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Integrability of orthogonal projections, and applications to Furstenberg sets ☆

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ABSTRACT

Let $\mathcal{G}(d,n)$ be the Grassmannian manifold of n-dimensional subspaces of $\mathbb{R}^d$, and let $\pi_V : \mathbb{R}^d \to V$ be the orthogonal projection. We prove that if μ is a compactly supported Radon measure on $\mathbb{R}^d$ satisfying the s-dimensional Frostman condition $\mu(B(x,r)) \leqslant Cr^s$ for all $x \in \mathbb{R}^d$ and $r > 0$, then

$$\int_{\mathcal{G}(d,n)} \|\pi_V \mu\|_{L^p(V)}^p \, d\gamma_{d,n}(V) < \infty, \qquad 1 \leqslant p < \frac{2d-n-s}{d-s}.$$

The upper bound for p is sharp, at least, for $d-1 \leqslant s \leqslant d$, and every $0 < n < d$.
Our motivation for this question comes from finding improved lower bounds on the Hausdorff dimension of (s,t)-Furstenberg sets. For $0 \leqslant s \leqslant 1$ and $0 \leqslant t \leqslant 2$, a set $K \subset \mathbb{R}^2$ is called an (s,t)-Furstenberg set if there exists a t-dimensional family $\mathcal{L}$ of affine lines in $\mathbb{R}^2$ such that $\dim_{\mathrm{H}}(K \cap \ell) \geqslant s$ for all $\ell \in \mathcal{L}$. As a consequence of our projection theorem in $\mathbb{R}^2$, we show that every (s,t)-Furstenberg set $K \subset \mathbb{R}^2$ with $1 < t \leqslant 2$ satisfies

$$\dim_{\mathrm{H}} K \geqslant 2s + (1-s)(t-1).$$

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This improves on previous bounds for pairs (s,t) with $s > \frac{1}{2}$ and $t \geqslant 1+\epsilon$ for a small absolute constant $\epsilon > 0$. We also prove a higher dimensional analogue of this estimate for codimension-1 Furstenberg sets in $\mathbb{R}^d$. As another corollary of our method, we obtain a δ-discretised sum-product estimate for (δ,s)-sets. Our bound improves on a previous estimate of Chen for every $\frac{1}{2} < s < 1$, and also of Guth-Katz-Zahl for $s \geqslant 0.5151$.

Contents

1. Introduction

This paper is concerned with the L^p regularity of orthogonal projections of fractal measures, with applications to (s,t)-Furstenberg sets. We introduce the following notation: $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ stands for the space of compactly supported Radon measures on $\mathbb{R}^d$, and $\mathcal{M}_s$ is the subset of those measures $\mu \in \mathcal{M}$ which satisfy an s-dimensional Frostman condition: there exists a constant $C > 0$ such that $\mu(B(x,r)) \leqslant Cr^s$ for all $x \in \mathbb{R}^d$ and $r > 0$. The Grassmannian manifold of n-dimensional subspaces in $\mathbb{R}^d$ is denoted $\mathcal{G}(d,n)$, and the $\mathcal{O}(d)$-invariant probability measure on $\mathcal{G}(d,n)$ is denoted $\gamma_{d,n}$. For $V \in \mathcal{G}(d,n)$, $\pi_V : \mathbb{R}^d \to V$ stands for the orthogonal projection onto V. Let us start with the following general question:

Question 1. *Let $0 < n < d$, and let $\mu \in \mathcal{M}_s$ for some $s > n$. For which values of $1 \leqslant p,q \leqslant \infty$ does it hold that*

$$\mathfrak{I}(p,q) := \left(\int_{\mathcal{G}(d,n)} \|\pi_V \mu\|_{L^p(V)}^q \, d\gamma_{d,n}(V) \right)^{1/q} < \infty? \tag{1.1}$$

The question is well-posed, since it is known since the works of Marstrand [21], Kaufman [17], and Mattila [23] that if $\mu \in \mathcal{M}_s$ with $s > n$, then $\pi_V \mu \ll \mathcal{H}^n|_V$ for $\gamma_{d,n}$ almost every plane $V \in \mathcal{G}(d,n)$, and in fact $\mathfrak{I}(2,2) \sim_{d,n} I_n(\mu)$, where $I_t(\mu)$ stands for the t-dimensional Riesz energy of μ. So, at least (1.1) holds for $p = q = 2$, for every $s > n$. This is not the best one can say: it follows easily from Falconer's Fourier analytic approach [8] and the Sobolev embedding theorem that if $I_s(\mu) < \infty$, then $\mathfrak{I}(2n/(2n-s), 2) < \infty$, see Section 3.1 for a few more details. Therefore, the answer to Question 1 (where we assume $\mu \in \mathcal{M}_s$ instead of $I_s(\mu) < \infty$) is positive for all pairs $(p,2)$ with $1 \leqslant p < 2n/(2n-s)$. For $s > 2n$, the correct interpretation of this is that $\mathfrak{I}(\infty, 2) < \infty$.

The results above only concern pairs of the form $(p,2)$, and the literature seems to be less complete for general pairs (p,q). Of course $\mathfrak{I}(p,q_1) \leqslant \mathfrak{I}(p,q_2)$ for $q_1 \leqslant q_2$ by Hölder's inequality, but this observation is unlikely to give any sharp results for $q_1 \neq q_2$. While studying problems related to Furstenberg sets (more on this in Section 1.1), we needed to understand pairs of the form (p,p). We show the following:

Theorem 1.2. *Let $\mu \in \mathcal{M}_s$ with $s > n$. Then $\mathfrak{I}(p,p) < \infty$ for $1 \leqslant p < (2d-n-s)/(d-s)$.*

The upper bound for "p" is sharp for $d \geqslant 2$, $0 < n < d$, and $d-1 \leqslant s \leqslant d$, as the next example demonstrates. We do not know how sharp Theorem 1.2 is for $n < s < d-1$. The simplest unknown case occurs for $d = 3, n = 1$, and $1 < s < 2$: what is the supremum of exponents $p \geqslant 1$ such that $\int_{\mathcal{G}(3,1)} \|\pi_L \mu\|_p^p \, d\gamma_{3,1}(L) < \infty$ for all $\mu \in \mathcal{M}_s(\mathbb{R}^3)$ with $1 < s < 2$?

Example 1.3. Fix $d \geqslant 2$, $0 < n < d$, and $d-1 \leqslant s < d$. Let $C \subset L_0 := \mathbb{R} \times \{\mathbf{0}\} \subset \mathbb{R}^d$ be an $(s-(d-1))$-regular Cantor set (take $C \subset [0,1] \times \{\mathbf{0}\}$ for concreteness), and let $\mu := \nu \times \mathcal{H}^{d-1}|_{\{\mathbf{0}\} \times B_{d-1}}$, where $\nu := \mathcal{H}^{s-d+1}|_C$, and $B_{d-1} \subset \mathbb{R}^{d-1}$ is the open unit ball. Then $\mu \in \mathcal{M}_s$.

Let $\delta > 0$, and let $\mathcal{G} \subset \mathcal{G}(d,n)$ be the δ-neighbourhood of the submanifold $\mathcal{G}_0 := \{V \in \mathcal{G}(d,n) : V \supset L_0\}$. We record that $\mathcal{G}_0$ is a $(d-n)(n-1)$-dimensional submanifold: the easiest way to get convinced is to note that the restriction "$V \supset L_0$" is equivalent to "$V^\perp \subset L_0^\perp$", and the set $\{W \in \mathcal{G}(d,d-n) : W \subset L_0^\perp\}$ is diffeomorphic to $\mathcal{G}(d-1,d-n)$, a manifold of dimension $(d-n)((d-1)-(d-n)) = (d-n)(n-1)$. Noting that $\gamma_{d,n}$ is an $n(d-n)$-regular measure (see [9, Proposition 4.1]), it follows that

$$\gamma_{d,n}(\mathcal{G}) \sim \delta^{n(d-n)} \cdot \delta^{-\dim \mathcal{G}_0} = \delta^{d-n}.$$

Now, let us consider the projections $\pi_V \mu$ for $V \in \mathcal{G}_0$, and eventually $V \in \mathcal{G}$. Note first that

$$C = \pi_{L_0}(\operatorname{spt}\mu) = \pi_{L_0}(\pi_V(\operatorname{spt}\mu)), \qquad V \in \mathcal{G}_0,$$

using that all the planes in $\mathcal{G}_0$ contain L_0. Therefore

$$\operatorname{spt}\pi_V\mu = \pi_V(\operatorname{spt}\mu) \subset B(1) \cap (\pi_{L_0}^{-1}(C) \cap V), \qquad V \in \mathcal{G}_0.$$

Recalling that C is $(s-d+1)$-regular, and $L_0 \subset V$, the set on the right is regular of dimension $(s-d+1)+(n-1) = n+s-d$. It can therefore be covered by $\sim \delta^{d-s-n}$ balls in V of radius δ. In particular, $\mathcal{H}^n(\operatorname{spt}\pi_V\mu) \lesssim \delta^{d-s}$. These arguments were carried for $V \in \mathcal{G}_0$, but the conclusion remains valid for $V \in \mathcal{G} = \mathcal{G}_0(\delta)$. Now a lower bound for $\|\pi_V\mu\|_{L^p(V)}$ follows from Hölder's inequality:

$$\|\pi_V\mu\|_{L^p(V)}^p \gtrsim \mathcal{H}^n(\operatorname{spt}\pi_V\mu)^{1-p} \gtrsim \delta^{(d-s)(1-p)} \qquad V \in \mathcal{G},\ p \geqslant 1.$$

Finally,

$$\int_{\mathcal{G}(d,n)} \|\pi_V\mu\|_{L^p(V)}^p \, d\gamma_{d,n}(V) \gtrsim \gamma_{d,n}(\mathcal{G}) \cdot \delta^{(d-s)(1-p)} \sim \delta^{d-n+(d-s)(1-p)}.$$

The right hand side stays bounded as $\delta \to 0$ only if $d-n+(d-s)(1-p) \geqslant 0$, or equivalently $p \leqslant (2d-n-s)/(d-s)$. This matches the upper bound in Theorem 1.2.

Remark 1.4. The generalisation of Example 1.3 to the case $s < d-1$ is not obvious. For $s \geqslant d-1$, the measure μ was defined as Hausdorff measure supported on a union of parallel $(d-1)$-planes (or pieces thereof, to be accurate). In the case $d=3$, $n=1$, and $1 < s < 2$ (for example) it might therefore seem natural to define $\mu := \mathcal{H}^s|_{C\times[0,1]}$, where $C \subset \mathbb{R}^2 \times \{0\}$ has $\mathcal{H}^{s-1}(C) = 1$. However, with this choice of "μ" it looks like

$$\int_{\mathcal{G}(3,1)} \|\pi_L\mu\|_{L^p(L)}^p \, d\gamma_{3,1}(L) < \infty, \qquad 1 \leqslant p < (3-s)/(2-s).$$

This upper bound for "p" is higher, for all $s \geqslant 1$, than the one predicted by Theorem 1.2.

Remark 1.5. In addition to the sharpness of Theorem 1.2 for $n < s < d-1$, another special case of Question 1 is worth highlighting: for $\mu \in \mathcal{M}_s(\mathbb{R}^2)$ with $s > 1$, determine the supremum of exponents $p \geqslant 1$ such that $\mathfrak{I}(p,1) < \infty$. This is closely related to the question Peres and Schlag raise in [34, §9.2(ii)]. More precisely, they ask for the value of $p(s) := \sup\{p \geqslant 1 : \pi_L\mu \in L^p \text{ for a.e. } L \in \mathcal{G}(2,1), \text{ for all } \mu \in \mathcal{M}_s(\mathbb{R}^2)\}$. We do not even have a good guess for the right answer. Measures supported on concentric unions of circles give one upper bound for $p(s)$, and measures supported on Furstenberg sets give another one. These upper bounds do not coincide.

Remark 1.6. While the problem regarding $\mathfrak{I}(p,1)$ seems difficult, and most likely unsolved, Theorem 1.2 may be known to experts in harmonic analysis: it is essentially an $L^p \to L^{p,\alpha}$ estimate for the $(d-n)$-plane transform, and there is a formidable amount of literature on estimating this operator. For the pairs $(d,n) = (d,1)$, $d \geqslant 2$, one could, with a little effort, deduce Theorem 1.2 from the work of Littman [19], by first expressing the $(d-1)$-plane transform (also known as the Radon transform) as an averaging operator over the $(d-1)$-dimensional paraboloid in $\mathbb{R}^d$, see the identities (2.1) and (2.9) in Christ's paper [2], and eventually exploiting the curvature of the paraboloid, as Littman does.

For more general dimensions and co-dimensions, Strichartz [39, Theorem 2.2] proves $L^p \to L^{p,\alpha}$ estimates for the n-plane transform in $\mathbb{R}^d$, but only for $1 < p \leqslant 2$ (there is a good reason, see Remark 3.5). Theorem 1.2 is also closely related to the papers of Drury [5], D. Oberlin and Stein [27], and D. Oberlin [28,29]. In these works, the authors prove sharp L^p to L^q estimates for the Radon transform, but as far as we can see, they do not contain the L^p to L^p-Sobolev result we need for our purposes. Mixed norm estimates for Radon transforms are intimately connected with Kakeya and Besicovitch (n,k)-set problems, and there is a wealth of literature for $d \geqslant 3$, see for example [11,18,30,35,40]. Smoothness and integrability estimates for Radon transforms are also of interest to mathematicians working on inverse problems: see the book [26] by Natterer, and in particular the bibliographical notes at the end of Section 2. In summary, there is a nonzero probability that Theorem 1.2 is covered by existing literature, but we could not easily find it, and in any case our proof is self-contained and fairly elementary.

1.1. Applications

We then move to the applications which motivate Question 1 for the pairs (p,p). The main one concerns *Furstenberg* (s,t)*-sets*, defined as follows. A set $K \subset \mathbb{R}^2$ is called an (s,t)-Furstenberg set if there exists a family $\mathcal{L}$ of affine lines with $\dim_{\mathrm{H}} \mathcal{L} = t$ such that $\dim_{\mathrm{H}}(K \cap \ell) \geqslant s$ for all $\ell \in \mathcal{L}$. Here the dimension "$\dim_{\mathrm{H}} \mathcal{L}$" is defined by viewing $\mathcal{L}$ as a subset of the metric space $(\mathcal{A}(2,1), d_{\mathcal{A}})$, the *affine Grassmannian* of all lines in the plane. We postpone the precise definition of the metric $d_{\mathcal{A}}$ to Section 2, see (2.2).

The case $t = 1$ has attracted the most attention: Wolff [41] introduced the problem in the late 90s and showed that every $(s,1)$-Furstenberg set $K \subset \mathbb{R}^2$, $0 < s \leqslant 1$, satisfies

$$\dim_{\mathrm{H}} K \geqslant \max\{2s, \tfrac{1}{2} + s\}. \tag{1.7}$$

Wolff also conjectured that the sharp estimate should be $\dim_{\mathrm{H}} K \geqslant \frac{1}{2} + \frac{3s}{2}$. In part relying on the work of Katz and Tao [16], Bourgain in 2003 managed to improve on Wolff's estimate by an "ϵ" in the case $s = \frac{1}{2}$. For $\frac{1}{2} < s < 1$, a similar ϵ-improvement was achieved in 2021 by the second author and Shmerkin [33], partly relying on the earlier paper [32]. In fact, [33] established that $\dim_{\mathrm{H}} K \geqslant 2s + \epsilon(s,t)$ for Furstenberg (s,t)-sets with $0 < s < 1$ and $t \in (s,2]$. For $0 < s \leqslant \frac{1}{2} - \epsilon$, Wolff's estimate remains the strongest one, although an ϵ-improvement for the *packing dimension* of s-Furstenberg sets in this region of parameters was obtained by Shmerkin [36] in 2020.

For more general $t \in [0,2]$, lower bounds for Furstenberg (s,t)-sets have been recently obtained by Molter and Rela [25], Héra [13], Héra, Máthé, and Keleti [14], Lutz and Stull [20], and Héra, Shmerkin, and Yavicoli [15]. The best previous bounds for the number

$$\gamma(s,t) := \inf\{\dim_{\mathrm{H}} K : K \subset \mathbb{R}^2 \text{ is an } (s,t)\text{-Furstenberg set}\}$$

are the following (combining contributions from all the papers cited above):

$$\gamma(s,t) \geqslant \begin{cases} s+t & \text{for } s \in (0,1] \text{ and } t \in [0,s], \\ 2s+\epsilon(s,t) & \text{for } s \in (0,1] \text{ and } t \in (s,2s], \\ s+\frac{t}{2} & \text{for } s \in (0,1] \text{ and } t \in (2s,2]. \end{cases}$$

Our new result concerns the "high dimensional" region where $s > \frac{1}{2}$ and $t > 1$:

Theorem 1.8. *Let $0 < s \leqslant 1$ and $1 < t \leqslant 2$. Then every (s,t)-Furstenberg set $K \subset \mathbb{R}^2$ satisfies*

$$\dim_{\mathrm{H}} K \geqslant 2s + (1-s)(t-1). \tag{1.9}$$

More generally, every $(d-1,s,t)$-Furstenberg set $K \subset \mathbb{R}^d$, with $d \geqslant 2$, $1 < t \leqslant d$ and $0 < s \leqslant d-1$ satisfies

$$\dim_{\mathrm{H}} K \geqslant (2s+2-d) + \frac{(t-1)(d-1-s)}{d-1}. \tag{1.10}$$

We postpone the definition of $(d-1,s,t)$-Furstenberg sets for a moment, see Section 1.1.1. The estimate (1.9) is stronger than the bound $s+t/2$, due to Héra [13], in the range $s > \frac{1}{2}$ and $t > 1$, and also improves on the bound $2s+\epsilon(s,t)$ for $(1-s)(t-1) > \epsilon(s,t)$ (the constant $\epsilon(s,t) > 0$ is very small). We derive Theorem 1.8 as a corollary of a following δ-discretised incidence result, which also gives some information in higher dimensions. To state the result, we first define the notion of (δ,s,C)-sets:

Definition 1.11 *((δ,s,C)-set).* Let $0 \leqslant s < \infty$, $0 < \delta < 1$, and $C > 0$. Given a metric space (X,d), a bounded set $P \subset X$ is called a (δ,s,C)-set if for every $\delta \leqslant r \leqslant 1$ and every ball $B \subset X$ of radius r we have

$$|P \cap B|_\delta \leqslant C \cdot |P|_\delta \cdot r^s.$$

Here $|A|_\delta$ denotes the δ-covering number of A, i.e. the minimal number of δ-balls needed to cover A (we set $|A|_\delta := \infty$ if A cannot be covered by finitely many δ-balls).

In the following, if $A \subset \mathbb{R}^d$, and $r > 0$, then $A(r) := \{x \in \mathbb{R}^d : \mathrm{dist}(x, A) \leqslant r\}$.

Theorem 1.12. *Let $0 < n < d$ and $C, C_F \geqslant 1$. Let $\mathcal{V} \subset \mathcal{A}(d,n)$ be a δ-separated set of n-planes, and let $P \subset B(1) \subset \mathbb{R}^d$ be a δ-separated (δ, t, C_F)-set with $t > d-n$. For $r > 0$ let $\mathcal{I}_r(P, \mathcal{V}) = \{(p, V) \in P \times \mathcal{V} : p \in V(r)\}$. Then, for every $\varepsilon > 0$ we have*

$$|\mathcal{I}_{C\delta}(P, \mathcal{V})| \lesssim_{C,d,\varepsilon,t} \delta^{-\varepsilon} \cdot C_F \cdot |P| \cdot |\mathcal{V}|^{n/(d+n-t)} \cdot \delta^{n(t+1-d)(d-n)/(d+n-t)}.$$

To derive Theorem 1.8 from Theorem 1.12, the incidence result needs to be applied to the dual set of (a suitable discretisation of) "$\mathcal{L}$", the t-dimensional set of lines appearing in the definition of (s,t)-Furstenberg sets. While it is unlikely that Theorem 1.8 is sharp for any $s \in (0,1)$ or $t \in [1,2)$, Theorem 1.12 is fairly sharp in the plane, essentially because the set $\mathcal{V}$ is "only" assumed to be δ-separated. This matter is discussed further in Section 5, see Proposition 5.2 and Remark 5.3.

Theorem 1.12, or rather its dual version, also allows us to make progress on the *δ-discretised sum-product problem* in the "supercritical" range $t > \frac{1}{2}$:

Corollary 1.13. *Let $\delta \in (0,1]$, $s, t, t' \in [0,1]$ with $t + t' > 1$, and $c, c' > 0$. Let $A, B, C \subset [1,2]$ be δ-separated sets such that $|A| = \delta^{-s}$, B is a (δ, t, c)-set and C is a (δ, t', c')-set. Then,*

$$\max\{|A + B|_\delta, |A \cdot C|_\delta\} \gtrsim_{\alpha,s,t,t',c,c'} \delta^{-\alpha}|A|, \qquad \alpha < \tfrac{(t+t'-1)(1-s)}{2}.$$

We are grateful to Josh Zahl for telling us that Corollary 1.13 follows from Theorem 1.12 combined with an argument of Elekes [6], see Section 6.3 for the details. Corollary 1.13 applied with $A = B = C$ (and assuming that A is a (δ, t)-set with $t \in (\frac{1}{2}, 1)$) improves on recent results of Chen [1] for every $t \in (\frac{1}{2}, 1)$, and of Guth, Katz, and Zahl [12] for $1 > t > (\sqrt{1113} - 21)/24 \approx 0.5151$. We refer the reader to these papers for more background and references on the δ-discretised sum-product problem. Since $(2t-1)(1-s)/2 > 0$ for $t \in (\frac{1}{2}, 1)$ and $s \in (0,1)$, if we assume that $B = C$ and B is a (δ, t)-set with $t \in (\frac{1}{2}, 1)$, Corollary 1.13 also implies that $\max\{|A+B|_\delta, |A \cdot B|_\delta\} \gg |B|$ in cases where A is substantially smaller than B (to be precise, this works when $s > 1/(3-2t)$; note that $1/(3-2t) < t$ for $t \in (\frac{1}{2}, 1)$, so the range $s \in (1/(3-2t), t)$ is non-empty).

1.1.1. Higher dimensional Furstenberg sets

Theorem 1.8 mentions the notion of (n,s,t)-Furstenberg sets in $\mathbb{R}^d$. These are defined just like (s,t)-Furstenberg sets, except that the set $\mathcal{L} \subset \mathcal{A}(2,1)$ is replaced by a t-dimensional set $\mathcal{V} \subset \mathcal{A}(d,n)$ of affine n-planes. Thus, a set $K \subset \mathbb{R}^d$ is called an (n,s,t)-Furstenberg set if there exists a family $\mathcal{V} \subset \mathcal{A}(d,n)$ with $\dim_{\mathrm{H}} \mathcal{V} = t$ such that $\dim_{\mathrm{H}}(K \cap V) \geqslant s$ for all $V \in \mathcal{V}$. The dimension "$\dim_{\mathrm{H}} \mathcal{V}$" is defined relative to the metric on $\mathcal{A}(d,n)$, see Section 2. Since Theorem 1.8 is deduced via duality from Theorem 1.12, we only obtain information about the case $n = d-1$.

Furstenberg (n,s,t)-sets have been studied in many of the papers cited above, see [13–15]. Additionally, finite field versions of (n,s,t)-Furstenberg sets in $\mathbb{F}_p^d$ have been considered by Ellenberg and Erman [7], Dhar, Dvir, and Lund [3], and Zhang [42]. We also mention the paper of Zhang [43], where the author studies a discrete variant of the Furstenberg set problems in $\mathbb{R}^d$.

We only discuss the existing bounds in the case $n = d-1$. Héra in [13] proves that every $(d-1,s,t)$-Furstenberg set $K \subset \mathbb{R}^d$ with $(s,t) \in (d-2, d-1] \times (0,d]$ satisfies $\dim_{\mathrm{H}} K \geqslant s + t/d$. In [14], Héra, Máthé, and Keleti prove the lower bound $\dim_{\mathrm{H}} K \geqslant 2s - d + 1 + \min\{t,1\}$ for all $(s,t) \in (0,d-1] \times (0,d]$. Clearly (1.10) improves on the H-K-M bound for all $t \in (1,d]$. One may calculate that (1.10) also improves on Héra's bound for $(s,t) \in (d-2+\frac{1}{d}, d-1] \times (1,d]$.

1.2. Outline of the paper

The proof of Theorem 1.2 is conceptually quite straightforward: it is based on complex interpolation between the cases $s = n$ and $s = d$. This argument is heavily influenced by the paper [39] of Strichartz. The technical details nevertheless take some work, see Section 3. Section 2 only contains some preliminaries.

Theorem 1.8 on $(d-1,s,t)$-Furstenberg sets is reduced to the incidence estimate in Theorem 1.12 by applying point-plane duality, and standard discretisation arguments. The details are contained in Section 6. The proof of Theorem 1.12 is carried out in Section 4. The idea is easiest to explain in the plane. Imagine that $P \subset \mathbb{R}^2$ is a δ-separated (δ,t)-set (see Definition 1.11) with $1 < t \leqslant 2$, and let $\mathcal{L} \subset \mathcal{A}(2,1)$ be a δ-separated line family with excessively many δ-incidences with P. Let $\mu \in \mathcal{M}_t$ with $\operatorname{spt}\mu = P(\delta)$. If the word "excessive" is interpreted as the serious failure of Theorem 1.12, then it turns out that many *radial projections* $\rho_x\mu$ of μ relative to base points $x \in \operatorname{spt}\mu = P(\delta)$ are singular. (The reader should be warned that $\rho_x\mu$ is not precisely the push-forward of μ under $y \mapsto \rho_x(y)$, see (4.7) for the proper definition.)

This sounds like a contradiction: a result of the second author [31] says that the radial projections of a t-dimensional measure, $t > 1$, relative to its own base points are (typically) absolutely continuous with a density in L^p, for some $p > 1$. The result in [31] is proved via relating the radial and orthogonal projections of μ by the following formula:

$$\int \|\rho_x\mu\|_{L^p(S^1)}^p \, d\mu(x) = \int_{S^1} \|\pi_e\mu\|_{L^{p+1}(\mathbb{R})}^{p+1} \, d\mathcal{H}^1(e).$$

For a higher dimensional generalisation, see (4.18). With this identity in hand, we may estimate the right hand side by appealing to Theorem 1.2: it is finite for all $p+1 < (3-t)/(2-t)$, or equivalently $p < 1/(2-t)$. Pitting this information against the hypothetical singularity of the radial projections $\rho_x\mu$ yields Theorem 1.12. A similar approach also works in higher dimensions and co-dimensions: the details can be found in Section 4.

As we already mentioned above, Section 5 contains a family of examples indicating the sharpness of Theorem 1.12. These examples will also indicate where the numerology in the lower bound (1.9) comes from.

1.3. Acknowledgments

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2. Preliminaries

We will write $f \lesssim g$ as an abbreviation for the inequality $f \leqslant Cg$, where $C > 0$ is an absolute constant. If the constant C depends on a parameter a, we will write $f \lesssim_a g$. Furthermore, $f \sim g$ and $f \sim_a g$ will denote $g \lesssim f \lesssim g$ and $g \lesssim_a f \lesssim_a g$, respectively.

In addition to the notations "$f \lesssim g$" and "$f \sim g$", we will also employ "$f \lessapprox g$" and "$f \approx g$". The notation $f \lessapprox g$ refers to an inequality of the form $f \leqslant C \cdot (\log(1/\delta))^C \cdot g$, where $C > 0$ is an absolute constant, and $\delta > 0$ is a parameter (always a "scale") which will be clear from context. The two-sided inequality $f \lessapprox g \lessapprox f$ is abbreviated to $f \approx g$.

The notation $B(x, r)$ stands for the closed ball of radius $r > 0$ around x. Usually $x \in \mathbb{R}^d$, in which case $B(x, r)$ denotes the usual Euclidean ball. Occasionally, x will belong to another metric space (e.g., the Grassmannian $\mathcal{G}(d, n)$, or the circle S^1). In such cases $B(x, r)$ denotes the metric ball. Sometimes we will write $B(r)$ instead of $B(0, r)$.

Our main result on incidences, Theorem 1.12, has been formulated in terms of (δ, s, C)-*sets*. We recall (from Definition 1.11) that a bounded set $P \subset X$ in a metric space (X, d) is called a (δ, s, C)-set if

$$|P \cap B(x, r)|_\delta \leqslant C \cdot |P|_\delta \cdot r^s, \qquad x \in X,\ \delta \leqslant r \leqslant 1. \tag{2.1}$$

If the value of the constant $C > 0$ is irrelevant, we may also talk casually about (δ, s)-sets. For more information about basic properties of (δ, s)-sets, see [33, Section 2.1]. Our notion of (δ, s)-sets is not entirely canonical: an alternative common definition is where (2.1) is replaced by $|P \cap B(x, r)|_\delta \leqslant (r/\delta)^s$. The definitions coincide when $|P|_\delta \sim \delta^{-s}$. One difference between the definitions is worth noting: our definition implies that if P is a non-empty (δ, s, C)-set, then $|P|_\delta \geqslant \delta^{-s}/C$. This follows from (2.1) applied to any ball $B(x, \delta)$ with $x \in P$. In contrast, the alternative definition $|P \cap B(x, r)|_\delta \leqslant (r/\delta)^s$ rather implies an upper bound $|P|_\delta \leqslant \delta^{-s}$, at least if $\operatorname{diam}(P) \leqslant 1$.

In the paper we will only consider (δ, s)-sets in the Euclidean space $(\mathbb{R}^d, |\cdot|)$, and in the affine Grassmannian $(\mathcal{A}(d, n), d_{\mathcal{A}})$. The metric $d_{\mathcal{A}}$ is defined as in [22, §3.16]: given $V, W \in \mathcal{A}(d, n)$, let $V_0, W_0 \in \mathcal{G}(d, n)$ and $a \in V_0^\perp$, $b \in W_0^\perp$, be the unique n-planes and vectors such that $V = V_0 + a$ and $W = W_0 + b$. The distance between V and W is given by

$$d_{\mathcal{A}}(V, W) := \|\pi_{V_0} - \pi_{W_0}\|_{op} + |a - b|, \tag{2.2}$$

where $\|\cdot\|_{op}$ denotes the operator norm. Note that $\mathcal{G}(d, n)$ can be seen as a submanifold of $\mathcal{A}(d, n)$, and the restriction of $d_{\mathcal{A}}$ to $\mathcal{G}(d, n) \times \mathcal{G}(d, n)$ defines a metric on $\mathcal{G}(d, n)$.

For a set $A \subset \mathbb{R}^d$ and $\delta > 0$, $A(\delta)$ will denote the δ-neighbourhood of A.

3. L^p-regularity of projections

3.1. Background

Let $0 < n < d$, let $\mathcal{G}(d, n)$ be the Grassmannian of n-dimensional subspaces of $\mathbb{R}^d$, and let $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ be the family of compactly supported Radon measures on $\mathbb{R}^d$. In this section we investigate the L^p-regularity of the projections of s-dimensional Frostman measures $\mu \in \mathcal{M}$ to planes $V \in \mathcal{G}(d, n)$.

It is classical that if $s > n$, and $\mu \in \mathcal{M}$ satisfies the s-dimensional Frostman condition $\mu(B(x, r)) \lesssim r^s$ for balls $B(x, r) \subset \mathbb{R}^d$, then

$$\int_{\mathcal{G}(d,n)} \|\pi_V \mu\|_2^2 \, d\gamma_{d,n}(V) < \infty,$$

where $\gamma_{d,n}$ is the $\mathcal{O}(d)$-invariant probability measure on $\mathcal{G}(d, n)$. This can be easily deduced from the potential theoretic method due to Kaufman [17] in $\mathbb{R}^2$ and Mattila [23] in higher dimensions, or see [22, Theorem 9.7] for a textbook reference. In fact, a little more is known: if the s-dimensional Riesz energy $I_s(\mu)$ is finite, $s \geqslant n$ (in particular: if $\mu(B(x, r)) \lesssim r^t$ for some $t > s$), then $\gamma_{d,n}$ almost every projection $\pi_V \mu$ lies in the fractional Sobolev space $H^{(s-n)/2}(V) \cong H^{(s-n)/2}(\mathbb{R}^n)$, and

$$\int_{\mathcal{G}(d,n)} \int_V |\widehat{\pi_V \mu}(\xi)|^2 |\xi|^{s-n} \, d\mathcal{H}^n(\xi) \, d\gamma_{d,n}(V) \lesssim I_s(\mu). \tag{3.1}$$

This approach via Fourier transforms was pioneered by Falconer [8], and the estimate (3.1) can be found for example in [24, (5.14)]. By the Sobolev embedding theorem [4, Theorem 6.5], it follows for that $\pi_V \mu$ has a density in $L^{p^\star}$ for $\gamma_{d,n}$ a.e. $V \in \mathcal{G}(d, n)$, with $p^\star := p^\star(n, s) := 2n/(2n - s)$, and indeed

$$\int_{\mathcal{G}(d,n)} \|\pi_V \mu\|_{L^{p^\star(n,s)}(V)}^2 \, d\gamma_{d,n}(V) \lesssim I_s(\mu), \qquad n \leqslant s < 2n. \tag{3.2}$$

For $2n < s < d$, one can even deduce that $\pi_V \mu \in C_c(V)$ for $\gamma_{d,n}$ a.e. $V \in \mathcal{G}(d, n)$, and $V \mapsto \|\pi_V \mu\|_{L^\infty(V)} \in L^2(\mathcal{G}(d, n))$, see the proof of [24, Theorem 5.4(c)] applied to $\pi_V \mu$.

3.2. New results

We do not know how sharp the facts from Section 3.1 are under the hypothesis $I_s(\mu) < \infty$, but they are certainly unsatisfactory under the s-Frostman assumption $\mu(B(x,r)) \lesssim r^s$. To see this, consider the situation in $\mathbb{R}^2$. If $\mu \in \mathcal{M}(\mathbb{R}^2)$ with $\mu(B(x,r)) \lesssim r^t$ for some $1 < t \leqslant 2$, then one may deduce from the "mixed norm estimate" (3.2) that $L \mapsto \|\pi_L \mu\|_{2/(2-s)} \in L^2(\mathcal{G}(2,1))$ for every $s < t$. It is reasonable that the exponent $2/(2-s)$ tends to infinity as $s,t \to 2$, but it is unsatisfactory that the exponent "2" in "$L^2(\mathcal{G}(2,1))$" stays constant. Indeed, for $t = 2$, trivially $\pi_L \mu \in L^\infty$ for **every** $L \in \mathcal{G}(2,1)$, or in other words $L \mapsto \|\pi_L \mu\|_\infty \in L^\infty(\mathcal{G}(2,1))$. Therefore, one would expect that there exists an exponent $p(s) \in [2,\infty)$ such that $p(s) \to \infty$ as $s \to 2$, and $L \mapsto \|\pi_L \mu\|_{p(s)} \in L^{p(s)}(\mathcal{G}(2,1))$ for every s-Frostman measure $\mu \in \mathcal{M}(\mathbb{R}^2)$. This is a special case of the theorem below:

Theorem 3.3. *Let $0 < n < d$, and let $\mu \in \mathcal{M}(\mathbb{R}^d)$ with $\operatorname{spt}\mu \subset B(1)$ satisfying the Frostman condition $\mu(B(x,r)) \leqslant C_F r^s$ for some $C_F \geqslant 1$, $s > n$, and for all balls $B(x,r) \subset \mathbb{R}^d$. Then,*

$$\int_{\mathcal{G}(d,n)} \|\pi_V \mu\|_p^p \, d\gamma_{d,n}(V) \lesssim_{d,p,s} C_F, \qquad 2 \leqslant p < \frac{2d-n-s}{d-s}. \tag{3.4}$$

Remark 3.5. Theorem 3.3 can be viewed as an L^p to L^p-Sobolev estimate for the $(d-n)$-plane transform, and there is plenty of existing literature on this topic. The most relevant reference is the paper [39] by Strichartz. Using complex interpolation between H^1 and L^2, he proves in [39, Theorem 2.2] the following inequality for $f \in \mathcal{S}(\mathbb{R}^d)$:

$$\left(\int_{\mathcal{G}(d,n)} \|\pi_V f\|_{p,(d-n)/q}^q \, d\gamma_{d,n}(V) \right)^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \qquad 1 < p \leqslant 2.$$

Here $1/p + 1/q = 1$. This looks a little like (3.4), with two main differences: (i) we are interested in exponents $p > 2$, and (ii) we want to see the L^p-norm of $\pi_V \mu$ on the left hand side, instead of an L^p-Sobolev norm. The main reason why Strichartz' estimates are restricted to the range $1 < p \leqslant 2$ is that while the $(d-n)$-plane transform maps L^1 to L^1, and even H^1 to H^1, it fails to map L^∞ to L^∞. This would be the desirable right endpoint of interpolation in the range $2 \leqslant p < \infty$. We will (morally) fix the issue by considering a "localised" $(d-n)$-plane transform, which maps L^p to L^p for every $1 \leqslant p \leqslant \infty$: such localised estimates are good enough to yield information about compactly supported measures. The point (ii) is fairly minor: if T is an operator which commutes with fractional Laplacians, such as the $(d-n)$-plane transform, then every estimate of the form $\|Tf\|_{p,\alpha} \leqslant C\|f\|_p$ implies an estimate of the form $\|Tf\|_p \leqslant C\|(-\triangle)^{-\alpha/2} f\|_p$. Eventually, the latter kind of estimate will be applied with $f = \mu$ to reach (3.4).

3.2.1. Fractional Laplacians

The fractional Laplacian operator "$(-\triangle)^s$" already appeared in the discussion above, and will also be used extensively in the arguments below. For a thorough introduction, see [38, Chapter V]. Here we just mention the basic definitions, and the facts we will need. Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions on $\mathbb{R}^d$, and let $f \in \mathcal{S}$. Then also $\hat{f} \in \mathcal{S}$. If $s \in \mathbb{C}$ with $\operatorname{Re} s > -d/2$, the function

$$\xi \mapsto (2\pi|\xi|)^{2s} \hat{f}(\xi) \tag{3.6}$$

is locally integrable, and has polynomial growth, so in particular it defines a tempered distribution. Here $r^{u+iv} = r^u r^{iv}$ for $r \geqslant 0$. By definition, $(-\triangle)^s f$ is the tempered distribution whose Fourier transform is the function defined in (3.6). Thus,

$$\widehat{(-\triangle)^s f} = (2\pi|\cdot|)^{2s} \hat{f}, \qquad f \in \mathcal{S}.$$

For $\operatorname{Re} s \geqslant 0$, clearly $(2\pi|\cdot|)^{2s}\hat{f} \in L^1 \cap L^2$ for $f \in \mathcal{S}$, so $(-\triangle)^s f$ is represented by a continuous L^2-function by Plancherel and the Fourier inversion theorem. For $s \in (0, d)$ and $f \in \mathcal{S}$, we will need to know that $(-\triangle)^{-s/2} f$ is the function represented by the *Riesz potential*

$$V_s(f)(x) = c_s \int \frac{f(y)\, dy}{|x-y|^{d-s}}, \qquad x \in \mathbb{R}^d. \tag{3.7}$$

Here $c_s = \pi^{d/2}\Gamma(s/2)/\Gamma((d-s)/2) > 0$. This follows from [38, Chapter V, Lemma 2]. The function $V_s(f)$ is continuous if $f \in \mathcal{S}$ and $s \in (0, d)$.

Finally, we will need the following fact about $(-\triangle)^{iv} f$ for $v \in \mathbb{R}$:

$$\|(-\triangle)^{iv} f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,v} \|f\|_{L^p(\mathbb{R}^d)}, \qquad f \in \mathcal{S}(\mathbb{R}^d),\ 1 < p < \infty, \tag{3.8}$$

where $C_{p,v} \geqslant 1$ grows polynomially in $|v|$ (for $p \in (1, \infty)$ fixed). In fact, $f \mapsto (-\triangle)^{iv} f$ is a *Calderón-Zygmund operator*. This follows from the Hörmander-Mihlin multiplier theorem, see [10, Theorem 5.2.7 + Example 5.2.9]. In particular, $(-\triangle)^{iv} f \in L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$, when $f \in \mathcal{S}$, and $v \in \mathbb{R}$.

3.3. Proof of Theorem 3.3

We then turn to the details of Theorem 3.3. It will be convenient to parametrise the projections $\pi_V \mu$ as follows. Let $\mathcal{O}(d)$ be the orthogonal group, and let $\pi_0(x_1, \ldots, x_d) := (x_1, \ldots, x_n)$ be the projection to the n first coordinates. Note that

$$\pi_0^*(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^d, \qquad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

For a complex Borel measure μ on $\mathbb{R}^d$, and $\mathfrak{g} \in \mathcal{O}(d)$, we define

$$\pi_{\mathfrak{g}}\mu := \pi_0(\mathfrak{g}^*\mu),$$

where $\mathfrak{g}^*$ is the adjoint of $\mathfrak{g}$ (or the inverse, since $\mathfrak{g}^* = \mathfrak{g}^{-1}$ for $\mathfrak{g} \in \mathcal{O}(d)$). Of course the definition $\pi_{\mathfrak{g}}\mu$ above also extends to functions $f \in L^1(\mathbb{R}^d)$, and then $\pi_{\mathfrak{g}} f \in L^1(\mathbb{R}^n)$. We record the following useful formula for the Fourier transforms:

$$\widehat{\pi_{\mathfrak{g}}\mu}(\xi) = \hat{\mu}(\mathfrak{g}\pi_0^*(\xi)) =: \hat{\mu}(\mathfrak{g}\xi), \qquad \xi \in \mathbb{R}^n,\ \mathfrak{g} \in \mathcal{O}(d). \tag{3.9}$$

The second equation means that we have identified $\xi \in \mathbb{R}^n$ and $\pi_0^*(\xi) \in \mathbb{R}^d$, and we will use this abbreviation in the sequel.

It is very well-known that if $f \in \mathcal{S}(\mathbb{R}^d)$, then the projections $\pi_{\mathfrak{g}} f$ lie (quantitatively) in a certain homogeneous L^2-Sobolev space for almost every $\mathfrak{g} \in \mathcal{O}(d)$. In fact:

$$\int_{\mathcal{O}(d)} \|\pi_{\mathfrak{g}} f\|_{2,(d-n)/2} \, d\mathfrak{g} \lesssim \|f\|_2, \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{3.10}$$

This formula is essentially based on the Plancherel formula and the identity

$$\int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} |x|^{d-n} f(\mathfrak{g}x) \, dx \, d\mathfrak{g} = c(d,n) \int_{\mathbb{R}^d} f(x) \, dx, \qquad f \in L^1(\mathbb{R}^d), \tag{3.11}$$

see [24, (24.2)]. We will need a slight variant of (3.10), so we include the full details below:

Lemma 3.12. *Let* $0 < n < d$, $\psi \in C_c^\infty(\mathbb{R}^d)$, $z \in \mathbb{C}$ *with* $\operatorname{Re} z \in [0,1]$, *and let* T_z *be the operator*

$$T_z f(\mathfrak{g}, x) := \pi_{\mathfrak{g}}(\psi(-\triangle)^{(1-z)(d-n)/4} f)(x), \qquad (\mathfrak{g}, x) \in \mathcal{O}(d) \times \mathbb{R}^n, \tag{3.13}$$

defined for $f \in \mathcal{S}(\mathbb{R}^d)$, *and taking values in measurable functions on* $\mathcal{O}(d) \times \mathbb{R}^n$. *Then,*

$$\|T_z f\|_{L^2(\mathcal{O}(d)\times\mathbb{R}^n)} \lesssim_{\psi,d,n} \|f\|_{L^2(\mathbb{R}^d)}, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

with bounds independent of $\operatorname{Re} z \in [0,1]$.

Proof. Fix $f \in \mathcal{S}(\mathbb{R}^d)$. Clearly $\psi(-\triangle)^{(1-z)(d-n)/4} f \in C_c(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, so the Fourier transform formula (3.9) is available. We write $h_z(\xi) := (2\pi|\xi|)^{(1-z)(d-n)/2}$ for the symbol of $(-\triangle)^{(1-z)(d-n)/4}$, and we abbreviate $\varphi := \hat{\psi}$. Then,

$$\widehat{T_z f}(\mathfrak{g}, \xi) = (\varphi * (h_z \hat{f}))(\mathfrak{g}\xi), \qquad \xi \in \mathbb{R}^n,\ \mathfrak{g} \in \mathcal{O}(d),$$

where $\widehat{T_z f}(\mathfrak{g}, \xi)$ is the Fourier transform of $x \mapsto T_z f(\mathfrak{g}, x) \in L^1(\mathbb{R}^n)$. With this formula in hand, we may apply the Plancherel identity for every fixed $\mathfrak{g} \in \mathcal{O}(d)$:

$$\|T_z f\|^2_{L^2(\mathcal{O}(d)\times\mathbb{R}^n)} = \int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} |(\varphi * (h_z \hat{f}))(\mathfrak{g}\xi)|^2 \, d\xi \, d\mathfrak{g}. \tag{3.14}$$

Next, we claim that if $f \in L^2(\mathbb{R}^d)$ is arbitrary (and not only $f \in \mathcal{S}(\mathbb{R}^d)$), then $\xi \mapsto (\varphi * (h_z \hat{f}))(\mathfrak{g}\xi) \in L^2(\mathbb{R}^n)$ for almost every $\mathfrak{g} \in \mathcal{O}(d)$, and in fact

$$\int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} |(\varphi * (h_z \hat{f}))(\mathfrak{g}\xi)|^2 \, d\xi \, d\mathfrak{g} \lesssim_{d,n,\psi} \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{3.15}$$

This follows from the next computation:

$$\begin{aligned}
\int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} |(\varphi * (h_z \hat{f}))(\mathfrak{g}\xi)|^2 \, d\xi \, d\mathfrak{g} &\lesssim_\psi \int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} (|\varphi| * |h_z \hat{f}|^2)(\mathfrak{g}\xi) \, d\xi \, d\mathfrak{g} \\
&= \int_{\mathbb{R}^d} |\varphi(y)| \int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} (2\pi|\mathfrak{g}\xi - y|)^{(1-\operatorname{Re} z)(d-n)} |\hat{f}(\mathfrak{g}\xi - y)|^2 \, d\xi \, d\mathfrak{g} \, dy \\
&\overset{(3.11)}{\sim_{d,n}} \int_{\mathbb{R}^d} |\varphi(y)| \int_{\mathbb{R}^d} |\xi|^{n-d} |\xi - y|^{(1-\operatorname{Re} z)(d-n)} |\hat{f}(\xi - y)|^2 \, d\xi \, dy \\
&\overset{\xi \mapsto x+y}{=} \int_{\mathbb{R}^d} |\varphi(y)| \int_{\mathbb{R}^d} |x+y|^{n-d} |x|^{(1-\operatorname{Re} z)(d-n)} |\hat{f}(x)|^2 \, dx \, dy \\
&= \int_{\mathbb{R}^d} |\hat{f}(x)|^2 |x|^{(1-\operatorname{Re} z)(d-n)} \int_{\mathbb{R}^d} |\varphi(y)||x+y|^{n-d} \, dy \, dx \lesssim \int_{\mathbb{R}^d} |\hat{f}(x)|^2 dx.
\end{aligned}$$

The final inequality follows from the estimates $(1 - \operatorname{Re} z)(d-n) \leqslant d - n$ and

$$\int_{\mathbb{R}^d} |\varphi(y)||x+y|^{n-d} \, dy \lesssim_\psi |x|^{n-d},$$

using the rapid decay of $\varphi = \hat{\psi}$, and recalling that $n < d$. In particular, a combination of (3.14)-(3.15) for $f \in \mathcal{S}(\mathbb{R}^d)$ completes the proof of the lemma. □

By Lemma 3.12, and the density of $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, the operators

$$T_z \colon (\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{L^2(\mathbb{R}^d)}) \to L^2(\mathcal{O}(d) \times \mathbb{R}^n), \qquad \operatorname{Re} z \in [0,1],$$

have unique extensions to operators $L^2(\mathbb{R}^d) \to L^2(\mathcal{O}(d) \times \mathbb{R}^n)$. We keep denoting these operators with the same symbol T_z. We record that the extensions continue to have the following concrete representation: if $\operatorname{Re} z \in [0,1]$, $f \in L^2(\mathbb{R}^d)$, and $G \in L^2(\mathcal{O}(d) \times \mathbb{R}^n)$, then

$$\int_{\mathcal{O}(d)\times\mathbb{R}^n} (T_z f)(\mathfrak{g},x)G(\mathfrak{g},x)\,dx\,d\mathfrak{g} = \int_{\mathcal{O}(d)}\int_{\mathbb{R}^n} (\varphi * h_z\hat{f})(\mathfrak{g}\xi)\widehat{G}(\mathfrak{g},\xi)\,d\xi\,d\mathfrak{g}. \tag{3.16}$$

Indeed, by the definition of the "abstract" extension $T_z\colon L^2(\mathbb{R}^d) \to L^2(\mathcal{O}(d)\times\mathbb{R}^n)$, if $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ is a sequence of Schwartz functions converging to f in $L^2(\mathbb{R}^d)$, then

$$\begin{aligned}\int_{\mathcal{O}(d)\times\mathbb{R}^n} (T_z f)(\mathfrak{g},x)G(\mathfrak{g},x)\,dx\,d\mathfrak{g} &= \lim_{j\to\infty}\int_{\mathcal{O}(d)}\int_{\mathbb{R}^n} (T_z f_j)(\mathfrak{g},x)G(\mathfrak{g},x)\,dx\,d\mathfrak{g}\\ &= \lim_{j\to\infty}\int_{\mathcal{O}(d)}\int_{\mathbb{R}^n} (\varphi * h_z\widehat{f_j})(\mathfrak{g}\xi)\widehat{G}(\mathfrak{g},\xi)\,d\xi\,d\mathfrak{g},\end{aligned}$$

where the final equation is due to Plancherel (for those a.e. $\mathfrak{g} \in \mathcal{O}(d)$ such that $G(\mathfrak{g},\cdot) \in L^2(\mathbb{R}^n)$). But then we may apply the inequality (3.15) to the differences $f - f_j \in L^2(\mathbb{R}^d)$ to conclude that the limit on the right equals the right hand side of (3.16).

Using the representation (3.16), it is not difficult to check (using Morera's theorem) that the family $\{T_z\}_{\operatorname{Re} z\in[0,1]}$ is analytic in the usual sense that

$$z \mapsto F_{f,G}(z) := \int_{\mathcal{O}(d)\times\mathbb{R}^n} T_z(f)(\mathfrak{g},x)G(\mathfrak{g},x)\,dx\,d\mathfrak{g} = \int_{\mathcal{O}(d)}\int_{\mathbb{R}^n} (\varphi * h_z\hat{f})(\mathfrak{g}\xi)\widehat{G}(\mathfrak{g},\xi)\,d\xi\,d\mathfrak{g}$$

is analytic for $\operatorname{Re} z \in (0,1)$, and continuous for $\operatorname{Re} z \in [0,1]$, for all simple functions $f\colon \mathbb{R}^d \to \mathbb{C}$ and $G\colon \mathcal{O}(d)\times\mathbb{R}^n \to \mathbb{C}$ (continuity follows from dominated convergence, which is justified by repeating the estimates below (3.15)). The map $F_{f,G}$ is also bounded for $\operatorname{Re} z \in [0,1]$, as a consequence of the uniform $L^2(\mathbb{R}^d) \to L^2(\mathcal{O}(d)\times\mathbb{R}^n)$-boundedness of the operators T_z. These are the hypotheses needed to apply Stein's interpolation theorem [37], or see [10, Theorem 1.3.7] for a textbook reference. The details are contained in the next proposition.

Proposition 3.17. *Let $0 < n < d$, $2 \leqslant p < \infty$, and $(p-2)/p < \theta \leqslant 1$. Then, the operator T_θ has a bounded extension to $L^p(\mathbb{R}^d)$. More precisely, if $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, then $\|T_\theta f\|_{L^p(\mathcal{O}(d)\times\mathbb{R}^n)} \lesssim_{p,\theta} \|f\|_{L^p(\mathbb{R}^d)}$.*

Proof. Fix $2 \leqslant p < \infty$ and $(p-2)/p < \theta \leqslant 1$. Then, define $p_\infty \in [p,\infty)$ as the solution to

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_\infty}.$$

Note that if p and θ are related as above, then $\theta = (p_\infty/p)\cdot(p-2)/(p_\infty - 2)$, and this expression takes all values on the interval $((p-2)/p, 1]$ as p_∞ ranges in $[p,\infty)$.

We write $\overline{T}_z := e^{z^2}\cdot T_z$. Since $z \mapsto e^{z^2}$ is a bounded analytic function on $\operatorname{Re} z \in [0,1]$, the operators $\overline{T}_z$ have all the good properties of the operators T_z, but this (standard) trick helps to establish the following: the operators $\overline{T}_{1+ir}$ are uniformly bounded $L^2(\mathbb{R}^d)\cap L^{p_\infty}(\mathbb{R}^d) \to L^{p_\infty}(\mathcal{O}(d)\times\mathbb{R}^n)$ for $r\in\mathbb{R}$. We first verify this for Schwartz functions, so fix $f\in\mathcal{S}(\mathbb{R}^d)$. Then we have the explicit expression (3.13) for the operators T_{1+ir}, which allows us to estimate as follows:

$$\begin{aligned}\|\overline{T}_{1+ir}f\|^{p_\infty}_{L^{p_\infty}(\mathcal{O}(d)\times\mathbb{R}^n)} &\leqslant e^{(1-r^2)p_\infty}\int\limits_{\mathcal{O}(d)} \|\pi_{\mathfrak{g}}(\psi(-\triangle)^{-ir(d-n)/4}f)\|^{p_\infty}_{L^{p_\infty}(\mathbb{R}^n)}\,d\mathfrak{g}\\ &\lesssim_\psi e^{(1-r^2)p_\infty}\|(-\triangle)^{-ir(d-n)/4}f\|^{p_\infty}_{L^{p_\infty}(\mathbb{R}^d)}\\ &\lesssim_{p_\infty} \mathrm{poly}(|r|)\cdot e^{-r^2p_\infty}\|f\|^{p_\infty}_{L^{p_\infty}(\mathbb{R}^d)} \lesssim \|f\|^{p_\infty}_{L^{p_\infty}(\mathbb{R}^d)}.\end{aligned}$$

The "localisation" by the fixed bump function $\psi\in C_c^\infty(\mathbb{R}^d)$ was crucial to pass from the first line to the second: the maps $f\mapsto \pi_{\mathfrak{g}}f$ are not bounded $L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^n)$ for any $p>1$, but the maps $f\mapsto\pi_{\mathfrak{g}}(\psi f)$ are bounded on all L^p-spaces by an application of Hölder's inequality. As another remark, the "poly($|r|$)" factor reflects the $L^{p_\infty}(\mathbb{R}^d)\to L^{p_\infty}(\mathbb{R}^d)$ boundedness of the imaginary fractional Laplacian $(-\triangle)^{-ir(d-n)/4}$, recall (3.8). The mitigation of this factor was the only reason to introduce the factor e^{z^2}.

It remains to argue that the same estimate holds for $f\in L^2(\mathbb{R}^d)\cap L^{p_\infty}(\mathbb{R}^d)$. Pick a sequence $\{f_i\}_{i\in\mathbb{N}}\subset\mathcal{S}(\mathbb{R}^d)$ which converges to f in both $L^2(\mathbb{R}^d)$ and $L^{p_\infty}(\mathbb{R}^d)$. Then, for $r\in\mathbb{R}$, the functions $T_{1+ir}(f_i)$ converge to $T_{1+ir}(f)$ in $L^2(\mathcal{O}(d)\times\mathbb{R}^n)$, so after passing to a subsequence, we may assume that $T_{1+ir}(f_i)\to T_{1+ir}(f)$ almost everywhere. Then, by Fatou's lemma,

$$\begin{aligned}&\int\limits_{\mathcal{O}(d)}\int\limits_{\mathbb{R}^n}|(T_{1+ir}f)(\mathfrak{g},x)|^{p_\infty}\,dx\,d\mathfrak{g}\\ &\quad\leqslant \liminf_{i\to\infty}\|T_{1+ir}(f_i)\|^{p_\infty}_{L^{p_\infty}(\mathcal{O}(d)\times\mathbb{R}^n)} \lesssim_{p_\infty} \liminf_{i\to\infty}\|f_i\|^{p_\infty}_{L^{p_\infty}(\mathbb{R}^d)} = \|f\|^{p_\infty}_{L^{p_\infty}(\mathbb{R}^d)}.\end{aligned}\tag{3.18}$$

Hence $T_{1+ir}f\in L^{p_\infty}(\mathcal{O}(d)\times\mathbb{R}^n)$, and $\|T_{1+ir}f\|_{L^{p_\infty}(\mathcal{O}(d)\times\mathbb{R}^n)}\lesssim_{p_\infty}\|f\|_{L^{p_\infty}(\mathbb{R}^d)}$.

We have now verified all the hypotheses of Stein's interpolation theorem, as stated in [10, Theorem 1.3.7], for the operator family $\{\overline{T}_z\}_{\mathrm{Re}\,z\in[0,1]}$. The conclusion is that

$$\|T_\theta f\|_{L^p(\mathcal{O}(d)\times\mathbb{R}^n)}\leqslant\|\overline{T}_\theta f\|_{L^p(\mathcal{O}(d)\times\mathbb{R}^n)}\lesssim_{p_\infty}\|f\|_{L^p(\mathbb{R}^d)}$$

for all simple functions f on $\mathbb{R}^d$. Since the choice of p_∞ only depends on p,θ, the notation $\lesssim_{p_\infty}$ is equivalent to $\lesssim_{p,\theta}$. The extension of the bound above for $f\in L^2(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ follows as in (3.18), so the proof of the proposition is complete. □

We are then ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let $n<s\leqslant d$, and let $\mu\in\mathcal{M}(\mathbb{R}^d)$ satisfy the assumptions of the theorem: $\operatorname{spt}\mu\subset B(1)$ and $\mu(B(x,r))\leqslant C_Fr^s$ for some constant $C_F>0$, and for all balls $B(x,r) \subset \mathbb{R}^2$. We assume[1] in addition (qualitatively) that $\mu \in C^\infty(\mathbb{R}^d)$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a function satisfying $\mathbf{1}_{B(1)} \leqslant \psi \leqslant \mathbf{1}_{B(2)}$, so $\mu = \psi\mu$. We abbreviate $\varphi := \hat{\psi} \in \mathcal{S}(\mathbb{R}^d)$.

Now, fix $2 \leqslant p < (2d-n-s)/(d-s)$ and $\epsilon \in (0,1)$, where ϵ is chosen sufficiently small so to satisfy the hypotheses of Proposition 3.20 below (it will then only depend on d,p,s, as per Proposition 3.20). Then set

$$\alpha := (1-\epsilon)\frac{d-n}{p} < \frac{d-n}{p}.$$

The rationale for this choice of "α" will be that if "θ" solves $(1-\theta)(d-n)/2 = \alpha$, then

$$\theta = \frac{p-2}{p} + \frac{2\epsilon}{p} \quad \Longrightarrow \quad \frac{p-2}{p} < \theta < 1, \tag{3.19}$$

and Proposition 3.17 will be applicable with this "θ". Note also that $(2\pi|\xi|)^\alpha = h_\theta(\xi)$ with the notation used in formula (3.16).

Let $q \geqslant 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Fixing also a simple function $G\colon \mathcal{O}(d) \times \mathbb{R}^n \to \mathbb{C}$ with $\|G\|_{L^q(\mathcal{O}(d)\times\mathbb{R}^n)} \leqslant 1$, we write

$$\begin{aligned}
\int\limits_{\mathcal{O}(d)} \int\limits_{\mathbb{R}^n} (\pi_{\mathfrak{g}}\mu)(x) G(\mathfrak{g},x)\, dx\, d\mathfrak{g} &= \int\limits_{\mathcal{O}(d)} \int\limits_{\mathbb{R}^n} (\pi_{\mathfrak{g}}(\psi\mu))(x) G(\mathfrak{g},x)\, dx\, d\mathfrak{g} \\
&= \int\limits_{\mathcal{O}(d)} \int\limits_{\mathbb{R}^n} (\varphi * \hat{\mu})(\mathfrak{g}\xi)\widehat{G}(\mathfrak{g},\xi)\, d\xi\, d\mathfrak{g} \\
&= \int\limits_{\mathcal{O}(d)} \int\limits_{\mathbb{R}^n} (\varphi * h_\theta \widehat{V_\alpha(\mu)})(\mathfrak{g}\xi)\widehat{G}(\mathfrak{g},\xi)\, d\xi\, d\mathfrak{g},
\end{aligned}$$

where

$$V_\alpha(\mu)(x) = (-\triangle)^{-\alpha/2}\mu(x) = c_\alpha \int \frac{\mu(y)\, dy}{|x-y|^{d-\alpha}}, \qquad x \in \mathbb{R}^d,$$

is the Riesz potential of μ with index α, recall (3.7). Note that

$$\alpha = (1-\epsilon)\frac{d-n}{p} \text{ and } p \geqslant 2 \quad \Longrightarrow \quad d - \alpha = \frac{d(p-1+\epsilon) + (1-\epsilon)n}{p} \geqslant \frac{d+n}{2} > \frac{d}{2},$$

so the smoothness and compact support of μ imply $V_\alpha(\mu)(x) \leqslant O((1+|x|)^{-d/2-\kappa})$ for some $\kappa > 0$, assuming that $\epsilon > 0$ in the definition of "α" is chosen sufficiently small. In

[1] That is, we convolve μ with an approximate identity φ_δ, so that the resulting function is $C^\infty(\mathbb{R}^d)$. Obviously, our estimates will not depend on δ. For notation's sake, we will not make this explicit, and we will simply make the qualitative assumption above.

particular, $V_\alpha(\mu) \in L^2(\mathbb{R}^d)$. This permits us to use the representation formula (3.16) for the operator T_θ with the choices $f := V_\alpha(\mu)$ and "θ" as in (3.19):

$$\int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} (\pi_{\mathfrak{g}}\mu)(x) G(\mathfrak{g}, x)\, dx\, d\mathfrak{g} = \int_{\mathcal{O}(d)\times\mathbb{R}^n} T_\theta(V_\alpha(\mu))(\mathfrak{g}, x) G(\mathfrak{g}, x)\, dx\, d\mathfrak{g}.$$

The operator T_θ is bounded $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \to L^p(\mathcal{O}(d) \times \mathbb{R}^n)$ for this "θ" by Proposition 3.17, so we conclude that

$$\left| \int_{\mathcal{O}(d)} \int_{\mathbb{R}^n} (\pi_{\mathfrak{g}}\mu)(x) G(\mathfrak{g}, x)\, dx\, d\mathfrak{g} \right| \lesssim \|V_\alpha(\mu)\|_{L^p(\mathbb{R}^d)} \|G\|_{L^q(\mathcal{O}(d)\times\mathbb{R}^n)} \leqslant \|V_\alpha(\mu)\|_{L^p(\mathbb{R}^d)}.$$

The proof of Theorem 3.3 is now completed by showing that $\|V_\alpha(\mu)\|_{L^p(\mathbb{R}^d)} \lesssim_{d,p,s} C_F$ with the choice $\alpha = (1-\epsilon)(d-n)/p$, if $\epsilon > 0$ small enough, depending on d, p, s. This follows from [29, (3.1)], but that argument is based on interpolation, and we give an elementary proof in Proposition 3.20 for completeness. This concludes the proof of Theorem 3.3. $\square$

Proposition 3.20. *Let $d \geqslant 2$, $n \geqslant 1$, $n < s \leqslant d$, and let $\mu \in \mathcal{M}(\mathbb{R}^d)$ satisfy $\mu(B(x,r)) \leqslant C_F r^s$ for all balls $B(x,r) \subset \mathbb{R}^d$, and $\operatorname{spt}\mu \subset B(1)$. Let $2 \leqslant p < (2d-n-s)/(d-s)$. Then, if $\epsilon \in (0,1)$ is small enough, depending only on d, p, s, and $\alpha := (1-\epsilon)(d-n)/p$, we have*

$$\|V_\alpha(\mu)\|_p \sim_\alpha \left[\int \left(\int \frac{d\mu(y)}{|x-y|^{d-\alpha}} \right)^p dx \right]^{1/p} \lesssim_{d,p,s} C_F. \tag{3.21}$$

Proof. Fix $2 \leqslant p < (2d-s-n)/(d-s)$. Fix also $x \in \mathbb{R}^d$ and $\epsilon > 0$ (whose value will eventually depend on d, p, s), and start by decomposing the inner integral as

$$\begin{aligned} \left(\int \frac{d\mu(y)}{|x-y|^{d-\alpha}} \right)^p &\lesssim \left(\sum_{j \geqslant 0} 2^{j(d-\alpha)} \mu(B(x, 2^{-j+2})) \right)^p \\ &\lesssim_{\epsilon,p} \sum_{j \geqslant 0} 2^{j(dp+\epsilon-\alpha p)} \mu(B(x, 2^{-j+2}))^p. \end{aligned}$$

The second inequality is a consequence of Hölder's inequality with exponent $p > 1$, after introducing artificially the factors $2^{\epsilon j/p}$ and $2^{-\epsilon j/p}$. The choice of $\epsilon > 0$ will eventually just depend on d, p, s, so "$\lesssim_\epsilon$" means the same as "$\lesssim_{d,p,s} 1$". We may restrict to indices $j \geqslant 0$ by the assumption $\operatorname{spt}\mu \subset B(1)$. Plugging the inequality above to the left hand side of (3.21) yields

$$\int \left(\int \frac{d\mu(y)}{|x-y|^{d-\alpha}} \right)^p dx \lesssim_{\epsilon,p} \sum_{j \geqslant 0} 2^{j(dp+\epsilon-\alpha p)} \int \mu(B(x, 2^{-j+2}))^p\, dx. \tag{3.22}$$

To treat the remaining integral, we make the following claim, for $\delta = 2^{-j+2} \in 2^{-\mathbb{N}}$:

$$\int \mu(B(x,\delta))^p \, dx \lesssim_{d,p} C_F^p \cdot \delta^{d-s+ps}. \tag{3.23}$$

To prove (3.23), we decompose μ as follows: for $i \geqslant 0$, let $\mathcal{Q}_i \subset \overline{\mathcal{D}}_\delta(\mathbb{R}^d)$ be the collection of those closed dyadic δ-cubes with the property

$$2^{-i-1} \cdot C_F \delta^s \leqslant \mu(Q) \leqslant 2^{-i} \cdot C_F \delta^s, \qquad Q \in \mathcal{Q}_i.$$

Further, let μ_i be the restriction of μ to $\cup \mathcal{Q}_i$. Clearly $\mu \leqslant \sum_{i \geqslant 0} \mu_i$, and $\mu_i(B(x,\delta)) \lesssim 2^{-i} \cdot C_F \delta^s$ for all $x \in \mathbb{R}^d$. For $\epsilon > 0$ arbitrary, it follows that

$$\begin{aligned}
\int \mu(B(x,\delta))^p \, dx &\leqslant \int \left(\sum_{i \geqslant 0} \mu_i(B(x,\delta)) \right)^p dx \\
&\lesssim_{\epsilon,p} \sum_{i \geqslant 0} 2^{i\epsilon} \int \mu_i(B(x,\delta))^p \, dx \\
&\lesssim C_F^p \cdot \delta^{ps} \cdot \sum_{i \geqslant 0} 2^{i(\epsilon - p)} \cdot \mathcal{H}^d(\{x \in \mathbb{R}^d : B(x,\delta) \cap \operatorname{spt} \mu_i \neq \emptyset\}).
\end{aligned}$$

Recall that $\operatorname{spt} \mu_i$ consists of the union of the cubes $Q \in \overline{\mathcal{D}}_\delta(\mathbb{R}^d)$, which satisfy $\mu(Q) \sim 2^{-i} \cdot C_F \delta^s$. Since $\|\mu\| = \mu(B(1)) \lesssim C_F$, we have $\operatorname{card} \mathcal{Q}_i \lesssim 2^i \cdot \delta^{-s}$, and consequently

$$\mathcal{H}^d(\{x \in \mathbb{R}^d : B(x,\delta) \cap \operatorname{spt} \mu_i \neq \emptyset\}) \lesssim \delta^d \cdot (\operatorname{card} \mathcal{Q}_i) \lesssim 2^i \cdot \delta^{d-s}.$$

Therefore, since $1 + \epsilon - p < 0$ (recall that $p \geqslant 2$), we have

$$\int \mu(B(x,\delta))^p \, dx \lesssim_{\epsilon,p} C_F^p \cdot \delta^{d-s+ps} \cdot \sum_{i \geqslant 0} 2^{i(1+\epsilon-p)} \lesssim_p C_F^p \cdot \delta^{d-s+ps},$$

as claimed in (3.23).

Inserting the inequality (3.23) into (3.22) now yields

$$\int \left(\int \frac{d\mu(y)}{|x-y|^{d-\alpha}} \right)^p dx \lesssim_{d,\epsilon,p} C_F^p \cdot \sum_{j \geqslant 0} 2^{j(dp+\epsilon-\alpha p - d + s - ps)}.$$

The geometric series is summable if and only if $dp + \epsilon - \alpha p - d + s - ps < 0$. Recalling that $\alpha = (1-\epsilon)(d-n)/p$, this amounts to

$$p < \frac{(1-\epsilon)(d-n) + d - s - \epsilon}{d-s}.$$

Since we assumed that $p < (2d-n-s)/(d-s)$, this is true with $\epsilon > 0$ small enough, depending only on d, p, s. $\square$

4. The incidence estimate

In this section we prove Theorem 1.12, which we recall.

Theorem 4.1. *Let $0 < n < d$ and $C, C_F \geqslant 1$. Let $\mathcal{V} \subset \mathcal{A}(d,n)$ be a δ-separated set of n-planes, and let $P \subset B(1) \subset \mathbb{R}^d$ be a δ-separated (δ, t, C_F)-set with $t > d-n$. For $r > 0$ let $\mathcal{I}_r(P, \mathcal{V}) = \{(p, V) \in P \times \mathcal{V} : p \in V(r)\}$. Then, for every $\varepsilon > 0$ we have*

$$|\mathcal{I}_{C\delta}(P,\mathcal{V})| \lesssim_{C,d,\varepsilon,t} \delta^{-\varepsilon} \cdot C_F \cdot |P| \cdot |\mathcal{V}|^{n/(d+n-t)} \cdot \delta^{n(t+1-d)(d-n)/(d+n-t)}. \tag{4.2}$$

Pigeonholing. We start off by finding subfamilies P_1 and $\mathcal{V}_1$ which have a uniform number of incidences. For $V \in \mathcal{A}(d,n)$, set $N_V := |P \cap V(C\delta)|$. Note that since $P \subset B(1)$ is δ-separated, we have $N_V \lesssim \delta^{-d}$ for every $V \in \mathcal{A}(d,n)$. By the pigeonhole principle, there exists a number $N \in \mathbb{N}$ and a subfamily $\mathcal{V}_1 \subset \mathcal{V}$ such that

$$\tfrac{N}{2} \leqslant N_V \leqslant N \text{ for all } V \in \mathcal{V}_1, \quad \text{and} \quad N \cdot |\mathcal{V}_1| \approx |\mathcal{I}_{C\delta}(P,\mathcal{V})|. \tag{4.3}$$

The implicit constants behind the "$\approx$" notation here are allowed to depend on "d". For $p \in P$, set

$$M_p := |\mathcal{V}^p| := |\{V \in \mathcal{V}_1 : p \in V(C\delta)\}|. \tag{4.4}$$

Using the pigeonhole principle once more, we find a number $M \in \mathbb{N}$ and a subfamily $P_1 \subset P$ so that

$$\tfrac{M}{2} \leqslant M_p \leqslant M \text{ for all } p \in P_1, \quad \text{and} \quad M \cdot |P_1| \approx |\mathcal{I}_{C\delta}(P,\mathcal{V})|. \tag{4.5}$$

Lower bounds for radial projections. Later on, we will apply Theorem 1.2 to the following density:

$$\mu(y) := \frac{1}{|P|} \sum_{p \in P} \varphi_\delta(p - y), \qquad y \in \mathbb{R}^d. \tag{4.6}$$

Here $\varphi_\delta = (C\delta)^{-d}\varphi(\cdot/(C\delta)) \in C_c^\infty(\mathbb{R}^d)$ is a non-negative radial function satisfying $\varphi_\delta(x) = (C\delta)^{-d}$ for $x \in B(3C\delta)$, $\operatorname{spt}\varphi_\delta \subset B(4C\delta)$, and $\operatorname{Lip}(\varphi_\delta) \leqslant (C\delta)^{-d-1}$. We will abuse notation and denote by μ also the measure given by the density above. It is easy to check that $\mu(\mathbb{R}^d) \sim 1$, and also it follows from the (δ, t, C_F)-set property of P that μ is a t-Frostman measure with constant $\sim C_F$, i.e. $\mu(B(x,r)) \lesssim C_F r^t$ for all $x \in \mathbb{R}^d$ and $r > 0$.

Now fix $x \in \mathbb{R}^d$. Since μ has continuous density, we may define another continuous density μ_x on $\mathcal{G}(d,n)$ by the following formula:

$$\mu_x(\mathbf{V}) := \int_{x+\mathbf{V}} \mu(y)\, d\mathcal{H}^n(y), \qquad \mathbf{V} \in \mathcal{G}(d,n). \tag{4.7}$$

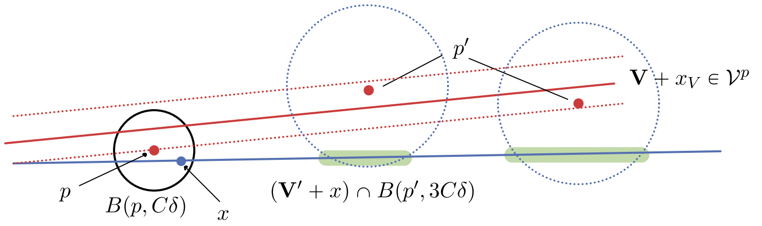

Fig. 1. The proof of Lemma 4.9.

In this section, we will keep the notational convention that affine n-planes are denoted V, V' and n-dimensional subspaces $\mathbf{V}, \mathbf{V}'$. For every $V \in \mathcal{V}^p$, as in (4.4), there exists a unique n-dimensional subspace $\mathbf{V} \in \mathcal{G}(d,n)$ and a point $x_V \in B(p, C\delta)$ so that $V = \mathbf{V} + x_V$. While the subspaces $\mathbf{V} \in \mathcal{G}(d,n)$ obtained in this way need not be δ-separated, it is easy to find $(\delta/2)$-separated subset of cardinality comparable to $|\mathcal{V}^p| \sim M$.

Lemma 4.8. *For every $p \in P_1$, there exists a $(\delta/2)$-separated subset $\mathcal{V}_0^p \subset \{\mathbf{V} : \mathbf{V} + x_V \in \mathcal{V}^p\}$ such that $|\mathcal{V}_0^p| \sim_d |\mathcal{V}^p| \sim M$.*

We leave the details to the reader, and turn to proving a lower bound for the integral of the density "μ" along certain (affine) n-planes:

Lemma 4.9. *Let $x \in P_1(\delta/10)$, so $|x - p| \leqslant \delta/10$ for some $p \in P_1$. Let $\mathbf{V} \in \mathcal{V}_0^p$, and $\mathbf{V}' \in B(\mathbf{V}, \delta/10) \subset \mathcal{G}(d,n)$. Then,*

$$\int_{\mathbf{V}'+x} \mu(y)\, d\mathcal{H}^n(y) \gtrsim_d N \cdot \frac{\delta^{n-d}}{C|P|}. \tag{4.10}$$

Proof. The proof is depicted in Fig. 1. By definition of $\mathbf{V} \in \mathcal{V}_0^p$, there exists a vector $x_V \in B(p, C\delta)$ such that $\mathbf{V} + x_V \in \mathcal{V}^p$. This plane is drawn in red. Since $\mathbf{V} + x_V \in \mathcal{V}^p \subset \mathcal{V}_1$, recall (4.3), the $C\delta$-neighbourhood $(\mathbf{V} + x_V)(C\delta)$ contains a subset $P_V \subset P$ with $|P_V| = N_V \sim N$. Two elements of P_V are drawn in red. The density "μ" then satisfies

$$\mu(y) \gtrsim (C|P|\delta^d)^{-1}, \qquad y \in B(p', 3C\delta),\ p' \in P_V, \tag{4.11}$$

by the definition of μ in (4.6). Finally, if $\mathbf{V}' \in B(\mathbf{V}, \delta/10)$ and $x \in B(p, \delta/10)$ (as in the statement), then the plane $\mathbf{V}' + x$, drawn in blue, remains close to $\mathbf{V} + x_V$ inside $B(1)$: in particular

$$\mathcal{H}^n((\mathbf{V}' + x) \cap B(p', 3C\delta)) \gtrsim_d \delta^n, \qquad p' \in P_V. \tag{4.12}$$

Two of the intersections $(\mathbf{V}' + x) \cap B(p', 3C\delta)$ are drawn in green. Now (4.10) follows by combining (4.11)-(4.12), and recalling that $|P_V| \sim N$. $\square$

Lemma 4.13. *Let $x \in P_1(\delta/10)$, let μ be as in (4.6) and μ_x be as in (4.7). Then, for $q \geqslant 1$,*

$$\|\mu_x\|^q_{L^q(\mathcal{G}(d,n))} \gtrsim_d M \cdot \delta^{n(d-n)} \left(N \cdot \frac{\delta^{n-d}}{C|P|}\right)^q. \tag{4.14}$$

Proof. Fix $x \in P_1(\delta/10)$. By definition,

$$\|\mu_x\|^q_{L^q(\mathcal{G}(d,n))} = \int_{\mathcal{G}(d,n)} \left| \int_{\mathbf{V}'+x} \mu(y)\, d\mathcal{H}^n(y) \right|^q d\gamma_{d,n}(\mathbf{V}'). \tag{4.15}$$

We will use the well-known fact, see [9, Proposition 4.1], that

$$\gamma_{d,n}(B(\mathbf{V},r)) \gtrsim_d r^{n(d-n)}, \qquad \mathbf{V} \in \mathcal{G}(d,n),\ 0 < r \leqslant 1. \tag{4.16}$$

Since $x \in P_1(\delta/10)$, we may find $p \in P_1$ with $|x-p| \leqslant \delta/10$. Recall from Lemma 4.8 that $|\mathcal{V}_0^p| \sim M$, and the subspaces in $\mathcal{V}_0^p$ are $(\delta/2)$-separated, so in particular the balls $B(\mathbf{V},\delta/10)$ with $\mathbf{V} \in \mathcal{V}_0^p$ are disjoint. We may then estimate the right hand side of (4.15):

$$\begin{aligned}(4.15) &\geqslant \sum_{\mathbf{V}\in\mathcal{V}_0^p} \int_{B(\mathbf{V},\delta/10)} \left| \int_{\mathbf{V}'+x} \mu(y)\, d\mathcal{H}^n(y)\right|^q d\gamma_{d,n}(\mathbf{V}') \\ &\overset{(4.10)}{\gtrsim_d} \sum_{\mathbf{V}\in\mathcal{V}_0^p} \gamma_{d,n}(B(\mathbf{V},\tfrac{\delta}{10})) \cdot \left(N \cdot \frac{\delta^{n-d}}{C|P|}\right)^q \overset{(4.16)}{\gtrsim_d} M \cdot \delta^{n(d-n)} \left(N \cdot \frac{\delta^{n-d}}{C|P|}\right)^q.\end{aligned}$$

This proves the lemma. □

Upper bounds for radial projections. During the remainder of the section, we will write V, V' for elements of $\mathcal{G}(d,n)$, since elements of $\mathcal{A}(d,n)$ no longer appear here. The following identity is useful for computing an upper bound for the L^q norm of μ_x. In the planar case, this is essentially [31, Lemma 3.1].

Lemma 4.17. *Let $q \geqslant 1$. With the notation as above,*

$$\int \|\mu_x\|^q_{L^q(\mathcal{G}(d,n))}\, d\mu(x) = \int_{\mathcal{G}(d,n)} \|\pi_{V^\perp}\mu\|^{q+1}_{L^{q+1}(V^\perp)}\, d\gamma_{d,n}(V). \tag{4.18}$$

Proof. Let $V \in \mathcal{G}(d,n)$. Since $\mu \in C_c(\mathbb{R}^d)$, also the push-forward measure $\pi_{V^\perp}\mu$ has a continuous compactly supported density on $V^\perp$, and

$$\mu_x(V) = \int_{x+V} \mu(y)\, d\mathcal{H}^n(y) = (\pi_{V^\perp}\mu)(\pi_{V^\perp}(x)), \qquad x \in \mathbb{R}^d. \tag{4.19}$$

Writing $x = \pi_V(x) + \pi_{V^\perp}(x) = v + v^\perp$ for a fixed plane $V \in \mathcal{G}(d,n)$, and using Fubini's theorem in $\mathbb{R}^d = V \times V^\perp$, we may now compute as follows:

$$\int \|\mu_x\|^q_{L^q(\mathcal{G}(d,n))}\,d\mu(x) \overset{(4.19)}{=} \iint\limits_{\mathcal{G}(d,n)} (\pi_{V^\perp}\mu)(\pi_{V^\perp}(x))^q\,d\gamma_{d,n}(V)\,d\mu(x)$$

$$= \int\limits_{\mathcal{G}(d,n)}\int\limits_{V^\perp}\int\limits_{V} (\pi_{V^\perp}\mu)(v^\perp)^q \mu(v+v^\perp)\,d\mathcal{H}^n(v)\,d\mathcal{H}^{d-n}(v^\perp)\,d\gamma_{d,n}(V)$$

$$= \int\limits_{\mathcal{G}(d,n)}\int\limits_{V^\perp} (\pi_{V^\perp}\mu)(v^\perp)^q \left(\int\limits_{V} \mu(v+v^\perp)\,d\mathcal{H}^n(v)\right)\,d\mathcal{H}^{d-n}(v^\perp)\,d\gamma_{d,n}(V)$$

$$= \int\limits_{\mathcal{G}(d,n)}\int\limits_{V^\perp} (\pi_{V^\perp}\mu)(v^\perp)^{q+1}\,d\mathcal{H}^{d-n}(v^\perp)\,d\gamma_{d,n}(V) = \int\limits_{\mathcal{G}(d,n)} \|\pi_{V^\perp}\mu\|^{q+1}_{L^{q+1}(V^\perp)}\,d\gamma_{d,n}(V).$$

This completes the proof of the lemma. □

We are now ready to prove Theorem 1.12.

Proof of Theorem 1.12. Let $g : \mathcal{G}(d,d-n) \to \mathbb{R}$ be the map $W \mapsto \|\pi_W\mu\|^{q+1}_{L^{q+1}}(W)$, and let $f : \mathcal{G}(d,n) \to \mathcal{G}(d,d-n)$ be the map which sends V to its orthogonal complement $W = V^\perp \in \mathcal{G}(d,d-n)$. Then we can rewrite the right hand side of (4.18) as

$$\int\limits_{\mathcal{G}(d,n)} (g\circ f)(V)\,d\gamma_{d,n}(V) = \int\limits_{\mathcal{G}(d,d-n)} g(W)\,d(f\gamma_{d,n})(W)$$
$$= \int\limits_{\mathcal{G}(d,d-n)} \|\pi_W\mu\|^{q+1}_{L^{q+1}(W)}\,d\gamma_{d,d-n}(W). \tag{4.20}$$

In the last equality we used the fact that $f\gamma_{d,n}$ defines an $\mathcal{O}(d)$-invariant probability measure on $\mathcal{G}(d,d-n)$, so $f\gamma_{d,n} = \gamma_{d,d-n}$ (see [22, (3.10)]).

Recall that the density μ defines a Radon measure satisfying the t-Frostman condition with constant $\sim C_F$, that is, $\mu \in \mathcal{M}_t$ and $\mu(B(x,r)) \lesssim C_F r^t$ for all $x \in \mathbb{R}^d$ and $r > 0$. Hence, from Theorem 1.2 we find that the integral on the right hand side of (4.20) is finite whenever

$$q + 1 < \tfrac{2d-(d-n)-t}{d-t} \quad \iff \quad q < \tfrac{n}{d-t}.$$

Since $\mu(P_1(\delta/10)) \sim_C |P_1|/|P|$, we may compute

$$M\cdot\delta^{n(d-n)}\left(N\cdot\frac{\delta^{n-d}}{C|P|}\right)^q\cdot\frac{|P_1|}{|P|} \overset{(4.14)}{\lesssim_{C,d}} \int\limits_{P_1(\delta/10)} \|\mu_x\|^q_{L^q(\mathcal{G}(d,n))} d\mu(x)$$

$$\overset{(4.18)}{\leqslant} \int_{\mathcal{G}(d,n)} \|\pi_{V^\perp}\mu\|_{L^{q+1}(V^\perp)}^{q+1} d\gamma_{d,n}(V) \overset{\text{Theorem 3.3}}{\lesssim_{d,q,t}} C_F$$

for any $q < n/(d-t)$. Recall from (4.3) and (4.5) that $\frac{|\mathcal{I}_{C\delta}(P,\mathcal{V})|}{|\mathcal{V}_1|} \approx N$ and that $M \approx \frac{|\mathcal{I}_{C\delta}(P,\mathcal{V})|}{|P_1|}$. Hence,

$$\frac{|\mathcal{I}_{C\delta}(P,\mathcal{V})|}{|P|} \cdot \delta^{n(d-n)} \cdot \left(\frac{|\mathcal{I}_{C\delta}(P,\mathcal{V})|}{|\mathcal{V}_1|} \cdot \frac{\delta^{n-d}}{C|P|}\right)^q \approx M \cdot \delta^{n(d-n)} \left(N \cdot \frac{\delta^{n-d}}{C|P|}\right)^q \frac{|P_1|}{|P|} \lesssim_{C,d,q,t} C_F$$

for any $q < n/(d-t)$. If we now rearrange the equation above, and use the obvious inequalities $|\mathcal{V}_1| \leqslant |\mathcal{V}|$ and $C_F^{1/(q+1)} \leqslant C_F$, we obtain

$$|\mathcal{I}_{C\delta}(P,\mathcal{V})| \lessapprox c(C,d,q,t) \cdot C_F \cdot |P| \cdot |\mathcal{V}|^{q/(q+1)} \cdot \delta^{(q-n)(d-n)/(q+1)}.$$

Recall that "$\lessapprox$" hides a factor of the form $C_d \log(\delta^{-1})^{C_d}$ for some dimensional constant C_d. Choosing q close enough to $n/(d-t)$, depending only on ε and C_d, we have

$$C_d \log(\delta^{-1})^{C_d} \delta^{(q-n)(d-n)/(q+1)} \lesssim_{d,\varepsilon,t} \delta^{n(t+1-d)(d-n)/(d+n-t)-\varepsilon}.$$

Thus,

$$|\mathcal{I}_{C\delta}(P,\mathcal{V})| \lesssim_{C,d,\varepsilon,t} \delta^{-\varepsilon} \cdot C_F \cdot |P| \cdot |\mathcal{V}|^{q/(q+1)} \cdot \delta^{n(t+1-d)(d-n)/(d+n-t)}.$$

Finally, note that the factor $|\mathcal{V}|^{q/(q+1)}$ is increasing in q, and so $|\mathcal{V}|^{q/(q+1)} \leqslant |\mathcal{V}|^{n/(d+n-t)}$. Together with the estimate above, this gives (4.2). $\square$

5. Sharpness of the incidence estimate

In this section we construct a family of examples showing that exponent in Theorem 1.12 is sharp in the plane. More precisely, we consider the following family of problems, for each pair of parameters $s \in [0,1]$ and $t \in [1,2]$: let $P \subset [0,1]^2$ be a (δ,t,C)-set with $t > 1$, and for some fixed constant $C > 1$. Assume that $\mathcal{L}_{s,t} \subset \mathcal{A}(2,1)$ is a δ-separated family of lines with the property that every $p \in P$ is δ-incident to at least δ^{-s} lines in $\mathcal{L}_{s,t}$: in other words the collections

$$\mathcal{L}(p) := \mathcal{L}^\delta(p) := \{\ell \in \mathcal{L}_{s,t} : p \in \ell(\delta)\}, \qquad p \in P,$$

satisfy $|\mathcal{L}(p)| \geqslant \delta^{-s}$ for all $p \in P$. How many lines are there in $\mathcal{L}_{s,t}$? Theorem 1.12 yields a lower bound, which (of course!) matches the numerology of Theorem 1.8:

$$|\mathcal{L}_{s,t}| \gtrsim_{C,\epsilon,t} \delta^{-\epsilon} \cdot \delta^{-2s-(1-s)(t-1)}. \tag{5.1}$$

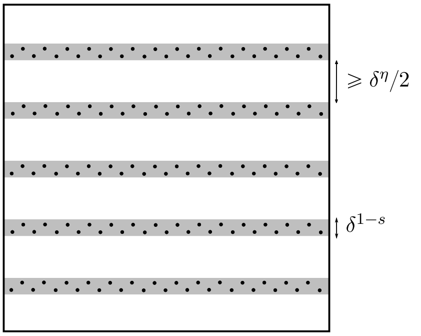

Fig. 2. The construction in Proposition 5.2.

This is not surprising, since Theorem 1.8 is proven by applying Theorem 1.12, see the next section. While it is highly unlikely that Theorem 1.8 is sharp, the lower bound (5.1) is sharp for every $s \in [0,1]$ and $t \in [1,2]$:

Proposition 5.2. *For every $s \in [0,1]$ and $t \in [1,2]$, there exists*

(1) a δ-separated (δ,t)-set $P \subset [0,1]^2$, and
(2) a $c\delta$-separated set $\mathcal{L}_{s,t} \subset \mathcal{A}(2,1)$, where $c > 0$ is an absolute constant, such that

$$|\mathcal{L}_{s,t}| \lesssim \delta^{-2s-(1-s)(t-1)} \quad \text{and} \quad |\mathcal{L}(p)| \gtrsim \delta^{-s} \text{ for all } p \in P.$$

All the implicit constants in Proposition 5.2 are absolute, and the (δ,t)-set P is, more precisely, a (δ,t,C)-set for an absolute constant $C > 0$.

Remark 5.3. How can (5.1) be sharp, while Theorem 1.8 is quite likely not? The reason is simple: in the context of Theorem 1.8, the line family $\mathcal{L}_{s,t}$ has better separation properties than the family $\mathcal{L}_{s,t}$ in Proposition 5.2. More precisely, Theorem 1.8 is roughly equivalent to the following discretised statement: *if $P \subset [0,1]^2$ is a δ-separated (δ,t)-set, and every point $p \in P$ is δ-incident to a (δ,s)-set of lines $\mathcal{L}(p) \subset \mathcal{L}_{s,t}$, then $|\mathcal{L}_{s,t}| \gtrapprox \delta^{-2s-(1-s)(t-1)}$.* Now, the assumption that $\mathcal{L}(p)$ is a (δ,s)-set implies that $|\mathcal{L}(p)| \gtrsim \delta^{-s}$ (as we also assume in Proposition 5.2), but it contains more information on the separation of the lines in $\mathcal{L}(p)$. Proposition 5.2 shows that this information is needed to improve on the bound $2s + (1-s)(t-1)$ in Theorem 1.8, for every $s \in (0,1)$ and $t \in [1,2)$.

We then begin the proof of Proposition 5.2. For brevity of notation, we write

$$\eta = \eta(s,t) = (1-s)(t-1), \qquad s \in [0,1],\ t \in [1,2].$$

Consider $\frac{1}{2}\delta^{-\eta}$ horizontal tubes of width δ^{1-s} and length 1, evenly distributed inside the unit cube (see Fig. 2). We will denote the family of these tubes by $\mathcal{C}$. Note that the sum of widths of tubes in $\mathcal{C}$ is equal to

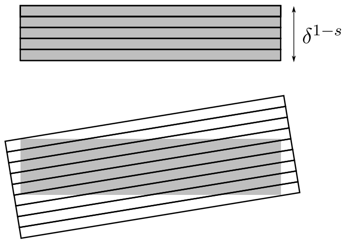

Fig. 3. In the definition of $\mathcal{L}_C$, we choose for every $e \in \Sigma \subset B(e_1, \delta^{1-s})$ a δ-net of lines $\mathcal{L}_C$ intersecting C, with direction e. For $e \in \Sigma$ fixed, there are $\sim \delta^{-s}$ lines in $\mathcal{L}_C$ with direction e. This is trivial if $e = e_1$ (first picture), and takes some easy trigonometry for general $e \in \Sigma$ (second picture).

$$\tfrac{1}{2} \cdot \delta^{1-s-\eta} = \tfrac{1}{2} \cdot \delta^{(2-t)(1-s)} \leqslant \tfrac{1}{2}.$$

Thus, the separation between the tubes is bounded from below by $|\mathcal{C}|^{-1}/2 = \delta^{\eta}/2$. It is also worth pointing out that this separation is at least as large as the width δ^{1-s} of the tubes (up to a constant), since $\delta^{\eta} = \delta^{(1-s)(t-1)} \geqslant \delta^{1-s}$.

Inside each $C \in \mathcal{C}$ we place $\sim \delta^{-t+\eta}$ points, distributed uniformly, see Fig. 2. We denote the sets so obtained P_C, $C \in \mathcal{C}$. With this definition, the points in P_C are (at least) δ-separated, since

$$|P_C|\,\delta^2 = \delta^{-t+\eta+2} \leqslant \mathcal{H}^2(C) = \delta^{1-s},$$

where the inequality follows from the fact that $-t + \eta + 1 + s \geqslant 0$.

Setting $P := \bigcup_{C\in\mathcal{C}} P_C$, we see that $|P| \sim \delta^{-t+\eta} \cdot |\mathcal{C}| \sim \delta^{-t}$. This was just a preliminary observation to convince the reader that P might be a (δ, t)-set, as we will prove a little later. One useful property of P_C, $C \in \mathcal{C}$, is that given a ball B with radius $\delta \leqslant r \leqslant 1$ we have

$$|P_C \cap B| \lesssim \frac{\mathcal{H}^2(C \cap B)}{\mathcal{H}^2(C)}\,|P_C| + 1 \sim \delta^{-t+\eta-1+s}\,\mathcal{H}^2(C \cap B) + 1. \tag{5.4}$$

Before proving that P is a (δ, t)-set, we define the family of lines $\mathcal{L}_{s,t}$, and verify the properties stated in Proposition 5.2(2). First, we define an appropriate set of directions $\Sigma \subset S^1$. Let $e_1 = (1, 0) \in S^1$ and let $\Sigma \subset B(e_1, \delta^{1-s}) \subset S^1$ be a δ-net, so that $|\Sigma| \sim \delta^{-s}$. For every thick horizontal tube $C \in \mathcal{C}$ we define $\mathcal{L}_C$ to be a $c\delta$-net among those lines in $\mathcal{A}(2, 1)$ which have directions in Σ and which intersect C. It follows from elementary geometry that for each fixed direction $e \in \Sigma$ there are $\sim \delta^{-s}$ lines in $\mathcal{L}_C$ with direction e (see Fig. 3). Hence,

$$|\mathcal{L}_C| \lesssim \delta^{-s}|\Sigma| \sim \delta^{-2s}.$$

We then set

$$\mathcal{L}_{s,t} = \bigcup_{C\in\mathcal{C}} \mathcal{L}_C,$$

so that

$$|\mathcal{L}_{s,t}| \leqslant |\mathcal{C}| \cdot |\mathcal{L}_C| \lesssim \delta^{-2s-\eta},$$

as claimed in Proposition 5.2(2).

Observe that for every fixed $e \in \Sigma$ and $p \in P_C$, some line in $\mathcal{L}_C$ with direction e is δ-incident to p. Therefore, $|\mathcal{L}(p)| \gtrsim \delta^{-s}$ for every $p \in P$, as claimed in Proposition 5.2(2).

To complete the proof of Proposition 5.2, it remains to verify that P is a (δ, t)-set.

Lemma 5.5. *For any ball B with radius δ^α, $0 \leqslant \alpha \leqslant 1$, we have*

$$|P \cap B| \lesssim \delta^{\alpha t - t} \sim \delta^{\alpha t} |P|. \tag{5.6}$$

Proof. Let $0 \leqslant \alpha \leqslant 1$, and let B be a ball of radius $r(B) = \delta^\alpha$ that intersects P. There are three cases to consider.

Case $1 - s < \alpha \leqslant 1$. Note that the radius of B is smaller than the width of the tubes in $\mathcal{C}$, so B intersects at most 3 tubes from $\mathcal{C}$. Let $C \in \mathcal{C}$ be one of these tubes. Note that $\mathcal{H}^2(C \cap B) \lesssim \delta^{2\alpha}$, and consequently

$$|P_C \cap B| \overset{(5.4)}{\lesssim} \delta^{-t+\eta-1+s}\, \mathcal{H}^2(C \cap B) + 1 \lesssim \delta^{2\alpha-t+\eta-1+s} + 1.$$

We need to check if the right hand side is bounded by $\delta^{\alpha t - t}$. The bound $1 \leqslant \delta^{\alpha t - t}$ is trivial, since $\alpha \leqslant 1$. So we only need to bound $\delta^{2\alpha-t+\eta-1+s}$. This amounts to verifying that

$$2\alpha + \eta - 1 + s - \alpha t \geqslant 0 \quad \Longleftrightarrow \quad (1 - s - \alpha)(t - 2) \geqslant 0.$$

This is true because we assume $\alpha \geqslant 1 - s$ and $t \leqslant 2$. This shows (5.6) for $1 - s < \alpha \leqslant 1$.

Case $\eta \leqslant \alpha \leqslant 1 - s$. Note that $\eta = (1-s)(t-1) \leqslant 1 - s$, so $[\eta, 1-s] \neq \emptyset$. Recall that the separation between the tubes in $\mathcal{C}$ was at least $\delta^\eta/2$. Since $r(B) \leqslant \delta^\eta$, it follows that B intersects at most 3 tubes from $\mathcal{C}$. Let $C \in \mathcal{C}$ be one of these tubes. Observe that, since the radius of B is larger than the width of C, we have

$$\mathcal{H}^2(C \cap B) \lesssim \delta^\alpha \mathcal{H}^2(C) = \delta^{\alpha+1-s}.$$

Hence,

$$|P_C \cap B| \overset{(5.4)}{\lesssim} \delta^{-t+\eta-1+s}\, \mathcal{H}^2(C \cap B) + 1 \lesssim \delta^{-t+\eta+\alpha} + 1.$$

It is, again, clear that $1 \leqslant \delta^{\alpha t - t}$. So we only need to check that

$$\delta^{-t+\eta+\alpha} \leqslant \delta^{\alpha t - t} \quad \Longleftrightarrow \quad \eta + \alpha - \alpha t \geqslant 0 \quad \Longleftrightarrow \quad (1 - s - \alpha)(t - 1) \geqslant 0.$$

This is true because $t \geqslant 1$ and $1 - s \geqslant \alpha$.

Case $0 \leqslant \alpha \leqslant \eta$. Note that, in particular, $\alpha \leqslant 1 - s$ holds in this case. Observe that since the tubes in $\mathcal{C}$ are $(\delta^\eta/2)$-separated, B intersects $\lesssim \delta^{\alpha-\eta}$ tubes in $\mathcal{C}$.

As in the previous case, for every tube $C \in \mathcal{C}$ we have $\mathcal{H}^2(C \cap B) \lesssim \delta^{\alpha+1-s}$. Thus,

$$|P \cap B| = \sum_{C \in \mathcal{C}} |P_C \cap B| \overset{(5.4)}{\lesssim} \sum_{C \in \mathcal{C}} \delta^{-t+\eta-1+s} \mathcal{H}^2(C \cap B) + |\{C \in \mathcal{C} : C \cap B \neq \emptyset\}|$$

$$\lesssim \delta^{\alpha-\eta} \delta^{-t+\eta+\alpha} + \delta^{\alpha-\eta} = \delta^{2\alpha-t} + \delta^{\alpha-\eta}. \quad (5.7)$$

Clearly $\delta^{2\alpha-t} \leqslant \delta^{\alpha t-t}$, since $t \leqslant 2$. It remains to show that $\delta^{\alpha-\eta} \leqslant \delta^{\alpha t-t}$. In fact, it even turns out that $\delta^{\alpha-\eta} \leqslant \delta^{2\alpha-t}$, or equivalently $\alpha + \eta \leqslant t$. Since $\alpha \leqslant \eta$ in the current case, we have $\alpha + \eta \leqslant 2\eta$, so it suffices to show that $2\eta \leqslant t$. Recalling once more that $\eta = (1 - s)(t - 1)$, this is equivalent to

$$(2 - t) + 2s(t - 1) \geqslant 0.$$

This is true for every $s \in [0, 1]$ and $t \in [1, 2]$. This completes the proof of (5.6), and hence that of Proposition 5.2. □

6. Application to Furstenberg sets

In this section we prove Theorem 1.8, which states that every $(d-1, s, t)$-Furstenberg set $K \subset \mathbb{R}^d$, with $1 < t \leqslant d$ and $0 < s \leqslant d - 1$ satisfies

$$\dim_{\mathrm{H}} K \geqslant (2s + 2 - d) + \frac{(t - 1)(d - 1 - s)}{d - 1}. \quad (6.1)$$

First, we define δ-discretised Furstenberg sets.

Definition 6.2. We say that $F \subset B(2) \subset \mathbb{R}^d$ is a δ-*discretised* (n, s, t)-*Furstenberg set* if

- there exists a δ-separated (δ, t)-set of n-planes $\mathcal{V} \subset \mathcal{A}(d, n)$,
- $F = \bigcup_{V \in \mathcal{V}} F_V$, where each F_V is a union of δ-balls,
- F_V is a (δ, s)-set contained in $V(2\delta)$.

We will use the following lemma due to Héra, Shmerkin, and Yavicoli [15, Lemma 3.3].

Lemma 6.3. *Suppose that every δ-discretised (n, s, t)-Furstenberg set, $\delta \in (0, 1]$, has Lebesgue measure $\gtrsim \delta^{d-\alpha}$. Then every (n, s, t)-Furstenberg set has Hausdorff dimension at least α.*

The lemma above was proved in [15] only for $n = 1$, but the proof for $1 < n < d$ is virtually the same. Now, to prove Theorem 1.8 it suffices to show that every δ-discretised $(d-1, s, t)$-Furstenberg set F, with $1 < t \leqslant d$ and $0 < s \leqslant d-1$, satisfies

$$\mathcal{H}^d(F) \gtrsim \delta^{d-\alpha}$$

for any $\alpha < \alpha_0 := (2s + 2 - d) + \frac{(t-1)(d-1-s)}{d-1}$. Actually, we will prove a slightly stronger result.

Proposition 6.4. *Assume that $t \in (1, d]$, $s \in (0, d-1]$, and $\mathfrak{c} > 0$. Let $\mathcal{V} \subset \mathcal{A}(d, d-1)$ be a δ-separated (δ, t)-set, with $\delta \in (0, 1]$. For each $V \in \mathcal{V}$ let $F_V \subset V(2\delta) \cap B(2)$ be a union of at least $\mathfrak{c}\delta^{-s}$ disjoint δ-balls. If $F = \bigcup_{V \in \mathcal{V}} F_V$, then for any $\alpha < \alpha_0$*

$$\mathcal{H}^d(F) \gtrsim \delta^{d-\alpha}, \tag{6.5}$$

with implicit constant depending on $\alpha, \mathfrak{c}, d, t$.

Note that compared to the definition of δ-discretised $(d-1, s, t)$-Furstenberg sets, we do not need to assume that F_V is a (δ, s)-set; the cardinality estimate for the number of δ-balls is sufficient. Of course, every δ-discretised $(d-1, s, t)$-Furstenberg set satisfies the assumptions of Proposition 6.4 because our definition of (δ, s)-sets implies the desired cardinality lower bound.

The proof of Proposition 6.4 can be summarized as follows: use point-plane duality and apply Theorem 1.12. We provide the details below.

6.1. Duality

Consider a map $\mathbf{D} : \mathbb{R}^d \to \mathcal{A}(d, d-1)$ given by

$$(x_1, \ldots, x_d) \mapsto \left\{ (y_1, \ldots, y_{d-1}, \sum_{i=1}^{d-1} x_i y_i + x_d) \ : \ (y_1, \ldots, y_{d-1}) \in \mathbb{R}^{d-1} \right\}.$$

The image of $\mathbf{D}$ consists of all the $(d-1)$-planes that do not contain a translate of the vertical line $\{(0, \ldots, 0, y_d) \ : \ y_d \in \mathbb{R}\}$, or equivalently, the $(d-1)$-planes whose orthogonal projection to the horizontal plane $\mathbf{D}(0) = \mathbb{R}^{d-1} \times \{0\}$ is the whole plane.

A direct computation shows that

$$d_{\mathcal{A}}(\mathbf{D}(x), \mathbf{D}(y)) = \left| \frac{(x_1, \ldots, x_{d-1}, -1)}{|(x_1, \ldots, x_{d-1}, -1)|} - \frac{(y_1, \ldots, y_{d-1}, -1)}{|(y_1, \ldots, y_{d-1}, -1)|} \right| + \left| \frac{x_d}{|(x_1, \ldots, x_{d-1}, -1)|} - \frac{y_d}{|(y_1, \ldots, y_{d-1}, -1)|} \right|.$$

Hence, for any given $0 < R < \infty$ the restriction of $\mathbf{D}$ to $B(R)$ is bilipschitz onto its image, with bilipschitz constant depending only on R and d. In particular, $\mathbf{D}$ is injective. Write $\mathbf{D}(0) = \mathbb{R}^{d-1} \times \{0\} =: V_0$, and observe that there exists a dimensional constant $0 < r_d < 1$ such that $B(V_0, r_d) \subset \mathbf{D}(B(1))$.

Consider now the map $\mathbf{D}^* : \operatorname{im} \mathbf{D} \to \mathbb{R}^d$ defined by

$$V = \mathbf{D}(x_1, \ldots, x_d) \mapsto (-x_1, \ldots, -x_{d-1}, x_d).$$

In other words, $\mathbf{D}^*$ is the inverse of $\mathbf{D}$ composed with reflection over the vertical line. The map $\mathbf{D}^*$ was defined this way in order to preserve the incidence relation: for $x \in \mathbb{R}^d$ and $V \in \operatorname{im} \mathbf{D}$, it holds

$$x \in V \quad \Longleftrightarrow \quad \mathbf{D}^*(V) \in \mathbf{D}(x). \tag{6.6}$$

Indeed, $x \in V = \mathbf{D}(y_1, \ldots, y_d)$ is equivalent to $x_d = \sum_{i=1}^{d-1} x_i y_i + y_d$, which is equivalent to $y_d = \sum_{i=1}^{d-1}(-y_i)x_i + x_d$, which is equivalent to $\mathbf{D}^*(V) = (-y_1, \ldots, -y_{d-1}, y_d) \in \mathbf{D}(x)$. Note that the restriction of $\mathbf{D}^*$ to $B(V_0, r_d) \subset \mathbf{D}(B(1))$ is bilipschitz onto its image, by our earlier remarks, and that $\mathbf{D}^*(B(V_0, r_d)) \subset \mathbf{D}^*(\mathbf{D}(B(1))) = B(1)$. We will also need the following quantitative version of (6.6).

Lemma 6.7. *For $x \in B(2)$ and $V \in B(V_0, r_d)$ we have*

$$\frac{\operatorname{dist}(\boldsymbol{D}^*(V), \boldsymbol{D}(x))}{3} \leqslant \operatorname{dist}(x, V) \leqslant 3 \operatorname{dist}(\boldsymbol{D}^*(V), \boldsymbol{D}(x)). \tag{6.8}$$

Proof. Let $p = (p_1, \ldots, p_d) \in B(1)$ be the unique point such that $V = \mathbf{D}(p)$. A direct computation yields

$$\operatorname{dist}(x, \mathbf{D}(p)) = \frac{\left|p_d - x_d + \sum_{i=1}^{d-1} x_i p_i\right|}{|(p_1, \ldots, p_{d-1}, -1)|},$$

and

$$\operatorname{dist}(\mathbf{D}^*(V), \mathbf{D}(x)) = \frac{\left|p_d - x_d + \sum_{i=1}^{d-1} x_i p_i\right|}{|(x_1, \ldots, x_{d-1}, -1)|}.$$

Since $1 \leqslant |(p_1, \ldots, p_{d-1}, -1)| \leqslant 2$ and $1 \leqslant |(x_1, \ldots, x_{d-1}, -1)| \leqslant 3$, (6.8) follows. □

Let $F \subset B(2)$ and $\mathcal{V} \subset \mathcal{A}(d, d-1)$ be as in Proposition 6.4, and let $P \subset F$ be a maximal δ-separated subset of F. Evidently each plane $V \in \mathcal{V}$ intersects $B(3)$, so $\mathcal{V} \subset B(V_0, 7)$. After this observation, a few standard steps allow us to reduce to the case $\mathcal{V} \subset B(V_0, r_d) \subset \mathbf{D}(B(1))$. In particular $\mathbf{D}^*$ is C_d-bilipschitz on $\mathcal{V}$.

We now define

$$\mathcal{V}_D := \mathbf{D}(P) := \{\mathbf{D}(p) : p \in P\} \subset \mathcal{A}(d, d-1) \quad \text{and} \quad P_D := \mathbf{D}^*(\mathcal{V}) \subset B(1). \tag{6.9}$$

Observe that since P is δ-separated, and $P \subset B(2)$, the collection $\mathcal{V}_D$ is $c\delta$-separated for some $c = c_d > 0$, by the local bilipschitz property of $\mathbf{D}$. Also, since $\mathcal{V}$ was assumed to be a δ-separated (δ, t)-set, $P_D \subset B(1)$ is a $c\delta$-separated (δ, t)-set (with explicit and implicit constants depending on "d" only).

6.2. Applying the incidence bound

We wish to apply Theorem 1.12 with $\mathcal{V}_D$ and P_D as above. Recall that

- $\mathcal{V}_D$ is $c\delta$ separated,
- $P_D \subset B(1)$ is a $c\delta$-separated (δ, t)-set.

Moreover, by (6.8) and the assumptions on $\mathcal{V}$ and F, for each $p \in P_D$ there exists a $c\delta$-separated set $\mathcal{V}_D(p) \subset \mathcal{V}_D$ such that $|\mathcal{V}_D(p)| \geqslant \mathfrak{c}\delta^{-s}$, and for each $V \in \mathcal{V}_D(p)$ we have $\operatorname{dist}(p, V) \leqslant 6\delta = (6/c) \cdot c\delta$. This numerology places us in a position to apply Theorem 1.12 at scale $\delta' := c\delta$, with "thickening" constant $C := 6/c \sim_d 1$. To simplify notation, we omit the apostrophe, and write "δ" in place of "δ'".

Proof of Proposition 6.4. Applying Theorem 1.12 to $\mathcal{V}_D$, P_D, and some small $\varepsilon > 0$, we arrive at

$$|\mathcal{I}_{C\delta}(P_D, \mathcal{V}_D)| \lesssim_{d,\varepsilon,t} \delta^{-\varepsilon} \cdot |P_D| \cdot |\mathcal{V}_D|^{(d-1)/(2d-t-1)} \cdot \delta^{(d-1)(t+1-d)/(2d-t-1)}.$$

Noting that each $p \in P_D$ is $C\delta$-incident to the $\geqslant \mathfrak{c}\delta^{-s}$ planes $\mathcal{V}_D(p) \subset \mathcal{V}_D$, we get that

$$\mathfrak{c}\delta^{-s}|P_D| \lesssim_{d,\varepsilon,t} \delta^{-\varepsilon} \cdot |P_D| \cdot |\mathcal{V}_D|^{(d-1)/(2d-t-1)} \cdot \delta^{(d-1)(t+1-d)/(2d-t-1)}.$$

Setting $\varepsilon_0 := \varepsilon(2d-t-1)/(d-1)$ we arrive at

$$|\mathcal{V}_D| \gtrsim_{\mathfrak{c},d,\varepsilon,t} \delta^{-t-1+d-s(2d-t-1)/(d-1)+\varepsilon_0}.$$

Recall from (6.9) that $|P| \geqslant |\mathcal{V}_D|$, where P is a maximal δ-separated subset of F, and F is a union of δ-balls. Hence,

$$\mathcal{H}^d(F) \gtrsim |P| \cdot \delta^d \gtrsim_{\mathfrak{c},d,\varepsilon,t} \delta^{d-t-1+d-s(2d-t-1)/(d-1)+\varepsilon_0}.$$

A simple computation shows that

$$t + 1 - d + \frac{s(2d-t-1)}{d-1} = (2s + 2 - d) + \frac{(t-1)(d-1-s)}{d-1} = \alpha_0,$$

and since we may choose ε arbitratrily small, we get (6.5). $\square$

6.3. Application to the sum-product problem

In this short section, we derive Corollary 1.13 from Proposition 6.4. Recall that Corollary 1.13 claims the following: if $A \subset [1,2]$ is a δ-separated set with $|A| = \delta^{-s}$, $B \subset [1,2]$ is a δ-separated (δ, t, c)-set, $C \subset [1,2]$ is a δ-separated (δ, t', c')-set, and $t + t' > 1$, then for any $\varepsilon > 0$

$$\max\{|A+B|_\delta, |A \cdot C|_\delta\} \gtrsim_{\varepsilon,s,t,t'c,c'} \delta^{-(t+t'-1)(1-s)/2+\varepsilon}|A|. \tag{6.10}$$

Given Proposition 6.4, this follows from a well-known argument of Elekes [6], repeated below. Consider the δ-neighbourhood

$$F := [(A+B) \times (A \cdot C)](\delta) \subset \mathbb{R}^2.$$

Consider also the family of planar lines

$$\mathcal{L} := \{y = cx - bc : b \in B,\, c \in C\}.$$

Thus $\mathcal{L}$ contains $|B|$ lines for every fixed slope $c \in C$, and in total $|\mathcal{L}| = |B| \cdot |C|$. It is not hard to check that $\mathcal{L}$ is a $c_0\delta$-separated $(\delta, t+t', c_1)$-set of lines, where $c_0 > 0$ is absolute, and $c_1 > 0$ only depends on c, c'. To give a few more details, if $(a,b) \mapsto \mathbf{D}(a,b) := \{y = ax + b : x \in \mathbb{R}\}$ is the duality map $\mathbb{R}^2 \to \mathcal{A}(2,1)$, then $\mathcal{L} = \mathbf{D}(\{(c, -bc) : b \in B,\, c \in C\})$. Here $\{(c,-bc) : b \in B,\, c \in C\} \subset \mathbb{R}^2$ is a $(\delta, t+t', c_1')$-set, since it is the image of the $(\delta, t+t', c_1'')$-set $C \times B \subset [1,2]^2$ under $(x,y) \mapsto R(x,y) = (x, -xy)$, which is bilipschitz on $[1,2]^2$.

Now observe that if $\ell = \{(x, cx - bc) : x \in \mathbb{R}\} \in \mathcal{L}$, then ℓ contains the set

$$F_\ell := \{(a+b, ac) : a \in A\} \subset (A+B) \times (A \cdot C) \subset F.$$

The set F_ℓ is an affine copy of A, and it is easy to see that it is δ-separated and satisfies $|F_\ell| = |A| = \delta^{-s}$, for every $\ell \in \mathcal{L}$. Since F contains the union of (the δ-neighbourhoods of) the sets F_ℓ for $\ell \in \mathcal{L}$, it follows from Proposition 6.4 that

$$\delta^2 \cdot |A+B|_\delta \cdot |A \cdot C|_\delta \sim \mathcal{L}^2(F) \gtrsim_{\alpha,s,t,t',c,c'} \delta^{2-\alpha}, \qquad \alpha < 2s + (t+t'-1)(1-s).$$

This yields (6.10), and therefore Corollary 1.13.

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