5 P}, \quad (\text{F38})$$

$$\begin{aligned} \bar{\kappa}_{i,0}^{(q)} &\equiv \sum_{q'}^{\{B,Q,S\}} \kappa_{i,0,q'} (\kappa^{-1})_{q'q} \\ &= \frac{8}{17 \sigma_{\text{tot}}} \sum_{q'}^{\{B,Q,S\}} (\kappa^{-1})_{q'q} \frac{P_i}{P} \left( q'_i - \frac{77}{128} \mathbf{c}_{q'} \right), \end{aligned} \quad (\text{F39})$$

$$\bar{\kappa}_{i,1}^{(q)} = \frac{26}{17 \beta \sigma_{\text{tot}}} \sum_{q'}^{\{B,Q,S\}} (\kappa^{-1})_{q'q} \frac{P_i}{P} (q'_i - \mathbf{c}_{q'}) (1 - \delta_{iN_{\text{spec}}}). \quad (\text{F40})$$

For the relaxation times defined in Eqs. (116) and (119) we get

$$\tau_\pi = \frac{5}{3} \frac{1}{\sigma_{\text{tot}} \beta P}, \quad (\text{F41})$$

$$\begin{aligned} \tau_{q'q} &= \frac{9}{68 \beta \sigma_{\text{tot}}^2 P} \sum_{q''}^{\{B,Q,S\}} (\kappa^{-1})_{q''q} \left( \frac{784}{153} \mathbf{c}_{q''q'} - \frac{4741}{2448} \mathbf{c}_{q''} \mathbf{c}_{q'} \right) \\ &\xrightarrow{\text{single}} \frac{9}{4 \beta \sigma_{\text{tot}} P}. \end{aligned} \quad (\text{F42})$$

We again note that also the relaxation-time matrix is symmetric,  $\tau_{q'q} = \tau_{qq'}$ . It is further apparent that the shear

relaxation time is equivalent to the value derived in Ref. [21]. In the case where only one conserved particle species is present (see above), such an equivalence is also recovered for the diffusive relaxation time,  $\tau_q = 27/(64\beta\sigma_{\text{tot}}^2 P\kappa) = 9/(4\beta\sigma_{\text{tot}} P)$ , as indicated with the notation “ $\xrightarrow{\text{single}}$ ”. In the following we will keep this notation, and continue with the second-order coefficients. For the coefficients in the shear-stress tensor equation, defined in Appendix C 3, we get

$$\delta_{\pi\pi} = \frac{4}{3}\tau_\pi, \quad (\text{F43})$$

$$\tau_{\pi\pi} = \frac{10}{7}\tau_\pi, \quad (\text{F44})$$

$$\begin{aligned} \ell_{\pi V}^{(q)} &= \frac{52}{51} \frac{1}{\beta^2 \sigma_{\text{tot}}^2 P} \sum_{q'}^{\{B,Q,S\}} (\kappa^{-1})_{q'q} (\mathbf{c}_{q'} - q'_{N_{\text{spec}}}) \\ &\times \frac{P N_{\text{spec}}}{P} \xrightarrow{\text{single}} 0, \end{aligned} \quad (\text{F45})$$

$$\tau_{\pi V}^{(q)} = 4\beta \frac{\partial}{\partial \beta} \ell_{\pi V}^{(q)} - 7\ell_{\pi V}^{(q)} \xrightarrow{\text{single}} 0, \quad (\text{F46})$$

$$\lambda_{\pi V}^{(q,q')} = \frac{\partial}{\partial \alpha_{q'}} \ell_{\pi V}^{(q)} + \frac{\beta}{4} \mathbf{c}_{q'} \frac{\partial}{\partial \beta} \ell_{\pi V}^{(q)} - 2\mathbf{c}_{q'} \ell_{\pi V}^{(q)} \xrightarrow{\text{single}} 0. \quad (\text{F47})$$

For the coefficients in the vector equations of motion (see Appendix C 2) we derive:

$$\begin{aligned} \delta_{VV}^{(q',q)} &= \frac{1}{\beta \sigma_{\text{tot}}^2 P} \sum_{q''}^{\{B,Q,S\}} (\kappa^{-1})_{q''q} \left[ \frac{640}{867} \mathbf{c}_{q'q''} - \frac{17551}{55488} \mathbf{c}_{q'} \mathbf{c}_{q''} + \frac{52}{289} (\mathbf{c}_{q'q''} - \mathbf{c}_{q'} \mathbf{c}_{q''}) \frac{4P}{\beta} \left( \mathcal{T}_{00} + \frac{\beta}{4} \sum_{q'''}^{\{B,Q,S\}} \mathcal{T}_{0q'''} \mathbf{c}_{q'''} \right) \right] \\ &- \frac{4}{\beta \sigma_{\text{tot}}^2} \sum_{q''}^{\{B,Q,S\}} \left\{ \frac{\partial}{\partial \alpha_{q''}} \left[ \sum_{q'''}^{\{B,Q,S\}} (\kappa^{-1})_{q''q'''} \left( \frac{196}{289} \mathbf{c}_{q'q'''} - \frac{1109}{2312} \mathbf{c}_{q'} \mathbf{c}_{q'''} \right) \right] \right\} \\ &+ \frac{162}{289} \mathbf{c}_{q'} \frac{\partial}{\partial \alpha_{q''}} \left[ \sum_{q'''}^{\{B,Q,S\}} (\kappa^{-1})_{q''q'''} (\mathbf{c}_{q'q'''} - \mathbf{c}_{q'} \mathbf{c}_{q'''}) \right] \left( \mathcal{T}_{q''0} + \frac{\beta}{4} \sum_{q'''}^{\{B,Q,S\}} \mathcal{T}_{q''q'''} \mathbf{c}_{q'''} \right) \xrightarrow{\text{single}} \tau_q. \end{aligned} \quad (\text{F48})$$

$$\lambda_{VV}^{(q',q)} = \frac{9}{68\beta\sigma_{\text{tot}}^2 P} \sum_{q''}^{\{B,Q,S\}} (\kappa^{-1})_{q''q} \left( \frac{2768}{765} \mathbf{c}_{q''q'} - \frac{20879}{12240} \mathbf{c}_{q''} \mathbf{c}_{q'} \right) \xrightarrow{\text{single}} \frac{3}{5} \tau_q, \quad (\text{F49})$$

$$\ell_{V\pi}^{(q)} = \frac{9}{80\sigma_{\text{tot}} P} \mathbf{c}_{q'} \xrightarrow{\text{single}} \frac{\beta}{20} \tau_q, \quad (\text{F50})$$

$$\tau_{V\pi}^{(q)} = \ell_{V\pi}^{(q)} \xrightarrow{\text{single}} \frac{\beta}{20} \tau_q, \quad (\text{F51})$$

$$\lambda_{V\pi}^{(q',q)} = \frac{28}{85} \frac{1}{\sigma_{\text{tot}} P} \left( \mathbf{c}_{qq'} - \frac{295}{448} \mathbf{c}_q \mathbf{c}_{q'} \right) \xrightarrow{\text{single}} \frac{\beta}{20} \tau_q. \quad (\text{F52})$$

Note that the coefficient  $\tau_{n\pi}$  from Denicol (2012) [21] was defined as  $\tau_{n\pi} \stackrel{\text{single}}{\equiv} \frac{1}{\epsilon+P} \tau_{V\pi}^{(q')}$ , which then yields  $\tau_{n\pi} = \frac{\beta}{80} \tau_q$ . Therefore, in the single-component limit we retrieve the same coefficients as in Ref. [21].

## APPENDIX G: INVERTING THE COLLISION MATRIX

For the calculation of the transport coefficients, the inverse of the collision matrix  $\mathcal{A}^{(\ell)} \equiv (\mathcal{A}_{ij,rm}^{(\ell)})$  from Eq. (90) must be calculated. In the tradition of Ref. [21], we provide a detailed discussion of the derivation of the linearized collision term (see Appendix E) and its inverse. In this section, we show that the collision matrix is singular in the cases  $\ell = 0$  and  $\ell = 1$  due to the conservation of energy-momentum and charge.

While in the single-component system the construction of the inverse was immediately clear [21], in the case of a multicomponent system such a construction is not obvious.

The conservation equations (73) for the various charges imply that certain moments of the Boltzmann equation vanish,

$$\sum_{i=1}^{N_{\text{spec}}} q_i C_{i,0} = \sum_{i=1}^{N_{\text{spec}}} q_i \int dK_i k_i^\nu \partial_\nu f_{i,\mathbf{k}} \equiv 0. \quad (\text{G1})$$

Similarly, projections of the conservation law (74) for energy and momentum give

$$\sum_{i=1}^{N_{\text{spec}}} C_{i,1} = \sum_{i=1}^{N_{\text{spec}}} \int dK_i E_{i,\mathbf{k}} k_i^\nu \partial_\nu f_{i,\mathbf{k}} \equiv 0, \quad (\text{G2})$$



$$\sum_{i=1}^{N_{\text{spec}}} C_{i,0}^{(\mu)} = \sum_{i=1}^{N_{\text{spec}}} \int d\mathbf{K}_i k_i^{(\mu)} k_i^\nu \partial_\nu f_{i,\mathbf{k}} \equiv 0. \quad (\text{G3})$$

These relations imply that a subset of row (or column) vectors of the collision matrices  $\mathcal{A}^{(\ell)}$  are linearly dependent. Since the irreducible moments  $\rho_{i,r}^{\mu_1 \dots \mu_\ell}$  are in principle independent of each other, Eq. (90) implies that the following linear combinations of the elements of the collision matrix must vanish:

$$0 = \sum_{i=1}^{N_{\text{spec}}} q_i \mathcal{A}_{is,1n}^{(0)}, \quad (\text{G4})$$

$$0 = \sum_{i=1}^{N_{\text{spec}}} \mathcal{A}_{is,2n}^{(0)}, \quad (\text{G5})$$

$$0 = \sum_{i=1}^{N_{\text{spec}}} \mathcal{A}_{is,1n}^{(1)}. \quad (\text{G6})$$

This means that the conservation laws render the collision matrix for the scalar moments,  $\mathcal{A}^{(0)}$ , and the vector moments,

$\mathcal{A}^{(1)}$ , singular. We note that the tensor moments ( $\ell = 2$ ) are not affected, and thus  $\mathcal{A}^{(2)}$  is in principle regular. For a meaningful fluid-dynamical theory, an equivalent description of the above discussed collision matrices has to be found such that they are rendered invertible, and at the same time their microscopic information is not altered. The linear dependence further implies that  $N_q + 4$  equations need to be removed from the set (83), (85), and (87) of equations of motion, i.e.,  $N_q + 1$  scalar moments  $\rho_{i,r}$  and one vector moment  $\rho_{i,r}^\mu$  (three equations). In the case of the vector moments, the choice of the frame provides a relationship between the vector moments, and it allows us to eliminate one of them from the equations of motion entirely.

For the Landau frame, from Eq. (70) we have

$$\rho_{N_{\text{spec}},1}^\mu = - \sum_{i=1}^{N_{\text{spec}}-1} \rho_{i,1}^\mu, \quad (\text{G7})$$

while in the Eckart frame we could impose via Eq. (B12)  $\rho_{N_{\text{spec}},1}^\mu = -h_q \sum_{i=1}^{N_{\text{spec}}} q_i \rho_{i,0}^\mu - \sum_{i=1}^{N_{\text{spec}}-1} \rho_{i,1}^\mu$ . As before, we proceed in the Landau frame. With the help of Eq. (G7), we can write:

$$\begin{aligned} \sum_{s=1}^{N_{\text{spec}}} \sum_{n=0}^{N_1} \mathcal{A}_{is, rn}^{(1)} \rho_{s,n}^\mu &= \sum_{n=0, \neq 1}^{N_1} \sum_{s=1}^{N_{\text{spec}}} \mathcal{A}_{is, r0}^{(1)} \rho_{s,0}^\mu + \sum_{s=1}^{N_{\text{spec}}-1} \mathcal{A}_{is, r1}^{(1)} \rho_{s,1}^\mu + \mathcal{A}_{iN_{\text{spec}}, r1}^{(1)} \rho_{N_{\text{spec}},1}^\mu \\ &= \sum_{n=0, \neq 1}^{N_1} \sum_{s=1}^{N_{\text{spec}}} \mathcal{A}_{is, r0}^{(1)} \rho_{s,0}^\mu + \sum_{s=1}^{N_{\text{spec}}-1} \underbrace{(\mathcal{A}_{is, r1}^{(1)} - \mathcal{A}_{iN_{\text{spec}}, r1}^{(1)})}_{(\star)} \rho_{s,1}^\mu \stackrel{!}{=} \sum_{n=0}^{N_1} \sum_{s=1}^{N_{\text{spec}}} \tilde{\mathcal{A}}_{is, rn}^{(1)} \rho_{s,n}^\mu. \end{aligned} \quad (\text{G8})$$

First, we note that the term marked with  $(\star)$  vanishes for  $s = N_{\text{spec}}$ . Therefore, we can extend the last sum to run up to  $s = N_{\text{spec}}$ . In the last step of the above equation, we introduced the matrix  $\tilde{\mathcal{A}}^{(1)}$ , which is a reduced version of the matrix  $\mathcal{A}^{(1)}$  with the row corresponding to  $r = 1$  and  $i = N_{\text{spec}}$  and the column corresponding to  $n = 1$  and  $s = N_{\text{spec}}$ , respectively, removed, i.e., its elements read:

$$\tilde{\mathcal{A}}_{is, rn}^{(1)} = \mathcal{A}_{is, rn}^{(1)} \text{ for } n \neq 1, \text{ and } \tilde{\mathcal{A}}_{is, r1}^{(1)} = \mathcal{A}_{is, r1}^{(1)} - \mathcal{A}_{iN_{\text{spec}}, r1}^{(1)}. \quad (\text{G9})$$

Then,  $\tilde{\mathcal{A}}^{(1)}$  is an  $(N_{\text{spec}} \cdot N_1 - 1)$ -dimensional, rectangular, regular matrix.

An explicit example for the construction of the corrected matrix is given in Appendix F in the case of an ultrarelativistic gas mixture. We further note that, we are free to choose which irreducible moment  $\rho_{i,1}^\mu$  we remove from the set of equations of motion, and thus we could have chosen any line and associated column (corresponding to that particular moment) to be removed (e.g.,  $\rho_{\lambda,1}^\mu$  instead of  $\rho_{N_{\text{spec}},1}^\mu$ ).

Once the reduced collision matrix  $\tilde{\mathcal{A}}^{(1)}$  is obtained it can be inverted. This yields the reduced inverse  $\tilde{\tau}^{(1)}$  of dimension  $N_{\text{spec}} \cdot N_1 - 1$ . In order to make the inverse  $\tau^{(1)}$  equivalent to the one of dimension  $N_{\text{spec}} \cdot N_1$  introduced in Eq. (93), and in order to keep a simple notation regarding the summations over the indices  $r$  and  $s$  in all equations following that definition [e.g., Eqs. (95) ff.], we add zero-element row(s) and column(s), which correspond to the ones originally removed from the matrix  $\tilde{\mathcal{A}}^{(1)}$ . For instance, in the case of the vector moments this means that we add a zero row for  $r = 1$  and  $i = N_{\text{spec}}$ , and a zero column for  $n = 1$  and  $s = N_{\text{spec}}$ . Due to the zero-element row and column, the removed irreducible moment  $\rho_{N_{\text{spec}},1}^\mu$ , even though it formally still appears in the equations following Eq. (95), effectively does not contribute anymore.

The procedure is analogous for the removal of the  $N_q + 1$  scalar moments. The energy- and charge-conservation laws provide  $N_q + 1$  relations for the linear dependence of the row vectors of matrix  $\mathcal{A}^{(0)}$ . These are given by Eqs. (G4) and (G5). In order to remove the corresponding moments, we impose the Landau matching conditions,

$$\sum_{i=1}^{N_{\text{spec}}} \rho_{i,2} = 0, \quad \text{and} \quad \sum_{i=1}^{N_{\text{spec}}} q_i \rho_{i,1} = 0. \quad (\text{G10})$$

The corrected matrix  $\tilde{\mathcal{A}}^{(0)}$  is then an  $(N_{\text{spec}} \cdot N_0 - N_q - 1)$ -dimensional, rectangular, regular matrix and its elements can be written as:

$$\begin{aligned} \tilde{\mathcal{A}}_{is,rm}^{(0)} &= \mathcal{A}_{is,rm}^{(0)} \quad \text{for } n \neq 1, 2, & \tilde{\mathcal{A}}_{is,r1}^{(0)} &= \mathcal{A}_{is,r1}^{(0)} - \frac{q_s}{q_\lambda} \mathcal{A}_{i\lambda,r1}^{(0)}, \\ \text{and } \tilde{\mathcal{A}}_{is,r2}^{(0)} &= \mathcal{A}_{is,r2}^{(0)} - \mathcal{A}_{i\lambda,r2}^{(0)}, & & \end{aligned} \quad (\text{G11})$$

where again  $i \neq \lambda$  for  $r = 1, 2$ , and  $s \neq \lambda$  for  $n = 1, 2$ . It should be noted that the  $N_q$  charge-conservation laws and the  $N_q$  matching conditions associated to the net-charge densities only allow for the removal of moments corresponding to species  $\lambda$  with nonvanishing charge of type  $q$ ,  $q_\lambda \neq 0$ . This can be understood by noting that the above relations (G11) are not well defined when  $q_\lambda = 0$ . In order to simplify the notation in this work, we construct the inverse  $\tau^{(0)}$  as an  $(N_{\text{spec}} \cdot N_0)$ -dimensional, rectangular matrix, which is the inverse of  $\tilde{\mathcal{A}}^{(0)}$  and contains zero elements for the rows and columns which were removed from  $\mathcal{A}^{(0)}$ .

- 
- [1] J. E. Bernhard, J. S. Moreland, and S. A. Bass, *Nat. Phys.* **15**, 1113 (2019).
- [2] J. Auvinen, K. J. Eskola, P. Huovinen, H. Niemi, R. Paatelainen, and P. Petreczky, *Phys. Rev. C* **102**, 044911 (2020).
- [3] G. Nijs, W. van der Schee, U. Gürsoy, and R. Snellings, *Phys. Rev. C* **103**, 054909 (2021).
- [4] D. Everett *et al.* (JETSCAPE Collaboration), *Phys. Rev. C* **103**, 054904 (2021).
- [5] J. E. Parkkila, A. Onnerstad, and D. J. Kim, *Phys. Rev. C* **104**, 054904 (2021).
- [6] W. Israel and J. M. Stewart, *Ann. Phys. (N.Y.)* **118**, 341 (1979).
- [7] H. Grad, *Commun. Pure Appl. Math.* **2**, 331 (1949); **2**, 325 (1949).
- [8] I. Müller, *Z. Phys.* **198**, 329 (1967); *Living Rev. Relativity* **2**, 1 (1999).
- [9] M. Prakash, M. Prakash, R. Venugopalan, and G. Welke, *Phys. Rep.* **227**, 321 (1993).
- [10] A. Monnai and T. Hirano, *Nucl. Phys. A* **847**, 283 (2010).
- [11] A. Monnai and T. Hirano, *J. Phys. Conf. Ser.* **270**, 012042 (2011).
- [12] L. Onsager, *Phys. Rev.* **37**, 405 (1931).
- [13] L. Onsager, *Phys. Rev.* **38**, 2265 (1931).
- [14] Y. Kikuchi, K. Tsumura, and T. Kunihiro, *Phys. Rev. C* **92**, 064909 (2015).
- [15] M. Greif, J. A. Fotakis, G. S. Denicol, and C. Greiner, *Phys. Rev. Lett.* **120**, 242301 (2018).
- [16] J. A. Fotakis, M. Greif, C. Greiner, G. S. Denicol, and H. Niemi, *Phys. Rev. D* **101**, 076007 (2020).
- [17] J.-B. Rose, M. Greif, J. Hammelmann, J. A. Fotakis, G. S. Denicol, H. Elfner, and C. Greiner, *Phys. Rev. D* **101**, 114028 (2020).
- [18] J. A. Fotakis, O. Soloveva, C. Greiner, O. Kaczmarek, and E. Bratkovskaya, *Phys. Rev. D* **104**, 034014 (2021).
- [19] G. S. Denicol and H. Niemi, *Nucl. Phys. A* **904-905**, 369c (2013).
- [20] Z. Wolff and D. Molnar, *Phys. Rev. C* **96**, 044909 (2017).
- [21] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, *Phys. Rev. D* **85**, 114047 (2012); **91**, 039902(E) (2015).
- [22] S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory: Principles and Applications* (North Holland, Amsterdam, 1980).
- [23] C. Cercignani and G. M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications* (Birkhäuser, Basel, 2002).
- [24] P. Chakraborty and J. I. Kapusta, *Phys. Rev. C* **83**, 014906 (2011).
- [25] M. Albright and J. I. Kapusta, *Phys. Rev. C* **93**, 014903 (2016).
- [26] F. Jüttner, *Ann. Phys. (N.Y.)* **339**, 856 (1911); *Z. Phys.* **47**, 542 (1928).
- [27] G. S. Denicol, E. Molnár, H. Niemi, and D. H. Rischke, *Eur. Phys. J. A* **48**, 170 (2012).
- [28] L. D. Landau and E. M. Lifshitz, *Fluid Dynamics*, 3rd ed. (Butterworth-Heinemann, Oxford, 1987).
- [29] C. Eckart, *Phys. Rev.* **58**, 919 (1940).
- [30] A. El, I. Bouras, F. Lauciello, Z. Xu, and C. Greiner, *arXiv:1103.4038*.
- [31] A. El, I. Bouras, C. Wesp, Z. Xu, and C. Greiner, *Eur. Phys. J. A* **48**, 166 (2012).
- [32] G. S. Denicol, H. Niemi, I. Bouras, E. Molnar, Z. Xu, D. H. Rischke, and C. Greiner, *Phys. Rev. D* **89**, 074005 (2014).
- [33] H. Struchtrup, *Phys. Fluids* **16**, 3921 (2004).
- [34] A. Monnai, B. Schenke, and C. Shen, *Phys. Rev. C* **100**, 024907 (2019).
- [35] J. Noronha-Hostler, P. Parotto, C. Ratti, and J. M. Stafford, *Phys. Rev. C* **100**, 064910 (2019).
- [36] E. Molnár, H. Niemi, G. S. Denicol, and D. H. Rischke, *Phys. Rev. D* **89**, 074010 (2014).