JYX JYVÄSKYLÄN YLIOPISTO UNIVERSITY OF JYVÄSKYLÄ

This is a self-archived version of an original article. This version may differ from the original in pagination and typographic details.

Author(s): Lassas, Matti; Liimatainen, Tony; Potenciano-Machado, Leyter; Tyni, Teemu

Title: Uniqueness, reconstruction and stability for an inverse problem of a semi-linear wave equation

Year: 2022

Version: Published version

Copyright: © 2022 The Author(s). Published by Elsevier Inc.

Rights: CC BY 4.0

Rights url: https://creativecommons.org/licenses/by/4.0/

Please cite the original version:

Lassas, M., Liimatainen, T., Potenciano-Machado, L., & Tyni, T. (2022). Uniqueness, reconstruction and stability for an inverse problem of a semi-linear wave equation. Journal of Differential Equations, 337, 395-435. https://doi.org/10.1016/j.jde.2022.08.010

Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 337 (2022) 395–435

Journal of Differential Equations

www.elsevier.com/locate/jde

Uniqueness, reconstruction and stability for an inverse problem of a semi-linear wave equation

Matti Lassas [a], Tony Liimatainen [b], Leyter Potenciano-Machado [b,*], Teemu Tyni [a]

[a] *Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland*
[b] *Department of Mathematics and Statistics, University of Jyväskylä, Jyväskylä, Finland*

Received 7 April 2022; revised 20 July 2022; accepted 14 August 2022

Abstract

We consider the recovery of a potential associated with a semi-linear wave equation on $\mathbb{R}^{n+1}$, $n \geq 1$. We show that an unknown potential $a(x,t)$ of the wave equation $\square u + au^m = 0$ can be recovered in a Hölder stable way from the map $u|_{\partial\Omega\times[0,T]} \mapsto \langle \psi, \partial_\nu u|_{\partial\Omega\times[0,T]}\rangle_{L^2(\partial\Omega\times[0,T])}$. This data is equivalent to the inner product of the Dirichlet-to-Neumann map with a measurement function ψ. We also prove similar stability result for the recovery of a when there is noise added to the boundary data. The method we use is constructive and it is based on the higher order linearization. As a consequence, we also get a uniqueness result. We also give a detailed presentation of the forward problem for the equation $\square u + au^m = 0$.
© 2022 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

0. Introduction

In this paper we study an inverse boundary value problem for a non-linear wave equation. The inverse problems we study are the uniqueness and stability of recovering an unknown potential $a \in C^\infty(\Omega \times \mathbb{R})$ of the non-linear wave equation

* Corresponding author.
E-mail addresses: matti.lassas@helsinki.fi (M. Lassas), tony.liimatainen@helsinki.fi (T. Liimatainen), leyter.m.potenciano@gmail.com (L. Potenciano-Machado), teemu.tyni@helsinki.fi (T. Tyni).

https://doi.org/10.1016/j.jde.2022.08.010
0022-0396/© 2022 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

$$\begin{cases} \Box u(x,t) + a(x,t)u(x,t)^m = 0 & \text{in } \Omega \times [T_1, T_2], \\ u = f & \text{on } \partial\Omega \times [T_1, T_2], \\ u\big|_{t=T_1} = 0, \quad \partial_t u\big|_{t=T_1} = 0 & \text{on } \Omega \end{cases} \tag{1}$$

from the Dirichlet-to-Neumann map (DN map) of the equation. Here $m \geq 2$ is an integer, Ω is an open and bounded subset of $\mathbb{R}^n$, $T_1 < T_2$, and $\Box$ is the standard wave operator $\partial_t^2 - \Delta$ in $\mathbb{R}^{n+1}$. We assume that the potential $a = a(x,t)$ can depend on the time variable t.

Inverse problems for Equation (1) are natural counterparts to the widely studied inverse problems for the linear operator $\Box u + au$. We refer to [29] for inverse problems for linear wave equations. Equations of the type (1) arise for example in quantum mechanics in the context of the Klein-Gordon equation.

We will show that the boundary value problem for Equation (1) has a unique small solution u for sufficiently small boundary data $f \in H^{s+1}(\partial\Omega \times [T_1, T_2])$, where $s \in \mathbb{N}$ and $s + 1 > (n+1)/2$. Precisely this means that there is $\varepsilon > 0$ and $\delta > 0$ such that whenever $\|f\|_{H^{s+1}(\partial\Omega\times[T_1,T_2])} \leq \varepsilon$, there is a unique solution u_f to (1) with norm smaller than in δ in the *energy space* E^{s+1}

$$E^{s+1} = \bigcap_{0 \leq k \leq s+1} C^k([T_1, T_2]; H^{s+1-k}(\Omega)).$$

Here H^{s+1} is the standard Sobolev space. We will call u_f the unique small solution. The DN map Λ is then defined by using the unique small solution by the usual assignment,

$$\Lambda : H^{s+1}(\partial\Omega \times [T_1, T_2]) \to H^s(\partial\Omega \times [T_1, T_2]), \quad f \mapsto \partial_\nu u_f|_{\partial\Omega\times[T_1,T_2]}.$$

Here ∂_ν denotes the normal derivative on the boundary $\partial\Omega \times [T_1, T_2]$. See Section 1 for details on well-posedness. We mention that the conditions imposed on s and n are necessary to use Sobolev embedding theorems.

Let us briefly mention some results on inverse problems for linear equations. In the case where the underlying equation is linear and elliptic, a standard example is from the pioneering work of Calderón [10], known nowadays as Calderón's inverse problem. This problem was solved in the fundamental papers by Sylvester and Uhlmann [58], in the three and higher dimensional case, and Nachman [48] and Astala and Päivärinta [4], in the two dimensional case. For a gentle introduction to Calderón's problem and related topics, see for instance [22,31,59] and the references therein. Numerical techniques for the problem are discussed in [47,52]. For the linear hyperbolic equation, the results on uniqueness and their corresponding quantitative versions have been studied using Carleman estimates and the complex geometric optics, see [9,25,49].

Uniqueness results for inverse problem for the wave equation with vanishing initial data are obtained using the boundary control method, originated by Belishev and Kurylev [5,6], that combines the wave propagation and controllability results, see also [29]. The boundary control method allows also an effective numerical algorithm [15]. Recent geometrical results on determining Riemannian manifolds with partial data or with general operators are considered in [2,23,27,32,33,38,41,44]. The boundary control method has been applicable only in the cases where the coefficients of the equation are time-independent, or when the lower order terms are real analytic in the time variable [18].

Inverse problems for linear wave equations with lower order terms depending on the time variable have been considered in [19,54,56]. These methods apply propagation of singularities along bicharacteristics to determine the integrals of the coefficients along rays. These results are closely related to the methods used in this paper with the significant difference that in these results one has to assume that the complete Dirichlet-to-Neumann operator or a scattering operator is known.

A recent observation by Kurylev, Lassas and Uhlmann [37] is that a non-linearity in the studied equation can be used as a beneficial tool in a corresponding inverse problem. By exploiting the non-linearity, some still unsolved inverse problems for linear hyperbolic equations have been solved for their non-linear counterparts. For the scalar wave equation with a quadratic non-linearity, they in [37] proved that local measurements determine the global topology, differentiable structure and the conformal class of the metric g on a globally hyperbolic $3+1$-dimensional Lorentzian manifold. Following this observation, there has been a surge of interesting results for inverse problems for non-linear equations. The authors of [46] studied inverse problems for general semi-linear wave equations on Lorentzian manifolds, and in [45] they studied the analogous problem for the Einstein-Maxwell equations. Recently, inverse problems for non-linear equations using the non-linearity as a tool, have been studied in [3,11–13,16,17,20,21,30,34–36,39,42,43,51,57,60,61]. The works mentioned above use the so-called *higher order linearization* method, which we will explain later.

In this work we continue to use the non-linearity as a tool to prove a stability estimate for the described inverse problem for Equation (1). Our main result is a Hölder stability estimate for recovering an unknown potential a in the inverse problem for Equation (1). We also present a constructive way to approximate a in the presence of additive noise. We do not assume that the noise is a linear mapping. The main idea is to use the non-linearity to approximate "virtual sources" which are multiplied by the unknown potential.

We present our main results next. We denote the *lateral boundary* $\partial\Omega \times [T_1, T_2]$ by

$$\Sigma = \partial\Omega \times [T_1, T_2]$$

or by Σ^{T_1,T_2} if we wish to emphasize the corresponding time interval. Due to the finite propagation speed of solutions to the wave equation, there are natural limitations on the regions of $\mathbb{R}^{n+1}$ where we can obtain information in the inverse problem. We consider recovering the potential $a(x,t)$ in a compact set

$$W \subset \Omega \times [t_1, t_2], \quad \text{for} \quad t_1 < t_2. \tag{2}$$

It is always possible to choose the time-interval $[T_1, T_2]$ for the measurements large enough, so that the potential can be recovered in the set W. Let us denote

$$d := 2\inf\left\{r > 0 \mid \Omega \subset B_r(x), \text{ for some } x \in \mathbb{R}^n\right\}, \tag{3}$$

where $B_r(x)$ is the ball of radius r centered at $x \in \mathbb{R}^n$. By Jung's theorem, $\mathrm{diam}(\Omega) \le d \le \mathrm{diam}(\Omega)\sqrt{2n/(n+1)}$. We can then choose a small $\lambda > 0$ and

$$T_2 \ge t_2 + d + \lambda, \quad T_1 \le t_1 - d - \lambda. \tag{4}$$

With T_1 and T_2 satisfying the above conditions, we may send waves to W from the lateral boundary and measure signals from W at the lateral boundary. We emphasize that we do not assume that the potential a is compactly supported in time. Especially, the potential a can be time-independent.

Finally, since we are interested in stability of the inverse problem, we require *a priori* bound on the norm of the potential a. Given $s \geq 0$ and $L > 0$, let us introduce the class of admissible potentials as follows

$$\mathcal{A}(L,s) := \left\{ a \in C^{s+1}(\Omega \times [0,T]) \mid \|a\|_{C^{s+1}} \leq L \right\}. \tag{5}$$

We note that by a change of variables $\theta : [T_1, T_2] \to [0, T]$, given by $\theta(t) = t - T_1$, we can without loss of generality consider the wave equation on the time-interval $[0, T]$ instead of $[T_1, T_2]$. From this point onwards, we assume that $t_1, t_2 \in \mathbb{R}$ satisfy

$$t_2 > t_1 \geq d + \lambda$$

and that $T > 0$ is chosen so that $T \geq t_2 + d + \lambda$. Here d and λ are as in (3) and (4).

Our first result is the following

Theorem 1 *(Uniqueness). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Let $L > 0$, $m \geq 2$ be an integer and $s + 1 > (n+1)/2$. There is a measurement function $\psi \in L^2(\Sigma)$ such that the following is true: Assume that $a \in \mathcal{A}(L,s)$. Let u_f be the solution to (1) for small enough $f \in H^{s+1}(\Sigma)$.*

Then the real-valued non-linear map

$$\lambda_\psi : f \mapsto \langle \psi, \partial_\nu u_f \rangle_{L^2(\Sigma)}$$

determines $a(x,t)$ uniquely in W.

The *measurement function* $\psi \in L^2(\Sigma)$ appearing in the statement of Theorem 1 is the restriction of a solution of the following backwards wave equation to the lateral boundary:

$$\begin{cases} \Box v_0 = 0, & \text{in } \mathbb{R}^n \times [0,T], \\ v_0 \equiv 1, & \text{in } \Omega \times [t_1, t_2], \\ v_0\big|_{t=T} = \partial_t v_0\big|_{t=T} = 0, & \text{in } \Omega, \\ v_0 \in C_c^\infty(\Sigma). \end{cases} \tag{6}$$

We will construct a suitable solution $v_0 \in C_c^\infty(\mathbb{R}^{n+1})$ to (6) explicitly in Appendix B. The measurement function is defined as the restriction

$$\psi := v_0|_\Sigma \in C_c^\infty(\Sigma) \tag{7}$$

to the lateral boundary Σ. The measurement function will be used in an integration by parts argument to cut off any contribution coming to the integral from the top $\Omega \times \{t = T\}$ of the time-cylinder $\Omega \times [0, T]$. We denote

$$\widetilde{\Sigma} = \Sigma^{0,T} \cap \operatorname{supp}(v_0).$$

Our main result is that reconstruction of $a(x,t)$ from the non-linear map λ_ψ is Hölder stable.

Theorem 2 *(Stability estimate with one dimensional measurements). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Let $W \subset \Omega \times [t_1, t_2]$ be the compact set defined in (2), where t_1 and t_2 are as in (4). Let $L > 0$, $m \geq 2$ be an integer, $r \in \mathbb{R}$ with $r \leq s \in \mathbb{N}$, $s+1 > (n+1)/2$ and $L > 0$. Assume that for $j = 1, 2$, the functions $a_j \in \mathcal{A}(L, s)$ and ψ are as in (7). Additionally, only for the case $n \geq 2$ assume that $a_1 = a_2$ on $\partial\Omega \times [0, T]$. Let $\Lambda_j : H^{s+1}(\Sigma) \to H^r(\widetilde{\Sigma})$ be the Dirichlet-to-Neumann maps of the non-linear wave equation (1).*

Let $\varepsilon_0 > 0$, $M > 0$ and $\delta \in (0, M)$ be such that

$$|\langle \psi, \Lambda_1(f) - \Lambda_2(f)\rangle_{L^2(\widetilde{\Sigma})}| \leq \delta \tag{8}$$

for all $f \in H^{s+1}(\Sigma)$ with $\|f\|_{H^{s+1}(\Sigma)} \leq \varepsilon_0$. Then

$$\|a_1|_W - a_2|_W\|_{L^\infty(W)} \leq C\delta^{\sigma(s)}, \tag{9}$$

where

$$\sigma(s) = \begin{cases} \frac{m-1}{(2m-1)(m(s+2)+1)}, & n = 1, \\ \frac{m-1}{2n(2m-1)(m(s+2)+1)}, & n \geq 2. \end{cases} \tag{10}$$

Theorem 1 follows from Theorem 2 by letting $\delta \to 0$. Note that in the theorem we assume that our boundary values may be supported on all of Σ. However, we only assume that the measurements are made on a smaller subset $\widetilde{\Sigma} = \Sigma \cap \mathrm{supp}(v_0)$ of Σ.

In fact, we emphasize that to recover the potential $a \in C^{s+1}(W)$ in a stable way it is sufficient to make *one dimensional measurements*

$$\lambda_\psi : f \mapsto \langle \psi, \Lambda(f)\rangle_{L^2(\widetilde{\Sigma})} \in \mathbb{R}$$

on $\widetilde{\Sigma}$. Here ψ can be considered as an instrument function that models the measurement instrument that is used to do observations on the solution u. Note that ψ is a smooth function that is a constant on $\partial\Omega \times [\lambda, T - \lambda]$. This means that $a(x, t)$ can be recovered from *low resolution measurements* if we can accurately control the source f.

Corollary 1. *Let us adopt assumptions and notations of Theorem 2. Instead of condition (8), suppose that*

$$\|\Lambda_1(f) - \Lambda_2(f)\|_{H^r(\widetilde{\Sigma})} \leq \delta$$

for all $f \in H^{s+1}(\Sigma)$ with $\|f\|_{H^{s+1}(\Sigma)} \leq \varepsilon_0$. Then the stability estimate (9) is valid.

There are stability estimates for the recovery of the potentials a and b of the corresponding linear drift wave equation $\Box u + b\partial_t u + au = 0$, see for example [26], where the authors obtained a local Hölder stability result for this problem when given measurements on a part Γ_0 of the lateral boundary Σ. In a related spirit, one might ask is it possible to recover a Riemannian metric g when given the Dirichlet to Neumann map for the equation $(\partial_t^2 - \Delta_g)u = 0$. Some earlier results in this direction are based on Tataru's unique continuation principle. In this case, stability estimates are of logarithmic type, see e.g. [8]. However, later these results have been improved by using different techniques. In [55] it was shown that a simple Riemannian metric g can be recovered in a Hölder stable way from the DN map.

We also consider the question of reconstruction of the unknown potential when there is possibly noise $\mathcal{E}$ involved in the measurements. Assume that for all boundary data f we are given the noisy measurement $\Lambda(f)+\mathcal{E}(f)$. As $f\mapsto\Lambda(f)$ is a nonlinear operator, it is natural to assume that the noise term $f\mapsto\mathcal{E}(f)$ can also be non-linear. We assume that the noise is a bounded, possibly non-linear, mapping $H^{s+1}(\Sigma)\to H^r(\Sigma)$, $r\in\mathbb{R}$ and $r\le s$. Allowing $r\le s$ is natural since in general the noise can not be expected to be as smooth as the measurements collected by the Dirichlet-to-Neumann map.

We present our reconstruction and stability results with noise in $\mathbb{R}^{1+1}$. The general case of $\mathbb{R}^{n+1}$, $n\ge 2$ will be given in Proposition 4, see Section 3. The reason is that the statement for $n\ge 2$ involves a Radon transformation. We write $\varepsilon=0$ for the condition $\varepsilon_1=\cdots=\varepsilon_m=0$.

Theorem 3 *(Reconstruction, $n=1$). Let $\Omega\subset\mathbb{R}$ be an interval. Let $m\ge 2$ be an integer, $r\in\mathbb{R}$ with $r\le s\in\mathbb{N}$, $s+1>(1+1)/2$ and $L>0$. Assume that $a\in\mathcal{A}(L,s)$ and let $\Lambda: H^{s+1}(\Sigma^T)\to H^r(\widetilde{\Sigma})$ be the Dirichlet-to-Neumann map of the non-linear wave equation (1). Assume also that $\mathcal{E}: H^{s+1}(\Sigma)\to H^r(\Sigma)$.*

Let $\varepsilon_0>0$, $M>0$ and $\delta\in(0,M)$ be such that

$$\|\mathcal{E}(f)\|_{H^r(\Sigma^T)}\le\delta,$$

for all $f\in H^{s+1}(\Sigma)$ with $\|f\|_{H^{s+1}(\Sigma)}\le\varepsilon_0$.

Then there exist $\tau\ge 1$, $\varepsilon_1,\dots,\varepsilon_m>0$ and a finite family of functions $\{H_j^{\tau,(x_0,t_0)}\}\subset H^{s+1}(\Sigma^T)$ where $j=1,\dots,m$; such that

$$\begin{aligned}\sup_{(x_0,t_0)\in\Omega\times[0,T]}\Bigg|&a(x_0,t_0)\\&+\frac{1}{2\pi}D^m_{\varepsilon_1,\dots,\varepsilon_m}\Big|_{\varepsilon=0}\int_{\widetilde{\Sigma}}\psi\,(\Lambda+\mathcal{E})(\varepsilon_1H_1^{\tau,(x_0,t_0)}+\cdots+\varepsilon_mH_m^{\tau,(x_0,t_0)})dS\Bigg|\\\le C\delta^{\sigma(s)}&\end{aligned}\tag{11}$$

for all $(x_0,t_0)\in\mathbb{R}^{1+1}$. Here $\sigma(s)$ and C are as in Theorem 2 and the measurement function ψ is as in (7). The finite difference operator $D^m_{\varepsilon_1,\dots,\varepsilon_m}\big|_{\varepsilon=0}$ is defined in (25).

In higher dimensions the situation is somewhat different. Using a similar approach as in $1+1$ dimensions, we get an estimate similar to (11) for the Radon transform $\mathcal{R}(a)$ in place of a, see Proposition 4 in Section 3. The knowledge of $\mathcal{R}(a)$ allows us to get information of the unknown potential in a negative Sobolev index (by using the Fourier slice theorem, see Section 3.1). Then Theorem 2 for higher dimensions $n\ge 2$ follows by combining this fact with a standard interpolation argument. In fact, the term $1/(2n)$ in the exponent $\sigma(s)$ in (10) comes from this interpolation step. We point out that the condition $a_1=a_2$ on $\partial\Omega\times[0,T]$ in Theorem 2 is a technical assumption we need to operate with Radon transformation on Ω when $\dim(\Omega)\ge 2$. With more careful analysis the assumption can likely be removed. The assumption is used only in Lemma 8. The definition of the Radon transform and its relevant properties can be found in Section 3.

Let us explain how we prove Theorem 2. The proof is based on the higher order linearization method, which was used in many of the works mentioned earlier. We now explain this method. We will also use an integration by parts argument introduced in the study of partial data inverse problem for non-linear elliptic equations in [35,43]. Similar argument was also used recently in [24].

We first explain how we can recover the potential a from the DN map Λ of the equation (1). Let us consider the case $m=2$. Let $f_1, f_2 \in H^{s+1}(\Sigma)$, and let us denote by $u_{\varepsilon_1 f_1+\varepsilon_2 f_2}$ the solution to (1) with boundary data $\varepsilon_1 f_1+\varepsilon_2 f_2$, where $\varepsilon_1, \varepsilon_2$ are sufficiently small parameters. By taking the mixed derivative of the equation (1) with respect to the parameters ε_1 and ε_2, and of the solution $u_{\varepsilon_1 f_1+\varepsilon_2 f_2}$, we see that

$$w:=\frac{\partial}{\partial \varepsilon_1}\frac{\partial}{\partial \varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} u_{\varepsilon_1 f_1+\varepsilon_2 f_2}$$

solves

$$\Box w=-2 a v_1 v_2 \tag{12}$$

with zero initial and Dirichlet boundary data. Here the functions v_j solve

$$\begin{cases}\Box v_j=0, & \text{in } \Omega\times[0,T],\\ v_j=f_j, & \text{on } \partial\Omega\times[0,T],\\ v_j\big|_{t=0}=\partial v_j\big|_{t=0}=0, & \text{in } \Omega,\end{cases}$$

for $j=1,2$. This way we have produced new linear equations from the non-linear equation (1). Studying these new equations in inverse problems for non-linear equations is known as the higher order linearization method.

If we assume that the DN map Λ is known, then the normal derivative of w is also known on Σ since

$$\partial_\nu w=\partial^2_{\varepsilon_1\varepsilon_2}|_{\varepsilon_1=\varepsilon_2=0}\Lambda(\varepsilon_1 f_1+\varepsilon_2 f_2).$$

We let v_0 be an auxiliary function solving $\Box v_0=0$ with $v_0|_{t=T}=\partial_t v_0|_{t=T}=0$ in Ω. The function v_0 will compensate the fact we know $\partial_\nu w$ only on the lateral boundary Σ. Then, by multiplying (12) by v_0, and integrating by parts on $\Omega\times[0,T]$, we have the integral identity

$$\int_\Sigma v_0\partial^2_{\varepsilon_1\varepsilon_2}|_{\varepsilon_1=\varepsilon_2=0}\Lambda(\varepsilon_1 f_1+\varepsilon_2 f_2)\mathrm{d}S=\int_{\Omega\times[0,T]} v_0\Box w\,\mathrm{d}x\,\mathrm{d}t=-2\int_{\Omega\times[0,T]} a v_0 v_1 v_2\,\mathrm{d}x\,\mathrm{d}t.$$

Thus the integral

$$\int_{\Omega\times[0,T]} a v_0 v_1 v_2\,\mathrm{d}x\,\mathrm{d}t \tag{13}$$

is known from the knowledge of the DN map Λ. Since v_1 and v_2 were arbitrary solutions to $\Box v = 0$, we may choose suitable solutions v_1 and v_2 so that the products of the form $v_0 v_1 v_2$ become dense in $L^1(\Omega \times [0,T])$. This recovers a.

Heuristically, in $1+1$ dimensions it would be sufficient to have $v_1 = \delta((x-x_0)-(t-t_0))$ and $v_2 = \delta((x-x_0)+(t-t_0))$ to recover $a(x_0,t_0)$ for $(x_0,t_0) \in \mathbb{R}^{1+1}$. Here δ is the 1-dimensional delta function. In this case $v_1 v_2$ is the delta function $\delta_{(x_0,t_0)}$ of $\mathbb{R}^{1+1}$ with mass at (x_0,t_0). However, since our theorems regard relatively smooth data, we will instead use *approximate delta functions*. In higher dimensions, different choices of v_1 and v_2 reduce the integral (13) to a Radon transformation of a on $\mathbb{R}^n$, which is stably invertible.

Instead of differentiating equation (1), to obtain stability we will take the mixed finite difference $D^2_{\varepsilon_1,\varepsilon_2}$ of $u_{\varepsilon_1 f_1+\varepsilon_2 f_2}$. See Appendix C for a definition of $D^2_{\varepsilon_1,\varepsilon_2}$ and higher order finite differences. In this case, we have the following integral identity

$$\int_{\Omega\times[0,T]} a v_0 v_1 v_2 \mathrm{d}x\,\mathrm{d}t = \int_{\Sigma} v_0 D^2_{\varepsilon_1,\varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} \Lambda(\varepsilon_1 f_1 + \varepsilon_2 f_2)\mathrm{d}S$$
$$+ \frac{1}{\varepsilon_1\varepsilon_2} \int_{\Omega\times[0,T]} v_0 \Box \widetilde{\mathcal{R}} \mathrm{d}x\,\mathrm{d}t,$$

where $\widetilde{\mathcal{R}} = \mathcal{O}_{E^{s+2}}(\langle\varepsilon_1,\varepsilon_2\rangle^3)$ in an energy space norm, for details see (15) and (26)–(27). Here we denote by $\langle\varepsilon_1,\varepsilon_2\rangle^3$ homogeneous polynomials of order 3 in ε_1 and ε_2. Stability result Theorem 2 will follow by optimizing in ε_1 and ε_2 and parameters related to the solutions v_1 and v_2. The proof of Theorem 3 follows from a similar argument.

The paper is organized as follows. In Section 1, we lay out the basic properties for semilinear hyperbolic equations that we use. This includes the well-posedness of the boundary value problem for the equation (1). We also calculate formulas for the second order finite differences of solutions to (1) in Section 1. In Section 2, we prove Theorems 2 and 3 in $1+1$ dimensions, and in Section 3 we prove these theorems in higher dimensions. We have placed some proofs in the appendices.

1. Forward problem and definition of the DN map

In this section we study the existence of solutions to the boundary value problem of a nonlinear wave equation in $\mathbb{R}^{n+1}$:

$$\begin{cases} \Box u + a u^m = 0, & \text{in } \Omega\times[0,T], \\ u = f, & \text{on } \partial\Omega\times[0,T], \\ u\big|_{t=0} = \partial_t u\big|_{t=0} = 0, & \text{in } \Omega. \end{cases}$$

Let Ω be an open subset of $\mathbb{R}^n$ with smooth boundary. Let $s\in\mathbb{N}$ and let us denote for the sake of brevity

$$X^s(\Omega) := C([0,T]; H^s(\Omega)) \cap C^s([0,T]; L^2(\Omega)).$$

If there is no danger of misunderstanding, we simply denote $X^s(\Omega)$ by X^s, or just by X if the index s is additionally known from the context. The norm of the Banach space $X^s(\Omega)$ is given by

$$\|f\|_{X^s} := \sup_{0<t<T}\Big(\|f(\,\cdot\,,t)\|_{H^s(\Omega)} + \|\partial_t^s f(\,\cdot\,,t)\|_{L^2(\Omega)}\Big).$$

To prove existence of small solutions for the non-linear wave equation, we consider the linear initial-boundary value problem

$$\begin{cases} \Box u = F, & \text{in } \Omega\times[0,T],\\ u = f, & \text{on } \partial\Omega\times[0,T],\\ u\big|_{t=0} = \psi_0, \quad \partial_t u\big|_{t=0} = \psi_1, & \text{in } \Omega \end{cases}$$

for the linear wave operator. The standard *compatibility conditions of order s* for this problem are given as

$$\begin{aligned} f|_{t=0} &= \psi_0|_{\partial\Omega}, \quad \partial_t f|_{t=0} = \partial_t u|_{\partial\Omega\times\{0\}} = \psi_1|_{\partial\Omega},\\ \partial_t^2 f|_{t=0} &= \partial_t^2 u|_{\partial\Omega\times\{0\}} = \Delta\psi_0|_{\partial\Omega} + F|_{\partial\Omega\times\{0\}}, \end{aligned}$$

and similarly for the higher order derivatives up to order s. These conditions guarantee that at the boundary $\partial\Omega$ the initial data (ψ_0,ψ_1) matches with the corresponding boundary condition f, see [29, Section 2.3.7]. Especially, if $\partial_t^k f|_{t=0}=0$ for all $k=0,\dots,s$, and $F\equiv 0$ and $\psi_0\equiv\psi_1\equiv 0$, then the compatibility conditions of order s are true. We will use the following result from the book [29, Theorem 2.45], see also [40].

Proposition 1 *(Existence and estimates for linear equation [29]). Let $s\in\mathbb{N}$ and $0<T<\infty$. Assume that $F\in L^1([0,T];H^s(\Omega))$, $\partial_t^s F\in L^1([0,T];L^2(\Omega))$, $\psi_0\in H^{s+1}(\Omega)$, $\psi_1\in H^s(\Omega)$ and $f\in H^{s+1}(\Sigma)$. If all the compatibility conditions up to the order s are satisfied, then the problem*

$$\begin{cases} \Box u = F, & \textit{in } \Omega\times[0,T],\\ u = f, & \textit{on } \partial\Omega\times[0,T],\\ u\big|_{t=0} = \psi_0, \quad \partial_t u\big|_{t=0} = \psi_1, & \textit{in } \Omega \end{cases} \tag{14}$$

has a unique solution u satisfying

$$u\in X^{s+1}(\Omega) \textit{ and } \partial_\nu u|_\Sigma \in H^s(\Sigma).$$

Moreover, we have the following estimate for all $t\in[0,T]$

$$\begin{aligned} &\|u(\,\cdot\,,t)\|_{H^{s+1}(\Omega)} + \|\partial_t^{s+1}u(\,\cdot\,,t)\|_{L^2(\Omega)} + \|\partial_\nu u\|_{H^s(\Sigma)}\\ &\quad\le c_T\Big(\|F\|_{L^1([0,T];H^s(\Omega))} + \|\partial_t^s F\|_{L^1([0,T];L^2(\Omega))}\\ &\qquad\qquad + \|\psi_0\|_{H^{s+1}(\Omega)} + \|\psi_1\|_{H^s(\Omega)} + \|f\|_{H^{s+1}(\Sigma)}\Big). \end{aligned}$$

Let us define the *energy spaces* E^s (see e.g. [14, Definition 3.5 on page 596]) of functions in $\Omega \times [0, T] \subset \mathbb{R}^{n+1}$:

$$E^s = \bigcap_{0 \le k \le s} C^k([0, T]; H^{s-k}(\Omega)).$$

These spaces are equipped with the norm

$$\|u\|_{E^s} = \sup_{0<t<T} \sum_{0 \le k \le s} \|\partial_t^k u(\cdot, t)\|_{H^{s-k}(\Omega)}. \tag{15}$$

The reason why we are considering the spaces E^s is that if $s > (n+1)/2$, then E^s is an algebra (see e.g. [14]) and we have the norm estimate

$$\|uv\|_{E^s} \le C_s \|u\|_{E^s} \|v\|_{E^s}, \text{ for all } u, v \in E^s.$$

We record the following consequence of Proposition 1, which we will use extensively. We have placed its proof in the Appendix A.

Corollary 2. *Adopt the notation and assumptions of Proposition 1. Assume in addition that*

$$\partial_t^k F \in L^1([0, T]; H^{s-k}(\Omega)), \quad k = 0, 1, \dots, s.$$

Then the solution u to (14) satisfies

$$u \in E^{s+1}(\Omega) \text{ and } \partial_\nu u|_\Sigma \in H^s(\Sigma)$$

and

$$\begin{aligned} \|u\|_{E^{s+1}} + \|\partial_\nu u\|_{H^s(\Sigma)} \le c_{s,T} \Big(&\sum_{0 \le k \le s} \|\partial_t^k F\|_{L^1([0,T]; H^{s-k}(\Omega))} \\ &+ \|\psi_0\|_{H^{s+1}(\Omega)} + \|\psi_1\|_{H^s(\Omega)} + \|f\|_{H^{s+1}(\Sigma)} \Big). \end{aligned} \tag{16}$$

The proofs of the following results are quite standard, we postpone them to the Appendix A for the interested reader.

Lemma 1. *Let $s + 1 > (n+1)/2$ and $L > 0$. Suppose that $a \in \mathcal{A}(L, s)$. There is $\kappa > 0$ and $\rho > 0$ such that if $f \in H^{s+1}(\Sigma)$ satisfies $\|f\|_{H^{s+1}} \le \kappa$ and $\partial_t^k f|_{t=0} = 0$, $k = 0, \dots, s$, on $\partial\Omega$, there is a unique solution to*

$$\begin{cases} \Box u + a u^m = 0, & \text{in } \Omega \times [0, T], \\ u = f, & \text{on } \partial\Omega \times [0, T], \\ u\big|_{t=0} = \partial_t u\big|_{t=0} = 0, & \text{in } \Omega \end{cases} \tag{17}$$

in the ball

$$B_\rho(0) := \{u \in E^{s+1} \mid \|u\|_{E^{s+1}} < \rho\} \subset E^{s+1}.$$

Furthermore, the solution satisfies the estimate

$$\|u\|_{E^{s+1}} \leq C_0 \|f\|_{H^{s+1}(\Sigma)}, \tag{18}$$

where $C_0 > 0$ is a constant depending only on L, s and T.

We are ready to consider an inverse problem for the non-linear hyperbolic equation

$$\begin{cases} \Box u + au^m = 0, & \text{in } \Omega \times [0,T], \\ u = f, & \text{on } \partial\Omega \times [0,T], \\ u\big|_{t=0} = \partial_t u\big|_{t=0} = 0, & \text{in } \Omega. \end{cases}$$

Our measurement data is the Dirichlet-to-Neumann map Λ, which is a map from a small ball in $H^{s+1}(\Sigma)$ into $H^s(\Sigma)$ and is defined as follows.

Definition 1 *(Dirichlet-to-Neumann map).* Let Ω be an open subset of $\mathbb{R}^n$ and let $s+1 > (n+1)/2$, $s \in \mathbb{N}$. Let $\rho > 0$ be such that for all f with $\|f\|_{H^{s+1}(\Sigma)} < \kappa$ and $\partial_t^k f|_{t=0} = 0$, $k = 0, \dots, s$, the problem (1) has a unique solution $u \in E^{s+1}$ satisfying $\|u\|_{E^{s+1}} < \rho$. The Dirichlet-to-Neumann map Λ is the map $\{f \in H^{s+1}(\Sigma) : \|f\|_{H^{s+1}(\Sigma)} < \kappa\} \to H^s(\Sigma)$ given as

$$\Lambda(f) = \partial_\nu u \text{ on } \Sigma, \quad f \in H^{s+1}(\Sigma),\ \|f\|_{H^{s+1}(\Sigma)} < \kappa,$$

where u is the unique solution to (1) with $\|u\|_{E^{s+1}} < \rho$.

We end this section by an expansion formula for a family of solutions depending on small parameters.

Proposition 2. *Let $s+1 > (n+1)/2$ and $L > 0$. Suppose that $a \in \mathcal{A}(L,s)$. There is $\kappa > 0$ and $\rho > 0$ with the following property: If $f_j \in H^{s+1}(\Sigma)$ and $\varepsilon_j > 0$ satisfy $\|\varepsilon_1 f_1 + \cdots \varepsilon_m f_m\|_{H^{s+1}} \leq \kappa$ and $\partial_t^k f_j|_{t=0} = 0$ on $\partial\Omega$, $k = 0, \dots, s$, $j = 1, \dots, m$, then there exists a unique solution u to*

$$\begin{cases} \Box u + au^m = 0, & \text{in } \Omega \times [0,T], \\ u = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_m f_m, & \text{on } \partial\Omega \times [0,T], \\ u\big|_{t=0} = 0, \quad \partial_t u\big|_{t=0} = 0, & \text{in } \Omega \end{cases} \tag{19}$$

in the ball

$$B_\rho(0) := \{u \in E^{s+1} \mid \|u\|_{E^{s+1}} < \rho\} \subset E^{s+1}.$$

The solution satisfies the estimate

$$\|u\|_{E^{s+1}} \leq C_0 \|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}, \tag{20}$$

where $C_0 > 0$ is a constant depending only on s, T and L. Furthermore, u has the following expansion in $\varepsilon_1, \dots, \varepsilon_m$ in terms of the multinomial coefficients

$$u = \epsilon_1 v_1 + \cdots + \epsilon_m v_m + \sum_{k_1, k_2, \dots, k_m} \binom{m}{k_1, k_2, \cdots, k_m} \epsilon_1^{k_1} \cdots \epsilon_m^{k_m} w_{(k_1, \dots, k_m)} + \mathcal{R}.$$

Here for $j = 1, \dots, m$ the functions v_j satisfy

$$\begin{cases} \Box v_j = 0, & \text{in } \Omega \times [0, T], \\ v_j = f_j, & \text{on } \partial\Omega \times [0, T], \\ v_j\big|_{t=0} = 0, \quad \partial_t v_j\big|_{t=0} = 0, & \text{in } \Omega \end{cases} \tag{21}$$

and for $k_j \in \{1, \dots, m\}$ the functions $w_{k_1, \dots, k_m}$ satisfy

$$\begin{cases} \Box w_{k_1, \dots, k_m} + a v_1^{k_1} \cdots v_m^{k_m} = 0, & \text{in } \Omega \times [0, T], \\ w_{k_1, \dots, k_m} = 0, & \text{on } \partial\Omega \times [0, T], \\ w_{k_1, \dots, k_m}\big|_{t=0} = 0, \quad \partial_t w_{k_1, \dots, k_m}\big|_{t=0} = 0, & \text{in } \Omega \end{cases} \tag{22}$$

and

$$\begin{aligned} \|\mathcal{R}\|_{E^{s+2}} &\le c(s, T) \|a\|_{E^{s+1}}^2 \|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}^{2m-1}, \\ \|\Box \mathcal{R}\|_{E^{s+1}} &\le C(s, T) \|a\|_{E^{s+1}}^2 \|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}^{2m-1}. \end{aligned} \tag{23}$$

Proof. First, inequality (20) immediately follows from (18). Then we note that $\mathcal{F} = u - (\varepsilon_1 v_1 + \varepsilon_2 v_2 + \cdots + \varepsilon_m v_m)$ satisfies

$$\begin{cases} \Box \mathcal{F} = -a u^m, & \text{in } \Omega \times [0, T], \\ \mathcal{F} = 0, & \text{on } \partial\Omega \times [0, T], \\ \mathcal{F}\big|_{t=0} = 0, \partial_t \mathcal{F}\big|_{t=0} = 0, & \text{in } \Omega. \end{cases}$$

Hence, by (20) and by using the energy estimate from Corollary 2, one obtains

$$\|\mathcal{F}\|_{E^{s+2}} \le C(s, T) \|a u^m\|_{E^{s+1}} \le C(s, T) \|a\|_{E^{s+1}} \|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}^m. \tag{24}$$

Here we have used that E^{s+1} is an algebra and the estimate (20):

$$\|u\|_{E^{s+1}} \le C \|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|_{H^{s+1}(\Sigma)}.$$

One step further, taking into account (22), the function $\mathcal{R}$ given by

$$\mathcal{R} := u - (\epsilon_1 v_1 + \cdots + \epsilon_m v_m) - \sum_{k_1, k_2, \dots, k_m} \binom{m}{k_1, k_2, \cdots, k_m} \epsilon_1^{k_1} \cdots \epsilon_m^{k_m} w_{k_1, \dots, k_m}$$

satisfies

$$\begin{cases} \Box \mathcal{R} = -au^m + a(\varepsilon_1 v_1 + \varepsilon_2 v_2 + \cdots + \varepsilon_m v_m)^m, & \text{in } \Omega \times [0,T], \\ \mathcal{R} = 0, & \text{on } \partial\Omega \times [0,T], \\ \mathcal{R}\big|_{t=0} = 0, \quad \partial_t \mathcal{R}\big|_{t=0} = 0, & \text{in } \Omega. \end{cases}$$

Using this identity together with the estimate (20) and (24), we obtain

$$\begin{aligned} \|\Box \mathcal{R}\|_{E^{s+1}} &\le C(s,T)\|-au^m + a(\varepsilon_1 v_1 + \cdots + \varepsilon_m v_m)^m\|_{E^{s+1}} \\ &= C(s,T)\|a(u - (\varepsilon_1 v_1 + \cdots \varepsilon_m v_m))P_{m-1}(u, \varepsilon_1 v_1 + \cdots + \varepsilon_m v_m)\|_{E^{s+1}} \\ &\le C(s,T)\|a\|_{E^{s+1}}\|\mathcal{F}\|_{E^{s+2}}\|P_{m-1}(u, \varepsilon_1 v_1 + \cdots + \varepsilon_m v_m)\|_{E^{s+1}} \\ &\le C(s,T,m)\|a\|^2_{E^{s+1}}\|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|^m_{H^{s+1}(\Sigma)} \\ &\qquad \times \Big(\sum_{l=0}^{m-1}\|u^{m-1-l}(\varepsilon_1 v_1 + \cdots + \varepsilon_m v_m)^l\|_{E^{s+1}}\Big), \\ &\le C(s,T,m)\|a\|^2_{E^{s+1}}\|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}. \end{aligned}$$

Here we wrote

$$u^m - v^m = (u - v)P_{m-1}(u, v),$$

where $P_{m-1}(a,b) = \sum_{k=0}^{m-1} a^{m-1-k}b^k$. In the last inequality we used (20). Thus it follows from the energy estimate (16) that

$$\|\mathcal{R}\|_{E^{s+2}} \le c(s,T,m)\|a\|^2_{E^{s+1}}\|\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}. \quad \Box$$

We next derive the integral identity (28) below, which relates the unknown potential with the Dirichlet-to-Neumann map. To this end, note that the finite differences $D^m_{\varepsilon_1,\ldots,\varepsilon_m}$ of the solution $u = u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m}$ of (19) satisfy

$$D^m_{\varepsilon_1,\ldots,\varepsilon_m}\big|_{\varepsilon=0}\Box u = -m!\,a v_1 \cdots v_m + D^m_{\varepsilon_1,\ldots,\varepsilon_m}\big|_{\varepsilon=0}\Box \mathcal{R},$$

where we used (22) with $k_1 = \cdots = k_m = 1$. Here we write $\varepsilon = 0$ when $\varepsilon_1 = \ldots = \varepsilon_m = 0$. For more details, we refer the reader to Appendix C. The finite difference is defined as usual by

$$D^m_{\varepsilon_1,\ldots,\varepsilon_m}\big|_{\varepsilon=0}u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m} = \frac{1}{\varepsilon_1 \cdots \varepsilon_m}\sum_{\sigma \in \{0,1\}^m}(-1)^{|\sigma|+m}u_{\sigma_1\varepsilon_1 f_1 + \ldots + \sigma_m \varepsilon_m f_m}. \tag{25}$$

For example, when $m = 2$, we have

$$D^2_{\epsilon_1,\epsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}u := \frac{1}{\epsilon_1\epsilon_2}\left(u_{\varepsilon_1 f_1 + \varepsilon_2 f_2} - u_{\varepsilon_1 f_1} - u_{\varepsilon_2 f_2}\right).$$

Here we used the fact that the solution to (19) with $\varepsilon_1 = \varepsilon_2 = 0$ is identically zero.

Let v_0 be an auxiliary function solving $\Box v_0 = 0$ with $v_0|_{t=T} = \partial_t v_0|_{t=T} = 0$ in Ω. By integrating by parts and using (22), we obtain

$$\begin{aligned}
&\int_\Sigma v_0 D^m_{\varepsilon_1,\dots,\varepsilon_m}\big|_{\varepsilon=0}\Lambda(\varepsilon_1 f_1+\cdots+\varepsilon_m f_m)\mathrm{d}S\\
&\quad=\int_\Sigma v_0 D^m_{\varepsilon_1,\dots,\varepsilon_m}\big|_{\varepsilon=0}\partial_\nu u_{\varepsilon_1 f_1+\cdots+\varepsilon_m f_m}\mathrm{d}S\\
&\quad=m!\int_{\Omega\times[0,T]} v_0\Box w_{1,1,\dots,1}\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon_1\cdots\varepsilon_m}\int_{\Omega\times[0,T]} v_0\Box\widetilde{\mathcal{R}}\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Here we denoted

$$\widetilde{\mathcal{R}}:=\varepsilon_1\varepsilon_2\dots\varepsilon_m D^m_{\varepsilon_1,\dots,\varepsilon_m}\big|_{\varepsilon=0}\mathcal{R}, \tag{26}$$

and $\widetilde{\mathcal{R}}$ satisfies

$$\begin{aligned}
\|\widetilde{\mathcal{R}}\|_{E^{s+2}}&\le c(s,T)\|a\|^2_{E^{s+1}}\sum_{\sigma\in\{0,1\}^m}\|\sigma_1\varepsilon_1 f_1+\cdots+\sigma_m\varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)},\\
\|\Box\widetilde{\mathcal{R}}\|_{E^{s+1}}&\le C(s,T)\|a\|^2_{E^{s+1}}\sum_{\sigma\in\{0,1\}^m}\|\sigma_1\varepsilon_1 f_1+\cdots+\sigma_m\varepsilon_m f_m\|^{2m-1}_{H^{s+1}(\Sigma)}.
\end{aligned}\tag{27}$$

Above we used the notation $\sigma=(\sigma_1,\dots,\sigma_m)$. Summarizing, we have arrived to the following integral identity which connects the potential a with the DN-map Λ.
Integral identity:

$$\begin{aligned}
-m!\int_{\Omega\times[0,T]} a v_0 v_1 v_2\cdots v_m\mathrm{d}x\,\mathrm{d}t&=\int_\Sigma v_0 D^m_{\varepsilon_1,\dots,\varepsilon_m}\big|_{\varepsilon=0}\Lambda(\varepsilon_1 f_1+\cdots+\varepsilon_m f_m)\mathrm{d}S\\
&\quad-\frac{1}{\varepsilon_1\varepsilon_2\cdots\varepsilon_m}\int_{\Omega\times[0,T]} v_0\Box\widetilde{\mathcal{R}}\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{28}$$

We will use this identity several times throughout the text. Recall that $\varepsilon=0$ means $\varepsilon_1=\cdots=\varepsilon_m=0$.

Remark 1. By taking $\varepsilon_j\to 0$, the integral identity (28) implies

$$\int_{\Omega\times[0,T]} a v_0 v_1 v_2\cdots v_m\mathrm{d}x\,\mathrm{d}t=-\frac{1}{m!}\int_\Sigma \psi\,\partial_{\varepsilon_1}\partial_{\varepsilon_2}\dots\partial_{\varepsilon_m}\big|_{\varepsilon=0}\Lambda(\varepsilon_1 f_1+\cdots+\varepsilon_m f_m)\mathrm{d}S,$$

where $\psi=v_0|_\Sigma$ is a measurement function and v_j, $j=0,1,2,\dots,m$, are the solutions of the linearized equation $\Box v_j=0$. We note that similar identities are encountered in the study of inverse problems for elliptic equations, e.g. $\Delta U(x)+q(x)U(x)^m=0$, $U|_{\partial\Omega}=f$, with the solutions V_j of the linearized equation $\Delta V_j(x)=0$, see [43]. As the constant function $V_0=1$ satisfies the linearized equation, one could use a similar approach to the one used in this paper to study the above non-linear elliptic equation with the one-dimensional boundary map $f\mapsto\langle\Psi,\partial_\nu U|_{\partial\Omega}\rangle_{L^2(\partial\Omega)}$ and the measurement function $\Psi=1$. However, these considerations are outside the context of this paper.

2. Proofs of the main results

2.1. *Proof of Theorem 2 in* $1+1$ *dimensions with* $m=2$

We prove Theorem 2 in $\mathbb{R}^{1+1}$ with $m=2$ and $m>2$ separately. In this section we consider only the case $m=2$. The proof for the case $m>2$ is very similar, but it uses some definitions we will introduce only in Section 3. We give a proof for the case $m>2$ at the end of Section 3. The proof will be divided into three steps.

Step 1. Let $\varepsilon_j>0$, $j=1,2$, and $f_j\in H^{s+1}(\Sigma)$ be functions that satisfy $\partial_t^k f_j|_{t=0}=0$, $k=0,\dots,s$, on $\partial\Omega$. Suppose also that $\|\varepsilon_1 f_1+\varepsilon_2 f_2\|_{H^{s+1}(\Omega\times[0,T])}\le\kappa$ for $\kappa>0$ small enough, as in Lemma 1. Then, for $l=1,2$, and according to Proposition 2, we have that the problem

$$\begin{cases}\Box u_l+a_l u_l^2=0, & \text{in } \Omega\times[0,T],\\ u_l=\varepsilon_1 f_1+\varepsilon_2 f_2, & \text{on } \partial\Omega\times[0,T],\\ u_l\big|_{t=0}=0,\quad \partial_t u_l\big|_{t=0}=0, & \text{in } \Omega\end{cases}\tag{29}$$

has a unique solution u_l with an expansion of the form

$$u_l=\varepsilon_1 v_{l,1}+\varepsilon_2 v_{l,2}+2\varepsilon_1\varepsilon_2 w_{l,(1,1)}+\varepsilon_1^2 w_{l,(2,0)}+\varepsilon_2^2 w_{l,(0,2)}+\mathcal{R}_l,$$

where $v_{l,j}$ and $w_{l,(k_1,k_2)}$, $l,k_1,k_2=1,2$, solve (21) and (22) with a replaced with a_l. Note that since the equation for $v_{l,j}$ is independent of a_l, we have by the uniqueness of solutions that

$$v_{1,j}=v_{2,j}=:v_j,\qquad j=1,2.$$

By (23), the correction term $\mathcal{R}_l$ satisfies

$$\|\Box\mathcal{R}_l\|_{E^{s+1}}\le C(s,T)\|a_l\|^2_{E^{s+1}}\|\varepsilon_1 f_1+\varepsilon_2 f_2\|^3_{H^{s+1}(\Sigma)},\qquad l=1,2.\tag{30}$$

We have that the mixed second difference $D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}$ of u_l is

$$D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}u_l=2w_{l,(1,1)}+\frac{1}{\varepsilon_1\varepsilon_2}\mathcal{R}_l,\qquad l=1,2.$$

Consequently

$$\Box D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}u_l=-2a_l v_1 v_2+\frac{1}{\varepsilon_1\varepsilon_2}\Box\widetilde{\mathcal{R}}_l,\qquad l=1,2,\tag{31}$$

where $\widetilde{\mathcal{R}}_l:=\varepsilon_1\varepsilon_2 D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}\mathcal{R}_l$ similarly as in (26).

As the first step we derive a useful integral identity which relates the DN maps Λ_1 and Λ_2 with the information of the unknown potentials a_1 and a_2 in $\Omega\times[0,T]$. We recall that v_0 is an auxiliary function given in (6). Combining (31) and the fact that v_0 satisfies $\Box v_0=0$ with $v_0|_{t=T}=\partial_t v_0|_{t=T}=0$ in Ω, we get

$$
\begin{aligned}
&\int_{\Sigma} v_0 D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}[(\Lambda_1-\Lambda_2)(\varepsilon_1 f_1+\varepsilon_2 f_2)]\,\mathrm{d}S\\
&\quad=\int_{\Sigma} v_0 D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}[\partial_\nu u_1-\partial_\nu u_2]\,\mathrm{d}S\\
&\quad=\int_{\partial\Omega\times[0,T]} v_0\partial_\nu\left[D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}(u_1-u_2)\right]\mathrm{d}S\\
&\quad=\int_{\Omega\times[0,T]} v_0\left[\Box(D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}(u_1-u_2))\right]\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_{\Omega\times[0,T]}(\Box\, v_0)\,D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}(u_1-u_2)\mathrm{d}x\,\mathrm{d}t\\
&\quad=-2\int_{\Omega\times[0,T]} v_0(a_1-a_2)v_1v_2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\epsilon_1\epsilon_2}\int_{\Omega\times[0,T]} v_0\,\Box(\widetilde{\mathcal{R}}_1-\widetilde{\mathcal{R}}_2)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{32}
$$

We remark that this identity is a consequence of manipulating identity (28) applied with $a=a_j$ and $u=u_j$, $j=1,2$. In what follows we denote by $\widetilde{H}^{-r}(\Sigma)$ and $\widetilde{H}^{-(s+1)}(\Omega\times[0,T])$ the dual spaces of $H^r(\Sigma)$ and $H^{s+1}(\Omega\times[0,T])$, respectively, endowed with the following norms

$$
\begin{aligned}
\|w\|_{\widetilde{H}^{-r}(\Sigma)} &:= \sup_{v\in H^r(\Sigma),\ \|v\|_{H^r(\Sigma)}\le 1} |\langle v,w\rangle_{L^2(\Sigma)}|,\\
\|w\|_{\widetilde{H}^{-(s+1)}(\Omega\times[0,T])} &:= \sup_{v\in H^{s+1}(\Omega\times[0,T]),\ \|v\|_{H^{s+1}(\Omega\times[0,T])}\le 1} |\langle v,w\rangle_{L^2(\Omega\times[0,T])}|.
\end{aligned}
$$

See for example [1] for more details. Below, the constant $\mathcal{Z}=1$, if it is assumed that

$$|\langle v_0,\Lambda_1(f)-\Lambda_2(f)\rangle_{L^2(\widetilde{\Sigma})}|\le\delta,$$

or $\mathcal{Z}=\|v_0\|_{\widetilde{H}^{-r}(\Sigma)}$, if it is assumed that

$$\|\Lambda_1(f)-\Lambda_2(f)\|_{H^r(\widetilde{\Sigma})}\le\delta.$$

These two different assumptions correspond to assumptions in Theorem 2 and Corollary 1, respectively.

As an immediate consequence of the integral identity (32) we obtain

$$
\begin{aligned}
&2\left|\langle v_0(a_1-a_2),v_1v_2\rangle_{L^2(\Omega\times[0,T])}\right|\\
&\le\left|\langle v_0,D^2_{\varepsilon_1,\varepsilon_2}(\Lambda_1-\Lambda_2)(\varepsilon_1f_1+\varepsilon_2f_2)\rangle_{L^2(\Sigma)}\right|+\varepsilon_1^{-1}\varepsilon_2^{-1}\left|\langle v_0,\Box(\widetilde{\mathcal{R}}_1-\widetilde{\mathcal{R}}_2)\rangle_{L^2(\Omega\times[0,T])}\right|\\
&\le 4\,\varepsilon_1^{-1}\varepsilon_2^{-1}\left|\langle v_0,(\Lambda_1-\Lambda_2)(\varepsilon_1f_1+\varepsilon_2f_2)\rangle_{L^2(\Sigma)}\right|\\
&\quad+\varepsilon_1^{-1}\varepsilon_2^{-1}\left|\langle v_0,\Box(\widetilde{\mathcal{R}}_1-\widetilde{\mathcal{R}}_2)\rangle_{L^2(\Omega\times[0,T])}\right|
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&\leq 4\,\delta\,\varepsilon_1^{-1}\,\varepsilon_2^{-1}\,\mathcal{Z} + \varepsilon_1^{-1}\,\varepsilon_2^{-1}\,\|\Box(\widetilde{\mathcal{R}}_1 - \widetilde{\mathcal{R}}_2)\|_{E^{s+1}}\,\|v_0\|_{\widetilde{H}^{-(s+1)}(\Omega\times[0,T])} \\
&\leq \varepsilon_1^{-1}\,\varepsilon_2^{-1}\,(\mathcal{Z} + \|v_0\|_{\widetilde{H}^{-(s+1)}(\Omega\times[0,T])}) \\
&\qquad \times \Big(4\,\delta + C(s,T)(\|a_1\|^2_{E^{s+1}} + \|a_2\|^2_{E^{s+1}})(\varepsilon_1\|f_1\|_{H^{s+1}(\Sigma)} + \varepsilon_2\|f_2\|_{H^{s+1}(\Sigma)})^3\Big) \\
&\leq C\,\varepsilon_1^{-1}\,\varepsilon_2^{-1}\Big(\delta + (\varepsilon_1\|f_1\|_{H^{s+1}(\Sigma)} + \varepsilon_2\|f_2\|_{H^{s+1}(\Sigma)})^3\Big),
\end{aligned}
$$

where we denoted

$$
C = \max\Big\{4, C(s,T)(\|a_1\|^2_{E^{s+1}} + \|a_2\|^2_{E^{s+1}})\Big\}\,(\mathcal{Z} + \|v_0\|_{\widetilde{H}^{-(s+1)}(\Omega\times[0,T])}).
$$

Above we have also used (30) and that $E^{s+1} \subset H^{s+1}(\Omega\times[0,T])$ to bound the term $\Box(\widetilde{\mathcal{R}}_1 - \widetilde{\mathcal{R}}_2)$ in E^{s+1}.

Step 2. The second step is to suitably choose the functions f_1 and f_2 so that they allow us to obtain information about $a_1 - a_2$ from the integral estimate (33). In this step, we shall need the following two technical results.

Lemma 2. *Let $\alpha > 0$, $\gamma \geq 0$ and $\tau \geq 1$. Let $\chi_\alpha \in C_c^\infty(\mathbb{R})$ be a cut-off function supported on $[-\alpha,\alpha]$, $|\chi_\alpha| \leq 1$. Consider the function $H \in C_c^\infty(\mathbb{R})$ defined by*

$$
H(l) = \chi_\alpha(l)\tau^{1/2}\mathrm{e}^{-\frac{1}{2}\tau l^2}, \quad l \in \mathbb{R}.
$$

Let $(x_0,t_0) \in \mathbb{R}^2$ and define

$$
\begin{aligned}
H_1^{\tau,(x_0,t_0)}(x,t) &:= H\big((x-x_0)-(t-t_0)\big), \\
H_2^{\tau,(x_0,t_0)}(x,t) &:= H\big((x-x_0)+(t-t_0)\big).
\end{aligned}
$$

The following estimate holds

$$
\|H_1^{\tau,(x_0,t_0)}\|_{H^\gamma(\Sigma)} + \|H_2^{\tau,(x_0,t_0)}\|_{H^\gamma(\Sigma)} \leq C\,\tau^{\frac{\gamma+1}{2}}.
$$

The constant C is independent of $(x_0,t_0) \in \mathbb{R}^2$.

Proof. Let $(x_0,t_0) \in \mathbb{R}^2$ and $\beta_1,\beta_2 \in \mathbb{N}$. Let us write

$$
F(x,t) = H\big((x-x_0)+(t-t_0)\big).
$$

We have for all $\tau \geq 1$ that

$$
\begin{aligned}
&\|\partial_x^{\beta_1}\partial_t^{\beta_2}F\|^2_{L^2(\Omega\times[0,T])}=\tau\int\limits_\Omega\int\limits_0^T\left[\partial_x^{\beta_1}\partial_t^{\beta_2}\left(\chi_\alpha(x-x_0+t-t_0)\mathrm{e}^{-\frac{\tau}{2}(x-x_0+t-t_0)^2}\right)\right]^2\mathrm{d}t\,\mathrm{d}x\\
&\le C\tau\tau^{2(\beta_1+\beta_2)}\int\limits_\Omega\int\limits_0^T|\chi_\alpha(x-x_0+t-t_0)|^2\,(x-x_0+t-t_0)^{2(\beta_1+\beta_2)}\mathrm{e}^{-\tau(x-x_0+t-t_0)^2}\mathrm{d}t\,\mathrm{d}x\\
&=C\tau\tau^{2(\beta_1+\beta_2)}\int\limits_\Omega\int\limits_{x-x_0-t_0}^{x-x_0+T-t_0}|\chi_\alpha(h)|^2\,h^{2(\beta_1+\beta_2)}\mathrm{e}^{-\tau h^2}\mathrm{d}h\,\mathrm{d}x\\
&\le C\tau\tau^{2(\beta_1+\beta_2)}\int\limits_\Omega\int\limits_{-\infty}^{\infty}|\chi_\alpha(h)|^2\,h^{2(\beta_1+\beta_2)}\mathrm{e}^{-\tau h^2}\mathrm{d}h\,\mathrm{d}x\\
&\le C\tau\tau^{2(\beta_1+\beta_2)}\tau^{-(\beta_1+\beta_2)-1/2}\int\limits_\Omega\mathrm{d}x=C_\Omega\tau^{(\beta_1+\beta_2)+1/2}.
\end{aligned}
$$

Here in the second line we used the fact that the largest power of τ in the calculation happens when all the derivatives hit the exponential and none the cut-off function. Therefore, when $\tau\ge 1$, we may absorb the other terms implicit in the calculation to the constant C. We also made a change of variables

$$
h=x-x_0+t-t_0
$$

in the integral in the variable t, while considering x is fixed. We also used

$$
\int\limits_{\mathbb{R}}h^{2(\beta_1+\beta_2)}e^{-\tau h^2}\mathrm{d}h\sim\tau^{-(\beta_1+\beta_2)-1/2}.
$$

Thus, we have

$$
\|F\|^2_{H^{\beta_1+\beta_2}(\Omega\times[0,T])}\le C_\Omega\tau^{(\beta_1+\beta_2)+1/2}.
$$

By a standard interpolation argument between Sobolev spaces, see for instance [7, Theorem 6.2.4/6.4.5], we then obtain for all $\gamma\ge 0$ that

$$
\|F\|^2_{H^\gamma(\Omega\times[0,T])}\le C\tau^{\gamma+1/2}.
$$

Finally, by using the trace theorem we have

$$
\|F\|^2_{H^\gamma(\Sigma)}\le C\|F\|^2_{H^{\gamma+1/2}(\Omega\times[0,T])}\le C\tau^{\gamma+1}.
$$

Similar argument yields the same estimate for $H_1^{\tau,(t_0,x_0)}$. This completes the proof. □

The Lipschitz semi-norm of a Lipschitz function f by

$$\|f\|_{\mathrm{Lip}} := \inf\{c \geq 0 \mid |f(x) - f(y)| \leq c|x - y|\}.$$

We prove next a couple of lemmas.

Lemma 3. *Let $\tau > 0$. Let $b \in \mathrm{Lip}(\mathbb{R}^2)$ be compactly supported. The following estimate*

$$\left| b(x_0, t_0) - \frac{\tau}{\pi} \int\limits_{\mathbb{R}^2} b(x,t) \mathrm{e}^{-\tau((x-x_0)^2+(t-t_0)^2)} \mathrm{d}x\, \mathrm{d}t \right| \leq \frac{\sqrt{\pi}}{2} \|b\|_{\mathrm{Lip}} \tau^{-1/2}$$

holds true for all $(x_0, t_0) \in \mathbb{R}^2$. In particular, the integral on the left converges uniformly to b when $\tau \to \infty$.

Proof. Without loss of generality we prove the estimate when $(x_0, t_0) = (0, 0)$, because it can be later applied to $b(x + x_0, t + t_0)$ in place of $b(x, t)$. Using polar coordinates, one can see that $\int_{\mathbb{R}^2} \mathrm{e}^{-(x^2+t^2)} \mathrm{d}x\, \mathrm{d}t = \pi$ and $\int_{\mathbb{R}^2} 2\sqrt{x^2 + t^2}\, \mathrm{e}^{-(x^2+t^2)} \mathrm{d}x\, \mathrm{d}t = \pi^{3/2}$. Additionally, note that $\left|b(0, 0) - b(\tau^{-1/2}x, \tau^{-1/2}t)\right| \leq \|b\|_{\mathrm{Lip}} \tau^{-1/2} |(x, t)|$ for all $(x, t) \in \mathbb{R}^2$. Thus we deduce

$$\begin{aligned}
&\left| b(0,0) - \frac{\tau}{\pi} \int\limits_{\mathbb{R}^2} b(x,t) \mathrm{e}^{-\tau(x^2+t^2)} \mathrm{d}x\, \mathrm{d}t \right| \\
&= \left| b(0,0) - \frac{1}{\pi} \int\limits_{\mathbb{R}^2} b(\tau^{-1/2}x, \tau^{-1/2}t) \mathrm{e}^{-(x^2+t^2)} \mathrm{d}x\, \mathrm{d}t \right| \\
&= \left| \frac{1}{\pi} \int\limits_{\mathbb{R}^2} \left(b(0,0) - b(\tau^{-1/2}x, \tau^{-1/2}t) \right) \mathrm{e}^{-(x^2+t^2)} \mathrm{d}x\, \mathrm{d}t \right| \\
&\leq \frac{\tau^{-1/2}}{\pi} \|b\|_{\mathrm{Lip}} \int\limits_{\mathbb{R}^2} |(x,t)| e^{-(x^2+t^2)} \mathrm{d}x\, \mathrm{d}t = \frac{\sqrt{\pi}}{2} \|b\|_{\mathrm{Lip}} \tau^{-1/2}. \quad \square
\end{aligned}$$

The next lemma generalizes Lemma 3. We need it to analyze for the case when (x_0, t_0) lies on Σ. If we take $x_1 = x_0$ then the result in Lemma 4 is exactly the same as in Lemma 3.

Lemma 4. *Let $\tau > 0$, $x_0, x_1, t_0 \in \mathbb{R}$ and assume $x_0 \geq x_1$. Let $b : [x_1, \infty[\times \mathbb{R} \to \mathbb{R}$ be Lipschitz. Define for $s \leq 0$*

$$\Theta(s) := \frac{1}{\sqrt{\pi}} \int\limits_{s}^{\infty} e^{-x^2} dx$$

and note that $\Theta(s, \tau) \in [1/2, 1]$. The following estimate

$$\left| b(x_0,t_0) - \frac{\tau}{\pi\Theta(\sqrt{\tau}(x_1-x_0))} \int\limits_{\mathbb{R}^2\cap\{x\geq x_1\}} b(x,t)\mathrm{e}^{-\tau((x-x_0)^2+(t-t_0)^2)}\mathrm{d}x\,\mathrm{d}t \right| \leq \sqrt{\pi}\,\|b\|_{\mathrm{Lip}}\,\tau^{-1/2}$$

holds true for all $(x_0,t_0)\in\mathbb{R}^2\cap\{x\geq x_1\}$*. In particular, the integral on the left converges uniformly to* b *as* $\tau\to\infty$*.*

Proof. Without loss of generality we may assume that $x_1=t_0=0$ and $x_0\geq 0$. To begin, recall the identities

$$\int\limits_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi} \quad \text{and} \quad \int\limits_{\mathbb{R}^2} \sqrt{x^2+t^2}\,\mathrm{e}^{-(x^2+t^2)}\mathrm{d}x\,\mathrm{d}t = \frac{\pi^{3/2}}{2}.$$

We calculate

$$\begin{aligned}\frac{\tau}{\pi}\int\limits_{0}^{\infty}\int\limits_{-\infty}^{\infty} b(x,t)e^{-\tau[(x-x_0)^2+t^2]}\mathrm{d}t\,\mathrm{d}x &= \frac{\tau}{\pi}\int\limits_{-x_0}^{\infty}\int\limits_{-\infty}^{\infty} b(x+x_0,t)e^{-\tau[x^2+t^2]}\mathrm{d}t\,\mathrm{d}x \\ &= \frac{\tau}{\pi}\int\limits_{-x_0}^{\infty}\int\limits_{-\infty}^{\infty} \big(b(x+x_0,t)-b(x_0,0)\big)e^{-\tau[x^2+t^2]}\mathrm{d}t\,\mathrm{d}x \qquad (34)\\ &\quad + \frac{\tau}{\pi}b(x_0,0)\int\limits_{-x_0}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\tau[x^2+t^2]}\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Here we see that

$$\frac{\tau}{\pi}b(x_0,0)\int\limits_{-x_0}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\tau[x^2+t^2]}\mathrm{d}t\,\mathrm{d}x = b(x_0,0)\frac{1}{\sqrt{\pi}}\int\limits_{-x_0\sqrt{\tau}}^{\infty} e^{-x^2}\mathrm{d}x = \Theta(-\sqrt{\tau}x_0)b(x_0,0).$$

Thus, using (34) and the fact that b is Lipschitz, we can estimate

$$\begin{aligned}&\left|\Theta(-\sqrt{\tau}x_0)b(x_0,0) - \frac{\tau}{\pi}\int\limits_{0}^{\infty}\int\limits_{-\infty}^{\infty} b(x,t)e^{-\tau[(x-x_0)^2+t^2]}\mathrm{d}t\,\mathrm{d}x\right| \\ &\leq \frac{\tau}{\pi}\int\limits_{-x_0}^{\infty}\int\limits_{-\infty}^{\infty} \big|b(x+x_0,t)-b(x_0,0)\big|e^{-\tau[x^2+t^2]}\mathrm{d}t\,\mathrm{d}x \\ &\leq \frac{1}{\pi}\int\limits_{-\sqrt{\tau}x_0}^{\infty}\int\limits_{-\infty}^{\infty} \big|b(\tau^{-\frac{1}{2}}x+x_0,\tau^{-\frac{1}{2}}t)-b(x_0,0)\big|e^{-[x^2+t^2]}\mathrm{d}t\,\mathrm{d}x\end{aligned}$$

$$\leq \frac{\tau^{-\frac{1}{2}}}{\pi}\|b\|_{\mathrm{Lip}}\int_{\mathbb{R}^2}\sqrt{x^2+t^2}\,e^{-[x^2+t^2]}\mathrm{d}t\,\mathrm{d}x$$

$$\leq \frac{\sqrt{\pi}}{2}\|b\|_{\mathrm{Lip}}\tau^{-\frac{1}{2}}.$$

Finally, dividing the above inequality by $\Theta(-\sqrt{\tau}x_0)$, and observing that Θ is monotone and satisfies $\Theta(0)=\frac{1}{2}$ and $\Theta(s)\to 1$ as $s\to-\infty$, we have the claim. $\square$

Let $(x_0,t_0)\in W$, where W is as in (2). For $j=1,2$ consider H_j as in Lemma 2. Note that

$$\Box H_j = 0. \tag{35}$$

We choose

$$v_j = H_j \text{ and } f_j = H_j|_{\Sigma}, \quad j=1,2,$$

where $H_j = H_j^{\tau,(x_0,t_0)}$ is as in Lemma 2 with $\gamma = s+1$ and the cut-off function χ_α so that $\chi_\alpha(0)=1$. We assume that $\alpha<\lambda/2$, where $\lambda>0$ as in (4). In this case f_j vanishes near $\{t=0\}$, and hence $\partial_t^k\big|_{t=0} f_j = 0$, $k=1,\dots,s$, on $\partial\Omega$. Similarly, we will have that the product $H_1H_2=0$ on Σ, since H_1H_2 is supported in a ball of radius $\sqrt{2}\alpha$ centered at (x_0,t_0). From here, we distinguish two cases.

Case 1. When $x_0\in\Omega$. In this case, substituting the choices of v_j into inequality (33), and using Lemma 3 with

$$b(x,t) := v_0(a_1-a_2)\chi_\alpha(x-x_0-(t-t_0))\chi_\alpha(x-x_0+(t-t_0)),$$

we get

$$\begin{aligned}
|(v_0(a_1-a_2))(x_0,t_0)| &\leq \frac{1}{\pi}\left|\int_{\Omega\times[0,T]} v_0(a_1-a_2)H_1H_2\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad + \left|(v_0(a_1-a_2))(x_0,t_0) - \frac{1}{\pi}\int_{\Omega\times[0,T]} v_0(a_1-a_2)H_1H_2\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leq C_{\Omega,T,a_j,\chi_\alpha}\left(2\tau^{-1/2}+\frac{\delta}{2}\varepsilon_1^{-1}\varepsilon_2^{-1}+\varepsilon_1^{-1}\varepsilon_2^{-1}(\varepsilon_1+\varepsilon_2)^3\tau^{\frac{3}{2}s+3}\right)\|v_0\|_{C^1} \\
&\leq \frac{C_{\Omega,T,a_j,\chi_\alpha}M}{\kappa^3}\left(2\tau^{-1/2}+\frac{\kappa^3\delta}{2M}\varepsilon_1^{-1}\varepsilon_2^{-1}+\epsilon_1^{-1}\varepsilon_2^{-1}(\varepsilon_1+\varepsilon_2)^3\tau^{\frac{3}{2}s+3}\right)\|v_0\|_{C^1}.
\end{aligned} \tag{36}$$

Case 2. When $x_0\in\partial\Omega$. Let $x_1\geq x_0$. In this case, motivated by Lemma 4, instead of H_j, we normalize H_j by a constant as

$$\widetilde{H}_j(x,t) := \left[\Theta(\sqrt{\tau}(x_1 - x_0))\right]^{-1/2} H_j(x,t), \quad j = 1, 2.$$

Since $1/2 \le \Theta(s) \le 1$ for all $s \le 0$, the estimates obtained in Lemma 2 still remain valid for $\widetilde{H}_j$ in place of H_j. Moreover, by (35), we also have

$$\Box \widetilde{H}_j = 0.$$

Thus, choosing

$$v_j = \widetilde{H}_j \text{ and } f_j = \widetilde{H}_j|_\Sigma, \quad j = 1, 2,$$

and substituting these choices into inequality (33), and using now Lemma 4 with

$$b(x,t) := v_0(a_1 - a_2)\chi_\alpha(x - x_0 - (t - t_0))\chi_\alpha(x - x_0 + (t - t_0)),$$

we have

$$\begin{aligned} |(v_0(a_1 - a_2))(x_0, t_0)| &\le \pi^{-1}\Theta^{-1}(\sqrt{\tau}(x_1 - x_0)) \left| \int\limits_{\Omega \times [0,T]} v_0(a_1 - a_2) H_1 H_2 \,\mathrm{d}x\,\mathrm{d}t \right| \\ &+ \left| (v_0(a_1 - a_2))(x_0, t_0) - \pi^{-1}\Theta^{-1}(\sqrt{\tau}(x_1 - x_0)) \int\limits_{\Omega \times [0,T]} v_0(a_1 - a_2) H_1 H_2 \,\mathrm{d}x\,\mathrm{d}t \right| \\ &\le C_{\Omega,T,a_j,\chi_\alpha} \left(2\tau^{-1/2} + \frac{\delta}{2} \varepsilon_1^{-1}\varepsilon_2^{-1} + \varepsilon_1^{-1}\varepsilon_2^{-1}(\varepsilon_1 + \varepsilon_2)^3 \tau^{\frac{3}{2}s+3} \right) \|v_0\|_{C^1} \\ &\le \frac{C_{\Omega,T,a_j,\chi_\alpha} M}{\kappa^3} \left(2\tau^{-1/2} + \frac{\kappa^3\delta}{2M} \varepsilon_1^{-1}\varepsilon_2^{-1} + \epsilon_1^{-1}\varepsilon_2^{-1}(\varepsilon_1 + \varepsilon_2)^3 \tau^{\frac{3}{2}s+3} \right) \|v_0\|_{C^1}. \end{aligned} \tag{37}$$

In the last inequalities of (36) and (37), we scaled δ by a constant κ^3/M, which we without loss of generality assume is < 1. This scaling is purely technical and will be clarified in Lemma 5.

Step 3. Our last step is optimizing τ, ε_1 and ε_2 in terms of δ to get the right hand side of (36) as small as possible. The constants 2 and $1/2$ in front of the terms with $\tau^{-1/2}$ and $\delta\epsilon_1^{-1}\epsilon_2^{-1}$ as a factor are used only to simplify the formulas. We begin by setting

$$\epsilon_1 = \epsilon_2 = \epsilon.$$

Note that we have

$$\varepsilon \|f_j\|_{H^{s+1}(\Sigma)} \sim \varepsilon\tau^{\frac{s+2}{2}}, \quad j = 1, 2. \tag{38}$$

To guarantee the unique solvability of the non-linear wave equations (29), we require the quantities on the right-hand side of (38) is bounded by κ as in Lemma 1. The following Lemma 5 shows how to optimally choose the parameters λ and ε of the inverse problem given a priori bounds κ and δ of the forward problem, while keeping the size of the sources $\varepsilon_j f_j$ small in $H^{s+1}(\Sigma)$.

Lemma 5. *For any given $\delta \in (0, M)$ and $\kappa \in (0, 1)$ small enough we find $\varepsilon(\delta, \kappa) = \varepsilon$ and $\tau(\delta, \kappa) = \tau \geq 1$ such that*

$$f(\epsilon, \tau) := 2\tau^{-1/2} + \frac{\kappa^3 \delta}{2M}\epsilon^{-2} + 8\epsilon\, \tau^{\frac{3}{2}s+3} \leq C_{s,M,\kappa}\, \delta^{\frac{1}{6s+15}}$$

and we also have

$$\varepsilon\tau^{\frac{s+2}{2}} \leq \kappa.$$

The constant $C_{s,M,\kappa}$ is independent of δ.

Proof. To simplify notation, let $\widehat{s} := 3s/2 + 3$ and $\gamma_0 = \kappa^3/M$. A direct computation shows that

$$\partial_\epsilon f = -(\gamma_0\delta)\epsilon^{-3} + 8\tau^{\hat{s}}, \quad \partial_\tau f = -\tau^{-3/2} + 8\epsilon\hat{s}\tau^{\hat{s}-1}.$$

Making $\partial_\epsilon f = \partial_\tau f = 0$, we obtain the critical points of f, namely

$$\tau = (8\widehat{s})^{-\frac{6}{4\widehat{s}+3}}(\frac{1}{8}\gamma_0\delta)^{-\frac{2}{4\widehat{s}+3}}, \quad \epsilon = (8\widehat{s})^{\frac{2\widehat{s}}{4\widehat{s}+3}}(\frac{1}{8}\gamma_0\delta)^{\frac{2\widehat{s}+1}{4\widehat{s}+3}}. \tag{39}$$

With these choices of τ and ϵ, one can check that $\tau^{-1/2}$, $(\gamma_0\delta)\epsilon^{-2}$ and $\epsilon\tau^{\widehat{s}}$ are all bounded by $C_s\,(\frac{1}{8}\gamma_0\delta)^{\frac{1}{4\widehat{s}+3}}$. Also, $\tau \geq 1$ for κ small enough.

Furthermore, since

$$\varepsilon\tau^{\frac{\widehat{s}}{3}} = (\frac{1}{8}\gamma_0\delta)^{1/3},$$

we have that

$$\varepsilon\tau^{\frac{s+2}{2}} \leq \frac{\kappa}{2} < \kappa$$

for any $0 < \delta < M$. This finishes the proof. □

Equation (39) in the proof of Lemma 5 also shows how to choose the parameters τ and ε depending on δ and κ. We also see that $\varepsilon\|f_j\|_{H^{s+1}(\Sigma)} \leq \kappa$.

Continuing from (36) by putting $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and then applying Lemma 5 we finally obtain

$$\begin{aligned}
&|(v_0(a_1 - a_2))(x_0, t_0)| \\
&\quad \leq \frac{C_{\Omega,T,a_j,\chi_\alpha} M}{\kappa^3}\left(2\tau^{-1/2} + \frac{\kappa^3\delta}{2M}\varepsilon_1^{-1}\varepsilon_2^{-1} + \varepsilon_1^{-1}\varepsilon_2^{-1}(\varepsilon_1+\varepsilon_2)^3\,\tau^{\frac{3}{2}s+3}\right)\|v_0\|_{C^1} \\
&\quad = \frac{C_{\Omega,T,a_j,\chi_\alpha} M}{\kappa^3}\left(2\tau^{-1/2} + \frac{\kappa^3\delta}{2M}\varepsilon^{-2} + 8\varepsilon\,\tau^{\frac{3}{2}s+3}\right)\|v_0\|_{C^1} \leq C\delta^{\frac{1}{6s+15}}.
\end{aligned} \tag{40}$$

Recall that v_0 satisfies (6). In particular $v_0(x_0, t_0) = 1$. This finishes the proof of Theorem 2. Moreover, by letting $\delta \to 0$ we obtain Theorem 1. □

2.1.1. Proof of Theorem 3 in $1+1$ *dimensions with* $m=2$

We complete the proof of Theorem 3 in the case $1+1$ dimensions with $m=2$. The proof follows from similar arguments we used in the previous section. Let us consider any point $(x_0,t_0)\in W$, where W is as in (2), and let v_0, H_1 and H_2 be as in (6) and Lemma 2 respectively. Let us also set $v_j=H_j$, $f_j=H_j|_\Sigma$, $j=1,2$, as before. Then, as in the proof of Theorem 2 in $1+1$ dimensions with $m=2$, we have

$$\begin{aligned}
&\left|2v_0a(x_0,t_0)+\frac{1}{\pi}D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}\int_\Sigma v_0(\Lambda+\mathcal{E})(\varepsilon_1 f_1+\varepsilon_2 f_2)\mathrm{d}S\right|\\
&\quad\le\left|2v_0a(x_0,t_0)-\frac{2}{\pi}\int_{\Omega\times[0,T]}v_0av_1v_2\mathrm{d}x\,\mathrm{d}t\right|\\
&\qquad+\left|\int_{\Omega\times[0,T]}v_0\frac{1}{\varepsilon_1\varepsilon_2}\Box\widetilde{\mathcal{R}}\mathrm{d}x\,\mathrm{d}t\right|+\left|D^2_{\varepsilon_1,\varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}\int_\Sigma v_0\mathcal{E}(\varepsilon_1 f_1+\varepsilon_2 f_2)\mathrm{d}S\right|\\
&=:I_1+I_2+I_3.
\end{aligned}$$

By using Lemma 3 on the first term I_1 we obtain

$$I_1\le C_{\Omega,T,a}\tau^{-1/2}.$$

The third term is estimated simply by

$$I_3\le C_{\Omega,T,a}\frac{\delta}{\varepsilon_1\varepsilon_2}\|v_0\|_{\widetilde{H}^{-r}(\Sigma)}.$$

The remaining term I_2 can be estimated by using (23) as

$$\begin{aligned}
I_2&\le\frac{C_{\Omega,T,a}}{\varepsilon_1\varepsilon_2}\|\Box\widetilde{\mathcal{R}}\|_{E^{s+1}}\|v_0\|_{\widetilde{H}^{-s-1}(\Sigma)}\\
&\le\frac{C_{\Omega,T,a}}{\varepsilon_1\varepsilon_2}\left(\varepsilon_1\|H_1\|_{H^{s+1}(\Sigma)}+\varepsilon_2\|H_2\|_{H^{s+1}(\Sigma)}\right)^3\|v_0\|_{\widetilde{H}^{-(s+1)}(\Sigma)}\\
&\le\frac{C_{\Omega,T,a}}{\varepsilon_1\varepsilon_2}(\varepsilon_1+\varepsilon_2)^3\tau^{3s/2+3}\|v_0\|_{\widetilde{H}^{-(s+1)}(\Sigma)}.
\end{aligned}$$

Combining everything and changing the constant if necessary, we have that

$$I_1+I_2+I_3\le C_{\Omega,T,a}\left(2\tau^{-1/2}+\frac{\kappa^3\delta}{2M}\epsilon_1^{-1}\epsilon_2^{-1}+\epsilon_1^{-1}\epsilon_2^{-1}\left(\varepsilon_1+\varepsilon_2\right)^3\tau^{\frac{3}{2}s+3}\right),$$

which is the same estimate as in equation (36). Choosing now $\varepsilon_1=\varepsilon_2=\varepsilon$ and optimizing by using Lemma 5 we have the claimed estimate. This completes the proof of Theorem 3. $\square$

3. Proofs of the main results in dimensions $n+1$, $n \geq 2$, and $m \geq 2$

Here we finish the proof of Theorem 2. To do that, we need to first discuss Radon transformation.

3.1. Radon transform

Let f be a function on $\mathbb{R}^n$, which is integrable on each hyperplane in $\mathbb{R}^n$. Each hyperplane can be expressed as the set of solutions x to the equation $x \cdot \theta = \eta$, where $\theta \in S^{n-1}$ is the unit normal of the hyperplane, and $\eta \in \mathbb{R}$. The Radon transform of f is defined by

$$(\boldsymbol{R} f)(\theta, \eta) = \int\limits_{x \cdot \theta = \eta} f(x) \mathrm{d}x = \int\limits_{y \in \theta^\perp} f(\eta\theta + y) \mathrm{d}y$$

whenever the integral is finite. Here $\theta^\perp$ denotes the set of orthogonal vectors to θ. We remark that if a function is supported in a ball of radius M in $\mathbb{R}^n$, then its Radon transformation is supported in its η variable in $[-M, M]$.

There is a natural relation between f and its Radon transform on the Fourier side. This is usually called the *Fourier slice theorem*. This result holds for smooth and compactly supported functions, see for instance [50, Theorem 1.1], but also for a much larger class of functions:

Proposition 3 *(Fourier slice theorem [53, Lemma 4.5]). Let $f \in L^p(\mathbb{R}^n)$ with $1 < p < n/(n-1)$. Then for almost all $\theta \in S^{n-1}$ one has*

$$\mathcal{F}_{\eta \to \sigma}\big((\boldsymbol{R} f)(\theta, \eta)\big)(\sigma) = (2\pi)^{\frac{n-1}{2}} \widehat{f}(\sigma\theta), \quad \textit{for a.e. } \sigma \in \mathbb{R}.$$

Here $\mathcal{F}_{\eta \to \sigma}$ denotes the one dimensional Fourier transform with respect to η and the hat-notation $\widehat{f}$ is used to denote the n-dimensional Fourier transform. More precisely,

$$\mathcal{F}_{\eta \to \sigma}\big((\boldsymbol{R} f)(\theta, \eta)\big)(\sigma) = \int\limits_{\mathbb{R}} e^{-i\eta\sigma} (\boldsymbol{R} f)(\theta, \eta) \mathrm{d}\eta, \quad \widehat{f}(\xi) = \int\limits_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \mathrm{d}x.$$

Using the Fourier slice theorem one can show that the Sobolev $H^{-\beta}$ norm of a function can be estimated by the L^2 norm of its Radon transform, if the Sobolev index is $\beta \geq (n-1)/2$. The following lemma is a special case of [50, Theorem 5.1], but we give a proof for it for the convenience of the reader in Appendix B. When we apply the lemma, the function f there will be the difference of the potentials a_1 and a_2 extended by zero outside Ω.

Lemma 6. *Let $\beta \geq (n-1)/2$. Let $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 < p < n/(n-1)$; with* $\operatorname{supp} f \subset B_M(0)$ *for some $M > 0$. Consider $F \in L^2(S^{n-1} \times [-M, M])$ and assume that there exists a constant $C_0 > 0$ such that*

$$|(\boldsymbol{R} f)(\theta, \eta)| \leq C_0 F(\theta, \eta), \quad a.e\ (\theta, \eta) \in S^{n-1} \times [-M, M].$$

Then we have the following estimate

$$\|f\|_{H^{-\beta}(\mathbb{R}^n)} \leq (2\pi)^{1/2} C_0 \|F\|_{L^2(S^{n-1}\times[-M,M])}.$$

Here C_0 is independent of θ and η.

Lemma 7. *Let $\alpha > 0$, $\gamma \geq 0$ and $\tau \geq 1$. Let $\chi_\alpha \in C_c^\infty(\mathbb{R})$ be a cutoff function supported on $[-\alpha, \alpha]$, $|\chi_\alpha| \leq 1$. Consider $H \in C_c^\infty(\mathbb{R})$ defined by*

$$H(l) = \chi_\alpha(l)\tau^{1/2}\mathrm{e}^{-\frac{1}{2}\tau l^2}.$$

In addition, consider $t_0 \in \mathbb{R}$, $\eta \in \mathbb{R}$ and $\theta \in S^{n-1}$, and define

$$\begin{aligned} H_1^{\tau,(t_0,\theta,\eta)}(x,t) &:= H(x\cdot\theta - t - (\eta - t_0)), \\ H_2^{\tau,(t_0,\theta,\eta)}(x,t) &:= H(-x\cdot\theta - t + (\eta + t_0)). \end{aligned}$$

The following estimate holds

$$\|H_1^{\tau,(t_0,\theta,\eta)}\|_{H^\gamma(\Sigma)} + \|H_2^{\tau,(t_0,\theta,\eta)}\|_{H^\gamma(\Sigma)} \leq C\tau^{\frac{\gamma+1}{2}},$$

where the implicit constant is independent of t_0, θ and η.

The proof of this lemma is similar to Lemma 2 and can be found in Appendix B.

Let $\Omega \subset \mathbb{R}^n$. We write $\mathcal{R}(G)$ for the *partial Radon transformation* of a function $G = G(x,t) \in \Omega \times \mathbb{R}$, in its spatial variable x:

$$\mathcal{R}(G)(t,\theta,\eta) = \int_{x\cdot\theta=\eta} G(x,t)\mathrm{d}x, \quad \theta \in S^{n-1}, \ \eta \in \mathbb{R}.$$

Lemma 8. *Let $G \in Lip_c(\mathbb{R}^{n+1})$. Let $t_0 \in \mathbb{R}$ and $\tau > 0$. There exists $C > 0$ (depending only on* $\operatorname{supp} G$*) such that the following estimate*

$$\begin{aligned} &\left|\mathcal{R}(G)(t_0,\theta,\eta) - \frac{\tau}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^n} G(x,t)\mathrm{e}^{-\tau((x\cdot\theta-\eta)^2+(t-t_0)^2)}\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \frac{\sqrt{\pi}}{2} C\,\|G\|_{Lip}\,\tau^{-1/2} \end{aligned}$$

holds. Here C is independent of $\theta \in S^{n-1}$ and $\eta \in \mathbb{R}$.

Proof. Let $\theta \in S^{n-1}$ and $\eta \in \mathbb{R}$. We write any $x \in \mathbb{R}^n$ as

$$x = s'\theta + y, \quad s' = x\cdot\theta \text{ and } y \in \theta^\perp.$$

By making the change of variables $x \mapsto (y, s')$ with $\mathrm{d}x = \mathrm{d}y\,\mathrm{d}s'$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^n} G(x,t)\,\mathrm{e}^{-\tau((x\cdot\theta-\eta)^2+(t-t_0)^2)}\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{y\in\theta^\perp} G(s'\theta+y,t)\mathrm{e}^{-\tau((s'-\eta)^2+(t-t_0)^2)}\mathrm{d}y\,\mathrm{d}s'\right)\mathrm{d}t\\
&=\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \mathrm{e}^{-\tau((s'-\eta)^2+(t-t_0)^2)}\int_{y\in\theta^\perp} G(s'\theta+y,t)\mathrm{d}y\,\mathrm{d}s'\right)\mathrm{d}t\\
&=\int_{\mathbb{R}^2}\mathcal{R}(G)(t,\theta,s')\,\mathrm{e}^{-\tau((s'-\eta)^2+(t-t_0)^2)}\mathrm{d}s'\,\mathrm{d}t.
\end{aligned}$$

The result will follow by applying Lemma 3 if we can show that $\mathcal{R}(G)(\cdot,\theta,\cdot)$ is uniformly Lipschitz in $\mathbb{R}^2$ for all $\theta\in S^{n-1}$. But this follows, since G is compactly supported and Lipschitz. □

3.2. *Proof of Theorem 2 in $n+1$ dimensions with $m\geq 2$*

The proof is quite similar to the one in $1+1$ dimension with $m=2$. The main difference between the proofs is that instead of having a pointwise estimate of the function $v_0(a_1-a_2)$, see (36), we obtain estimates for the partial Radon transformation of this function when $n\geq 2$. Here v_0 satisfies $\square v_0=0$ as before, see (6).

We have the integral identity (28)

$$\begin{aligned}
-m!\int_{\Omega\times[0,T]} a\,v_0v_1v_2\cdots v_m\,\mathrm{d}x\,\mathrm{d}t &= \int_{\Sigma} v_0\,D^m_{\varepsilon_1,\ldots,\varepsilon_m}\big|_{\varepsilon=0}\Lambda(\varepsilon_1 f_1+\cdots+\varepsilon_m f_m)\mathrm{d}S\\
&\quad -\frac{1}{\varepsilon_1\varepsilon_2\cdots\varepsilon_m}\int_{\Omega\times[0,T]} v_0\square\widetilde{\mathcal{R}}\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

It follows that we have an estimate similar to (33)

$$\begin{aligned}
&m!\left|\langle v_0(a_1-a_2),v_1\cdots v_m\rangle_{L^2(\Omega\times[0,T])}\right|\\
&\leq C\,\varepsilon_1^{-1}\cdots\varepsilon_m^{-1}\left(\delta+(\varepsilon_1\|f_1\|_{H^{s+1}(\Sigma)}+\cdots+\varepsilon_m\|f_m\|_{H^{s+1}(\Sigma)})^{2m-1}\right),
\end{aligned}\tag{41}$$

where

$$C=\max\left\{4,C(s,T)(\|a_1\|^2_{E^{s+1}}+\|a_2\|^2_{E^{s+1}})\right\}(1+\|v_0\|_{H^{-r}(\Sigma)}+\|v_0\|_{H^{-(s+1)}(\Omega\times[0,T])}).$$

We choose the boundary values as follows. Let $(x_0,t_0)\in W\subset\Omega\times[t_1,t_2]$. Here, $t_1,t_2\in\mathbb{R}$ are as in (2) and (4). Let $\theta\in S^{n-1}$ be arbitrary. Let us denote $x_0\cdot\theta=\eta\in\mathbb{R}$. By construction, hyperplanes of the form $\{x\in\mathbb{R}^n \mid (x-x_0)\cdot\theta=0\}$ intersect the point x_0 and span $\mathbb{R}^n$, so by varying $\theta\in S^{n-1}$ and $x_0\in W$ we are able to construct all hyperplanes intersecting W. For $j=1,2$, we choose

$$v_j = H_j \text{ and } f_j = H_j|_\Sigma, \quad j = 1, 2,$$

where $H_j = H_j^{\tau,(t_0,\theta,\eta)}$ are as in Lemma 7 with $\gamma = s+1$ and the cutoff function χ_α so that $\chi_\alpha(0) = 1$. Recall that

$$H_j^{\tau,(t_0,\theta,\eta)}(x,t) = H((t-t_0) + (-1)^j(x-x_0)\cdot\theta), \quad j = 1, 2.$$

By (3) and (4) we have for $t \leq 0$

$$|(x-x_0)\cdot\theta \pm (t-t_0)| \geq |t-t_0| - |x-x_0| \geq d + \lambda - d = \lambda$$

which means that if $\alpha > 0$ is small enough, then f_j vanishes near $\{t = 0\}$, and thus $\partial_t^k\big|_{t=0} f_j = 0$, $k = 1, \ldots, s$, on $\partial\Omega$. Moreover, for $t \geq T$

$$|(x-x_0)\cdot\theta \pm (t-t_0)| \geq |t-t_0| - |x-x_0| \geq t_2 + d + \lambda - t_2 - d = \lambda,$$

so in fact

$$\operatorname{supp}(H_j^{\tau,(t_0,\theta,\eta)}(x,\cdot)) \subset (0,T) \tag{42}$$

for all $x \in \Omega$.

For $j = 3, \ldots, m$, we let $\tau_0 > 0$ and we choose

$$v_j = \tau_0^{-1/2} H_1^{\tau_0,(t_0,\theta,\eta)} \text{ and } f_j = \tau_0^{-1/2} H_1^{\tau_0,(t_0,\theta,\eta)}|_\Sigma.$$

Let us write

$$\overline{v} = v_0 v_3 \cdots v_m.$$

Note that $\overline{v}(x,t_0) = 1$ if $x\cdot\theta = \eta$. Recall the definition of being an admissible potential in (5). Let us set

$$\mathbf{a} = \mathbf{1}_{\Omega\times[0,T]}(a_1 - a_2),$$

where $\mathbf{1}_{\Omega\times[0,T]}$ stands for the characteristic function associated with the set $\Omega\times[0,T]$. It is clear that $\mathbf{a}$ is compactly supported in $\overline{\Omega}\times[0,T]$. Since $a_1 = a_2$ on $\partial\Omega\times[0,T]$ and $a_1, a_2 \in C^{s+1}(\overline{\Omega}\times[0,T])$, by assumptions, it follows that $\mathbf{a}$ is uniformly Lipschitz in $\mathbb{R}^n\times[0,T]$. Substituting these choices of v_j into inequality (33), and using Lemma 8 with

$$G(x,t) = \overline{v}(x,t_0)\mathbf{a}(x,t_0)\chi_\alpha(x\cdot\theta - t - (\eta - t_0))\chi_\alpha(-x\cdot\theta - t + (\eta + t_0))$$

we get,

$$
\begin{aligned}
|\mathcal{R}(G)(t_0,\theta,\eta)| &\le \frac{1}{\pi}\left|\int\limits_{\Omega\times[0,T]} \overline{v}(a_1-a_2)H_1\,H_2\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad+\left|\mathcal{R}(G)(t_0,\theta,\eta)-\frac{1}{\pi}\int\limits_{\Omega\times[0,T]} \overline{v}(a_1-a_2)H_1\,H_2\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\le C\left(2\tau^{-1/2}+\epsilon_1^{-1}\cdots\epsilon_m^{-1}\Big(\delta+(\varepsilon_1+\cdots+\varepsilon_m)^{2m-1}\,(\tau^{\frac{s+2}{2}})^{2m-1}\Big)\right)\|\overline{v}\|_{C^1}\,.
\end{aligned}
\tag{43}
$$

Here, in applying Lemma 8, we used that G is globally Lipschitz because $\mathbf{a}$ is Lipschitz over $\mathbb{R}^n\times[0,T]$ and by (42) over $\mathbb{R}^{n+1}$. Moreover the integral over the set $\Omega\times[0,T]$ was trivially extended to $\mathbb{R}^{n+1}$. Thus, we were able to apply Lemma 8.

Since $\overline{v}(t_0,\theta,\eta)=1$, we have by the definition of the Radon transform that

$$
\begin{aligned}
\mathcal{R}(G)(t_0,\theta,\eta) &= \int\limits_{x\cdot\theta=\eta} G(x,t_0)\mathrm{d}x \\
&= \int\limits_{x\cdot\theta=\eta} \overline{v}(x,t_0)\mathbf{a}(x,t_0)\,\chi_\alpha(0)\chi_\alpha(0)\mathrm{d}x \\
&= \int\limits_{x\cdot\theta=\eta} \mathbf{a}(x,t_0)\mathrm{d}x=\mathcal{R}(\mathbf{a})(t_0,\theta,\eta).
\end{aligned}
$$

By using this identity and (43) we obtain

$$
\begin{aligned}
&|\mathcal{R}(\mathbf{a})(t_0,\theta,\eta)| \\
&\le \widetilde{C}\left(2\tau^{-1/2}+\epsilon_1^{-1}\cdots\epsilon_m^{-1}\Big(\delta+(\varepsilon_1+\cdots+\varepsilon_m)^{2m-1}\,(\tau^{\frac{s+2}{2}})^{2m-1}\Big)\right)\|\overline{v}\|_{C^1} \\
&=: C(\varepsilon_j,\tau,\delta).
\end{aligned}
\tag{44}
$$

Let us choose $F\in L^2(S^{n-1}\times[-M,M])$, $F\equiv 1$, where $\mathrm{supp}(a_j)\subset B_M(0)$ for $j=1,2$. Applying Lemma 6 with $f(\,\cdot\,)=\mathbf{a}(\,\cdot\,,t_0)$ and $\beta=(n-1)/2$, we obtain

$$
\|\mathbf{a}(\,\cdot\,,t_0)\|_{H^{-(n-1)/2}(\mathbb{R}^n)}\le(2\pi)^{1/2}\,C(\varepsilon_j,\tau,\delta)\,C_{\mathrm{supp}(a_j)}.
$$

The identical embedding operator $\mathbf{E}:H_0^s(\Omega)\to H^s(\mathbb{R}^n)$ is continuous for all $s\ge 0$. Its dual operator is the restriction operator $\mathbf{R}:H^{-s}(\mathbb{R}^n)\to H^{-s}(\Omega)$, $\mathbf{R}u=u|_\Omega$, which is also continuous:

$$
\|\mathbf{a}(\,\cdot\,,t_0)\|_{H^{-(n-1)/2}(\Omega)}\le C_{\mathbf{R}}\,\|\mathbf{a}(\,\cdot\,,t_0)\|_{H^{-(n-1)/2}(\mathbb{R}^n)}.
$$

The above facts can be found, for instance, from [28]. By the a priori bound $\|a_j\|_{C^{s+1}}\le L$ on the potentials, $j=1,2$, we know that

$$
\|\mathbf{a}(\,\cdot\,,t_0)\|_{H^{s+1}(\Omega)}\le 2L.
$$

Let $l \in (0,1)$. Using [28, Proposition 2.4, Corollary 2.10 and Proposition 2.11], we can interpolate the Sobolev spaces $H^{-(n-1)/2}(\Omega)$ and $H^{s+1}(\Omega)$ up to $H^{\gamma_l}(\Omega)$, where

$$\gamma_l = -l(n-1)/2 + (1-l)(s+1), \quad l \in (0,1).$$

Particularly, for all $l \in (0, 1/(2n)]$, we have $\gamma_l > n/2$. For $l \in (0, 1/(2n)]$, after possibly redefining constants, we have

$$\begin{aligned}\|\mathbf{a}(\cdot, t_0)\|_{L^\infty(\Omega)} &\le c_{n,s,l}\|\mathbf{a}(\cdot, t_0)\|_{H^{-(n-1)l/2+(s+1)(1-l)}(\Omega)}\\ &\le c_{n,s,l}\|\mathbf{a}(\cdot, t_0)\|^{l}_{H^{-(n-1)/2}(\Omega)}\|\mathbf{a}(\cdot, t_0)\|^{1-l}_{H^{s+1}(\Omega)}\\ &\le \widetilde{C}\left(2\tau^{-1/2} + \epsilon_1^{-1}\cdots\epsilon_m^{-1}\Big(\delta + (\varepsilon_1+\cdots+\varepsilon_m)^{2m-1}\,(\tau^{\frac{s+2}{2}})^{2m-1}\Big)\right)^{l}\|\overline{v}\|^{l}_{C^1}.\end{aligned}$$

The above estimate is uniform in $t_0 \in \mathbb{R}$. Therefore, together with Sobolev embedding (Morrey embedding),

$$\|\mathbf{a}(\cdot, t_0)\|_{L^\infty(\Omega)} \le c_{n,s,l}\|\mathbf{a}(\cdot, t_0)\|_{H^{\gamma_l}(\Omega)}, \quad l \in (0, 1/(2n)],$$

we obtain

$$\begin{aligned}&\|a_1 - a_2\|_{L^\infty(W)}\\ &\quad\le \widetilde{C}\left(2\tau^{-1/2} + \epsilon_1^{-1}\cdots\epsilon_m^{-1}\Big(\frac{\kappa^{2m-1}\delta}{mM} + \frac{(\varepsilon_1+\cdots+\varepsilon_m)^{2m-1}}{m-1}\,(\tau^{\frac{s+2}{2}})^{2m-1}\Big)\right)^{l}\|\overline{v}\|^{l}_{C^1}\end{aligned}$$

for all $l \in (0, 1/(2n)]$. Here we made a similar scaling of δ as we did in deriving the inequality (36).

As before, we choose $\varepsilon_1 = \ldots = \varepsilon_m = \varepsilon$. Then the last step is to optimize in τ and ε as was done in Lemma 5. We recall, that the quantity

$$\varepsilon\|f_j\|_{H^{s+1}(\Sigma)} \sim \varepsilon\tau^{\frac{s+2}{2}}$$

should be bounded by κ. The next lemma generalizes Lemma 5:

Lemma 9. *For any given $\delta \in (0, M)$ and $\kappa \in (0,1)$ small enough we find $\varepsilon(\delta,\kappa) = \varepsilon$ and $\tau(\delta,\kappa) = \tau \ge 1$ such that*

$$f(\epsilon,\tau) := 2\tau^{-1/2} + \frac{\kappa^{2m-1}\delta}{mM}\epsilon^{-m} + \frac{m^{2m-1}}{m-1}\epsilon^{m-1}\,\tau^{\frac{s+2}{2}(2m-1)} \le C_{s,M,\kappa}\,\delta^{\frac{m-1}{(2m-1)(m(s+2)+1)}}$$

and we also have

$$\varepsilon\tau^{\frac{s+2}{2}} \le \kappa.$$

The constant $C_{s,M,\kappa}$ is independent of δ.

We leave the proof of this Lemma to Appendix B. Finally, we choose the interpolation parameter $l = 1/(2n)$. This finishes proof of Theorem 4. $\square$

3.3. *Proof of Theorems 2 and 3 in $1+1$ dimensions with $m > 2$*

We have not yet proven Theorems 2 and 3 in the case the dimension is $1+1$ and $m > 2$. The proofs are almost identical to the case the dimension is $\dim(\Omega) = 1$ and $m = 2$. We first consider Theorem 2.

Using the integral identity (28) we arrive at (41), which reads:

$$\begin{aligned} &m! \left| \langle v_0(a_1 - a_2), v_1 \cdots v_m \rangle_{L^2(\Omega\times[0,T])} \right| \\ &\le C\, \varepsilon_1^{-1} \cdots \varepsilon_m^{-1} \left(\delta + (\varepsilon_1 \| f_1 \|_{H^{s+1}(\Sigma)} + \cdots + \varepsilon_m \| f_m \|_{H^{s+1}(\Sigma)})^{2m-1} \right), \end{aligned} \tag{45}$$

where $\varepsilon_j > 0$ are small parameters and v_j solves (21) for $j = 1, \dots, m$:

$$\begin{cases} \square v_j = 0, & \text{in } \Omega \times [0, T], \\ v_j = f_j, & \text{on } \partial\Omega \times [0, T], \\ v_j\big|_{t=0} = 0, \quad \partial_t v_j\big|_{t=0} = 0, & \text{in } \Omega. \end{cases}$$

Let $(x_0, t_0) \in W$, where W is as in (2). For $j = 1, 2$, we choose

$$v_j = H_j \text{ and } f_j = H_j|_\Sigma, \quad j = 1, 2,$$

where $H_j = H_j^{\tau,(x_0,t_0)}$ is as in Lemma 2 with $\gamma = s+1$ and the cut-off function χ_α so that $\chi_\alpha(0) = 1$. We assume that $\alpha > 0$ is small enough, that is, $\alpha < \lambda$, so that f_j vanishes near $\{t = 0\}$, and hence $\partial_t^k\big|_{t=0} f_j = 0$, $k = 1, \dots, s$, on $\partial\Omega$. For $j = 3, \dots, m$, we let $\tau_0 > 0$ and we choose

$$v_j = \tau_0^{-1/2} H_1^{\tau,(t_0,\theta,s)} \text{ and } f_j = \tau_0^{-1/2} H_1^{\tau,(t_0,\theta,s)}|_\Sigma.$$

Let us write

$$\overline{v} = v_0 v_3 \cdots v_m.$$

Substituting this choice of v_j into inequality (45), and using Lemma 3 with

$$b(x,t) := \overline{v}\, \mathbf{a}\, \chi_\alpha(x - x_0 - (t - t_0))\chi_\alpha(x - x_0 + (t - t_0)),$$

we deduce the following point-wise estimate by combining the steps to derive (40) and (44):

$$\begin{aligned} &|(\overline{v}\mathbf{a})(t_0, x_0)| \\ &\le C \left(2\tau^{-1/2} + \epsilon_1^{-1} \cdots \epsilon_m^{-1} \left(\delta + (\varepsilon_1 + \cdots + \varepsilon_m)^{2m-1} (\tau^{\frac{s+2}{2}})^{2m-1} \right) \right) \|\overline{v}\|_{C^1}. \end{aligned}$$

The constant $C > 0$ is independent of $(t_0, x_0) \in W$. The proof is completed by choosing $\varepsilon_1 = \cdots = \varepsilon_m = \varepsilon$ and optimizing all the involved parameters as in Lemma 9.

The proof Theorem 3 in $1+1$ dimensions with $m>2$ follows in a similar way by using the computations from this section and Section 2.1.1. We omit the details. □

As we already mentioned, in dimensions $n+1$ with $n \geq 2$ we initially recover the Radon transform of the unknown potential, see (44). For this reason, in $\mathbb{R}^{n+1}$ with $n \geq 2$, it is natural to formulate a reconstruction result in terms of the Radon transformation. This result is analogous to Theorem 3.

Proposition 4 *(Stability with noise and reconstruction $n \geq 2$). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Let $L>0$ and $m \geq 2$ be an integer, $r \in \mathbb{R}$ and $r \leq s \in \mathbb{N}$ and $s+1 > (n+1)/2$. Assume that $a \in \mathcal{A}(L,s)$ and $a=0$ on $\partial\Omega \times [0,T]$. Let $\Lambda : H^{s+1}(\Sigma^T) \to H^r(\Sigma^T)$ be the Dirichlet-to-Neumann map of the non-linear wave equation (1). Assume also that $\mathcal{E} : H^{s+1}(\Sigma^T) \to H^r(\Sigma^T)$.*

Let $\varepsilon_0 > 0$, $M > 0$, $0 < T < \infty$ and $\delta \in (0, M)$ be such that

$$\|\mathcal{E}(f)\|_{H^r(\Sigma^T)} \leq \delta,$$

for all $f \in H^{s+1}(\Sigma^T)$ with $\|f\|_{H^{s+1}(\Sigma^T)} \leq \varepsilon_0$.

There are $\tau \geq 1$, $\varepsilon_1, \ldots, \varepsilon_m > 0$ and a finite family of functions $\{H_j^{\tau,Q}\} \subset H^{s+1}(\Sigma^T)$ where $j=1,\ldots,m$, and $Q \in \mathbb{R} \times (S^{n-1} \times \mathbb{R})$, such that

$$\sup_{Q \in \mathbb{R}\times(S^{n-1}\times\mathbb{R})} \Big| \mathcal{R}(a)(Q) + \frac{1}{m!\pi} D^m_{\varepsilon_1\cdots\varepsilon_m}\Big|_{\varepsilon=0} \int_{\widetilde{\Sigma}} \psi(\Lambda+\mathcal{E})(\varepsilon_1 H_1^{\tau,Q} + \cdots + \varepsilon_m H_m^{\tau,Q}) dS \Big| \leq C\delta^{2n\sigma(s)}.$$

The exponent $\sigma = \sigma(s)$ and the constant C are as in Theorem 2. The measurement function ψ is as in (7).

We conclude this section by noting that we expect it to be possible to remove the auxiliary measurement function ψ from the proofs in odd dimensions n. The function v_0 was essentially only used to accomplish the integration by parts argument, which led to the integral identity (28). Recall that the potentials a we consider are compactly supported. Therefore, if n is odd, the Huygens principle implies that the terms that depend on a in the expansion of a solution u with respect to the parameters $\varepsilon_1, \ldots, \varepsilon_m$, will exit Ω before time T. Thus using v_0 is not necessary.

Data availability

No data was used for the research described in the article.

Acknowledgments

M. L. was supported by Academy of Finland, grants 320113, 318990, and 312119. L. P-M. and T. L. were supported by the Academy of Finland (Centre of Excellence in Inverse Modeling and Imaging, grant numbers 284715 and 309963) and by the European Research Council under Horizon 2020 (ERC CoG 770924). T. T. was supported by the Academy of Finland (Centre of Excellence in Inverse Modeling and Imaging, grant number 312119).

Appendix A. Proofs related to the forward problem

We collect the proofs of the results for the forward problem of Section 1 here, since they are quite standard.

Proof of Corollary 2. By Proposition 1 we have that $u \in X^{s+1}$. Let $l \in \{1, \ldots, n\}$ and let us denote for simplicity $\partial = \partial_{x^l}$. Then $\tilde{u} = \partial u$ satisfies

$$\begin{cases} \Box \tilde{u} = \partial F, & \text{in } \Omega \times [0,T], \\ \tilde{u} = \partial f, & \text{on } \partial\Omega \times [0,T], \\ \tilde{u}\big|_{t=0} = \partial \psi_0, \quad \partial_t \tilde{u}\big|_{t=0} = \partial \psi_1, & \text{in } \Omega. \end{cases}$$

Here we have that $\partial F \in L^1([0,T]; H^{s-1}(\Omega))$, $\partial \psi_0 \in H^s(\Omega)$, $\partial \psi_1 \in H^{s-1}(\Omega)$ and $\partial f \in H^s(\Sigma)$. We apply Proposition 1 for $\tilde{u}$ and with s replaced by $s-1$. For this, it will be needed that $\partial_t^{s-1} \partial F \in L^1([0,T], L^2(\Omega))$, which is satisfied by the additional assumption $\partial_t^{s-1} F \in L^1([0,T]; H^1(\Omega))$. Therefore, by Proposition 1, we have that

$$\tilde{u} \in X^s = C([0,T]; H^s(\Omega)) \cap C^s([0,T]; L^2(\Omega))$$

and

$$\begin{aligned} \|\partial_t^s \tilde{u}(\cdot, t)\|_{L^2(\Omega)} \leq c_T \Big(& \|\partial F\|_{L^1([0,T]; H^{s-1}(\Omega))} + \|\partial_t^{s-1} \partial F\|_{L^1([0,T]; L^2(\Omega))} \\ & + \|\partial \psi_0\|_{H^s(\Omega)} + \|\partial \psi_1\|_{H^{s-1}(\Omega)} + \|\partial f\|_{H^s(\Sigma)} \Big) \\ \leq c_T \big(& \|F\|_{L^1([0,T]; H^s(\Omega))} + \|\partial_t^{s-1} F\|_{L^1([0,T]; H^1(\Omega))} \\ & + \|\psi_0\|_{H^{s+1}(\Omega)} + \|\psi_1\|_{H^s(\Omega)} + \|f\|_{H^{s+1}(\Sigma)} \big). \end{aligned} \tag{46}$$

Since we also have $u \in X^{s+1}$, the above yields that

$$u \in C^s([0,T]; H^1(\Omega)).$$

Repeating the argument several times by taking derivatives $\partial_{x_{l_1}} \partial_{x_{l_2}} \cdots \partial_{x_{l_k}}$ of u and the initial and boundary values for all $k = 2, \ldots, s$, we obtain

$$u \in C^{s+1-k}([0,T]; H^k(\Omega)), \quad k = 0, \ldots, s+1.$$

Summing up the corresponding estimates similar to (46), we have (16). □

Next we prove Lemma 1 which states that for sufficiently small initial and boundary data there exists a unique small solution to the non-linear wave equation

$$\begin{cases} \Box u + au^m = 0, & \text{in } \Omega \times [0,T], \\ u = f, & \text{on } \partial\Omega \times [0,T], \\ u\big|_{t=0} = \partial_t u\big|_{t=0} = 0, & \text{in } \Omega. \end{cases} \tag{47}$$

(This equation is the equation (17) in Lemma 1.)

Proof of Lemma 1. We prove the existence and uniqueness of small solutions to (47) by using the Banach fixed-point theorem. See for example [62] for the latter. For this purpose, we define a contraction mapping $\Theta : B_\rho(0) \to B_\rho(0)$ as follows. Let first $F \in E^{s+1}$. Then we certainly have

$$\partial_t^k F \in L^1([0,T]; H^{s-k}(\Omega)), \quad k = 0, 1, \dots, s.$$

Assume also that $\partial_t^k F|_{t=0} = 0$, $k = 0, \dots, s$. Let $f \in H^{s+1}(\Sigma)$ be a function satisfying $\partial_t^k f|_{t=0} = 0$, $k = 0, \dots, s$. Now, Corollary 2 implies that the linear problem

$$\begin{cases} \Box u = F, & \text{in } \Omega \times [0,T], \\ u = f, & \text{on } \partial\Omega \times [0,T], \\ u\big|_{t=0} = \partial_t u\big|_{t=0} = 0, & \text{on } \partial\Omega \end{cases} \tag{48}$$

has a unique solution $u \in E^{s+1}$. Let

$$S : E^{s+1} \cap \{\partial_t^k F|_{t=0} = 0,\ k = 0, \dots, s\} \to E^{s+1} \cap \{\partial_t^k F|_{t=0} = 0,\ k = 0, \dots, s\}$$

be the source-to-solution map which takes F to the corresponding solution u of (48) (with f fixed) as $F \mapsto u$. We then define a new non-linear mapping $\Theta : B_\rho(0) \to B_\rho(0)$ on a ball $B_\rho(0) \subset E^{s+1} \cap \{\partial_t^k F|_{t=0} = 0,\ k = 0, \dots, s\}$ via the formula

$$\Theta(u) = S(-au^m), \tag{49}$$

where $\rho > 0$ will be fixed later. The Banach space E^{s+1} is an algebra since $s + 1 > (n+1)/2$. We also have $\partial_t^k(-au^m)|_{t=0} = 0$, $k = 0, \dots, s$. We conclude that the map Θ is well-defined.

Let us verify that Θ defined by (49) is a contraction from a small ball into itself after we have chosen κ and ρ small enough. The energy estimate (16) shows that for $u \in B_\rho(0)$ we have

$$\begin{aligned} \|\Theta(u)\|_{E^{s+1}} = \|S(-au^m)\|_{E^{s+1}} &\le c_{s,T}\left(\|f\|_{H^{s+1}(\Sigma)} + \|au^m\|_{E^s}\right) \\ &\le C_{s,T}\left(\kappa + \|a\|_{C^s}\|u\|_{E^s}^m\right) \le C_{s,T}\left(\kappa + \|a\|_{C^s}\rho^m\right). \end{aligned}$$

So, if ρ and κ are chosen so that

$$0 < \rho^{m-1} < \frac{1}{2C_{s,T}L} \quad \text{and} \quad 0 < \kappa \le \frac{\rho}{2C_{s,T}},$$

then we have that $\|\Theta(u)\|_X < \rho$, giving $\Theta : B_\rho(0) \to B_\rho(0)$.

To show that Θ is a contraction mapping, let $u, v \in B_\rho(0)$. Then the function $S(-au^m) - S(-av^m)$ solves

$$\begin{cases} \Box\big(S(-au^m) - S(-av^m)\big) = -au^m - av^m, & \text{in } \Omega \times [0,T], \\ S(-au^m) - S(-av^m) = 0, & \text{on } \partial\Omega \times [0,T], \\ \big(S(-au^m) - S(-av^m)\big)\big|_{t=0} = \partial_t\big(S(-au^m) - S(-av^m)\big)\big|_{t=0} = 0, & \text{in } \Omega. \end{cases}$$

Consequently, we have by using the energy estimate (16) again that

$$\begin{aligned} \|\Theta(u) - \Theta(v)\|_{E^{s+1}} = \|S(-au^m) - S(-av^m)\|_{E^{s+1}} &\le c_{s,T}\|au^m - av^m\|_{E^s} \\ &\le C_{s,T}\|a\|_{C^s}\|u-v\|_{E^{s+1}}\|P_{m-1}(u,v)\|_{E^{s+1}} \\ &\le C_{s,T}m\|a\|_{C^s}\rho^{m-1}\|u-v\|_{E^{s+1}}. \end{aligned}$$

Here we expanded

$$u^m - v^m = (u-v)P_{m-1}(u,v),$$

where $P_{m-1}(a,b) = \sum_{k=0}^{m-1} a^{m-1-k}b^k$. Redefining $\rho > 0$ by

$$\rho^{m-1} < \frac{1}{2mC_{s,T}L}$$

yields

$$\|\Theta(u) - \Theta(v)\|_X \le \widetilde{C}\|u-v\|_X.$$

Here $\widetilde{C} < 1/2$. Thus $\Theta : B_\rho(0) \to B_\rho(0)$ is a contraction as claimed.

To finish the proof, note that if the Banach fixed-point iteration is started at $u_0 = 0$, we have an estimate for a fixed point u in terms of $u_1 := \Theta(0)$ as follows:

$$\|u\|_{E^{s+1}} \le \frac{1}{1-\widetilde{C}}\|u_1\|_{E^{s+1}} \le 2C_{s,T}\|f\|_{H^{s+1}(\Sigma)}.$$

Here we once again used the energy estimate for u_1 together with the fact that $\Theta(0)$ corresponds to a solution of the linear problem with no source. The first inequality follows from a simple argument using geometric sum, see e.g. [62, Theorem 1.A]. $\Box$

Appendix B. Auxiliary results

B.1. Construction of the measurement function

Here we construct a measurement function ψ by finding a function $v_0 \in C^\infty(\mathbb{R}^{n+1})$ satisfying (6). Let us denote

$$\begin{aligned} \alpha &:= (t_2 - t_1 + d)/2, \\ t_0 &:= (t_2 + t_1)/2, \end{aligned}$$

where t_1, t_2 and d were defined in (3) and (4). By definition of d there exists $x_0 \in \mathbb{R}^n$ such that $\Omega \subset B_{d/2}(x_0)$. Let us pick a smooth cut-off function $\chi_\alpha(l) \in C_0^\infty(\mathbb{R})$ such that

$$\chi_\alpha(l) = \begin{cases} 1, & \text{when } |l| \le \alpha, \\ 0, & \text{when } |l| > \alpha + \varepsilon, \end{cases}$$

where $0 < \varepsilon < \lambda$ and λ is as in (4). Let us define

$$v_0(x,t) := \chi_\alpha((x - x_0)\cdot\theta - (t - t_0))$$

for some $\theta \in S^{n-1}$ (and $\theta = \pm 1$ when $n = 1$). Clearly v_0 satisfies the wave equation $\Box v_0 = 0$ in $\mathbb{R}^n \times \mathbb{R}$.

It is also straightforward to verify that $v_0(x,t) = 1$ if $(x,t) \in \Omega \times [t_1, t_2]$ and $v_0(x,t) = 0$ in Ω, when $t \in [T - r, T]$ for all $0 < r < \varepsilon$. In particular, $v_0 = \partial_t v_0 = 0$ in $\Omega \times]T - r, T]$ for all $r \ge 0$ small enough. We may now set $\psi := v_0|_\Sigma$.

B.2. Proofs of lemmata in higher dimensions with $m \ge 2$

Proof of Lemma 6. The proof is a direct consequence of the Fourier slice theorem. Since $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 < p < n/n - 1$, is compactly supported, we may apply the L^p-version of Proposition 3 of the Fourier slice theorem to f. We now compute the $H^{-\beta}$ norm of f by using polar coordinates on the Fourier side: $\xi \mapsto (\sigma, \theta)$ where $\xi = \sigma\theta$ with $\sigma = |\xi|$ and $\theta = \xi/|\xi|$, and so $\mathrm{d}\xi = \sigma^{n-1}\mathrm{d}\sigma\,\mathrm{d}\theta$. By using Plancherel and Fourier slice theorems, combined with the condition $\beta \ge (n-1)/2$, we obtain

$$\begin{aligned}
(2\pi)^{n-1}\|f\|^2_{H^{-\beta}(\mathbb{R}^n)} &= (2\pi)^{n-1}\int_{\mathbb{R}^n}(1+|\xi|^2)^{-\beta}|\widehat{f}(\xi)|^2\mathrm{d}\xi \\
&= (2\pi)^{n-1}\int_{S^{n-1}}\int_0^\infty (1+\sigma^2)^{-\beta}|\widehat{f}(\sigma\theta)|^2\sigma^{n-1}\mathrm{d}\sigma\,\mathrm{d}\theta \\
&= \int_{S^{n-1}}\int_0^\infty (1+\sigma^2)^{-\beta}|\mathcal{F}_{\eta\to\sigma}\big(\boldsymbol{R}f(\theta,\eta)\big)(\sigma)|^2\sigma^{n-1}\mathrm{d}\sigma\,\mathrm{d}\theta \\
&\le \int_{S^{n-1}}\int_0^\infty (1+\sigma^2)^{\frac{n-1}{2}-\beta}|\mathcal{F}_{\eta\to\sigma}\big(\boldsymbol{R}f(\theta,\eta)\big)(\sigma)|^2\mathrm{d}\sigma\,\mathrm{d}\theta \\
&\le \int_{S^{n-1}}\int_{\mathbb{R}} |\mathcal{F}_{\eta\to\sigma}\big(\boldsymbol{R}f(\theta,\eta)\big)(\sigma)|^2\mathrm{d}\sigma\,\mathrm{d}\theta \\
&= (2\pi)^n\int_{S^{n-1}}\int_{\mathbb{R}} |\boldsymbol{R}f(\theta,\eta)|^2\mathrm{d}\eta\,\mathrm{d}\theta \\
&\le (2\pi)^n C_0^2\int_{S^{n-1}}\int_{-M}^{M} |F(\theta,\eta)|^2\mathrm{d}\eta\,\mathrm{d}\theta \\
&= (2\pi)^n C_0^2\|F\|^2_{L^2(S^{n-1}\times[-M,M])}.
\end{aligned}$$

Here in the second to last inequality we used that Radon transformation of f is supported in $[-M, M]$ in its variable η, since $\operatorname{supp}(f) \subset B_M(0)$. $\square$

Proof of Lemma 7. Let $\theta \in S^{n-1}$ and $c_0 \in \mathbb{R}$. Let us write $c_0 = t_0 - \eta$ and $F(x,t) = H(x \cdot \theta - t + c_0)$. The proof is almost the same as that of Lemma 2. We however sketch a proof to help the reader to see why the constant C is independent of t_0, θ and η.

Let β_1 be a multi-index and $\beta_2 \in \mathbb{N}$. Then for all τ large enough we obtain

$$
\begin{aligned}
\|\partial_x^{|\beta_1|}\partial_t^{\beta_2} F\|^2_{L^2(\Omega\times[0,T])} &= \tau \int_\Omega \int_0^T \left[\partial_x^{|\beta_1|}\partial_t^{\beta_2}\left(\chi_\alpha(x\cdot\theta - t + c_0)\, \mathrm{e}^{-\frac{\tau}{2}(x\cdot\theta - t + c_0)^2}\right)\right]^2 \mathrm{d}t\, \mathrm{d}x \\
&\leq C\tau\tau^{2(|\beta_1|+\beta_2)} \int_\Omega \int_0^T |\chi_\alpha(x\cdot\theta - t + c_0)|^2\, (x\cdot\theta - t + c_0)^{2(|\beta_1|+\beta_2)} \mathrm{e}^{-\tau(x\cdot\theta - t + c_0)^2} \mathrm{d}t\, \mathrm{d}x \\
&= C\tau\tau^{2(|\beta_1|+\beta_2)} \int_\Omega \int_{x\cdot\theta + c_0}^{x\cdot\theta - T + c_0} |\chi_\alpha(h)|^2\, h^{2(|\beta_1|+\beta_2)} \mathrm{e}^{-\tau h^2} \mathrm{d}h\, \mathrm{d}x \\
&\leq C\tau\tau^{2(|\beta_1|+\beta_2)} \int_\Omega \int_{-\infty}^{\infty} |\chi_\alpha(h)|^2\, h^{2(|\beta_1|+\beta_2)} \mathrm{e}^{-\tau h^2} \mathrm{d}h\, \mathrm{d}x \\
&\leq C\tau\tau^{2(|\beta_1|+\beta_2)}\tau^{-(|\beta_1|+\beta_2)-1/2} \int_\Omega dx = C\tau^{|\beta_1|+\beta_2+1/2},
\end{aligned}
$$

where C is independent of $c_0 = t_0 - \eta$ and θ. Here we made a change of variables

$$
h = x\cdot\theta - t + c_0
$$

in the integral in the variable t, while considering x is fixed. We also absorbed terms of lower order powers of τ into the constant C (see proof of Lemma 2 for an explanation), and used $\int_{\mathbb{R}} h^{2(|\beta_1|+\beta_2)} e^{-\tau h^2} dh \sim \tau^{-(|\beta_1|+\beta_2)-1/2}$.

By interpolation, see e.g. [7, Theorem 6.2.4/6.4.5], we obtain for all $\gamma \geq 0$

$$
\|F\|^2_{H^\gamma(\Omega\times[0,T])} \leq C\tau^{\gamma+1/2}.
$$

By trace theorem, we obtain

$$
\|F\|^2_{H^\gamma(\Sigma)} \leq C\|F\|^2_{H^{\gamma+1/2}(\Omega\times[0,T])} \leq C\tau^{\gamma+1}.
$$

Similar argument yields the same estimate for $H_2^{\tau,(t_0,\theta,\eta)}$. This completes the proof. $\square$

Proof of Lemma 9. To simplify notation, let $\widehat{s} := (2m-1)(s+2)/2$ and $\gamma_0 = \kappa^{2m-1}/M$. A direct computation shows that

$$\partial_\epsilon f = -(\gamma_0\delta)\epsilon^{-m-1} + m^{2m-1}\varepsilon^{m-2}\tau^{\hat{s}}, \quad \partial_\tau f = -\tau^{-3/2} + m^{2m-1}\widehat{s}\epsilon^{m-1}\tau^{\hat{s}-1}.$$

Making $\partial_\epsilon f = \partial_\tau f = 0$, we obtain the critical points of f, namely

$$\tau = \left(\frac{m^{2m-1}\widehat{s}}{m-1}\right)^{-\dfrac{2(2m-1)}{2m(\widehat{s}+1)-1}} (m^{1-2m}\gamma_0\delta)^{-\dfrac{2(m-1)}{2m(\widehat{s}+1)-1}},$$

$$\varepsilon = \left(\frac{m^{2m-1}\widehat{s}}{m-1}\right)^{\dfrac{2\widehat{s}}{2m(\widehat{s}+1)-1}} (m^{1-2m}\gamma_0\delta)^{\dfrac{2\widehat{s}+1}{2m(\widehat{s}+1)-1}}.$$

With these choices of τ and ϵ, one can check that $\tau^{-1/2}$, $(\gamma_0\delta)\epsilon^{-m}$ and $\epsilon^{m-1}\tau^{\widehat{s}}$ are all bounded by $C_{s,m}(\gamma_0\delta)^{(m-1)/[(2m-1)(m(s+2)+1)]}$. It is also straightforward to verify that $\tau \geq 1$ for κ small enough.

Furthermore, since

$$\varepsilon\tau^{\widehat{s}/(2m-1)} = (m^{1-2m}\gamma_0\delta)^{1/(2m-1)},$$

we have that

$$\varepsilon\tau^{\frac{s+2}{2}} \leq \frac{\kappa}{m} < \kappa$$

for any $0 < \delta < M$. This finishes the proof. $\square$

Appendix C. Higher order finite differences

Let us define the higher order finite difference operator by

$$D^m_{\varepsilon_1,\ldots,\varepsilon_m}\big|_{\varepsilon=0} u_{\varepsilon_1 f_1+\cdots+\varepsilon_m f_m} = \frac{1}{\varepsilon_1\cdots\varepsilon_m}\sum_{\sigma\in\{0,1\}^m}(-1)^{|\sigma|+m}u_{\sigma_1\varepsilon_1 f_1+\ldots+\sigma_m\varepsilon_m f_m},$$

where the sum is over all combinations of $\{0, 1\}$ of length m. Then for the solution u of (19) we have

$$\begin{aligned}\Box u_{\varepsilon_1 f_1+\ldots+\varepsilon_m f_m} &= -a\sum_{k_1,\ldots,k_m}\binom{m}{k_1,\ldots,k_m}\varepsilon_{k_1}\cdots\varepsilon_{k_m}v_{k_1}\cdots v_{k_m} + \Box\mathcal{R} \\ &= -a(\varepsilon_1 v_1+\ldots+\varepsilon_m v_m)^m + \Box\mathcal{R}.\end{aligned}$$

Applying the finite difference operator to this reduces to the following algebraic identity about numbers.

Lemma 10. *Let $x_1, \ldots, x_m \in \mathbb{R}$. Then*

$$I(x_1,\ldots,x_m) := \sum_{\sigma\in\{0,1\}^m}(-1)^{m+|\sigma|}(\sigma_1 x_1+\ldots+\sigma_m x_m)^m = m!x_1\cdots x_m. \tag{50}$$

Proof. Let $j \in \{1, \ldots, m\}$ and split the summation in (50) with respect to

$$\sigma' = (\sigma_1, \ldots, \sigma_{j-1}, 1, \sigma_{j+1}, \ldots, \sigma_m),$$
$$\sigma'' = (\sigma_1, \ldots, \sigma_{j-1}, 0, \sigma_{j+1}, \ldots, \sigma_m).$$

Since $|\sigma'| = |\sigma''| + 1$, we have

$$\begin{aligned}
&\sum_{\sigma \in \{0,1\}^m} (-1)^{m+|\sigma|}(\sigma_1 x_1 + \ldots + \sigma_j x_j + \ldots + \sigma_m x_m)^m \\
&\quad = - \sum_{\sigma \in \{0,1\}^{m-1}} (-1)^{m+|\sigma|}(\sigma_1 x_1 + \ldots + 1 \cdot x_j + \ldots + \sigma_m x_m)^m, \\
&\qquad + \sum_{\sigma \in \{0,1\}^{m-1}} (-1)^{m+|\sigma|}(\sigma_1 x_1 + \ldots + 0 \cdot x_j + \ldots + \sigma_m x_m)^m.
\end{aligned}$$

Then note that, if $x_j = 0$, the above implies $I(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_m) = 0$. Let us express I via the multinomial formula and write

$$I(x_1, \ldots, x_m) = I_1 + I_1^c,$$

where I_1 contains all terms of $I(x_1, \ldots, x_m)$ of the form $x_1^{p_1} \cdots x_m^{p_m}$ with $p_1 \geq 1$ and I_1^c contains the remaining terms of the form $x_1^0 x_2^{p_2} \cdots x_m^{p_m}$. Then, if $x_1 = 0$, we deduce that $0 = I(0, x_2, \ldots, x_m) = I_1^c$. Next, we split $I_1^c = I_2 + I_2^c$, where I_2 contains all terms of I_1 that have x_2 in them, similarly as before. Then $0 = I(x_1, 0, x_3, \ldots, x_m) = I_2^c$. Repeating this process, we deduce that all terms of $I(x_1, \ldots, x_m)$ that miss one of x_j, $j = 1, \ldots, m$, must cancel. Hence $I(x_1, \ldots, x_m) = c(m) x_1 \cdots x_m$ for a constant $c(m)$.

This term $c(m) x_1 \cdots x_m$ only appears in the sum I when $\sigma = (1, \ldots, 1)$. From the multinomial formula we then see that the constant $c(m) = m!$. $\square$

Using Lemma 10 we see that

$$D^m_{\varepsilon_1, \ldots, \varepsilon_m}\big|_{\varepsilon=0} \square u_{\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m} = -m! a v_1 \cdots v_m + D^m_{\varepsilon_1, \ldots, \varepsilon_m}\big|_{\varepsilon=0} \square \mathcal{R}.$$

References

[1] R. Adams, J. Fournier, Sobolev Spaces, Academic Press, 2003.

[2] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, M. Taylor, Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand's inverse boundary problem, Invent. Math. 158 (2004) 261–321.

[3] Y.M. Assylbekov, T. Zhou, Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media, J. Spectr. Theory 11 (1) (2021) 1–38.

[4] K. Astala, L. Päivärinta, Calderón's inverse conductivity problem in the plane, Ann. Math. 163 (2006) 265–299.

[5] M.I. Belishev, An approach to multidimensional inverse problems for the wave equation, Dokl. Akad. Nauk SSSR 297 (3) (1987) 524–527.

[6] M.I. Belishev, Y. Kurylev, To the reconstruction of a Riemannian manifold via its spectral data (BC-method), Commun. Partial Differ. Equ. 17 (1992) 767–804.

[7] J. Bergh, J. Löfstrom, Interpolation Spaces: an Introduction, Springer-Verlag, Berlin, 1976.

[8] R. Bosi, Y. Kurylev, M. Lassas, Reconstruction and stability in Gel'fand's inverse interior spectral problem, Anal. PDE (2022), https://doi.org/10.2140/apde.2022.15.273.

[9] A. Bukhgeim, M. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, Dokl. Akad. Nauk SSSR 260 (2) (1981) 269–272.
[10] P. Calderón, On an inverse boundary value problem, in: Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Rio de Janeiro, 1980, pp. 65–73.
[11] C.I. Cârstea, G. Nakamura, M. Vashisth, Reconstruction for the coefficients of a quasilinear elliptic partial differential equation, Appl. Math. Lett. 98 (2019) 121–127.
[12] X. Chen, M. Lassas, L. Oksanen, G.P. Paternain, Detection of Hermitian connections in wave equations with cubic non-linearity, J. Eur. Math. Soc. (JEMS) (2021).
[13] X. Chen, M. Lassas, L. Oksanen, G.P. Paternain, Inverse problem for the Yang-Mills equations, Commun. Math. Phys. 384 (2) (2021) 1187–1225.
[14] Y. Choquet-Bruhat, General Relativity and the Einstein Equations, OUP, Oxford, 2008.
[15] M. de Hoop, P. Kepley, L. Oksanen, Recovery of a smooth metric via wave field and coordinate transformation reconstruction, SIAM J. Appl. Math. 78 (2018) 1931–1953.
[16] M. de Hoop, G. Uhlmann, Y. Wang, Nonlinear responses from the interaction of two progressing waves at an interface, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 36 (2) (2019) 347–363.
[17] M. de Hoop, G. Uhlmann, Y. Wang, Nonlinear interaction of waves in elastodynamics and an inverse problem, Math. Ann. 376 (1–2) (2020) 765–795.
[18] G. Eskin, Inverse hyperbolic problems with time-dependent coefficients, Commun. Partial Differ. Equ. 32 (11) (2007) 1737–1758.
[19] A. Feizmohammadi, J. Ilmavirta, Y. Kian, L. Oksanen, Recovery of time dependent coefficients from boundary data for hyperbolic equations, J. Spectr. Theory (2020), https://doi.org/10.4171/JST/367.
[20] A. Feizmohammadi, L. Oksanen, An inverse problem for a semi-linear elliptic equation in Riemannian geometries, J. Differ. Equ. 269 (6) (2020) 4683–4719.
[21] A. Feizmohammadi, L. Oksanen, Recovery of zeroth order coefficients in non-linear wave equations, J. Inst. Math. Jussieu (2020) 1–27.
[22] C. Guillarmou, L. Tzou, The Calderón Inverse Problem in Two Dimensions, Math. Sci. Res. Inst. Publ., vol. 60, Cambridge Univ. Press, Cambridge, 2013.
[23] T. Helin, M. Lassas, L. Oksanen, T. Saksala, Correlation based passive imaging with a white noise source, J. Math. Pures Appl. 116 (9) (2018) 132–160.
[24] P. Hintz, G. Uhlmann, J. Zhai, An inverse boundary value problem for a semilinear wave equation on Lorentzian manifolds, Int. Math. Res. Not., rnab088, https://doi.org/10.1093/imrn/rnab088.
[25] O. Imanulov, M. Yamamoto, Global uniqueness and stability in determining coefficients of wave equations, Commun. Partial Differ. Equ. 26 (7–8) (2001) 1409–1425.
[26] V. Isakov, Z. Sun, Stability estimates for hyperbolic inverse problems with local boundary data, Inverse Probl. 8 (2) (1992) 193–206.
[27] H. Isozaki, L. Kurylev, M. Lassas, Conic singularities, generalized scattering matrix, and inverse scattering on asymptotically hyperbolic surfaces, J. Reine Angew. Math. 724 (2017) 53–103.
[28] D. Jerison, C. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.
[29] A. Kachalov, Y. Kurylev, M. Lassas, Inverse Boundary Spectral Problems, CRC Press, 2001.
[30] H. Kang, G. Nakamura, Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map, Inverse Probl. 18 (4) (2002) 1079.
[31] C. Kenig, M. Salo, Recent Progress in the Caderón Problem with Partial Data, Contemp. Math., vol. 615, Amer. Math. Soc., Providence, RI, 2014.
[32] Y. Kian, Y. Kurylev, M. Lassas, L. Oksanen, Unique recovery of lower order coefficients for hyperbolic equations from data on disjoint sets, J. Differ. Equ. 267 (4) (2019) 2210–2238.
[33] K. Krupchyk, Y. Kurylev, M. Lassas, Inverse spectral problems on a closed manifold, J. Math. Pures Appl. 90 (2008) 42–59.
[34] K. Krupchyk, G. Uhlmann, Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities, Math. Res. Lett. 27 (6) (2020) 1801–1824.
[35] K. Krupchyk, G. Uhlmann, A remark on partial data inverse problems for semilinear elliptic equations, Proc. Am. Math. Soc. 148 (2) (2020) 681–685.
[36] Y. Kurylev, M. Lassas, L. Oksanen, G. Uhlmann, Inverse problem for Einstein-scalar field equations, Duke Math. J. (2022), in press, https://doi.org/10.48550/arXiv.1406.4776, 2014.
[37] Y. Kurylev, M. Lassas, G. Uhlmann, Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations, Invent. Math. 212 (3) (2018) 781–857, https://doi.org/10.1007/s00222-017-0780-y.

[38] Y. Kurylev, L. Oksanen, G. Paternain, Inverse problems for the connection Laplacian, J. Differ. Geom. 110 (3) (2018) 457–494.

[39] R-Y. Lai, G. Uhlmann, Y. Yang, Reconstruction of the collision kernel in the nonlinear Boltzmann equation, SIAM J. Math. Anal. 53 (2021) 1049–1069.

[40] I. Lasiecka, J-L. Lions, R. Triggiani, Non homogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures Appl. 65 (2) (1986) 149–192.

[41] M. Lassas, Inverse problems for linear and non-linear hyperbolic equations, in: Proc. Int. Congress of Math., Vol. III, ICM 2018, Rio de Janeiro, Brazil, 2018, pp. 3739–3760.

[42] M. Lassas, T. Liimatainen, Y-H. Lin, M. Salo, Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations, Rev. Mat. Iberoam. 37 (4) (2021) 1553–1580.

[43] M. Lassas, T. Liimatainen, Y-H. Lin, M. Salo, Inverse problems for elliptic equations with power type nonlinearities, J. Math. Pures Appl. 145 (2021) 44–82.

[44] M. Lassas, L. Oksanen, Inverse problem for the Riemannian wave equation with Dirichlet data and Neumann data on disjoint sets, Duke Math. J. 163 (2014) 1071–1103.

[45] M. Lassas, G. Uhlmann, Y. Wang, Determination of vacuum space-times from the Einstein-Maxwell equations, arXiv preprint, arXiv:1703.10704, 2017.

[46] M. Lassas, G. Uhlmann, Y. Wang, Inverse problems for semilinear wave equations on Lorentzian manifolds, Commun. Math. Phys. 360 (2018) 555–609.

[47] J. Mueller, S. Siltanen, Linear and Nonlinear Inverse Problems with Practical Applications, SIAM, Philadelphia, 2012.

[48] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. Math. 143 (2) (1996) 71–96.

[49] A. Nachman, J. Sylvester, G. Uhlmann, An n-dimensional Borg-Levinson theorem, Commun. Math. Phys. 115 (4) (1988) 595–605.

[50] F. Natterer, The Mathematics of Computerized Tomography, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2001.

[51] L. Oksanen, M. Salo, P. Stefanov, G. Uhlmann, Inverse problems for real principal type operators, arXiv preprint, arXiv:2001.07599.

[52] S. Siltanen, J. Mueller, D. Isaacson, An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem, Inverse Probl. 6 (2000) 681–699.

[53] D. Solmon, Asymptotic formulas for the dual radon transform and applications, Math. Z. 195 (1987) 321–343.

[54] P. Stefanov, Inverse scattering problem for the wave equation with time-dependent potential, J. Math. Anal. Appl. 140 (1989) 351–362.

[55] P. Stefanov, G. Uhlmann, Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map, Int. Math. Res. Not. 17 (2005) 1047–1061.

[56] P. Stefanov, Y. Yang, The inverse problem for the Dirichlet-to-Neumann map on Lorentzian manifolds, Anal. PDE 11 (6) (2018) 1381–1414.

[57] Z. Sun, G. Uhlmann, Inverse problems in quasilinear anisotropic media, Am. J. Math. 119 (4) (1997) 771–797.

[58] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. Math. 125 (1) (1987) 153–169.

[59] G. Uhlmann, 30 years of Calderón's problem, in: Sémin. Équ. Dériv. Partielles, École Polytech, Palaiseau, 2014.

[60] G. Uhlmann, Y. Wang, Determination of space-time structures from gravitational perturbations, Commun. Pure Appl. Math. 73 (6) (2020) 1315–1367.

[61] Y. Wang, T. Zhou, Inverse problems for quadratic derivative nonlinear wave equations, Commun. Partial Differ. Equ. 44 (11) (2019) 1140–1158.

[62] E. Zeidler, Applied Functional Analysis, Springer-Verlag, New York, 1995.